Cyclic Products of Higher-Genus Szegö Kernels, Modular Tensors, and Polylogarithms

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A wealth of information on multiloop string amplitudes is encoded in fermionic two-point functions known as Szegö kernels. Here we show that cyclic products of any number of Szegö kernels on a Riemann surface of arbitrary genus may be decomposed into linear combinations of modular tensors on moduli space that carry all the dependence on the spin structure δ. The δ-independent coefficients in these combinations carry all the dependence on the marked points and are composed of the integration kernels of higher-genus polylogarithms. We determine the antiholomorphic moduli derivatives of the δ-dependent modular tensors.

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Introduction.—In the Ramond-Neveu-Schwarz (RNS) formulation of superstring theory, space-time supersymmetry is implemented via the Gliozzi-Scherk-Olive projection. On a Riemann surface world sheet of genus h the Gliozzi-Scherk-Olive projection is realized by summing over the 2^{2h} different spin structures of the world sheet fermions, consistently with modular invariance. At genus one, the Riemann relations between Jacobi ϑ functions and properties of modular forms [1,2] provide systematic tools for evaluating these spin structure sums explicitly [3–10]. At higher genus, however, carrying out spin structure sums and thus exposing the simplifications due to space-time supersymmetry presents a significant challenge which is often regarded as a drawback of the Ramond-Neveu-Schwarz formulation for evaluating superstring amplitudes.

In a recent paper, the authors made progress toward solving the problem of spin structure summations for the special case of even spin structures at genus two [11]; see also [7,10,12–15] for earlier work on this subject. It was shown in [11] that all the dependence on the spin structure of the cyclic product of an arbitrary number of world sheet fermion propagators, also known as Szegö kernels, may be reduced to the spin structure dependence of certain modular tensors which are locally holomorphic on Torelli space (the moduli space of Riemann surfaces endowed with a choice of canonical homology basis). Thanks to certain *trilinear relations* between these modular tensors, all spin structure dependence was further reduced to that of the well-known four-point functions. The restriction to genus two stems from the fact that the results of [11], including the existence of the trilinear relations, rely heavily on the fact that every genus-two Riemann surface is hyperelliptic which is generically not the case at higher genus.

In the present Letter, we shall consider cyclic products $C_\delta(\mathbf{z}) = C_\delta(z_1, \ldots, z_n)$ of n Szegö kernels,

$$C_\delta(z_1, \ldots, z_n) = S_\delta(z_1, z_2) S_\delta(z_2, z_3) \cdots S_\delta(z_n, z_1), \quad (1)$$

on a Riemann surface Σ of arbitrary genus h and even spin structure δ (encoding the parity-even part of string amplitudes), and an arbitrary number $n \geq 2$ of points $z_i \in \Sigma$. Generalizations of (1) to open chain products of Szegö kernels may be handled by similar methods and their study is deferred to future work. Throughout, the dependence on the moduli of Σ will be suppressed. The Szegö kernel $S_\delta(y, z)$ is a differential $(\frac{1}{2}, 0)$ form in both y and z which, for even spin structure δ and generic moduli, obeys the chiral Dirac equation [16,17],

$$\partial_{\bar{y}} S_\delta(y, z) = \pi \delta(y, z). \quad (2)$$

As the main result of this Letter, we completely disentangle the dependence of $C_\delta(\mathbf{z})$ on the points $z_i \in \Sigma$ from the dependence on the spin structure δ for arbitrary genus h and multiplicity n. Specifically, $C_\delta(\mathbf{z})$ is decomposed into the following linear combination:

$$C_\delta(\mathbf{z}) = F^{(0)}(\mathbf{z}) + \sum_{r=2}^{n} F^{(r)}_{I_1 \cdots I_r}(\mathbf{z}) C_\delta^{I_1 \cdots I_r}. \quad (3)$$

(i) $C_\delta^{I_1 \cdots I_r}$ are δ-dependent but z_i-independent modular tensors of rank r on Torelli space. (ii) $F^{(r)}_{I_1 \cdots I_r}(\mathbf{z})$ are

δ-independent functions that carry all the dependence of $C_\delta(\mathbf{z})$ on the points z_i. Their explicit form will be derived here and related to the construction of higher-genus polylogarithms of [18]. Therefore, any spin structure sum over $C_\delta(\mathbf{z})$ simplifies to a sum over the z_i-independent modular tensors $C_\delta^{I_1\cdots I_r}$.

We calculate the moduli variations of $C_\delta^{I_1\cdots I_r}$, identify components that are locally holomorphic in moduli, and thereby pave the way for their systematic evaluation for arbitrary genus $h \geq 3$ in future work.

The functions $F^{(r)}_{I_1\cdots I_r}(\mathbf{z})$ provide the natural mathematical setting in terms of which the integrands of higher-genus superstring amplitudes and their low-energy expansions may be organized. As such, they generalize the Parke-Taylor factors familiar at genus zero [19] and the Kronecker-Eisenstein kernels at genus one [8].

Abelian differentials and the Arakelov Green function.—The basic ingredients in our construction are convolutions of Abelian differentials and their complex conjugates as well as the Arakelov Green function to be reviewed below. Let Σ be a compact Riemann surface of genus h without boundary. Its first homology group $H_1(\Sigma,\mathbb{Z})$ supports an intersection pairing $\mathfrak{J}$ for which we choose a canonical basis of cycles $\mathfrak{A}_I$ and $\mathfrak{B}_J$ with $I,J=1,\ldots,h$ with intersection pairing $\mathfrak{J}(\mathfrak{A}_I,\mathfrak{B}_J)=\delta_{IJ}=-\mathfrak{J}(\mathfrak{B}_J,\mathfrak{A}_I)$ and $\mathfrak{J}(\mathfrak{A}_I,\mathfrak{A}_J)=\mathfrak{J}(\mathfrak{B}_I,\mathfrak{B}_J)=0$. A canonical basis of holomorphic Abelian differentials $\boldsymbol{\omega}_I$ is normalized on $\mathfrak{A}_J$ cycles and provides the periods Ω_{IJ} on the $\mathfrak{B}_J$ cycles:

$$\oint_{\mathfrak{A}_J}\boldsymbol{\omega}_I=\delta_{IJ}, \qquad \oint_{\mathfrak{B}_J}\boldsymbol{\omega}_I=\Omega_{IJ}. \tag{4}$$

The period matrix Ω is symmetric $\Omega^t=\Omega$ while its imaginary part $Y=\mathrm{Im}(\Omega)$ is positive definite. The matrices Y and Y^{-1} with components Y_{IJ} and Y^{IJ}, respectively, may be used to raise and lower indices I, J so that, adopting the Einstein summation convention, we denote $\boldsymbol{\omega}^I=Y^{IJ}\boldsymbol{\omega}_J$ and $\bar{\boldsymbol{\omega}}^I=Y^{IJ}\bar{\boldsymbol{\omega}}_J$. In terms of these differentials, and their expression $\boldsymbol{\omega}_I=\omega_I(z)dz$ in local complex coordinates $z,\bar z$, we may define a canonically normalized volume form $\boldsymbol{\kappa}$ on Σ,

$$\boldsymbol{\kappa}=\frac{i}{2h}\boldsymbol{\omega}_I\wedge\bar{\boldsymbol{\omega}}^I=\kappa(z)d^2z, \qquad \int_\Sigma\boldsymbol{\kappa}=1, \tag{5}$$

with coordinate volume form $d^2z=(i/2)dz\wedge d\bar z$. The Arakelov Green function $\mathcal{G}(x,y)=\mathcal{G}(y,x)$ is a single-valued function $\mathcal{G}\colon\Sigma\times\Sigma\to\mathbb{R}$ uniquely defined by [20] [the Dirac δ function is normalized by $\int d^2z\delta(z,y)f(z)=f(y)$],

$$\begin{aligned}&\partial_{\bar x}\partial_x\mathcal{G}(x,y)=-\pi\delta(x,y)+\pi\kappa(x),\\&\int_\Sigma\kappa(x)\mathcal{G}(x,y)=0,\end{aligned} \tag{6}$$

whose explicit construction via the prime form $E(x,y)$ may be found in [21]. Besides, its defining equations, $\mathcal{G}(x,y)$ also satisfies the following useful relations:

$$\begin{aligned}&\partial_{\bar x}\partial_y\mathcal{G}(x,y)=\pi\delta(x,y)-\pi\omega_I(x)\bar\omega^I(y),\\&\partial_x\partial_y\mathcal{G}(x,y)=-\partial_x\partial_y\ln E(x,y)+\pi\omega_I(x)\omega^I(y).\end{aligned} \tag{7}$$

Modular tensors.—Linear transformations with integer coefficients that act on $H_1(\Sigma,\mathbb{Z})$ by preserving the intersection pairing $\mathfrak{J}$ form the modular group $\mathrm{Sp}(2h,\mathbb{Z})$. An element $M\in\mathrm{Sp}(2h,\mathbb{Z})$ transforms the homology cycles $\mathfrak{A}_I$, $\mathfrak{B}_J$ by

$$\begin{pmatrix}\mathfrak{B}\\\mathfrak{A}\end{pmatrix}\to M\begin{pmatrix}\mathfrak{B}\\\mathfrak{A}\end{pmatrix}, \qquad M=\begin{pmatrix}A&B\\C&D\end{pmatrix}. \tag{8}$$

The modular transformation M acts on the period matrix by $\Omega\to(A\Omega+B)(C\Omega+D)^{-1}$ and on the Abelian differentials by its nonlinear $GL(h,\mathbb{C})$ representation:

$$\begin{aligned}&\boldsymbol{\omega}_J\to\boldsymbol{\omega}_{J'}R^{J'}_J, \qquad R=(C\Omega+D)^{-1},\\&\bar{\boldsymbol{\omega}}^I\to Q^I_{I'}\bar{\boldsymbol{\omega}}^{I'}, \qquad Q=C\Omega+D.\end{aligned} \tag{9}$$

Modular tensors $\mathcal{T}$ of arbitrary rank were defined in [22] (see also [23,24]) to transform as follows:

$$\mathcal{T}^{I_1\cdots I_r}{}_{J_1\cdots J_s}\to Q^{I_1}{}_{I'_1}\cdots Q^{I_r}{}_{I'_r}\mathcal{T}^{I'_1\cdots I'_r}{}_{J'_1\cdots J'_s}R^{J'_1}{}_{J_1}\cdots R^{J'_s}{}_{J_s}. \tag{10}$$

While the volume form $\boldsymbol{\kappa}$ and the Arakelov Green function $\mathcal{G}$ are invariant under the full modular group $\mathrm{Sp}(2h,\mathbb{Z})$, the Szegö kernel and its cyclic products transform via $S_\delta(x,y)\to S_{\tilde\delta}(x,y)$ and $C_\delta(\mathbf{z})\to C_{\tilde\delta}(\mathbf{z})$, where the spin structure $\delta=[\delta',\delta'']$ maps to $\tilde\delta=[\tilde\delta',\tilde\delta'']$ with

$$\begin{pmatrix}\tilde\delta''\\\tilde\delta'\end{pmatrix}=\begin{pmatrix}A&-B\\-C&D\end{pmatrix}\begin{pmatrix}\delta''\\\delta'\end{pmatrix}+\frac{1}{2}\mathrm{diag}\begin{pmatrix}AB^t\\CD^t\end{pmatrix}. \tag{11}$$

Accordingly, $S_\delta(x,y)$ and $C_\delta(\mathbf{z})$ are invariant under the congruence subgroup $\Gamma_h(2)=\{M\in\mathrm{Sp}(2h,\mathbb{Z})|M\equiv I_{2h\times2h}(\bmod 2)\}$ that preserves each spin structure. The z_i-independent building blocks $C_\delta^{I_1\cdots I_r}$ of $C_\delta(\mathbf{z})$, to be introduced below, furnish modular tensors of $\mathrm{Sp}(2h,\mathbb{Z})$ for which $\delta\to\tilde\delta$ according to (11),

$$C_\delta^{I_1\cdots I_r}\to Q^{I_1}{}_{I'_1}\cdots Q^{I_r}{}_{I'_r}C_{\tilde\delta}^{I'_1\cdots I'_r}, \tag{12}$$

or modular tensors of $\Gamma_h(2)$ for which $\tilde\delta=\delta$.

Descent procedure for $n=2$, 3.—In this section, we shall introduce a simple and systematic procedure by which $C_\delta(\mathbf{z})$ may be decomposed into a linear combination of modular tensors $C_\delta^{I_1\cdots I_r}$ on Torelli space. As a major simplification of the spin-structure summation in string amplitudes, the coefficients of these tensors containing all the dependence on the points z_i no longer depend on δ. The method is constructive and recursive and will be referred to as the *descent procedure*.

The two-point function, namely the case $n = 2$ of (1), offers the simplest such relation [25]:

$$C_\delta(x,y) = C_\delta^{IJ}\omega_I(x)\omega_J(y) + \partial_x\partial_y\mathcal{G}(x,y). \quad (13)$$

This equation may be deduced by using the Fay identities [2] along with the second relation of (7) resulting in the symmetric modular tensor,

$$C_\delta^{IJ} = -\frac{\partial^I\partial^J\vartheta[\delta](0)}{\vartheta[\delta](0)} - \pi Y^{IJ}, \quad (14)$$

of $\Gamma_h(2)$ with derivatives $\partial^I\vartheta[\delta](0) = (\partial/\partial\zeta_J)\vartheta[\delta](\zeta)|_{\zeta=0}$ in the Jacobian variety $\mathbb{C}^h/(\mathbb{Z}^h + \Omega\mathbb{Z}^h)$ of the genus-h surface [1,2]. We note that the tensorial transformation law $C_\delta^{IJ} \to Q_K^I Q_L^J C_{\tilde\delta}^{KL}$ with Q and $\tilde\delta$ given in (9) and (11), respectively, emerges only upon combining the transformations of the two terms on the right-hand side of (14).

The three-point function offers the lowest-order case solved by using the descent equations. We consider the problem at a generic point in moduli space, where the Szegö kernel satisfies Eq. (2), so that $C_\delta(1,2,3)$ satisfies the following Cauchy-Riemann equations,

$$\begin{aligned}
\bar\partial_1 C_\delta(1,2,3) &= \pi\big(\delta(1,2) - \delta(1,3)\big)C_\delta(2,3),\\
\bar\partial_2 C_\delta(1,2,3) &= \pi\big(\delta(2,3) - \delta(2,1)\big)C_\delta(1,3),\\
\bar\partial_3 C_\delta(1,2,3) &= \pi\big(\delta(3,1) - \delta(3,2)\big)C_\delta(1,2),
\end{aligned} \quad (15)$$

with $\bar\partial_j = \partial_{\bar z_j}$. The descent proceeds by solving the first equation as a function of z_1 with the help of the Arakelov Green function defined by (6). The solution is, however, not unique as the $\bar\partial_1$ operator acting on (1,0) differentials has a nontrivial kernel spanned by the holomorphic Abelian differentials of Σ. Thus, the general solution may be expressed as follows,

$$C_\delta(1,2,3) = \omega_I(1)C_\delta^I(2,3) - \big(\partial_1\mathcal{G}(1,2) - \partial_1\mathcal{G}(1,3)\big)C_\delta(2,3), \quad (16)$$

where $C_\delta^I(2,3)$ is independent of z_1. In view of the modular invariance of $C_\delta(\mathbf{z})$ and $\mathcal{G}(1,a)$, the coefficient $C_\delta^I(2,3)$ transforms as a modular tensor of $\Gamma_h(2)$, i.e., according to (12) with $r = 1$. The descent proceeds by evaluating the Cauchy-Riemann operators $\bar\partial_2$ and $\bar\partial_3$ on (16), using the second equation of (15), the relation $\bar\partial_2 C_\delta(2,3) = \pi\partial_2\delta(2,3)$, and the second relation in (7), and we obtain after some simplifications:

$$\begin{aligned}
\bar\partial_2 C_\delta^I(2,3) &= \pi\delta(2,3)C_\delta^{IJ}\omega_J(3) - \pi\bar\omega^I(2)C_\delta(2,3),\\
\bar\partial_3 C_\delta^I(2,3) &= -\pi\delta(2,3)C_\delta^{IJ}\omega_J(2) + \pi\bar\omega^I(3)C_\delta(2,3).
\end{aligned} \quad (17)$$

To solve the first equation of (17) in z_2 we first decompose $C_\delta(2,3)$ using (13)

$$\begin{aligned}
\bar\partial_2 C_\delta^I(2,3) &= \pi C_\delta^{JK}\big(\delta(2,3)\delta_J^I - \bar\omega^I(2)\omega_J(2)\big)\omega_K(3)\\
&\quad - \pi\bar\omega^I(2)\partial_2\partial_3\mathcal{G}(2,3).
\end{aligned} \quad (18)$$

Both lines of the right-hand side are separately integrable since their respective integrals over Σ vanish. The following convolutions involving Abelian differentials and the Arakelov Green function,

$$\begin{aligned}
\Phi^I{}_J(x) &= \int_\Sigma d^2z\mathcal{G}(x,z)\bar\omega^I(z)\omega_J(z),\\
\mathcal{G}^I(x,y) &= \int_\Sigma d^2z\mathcal{G}(x,z)\bar\omega^I(z)\partial_z\mathcal{G}(z,y),
\end{aligned} \quad (19)$$

solve the corresponding differential equations:

$$\begin{aligned}
\partial_{\bar x}\partial_x\Phi^I{}_J(x) &= \pi\kappa(x)\delta^I{}_J - \pi\bar\omega^I(x)\omega_J(x),\\
\partial_{\bar x}\partial_x\mathcal{G}^I(x,y) &= -\pi\bar\omega^I(x)\partial_x\mathcal{G}(x,y).
\end{aligned} \quad (20)$$

More succinctly, the special combination defined by

$$f^I{}_J(x,y) = \partial_x\Phi^I{}_J(x) - \partial_x\mathcal{G}(x,y)\delta^I{}_J, \quad (21)$$

solves the differential equation,

$$\partial_{\bar x}f^I{}_J(x,y) = \pi\delta^I{}_J\delta(x,y) - \pi\bar\omega^I(x)\omega_J(x), \quad (22)$$

so that the general solution to (18) is given as follows:

$$\begin{aligned}
C_\delta^I(2,3) &= \omega_J(2)C_\delta^{IJ}(3) + f^I{}_J(2,3)C_\delta^{JK}\omega_K(3)\\
&\quad + \partial_2\partial_3\mathcal{G}^I(2,3).
\end{aligned} \quad (23)$$

Finally, we determine the modular tensor $C_\delta^{IJ}(3)$ from the $\bar\partial_3$ derivative of (23) using the second equation in (17), and the following relations:

$$\begin{aligned}
\partial_{\bar y}f^I{}_J(x,y) &= -\pi\delta^I{}_J\big(\delta(x,y) - \omega_K(x)\bar\omega^K(y)\big),\\
\partial_{\bar y}\partial_x\mathcal{G}^I(x,y) &= -\pi f^I{}_J(x,y)\bar\omega^J(y).
\end{aligned} \quad (24)$$

We obtain an integrable differential equation,

$$\bar\partial_3 C_\delta^{IJ}(3) = \pi\big(\bar\omega^I(3)C_\delta^{JK} - \bar\omega^J(3)C_\delta^{IK}\big)\omega_K(3), \quad (25)$$

whose integral may be obtained in terms of Φ in (19),

$$\begin{aligned}
C_\delta^{IJ}(3) &= \omega_K(3)C_\delta^{IJK}\\
&\quad - C_\delta^{JK}\partial_3\Phi^I{}_K(3) + C_\delta^{IK}\partial_3\Phi^J{}_K(3),
\end{aligned} \quad (26)$$

where C_δ^{IJK} is a z_i-independent modular tensor. The relations (16), (23), and (26) give a formula for C_δ^{IJK},

$$C_\delta^{I_1 I_2 I_3} = \left(\prod_{i=1}^{3} \int_\Sigma d^2 z_i \bar{\omega}^{I_i}(z_i) \right) C_\delta(z_1, z_2, z_3), \quad (27)$$

which proves that C_δ^{IJK} inherits the total antisymmetry in I, J, K from total antisymmetry of $C_\delta(x,y,z)$ in x, y, z:

$$C_\delta^{IJK} = C_\delta^{[IJK]}. \quad (28)$$

Eliminating $C_\delta^I(2,3)$ and $C_\delta^{IJ}(3)$ from (16), (23), and (26) expresses $C_\delta(1,2,3)$ in the general form (3), where

$$\begin{aligned} F^{(0)}(\mathbf{z}) &= -\big(\partial_1 \mathcal{G}(1,2) - \partial_1 \mathcal{G}(1,3)\big)\partial_2\partial_3 \mathcal{G}(2,3) \\ &\quad + \omega_I(1)\partial_2\partial_3 \mathcal{G}^I(2,3), \\ F^{(2)}_{JK}(\mathbf{z}) &= \big(\omega_J(1)\omega_I(2) - \omega_I(1)\omega_J(2)\big)\partial_3 \Phi^I{}_K(3) \\ &\quad - \big(\partial_1 \mathcal{G}(1,2) - \partial_1 \mathcal{G}(1,3)\big)\omega_J(2)\omega_K(3) \\ &\quad + \omega_I(1) f^I{}_J(2,3)\omega_K(3), \\ F^{(3)}_{IJK}(\mathbf{z}) &= \omega_I(1)\omega_J(2)\omega_K(3). \end{aligned} \quad (29)$$

Clearly, all spin-structure dependence has been reduced to the modular tensors C_δ^{IJK} and C_δ^{IJ}, while all the dependence on the points z_1, z_2, z_3 enters via the δ-independent single-valued functions Φ, $\mathcal{G}$, and f.

Convolutions and modular tensors.—The descent procedure for higher n necessitates higher-rank generalizations of the tensors $\mathcal{G}^I(x,y)$ and $\Phi^I_J(x)$ in (19) obtained from the following convolutions of Arakelov Green functions and Abelian differentials with $r \geq 2$,

$$\begin{aligned} \Phi^{I_1 \cdots I_r}{}_J(x) &= \int_\Sigma d^2 z \mathcal{G}(x,z)\bar{\omega}^{I_1}(z)\partial_z \Phi^{I_2 \cdots I_r}{}_J(z), \\ \mathcal{G}^{I_1 \cdots I_r}(x,y) &= \int_\Sigma d^2 z \mathcal{G}(x,z)\bar{\omega}^{I_1}(z)\partial_z \mathcal{G}^{I_2 \cdots I_r}(z,y), \end{aligned} \quad (30)$$

which frequently occur in the combination,

$$f^{I_1 \cdots I_r}{}_J(x,y) = \partial_x \Phi^{I_1 \cdots I_r}{}_J(x) - \partial_x \mathcal{G}^{I_1 \cdots I_{r-1}}(x,y)\delta^{I_r}_J. \quad (31)$$

Conversely, $\partial\mathcal{G}$ and $\partial\Phi$ may be obtained as the trace and traceless parts of f. The functions Φ, $\mathcal{G}$, and f transform as modular tensors under $\mathrm{Sp}(2h,\mathbb{Z})$ and furnish the integration kernels in the recent construction of higher-genus polylogarithms [18] (see also [26–28] for different approaches to higher-genus polylogarithms in the mathematics literature). More specifically, the higher-genus polylogarithms in [18] are defined though iterated integrals over a flat connection whose entire dependence on marked points on Σ is expressible in terms of $\omega_I(x)$ and the tensor functions (31). One may recast the convolutions used to define Φ, $\mathcal{G}$, and f given in (30) in terms of recursive differential equations obtained for $r \geq 2$ from the trace or traceless part of

$$\begin{aligned} \partial_{\bar{x}} f^{I_1 \cdots I_r}{}_J(x,y) &= -\pi \bar{\omega}^{I_1}(x) f^{I_2 \cdots I_r}{}_J(x,y), \\ \partial_{\bar{y}} f^{I_1 \cdots I_r}{}_J(x,y) &= \pi \delta^{I_r}_J f^{I_1 \cdots I_{r-1}}{}_K(x,y)\bar{\omega}^K(y). \end{aligned} \quad (32)$$

Based on the tensor functions in this section, the descent procedure introduced for $n \leq 3$ may be extended to arbitrary n; see Appendix B in the Supplemental Material [17] for the case $n = 4$.

Descent procedure for any n.—In this section, we apply the descent procedure to the case of arbitrary $n \geq 3$. Inspection of the results (29) for the special case $n = 3$ (and Appendix B in the Supplemental Material [17] for $n = 4$) shows that both the differential relations between the various intermediate tensor functions $C_\delta^{I_1 \cdots I_j}(j+1,\ldots,n)$ and the recursive decomposition of $C_\delta(1,\ldots,n)$ expose a simple and important pattern that may be extended and proven for arbitrary n. While the differential relations can be found in Appendix C in the Supplemental Material [17], the integrated relations for $j = 1,\ldots,n-2$ are given by

$$\begin{aligned} &C_\delta^{I_1 \cdots I_{j-1}}(j,\ldots,n) \\ &= \omega_J(j) C_\delta^{I_1 \cdots I_{j-1} J}(j+1,\ldots,n) \\ &\quad + \sum_{i=1}^{j-1} f^{I_{j-1} I_{j-2} \cdots I_i}{}_J(j,j+1) C_\delta^{I_1 \cdots I_{i-1} J}(j+1,\ldots,n) \\ &\quad - \partial_j\big(\mathcal{G}^{I_{j-1} \cdots I_1}(j,j+1) - \mathcal{G}^{I_{j-1} \cdots I_1}(j,n)\big) C_\delta(j+1,\ldots,n), \end{aligned} \quad (33)$$

which successively express the dependence of $C_\delta(1,\ldots,n)$ on z_j in terms of $\omega_J(j)$ and the convolutions of the previous section. The cases $j = n$ and $j = n-1$ require separate formulas,

$$\begin{aligned} C_\delta^{I_1 \cdots I_{n-2}}(n-1,n) &= \omega_J(n-1) C_\delta^{I_1 \cdots I_{n-2} J}(n) \\ &\quad + \sum_{i=2}^{n-2} f^{I_{n-2} I_{n-3} \cdots I_i}{}_J(n-1,n) C_\delta^{I_1 I_2 \cdots I_{i-1} J}(n) \\ &\quad + f^{I_{n-2} I_{n-3} \cdots I_1}{}_J(n-1,n) C_\delta^{JK}\omega_K(n) \\ &\quad + \partial_{n-1}\partial_n \mathcal{G}^{I_{n-2} \cdots I_2 I_1}(n-1,n), \end{aligned} \quad (34)$$

as well as

$$\begin{aligned} C_\delta^{I_1 \cdots I_{n-1}}(n) &= \omega_J(n) C_\delta^{I_1 \cdots I_{n-1} J} + \sum_{\substack{1 \leq i \leq j \\ (i,j) \neq (1,n-1)}} (-1)^{i-1} \\ &\quad \times \partial_n \Phi^{I_1 I_2 \cdots I_{i-1} \shuffle I_{n-1} I_{n-2} \cdots I_{j+1}}{}_M(n) C_\delta^{I_i I_{i+1} \cdots I_j M}, \end{aligned} \quad (35)$$

where the shuffle of multi-indices $\vec{I}$, $\vec{J}$ is understood as

$$\Phi^{\vec{I}ш\vec{J}}{}_{M}(n) = \sum_{\vec{K}\in\vec{I}ш\vec{J}} \Phi^{\vec{K}}{}_{M}(n). \tag{36}$$

Combining (33) and (35) determines the z_i-dependent constituents $F^{(r)}_{I_1\cdots I_r}(\mathbf{z})$ of $C_\delta(\mathbf{z})$ in the notation of (3).

We conclude this section with integral representations of the modular tensors $C_\delta^{I_1\cdots I_n}$ in terms of $C_\delta(1,\ldots,n)$,

$$C_\delta^{I_1\cdots I_n} = \left(\prod_{i=1}^{n}\int_\Sigma d^2z_i\bar\omega^{I_i}(z_i)\right)C_\delta(z_1,\ldots,z_n), \tag{37}$$

which generalize (27) to arbitrary n and imply dihedral (anti)symmetry $C_\delta^{I_1I_2\cdots I_n} = C_\delta^{I_2\cdots I_nI_1} = (-1)^nC_\delta^{I_n\cdots I_2I_1}$.

Variations in moduli.—While the modular tensor $C_\delta^{I_1\cdots I_n}$ enters a term in the descent of $C_\delta(z_1,\ldots,z_n)$ that is meromorphic in the points $z_i\in\Sigma$, it is not, in general, meromorphic in moduli. We will study antiholomorphic derivatives in moduli through complex structure variations $\delta_{\bar w\bar w}$ which, in a conformal field theory setting, amount to an insertion of the stress tensor at a point $w\in\Sigma$ [29–32]. The antiholomorphic $\delta_{\bar w\bar w}$ variation of $C_\delta^{I_1\cdots I_n}$ may be constructed via the relations $\delta_{\bar w\bar w}\Omega_{IJ} = \delta_{\bar w\bar w}\omega_I(z) = \delta_{\bar w\bar w}S_\delta(x,y) = 0$,

$$\begin{aligned}\delta_{\bar w\bar w}\bar\Omega_{IJ} &= -2\pi i\bar\omega_I(w)\bar\omega_J(w),\\ \delta_{\bar w\bar w}\bar\omega^I(x) &= -\bar\omega^I(w)\partial_{\bar x}\partial_{\bar w}\mathcal{G}(x,w),\end{aligned} \tag{38}$$

so that we have $\delta_{\bar w\bar w}C_\delta(\mathbf{z}) = 0$. While the variation of the two-point function $\delta_{\bar w\bar w}C_\delta^{IJ} = \pi^2\bar\omega^I(w)\bar\omega^J(w)$ readily follows from (14), the variation $\delta_{\bar w\bar w}C_\delta^{I_1\cdots I_n}$ at $n\geq 3$ may be derived from the integral representation (37),

$$\begin{aligned}\delta_{\bar w\bar w}C_\delta^{I_1\cdots I_n} &= \pi\bar\omega^{I_1}(w)\partial_{\bar w}\left(F_\delta^{I_2|I_3\cdots I_n}(w) - F_\delta^{I_n|I_2\cdots I_{n-1}}(w)\right)\\ &\quad + \mathrm{cycl}(1,\ldots,n),\end{aligned} \tag{39}$$

in terms of w-dependent modular tensors F_δ of $\Gamma_h(2)$:

$$F_\delta^{I_2|I_3\cdots I_n}(w) = \int_\Sigma d^2z_2\mathcal{G}(w,2)\bar\omega^{I_2}(2)C_\delta^{I_3\cdots I_n}(2). \tag{40}$$

Total symmetrization in $n\geq 3$ indices $I_1,\ldots,I_n$ cancels the right-hand side of (39), implying that the left-hand side obeys

$$\delta_{\bar w\bar w}C_\delta^{(I_1\cdots I_n)} = 0, \tag{41}$$

and the totally symmetric modular tensor $C_\delta^{(I_1\cdots I_n)}$ is actually a holomorphic modular tensor on Torelli space.

Specialization to genus ≤ 2.—At genus one, each tensor $C_\delta^{I_1I_2\cdots I_n}\to C_\delta^{11\cdots 1}$ has a single component (which vanishes at odd rank n) and is a degree-two polynomial in $e_\delta\in\{\wp(\frac{1}{2}),\wp(\tau/2),\wp[(1+\tau)/2]\}$ at even n [7,10], where we set $\Omega_{11}=\tau$. The Weierstrass $\wp$ function at genus one is evaluated at the half-periods associated with the three even spin structures, and the coefficients of $e_\delta^2, e_\delta^1, e_\delta^0$ are combinations of holomorphic Eisenstein series; see Appendix D in the Supplemental Material [17] for examples at $n\leq 8$.

At genus two, the results of [11], translated into the language of ϑ functions, again organize the entire spin-structure dependence of $C_\delta(1,\ldots,n)$ into degree-two polynomials in the components of the z_i-independent, symmetric modular tensor of $\Gamma_{h=2}(2)$,

$$\mathfrak{L}_\delta^{IJ} = \frac{2\pi i}{10}\partial^{IJ}\ln\left(\frac{\Psi_{10}}{\vartheta[\delta](0)^{20}}\right), \tag{42}$$

where Ψ_{10} denotes the Igusa cusp form (a Siegel modular form of weight ten), and $\partial^{IJ} = \frac{1}{2}(1+\delta^{IJ})(\partial/\partial\Omega_{IJ})$ are the moduli derivatives. In the two-point function,

$$\begin{aligned}C_\delta(x,y) &= \omega_I(x)\omega_J(y)\mathfrak{L}_\delta^{IJ} - \wp(x,y),\\ \wp(x,y) &= \partial_x\partial_y\ln E(x,y) + \frac{2\pi i}{10}\partial^{IJ}\ln\Psi_{10},\end{aligned} \tag{43}$$

the modular tensor $\mathfrak{L}_\delta^{IJ}$ in (42) offers an alternative to capturing the δ dependence through the modular tensor C_δ^{IJ} in (14), and $\wp$ denotes the $\mathrm{Sp}(4,\mathbb{Z})$-invariant genus-two generalization of the Weierstrass function.

A first key result of [11] is that all spin-structure dependence of $C_\delta(\mathbf{z})$ at genus two and any multiplicity n may be reduced to a linear combination of tensor powers of $\mathfrak{L}_\delta^{IJ}$. A second result of [11] is that the tensor $\mathfrak{L}_\delta^{IJ}$ satisfies the *trilinear relations*, eliminating any tensor power higher than two and leading to the major simplification:

$$\begin{aligned}C_\delta^{I_1\cdots I_n} &= (\mathfrak{T}_n^{(2)})^{I_1\cdots I_n}_{J_1J_2J_3J_4}\mathfrak{L}_\delta^{J_1J_2}\mathfrak{L}_\delta^{J_3J_4}\\ &\quad + (\mathfrak{T}_n^{(1)})^{I_1\cdots I_n}_{J_1J_2}\mathfrak{L}_\delta^{J_1J_2} + (\mathfrak{T}_n^{(0)})^{I_1\cdots I_n}.\end{aligned} \tag{44}$$

The modular tensors $\mathfrak{T}_n^{(i)}$ of $\mathrm{Sp}(4,\mathbb{Z})$ are independent of z_i and δ but, just as $C_\delta^{I_1\cdots I_n}$, they are not necessarily locally holomorphic in moduli. Since spin-structure independent modular tensors of odd rank must vanish at genus two, the modular tensors $\mathfrak{T}_n^{(i)}$ and therefore $C_\delta^{I_1\cdots I_n}$ itself must vanish at genus two for odd values of n. Examples up to $n=4$ can be found in Appendix E in the Supplemental Material [17].

Conclusion and outlook.—The descent procedure described in this work reorganizes cyclic products of Szegö kernels such that their dependences on the points $z_1,\ldots,z_n$ and on the even spin structure δ are completely disentangled. The key results apply to Riemann surfaces of arbitrary genus which find an increasingly universal appearance in different areas of theoretical physics and mathematics.

First, the function space identified through the δ-independent building blocks of this work plays a crucial role in the explicit evaluation of multiparticle string amplitudes beyond genus one, as well as in bootstrap approaches to the construction of such amplitudes [33]. Their intimate connection with the higher-genus polylogarithms of [18] will strengthen the symbiosis between algebraic geometry, string perturbation theory, and particle physics, stimulating, for instance, the application to higher-genus surfaces in Feynman integrals [34–37].

Second, the descent procedure introduced here sheds further light on the cohomology structure of chiral blocks investigated in [38] as the cyclic products $C_\delta(\mathbf{z})$ are automatically part of the chiral-block structure [31,39]. Accordingly, the field-theory limit will translate our simplifications of $C_\delta(\mathbf{z})$ into new double-copy representations of multiloop amplitudes in supergravity theories, expressed via bilinears of gauge-theory building blocks [40,41].

Among the numerous directions of follow-up research, the results of this work suggest future investigations of the following problems: (a) to simplify the spin-structure sums in the proposal of [42] for the genus-three four-point superstring amplitude by means of the four-point results in Appendix B in the Supplemental Material [17]; (b) to explore generalizations of the structure obtained in (44) to higher genus, i.e., whether the δ dependence of $C_\delta^{I_1\cdots I_n}$ at arbitrary n and fixed $h \geq 3$ can still be reduced to *finitely* many tensor products of lower-rank tensors; (c) to explicitly compute the modular tensors $C_\delta^{I_1\cdots I_n}$ of $\Gamma_h(2)$, starting with C_δ^{IJK} at genus $h = 3$ and the components of C_δ^{IJKL} at $h = 2$ beyond the symmetrized ones in Appendix E in the Supplemental Material [17].

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