

Discrete subgroups of semisimple Lie groups, beyond lattices

Fanny Kassell^a

Abstract

Discrete subgroups of $SL(2, \mathbb{R})$ are well understood, and classified by the geometry of the corresponding hyperbolic surfaces. Discrete subgroups of higher-rank semisimple Lie groups, such as $SL(n, \mathbb{R})$ for $n > 2$, remain more mysterious. While lattices in this setting are rigid, there also exist more flexible, “thinner” discrete subgroups, which may have large and interesting deformation spaces, giving rise in particular to so-called higher Teichmüller theory. We survey recent progress in constructing and understanding such discrete subgroups from a geometric and dynamical viewpoint.

4.1 Introduction

Recall that a Lie group is a group which is also a differentiable manifold. All Lie groups considered in these notes will be assumed to be real linear Lie groups, i.e. closed subgroups of $GL(N, \mathbb{R})$ for some $N \in \mathbb{N}$, with finitely many connected components. We will be specifically interested in such Lie groups which are *noncompact*, since our goal is to study their *infinite* discrete subgroups.

We say that a Lie group G is *simple* if its Lie algebra is simple, i.e. nonabelian with no nonzero proper ideals; equivalently, all infinite closed normal subgroups of G have finite index in G and are nonabelian. Simple Lie algebras have been completely classified by É. Cartan, leading

^a CNRS and Laboratoire Alexander Grothendieck, Institut des Hautes Études Scientifiques, Université Paris-Saclay, 35 route de Chartres, 91440 Bures-sur-Yvette, France.
kassel@ihes.fr

to a classification of simple Lie groups up to local isomorphism. (Recall that two Lie groups G_1 and G_2 are said to be *locally isomorphic* if they have the same Lie algebra; equivalently, some finite cover of the identity component of G_1 is isomorphic to some finite cover of the identity component of G_2 .) Noncompact simple Lie groups come in several infinite families, given in Table 4.1, and 17 (up to local isomorphism) additional groups, called *exceptional* (see e.g. [90, Ch. X]).

	Noncompact simple Lie group G	Maximal compact subgroup K	$\text{rank}_{\mathbb{R}}(G)$
A	$\text{SL}(n, \mathbb{C})$	$\text{SU}(n)$	$n - 1$
B	$\text{SO}(2n + 1, \mathbb{C})$	$\text{SO}(2n + 1)$	n
C	$\text{Sp}(2n, \mathbb{C})$	$\text{Sp}(n)$	n
D	$\text{SO}(2n, \mathbb{C})$	$\text{SO}(2n)$	n
A I	$\text{SL}(n, \mathbb{R})$	$\text{SO}(n)$	$n - 1$
A II	$\text{SU}^*(2n)$	$\text{Sp}(n)$	$n - 1$
A III	$\text{SU}(p, q)$	$\text{S}(\text{U}(p) \times \text{U}(q))$	$\min(p, q)$
BD I	$\text{SO}(p, q)_0$	$\text{SO}(p) \times \text{SO}(q)$	$\min(p, q)$
D III	$\text{SO}^*(2n)$	$\text{U}(n)$	$\lfloor n/2 \rfloor$
C I	$\text{Sp}(2n, \mathbb{R})$	$\text{U}(n)$	n
C II	$\text{Sp}(p, q)$	$\text{Sp}(p) \times \text{Sp}(q)$	$\min(p, q)$

Table 4.1 *List of classical noncompact simple real linear Lie groups, up to local isomorphism. Here $n, p, q \geq 1$ are integers. For types A, A I, and A II we assume $n \geq 2$, for types D and D III we assume $n \geq 3$, and for type BD I we assume $(p, q) \notin \{(1, 1), (2, 2)\}$.*

We say that a Lie group G is *semisimple* if it is locally isomorphic to a direct product $G_1 \times \cdots \times G_\ell$ of simple Lie groups G_i , called the *simple factors* of G ; in that case, if G is connected and simply connected, then it is actually isomorphic to such a direct product $G_1 \times \cdots \times G_\ell$. For instance, $\text{SO}(2, 2)$ and $\text{SO}(4, \mathbb{C})$ are semisimple (they are locally isomorphic to $\text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R})$ and $\text{PSL}(2, \mathbb{C}) \times \text{PSL}(2, \mathbb{C})$, respectively). Any connected semisimple Lie group is the identity component (for the real topology) of the real points of some \mathbb{R} -algebraic group (see [45, § 2.14]).

Infinite discrete subgroups of semisimple Lie groups are important objects that appear in various areas of mathematics, such as geometry, complex analysis, differential equations, number theory, mathematical physics, ergodic theory, representation theory, etc. There are many motivations for studying these discrete subgroups. Let us mention three:

- (1) *Historical importance:* The study of second-order linear differential equations over \mathbb{C} , in particular by Fuchs, naturally led to the study of discrete subgroups of $\mathrm{PSL}(2, \mathbb{C})$, in particular by Poincaré, and to the celebrated Uniformisation Theorem: any closed Riemann surface of genus ≥ 2 is a quotient of the hyperbolic plane \mathbb{H}^2 by a discrete subgroup Γ of $\mathrm{PSL}(2, \mathbb{R})$. See e.g. [122] for details.
- (2) *Locally symmetric spaces:* Any discrete subgroup Γ of a noncompact semisimple Lie group G defines a Riemannian locally symmetric space $\Gamma \backslash G/K$, where K is a maximal compact subgroup of G . These locally symmetric spaces, which include real hyperbolic manifolds $\Gamma \backslash \mathbb{H}^n$ for $G = \mathrm{PO}(n, 1) = \mathrm{O}(n, 1)/\{\pm I\}$, are geometrically important. They naturally appear in representation theory and harmonic analysis, where symmetric spaces G/K play a central role (see e.g. [7]).
- (3) *Geometric structures on manifolds:* A modern point of view on geometry, which dates back to Klein's 1872 Erlangen program and which has been much developed in the twentieth century especially through the work of Ehresmann and Thurston, is to study manifolds that "locally look like" some "model spaces" with large "symmetry groups". Model spaces are typically homogeneous spaces $X = G/H$ where G is a real Lie group (often semisimple). Important examples include $X = G/K$ as above, but also $(X, G) = (\mathbb{RP}^n, \mathrm{PGL}(n+1, \mathbb{R}))$ (real projective geometry), $(\mathbb{CP}^n, \mathrm{PGL}(n+1, \mathbb{C}))$ (complex projective geometry), or $(\mathbb{H}^{p,q}, \mathrm{PO}(p, q+1))$ (pseudo-Riemannian hyperbolic geometry in signature (p, q)). See [78] for details.

An important class of discrete subgroups of noncompact semisimple Lie groups is the class of *lattices*, namely discrete subgroups of finite covolume for the Haar measure (see Section 4.2 below). They play an important role in several fields of mathematics, in addition to the above, such as:

- geometric group theory (lattices are finitely presented groups with many desirable properties — e.g. lattices of $\mathrm{SL}(n, \mathbb{R})$ for $n \geq 3$ have Kazhdan's property (T)),
- combinatorics (construction of expander graphs),
- number theory (arithmetic groups),
- ergodic theory (flows on $\Gamma \backslash G$) and homogeneous dynamics.

See e.g. [135] and references therein. In some of these settings (in particular ergodic theory and homogeneous dynamics), there is currently active

research aiming to extend, to classes of discrete subgroups of infinite covolume, classical results involving lattices. Infinite-index subgroups of arithmetic groups (and particularly those that are still Zariski-dense, named *thin groups* by Sarnak) have also attracted considerable interest recently, see e.g. [107].

In these notes, we will review a few properties of lattices, and then focus on the problem of finding other large classes of infinite discrete subgroups Γ of semisimple Lie groups G with desirable properties, including:

- (1) the existence of examples with interesting geometric interpretations,
- (2) a good control of the subgroups' behaviour under deformation,
- (3) interesting dynamics of Γ on certain homogeneous spaces of G .

These properties are typically invariant under replacing Γ by a finite-index subgroup. This will allow us to sometimes reduce to *torsion-free* Γ : indeed, the Selberg lemma [125, Lem. 8] states that any finitely generated subgroup of G admits a finite-index subgroup which is torsion-free.

4.2 Lattices

Let G be a noncompact semisimple Lie group. It admits a *Haar measure*, i.e. a nonzero Radon measure which is invariant under left and right multiplication; this measure is unique up to scaling.

Definition 4.2.1 A *lattice* of G is a discrete subgroup Γ of G such that the quotient $\Gamma \backslash G$ has finite volume for the measure induced by the Haar measure of G .

If Γ is a lattice of G , then the quotient $\Gamma \backslash G$ can be compact (in which case we say that Γ is a *cocompact* or *uniform* lattice) or not.

A fundamental result of Borel and Harish-Chandra [29, 30] states that G always admits both cocompact lattices and noncocompact lattices.

Borel's Density Theorem [28] states that lattices are Zariski-dense in G as soon as G is connected and has no compact simple factors. This means that if the set of real points of some \mathbb{R} -algebraic group contains a lattice of G , then it actually contains the whole of G .

We say that a lattice Γ of G is *irreducible* if for any noncompact, infinite-index, closed normal subgroup G' of G , the projection of Γ to G/G' is nondiscrete. (This is automatically satisfied if G is simple.)

4.2.1 Geometric interpretation

Lattices of G can be characterised by their action on the *Riemannian symmetric space* of G . Let us recall what this fundamental object is (see e.g. [66, 90] for details).

As mentioned in the introduction, G admits a maximal compact subgroup K . It is unique up to conjugation, and so the quotient G/K is uniquely defined. For instance, if $G = \mathrm{SL}(n, \mathbb{R})$, then $K = \mathrm{SO}(n)$ up to conjugation, and G/K identifies with the space of ellipsoids of \mathbb{R}^n of volume 1; if $G = \mathrm{SL}(n, \mathbb{C})$, then $K = \mathrm{SU}(n)$ up to conjugation. See Table 4.1 for further examples.

The group K is the set of fixed points of some involution θ of G , called a Cartan involution. This yields a splitting of the Lie algebra \mathfrak{g} of G as the direct sum of two linear subspaces, namely the subspace $\mathfrak{g}^{\mathrm{d}\theta}$ of fixed points of $\mathrm{d}\theta$ (which is the Lie algebra of K) and the subspace $\mathfrak{g}^{-\mathrm{d}\theta}$ of anti-fixed points of $\mathrm{d}\theta$. The tangent space $T_{eK}(G/K)$ to G/K at the origin identifies with $\mathfrak{g}^{-\mathrm{d}\theta}$, on which there is a natural K -invariant positive definite symmetric bilinear form, the Killing form. Pushing forward this bilinear form by elements of G yields a G -invariant Riemannian metric on G/K . With this metric, G/K has nonpositive sectional curvature and is a *symmetric space*: at every point, the geodesic symmetry sending $\exp(tv)$ to $\exp(-tv)$ (where v is a tangent vector) is an isometry.

Since K is compact, any discrete subgroup Γ of G acts properly discontinuously on G/K . The subgroup Γ is a lattice if and only if the quotient $\Gamma \backslash G/K$ has finite volume, which is equivalent to the action of Γ on G/K admitting a fundamental domain of finite volume.

4.2.2 Examples

The following fundamental example goes back to Minkowski.

Example 4.2.2 The group $\Gamma = \mathrm{SL}(n, \mathbb{Z})$ is a noncocompact lattice in $G = \mathrm{SL}(n, \mathbb{R})$.

Let us briefly explain how to see this, starting with the case $n = 2$.

For $n = 2$, the Riemannian symmetric space G/K is the hyperbolic plane $\mathbb{H}^2 \simeq \{z = x + iy \in \mathbb{C} \mid y = \mathrm{Im}(z) > 0\}$ with its G -invariant metric $\mathrm{d}s^2 = (\mathrm{d}x^2 + \mathrm{d}y^2)/y^2$, on which $G = \mathrm{SL}(2, \mathbb{R})$ acts by Möbius transformations: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$. It is an easy exercise to check that

$$\mathcal{D} := \left\{ z \in \mathbb{H}^2 \mid |\mathrm{Re}(z)| \leq \frac{1}{2} \quad \text{and} \quad |z| \geq 1 \right\}$$

(see Figure 4.1) is a finite-volume fundamental domain for the action of $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ on \mathbb{H}^2 . (Use that Γ is generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and that the G -invariant volume form on \mathbb{H}^2 is given by $\mathrm{dvol} = dx dy / (4y^2)$.) Therefore Γ is a lattice in $G = \mathrm{SL}(2, \mathbb{R})$. This lattice is not cocompact since for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ we have $\mathrm{Im}(\gamma \cdot i) = 1/(c^2 + d^2) \leq 1$, hence there exist points of \mathbb{H}^2 (e.g. ti with $t > 0$ large) that are arbitrarily far away from any point of the Γ -orbit of i in \mathbb{H}^2 .

For general $n \geq 2$, we can use the classical *Iwasawa decomposition* $G = NAK$, where N (resp. A) is the subgroup of $G = \mathrm{SL}(n, \mathbb{R})$ consisting of upper triangular unipotent (resp. positive diagonal) matrices and $K = \mathrm{SO}(n)$. This means that any element $g \in G$ can be written in a unique way as $g = nak$ where $n \in N$, $a \in A$, and $k \in K$. A finite-volume fundamental domain for the action of Γ on G/K is given by the *Siegel set* \mathcal{S} consisting of those elements of G/K of the form naK with $n \in N$ having all entries above the diagonal in $[-1/2, 1/2]$ and $a = \mathrm{diag}(a_1, \dots, a_n) \in A$ satisfying $|a_i/a_{i+1}| \geq \sqrt{3}/2$ for all $1 \leq i \leq n-1$.

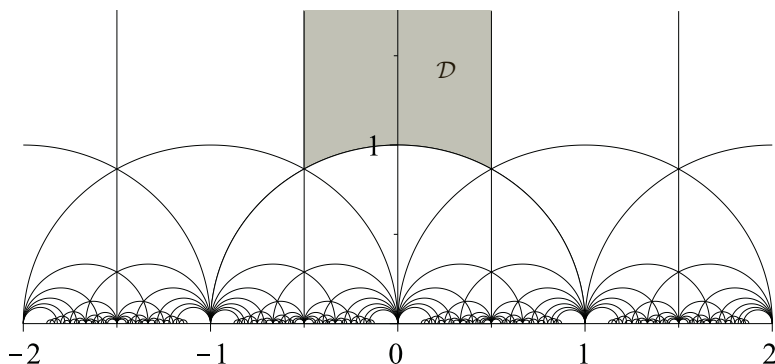


Figure 4.1 Fundamental domains for the action of $\mathrm{SL}(2, \mathbb{Z})$ on the upper half plane model of \mathbb{H}^2

Generalising Example 4.2.2, a fundamental result of Borel and Harish-Chandra [30] states that if \mathbf{G} is a semisimple \mathbb{Q} -algebraic group, then $\mathbf{G}_{\mathbb{Z}}$ is a lattice in $\mathbf{G}_{\mathbb{R}}$. Godement's cocompactness criterion (see e.g. [19, §2.8]) states that this lattice is cocompact if and only if it does not contain any nontrivial unipotent elements.

We now give a concrete example of a cocompact lattice (see [19, §2]).

Example 4.2.3 For $p, q \geq 1$ with $p + q = n \geq 3$, consider the block

diagonal matrix

$$J_{p,q} := \begin{pmatrix} I_p & 0 \\ 0 & -\sqrt{2} I_q \end{pmatrix}$$

and the Lie group $G := \mathrm{SO}(J_{p,q}, \mathbb{R}) \simeq \mathrm{SO}(p, q)$. Then $\Gamma := G \cap \mathrm{SL}(n, \mathbb{Z}[\sqrt{2}])$ is a cocompact lattice in G .

In order to see this, we can apply Weil's trick of "restriction of scalars". Namely, consider the automorphism σ of $\mathrm{SL}(n, \mathbb{Z}[\sqrt{2}])$ obtained by applying the Galois conjugation $x + \sqrt{2}y \mapsto x - \sqrt{2}y$ of $\mathbb{Q}[\sqrt{2}]$ to each entry. Let $J_{p,q}^\sigma$ be the image of $J_{p,q}$ under σ , and \mathbf{H} the semisimple algebraic subgroup of \mathbf{GL}_{2n} whose set $\mathbf{H}_{\mathbb{C}}$ of complex points consists of those block matrices of the form

$$h := \begin{pmatrix} a & 2b \\ b & a \end{pmatrix} \in \mathrm{GL}(2n, \mathbb{C})$$

with $\varphi_+(h) := a + \sqrt{2}b \in \mathrm{SO}(J_{p,q}, \mathbb{C})$ and $\varphi_-(h) := a - \sqrt{2}b \in \mathrm{SO}(J_{p,q}^\sigma, \mathbb{C})$. An elementary computation (or more abstractly the fact that the family of polynomial equations defining \mathbf{H} is invariant under σ) shows that \mathbf{H} is a \mathbb{Q} -algebraic group. We have isomorphisms

$$\begin{cases} \mathbf{H}_{\mathbb{R}} \stackrel{(\varphi_+, \varphi_-)}{\simeq} \mathrm{SO}(J_{p,q}, \mathbb{R}) \times \mathrm{SO}(J_{p,q}^\sigma, \mathbb{R}) = G \times \mathrm{SO}(J_{p,q}^\sigma, \mathbb{R}), \\ \mathbf{H}_{\mathbb{Z}} \stackrel{\varphi_+}{\simeq} \Gamma, \end{cases}$$

where $\mathrm{SO}(J_{p,q}^\sigma, \mathbb{R}) \simeq \mathrm{SO}(n)$ is compact. The group $\mathbf{H}_{\mathbb{Z}}$ is a lattice in $\mathbf{H}_{\mathbb{R}}$, hence Γ is a lattice in G . Moreover, $\mathbf{H}_{\mathbb{Z}}$ does not contain any nontrivial unipotent elements since φ_- takes Γ to a subgroup of a *compact* group, hence without nontrivial unipotent elements, and a homomorphism of algebraic groups takes unipotent elements to unipotent elements. Godevent's criterion then ensures that $\mathbf{H}_{\mathbb{Z}} \backslash \mathbf{H}_{\mathbb{R}}$ is compact, and so $\Gamma \backslash G$ is compact too.

In both Examples 4.2.2 and 4.2.3, the group Γ is *arithmetic* in G , i.e. there is a homomorphism $\pi : \mathbf{H} \rightarrow \mathbf{G}$ of semisimple \mathbb{Q} -algebraic groups such that $G = \mathbf{G}_{\mathbb{R}}$, such that the kernel of π in $\mathbf{H}_{\mathbb{R}}$ is compact, and such that Γ is commensurable to $\pi(\mathbf{H}_{\mathbb{Z}})$ (see [135]).

Nonarithmetic lattices are known to exist in $G = \mathrm{SO}(n, 1)$ for any $n \geq 2$: examples were constructed by Vinberg [129] for small n using reflection groups, then by Gromov and Piatetski-Shapiro [80] for any n . Later, different examples were constructed by Agol [1] and Belolipetsky–Thomson [13] (see also the very recent work [63]) in the form of lattices of $\mathrm{SO}(n, 1)$ whose systole (i.e. length of the shortest closed geodesic)

is arbitrarily small. (Due to a separability property later established in [20, Cor. 1.12], Agol's construction [1] actually works for any n .) Finitely many commensurability classes of nonarithmetic lattices are also known in $SU(2, 1)$ and $SU(3, 1)$ by Deligne–Mostow [57, 116] and Deraux–Parker–Paupert [58, 59]. It is an open question whether nonarithmetic lattices exist in $SU(n, 1)$ for $n > 3$.

On the other hand, in noncompact simple Lie groups which are not locally isomorphic to $SO(n, 1)$ or $SU(n, 1)$, all lattices are arithmetic (as a consequence of superrigidity, see Section 4.2.4).

4.2.3 Rank one versus higher rank

The *real rank* of a semisimple Lie group is an integer defined as follows.

Definition 4.2.4 The *real rank* of G , denoted $\text{rank}_{\mathbb{R}}(G)$, is the maximum dimension of a closed connected subgroup of G which is diagonalisable over \mathbb{R} ; equivalently, for noncompact G , it is the maximum dimension of a totally geodesic subspace of the Riemannian symmetric space G/K which is flat (i.e. of constant zero sectional curvature).

The real rank is invariant under local isomorphism, and the real rank of a product is the sum of the real ranks of the factors. We refer to Table 4.1 for the real ranks of the classical noncompact simple Lie groups. A compact Lie group has real rank 0.

The simple Lie groups of real rank 1 are, up to local isomorphism, $SO(n, 1)$, $SU(n, 1)$, $Sp(n, 1)$ for $n \geq 2$, and the exceptional group $F_{4(-20)}$. (Note that $\text{PSL}(2, \mathbb{R}) \simeq SO(2, 1)_0$ and $\text{PSL}(2, \mathbb{C}) \simeq SO(3, 1)_0$, where the subscript 0 denotes the identity components.)

Semisimple Lie groups G of real rank 1 are characterised by the fact that the sectional curvature of the corresponding Riemannian symmetric space G/K is everywhere < 0 . (In fact, the curvature is then *pinched*, i.e. contained in an interval of the form $[\alpha, \beta]$ where $\alpha \leq \beta < 0$.) This implies that the geodesic metric space G/K is *Gromov hyperbolic*, meaning that there exists $\delta \geq 0$ such that all geodesic triangles (a, b, c) of G/K are δ -thin: the side $[a, b]$ lies in the uniform δ -neighbourhood of the union $[b, c] \cup [c, a]$ of the other two sides (see Figure 4.2). On the other hand, when $r := \text{rank}_{\mathbb{R}}(G) \geq 2$, the Riemannian symmetric space G/K is only nonpositively curved, and not Gromov hyperbolic; its geometry is somewhat more complicated due to the presence of *flats* (i.e. isometric copies of Euclidean \mathbb{R}^r , where the curvature vanishes).

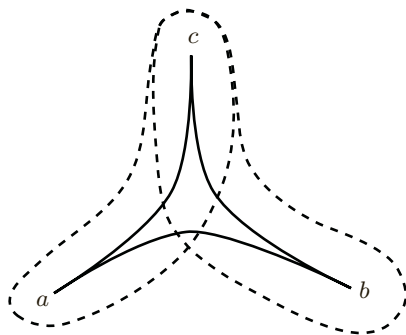


Figure 4.2 A δ -thin triangle in a geodesic metric space. The side $[a, b]$ is contained in the union of the uniform δ -neighbourhoods (indicated by dashes) of the sides $[b, c]$ and $[c, a]$.

There are a number of differences between lattices in real rank one and lattices in higher real rank.

One difference concerns hyperbolicity. Namely, if $\text{rank}_{\mathbb{R}}(G) = 1$, then

- any cocompact lattice Γ of G is *Gromov hyperbolic*, i.e. Γ acts properly discontinuously, by isometries, with compact quotient, on a Gromov hyperbolic proper geodesic metric space X (e.g. $X = G/K$);
- any noncocompact lattice Γ of G is *relatively hyperbolic* with respect to some collection \mathcal{P} of subgroups which are virtually (i.e. up to finite index) nilpotent: this means that Γ acts properly discontinuously by isometries on some visual Gromov hyperbolic proper metric space X (e.g. $X = G/K$), and with compact quotient on some closed subset of X of the form $X \setminus \bigcup_{P \in \mathcal{P}} B_P$ where each B_P is a P -invariant open horoball of X and $B_P \cap B_{P'} = \emptyset$ for $P \neq P'$ (see [89, § 4] for details).

On the other hand, if $\text{rank}_{\mathbb{R}}(G) \geq 2$, then lattices of G are never Gromov hyperbolic, nor relatively hyperbolic with respect to any collection of subgroups [12]. This follows from the fact that these groups are *metrically thick* in the sense of [12] (see [105] for cocompact lattices). In fact, if $\text{rank}_{\mathbb{R}}(G) \geq 2$, then any isometric action of a lattice Γ of G on a Gromov hyperbolic metric space X is “trivial” (i.e. admits a global fixed point in X or its boundary), unless it is obtained by projecting Γ to a rank-one factor of G [3, 87].

More generally, lattices Γ in simple Lie groups G with $\text{rank}_{\mathbb{R}}(G) \geq 2$ tend to have global fixed points when they act on various classes of spaces. For instance, any continuous action by affine isometries of Γ on a Hilbert space has a global fixed point. This property, which is equivalent

to Kazhdan's property (T) (see [135, Ch. 13]), is also satisfied by lattices in the rank-one Lie groups $\mathrm{Sp}(n, 1)$ with $n \geq 2$ or $F_{4(-20)}$. However, other fixed point properties actually distinguish higher rank from rank one. For instance, for any simple Lie group G with $\mathrm{rank}_{\mathbb{R}}(G) \geq 2$ and any σ -finite positive measure ν on a standard Borel space, any continuous affine isometric action of a lattice of G on $L^p(\nu)$ for $1 < p < +\infty$ has a global fixed point, by Bader–Furman–Gelander–Monod; on the other hand, by Pansu and Bourdon–Pajot, any cocompact lattice Γ in a simple Lie group G with $\mathrm{rank}_{\mathbb{R}}(G) = 1$ (and more generally, any Gromov hyperbolic group Γ) admits fixed-point-free affine isometric actions on $L^p(\Gamma)$ whose linear part is the regular representation, for any $p > 1$ large enough. See [4].

Another difference between real rank one and higher real rank concerns normal subgroups. Namely, if $\mathrm{rank}_{\mathbb{R}}(G) = 1$, then lattices of G have many normal subgroups (see Gromov [79]); in fact, if Γ is a lattice of G , then any countable group can be embedded into a quotient of Γ by some normal subgroup (this “universality” property holds for all relatively hyperbolic groups [2]). On the other hand, if $\mathrm{rank}_{\mathbb{R}}(G) \geq 2$, then all normal subgroups of an irreducible lattice Γ of G are finite or finite-index in Γ (this is Margulis's Normal Subgroups Theorem, see [112]).

Note that for an irreducible lattice Γ of G , the finite normal subgroups of Γ are easy to describe: for connected G , they are the subgroups of the finite abelian group $\Gamma \cap Z(G)$ (using Borel's Density Theorem [28]). On the other hand, much more effort is required to understand the finite-index normal subgroups of Γ . By [11], for $\Gamma = \mathrm{SL}(n, \mathbb{Z})$ with $n \geq 3$, any finite-index normal subgroup of Γ is a *congruence subgroup*, i.e. contains the kernel of the natural projection $\mathrm{SL}(n, \mathbb{Z}) \rightarrow \mathrm{SL}(n, \mathbb{Z}/m\mathbb{Z})$ for some $m \geq 1$; this is false for $\Gamma = \mathrm{SL}(2, \mathbb{Z})$. In general, it is conjectured that lattices of G have a slightly weaker form of this “Congruence Subgroup Property” if and only if $\mathrm{rank}_{\mathbb{R}}(G) \geq 2$: see [127].

We now discuss some rigidity results for representations of lattices inside G , which hold in particular for $\mathrm{rank}_{\mathbb{R}}(G) \geq 2$.

4.2.4 Deformations and rigidity

Let Γ be a discrete subgroup of G . We denote by $\mathrm{Hom}(\Gamma, G)$ the space of representations of Γ to G , endowed with the compact-open topology (if Γ admits a finite generating subset F , then this coincides with the topology of pointwise convergence on F).

By a *continuous deformation* of Γ in G we mean a continuous path

$(\rho_t)_{t \in [0,1]}$ in $\text{Hom}(\Gamma, G)$ where ρ_0 is the natural inclusion of Γ in G . Certain continuous deformations of Γ in G are considered *trivial*: namely, those of the form $\rho_t = g_t \rho_0(\cdot) g_t^{-1}$ where $(g_t)_{t \in [0,1]}$ is a continuous path in G (and g_0 is the identity element). In other words, if $\text{Hom}(\Gamma, G)/G$ denotes the quotient of $\text{Hom}(\Gamma, G)$ by the natural action of G by conjugation at the target, then the trivial deformations are those whose image in $\text{Hom}(\Gamma, G)/G$ are constant.

For $G = \text{PSL}(2, \mathbb{R}) \simeq \text{SO}(2, 1)_0$, torsion-free lattices Γ of G admit many nontrivial continuous deformations. Indeed, if Γ is noncocompact in G , then Γ is a nonabelian free group on finitely many generators $\gamma_1, \dots, \gamma_m$, and the natural inclusion $\rho_0 : \Gamma \hookrightarrow G$ can be continuously deformed by deforming independently the image of each γ_i ; the map $\rho \mapsto (\rho(\gamma_1), \dots, \rho(\gamma_m))$ yields an isomorphism $\text{Hom}(\Gamma, G) \simeq G^m$. If Γ is cocompact in G , then Γ identifies with the fundamental group of the closed hyperbolic surface $S := \Gamma \backslash \mathbb{H}^2$; the connected component of the natural inclusion ρ_0 in $\text{Hom}(\Gamma, G)$ consists entirely of injective and discrete representations [77], and its image in $\text{Hom}(\Gamma, G)/G$ is homeomorphic to \mathbb{R}^{6g-6} : it is the *Teichmüller space* of S .

On the other hand, a number of rigidity results have been proved for lattices in other noncompact semisimple Lie groups G , including local rigidity, Mostow rigidity, and Margulis superrigidity, which we now briefly state and comment on. See [67, 119] for details and references.

Local rigidity (Selberg, Calabi, Weil, Garland–Raghunathan) *Let G be a semisimple Lie group with no simple factors that are compact or locally isomorphic to $\text{PSL}(2, \mathbb{R})$ (resp. $\text{PSL}(2, \mathbb{K})$ with $\mathbb{K} = \mathbb{R}$ or \mathbb{C}). If Γ is a cocompact (resp. noncocompact) irreducible lattice of G , then any continuous deformation of Γ in G is trivial.*

Note that noncocompact lattices of $G = \text{PSL}(2, \mathbb{C})$ are not locally rigid: they can be deformed using Thurston's *hyperbolic Dehn surgery* theory. However, they do not admit nontrivial deformations sending unipotent elements to unipotent elements.

Local rigidity is an important ingredient in the proof of Wang's finiteness theorem, which states that if G is simple and not locally isomorphic to $\text{PSL}(2, \mathbb{K})$ with $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , then for any $v > 0$ there are only finitely many conjugacy classes of lattices of G with covolume $\leq v$.

Mostow rigidity (Mostow, Prasad, Margulis) *Let G, G' be connected semisimple Lie groups, with trivial centre, and with no simple factors that are compact or locally isomorphic to $\text{PSL}(2, \mathbb{R})$. If Γ and Γ' are*

irreducible lattices of G and G' , respectively, then any isomorphism between Γ and Γ' extends to a continuous isomorphism between G and G' .

This implies (see Section 4.2.1) that the fundamental group of any locally symmetric space $\Gamma \backslash G/K$ completely determines its geometry.

Margulis superrigidity (Margulis, Corlette, Gromov–Schoen, see e.g. [135, Th. 16.1.4]) *Let G be a noncompact semisimple Lie group which is connected, algebraically simply connected, and not locally isomorphic to the product of $\mathrm{SO}(n, 1)$ or $\mathrm{SU}(n, 1)$ with a compact Lie group. Then any irreducible lattice Γ of G is superrigid, in the sense that any representation $\rho : \Gamma \rightarrow \mathrm{GL}(d, \mathbb{R})$ (for any $d \geq 2$) continuously extends to G up to finite index and to bounded error.*

Here “ ρ continuously extends to G up to finite index and to bounded error” means that there exist a finite-index subgroup Γ' of Γ , a continuous homomorphism $\rho_G : G \rightarrow \mathrm{GL}(d, \mathbb{R})$, and a compact subgroup C of $\mathrm{GL}(d, \mathbb{R})$ centralising $\rho_G(G)$ such that $\rho(\gamma) \in \rho_G(\gamma)C$ for all $\gamma \in \Gamma'$. Under an appropriate assumption on the image of ρ , we can take C to be trivial. “Algebraically simply connected” is a technical assumption which is always satisfied up to passing to a finite cover: see [135, § 16.1].

Margulis used his superrigidity (over \mathbb{R} as above, but also over non-Archimedean local fields) to prove that if G is semisimple with no compact simple factors and if $\mathrm{rank}_{\mathbb{R}}(G) \geq 2$, then all irreducible lattices Γ of G are arithmetic in the sense of Section 4.2.2. The same conclusion holds when G is locally isomorphic to $\mathrm{Sp}(n, 1)$ with $n \geq 2$ or $F_{4(-20)}$, as superrigidity holds for these rank-one groups as well.

Margulis superrigidity was further extended by Zimmer into a rigidity result for cocycles, see [69]. This was the starting point of important new directions of research at the intersection of group theory and dynamics (see e.g. [73]), including the so-called *Zimmer program* (see [44, 68]). The idea of this program is the following: for a lattice Γ in a simple Lie group G with $\mathrm{rank}_{\mathbb{R}}(G) \geq 2$, Margulis superrigidity states that any linear representation of Γ essentially comes from a linear representation of G ; in particular, the minimal dimension of a finite-kernel linear representation of Γ is equal to the minimal dimension of a finite-kernel linear representation of G . Zimmer asked whether this last property has a non-linear analogue, for actions by diffeomorphisms of Γ on closed manifolds: namely, is the minimal dimension of a closed manifold on which Γ acts faithfully by diffeomorphisms equal to the minimal dimension of a closed manifold on which G (or a compact form of the complexification of G)

acts faithfully by diffeomorphisms? Brown, Fisher, and Hurtado have recently answered this question positively in many cases, building on new developments in dynamics and on recent strengthenings of Kazhdan's property (T): see [44, 68]. This has led to intense research activity around rigidity questions for actions by diffeomorphisms of higher-rank lattices on manifolds.

4.3 A change of paradigm

We just saw that many important rigidity results have been established for lattices since the 1960s, particularly in higher real rank, and that this topic is still very active. On the other hand, since the 1990s and early 2000s, there has been growing interest in *flexibility*: namely, there has been increasing effort to find and study infinite discrete subgroups of semisimple Lie groups which are more flexible than lattices, and which in certain cases can have large deformation spaces. Such discrete subgroups have been known to exist for a long time in real rank one, whereas the investigation of their analogues in higher real rank has gathered momentum only much more recently. We present a few examples below.

To be more precise, we are interested in infinite discrete subgroups Γ of semisimple Lie groups G that admit continuous deformations $(\rho_t)_{t \in [0,1]} \subset \text{Hom}(\Gamma, G)$ as in Section 4.2.4 which, not only are nontrivial, but also satisfy that each ρ_t is injective with discrete image, so that the $\rho_t(\Gamma)$ for $t > 0$ are still discrete subgroups of G isomorphic (but not conjugate) to Γ . An ideal situation is when the natural inclusion $\rho_0 : \Gamma \hookrightarrow G$ admits a full open neighbourhood in $\text{Hom}(\Gamma, G)$ consisting entirely of injective and discrete representations, with a nonconstant image in $\text{Hom}(\Gamma, G)/G$.

We are thus led, for given discrete subgroups Γ of G , to study subsets of $\text{Hom}(\Gamma, G)$ consisting of injective and discrete representations, and their images in the corresponding character varieties. In this framework, we discuss so-called *higher Teichmüller theory* in Section 4.3.4 below.

Remark 4.3.1 In the sequel, we go back and forth between two equivalent points of view: studying discrete subgroups Γ of G , or fixing an abstract group Γ_0 and studying the injective and discrete representations of Γ_0 into G (corresponding to the various ways of realising Γ_0 as a discrete subgroup of G). We sometimes allow ourselves to weaken “injective” into “finite-kernel”.

4.3.1 Examples in real rank one

Examples of flexible discrete subgroups in real rank one include classical Schottky groups (which are nonabelian free groups), quasi-Fuchsian groups (which are closed surface groups), as well as other discrete subgroups which are fundamental groups of higher-dimensional manifolds. We briefly review such examples, referring to [95, 113] for more details.

Schottky groups

For $n \geq 2$, let $X = \mathbb{H}^n$ be the real hyperbolic space of dimension n , with visual boundary $\partial_\infty X \simeq \mathbb{S}^{n-1}$. Concretely, choosing a symmetric bilinear form $\langle \cdot, \cdot \rangle_{n,1}$ of signature $(n, 1)$ on \mathbb{R}^{n+1} , we can realise X as the open subset

$$\mathbb{H}^n = \{[v] \in \mathbb{P}(\mathbb{R}^{n+1}) \mid \langle v, v \rangle_{n,1} < 0\} \quad (4.1)$$

of the real projective space $\mathbb{P}(\mathbb{R}^{n+1})$ and $\partial_\infty X$ as the boundary of X in $\mathbb{P}(\mathbb{R}^{n+1})$. The geodesics of X are then the nonempty intersections of X with projective lines of $\mathbb{P}(\mathbb{R}^{n+1})$, the geodesic copies of \mathbb{H}^{n-1} in X are the nonempty intersections of X with projective hyperplanes of $\mathbb{P}(\mathbb{R}^{n+1})$, and the isometry group $G = \text{Isom}(X)$ of X is $\text{PO}(n, 1) = \text{O}(n, 1)/\{\pm I\}$.

An *open disk* in $\partial_\infty X$ is the boundary at infinity of an open half-space of X , bounded by a geodesic copy of \mathbb{H}^{n-1} . (For $n = 2$, open disks are just open intervals in $\partial_\infty X \simeq \mathbb{S}^1$.)

For $m \geq 2$, choose $2m$ pairwise disjoint open disks B_1^\pm, \dots, B_m^\pm in $\partial_\infty X$, such that $\partial_\infty X \setminus \bigcup_{i=1}^m (B_i^- \cup B_i^+)$ has nonempty interior, and elements $\gamma_1, \dots, \gamma_m \in G$ such that $\gamma_i \cdot \text{Int}(\partial_\infty X \setminus B_i^-) = B_i^+$ for all i . Let Γ be the subgroup of G generated by $\gamma_1, \dots, \gamma_m$.

Claim 4.3.2 *The group Γ is a nonabelian free group with free generating subset $\{\gamma_1, \dots, \gamma_m\}$. It is discrete in G .*

Proof Consider any reduced word $\gamma = \gamma_{i_1}^{\sigma_1} \dots \gamma_{i_N}^{\sigma_N}$ in the alphabet $\{\gamma_1^{\pm 1}, \dots, \gamma_m^{\pm 1}\}$, where $1 \leq i_j \leq m$ and $\sigma_j \in \{\pm 1\}$ for all $1 \leq j \leq N$. Since $\gamma_{i_j}^{\sigma_j} \cdot \text{Int}(\partial_\infty X \setminus B_{i_j}^{-\text{sign}(\sigma_j)}) = B_{i_j}^{\text{sign}(\sigma_j)}$ for all j and since $B_{i_j}^{\text{sign}(\sigma_j)} \subset \text{Int}(\partial_\infty X \setminus B_{i_{j-1}}^{-\text{sign}(\sigma_{j-1})})$ for $j \geq 2$, we see that the element of Γ corresponding to γ sends $\partial_\infty X \setminus \bigcup_{i=1}^m (B_i^- \cup B_i^+)$ into the closure of $B_{i_1}^{\text{sign}(\sigma_1)}$ in $\partial_\infty X$. On the other hand, the set of elements $g \in G$ sending $\partial_\infty X \setminus \bigcup_{i=1}^m (B_i^- \cup B_i^+)$ into the closure of $\bigcup_{i=1}^m (B_i^- \cup B_i^+)$ in $\partial_\infty X$ is a closed subset of G that does not contain the identity element. \square

Such a group Γ is called a *Schottky group*. The proof of Claim 4.3.2 is based on the so-called *ping pong dynamics* of Γ on $\partial_\infty X$: imagine the

ping pong players are the generators $\gamma_1, \gamma_1^{-1}, \dots, \gamma_m, \gamma_m^{-1}$; the ping pong table is $\partial_\infty X$, which is divided into several open regions, namely the B_i^\pm and the “central region” $\text{Int}(\partial_\infty X \setminus \bigcup_i (B_i^- \cup B_i^+))$; the rules of the game are that each player $\gamma_i^{\pm 1}$ sends all regions but one (namely B_i^\mp) into a single region (namely B_i^\pm). The ping pong ball is a point which is initially in the central region. For any reduced word in the generators, we successively apply the corresponding ping pong players; the ball ends up in one of the B_i^\pm . We deduce that the element of Γ corresponding to this reduced word is nontrivial in Γ , and not too close to the identity in G .

Remark 4.3.3 Let $\mathcal{D} := \partial_\infty X \setminus \bigcup_{i=1}^m (B_i^- \cup B_i^+)$ and $\Omega := \text{Int}(\bigcup_{\gamma \in \Gamma} \gamma \cdot \mathcal{D})$. Then Ω is an open subset of $\partial_\infty X$ on which Γ acts properly discontinuously with fundamental domain \mathcal{D} .

(Here we have assumed that \mathcal{D} has nonempty interior; therefore $\Omega \neq \emptyset$ and Γ is *not* a lattice in G : it has infinite covolume for the Haar measure.) See e.g. [117] for beautiful illustrations in dimension two, for $X = \mathbb{H}^3$.

Since Schottky groups Γ are nonabelian free groups, they admit, as in Section 4.2.4, many nontrivial continuous deformations $(\rho_t)_{t \in [0,1)} \subset \text{Hom}(\Gamma, G)$, obtained by independently deforming the image of each generator γ_i . Some of these deformations $(\rho_t)_{t \in [0,1)}$ are “good” in the sense that for every $t \in [0,1)$, the group $\rho_t(\Gamma)$ still has a ping pong configuration analogous to that of Γ , hence ρ_t is injective with discrete image by arguing as in Claim 4.3.2. If the open disks B_1^\pm, \dots, B_m^\pm in the initial configuration have pairwise disjoint *closures* (i.e. Γ is a “strong” Schottky group), then all small deformations are “good”: the natural inclusion $\rho_0 : \Gamma \hookrightarrow G$ admits an open neighbourhood in $\text{Hom}(\Gamma, G)$ consisting entirely of injective and discrete representations, with a ping pong configuration analogous to that of Γ .

Quasi-Fuchsian groups

Quasi-Fuchsian groups are important infinite discrete subgroups of $\text{PSL}(2, \mathbb{C})$ which have been much studied (see [113]), and which are *not* lattices in $\text{PSL}(2, \mathbb{C})$. They are by definition the images of quasi-Fuchsian representations. Let us briefly recall what these are.

Let S be a closed orientable surface of genus $g \geq 2$. By the Uniformisation Theorem (see Section 4.1), there exist injective and discrete representations from the fundamental group $\pi_1(S)$ to $\text{PSL}(2, \mathbb{R})$. These representations form two connected components of $\text{Hom}(\pi_1(S), \text{PSL}(2, \mathbb{R}))$ [77], switched by conjugation by elements of $\text{PGL}(2, \mathbb{R}) \setminus \text{PSL}(2, \mathbb{R})$ (i.e. by orientation-reversing isometries of \mathbb{H}^2). The image of either of

these connected components in $\text{Hom}(\pi_1(S), \text{PSL}(2, \mathbb{R}))/\text{PSL}(2, \mathbb{R})$ identifies with the *Teichmüller space* of S , which is homeomorphic to \mathbb{R}^{6g-6} .

Now view $\text{PSL}(2, \mathbb{R})$ as a subgroup of $\text{PSL}(2, \mathbb{C})$. Recall that $\text{PSL}(2, \mathbb{C}) \simeq \text{PO}(3, 1)_0$ acts by isometries on the hyperbolic space \mathbb{H}^3 ; the subgroup $\text{PSL}(2, \mathbb{R}) \simeq \text{PO}(2, 1)_0$ preserves an isometric copy of \mathbb{H}^2 inside \mathbb{H}^3 . We see the injective and discrete representations $\rho : \pi_1(S) \rightarrow \text{PSL}(2, \mathbb{R})$ as representations with values in $\text{PSL}(2, \mathbb{C})$, called *Fuchsian*. They preserve a circle in $\partial_\infty \mathbb{H}^3$, namely the boundary $\partial_\infty \mathbb{H}^2$ of the isometric copy of \mathbb{H}^2 preserved by $\text{PSL}(2, \mathbb{R})$.

The Fuchsian representations admit an open neighbourhood in $\text{Hom}(\pi_1(S), \text{PSL}(2, \mathbb{C}))$ consisting entirely of injective and discrete representations, called *quasi-Fuchsian*. Each quasi-Fuchsian representation preserves a topological circle in $\partial_\infty \mathbb{H}^3$, but which may now be “wiggly” as in Figure 4.3. Quasi-Fuchsian representations form an open subset of $\text{Hom}(\pi_1(S), \text{PSL}(2, \mathbb{C}))$ which is dense in the set of injective and discrete representations; its image in $\text{Hom}(\pi_1(S), \text{PSL}(2, \mathbb{C}))/\text{PSL}(2, \mathbb{C})$ admits a natural parametrisation (due to Bers) by two copies of the Teichmüller space of S (hence by \mathbb{R}^{12g-12}). See e.g. [126] for details and references.

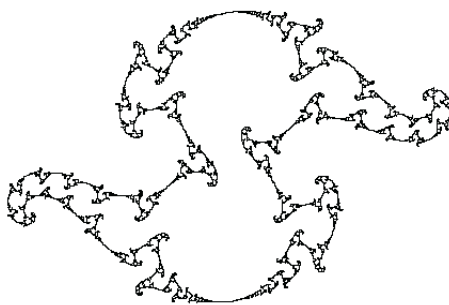


Figure 4.3 The limit set (an invariant topological circle) of a quasi-Fuchsian group in $\partial_\infty \mathbb{H}^3 \simeq \mathbb{C} \cup \{\infty\}$

Deformations of Fuchsian representations for higher-dimensional groups

Recall that $\text{PSL}(2, \mathbb{R}) \simeq \text{SO}(2, 1)_0$ and $\text{PSL}(2, \mathbb{C}) \simeq \text{SO}(3, 1)_0$. We now consider any integer $n \geq 2$ and let Γ be a cocompact lattice of $\text{SO}(n, 1)_0$. As above, we can see Γ as a discrete subgroup of $\text{SO}(n+1, 1)$ (which is not a lattice anymore). Interestingly, although all continuous deformations of

Γ in $\mathrm{SO}(n, 1)$ are trivial for $n \geq 3$ (by Mostow rigidity, see Section 4.2.4), there can exist nontrivial continuous deformations of Γ in $\mathrm{SO}(n + 1, 1)$. Such deformations were constructed in [94, 108] based on a construction of Thurston called *bending*.

The idea is the following. The cocompact lattice Γ of $\mathrm{SO}(n, 1)_0$ defines a closed hyperbolic manifold $M = \Gamma \backslash \mathbb{H}^n$ whose fundamental group $\pi_1(M)$ identifies with Γ . Suppose that M admits a closed totally geodesic embedded hypersurface N . Its fundamental group $\pi_1(N)$ is a subgroup of Γ contained in a copy of $\mathrm{SO}(n - 1, 1)$ inside $\mathrm{SO}(n, 1)$. In particular, the centraliser of $\pi_1(N)$ in $\mathrm{SO}(n + 1, 1)$ contains a one-parameter subgroup $(g_t)_{t \in \mathbb{R}}$ which is not contained in $\mathrm{SO}(n, 1)$.

If N separates M into two submanifolds M_1 and M_2 , then by van Kampen's theorem $\pi_1(M)$ is the amalgamated free product $\pi_1(M_1) *_{\pi_1(N)} \pi_1(M_2)$ of $\pi_1(M_1)$ and $\pi_1(M_2)$ over $\pi_1(N)$. Let $\rho_0 : \Gamma \rightarrow \mathrm{SO}(n + 1, 1)$ be the natural inclusion. A continuous deformation $(\rho_t)_{t \in [0, 1]} \subset \mathrm{Hom}(\Gamma, \mathrm{SO}(n + 1, 1))$ is obtained by defining ρ_t to be ρ_0 when restricted to $\pi_1(M_1)$ and $g_t \rho_0(\cdot) g_t^{-1}$ when restricted to $\pi_1(M_2)$ (these two representations coincide on $\pi_1(N)$).

Otherwise, $M' := M \setminus N$ is connected and $\pi_1(M)$ is an HNN extension of $\pi_1(M')$: it is generated by $\pi_1(M')$ and some element ν with the relations $\nu j_1(\gamma) \nu^{-1} = j_2(\gamma)$ for all $\gamma \in \pi_1(N)$, where $j_1 : \pi_1(N) \rightarrow \pi_1(M)$ and $j_2 : \pi_1(N) \rightarrow \pi_1(M)$ are the inclusions in $\pi_1(M)$ of the fundamental groups of the two sides of N . Let $\rho_0 : \Gamma \rightarrow \mathrm{SO}(n + 1, 1)$ be the natural inclusion. A continuous deformation $(\rho_t)_{t \in [0, 1]} \subset \mathrm{Hom}(\Gamma, \mathrm{SO}(n + 1, 1))$ is obtained by defining ρ_t to be ρ_0 when restricted to $\pi_1(M')$ and setting $\rho_t(\nu) := \nu g_t$ (the relations $\nu j_1(\gamma) \nu^{-1} = j_2(\gamma)$ for $\gamma \in \pi_1(N)$ are preserved since g_t centralises $\pi_1(N)$).

In either case, Johnson and Millson [94] observed that for small enough $t > 0$ the representation ρ_t has Zariski-dense image in $\mathrm{SO}(n + 1, 1)$; moreover, ρ_t is still injective and discrete for small t (see Section 4.4.2).

Remarks 4.3.4 (1) In this construction, ρ_t is not injective and discrete for all $t \in \mathbb{R}$. Indeed, the one-parameter subgroup $(g_t)_{t \in \mathbb{R}}$ takes values in a copy of $\mathrm{SO}(2)$ in $\mathrm{SO}(n + 1, 1)$, which centralises $\rho_0(\pi_1(N))$. For $t \in \mathbb{R}$ such that $g_t = -I$ in $\mathrm{SO}(2)$, the representation ρ_t takes values in $\mathrm{SO}(n, 1)$ but is not injective and discrete.

(2) The fact that ρ_t is injective and discrete for small t also follows from Maskit's combination theorems, which generalise the idea of ping pong to amalgamated free products and HNN extensions (see [113, § VIII.E.3]).

4.3.2 Ping pong in higher real rank

Examples of “flexible” discrete subgroups of higher-rank semisimple Lie groups G which are nonabelian free groups can be constructed by generalising the classical Schottky groups of Section 4.3.1 in various ways. Let us mention three geometric constructions.

Ping pong in projective space

The idea of the following construction goes back to Tits [128] in his proof of the Tits alternative. The construction was later studied in a more quantitative way by Benoist [15]. It works in any flag variety G/P where G is a noncompact semisimple Lie group and P a proper parabolic subgroup of G , but for simplicity we consider the projective space $\mathbb{P}(\mathbb{R}^d)$ which is a flag variety of $G = \mathrm{SL}(d, \mathbb{R})$, for $d \geq 3$. We fix a Riemannian metric $d_{\mathbb{P}(\mathbb{R}^d)}$ on $\mathbb{P}(\mathbb{R}^d)$.

An element $g \in G$ is said to be *biproximal* in $\mathbb{P}(\mathbb{R}^d)$ if it admits a unique complex eigenvalue of highest modulus and a unique complex eigenvalue of lowest modulus, and if these two eigenvalues (which are then necessarily real) have multiplicity 1; equivalently, g is conjugate to a block-diagonal matrix $\mathrm{diag}(t, A, s^{-1})$ where $t, s > 1$ and $A \in \mathrm{GL}(d-2, \mathbb{R})$ is such that the spectral radii of A and A^{-1} are $< t$ and $< s$, respectively (for instance, A could be the identity matrix). In this case, g has a unique attracting fixed point x_g^+ and a unique repelling fixed point x_g^- in $\mathbb{P}(\mathbb{R}^d)$, corresponding to the eigenspaces for the highest and lowest eigenvalues. More precisely, g has the following “North-South dynamics” on $\mathbb{P}(\mathbb{R}^d)$:

- it preserves a unique projective hyperplane X_g^+ (resp. X_g^-) of $\mathbb{P}(\mathbb{R}^d)$ containing x_g^+ (resp. x_g^-), corresponding to the sum of the generalised eigenspaces for the eigenvalues of nonminimal (resp. nonmaximal) modulus,
- for any $x \in \mathbb{P}(\mathbb{R}^d) \setminus X_g^\mp$ we have $g^{\pm k} \cdot x \rightarrow x_g^\pm$ as $k \rightarrow +\infty$, uniformly on compact sets.

In particular, if we fix $\varepsilon > 0$, then any large power of g sends the complement of the open uniform ε -neighbourhood $B_{g^{-1}}^\varepsilon$ of X_g^- into the closure of the open ball b_g^ε of radius ε centred at x_g^+ for $d_{\mathbb{P}(\mathbb{R}^d)}$, and g^{-1} has a similar behaviour after replacing $(B_{g^{-1}}^\varepsilon, X_g^-, b_g^\varepsilon, x_g^+)$ by $(B_g^\varepsilon, X_g^+, b_{g^{-1}}^\varepsilon, x_g^-)$ (see Figure 4.4).

Let $\gamma_1, \dots, \gamma_m \in G$ be biproximal elements which are “transverse” in the sense that $x_{\gamma_i}^+, x_{\gamma_i}^- \notin X_{\gamma_j}^+ \cup X_{\gamma_j}^-$ for all $1 \leq i \neq j \leq m$ (in other words, the configuration of pairs $(x_{\gamma_i}^\bullet, X_{\gamma_i}^\bullet)_{1 \leq i \leq m, \bullet \in \{+, -\}}$ with $x_{\gamma_i}^\bullet \in X_{\gamma_i}^\bullet$ is

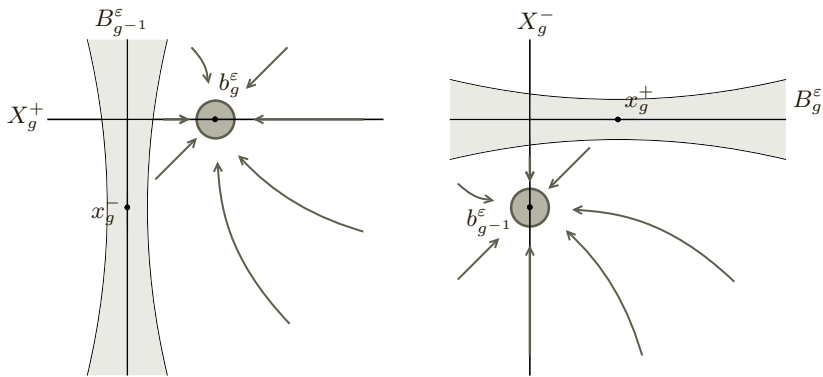


Figure 4.4 Left (resp. right) panel: the dynamics of a large power of g (resp. g^{-1}) on $\mathbb{P}(\mathbb{R}^d)$ for a biproximal element $g \in \mathrm{SL}(d, \mathbb{R})$

generic). Up to replacing each γ_i by a large power, we may assume that there exists $\varepsilon > 0$ such that $\mathbb{P}(\mathbb{R}^d) \setminus \bigcup_{i=1}^m (B_{\gamma_i}^\varepsilon \cup B_{\gamma_i^{-1}}^\varepsilon)$ has nonempty interior and such that for any $\alpha \neq \beta$ in $\{\gamma_1, \gamma_1^{-1}, \dots, \gamma_m, \gamma_m^{-1}\}$, the sets b_α^ε and B_β^ε have disjoint closures in $\mathbb{P}(\mathbb{R}^d)$ and α sends the interior of $\mathbb{P}(\mathbb{R}^d) \setminus B_{\alpha^{-1}}^\varepsilon$ into b_α^ε (see Figure 4.5 for $m = 2$).

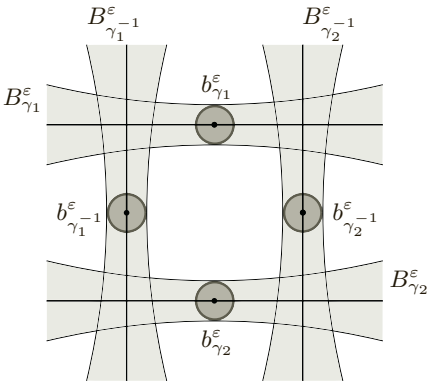


Figure 4.5 A ping pong configuration as in Claim 4.3.5

Let Γ be the subgroup of G generated by $\gamma_1, \dots, \gamma_m$. The following is analogous to Claim 4.3.2.

Claim 4.3.5 *The group Γ is a nonabelian free group with free generating subset $\{\gamma_1, \dots, \gamma_m\}$. It is discrete in G .*

Proof Consider any reduced word $\gamma = \gamma_{i_1}^{\sigma_1} \dots \gamma_{i_N}^{\sigma_N}$ in the alphabet $\{\gamma_1^{\pm 1}, \dots, \gamma_m^{\pm 1}\}$, where $1 \leq i_j \leq m$ and $\sigma_j \in \{\pm 1\}$ for all $1 \leq j \leq N$. Using the inclusions $\alpha \cdot \text{Int}(\mathbb{P}(\mathbb{R}^d) \setminus B_{\alpha^{-1}}^\varepsilon) \subset b_\alpha^\varepsilon$ for $\alpha = \gamma_{i_j}^{\sigma_j}$ and $b_\alpha^\varepsilon \subset \text{Int}(\mathbb{P}(\mathbb{R}^d) \setminus B_{\beta^{-1}}^\varepsilon)$ for $(\alpha, \beta) = (\gamma_{i_j}^{\sigma_j}, \gamma_{i_{j-1}}^{\sigma_{j-1}})$ with $j \geq 2$, we see that the element of Γ corresponding to γ sends $\mathbb{P}(\mathbb{R}^d) \setminus \bigcup_{i=1}^m (B_{\gamma_i}^\varepsilon \cup B_{\gamma_i^{-1}}^\varepsilon)$ into the closure of $b_{\gamma_{i_1}^{\sigma_1}}^\varepsilon$ (hence of $B_{\gamma_{i_1}^{\sigma_1}}^\varepsilon$) in $\mathbb{P}(\mathbb{R}^d)$. On the other hand, the set of elements $g \in G$ sending $\mathbb{P}(\mathbb{R}^d) \setminus \bigcup_{i=1}^m (B_{\gamma_i}^\varepsilon \cup B_{\gamma_i^{-1}}^\varepsilon)$ into the closure of $\bigcup_{i=1}^m (B_{\gamma_i}^\varepsilon \cup B_{\gamma_i^{-1}}^\varepsilon)$ in $\mathbb{P}(\mathbb{R}^d)$ is a closed subset of G that does not contain the identity element. \square

Similarly to the classical strong Schottky groups of Section 4.3.1, the group Γ admits nontrivial continuous deformations $(\rho_t)_{t \in [0,1)} \subset \text{Hom}(\Gamma, G)$, obtained by independently deforming the image of each generator γ_i ; moreover, there is a neighbourhood of the natural inclusion $\rho_0 : \Gamma \hookrightarrow G$ consisting entirely of injective and discrete representations.

Schottky groups with disjoint ping pong domains

In certain situations it is possible to construct discrete subgroups of G , with ping pong dynamics, for which the ping pong domains are pairwise disjoint, as in the case of the classical rank-one Schottky groups of Section 4.3.1. Achieving this disjointness may require using a slightly modified ping pong compared to Figures 4.4 and 4.5, allowing the attracting and repelling subsets of the generators to be larger than points.

Such a construction has been made in $G = \text{PGL}(2n, \mathbb{K})$, acting on the projective space $\mathbb{P}(\mathbb{K}^{2n})$, for $\mathbb{K} = \mathbb{R}$ or \mathbb{C} : the first examples were constructed by Nori in the 1980s, for $\mathbb{K} = \mathbb{C}$, then generalised by Seade and Verjovsky (see [124]); it was observed in [91] that the construction also works for $\mathbb{K} = \mathbb{R}$. The idea is to consider pairwise disjoint $(n-1)$ -dimensional projective subspaces $X_1^+, X_1^-, \dots, X_m^+, X_m^-$ of $\mathbb{P}(\mathbb{K}^{2n})$ and elements $\gamma_1, \dots, \gamma_m \in G$ such that for any $1 \leq i \leq m$ we have $\gamma_i^{\pm k} \cdot x \rightarrow X_i^\pm$ for all $x \in \mathbb{P}(\mathbb{K}^{2n}) \setminus X_i^\mp$ as $k \rightarrow +\infty$, uniformly on compact sets. Up to replacing each γ_i by a large power, we may assume that there exist tubular neighbourhoods B_i^\pm of X_i^\pm such that $B_1^+, B_1^-, \dots, B_m^+, B_m^-$ are pairwise disjoint, $\mathbb{P}(\mathbb{K}^{2n}) \setminus \bigcup_{i=1}^m (B_i^- \cup B_i^+)$ has nonempty interior, and $\gamma_i \cdot \text{Int}(\mathbb{P}(\mathbb{K}^{2n}) \setminus B_i^-) = B_i^+$ for all i . Then the subgroup Γ of G generated by $\gamma_1, \dots, \gamma_m$ is a nonabelian free group with free generating subset $\{\gamma_1, \dots, \gamma_m\}$. It is discrete in G , and it acts properly discontinuously on $\Omega := \text{Int}(\bigcup_{\gamma \in \Gamma} \gamma \cdot \mathcal{D})$ with fundamental domain \mathcal{D} , as in Remark 4.3.3. As for the classical strong Schottky groups of Section 4.3.1, there is a

neighbourhood of the natural inclusion $\rho_0 : \Gamma \hookrightarrow G$ consisting entirely of injective and discrete representations.

Crooked Schottky groups

Here is another ping pong construction, introduced and studied in [36].

Let $G = \mathrm{Sp}(2n, \mathbb{R})$ be the group of elements of $\mathrm{GL}(2n, \mathbb{R})$ that preserve the skew-symmetric bilinear form $\omega(v, v') = \sum_{i=1}^{2n} (-1)^i v_i v'_{2n+1-i}$ on \mathbb{R}^{2n} . A *symplectic basis* of \mathbb{R}^{2n} is a basis in which the matrix of ω is antidiagonal with entries $1, -1, \dots, 1, -1$; for instance, the canonical basis is a symplectic basis. To any symplectic basis (e_1, \dots, e_{2n}) of \mathbb{R}^{2n} we associate an open simplex $B = \mathbb{P}(\mathbb{R}^{>0}\text{-span}(e_1, \dots, e_{2n}))$ in $\mathbb{P}(\mathbb{R}^{2n})$, which we call a *symplectic simplex*. Its *dual* $B^* := \{[v] \in \mathbb{P}(\mathbb{R}^{2n}) \mid \mathbb{P}(v^\perp) \cap \overline{B} = \emptyset\}$ (where v^\perp denotes the orthogonal of v with respect to ω and \overline{B} the closure of B in $\mathbb{P}(\mathbb{R}^{2n})$) is still a symplectic simplex, associated to the symplectic basis $(e_{2n}, -e_{2n-1}, \dots, e_2, -e_1)$. Note that B and B^* are two of the 2^{2n-1} connected components of $\mathbb{P}(\mathbb{R}^{2n}) \setminus \bigcup_{i=1}^{2n} \mathbb{P}(e_i^\perp)$ (see Figure 4.6). We also make the elementary observation that for any symplectic simplices B_1 and B_2 , we have $B_1 \subset B_2^*$ if and only if $B_2 \subset B_1^*$.

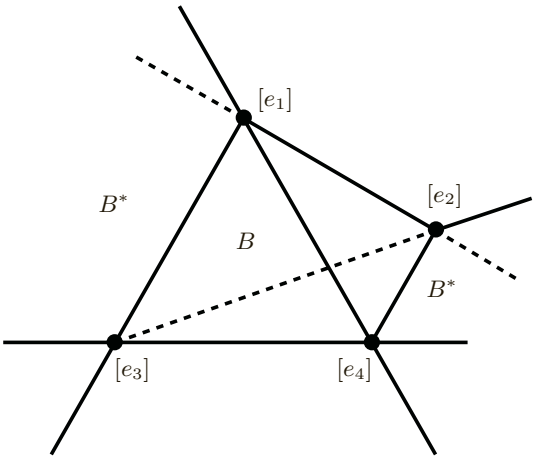


Figure 4.6 A symplectic simplex B of $\mathbb{P}(\mathbb{R}^4)$, associated to a symplectic basis (e_1, e_2, e_3, e_4) of \mathbb{R}^4 , and its dual B^*

Lemma 4.3.6 *For any $m \geq 2$, there exist $2m$ symplectic simplices B_1^\pm, \dots, B_m^\pm in $\mathbb{P}(\mathbb{R}^{2n})$ such that $B \subset B'^*$ for all $B \neq B'$ in $\{B_1^\pm, \dots, B_m^\pm\}$.*

Proof Choose any symplectic simplex B_1 . Note that any nonempty open subset of $\mathbb{P}(\mathbb{R}^{2n})$ contains a symplectic simplex; therefore, we can find a symplectic simplex B_2 such that $\overline{B_2} \subset B_1^*$. Moreover, any neighbourhood of the closure of a symplectic simplex meets the dual of the simplex; therefore $B_1^* \cap B_2^*$ is nonempty. By induction, we construct symplectic simplices B_1, \dots, B_{2m} such that $\overline{B_j} \subset \bigcap_{i=1}^{j-1} B_i^*$ for all $2 \leq j \leq 2m$. We then have $B_j \subset B_i^*$ for all $1 \leq i < j \leq 2m$. By the elementary observation above, we also have $B_i \subset B_j^*$ for all $1 \leq i < j \leq 2m$. We can then take $(B_i^+, B_i^-) := (B_i, B_{m+i})$ for all $1 \leq i \leq m$. \square

For $m \geq 2$, choose symplectic simplices B_1^\pm, \dots, B_m^\pm as in Lemma 4.3.6, and elements $\gamma_1, \dots, \gamma_m \in G$ such that $\gamma_i \cdot (B_i^-)^* = B_i^+$ for all $1 \leq i \leq m$ (these exist since G acts transitively on the set of symplectic simplices). A ping pong argument as in Claim 4.3.2 shows that the subgroup Γ of G generated by $\gamma_1, \dots, \gamma_m$ is a nonabelian free group with free generating subset $\{\gamma_1, \dots, \gamma_m\}$, and that it is discrete in G .

Moreover, there is an interesting counterpart of Remark 4.3.3, not in the projective space $\mathbb{P}(\mathbb{R}^{2n})$, but in the space $\text{Lag}(\mathbb{R}^{2n})$ of *Lagrangians* of $(\mathbb{R}^{2n}, \omega)$, i.e. of n -dimensional linear subspaces of \mathbb{R}^{2n} which are totally isotropic for ω . For this, we associate to any symplectic simplex $B = \mathbb{P}(\mathbb{R}^{>0}\text{-span}(e_1, \dots, e_{2n}))$ of $\mathbb{P}(\mathbb{R}^{2n})$ an open subset of $\text{Lag}(\mathbb{R}^{2n})$, namely $\mathcal{H}(B) := \{L \in \text{Lag}(\mathbb{R}^{2n}) \mid L \cap B \neq \emptyset\}$ where we see each $L \in \text{Lag}(\mathbb{R}^{2n})$ as an $(n-1)$ -dimensional projective subspace of $\mathbb{P}(\mathbb{R}^{2n})$. In [36] we prove the following remarkable property: for any symplectic simplex B , we have $\text{Lag}(\mathbb{R}^{2n}) = \overline{\mathcal{H}(B)} \sqcup \mathcal{H}(B^*)$. In other words, $\mathcal{H}(B)$ and $\mathcal{H}(B^*)$ are two open “half-spaces” of $\text{Lag}(\mathbb{R}^{2n})$, bounded by their common boundary $\partial\mathcal{H}(B) = \overline{\mathcal{H}(B)} \setminus \mathcal{H}(B) = \overline{\mathcal{H}(B^*)} \setminus \mathcal{H}(B^*)$. We observe [36] that these boundaries $\partial\mathcal{H}(B)$ are nice geometric objects which for $n=2$ coincide with the *crooked surfaces* of Frances [72] in the Einstein universe $\text{Ein}^3 \simeq \text{Lag}(\mathbb{R}^4)$. Remark 4.3.3 generalises as follows: consider symplectic simplices B_1^\pm, \dots, B_m^\pm and elements $\gamma_1, \dots, \gamma_m \in G$ as above. If we set $\mathcal{D} := \text{Lag}(\mathbb{R}^{2n}) \setminus \bigcup_{i=1}^m (\mathcal{H}(B_i^-) \cup \mathcal{H}(B_i^+))$, then the group Γ generated by $\gamma_1, \dots, \gamma_m$ acts properly discontinuously on $\Omega := \text{Int}(\bigcup_{\gamma \in \Gamma} \gamma \cdot \mathcal{D})$ with fundamental domain \mathcal{D} . We call Γ a *crooked Schottky group*.

As in the classical case of Section 4.3.1, continuous deformations of configurations of symplectic simplices yield nontrivial continuous deformations $(\rho_t)_{t \in [0,1]} \subset \text{Hom}(\Gamma, G)$ of crooked Schottky groups Γ for which each ρ_t is injective with discrete image. If the symplectic simplices B_1^\pm, \dots, B_m^\pm in the initial configuration have pairwise disjoint *closures* (i.e. Γ is a “strong” crooked Schottky group), then the natural inclusion

$\rho_0 : \Gamma \hookrightarrow G$ admits an open neighbourhood in $\text{Hom}(\Gamma, G)$ consisting entirely of injective and discrete representations.

4.3.3 Higher-rank deformations of Fuchsian representations

Inspired by the quasi-Fuchsian representations and their higher-dimensional analogues from Section 4.3.1, here is one strategy for constructing “flexible” infinite discrete subgroups, beyond nonabelian free groups, in semisimple Lie groups G of higher real rank. Consider a finitely generated group Γ_0 , an injective and discrete representation σ_0 of Γ_0 into a simple Lie group G' of real rank one, and a nontrivial Lie group homomorphism $\tau : G' \rightarrow G$. Consider the composed representation

$$\rho_0 : \Gamma_0 \xrightarrow{\sigma_0} G' \xrightarrow{\tau} G.$$

In some important cases (see e.g. Facts 4.5.6 and 4.5.7), there will be an open neighbourhood of ρ_0 in $\text{Hom}(\Gamma_0, G)$ consisting entirely of injective and discrete representations. The goal is then to deform ρ_0 nontrivially in $\text{Hom}(\Gamma_0, G)$ outside of $\text{Hom}(\Gamma_0, G')$, so as to obtain discrete subgroups of G that are isomorphic to Γ_0 but not conjugate to Γ_0 or any subgroup of G' (ideally Zariski-dense discrete subgroups of G).

This strategy works well, for instance, for $\Gamma_0 = \pi_1(S)$ where S is a closed orientable surface of genus ≥ 2 as in Section 4.3.1, and $G' = \text{SL}(2, \mathbb{R})$ or $\text{PSL}(2, \mathbb{R})$. Let us give three examples in this setting.

Barbot representations

For $d \geq 2$, consider the standard embedding $\tau : G' = \text{SL}(2, \mathbb{R}) \hookrightarrow G = \text{SL}(d, \mathbb{R})$, acting trivially on a $(d-2)$ -dimensional linear subspace of \mathbb{R}^d . Then there is a neighbourhood of ρ_0 in $\text{Hom}(\Gamma_0, G)$ consisting entirely of injective and discrete representations (see Section 4.5.1). Nontrivial continuous deformations $(\rho_t)_{t \in [0,1]} \subset \text{Hom}(\Gamma_0, G)$ exist; for $d = 3$, they were studied by Barbot, who particularly investigated [8] the case that the ρ_t take values in $\text{GL}(2, \mathbb{R}) \ltimes \mathbb{R}^2$, seen as the subgroup of G consisting of lower block-triangular matrices with blocks of size $(2, 1)$.

Hitchin representations

For $d \geq 2$, consider the irreducible embedding $\tau_d : G' = \text{PSL}(2, \mathbb{R}) \hookrightarrow G = \text{PSL}(d, \mathbb{R})$. It is unique modulo conjugation by $\text{PGL}(d, \mathbb{R})$, and given concretely as follows: identify \mathbb{R}^d with the vector space $\mathbb{R}[X, Y]_{d-1}$ of

real polynomials in two variables X, Y which are homogeneous of degree $d - 1$. The group $\mathrm{SL}(2, \mathbb{R})$ acts on $\mathbb{R}[X, Y]_{d-1}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot P \begin{pmatrix} X \\ Y \end{pmatrix} = P \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} X \\ Y \end{pmatrix} \right),$$

and this defines an irreducible representation $\mathrm{SL}(2, \mathbb{R}) \rightarrow \mathrm{SL}(\mathbb{R}[X, Y]_{d-1}) \simeq \mathrm{SL}(d, \mathbb{R})$, which is injective if d is even, and has kernel $\{\pm I\}$ if d is odd. It factors into an embedding $\tau_d : \mathrm{PSL}(2, \mathbb{R}) \hookrightarrow \mathrm{PSL}(d, \mathbb{R})$. In this setting, the following result was proved by Choi–Goldman [46] for $d = 3$, and by Labourie [109] and Fock–Goncharov [70] for general d (recall that the case $d = 2$ is due to Goldman [77]).

Theorem 4.3.7 *Let $\Gamma_0 = \pi_1(S)$ be a closed surface group and $\sigma_0 : \Gamma_0 \rightarrow \mathrm{PSL}(2, \mathbb{R})$ an injective and discrete representation. For any $d \geq 2$, the connected component of $\rho_0 := \tau_d \circ \sigma_0$ in $\mathrm{Hom}(\Gamma_0, \mathrm{PSL}(d, \mathbb{R}))$ consists entirely of injective and discrete representations.*

The image of this connected component in the $\mathrm{PSL}(d, \mathbb{R})$ -character variety of Γ_0 had previously been studied by Hitchin [92], and is now known as the *Hitchin component*. The corresponding representations are called *Hitchin representations*.

Rough sketch of the proofs of Theorem 4.3.7 The proof of Choi–Goldman [46] for $d = 3$ is geometric. The point is that the group $\tau_3(\mathrm{PSL}(2, \mathbb{R})) \simeq \mathrm{SO}(2, 1)_0$ preserves a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle_{2,1}$ of signature $(2, 1)$ on \mathbb{R}^3 ; in particular, it preserves the open subset

$$\Omega = \{[v] \in \mathbb{P}(\mathbb{R}^3) \mid \langle v, v \rangle_{2,1} < 0\}$$

of the projective plane $\mathbb{P}(\mathbb{R}^3)$, which is a model for the hyperbolic plane \mathbb{H}^2 (see (4.1)). This set Ω is *properly convex*: it is convex and bounded in some affine chart of $\mathbb{P}(\mathbb{R}^3)$ (e.g. it is the open unit disk in the affine chart $\{v_3 = 1\}$, see Figure 4.7, left). The group $\mathrm{PSL}(2, \mathbb{R})$ acts properly and transitively on Ω via τ_3 , hence Γ_0 acts properly discontinuously with compact quotient on Ω via $\rho_0 = \tau_3 \circ \sigma_0$. By work of Koszul, the set of representations through which Γ_0 acts properly discontinuously with compact quotient on some nonempty properly convex open subset of $\mathbb{P}(\mathbb{R}^3)$ is open in $\mathrm{Hom}(\Gamma_0, \mathrm{PSL}(3, \mathbb{R}))$. Choi and Goldman proved that this set is also closed. Therefore the entire connected component of ρ_0 consists of such representations, and they are injective and discrete.

The proofs of Labourie and Fock–Goncharov for general d are dynamical. They involve two key objects. The first one is the *Gromov boundary*

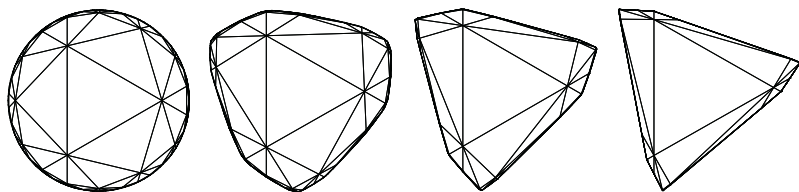


Figure 4.7 The left-most picture shows the projective model of \mathbb{H}^2 (an open disk in an affine chart of the projective plane $\mathbb{P}(\mathbb{R}^3)$), tiled by fundamental domains for the action of a triangle group T given by some injective and discrete representation $\rho_0 : T \rightarrow \mathrm{PO}(2, 1) \subset \mathrm{PGL}(3, \mathbb{R})$. (Note that T admits a finite-index subgroup which is a closed surface group $\Gamma_0 = \pi_1(S)$ as in the proof of Theorem 4.3.7.) The other pictures show the effect of a continuous deformation of ρ_0 in $\mathrm{Hom}(T, \mathrm{PGL}(3, \mathbb{R}))$: the open disk deforms into an invariant properly convex open subset of $\mathbb{P}(\mathbb{R}^3)$ which is not a disk anymore, and the tiling deforms as the action of T remains properly discontinuous and cocompact. These pictures are taken from [16].

$\partial_\infty \Gamma_0$ of Γ_0 : by definition, this is the visual boundary of a proper geodesic metric space on which Γ_0 acts properly discontinuously, by isometries, with compact quotient; in our situation, Γ_0 is a closed surface group and $\partial_\infty \Gamma_0$ is the visual boundary of \mathbb{H}^2 , namely a circle. The second key object is the space $\mathrm{Flags}(\mathbb{R}^d)$ of full flags ($V_1 \subset \cdots \subset V_{d-1} \subset \mathbb{R}^d$) of \mathbb{R}^d (where each V_i is an i -dimensional linear subspace of \mathbb{R}^d); this space $\mathrm{Flags}(\mathbb{R}^d)$ is compact with a transitive action of $G = \mathrm{PSL}(d, \mathbb{R})$, and may be thought of as a kind of “boundary” for G or its symmetric space. The point of the proof is then to show that for any Hitchin representation $\rho : \Gamma_0 \rightarrow G$, there exists a continuous, injective, ρ -equivariant “boundary map” $\xi : \partial_\infty \Gamma_0 \rightarrow \mathrm{Flags}(\mathbb{R}^d)$. (By ρ -equivariant we mean that $\xi(\gamma \cdot w) = \rho(\gamma) \cdot \xi(w)$ for all $\gamma \in \Gamma_0$ and all $w \in \partial_\infty \Gamma_0$.) The existence of such a boundary map ξ easily implies that ρ is injective and discrete: see the proof of Lemma 4.5.3.(3). Indeed, the idea is that the continuous, injective, equivariant map ξ “transfers”, to $\mathrm{Flags}(\mathbb{R}^d)$, the dynamics of the intrinsic action of Γ_0 on $\partial_\infty \Gamma_0$, which is a so-called *convergence action*: any sequence $(\gamma_k)_{k \in \mathbb{N}}$ of pairwise distinct elements of Γ_0 comes with some contraction in $\partial_\infty \Gamma_0$, hence (using ξ) the sequence $(\rho(\gamma_k))_{k \in \mathbb{N}}$ comes with some contraction in $\mathrm{Flags}(\mathbb{R}^d)$, and this prevents $(\rho(\gamma_k))_{k \in \mathbb{N}}$ from converging to the identity element of G .

We note that the existence of continuous, injective, equivariant boundary maps for Hitchin representations is obtained by an open-and-closed argument, using stronger properties satisfied by these maps (namely,

some uniform forms of contraction and transversality for Labourie, and a positivity property for Fock and Goncharov). \square

Maximal representations

For $n \geq 2$, consider the embedding $\tau : G' = \mathrm{PSL}(2, \mathbb{R}) \simeq \mathrm{SO}(2, 1)_0 \hookrightarrow \mathrm{SO}(2, n)$. Then the entire connected component of ρ_0 in $\mathrm{Hom}(\Gamma_0, G)$ consists of injective and discrete representations, as was proved by Burger, Iozzi, and Wienhard [38]. This is an example of a so-called *maximal component*: it consists of representations (called *maximal representations*) that maximise the *Toledo invariant* (a topological invariant generalising the Euler number, see e.g. [39, §5.1]).

4.3.4 Higher Teichmüller theory

We already encountered in Sections 4.2.4 and 4.3.1 the Teichmüller space of a closed surface S of genus ≥ 2 . It is a fundamental object in many areas of mathematics, which can be viewed both as a moduli space for marked complex structures on S or, via the Uniformisation Theorem, as a moduli space for marked hyperbolic structures on S . In this second point of view, the holonomy representation of the fundamental group $\Gamma_0 = \pi_1(S)$ naturally realises the Teichmüller space of S as a connected component of the G -character variety of Γ_0 for $G = \mathrm{PSL}(2, \mathbb{R})$, corresponding to the image, modulo conjugation by G at the target, of a connected component of $\mathrm{Hom}(\Gamma_0, G)$ consisting entirely of injective and discrete representations.

An interesting and perhaps surprising phenomenon, which has led to a considerable amount of research in the past twenty years, is that for certain semisimple Lie groups G of higher real rank, there also exist connected components of $\mathrm{Hom}(\Gamma_0, G)$ consisting entirely of injective and discrete representations, and which are nontrivial in the sense that they are not reduced to a single representation and its conjugates by G . The images in the G -character variety of these components are now called *higher(-rank) Teichmüller spaces*. We saw two examples in Section 4.3.3:

- *Hitchin components* when G is $\mathrm{PSL}(d, \mathbb{R})$, or more generally a real split simple Lie group; these are by definition components containing a Fuchsian representation $\rho_0 : \Gamma_0 \hookrightarrow \mathrm{PSL}(2, \mathbb{R}) \hookrightarrow G$, where $\mathrm{PSL}(2, \mathbb{R}) \hookrightarrow G$ is the so-called *principal embedding*;
- *maximal components* when G is $\mathrm{SO}(2, n)$, or more generally a simple Lie group of Hermitian type; these are by definition components of representations that maximise the Toledo invariant.

See [39, 41, 103, 120, 134] for details about these examples.

Towards a full list of higher Teichmüller spaces

Recently, new higher Teichmüller spaces were discovered in [22, 33, 83] when G is $O(p, q)$ with $p \neq q$ or an exceptional simple Lie group whose restricted root system is of type F_4 . These higher Teichmüller spaces consist of so-called Θ -positive representations, introduced by Guichard and Wienhard [85, 86]. Notions of positivity for Hitchin representations and maximal representations had been previously found by Fock–Goncharov [70] (based on Lusztig’s total positivity [111]) and Burger–Iozzi–Wienhard [38]; the notion of Θ -positivity encompasses them both. Together with Hitchin components and maximal components, these new Θ -positive components conjecturally (see [85]) form the full list of higher Teichmüller spaces.

Without entering into technical details, let us mention briefly the role of Higgs bundles in this conjectural classification. See [32, 34, 74] for details.

Let Σ be a Riemann surface homeomorphic to S . By definition, a G -Higgs bundle over Σ is a pair (E, φ) where E is a holomorphic $K_{\mathbb{C}}$ -bundle over Σ and φ (the *Higgs field*) is a holomorphic section of a certain natural bundle over Σ associated to E . (Here $K_{\mathbb{C}}$ is the complexification of a maximal compact subgroup K of G .) The *non-Abelian Hodge correspondence* of Hitchin, Donaldson, Corlette, Simpson, and others (see [75]), gives a homeomorphism between the G -character variety of $\pi_1(S)$ and the moduli space $\mathcal{M}_G(\Sigma)$ of so-called *polystable* G -Higgs bundles over Σ . This was used by Hitchin to define and study the Hitchin component.

Some of the connected components of the G -character variety of $\pi_1(S)$ can be distinguished using topological invariants. However, such invariants are not sufficient to distinguish them all in general. One fruitful approach is to use the fact, proved by Hitchin, that $(E, \varphi) \mapsto \|\varphi\|_{L^2(\Sigma)}^2$ defines a proper Morse function f from $\mathcal{M}_G(\Sigma)$ to $\mathbb{R}_{\geq 0}$; therefore, the connected components of $\mathcal{M}_G(\Sigma)$ can be studied by examining the local minima of f . The zero locus $f^{-1}(0)$ of f corresponds, in the G -character variety, to representations of $\pi_1(S)$ whose image lies in a compact subgroup of G ; in particular, these representations are not injective and discrete, and so connected components for which f has a local minimum of 0 cannot be higher Teichmüller spaces. This approach has already been successfully exploited to find and count almost all connected components of the G -character variety of $\pi_1(S)$ for simple G , including conjecturally all higher Teichmüller spaces: see [32, 33, 34, 74].

Similarities with the classical Teichmüller space

The study of higher Teichmüller spaces, or *higher Teichmüller theory*, has been very active in the past twenty years. In particular, striking similarities have been found between higher Teichmüller spaces and the classical Teichmüller space of S , including:

- associated notions of positivity (see above);
- for Hitchin components: the topology of $\mathbb{R}^{\dim(G)|\chi(S)|}$ (Hitchin);
- the proper discontinuity of the action of the mapping class group (Labourie, Wienhard);
- good systems of coordinates (Goldman, Fock–Goncharov, Bonahon–Dreyer, Strubel, Zhang);
- analytic Riemannian metrics invariant under the mapping class group (Bridgeman–Canary–Labourie–Sambarino, Pollicott–Sharp);
- natural maps to the space of geodesic currents on S (Labourie, Bridgeman–Canary–Labourie–Sambarino, Martone–Zhang, Ouyang–Tamburelli);
- versions of the collar lemma for the associated locally symmetric spaces (Lee–Zhang, Burger–Pozzetti, Beyrer–Pozzetti, Beyrer–Guichard–Labourie–Pozzetti–Wienhard);
- interpretations of higher Teichmüller spaces as moduli spaces of geometric structures on S or on closed manifolds fibering over S (Choi–Goldman, Guichard–Wienhard, Collier–Tholozan–Touliisse).

There are also conjectural interpretations of higher Teichmüller spaces as moduli spaces of “higher complex structures” on S (Fock–Thomas), as well as various approaches to see higher Teichmüller spaces as mapping-class-group-equivariant fiber bundles over the classical Teichmüller space of S (Labourie, Loftin, Alessandrini–Collier, Collier–Tholozan–Touliisse). We refer to [39, 103, 120, 134] for more details and references.

Higher higher Teichmüller spaces

Phenomena analogous to Theorem 4.3.7 have also been uncovered for fundamental groups of higher-dimensional manifolds, in two situations.

The first one is in the context of *convex projective geometry*, which is by definition the study of properly convex open subsets Ω of real projective spaces $\mathbb{P}(\mathbb{R}^d)$, as in Choi–Goldman’s proof of Theorem 4.3.7 for $d = 3$. Let $\Gamma_0 = \pi_1(M)$ where M is a closed topological manifold of dimension $n \geq 2$. Generalising Theorem 4.3.7, Benoist [17] proved that if Γ_0 does not contain an infinite nilpotent normal subgroup, then the set of representations through which Γ_0 acts properly discontinuously

with compact quotient on some properly convex open subset of $\mathbb{P}(\mathbb{R}^{n+1})$ is closed in $\text{Hom}(\Gamma_0, G)$ for $G = \text{PGL}(n+1, \mathbb{R})$. This set is also open in $\text{Hom}(\Gamma_0, G)$ by Koszul, and so it is a union of connected components of $\text{Hom}(\Gamma_0, G)$. It consists entirely of injective and discrete representations. Recent results of Marseglia and Cooper–Tillman extend this to some cases where M and the quotients of the properly convex sets are not necessarily closed (see [50]).

The second situation is in the context of *pseudo-Riemannian hyperbolic geometry*, which is by definition the study of pseudo-Riemannian manifolds (i.e. smooth manifolds with a smooth assignment, to each tangent space, of a nondegenerate quadratic form) which have constant negative sectional curvature. In signature (p, q) , such manifolds are locally modeled on the pseudo-Riemannian symmetric space $\mathbb{H}^{p,q} = \text{PO}(p, q+1)/\text{P}(\text{O}(p) \times \text{O}(q+1))$, which can be realised as an open set in projective space, namely $\{[v] \in \mathbb{P}(\mathbb{R}^{p+q+1}) \mid \langle v, v \rangle_{p,q+1} < 0\}$ where $\langle \cdot, \cdot \rangle_{p,q+1}$ is a symmetric bilinear form of signature $(p, q+1)$ on \mathbb{R}^{p+q+1} . For $q = 0$ we recover the real hyperbolic space \mathbb{H}^p , with its projective model (4.1), and for $q = 1$ the space $\mathbb{H}^{p,1}$ is the $(p+1)$ -dimensional *anti-de Sitter space* (a Lorentzian analogue of the real hyperbolic space). Let $\Gamma_0 = \pi_1(M)$ where M is a closed hyperbolic p -manifold, with holonomy $\sigma_0 : \Gamma_0 \rightarrow \text{O}(p, 1)$, and let $\tau : \text{O}(p, 1) \hookrightarrow G = \text{PO}(p, q+1)$ be the standard embedding. For $q = 1$, Barbot [9] proved that the connected component of $\rho_0 = \tau \circ \sigma_0$ in $\text{Hom}(\Gamma_0, G)$ consists entirely of injective and discrete representations (corresponding to holonomies of so-called *globally hyperbolic spatially compact* anti-de Sitter manifolds, studied in [114] for $p = 2$). This was recently extended in [21] to general $p \geq 2$ and $q \geq 1$. In fact, the following more general result is proved in [21]: for $\Gamma_0 = \pi_1(M)$ where M is any closed topological manifold of dimension $p \geq 2$, the set of so-called $\mathbb{H}^{p,q}$ -convex cocompact representations is a union of connected components in $\text{Hom}(\Gamma_0, G)$. These $\mathbb{H}^{p,q}$ -convex cocompact representations are injective and discrete representations with a nice geometric behaviour in $\mathbb{H}^{p,q}$ (see Section 4.5.2); they include the representations $\tau \circ \sigma_0 : \Gamma_0 = \pi_1(M) \rightarrow \text{O}(p, 1) \hookrightarrow G = \text{PO}(p, q+1)$ above where M is a closed hyperbolic manifold, but also other examples where M can be quite “exotic” (see [110, 115] for $q = 1$). These representations can have Zariski-dense image in G : see e.g. [21] for a bending argument as in Section 4.3.1.

In these two situations, there are connected components in $\text{Hom}(\Gamma_0, G)$ consisting entirely of injective and discrete representations, where Γ_0 is the fundamental group of an n -dimensional closed manifold with $n > 2$

and G is a semisimple Lie group with $\text{rank}_{\mathbb{R}}(G) \geq 2$. It is natural to call *higher-dimensional higher-rank Teichmüller spaces* (or *higher higher Teichmüller spaces* for short) the images of these components in the G -character variety of Γ_0 . It would be interesting in the future to investigate whether these higher higher Teichmüller spaces have any topological or geometric analogies with classical Teichmüller space or its higher-rank counterparts, as above. See also [134, § 14] for some further discussion.

4.4 Classes of discrete subgroups in real rank one

In Section 4.3 we saw various examples of “flexible” infinite discrete subgroups of semisimple Lie groups. We now present some general theory in which these examples fit, first in real rank one (this section), then in higher real rank (Section 4.5).

More precisely, throughout this section we consider a semisimple Lie group G with $\text{rank}_{\mathbb{R}}(G) = 1$. We discuss two important classes of finitely generated discrete subgroups of G that have received considerable attention, namely convex cocompact subgroups and geometrically finite subgroups. The inclusion relations between these classes and lattices of G are shown in Figure 4.8.

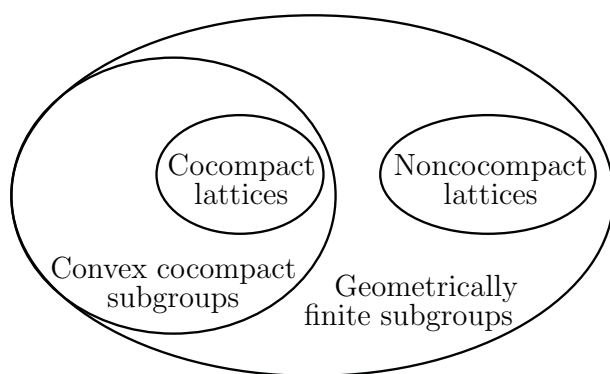


Figure 4.8 Inclusions between four important classes of discrete subgroups of G for $\text{rank}_{\mathbb{R}}(G) = 1$

4.4.1 Definitions

Consider, as in Section 4.2.1, the Riemannian symmetric space $X = G/K$, where K is a maximal compact subgroup of G . If $G = \mathrm{SO}(n, 1)$ (resp. $\mathrm{SU}(n, 1)$, resp. $\mathrm{Sp}(n, 1)$), then X is the n -dimensional hyperbolic space over \mathbb{R} (resp. \mathbb{C} , resp. the quaternions). If G is the exceptional group $F_{4(-20)}$, then X is the “hyperbolic plane over the octonions”.

There is a natural notion of convexity in X : any two points x, y of X are joined by a unique geodesic segment; we say that a subset \mathcal{C} of X is convex if this segment is contained in \mathcal{C} for all $x, y \in \mathcal{C}$. See [66, § 1.6] for more details. For any $\varepsilon > 0$ and any subset \mathcal{C} of X , we denote by $\mathcal{U}_\varepsilon(\mathcal{C})$ the uniform ε -neighbourhood of \mathcal{C} in X .

Definition 4.4.1 Suppose $\mathrm{rank}_{\mathbb{R}}(G) = 1$. A discrete subgroup Γ of G is *convex cocompact* (resp. *geometrically finite*) if it is finitely generated and there is a nonempty Γ -invariant convex subset \mathcal{C} of X such that the quotient $\Gamma \backslash \mathcal{C}$ is compact (resp. the quotient $\Gamma \backslash \mathcal{U}_\varepsilon(\mathcal{C})$ has finite volume for some $\varepsilon > 0$).

Alternatively (see Remark 4.3.1), given a group Γ_0 , we say that a representation $\rho : \Gamma_0 \rightarrow G$ is *convex cocompact* (resp. *geometrically finite*) if it has finite kernel and discrete, convex cocompact (resp. geometrically finite) image.

Remark 4.4.2 If there is a nonempty Γ -invariant convex subset \mathcal{C} of X such that $\Gamma \backslash \mathcal{C}$ is compact, then Γ is automatically finitely generated, by the Švarc–Milnor lemma (see e.g. [65, Th. 8.37]). Thus the assumption that Γ be finitely generated can be omitted in the definition of convex cocompactness. On the other hand, this assumption cannot be omitted in general in the definition of geometric finiteness: see [88].

Remark 4.4.3 Bowditch [31] gave several equivalent definitions of geometric finiteness. Here we use a variation on his definition F5, where the uniform bound on the orders of finite subgroups of Γ is replaced by the assumption that Γ be finitely generated. The two definitions are equivalent by [31] and the Selberg lemma [125, Lem. 8].

We now explain how Definition 4.4.1 can be rephrased in terms of a specific convex set in G/K . For this, we first recall the important notion of the *limit set* of a discrete subgroup of G .

Limit sets and convex cores

Let $\partial_\infty X$ be the visual boundary of $X = G/K$, i.e. the set of equivalence classes of geodesic rays in X for the equivalence relation “to remain at

bounded distance". There is a natural topology on $\overline{X} := X \sqcup \partial_\infty X$ that extends that of X and makes \overline{X} compact, and the action of G on X extends continuously to \overline{X} (see e.g. [66, § 1.7]). For instance, as in Sections 4.3.1 and 4.3.4, if $G = \mathrm{PO}(n, 1)$ and $X = \mathbb{H}^n$, then we can realise X as the open subset (4.1) of $\mathbb{P}(\mathbb{R}^{n+1})$ where some quadratic form of signature $(n, 1)$ is negative, and \overline{X} is then the closed subset of $\mathbb{P}(\mathbb{R}^{n+1})$ where the quadratic form is nonpositive, endowed with the topology from $\mathbb{P}(\mathbb{R}^{n+1})$ and the natural action of $G = \mathrm{PO}(n, 1)$.

Definition 4.4.4 Let Γ be a discrete subgroup of G . The *limit set* of Γ is the set Λ_Γ of accumulation points in \overline{X} of a Γ -orbit of X ; it is contained in $\partial_\infty X$ and does not depend on the choice of Γ -orbit. The *convex core* $\mathcal{C}_\Gamma^{\mathrm{cor}} \subset X$ of Γ is the convex hull of Λ_Γ in X (i.e. the smallest closed convex subset of X whose closure in \overline{X} contains Λ_Γ).

Note that Λ_Γ and $\mathcal{C}_\Gamma^{\mathrm{cor}}$ are both invariant under the action of Γ on \overline{X} .

The limit set Λ_Γ is nonempty if and only if Γ is infinite. This set has either at most two elements (in which case we say Γ is *elementary*), or infinitely many. If Γ is not elementary, then the action of Γ on Λ_Γ is minimal (all orbits are dense), and any nonempty Γ -invariant closed subset of $\partial_\infty X$ contains Λ_Γ (see e.g. [31, § 3.2]); in particular, any nonempty Γ -invariant closed convex subset of X contains the convex core $\mathcal{C}_\Gamma^{\mathrm{cor}}$. We deduce the following.

Fact 4.4.5 Suppose $\mathrm{rank}_\mathbb{R}(G) = 1$. A finitely generated infinite discrete subgroup Γ of G is convex cocompact (resp. geometrically finite) if and only if the quotient $\Gamma \backslash \mathcal{C}_\Gamma^{\mathrm{cor}}$ is compact and nonempty (resp. the quotient $\Gamma \backslash \mathcal{U}_\varepsilon(\mathcal{C}_\Gamma^{\mathrm{cor}})$ has finite volume for some $\varepsilon > 0$).

Remark 4.4.6 In our setting where $\mathrm{rank}_\mathbb{R}(G) = 1$, the group G acts transitively on $\partial_\infty X$. The stabilisers in G of points of $\partial_\infty X$ are the *proper parabolic subgroups* of G . Thus $\partial_\infty X$ is G -equivariantly homeomorphic to G/P where P is a proper parabolic subgroup of G .

4.4.2 Properties

Let us briefly mention a few useful properties of geometrically finite and convex cocompact representations.

Domains of discontinuity

We first observe that any discrete subgroup Γ of G acts properly discontinuously on the open subset $\Omega_\Gamma := \partial_\infty X \setminus \Lambda_\Gamma$ of $\partial_\infty X$, and in fact on

$X \cup \Omega_\Gamma$. Indeed, let \mathcal{C} be a nonempty Γ -invariant closed convex subset of X . One can check that the closest point projection from X to \mathcal{C} extends to a continuous Γ -equivariant map from $X \cup \Omega_\Gamma$ to \mathcal{C} . The fact that Γ acts properly discontinuously on $\mathcal{C} \subset X$ then implies that Γ acts properly discontinuously on $X \cup \Omega_\Gamma$.

If Γ is convex cocompact, then the quotient $\Gamma \backslash \Omega_\Gamma$ is compact (possibly empty), and $\Gamma \backslash (X \cup \Omega_\Gamma)$ is a compactification of $\Gamma \backslash X$.

If Γ is geometrically finite, then $\Gamma \backslash (X \cup \Omega_\Gamma)$ is not necessarily compact, but it has only finitely many topological ends, each of which is a “parabolic end”; this actually characterises geometric finiteness: see [31].

Deformations

Convex cocompactness is stable under small deformations:

Fact 4.4.7 *Suppose $\text{rank}_{\mathbb{R}}(G) = 1$. For any finitely generated group Γ_0 , the space of convex cocompact representations is open in $\text{Hom}(\Gamma_0, G)$.*

On the other hand, geometric finiteness is in general not stable under small deformations. If one restricts to small deformations that are *cuspidal preserving* (i.e. that keep parabolic elements parabolic), then stability holds for $G = \text{PO}(n, 1)$ when $n \leq 3$ or when all cusps have rank $\geq n - 2$, but not in general. See e.g. [82, App. B] for more details and references.

Homomorphisms

Convex cocompactness behaves well under Lie group homomorphisms:

Fact 4.4.8 *Suppose $\text{rank}_{\mathbb{R}}(G) = 1$. Let G' be another semisimple Lie group with $\text{rank}_{\mathbb{R}}(G') = 1$ and let $\tau : G' \rightarrow G$ be a Lie group homomorphism with compact kernel. For any finitely generated group Γ_0 and any representation $\sigma_0 : \Gamma_0 \rightarrow G'$, the composed representation $\tau \circ \sigma_0 : \Gamma_0 \rightarrow G$ is convex cocompact if and only if σ_0 is.*

4.4.3 Examples

- If Γ is a lattice in G , then $\Lambda_\Gamma = \partial_\infty X$ and $\mathcal{C}_\Gamma^{\text{cor}} = X$, and Γ is geometrically finite. If Γ is cocompact in G , then it is convex cocompact.
- Suppose $G = \text{PSL}(2, \mathbb{R}) \simeq \text{PO}(2, 1)_0$. Then every finitely generated discrete subgroup Γ of G is geometrically finite; Γ is convex cocompact if and only if the associated hyperbolic surface $\Gamma \backslash \mathbb{H}^2$ has no cusps.

Remark 4.4.9 On the other hand, for $G = \text{PO}(n, 1)$ with $n \geq 3$, there exist finitely generated discrete subgroups of G which are *not*

geometrically finite. The first examples were given by Bers for $n = 3$ (“singly degenerate” Kleinian groups, for which the domain of discontinuity Ω_Γ is simply connected): see [95, § 2].

- Any discrete subgroup of $G = \mathrm{PO}(n, 1)$ generated by the orthogonal reflections in the faces of a finite-sided right-angled polyhedron of \mathbb{H}^n is geometrically finite; it is convex cocompact if and only if no distinct facets of the polyhedron have closures meeting in $\partial_\infty \mathbb{H}^n$ (see [60, § 4]).
- The Schottky groups of Section 4.3.1 are geometrically finite; the strong Schottky groups (for which B_1^\pm, \dots, B_m^\pm have pairwise disjoint closures) are convex cocompact. Their limit sets are Cantor sets. The set Ω of Remark 4.3.3 is the domain of discontinuity $\Omega_\Gamma = \partial_\infty X \setminus \Lambda_\Gamma$ of Γ in $\partial_\infty X$ from Section 4.4.2.
- Any quasi-Fuchsian group $\Gamma = \rho(\pi_1(S))$ as in Section 4.3.1 is convex cocompact. The limit set Λ_Γ is a topological circle in $\partial_\infty \mathbb{H}^3$ (see Figure 4.3). The quotient $\Gamma \backslash \mathcal{C}_\Gamma^{\mathrm{cor}}$ is homeomorphic to $S \times [0, 1]$.
- The small deformations of cocompact lattices of $G' = \mathrm{SO}(n, 1)$ inside $G = \mathrm{SO}(n + 1, 1)$ from Section 4.3.1 are convex cocompact by Facts 4.4.7 and 4.4.8 (see also Remark 4.3.4.(2)).

4.4.4 A few characterisations of convex cocompactness

Preliminaries

Given a finitely generated group Γ_0 , we choose a finite generating subset F of Γ_0 and denote by $\mathrm{Cay}(\Gamma_0) = \mathrm{Cay}(\Gamma_0, F)$ the corresponding Cayley graph, with its metric $\mathrm{d}_{\mathrm{Cay}(\Gamma_0)}$.

As in Section 4.2.3, a group Γ_0 is called *Gromov hyperbolic* if it is finitely generated and acts properly discontinuously, by isometries, with compact quotient, on some Gromov hyperbolic proper geodesic metric space Y ; in that case, we can take Y to be $\mathrm{Cay}(\Gamma_0)$. As in the proof of Theorem 4.3.7, the *Gromov boundary* $\partial_\infty \Gamma_0$ of Γ_0 is then the visual boundary of Y , endowed with the action of Γ_0 extending that on Y . The Gromov boundary $\partial_\infty \Gamma_0$ does not depend on Y up to Γ_0 -equivariant homeomorphism. An important property is that the action of Γ_0 on $\partial_\infty \Gamma_0$ is a *convergence action*: for any sequence $(\gamma_k)_{k \in \mathbb{N}}$ of pairwise distinct elements of Γ_0 , up to passing to a subsequence, there exist $w^+, w^- \in \partial_\infty \Gamma_0$ such that $\gamma_k \cdot w \rightarrow w^+$ for all $w \in \partial_\infty \Gamma_0 \setminus \{w^-\}$, uniformly on compact sets. Moreover, any infinite-order element of Γ_0 has two fixed points in $\partial_\infty \Gamma_0$, one attracting and one repelling. The group Γ_0 is called *elementary* if it is finite (in which case $\partial_\infty \Gamma_0$ is empty) or

if it admits a finite-index subgroup which is cyclic (in which case $\partial_\infty \Gamma_0$ consists of two points). If Γ_0 is not elementary, then the set of attracting fixed points of infinite-order elements of Γ_0 is infinite and dense in $\partial_\infty \Gamma_0$. See e.g. [14] for details.

Examples 4.4.10 If Γ_0 is a nonabelian free group with finite free generating subset F , then Γ_0 is Gromov hyperbolic, $\text{Cay}(\Gamma_0)$ is a tree, and $\partial_\infty \Gamma_0$ is a Cantor set. If $\Gamma_0 = \pi_1(M)$ for some closed negatively-curved manifold M , then Γ_0 is Gromov hyperbolic, we can take Y to be the universal cover \widetilde{M} of M , and $\partial_\infty \Gamma_0 = \partial_\infty \widetilde{M}$. In particular, if $\Gamma_0 = \pi_1(S)$ for some closed orientable surface of genus ≥ 2 , then Γ_0 is Gromov hyperbolic and $\partial_\infty \Gamma_0$ is a circle (as in the proof of Theorem 4.3.7).

Remark 4.4.11 A Gromov hyperbolic group can never contain a subgroup isomorphic to \mathbb{Z}^2 or to a Baumslag–Solitar group $BS(m, n) := \langle a, t \mid t^{-1}a^mt = a^n \rangle$. Understanding how close this is to characterising Gromov hyperbolic groups is an important question in geometric group theory: see e.g. [76].

For any isometry g of a metric space (M, \mathbf{d}_M) , we define the *translation length* of g in M to be

$$\text{transl}_M(g) := \inf_{m \in M} \mathbf{d}_M(m, g \cdot m) \geq 0. \quad (4.2)$$

Finally, we denote by \mathbf{d}_X the metric on the Riemannian symmetric space $X = G/K$ (see Section 4.2.1). We fix a basepoint $x_0 \in X$, and a Riemannian metric $\mathbf{d}_{\partial_\infty X}$ on the visual boundary $\partial_\infty X$.

A few classical characterisations

Many interesting characterisations of convex cocompactness have been found by various authors including Beardon, Bowditch, Maskit, Sullivan, Thurston, Tukia, and others. We now give a few. We refer to [95, 96] for more details and references, as well as further characterisations (e.g. in terms of *conical limit points*). We also refer to [31, 95] for characterisations of geometric finiteness.

Theorem 4.4.12 *Suppose $\text{rank}_{\mathbb{R}}(G) = 1$. For any infinite group Γ_0 and any representation $\rho : \Gamma_0 \rightarrow G$, the following are equivalent:*

- (1) ρ is convex cocompact (Definition 4.4.1);
- (2) Γ_0 is finitely generated and ρ is a quasi-isometric embedding: there exist $c, c' > 0$ such that for any $\gamma \in \Gamma$,

$$\mathbf{d}_X(x_0, \rho(\gamma) \cdot x_0) \geq c \mathbf{d}_{\text{Cay}(\Gamma_0)}(e, \gamma) - c'; \quad (4.3)$$

- (3) Γ_0 is Gromov hyperbolic and ρ is well-displacing: there exist $c, c' > 0$ such that for any $\gamma \in \Gamma$,

$$\text{transl}_X(\rho(\gamma)) \geq c \text{transl}_{\text{Cay}(\Gamma_0)}(\gamma) - c'; \quad (4.4)$$

- (4) Γ_0 is Gromov hyperbolic and there exists a ρ -equivariant map

$$\xi : \partial_\infty \Gamma_0 \longrightarrow \partial_\infty X$$

which is continuous, injective, and dynamics-preserving (i.e. for any infinite-order element $\gamma \in \Gamma_0$, the image by ξ of the attracting fixed point of γ in $\partial_\infty \Gamma_0$ is an attracting fixed point of $\rho(\gamma)$ in $\partial_\infty X$);

- (5) Γ_0 is Gromov hyperbolic and there exists a ρ -equivariant map

$$\xi : \partial_\infty \Gamma_0 \longrightarrow \partial_\infty X$$

which is continuous, injective, and strongly dynamics-preserving (i.e. for any $(\gamma_k) \in \Gamma_0^{\mathbb{N}}$ and any $w^+, w^- \in \partial_\infty \Gamma_0$, if $\gamma_k \cdot w \rightarrow w^+$ for all $w \in \partial_\infty \Gamma_0 \setminus \{w^-\}$, then $\rho(\gamma_k) \cdot z \rightarrow \xi(w^+)$ for all $z \in \partial_\infty X \setminus \{\xi(w^-)\}$);

- (6) ρ has finite kernel, discrete image, and the action of Γ on $\partial_\infty X$ via ρ is expanding at $\Lambda_{\rho(\Gamma_0)}$, i.e. for any $z \in \Lambda_{\rho(\Gamma_0)}$, there exist a neighbourhood \mathcal{U} of z in $\partial_\infty X$ and an element $\gamma \in \Gamma_0$ such that

$$\inf_{z_1 \neq z_2 \text{ in } \mathcal{U}} \frac{\text{d}_{\partial_\infty X}(\rho(\gamma) \cdot z_1, \rho(\gamma) \cdot z_2)}{\text{d}_{\partial_\infty X}(z_1, z_2)} > 1. \quad (4.5)$$

Remarks 4.4.13 • Using the triangle inequality, one sees that condition (2) does not depend on the choice of basepoint $x_0 \in X$ (changing x_0 may change the values of c, c' but not their existence).

- One also sees that for Γ_0 with finite generating subset F , the reverse inequality $\text{d}_X(x_0, \rho(\gamma) \cdot x_0) \leq C \text{d}_{\text{Cay}(\Gamma_0)}(e, \gamma)$ to (4.3) holds for any representation $\rho : \Gamma_0 \rightarrow G$, with $C := \max_{f \in F} \text{d}_X(x_0, \rho(f) \cdot x_0)$.
- In condition (3) we cannot remove the assumption that Γ_0 be Gromov hyperbolic: for instance, there exist finitely generated infinite groups Γ_0 with only finitely many conjugacy classes [118], and for such Γ_0 any representation $\rho : \Gamma_0 \rightarrow G$ is well-displacing.
- In condition (4), *dynamics-preserving* implies that for any $\gamma \in \Gamma_0$ of infinite order, $\rho(\gamma)$ is a *hyperbolic* element of G (i.e. an element with two fixed points in $\partial_\infty X$, one attracting and one repelling). In condition (5), *strongly dynamics-preserving* means that ξ preserves the convergence action of Γ_0 on $\partial_\infty \Gamma_0$ mentioned above.

Sketches of proofs

Proof of (1) \Rightarrow (2): We may assume that the basepoint x_0 belongs to the convex core $\mathcal{C}_{\rho(\Gamma_0)}^{\text{cor}}$. By the Švarc–Milnor lemma (see e.g. [65, Th. 8.37]), if Γ_0 acts properly discontinuously, by isometries, with compact quotient, on a proper geodesic metric space M , then Γ_0 is finitely generated and any orbital map $\gamma \mapsto \gamma \cdot m$ is a quasi-isometric embedding: there exist $c, c' > 0$ such that $\mathbf{d}_M(m, \gamma \cdot m) \geq c \mathbf{d}_{\text{Cay}(\Gamma_0)}(e, \gamma) - c'$ for all $\gamma \in \Gamma_0$. We apply this to the convex core $M = \mathcal{C}_{\rho(\Gamma_0)}^{\text{cor}}$, endowed with the restriction of the metric \mathbf{d}_X . \square

Proof of (2) \Rightarrow (1): Since $\mathbf{d}_X(x_0, \rho(\gamma) \cdot x_0) \rightarrow +\infty$ as $\mathbf{d}_{\text{Cay}(\Gamma_0)}(e, \gamma) \rightarrow +\infty$, the representation ρ has finite kernel and discrete image.

The orbital map $\gamma \mapsto \rho(\gamma) \cdot x_0$ from Γ_0 to X extends to a map from $\text{Cay}(\Gamma_0)$ to X sending edges of $\text{Cay}(\Gamma_0)$ to geodesic segments of X . The fact that ρ is a quasi-isometric embedding implies the existence of $c, c' > 0$ such that any geodesic of $\text{Cay}(\Gamma_0)$ is sent to a (c, c') -quasigeodesic in X , and the Morse lemma (see e.g. [65, Th. 11.40 & 11.105]) states that (c, c') -quasigeodesics are uniformly close to actual geodesics in X . Therefore the orbit $\rho(\Gamma_0) \cdot x_0$ is *quasiconvex*: there exists a uniform neighbourhood \mathcal{U} of $\rho(\Gamma_0) \cdot x_0$ in X such that any geodesic segment between two points of $\rho(\Gamma_0) \cdot x_0$ is contained in \mathcal{U} . We conclude using the fact (see [31, Prop. 2.5.4]) that any quasiconvex subset of X lies at finite Hausdorff distance from its convex hull in X . \square

In order to prove (2) \Rightarrow (3), we consider, for any metric space (M, \mathbf{d}_M) and any isometry g of M , the *stable length*

$$\text{length}_M^\infty(g) := \lim_k \frac{1}{k} \mathbf{d}_M(m, g^k \cdot m) \geq 0$$

of g . It is an easy exercise to check, using the triangle inequality, that this limit exists (because the sequence $(\mathbf{d}_M(m, g^k \cdot m))_{k \in \mathbb{N}}$ is subadditive) and that it does not depend on the choice of $m \in M$. Note that

$$\text{length}_M^\infty(g) \leq \text{transl}_M(g). \tag{4.6}$$

Indeed, for any $m \in M$ and any $k \geq 1$ we have $\mathbf{d}_M(m, g^k \cdot m) \leq k \mathbf{d}_M(m, g \cdot m)$ by the triangle inequality. Dividing by k and passing to the limit yields $\text{length}_M^\infty(g) \leq \mathbf{d}_M(m, g \cdot m)$, and we conclude by taking an infimum over all $m \in M$ on the right-hand side.

Proof of (2) \Rightarrow (3): Applying (4.3) to γ^k instead of γ , dividing by k , and passing to the limit yields $\text{length}_X^\infty(\rho(\gamma)) \geq c \text{length}_{\text{Cay}(\Gamma_0)}^\infty(\gamma)$ for all

$\gamma \in \Gamma$. In order to obtain (4.4), it is sufficient to use (4.6) for $M = X$ and to check that

- (i) for $M = \text{Cay}(\Gamma_0)$, the inequality (4.6) is “almost” an equality: $\text{length}_{\text{Cay}(\Gamma_0)}^\infty(g) \geq \text{transl}_{\text{Cay}(\Gamma_0)}(g) - 8\delta$ where $\delta \geq 0$ is a hyperbolicity constant for $\text{Cay}(\Gamma_0)$ (i.e. all triangles of $\text{Cay}(\Gamma_0)$ are δ -thin).

Indeed, then (4.4) will hold with $c'' = 8\delta$. We note that actually

- (ii) for $M = X = G/K$, the inequality (4.6) is an equality.

Indeed, (ii) is based on the fact that X is a $\text{CAT}(0)$ space: any geodesic triangle of X is “at least as thin” as a triangle with the same side lengths in the Euclidean plane. Applying this to a geodesic triangle with vertices $m, g \cdot m, g^2 \cdot m$, we see that if m' is the midpoint of the geodesic segment $[m, g \cdot m]$ (so that $g \cdot m'$ is the midpoint of $[g \cdot m, g^2 \cdot m]$), then $d_X(m', g \cdot m') \leq d_X(m, g^2 \cdot m)/2$. By induction on k , we obtain that for any $m \in M$ and any $k \geq 1$, there exists $m_k \in M$ such that $d_X(m, g^{2^k} \cdot m) \geq 2^k d_X(m_k, g \cdot m_k) \geq 2^k \text{transl}_M(g)$. We conclude by dividing by 2^k and passing to the limit.

(i) can be proved in a similar way, replacing the $\text{CAT}(0)$ inequality $d_X(m', g \cdot m') \leq d_X(m, g^2 \cdot m)/2$ by the Gromov hyperbolicity inequality $d_X(m', g \cdot m') \leq d_X(m, g^2 \cdot m)/2 + 4\delta$ (see [51, Ch. 10, Prop. 5.1]). \square

Proof of (3) \Rightarrow (2): The Gromov hyperbolic group Γ_0 has the following property: there exist a finite subset S of Γ_0 and a constant $C' > 0$ such that for any $\gamma \in \Gamma_0$ we can find $s \in S$ with $\text{transl}_{\text{Cay}(\Gamma_0)}(s\gamma) \geq d_{\text{Cay}(\Gamma_0)}(e, \gamma) - C'$. (If Γ_0 is nonelementary, then we can take $S = \{\gamma_1^N, \gamma_1^{-N}, \gamma_2^N, \gamma_2^{-N}\}$ for some large N , where $\gamma_1, \gamma_2 \in \Gamma_0$ are infinite-order elements such that the attracting fixed points in $\partial_\infty \Gamma_0$ of $\gamma_1, \gamma_1^{-1}, \gamma_2$, and γ_2^{-1} are pairwise distinct: see e.g. [139, Lem. B.2].)

Given $\gamma \in \Gamma_0$, consider $s \in S$ as above. Applying (4.4) to $s\gamma$ yields $\text{transl}_X(\rho(s\gamma)) \geq c \text{transl}_{\text{Cay}(\Gamma_0)}(s\gamma) - c'' \geq c d_{\text{Cay}(\Gamma_0)}(e, \gamma) - (cC' + c'')$.

To conclude, we observe that

$$\text{transl}_X(g_1 g_2) \leq d_X(x_0, g_1 g_2 \cdot x_0) \leq d_X(x_0, g_1 \cdot x_0) + d_X(x_0, g_2 \cdot x_0)$$

for all $g_1, g_2 \in G$. Applying this to $(g_1, g_2) = (\rho(s), \rho(\gamma))$, we obtain (4.3) with $c' = cC' + c'' + \max_{s' \in S} d_X(x_0, \rho(s') \cdot x_0)$. \square

Proof of (1) \Rightarrow (5): We have seen in the proof of (1) \Rightarrow (2) that for any $m \in \mathcal{C}_{\rho(\Gamma_0)}^{\text{cor}}$, the orbital map $\gamma \mapsto \rho(\gamma) \cdot m$ is a quasi-isometry from Γ_0 to $\mathcal{C}_{\rho(\Gamma_0)}^{\text{cor}}$. It is a classical result in geometric group theory (see e.g.

[65, Th. 11.108]) that such a quasi-isometry extends to a Γ_0 -equivariant homeomorphism ξ from $\partial_\infty \Gamma_0$ to $\partial_\infty \mathcal{C}_{\rho(\Gamma_0)}^{\text{cor}}$. Here $\partial_\infty \mathcal{C}_{\rho(\Gamma_0)}^{\text{cor}}$ is a subset of $\partial_\infty X$ (namely the intersection of $\partial_\infty X$ with the closure of $\mathcal{C}_{\rho(\Gamma_0)}^{\text{cor}}$ in \overline{X}). Thus we can view ξ as a ρ -equivariant, continuous, injective map from $\partial_\infty \Gamma_0$ to $\partial_\infty X$, such that for any $(\gamma_k) \in \Gamma_0^{\mathbb{N}}$ and any $w^+, w^- \in \partial_\infty \Gamma_0$, if $\gamma_k \cdot w \rightarrow w^+$ for all $w \in \partial_\infty \Gamma_0 \setminus \{w^-\}$, then $\rho(\gamma_k) \cdot z \rightarrow \xi(w^+)$ for all $z \in \partial_\infty \mathcal{C}_{\rho(\Gamma_0)}^{\text{cor}} \setminus \{\xi(w^-)\}$.

In order to see that this last convergence holds for all $z \in \partial_\infty X \setminus \{\xi(w^-)\}$, one possibility is to use the fact (Cartan decomposition) that if we choose a point $x \in X$ and a geodesic line \mathcal{G} of X through x with endpoints $z_0^+, z_0^- \in \partial_\infty X$, then any element $\rho(\gamma_k) \in G$ can be written as $\rho(\gamma_k) = \kappa_k a_k \kappa'_k$ where $\kappa_k, \kappa'_k \in G$ fix x and $a_k \in G$ is a pure translation along \mathcal{G} towards z_0^+ . The subgroup of G fixing x is compact (it is conjugate to K); therefore, up to passing to a subsequence we may assume that $(\kappa_k)_{k \in \mathbb{N}}, (\kappa'_k)_{k \in \mathbb{N}}$ converge respectively to some $\kappa, \kappa' \in G$. Since ρ has finite kernel and discrete image, we have $a_k \cdot z \rightarrow z_0^+$ for all $z \in \partial_\infty X \setminus \{z_0^-\}$. Therefore $\rho(\gamma_k) \cdot z \rightarrow \kappa \cdot z_0^+$ for all $z \in \partial_\infty X \setminus \{\kappa'^{-1} \cdot z_0^-\}$. Necessarily $\kappa \cdot z_0^+ = \xi(w^+)$ and $\kappa'^{-1} \cdot z_0^- = \xi(w^-)$. \square

The implication (5) \Rightarrow (4) is immediate by considering, for any infinite-order element $\gamma \in \Gamma_0$, the sequence $(\gamma_k) := (\gamma^k) \in \Gamma_0^{\mathbb{N}}$. The implication (4) \Rightarrow (2) can be proved using flows as in Section 4.5.1 below (see Remark 4.5.5 and the implication (1) \Rightarrow (2) in Theorem 4.5.13).

Proof of (1) \Rightarrow (6): We treat the case that X is \mathbb{H}^n , seen as the open unit ball of \mathbb{R}^n for a Euclidean norm $\|\cdot\|$, that 0 belongs to $\mathcal{C}_{\rho(\Gamma_0)}^{\text{cor}}$, and that $\mathbf{d}_{\partial_\infty X}$ is the metric induced by $\|\cdot\|$ on the unit sphere $\partial_\infty X$ of \mathbb{R}^n .

We first observe that for any element $g \in G$ that does not fix 0, the closed subset

$$\mathcal{H}_g := \{x \in X \mid \|x\| \leq \|g \cdot x\|\} = \{x \in X \mid \mathbf{d}_X(0, x) \leq \mathbf{d}_X(0, g \cdot x)\}$$

of X is bounded by the bisector between 0 and $g^{-1} \cdot 0$. Moreover, for any neighbourhood \mathcal{V} in \overline{X} of the closure of \mathcal{H}_g in \overline{X} , the restriction of g to $\partial_\infty X \setminus \mathcal{V}$ is uniformly expanding in the sense that

$$\inf_{z_1 \neq z_2} \inf_{\text{in } \partial_\infty X \setminus \mathcal{V}} \frac{\mathbf{d}_{\partial_\infty X}(g \cdot z_1, g \cdot z_2)}{\mathbf{d}_{\partial_\infty X}(z_1, z_2)} > 1.$$

Indeed, one can check this when g is a pure translation along a geodesic of X through 0, and then conclude using the fact (Cartan decomposition) that any $g \in G$ can be written as $g = \kappa a \kappa'$ where $a \in G$ is such a pure translation and $\kappa, \kappa' \in G$ fix 0 and preserve $\|\cdot\|$.

Consider the Dirichlet domain of $\mathcal{C}_{\rho(\Gamma_0)}^{\text{cor}}$ centred at 0:

$$\mathcal{D} = \bigcap_{\gamma \in \Gamma_0} \mathcal{H}_{\rho(\gamma)} \cap \mathcal{C}_{\rho(\Gamma_0)}^{\text{cor}}.$$

It is compact by (1). Since Γ_0 acts properly discontinuously on $\mathcal{C}_{\rho(\Gamma_0)}^{\text{cor}}$ via ρ , the set \mathcal{F} of elements $\gamma \in \Gamma_0$ such that $\mathcal{D} \cap \rho(\gamma) \cdot \mathcal{D} \neq \emptyset$ and $\rho(\gamma) \cdot 0 \neq 0$ is finite. One easily checks that $\mathcal{D} = \bigcap_{\gamma \in \mathcal{F}} \mathcal{H}_{\rho(\gamma)} \cap \mathcal{C}_{\rho(\Gamma_0)}^{\text{cor}}$. For each $\gamma \in \mathcal{F}$, let $\mathcal{V}_{\rho(\gamma)}$ be a closed neighbourhood in \overline{X} of the closure of $\mathcal{H}_{\rho(\gamma)}$ in \overline{X} . If we choose these neighbourhoods small enough, then $\mathcal{D}' := \bigcap_{\gamma \in \mathcal{F}} \mathcal{V}_{\rho(\gamma)} \cap \mathcal{C}_{\rho(\Gamma_0)}^{\text{cor}}$ is still a compact subset of X , and so $\Lambda_{\rho(\Gamma_0)} \subset \bigcup_{\gamma \in \mathcal{F}} (\partial_\infty X \setminus \mathcal{V}_{\rho(\gamma)})$. We conclude using the fact, observed above, that (4.5) holds for $\mathcal{U} := \partial_\infty X \setminus \mathcal{V}_{\rho(\gamma)}$ for each $\gamma \in \mathcal{F}$. \square

Proof of (6) \Rightarrow (1): We again treat the case that X is \mathbb{H}^n , seen as the open unit ball of \mathbb{R}^n for a Euclidean norm $\|\cdot\|$, and that the metric $\mathbf{d}_{\partial_\infty X}$ is induced by $\|\cdot\|$. We denote by \mathbf{d}_{Euc} the Euclidean distance on \mathbb{R}^n associated to $\|\cdot\|$.

Suppose that (6) holds. Then $\Lambda_{\rho(\Gamma_0)}$ contains at least two points. (Indeed, by assumption $\rho(\Gamma_0)$ is an infinite discrete subgroup of G , hence $\Lambda_{\rho(\Gamma_0)}$ is nonempty; moreover, the expansion assumption prevents $\Lambda_{\rho(\Gamma_0)}$ from being a singleton, as follows e.g. from the classification of elementary discrete subgroups of G : see [31, Prop. 3.2.1].) Therefore $\mathcal{C}_{\rho(\Gamma_0)}^{\text{cor}}$ is nonempty. Moreover, one can check (e.g. using the Cartan decomposition as in the proof of (1) \Rightarrow (6) just above) that for any $z \in \Lambda_{\rho(\Gamma_0)}$, there exist a neighbourhood \mathcal{U} of z in \mathbb{R}^n (rather than just $\partial_\infty X$) and an element $\gamma \in \Gamma_0$ such that (4.5) holds for \mathbf{d}_{Euc} (rather than $\mathbf{d}_{\partial_\infty X}$).

Suppose by contradiction that the action of Γ_0 on $\mathcal{C}_{\rho(\Gamma_0)}^{\text{cor}}$ via ρ is *not* cocompact. Let $(\varepsilon_m)_{m \in \mathbb{N}}$ be a sequence of positive reals going to 0. For any m , the set $\mathcal{K}_m := \{x \in \mathcal{C}_{\rho(\Gamma_0)}^{\text{cor}} \mid \mathbf{d}_{\text{Euc}}(x, \Lambda_{\rho(\Gamma_0)}) \geq \varepsilon_m\}$ is compact, hence there exists a $\rho(\Gamma_0)$ -orbit contained in $\mathcal{C}_{\rho(\Gamma_0)}^{\text{cor}} \setminus \mathcal{K}_m$. By proper discontinuity of the action on $\mathcal{C}_{\rho(\Gamma_0)}^{\text{cor}}$, the supremum of $\mathbf{d}_{\text{Euc}}(\cdot, \Lambda_{\rho(\Gamma_0)})$ on this orbit is achieved at some point $x_m \in \mathcal{C}_{\rho(\Gamma_0)}^{\text{cor}}$, and by construction we have $0 < \mathbf{d}_{\text{Euc}}(\rho(\gamma) \cdot x_m, \Lambda_{\rho(\Gamma_0)}) \leq \mathbf{d}_{\text{Euc}}(x_m, \Lambda_{\rho(\Gamma_0)}) \leq \varepsilon_m$ for all $\gamma \in \Gamma_0$. Up to passing to a subsequence, we may assume that $(x_m)_{m \in \mathbb{N}}$ converges to some $z \in \Lambda_{\rho(\Gamma_0)}$. Consider a neighbourhood \mathcal{U} of z in \mathbb{R}^n and an element $\gamma \in \Gamma_0$ such that (4.5) holds for \mathbf{d}_{Euc} , and let $c > 1$ be the infimum in (4.5). For any $m \in \mathbb{N}$, there exists $z_m \in \Lambda_{\rho(\Gamma_0)}$ such that $\mathbf{d}_{\text{Euc}}(\rho(\gamma) \cdot x_m, \Lambda_{\rho(\Gamma_0)}) = \mathbf{d}_{\text{Euc}}(\rho(\gamma) \cdot x_m, \rho(\gamma) \cdot z_m)$. For large enough m we have $x_m, z_m \in \mathcal{U}$,

and so $d_{\text{Euc}}(\rho(\gamma) \cdot x_m, \Lambda_{\rho(\Gamma_0)}) \geq c d_{\text{Euc}}(x_m, z_m) \geq c d_{\text{Euc}}(x_m, \Lambda_{\rho(\Gamma_0)}) \geq c d_{\text{Euc}}(\rho(\gamma) \cdot x_m, \Lambda_{\rho(\Gamma_0)}) > 0$. This is impossible since $c > 1$. \square

4.5 Classes of discrete subgroups in higher real rank

We have seen in Section 4.4 two important classes of discrete subgroups of semisimple Lie groups G with $\text{rank}_{\mathbb{R}}(G) = 1$, namely convex cocompact subgroups and geometrically finite subgroups. These classes have been much studied, although many interesting questions remain open even in the case of $G = \text{PO}(n, 1)$ for $n \geq 4$ (see e.g. [95]).

We now turn to infinite discrete subgroups of semisimple Lie groups G for $\text{rank}_{\mathbb{R}}(G) \geq 2$. These discrete subgroups, beyond lattices, remain more mysterious, and very few general results are known (see [71] for a notable exception). Recently, an important class has emerged, namely the class of *Anosov subgroups*, which are by definition the images of the *Anosov representations* of Gromov hyperbolic groups introduced by Labourie [109] as part of his study of Hitchin representations (see Section 4.3.3). In fact, most examples in Section 4.3 are Anosov subgroups. We now discuss these subgroups, make the link with convex cocompactness, and mention some generalisations.

4.5.1 Anosov subgroups

Given a noncompact semisimple Lie group G , there are several possible types of Anosov subgroups of G , depending on the choice of one of the (finitely many) *flag varieties* G/P of G , where P is a proper parabolic subgroup of G . For simplicity, in these notes we consider $G = \text{PGL}(d, \mathbb{K})$ or $\text{SL}^{\pm}(d, \mathbb{K}) = \{g \in \text{GL}(d, \mathbb{K}) \mid \det(g) = \pm 1\}$ where $\mathbb{K} = \mathbb{R}$ or \mathbb{C} ; we take $P = P_i$ to be the stabiliser in G of an i -plane of \mathbb{K}^d , for some $1 \leq i \leq d-1$, so that $G/P_i = \text{Gr}_i(\mathbb{K}^d)$ is the Grassmannian of i -planes of \mathbb{K}^d .

Definition and first observations

Here is the original definition from Labourie, which appeared in [109] for surface groups $\Gamma_0 = \pi_1(S)$ and in [84] for general hyperbolic groups.

Definition 4.5.1 Let Γ_0 be an infinite Gromov hyperbolic group and $G = \text{PGL}(d, \mathbb{K})$ or $\text{SL}^{\pm}(d, \mathbb{K})$. For $1 \leq i \leq d-1$, a representation $\rho : \Gamma_0 \rightarrow G$ is P_i -Anosov if there exist ρ -equivariant maps $\xi_i : \partial_{\infty}\Gamma_0 \rightarrow G/P_i = \text{Gr}_i(\mathbb{K}^d)$ and $\xi_{d-i} : \partial_{\infty}\Gamma_0 \rightarrow G/P_{d-i} = \text{Gr}_{d-i}(\mathbb{K}^d)$ which

- are continuous,
- are *transverse*: $\xi_i(w) \oplus \xi_{d-i}(w') = \mathbb{K}^d$ for all $w \neq w'$ in $\partial_\infty \Gamma_0$;
- satisfy a uniform contraction property (Condition 4.5.4 below) which strengthens the dynamics-preserving condition of Theorem 4.4.12.(4).

By the *dynamics-preserving condition of Theorem 4.4.12.(4)* for ξ_i we mean that for any infinite-order element $\gamma \in \Gamma_0$, the image by ξ_i of the attracting fixed point of γ in $\partial_\infty \Gamma_0$ (see Section 4.4.4) is an attracting fixed point of $\rho(\gamma)$ in $\text{Gr}_i(\mathbb{K}^d)$.

We note that for an element $g \in G$, the property of admitting an attracting fixed point in G/P_i can be characterised in terms of eigenvalues, namely as $(\lambda_i - \lambda_{i+1})(g) > 0$ (Notation 4.5.10). In this case the attracting fixed point is unique and we say that g is *proximal* in $\text{Gr}_i(\mathbb{K}^d)$.

Remark 4.5.2 For our purposes, working with $\text{PGL}(d, \mathbb{K})$ or $\text{SL}^\pm(d, \mathbb{K})$ is equivalent. Indeed, a representation $\rho : \Gamma_0 \rightarrow \text{SL}^\pm(d, \mathbb{K})$ is P_i -Anosov if and only if its composition with the natural projection $\text{SL}^\pm(d, \mathbb{K}) \rightarrow \text{PGL}(d, \mathbb{K})$ is P_i -Anosov, and up to passing to a finite-index subgroup (which does not change the property of being P_i -Anosov) any representation $\rho : \Gamma_0 \rightarrow \text{PGL}(d, \mathbb{K})$ with Γ_0 Gromov hyperbolic lifts to $\text{SL}^\pm(d, \mathbb{K})$.

The uniform contraction property in Definition 4.5.1 is reminiscent of the condition defining Anosov flows in dynamics, which explains the terminology *Anosov representation*. Before stating it (Condition 4.5.4), let us make a few elementary observations that already follow from the fact that ξ_i and ξ_{d-i} are continuous, transverse, and dynamics-preserving.

Lemma 4.5.3 *If $\rho : \Gamma \rightarrow G$ is P_i -Anosov, then*

- (1) *the boundary maps ξ_i and ξ_{d-i} are unique, and compatible: $\xi_{\min(i, d-i)}(w) \subset \xi_{\max(i, d-i)}(w)$ for all $w \in \partial_\infty \Gamma_0$; the image of ξ_i is the proximal limit set of $\rho(\Gamma_0)$ in $\text{Gr}_i(\mathbb{K}^d)$, i.e. the closure in $\text{Gr}_i(\mathbb{K}^d)$ of the set of attracting fixed points of proximal elements of $\rho(\Gamma_0)$;*
- (2) *ξ_i and ξ_{d-i} are injective, hence they are homeomorphisms onto their images;*
- (3) *ρ has finite kernel and discrete image.*

By (3), the images of P_i -Anosov representations are infinite discrete subgroups of G ; we shall call them *P_i -Anosov subgroups*.

Proof (1) Recall from Section 4.4.4 that the subset of $\partial_\infty \Gamma_0$ consisting

of the attracting fixed points of infinite-order elements of Γ_0 is dense in $\partial_\infty \Gamma_0$. Since ξ_i and ξ_{d-i} are dynamics-preserving, they are uniquely determined on this subset, and compatible on this subset. By continuity, they are uniquely determined and compatible on all of $\partial_\infty \Gamma_0$. Moreover, the image of ξ_i is the proximal limit set of $\rho(\Gamma_0)$ in $\text{Gr}_i(\mathbb{K}^d)$.

(2) For any $w \neq w'$ in $\partial_\infty \Gamma_0$, the subspaces $\xi_i(w)$ and $\xi_{d-i}(w')$ are transverse by definition, whereas $\xi_i(w)$ and $\xi_{d-i}(w)$ are not by (1) above.

(3) Suppose Γ_0 is nonelementary. In order to show that ρ has finite kernel and discrete image, it is sufficient to consider an arbitrary sequence $(\gamma_k)_{k \in \mathbb{N}}$ of pairwise distinct points of Γ_0 and to check that $(\rho(\gamma_k))_{k \in \mathbb{N}}$ does not converge to the identity of G . Recall from Section 4.4.4 that the action of Γ_0 on $\partial_\infty \Gamma_0$ is a convergence action. Therefore, up to passing to a subsequence, there exist $w^+, w^- \in \partial_\infty \Gamma_0$ such that $\gamma_k \cdot w \rightarrow w^+$ for all $w \in \partial_\infty \Gamma_0 \setminus \{w^-\}$. By ρ -equivariance and continuity of ξ_i , we then have $\rho(\gamma_k) \cdot \xi_i(w) = \xi_i(\gamma_k \cdot w) \rightarrow \xi_i(w^+)$ for all $w \in \partial_\infty \Gamma_0 \setminus \{w^-\}$. Since $\partial_\infty \Gamma_0$ is infinite and ξ_i is injective, there exists $w \in \partial_\infty \Gamma_0 \setminus \{w^-\}$ such that $\xi_i(w) \neq \xi_i(w^+)$. The convergence $\rho(\gamma_k) \cdot \xi_i(w) \rightarrow \xi_i(w^+)$ then implies that $(\rho(\gamma_k))_{k \in \mathbb{N}}$ does not converge to the identity element of G . This shows that ρ has finite kernel and discrete image.

If Γ_0 is elementary, then it admits a finite-index subgroup Γ'_0 which is cyclic. The fact that ξ_i is dynamics-preserving implies that ρ is injective and discrete in restriction to Γ'_0 . From this one easily deduces that ρ has finite kernel and discrete image. \square

The uniform contraction condition

Let us state this condition in the original case considered by Labourie [109], where $\Gamma_0 = \pi_1(M)$ for some closed negatively curved manifold M . We denote by \widetilde{M} the universal cover of M , by T^1 the unit tangent bundle, and by $(\varphi_t)_{t \in \mathbb{R}}$ the geodesic flow on either $T^1(M)$ or $T^1(\widetilde{M})$. (For a general Gromov hyperbolic group Γ_0 , one should replace $T^1(\widetilde{M})$ by a certain *flow space* for Γ_0 , see [84] or [26, § 4.1].)

For simplicity, we take $G = \text{SL}^\pm(d, \mathbb{K})$ (see Remark 4.5.2). Any representation $\rho : \Gamma_0 \rightarrow G$ then determines a flat vector bundle

$$E^\rho = \Gamma_0 \backslash (T^1(\widetilde{M}) \times \mathbb{K}^d)$$

over $T^1(M) = \Gamma_0 \backslash T^1(\widetilde{M})$, where Γ_0 acts on $T^1(\widetilde{M}) \times \mathbb{K}^d$ by $\gamma \cdot (\tilde{x}, v) = (\gamma \cdot \tilde{x}, \rho(\gamma) \cdot v)$. The geodesic flow $(\varphi_t)_{t \in \mathbb{R}}$ on $T^1(M)$ lifts to a flow $(\psi_t)_{t \in \mathbb{R}}$ on E^ρ , given by $\psi_t \cdot [(\tilde{x}, v)] = [(\varphi_t \cdot \tilde{x}, v)]$.

Suppose, as in Definition 4.5.1, that there exist continuous, transverse, ρ -equivariant boundary maps $\xi_i : \partial_\infty \Gamma_0 \rightarrow \text{Gr}_i(\mathbb{K}^d)$ and $\xi_{d-i} : \partial_\infty \Gamma_0 \rightarrow$

$\mathrm{Gr}_{d-i}(\mathbb{K}^d)$. By transversality, for each $\tilde{x} \in T^1(\widetilde{M})$ we have a decomposition $\mathbb{K}^d = \xi_i(\tilde{x}^+) \oplus \xi_{d-i}(\tilde{x}^-)$, where $\tilde{x}^\pm = \lim_{t \rightarrow \pm\infty} \varphi_t \cdot \tilde{x} \in \partial_\infty \widetilde{M} \simeq \partial_\infty \Gamma_0$ are the forward and backward endpoints of the geodesic determined by \tilde{x} , and this defines a decomposition of the vector bundle E^ρ into the direct sum of two subbundles $E_i^\rho = \{[(\tilde{x}, v)] \mid v \in \xi_i(\tilde{x}^+)\}$ and $E_{d-i}^\rho = \{[(\tilde{x}, v)] \mid v \in \xi_{d-i}(\tilde{x}^-)\}$. This decomposition is invariant under the flow (ψ_t) . By definition, the representation ρ is P_i -Anosov if the following “dominated splitting” condition is satisfied.

Condition 4.5.4 *The flow $(\psi_t)_{t \in \mathbb{R}}$ uniformly contracts E_i^ρ with respect to E_{d-i}^ρ , i.e. given a continuous family $(\|\cdot\|_x)_{x \in T^1(M)}$ of norms on the fibers $E^\rho(x)$, there exist $C, C' > 0$ such that for any $t \geq 0$, any $x \in T^1(M)$, and any nonzero $v_i \in E_i^\rho(x)$ and $v_{d-i} \in E_{d-i}^\rho(x)$,*

$$\frac{\|\psi_t \cdot v_i\|_{\varphi_t \cdot x}}{\|\psi_t \cdot v_{d-i}\|_{\varphi_t \cdot x}} \leq e^{-Ct+C'} \frac{\|v_i\|_x}{\|v_{d-i}\|_x},$$

By compactness of $T^1(M)$, this condition does not depend on the choice of continuous family of norms $(\|\cdot\|_x)_{x \in T^1(M)}$ (changing the norms may change the values of C, C' but not their existence).

Remark 4.5.5 Guichard and Wienhard [84] showed that if there exist ρ -equivariant maps ξ_i and ξ_{d-i} which are continuous, transverse, and dynamics-preserving, and if the group $\rho(\Gamma_0)$ is *Zariski-dense* in G , then Condition 4.5.4 is automatically satisfied.

Properties

- P_i -Anosov is equivalent to P_{d-i} -Anosov, as the integers i and $d-i$ play a similar role in Definition 4.5.1 and Condition 4.5.4 (up to reversing the flow, which switches contraction and expansion). In particular, we may restrict to P_i -Anosov for $1 \leq i \leq d/2$.
- When $\mathrm{rank}_{\mathbb{R}}(G) = 1$ (i.e. $d = 2$ for $G = \mathrm{PGL}(d, \mathbb{K})$ or $\mathrm{SL}^\pm(d, \mathbb{K})$), there is only one proper parabolic subgroup P of G up to conjugation (see Remark 4.4.6), hence only one notion of Anosov. In that case, an infinite discrete subgroup of G is Anosov if and only if it is convex cocompact in the classical sense of Definition 4.4.1.
- When $\mathrm{rank}_{\mathbb{R}}(G) \geq 2$ (i.e. $d \geq 3$ for $G = \mathrm{PGL}(d, \mathbb{K})$ or $\mathrm{SL}^\pm(d, \mathbb{K})$), Anosov subgroups are *not* lattices of G (since Anosov subgroups are Gromov hyperbolic unlike lattices, see Section 4.2.3).
- Uniform contraction over a compact space as in Condition 4.5.4 is stable under small deformations, which implies the following analogue of Fact 4.4.7.

Fact 4.5.6 *Let G be a noncompact semisimple Lie group and P a proper parabolic subgroup of G . For any infinite Gromov hyperbolic group Γ_0 , the space of P -Anosov representations is open in $\text{Hom}(\Gamma_0, G)$.*

- Anosov representations behave well under Lie group homomorphisms: the following holds similarly to Fact 4.4.8. (We refer to Remark 4.4.13 for the notion of a hyperbolic element of G' .)

Fact 4.5.7 (see [84]) *Let G' be a semisimple Lie group with $\text{rank}_{\mathbb{R}}(G') = 1$ and let $\tau : G' \rightarrow \text{PGL}(d, \mathbb{K})$ be a Lie group homomorphism with compact kernel. For any Gromov hyperbolic group Γ_0 , any representation $\sigma_0 : \Gamma_0 \rightarrow G'$, and any $1 \leq i \leq d - 1$, the following are equivalent:*

- (1) *the representation $\tau \circ \sigma_0 : \Gamma_0 \rightarrow \text{PGL}(d, \mathbb{K})$ is P_i -Anosov;*
- (2) *σ_0 is convex cocompact (Definition 4.4.1) and $(\lambda_i - \lambda_{i+1})(\tau(g')) > 0$ for some hyperbolic element $g' \in G'$.*

In this case, τ induces an embedding $\partial_{\infty} \tau_i : G'/P' \hookrightarrow \text{Gr}_i(\mathbb{K}^d)$ (where G'/P' is the visual boundary of the symmetric space of G' , see Remark 4.4.6) and the boundary map of ρ_0 is the composition of the boundary map $\partial_{\infty} \Gamma_0 \rightarrow G'/P'$ of σ_0 (see Theorem 4.4.12) with $\partial_{\infty} \tau_i$. Moreover, by Fact 4.5.6 there is in that case a neighbourhood of $\tau \circ \sigma_0$ in $\text{Hom}(\Gamma_0, \text{PGL}(d, \mathbb{K}))$ consisting entirely of P_i -Anosov representations (hence with finite kernel and discrete image — see Lemma 4.5.3.(3)).

Examples in higher real rank

Many of the discrete subgroups in Section 4.3 were Anosov subgroups.

- Section 4.3.2: It follows from the work of Benoist [15] that the ping pong groups of Claim 4.3.5 are quasi-isometrically embedded (see Remark 4.5.11) in $\text{PGL}(d, \mathbb{R})$. They are in fact P_1 -Anosov: see [42, 98].
- Section 4.3.2: When they are defined by $B_1^{\pm}, \dots, B_m^{\pm}$ which have pairwise disjoint closures, the Schottky groups in $\text{PGL}(2n, \mathbb{K})$ of Nori and Seade–Verjovsky are P_n -Anosov (see [84]) and the crooked Schottky groups in $\text{Sp}(2n, \mathbb{R}) \subset \text{SL}(2n, \mathbb{R})$ are P_1 -Anosov (see [36]).
- Section 4.3.3: By Facts 4.5.6 and 4.5.7, the Barbot representations of closed surface groups into $\text{SL}(d, \mathbb{R})$ are P_1 -Anosov. The Hitchin representations into $\text{PSL}(d, \mathbb{R})$ are P_i -Anosov for all $1 \leq i \leq d - 1$: this is Labourie's original result from [109], where he introduced Anosov

representations. The maximal representations of closed surface groups into $\mathrm{SO}(2, n) \subset \mathrm{SL}(n + 2, \mathbb{R})$ are P_1 -Anosov (see [37, 83]). We refer to Figures 4.9 and 4.10 for some illustrations of boundary maps.

Remark 4.5.8 Being P_i -Anosov for all $1 \leq i \leq d - 1$ is the strongest possible form of Anosov; in this case, the various boundary maps $\xi_i : \partial_\infty \Gamma_0 \rightarrow \mathrm{Gr}_i(\mathbb{R}^d)$ for $1 \leq i \leq d - 1$ combine into a continuous, injective, ρ -equivariant boundary map $\xi : \partial_\infty \Gamma_0 \rightarrow \mathrm{Flags}(\mathbb{R}^d)$ as in the proof of Theorem 4.3.7.

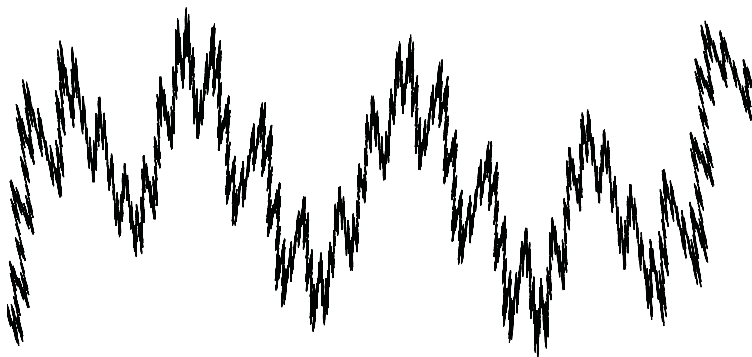


Figure 4.9 The image of the boundary map $\xi_1 : \partial_\infty \Gamma_0 \rightarrow \mathbb{P}(\mathbb{R}^3)$ of a representation $\rho : \Gamma_0 = \pi_1(S) \rightarrow \mathrm{SL}(3, \mathbb{R})$ which is a small deformation of $\Gamma_0 \xrightarrow{\sigma_0} \mathrm{SL}(2, \mathbb{R}) \xrightarrow{\tau} \mathrm{SL}(3, \mathbb{R})$, where σ_0 is injective and discrete and τ is the standard representation. This image is a topological circle in $\mathbb{P}(\mathbb{R}^3)$ which has Hölder, but not Lipschitz, regularity.

- Section 4.3.4: All (known) higher Teichmüller spaces consist of Anosov representations (see [22, 37, 83, 109]).

Remark 4.5.9 Not all Anosov representations of closed surface groups belong to higher Teichmüller spaces. For instance, the Barbot representations of $\pi_1(S)$ into $\mathrm{SL}(d, \mathbb{R})$ from Section 4.3.3 are P_1 -Anosov, but their connected component in $\mathrm{Hom}(\pi_1(S), \mathrm{SL}(d, \mathbb{R}))$ contains representations that are not injective and discrete.

- Section 4.3.4: The two known families of higher-dimensional higher-rank Teichmüller spaces that we mentioned for Gromov hyperbolic groups $\Gamma_0 = \pi_1(M)$ are P_1 -Anosov: for holonomies of convex projective structures, see [16], and for $\mathbb{H}^{p,q}$ -convex cocompact representations, see [10] (case $q = 1$) and [53, 54] (general case).

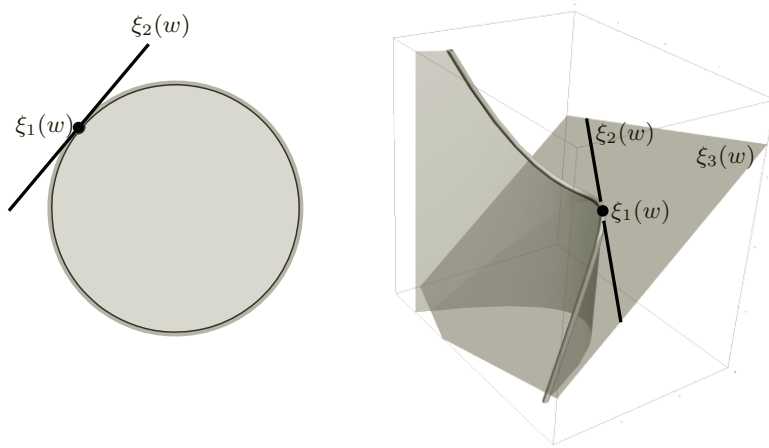


Figure 4.10 If $\xi = (\xi_1, \dots, \xi_{d-1}) : \partial_\infty \Gamma_0 \rightarrow \text{Flags}(\mathbb{R}^d)$ is the boundary map of a Hitchin representation $\rho : \Gamma_0 = \pi_1(S) \rightarrow \text{PSL}(d, \mathbb{R})$, then the image of ξ_1 is a C^1 curve in $\mathbb{P}(\mathbb{R}^d)$, and $\xi(w)$ is the osculating flag to this curve at the point $\xi_1(w)$ for all $w \in \partial_\infty \Gamma_0$. For $d = 3$ the curve is the boundary of the properly convex open subset of $\mathbb{P}(\mathbb{R}^3)$ preserved by ρ , while for $d = 4$ the curve is homotopically nontrivial in $\mathbb{P}(\mathbb{R}^4)$. This figure shows the curve $\xi_1(\partial_\infty \Gamma_0)$ and an osculating flag $\xi(w)$ when $\rho : \Gamma_0 \rightarrow \text{PSL}(2, \mathbb{R}) \hookrightarrow \text{PSL}(d, \mathbb{R})$ is Fuchsian, for $d = 3$ (left) and $d = 4$ (right); for $d = 4$, the curve is the so-called *twisted cubic* in $\mathbb{P}(\mathbb{R}^4)$, given by $t \mapsto (t, t^2, t^3)$ in some affine chart.

Interlude: eigenvalues and singular values

Before giving (in Theorem 4.5.13 below) some characterisations of Anosov representations that generalise Theorem 4.4.12, we introduce some notation and make a few preliminary observations.

Notation 4.5.10 For any $g \in \text{GL}(d, \mathbb{C})$, we denote by $\lambda_1(g) \geq \dots \geq \lambda_d(g)$ the logarithms of the moduli of the complex eigenvalues of g , and by $\mu_1(g) \geq \dots \geq \mu_d(g)$ the logarithms of the *singular values* of g (i.e. of the square roots of the eigenvalues of $\bar{g}^T g$, which are positive numbers). For any $1 \leq i < j \leq d$, this defines functions $\lambda_i - \lambda_j : \text{GL}(d, \mathbb{C}) \rightarrow \mathbb{R}_{\geq 0}$ and $\mu_i - \mu_j : \text{GL}(d, \mathbb{C}) \rightarrow \mathbb{R}_{\geq 0}$ which factor through $\text{PGL}(d, \mathbb{C})$.

As in Section 4.4.4, for any finitely generated group Γ_0 , we choose a finite generating subset of Γ_0 and denote by $\text{Cay}(\Gamma_0)$ the corresponding Cayley graph, with its metric $\mathbf{d}_{\text{Cay}(\Gamma_0)}$. We denote by $X = G/K$ the Riemannian symmetric space of G , with its metric \mathbf{d}_X , and fix a basepoint $x_0 \in X$. We denote the translation length as in (4.2).

Our starting point is the following (see Theorem 4.4.12 and its proof).

Remark 4.5.11 Let Γ_0 be a finitely generated group and $\rho : \Gamma_0 \rightarrow G$ a representation. Then

- ρ has finite kernel and discrete image if and only if $\mathbf{d}_X(x_0, \rho(\gamma) \cdot x_0) \rightarrow +\infty$ as $\mathbf{d}_{\text{Cay}(\Gamma_0)}(e, \gamma) \rightarrow +\infty$;
- ρ is called a *quasi-isometric embedding* if there exist $c, c' > 0$ such that $\mathbf{d}_X(x_0, \rho(\gamma) \cdot x_0) \geq c \mathbf{d}_{\text{Cay}(\Gamma_0)}(e, \gamma) - c'$ for all $\gamma \in \Gamma$;
- ρ is called *well-displacing* if there exist $c, c'' > 0$ such that $\text{transl}_X(\rho(\gamma)) \geq c \text{transl}_{\text{Cay}(\Gamma_0)}(\gamma) - c''$ for all $\gamma \in \Gamma$.

(As in Remarks 4.4.13, the inequality $\mathbf{d}_X(x_0, \rho(\gamma) \cdot x_0) \leq C \mathbf{d}_{\text{Cay}(\Gamma_0)}(e, \gamma)$ always holds for $C := \max_{f \in F} \mathbf{d}_X(x_0, \rho(f) \cdot x_0)$; it implies that the inequality $\text{transl}_X(\rho(\gamma)) \leq C \text{transl}_{\text{Cay}(\Gamma_0)}(\gamma)$ always holds too: see the proof of the implication (2) \Rightarrow (3) of Theorem 4.4.12.)

We now reinterpret Remark 4.5.11 using Notation 4.5.10. Let $\|\cdot\|_{\text{Euc}}$ be the standard Euclidean norm on \mathbb{R}^d . For $G = \text{PGL}(d, \mathbb{K})$ with $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , we can take $K = \text{PO}(d)$ or $\text{PU}(d)$ and $x_0 = eK \in G/K = X$, so that for any $g \in G$ lifting to $\hat{g} \in \text{GL}(d, \mathbb{K})$ with $|\det(\hat{g})| = 1$,

$$\begin{aligned}\mathbf{d}_X(x_0, g \cdot x_0) &= \|(\mu_1(\hat{g}), \dots, \mu_d(\hat{g}))\|_{\text{Euc}}, \\ \text{transl}_X(g) &= \|(\lambda_1(\hat{g}), \dots, \lambda_d(\hat{g}))\|_{\text{Euc}}.\end{aligned}$$

On the other hand, we have $\sum_{i=1}^d \mu_i(\hat{g}) = \sum_{i=1}^d \lambda_i(\hat{g}) = 0$, and on the linear hyperplane $\{v \in \mathbb{R}^d \mid \sum_{i=1}^d v_i = 0\}$ of \mathbb{R}^d the Euclidean norm $\|\cdot\|_{\text{Euc}}$ is equivalent to $\sum_{i=1}^{d-1} |v_i - v_{i+1}|$. In this setting we can therefore rewrite Remark 4.5.11 as follows.

Remark 4.5.12 Let Γ_0 be a finitely generated group and $\rho : \Gamma_0 \rightarrow G = \text{PGL}(d, \mathbb{K})$ a representation. Then

- ρ has finite kernel and discrete image if and only if $\sum_{i=1}^{d-1} (\mu_i - \mu_{i+1})(\rho(\gamma)) \rightarrow +\infty$ as $\mathbf{d}_{\text{Cay}(\Gamma_0)}(e, \gamma) \rightarrow +\infty$;
- ρ is a quasi-isometric embedding if and only if there exist $c, c' > 0$ such that $\sum_{i=1}^{d-1} (\mu_i - \mu_{i+1})(\rho(\gamma)) \geq c \mathbf{d}_{\text{Cay}(\Gamma_0)}(e, \gamma) - c'$ for all $\gamma \in \Gamma$;
- ρ is well-displacing if and only if there exist $c, c'' > 0$ such that $\sum_{i=1}^{d-1} (\lambda_i - \lambda_{i+1})(\rho(\gamma)) \geq c \text{transl}_{\text{Cay}(\Gamma_0)}(\gamma) - c''$ for all $\gamma \in \Gamma$.

Remark 4.5.12 should be kept in mind while reading Theorem 4.5.13.(2)–(3) below, as it explains how Anosov representations are refinements of quasi-isometric embeddings and well-displacing representations.

Characterisations

The following characterisations of Anosov representations were established by Kapovich–Leeb–Porti, Guéritaud–Guichard–Kassel–Wienhard, Bochi–Potrie–Sambarino, and Kassel–Potrie. More precisely, $(1) \Rightarrow (2)$ is easy and follows from a property of dominated splittings proved in [25]. The implication $(2) \Rightarrow (1)$ was proved in [101], with an alternative proof later given in [26]. The implication $(2) \Rightarrow (3)$ is easy and similar to the implication $(2) \Rightarrow (3)$ of Theorem 4.4.12 (note that $\lambda_i(\hat{g}) = \lim_k \mu_i(\hat{g}^k)/k$ for all $\hat{g} \in \mathrm{GL}(d, \mathbb{K})$). The implication $(3) \Rightarrow (2)$ was proved in [104]. The implications $(1) \Leftrightarrow (4) \Leftrightarrow (5)$ and $(1) \Rightarrow (6)$ were proved in [81] and [98], and $(6) \Rightarrow (1)$ was proved in [98]. We refer to [96, 99] for further characterisations (e.g. in terms of *conical limit points*).

We fix a basepoint $x_0 \in X = G/K$ and a Riemannian metric on $G/P_{i,d-i} = \mathrm{Flags}_{i,d-i}(\mathbb{R}^d) = \{(V_{\min(i,d-i)} \subset V_{\max(i,d-i)} \mid \dim(V_\bullet) = \bullet\}$. See Definition 4.5.1 for the notions of transversality and dynamics-preserving, and Theorem 4.4.12.(6) for expansion at the limit set. The notion of limit set that we use is discussed in the next section.

Theorem 4.5.13 *Let $G = \mathrm{PGL}(d, \mathbb{K})$ or $\mathrm{SL}^\pm(d, \mathbb{K})$ where $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , and let $1 \leq i \leq d - 1$. Let Γ_0 be a finitely generated infinite group and $\rho : \Gamma_0 \rightarrow G$ a representation. Then the following are equivalent:*

- (1) Γ_0 is Gromov hyperbolic and ρ is P_i -Anosov,
- (2) ρ is a quasi-isometric embedding “in the i -th direction”: there exist $c, c' > 0$ such that for any $\gamma \in \Gamma$,

$$(\mu_i - \mu_{i+1})(\rho(\gamma)) \geq c \, \mathrm{d}_{\mathrm{Cay}(\Gamma_0)}(e, \gamma) - c';$$

- (3) Γ_0 is Gromov hyperbolic and ρ is well-displacing “in the i -th direction”: there exist $c, c'' > 0$ such that for any $\gamma \in \Gamma$,

$$(\lambda_i - \lambda_{i+1})(\rho(\gamma)) \geq c \, \mathrm{transl}_{\mathrm{Cay}(\Gamma_0)}(\gamma) - c'';$$

- (4) Γ_0 is Gromov hyperbolic, there exist ρ -equivariant maps

$$\xi_\bullet : \partial_\infty \Gamma_0 \longrightarrow G/P_\bullet = \mathrm{Gr}_\bullet(\mathbb{K}^d),$$

for $\bullet \in \{i, d-i\}$, which are continuous, transverse, dynamics-preserving, and $(\mu_i - \mu_{i+1})(\rho(\gamma)) \rightarrow +\infty$ as $\mathrm{d}_{\mathrm{Cay}(\Gamma_0)}(e, \gamma) \rightarrow +\infty$;

- (5) Γ_0 is Gromov hyperbolic and there exist ρ -equivariant maps

$$\xi_\bullet : \partial_\infty \Gamma_0 \longrightarrow G/P_\bullet = \mathrm{Gr}_\bullet(\mathbb{K}^d),$$

for $\bullet \in \{i, d-i\}$, which are continuous, transverse, and strongly

dynamics-preserving (i.e. for any $(\gamma_k) \in \Gamma_0^{\mathbb{N}}$ and $w^+, w^- \in \partial_\infty \Gamma_0$, if $\gamma_k \cdot w \rightarrow w^+$ for all $w \in \partial_\infty \Gamma_0 \setminus \{w^-\}$, then $\rho(\gamma_k) \cdot z \rightarrow \xi_i(w^+)$ for all $z \in G/P_i$ transverse to $\xi_{d-i}(w^-) \in G/P_{d-i}$);

- (6) $(\mu_i - \mu_{i+1})(\rho(\gamma)) \rightarrow +\infty$ as $d_{\text{Cay}(\Gamma_0)}(e, \gamma) \rightarrow +\infty$, any two points of the limit set of $\rho(\Gamma_0)$ in $G/P_{i,d-i}$ are transverse, and the action of Γ_0 on $G/P_{i,d-i}$ via ρ is expanding at this limit set.

Remark 4.5.14 Recall that when $\text{rank}_{\mathbb{R}}(G) = 1$ (i.e. $d = 2$), an infinite discrete subgroup of G is Anosov if and only if it is convex cocompact in the classical sense of Definition 4.4.1. In that case, the flag variety $G/P_i = G/P_{d-i}$ identifies with the visual boundary $\partial_\infty X$ of $X = G/K$ (Remark 4.4.6) and conditions (2), (3), (4), (5), (6) of Theorem 4.5.13 are the same as conditions (2), (3), (4), (5), (6) of Theorem 4.4.12 (see Remark 4.5.12 and Lemma 4.5.3.(3)). On the other hand, when $\text{rank}_{\mathbb{R}}(G) \geq 2$, conditions (2) and (3) of Theorem 4.5.13 are strictly stronger than conditions (2) and (3) of Theorem 4.4.12 (see Remark 4.5.12).

As in Remarks 4.4.13, in condition (3) we cannot remove the assumption that Γ_0 be Gromov hyperbolic. See [104, § 4.4] for further discussion.

Limit sets

We now explain the notion of limit set used in Theorem 4.5.13.(6). It is based on an important decomposition of the noncompact semisimple Lie group G : the *Cartan decomposition* $G = K \exp(\mathfrak{a}^+)K$. We refer to [90] for the general theory for noncompact semisimple Lie groups G . For $G = \text{PGL}(d, \mathbb{K})$, as in Remark 4.5.12, we can take $K = \text{PO}(d)$ or $\text{PU}(d)$, and \mathfrak{a}^+ to be the set of diagonal matrices in $\mathfrak{g} = \{y \in M_d(\mathbb{K}) \mid \text{tr}(y) = 0\}$ whose entries $t_1, \dots, t_d \in \mathbb{R}$ are in nonincreasing order, with $t_1 + \dots + t_d = 0$; the Cartan decomposition can then be stated as follows.

Fact 4.5.15 Any $g \in \text{PGL}(d, \mathbb{K})$ can be written as $g = \kappa \exp(a) \kappa'$ for some $\kappa, \kappa' \in K$ and a unique $a \in \mathfrak{a}^+$; the entries of a are $\mu_1(\hat{g}), \dots, \mu_d(\hat{g})$ (see Notation 4.5.10) where $\hat{g} \in \text{GL}(d, \mathbb{K})$ is any lift of g with $|\det(\hat{g})| = 1$.

Proof By the polar decomposition, any element of $\text{GL}(d, \mathbb{R})$ (resp. $\text{GL}(d, \mathbb{C})$) can be written as the product of an orthogonal (resp. unitary) matrix and a positive semi-definite real symmetric (resp. Hermitian) matrix; on the other hand, any real symmetric (resp. Hermitian) matrix can be diagonalised by an orthogonal (resp. unitary) matrix. \square

Here is a useful consequence of the Cartan decomposition.

Lemma 4.5.16 *For $1 \leq i \leq d-1$ and a sequence (g_m) of points of $G = \mathrm{PGL}(d, \mathbb{K})$, consider the following two conditions:*

- (a) $(\mu_i - \mu_{i+1})(g_m) \rightarrow +\infty$,
- (b) *there exist $z^+ \in G/P_i$ and $z^- \in G/P_{d-i}$ such that $g_m \cdot z \rightarrow z^+$ for all $z \in G/P_i$ transverse to z^- .*

If (g_m) satisfies (a), then some subsequence of (g_m) satisfies (b). Conversely, if (g_m) satisfies (b), then it satisfies (a).

Proof Let $z_0^+ := \mathrm{span}(e_1, \dots, e_i) \in G/P_i$ and $z_0^- := \mathrm{span}(e_{i+1}, \dots, e_d) \in G/P_{d-i}$, where (e_1, \dots, e_d) is the canonical basis of \mathbb{K}^d . By Fact 4.5.15, for any m we can write $g_m = \kappa_m \exp(a_m) \kappa'_m$ where $\kappa_m, \kappa'_m \in K$ and $a_m \in \mathfrak{a}^+$ is diagonal; the entries of a_m are $\mu_1(\hat{g}_m), \dots, \mu_d(\hat{g}_m)$ where $\hat{g}_m \in \mathrm{GL}(d, \mathbb{K})$ is any lift of g_m with $|\det(\hat{g}_m)| = 1$.

(a) \Rightarrow (b): If $(\mu_i - \mu_{i+1})(g_m) \rightarrow +\infty$, then $a_m \cdot z \rightarrow z_0^+$ for all $z \in G/P_i$ transverse to z_0^- . Since K is compact, up to passing to a subsequence, we may assume that $(\kappa_m), (\kappa'_m)$ converge respectively to some $\kappa, \kappa' \in K$. Then $g_m \cdot z \rightarrow z^+ := \kappa \cdot z_0^+$ for all $z \in G/P_i$ transverse to $z^- := \kappa'^{-1} \cdot z_0^-$.

(b) \Rightarrow (a): If $(\mu_i - \mu_{i+1})(g_m)$ does not tend to $+\infty$, then up to passing to a subsequence it converges to some nonnegative real number, and one easily sees that the image by a_m of any open subset of G/P_i fails to converge to a point. Up to passing to a subsequence, we may assume that $(\kappa_m), (\kappa'_m)$ converge in K . Then the image by g_m of any open subset of G/P_i fails to converge to a point. \square

For (g_m) and z^+ as in condition (b) of Lemma 4.5.16, we say that z^+ is a *contraction point* for (g_m) in G/P_i . We then define the *limit set in G/P_i* of a discrete subgroup Γ of G to be the set of contraction points in G/P_i of sequences of elements of Γ . It is a closed Γ -invariant subset of G/P_i . When Γ is P_i -Anosov, it coincides with the proximal limit set of Γ in G/P_i , which is also the image of the boundary map $\xi_i : \partial_\infty \Gamma \rightarrow G/P_i$.

Similarly, sequences $(g_m) \in G^\mathbb{N}$ satisfying both $(\mu_i - \mu_{i+1})(g_m) \rightarrow +\infty$ and $(\mu_{d-i} - \mu_{d-i+1})(g_m) \rightarrow +\infty$ define contraction points in $G/P_{i,d-i} = \mathrm{Flags}_{i,d-i}(\mathbb{R}^d)$. This gives a notion of *limit set in $G/P_{i,d-i}$* of a discrete subgroup Γ of G , as considered in Theorem 4.5.13.(6).

We note that in the setting of Theorem 4.5.13.(6), the limit set of $\rho(\Gamma_0)$ in $G/P_{i,d-i}$ is nonempty. Indeed, $(\mu_{d-i} - \mu_{d-i+1})(g) = (\mu_i - \mu_{i+1})(g^{-1})$ for all $g \in G$, and so $(\mu_i - \mu_{i+1})(\rho(\gamma)) \rightarrow +\infty$ as $\mathbf{d}_{\mathrm{Cay}(\Gamma_0)}(e, \gamma) \rightarrow +\infty$ implies $(\mu_{d-i} - \mu_{d-i+1})(\rho(\gamma)) \rightarrow +\infty$ as $\mathbf{d}_{\mathrm{Cay}(\Gamma_0)}(e, \gamma) \rightarrow +\infty$.

Cocompact domains of discontinuity

We end this section by briefly mentioning a generalisation to Anosov representations of a nice feature of rank-one convex cocompact representations. Namely, we have seen in Section 4.4.2 that for $\text{rank}_{\mathbb{R}}(G) = 1$, if $X = G/K$ denotes the Riemannian symmetric space of G , then any convex cocompact subgroup Γ of G acts properly discontinuously, with compact quotient, on the open subset $\Omega_{\Gamma} := \partial_{\infty}X \setminus \Lambda_{\Gamma}$ of $\partial_{\infty}X$. In that case, $\partial_{\infty}X$ is the unique flag variety G/P of G with P a proper parabolic subgroup of G (Remark 4.4.6).

Guichard and Wienhard [84], inspired by work of Frances, generalised this picture to show that in certain situations, for certain proper parabolic subgroups P and Q of G , any P -Anosov subgroup Γ of G acts properly discontinuously, with compact quotient, on some open subset Ω of G/Q which is obtained by removing all points of G/Q that are “not transverse enough” (in some precise sense) to the limit set of Γ in G/P . This phenomenon was then investigated and described in full generality by Kapovich, Leeb, and Porti [100]. Let us give one concrete example.

Example 4.5.17 Let b be a nondegenerate symmetric bilinear form on \mathbb{R}^d with noncompact automorphism group $G := \text{Aut}(b) \subset \text{SL}^{\pm}(d, \mathbb{R})$. (If b is symmetric, then $G = \text{O}(p, q)$ for some $p, q \geq 1$; we require p and q to be distinct. If b is skew-symmetric, then $d = 2n$ is even and $G = \text{Sp}(2n, \mathbb{R})$.) Let Γ_0 be an infinite Gromov hyperbolic group, $\rho : \Gamma_0 \rightarrow G \subset \text{SL}^{\pm}(d, \mathbb{R})$ a P_1 -Anosov representation, and $\Lambda_{\rho(\Gamma_0)}$ the limit set of $\rho(\Gamma_0)$ in $\mathbb{P}(\mathbb{R}^d)$. Let \mathcal{L} be the space of maximal b -isotropic subspaces of \mathbb{R}^d . (It identifies with G/Q where Q is the stabiliser in G of a maximal b -isotropic subspace of \mathbb{R}^d .) Then Γ_0 acts properly discontinuously with compact quotient, via ρ , on

$$\Omega_{\rho(\Gamma_0)} := \mathcal{L} \setminus \bigcup_{z \in \Lambda_{\rho(\Gamma_0)}} \mathcal{L}_z,$$

where \mathcal{L}_z is the set of maximal b -isotropic subspaces of \mathbb{R}^d that contain the line z .

When b is skew-symmetric, i.e. $G = \text{Sp}(2n, \mathbb{R})$, the set \mathcal{L} is the space $\text{Lag}(\mathbb{R}^{2n})$ of *Lagrangians* of \mathbb{R}^{2n} . In this setting, if $\rho(\Gamma_0)$ is a “strong” crooked Schottky group as in Section 4.3.2, defined by $B_1^{\pm}, \dots, B_m^{\pm}$ with pairwise disjoint closures, then the set $\Omega_{\rho(\Gamma_0)}$ of Example 4.5.17 coincides with the set $\Omega = \text{Int}(\bigcup_{\gamma \in \Gamma_0} \rho(\gamma) \cdot \mathcal{D}) \subset \text{Lag}(\mathbb{R}^{2n})$ of Section 4.3.2.

4.5.2 Anosov representations and convex cocompactness

Recall that when $\text{rank}_{\mathbb{R}}(G) = 1$, Anosov representations coincide with convex cocompact representations in the classical sense of Definition 4.4.1. When $\text{rank}_{\mathbb{R}}(G) \geq 2$, Theorem 4.5.13 shows that Anosov representations have a number of similarities, in terms of their dynamics, with rank-one convex cocompact representations: see Remark 4.5.14. Another similarity, of a more geometric nature, is the existence of cocompact domains of discontinuity as in Example 4.5.17: given an Anosov representation $\rho : \Gamma_0 \rightarrow G$, such a domain of discontinuity $\Omega \subset G/Q$ yields, by taking the quotient, a closed manifold $\rho(\Gamma_0) \backslash \Omega$ locally modeled on G/Q , whose geometry can be quite interesting (see [134, § 5]). These manifolds $\rho(\Gamma_0) \backslash \Omega$ do not satisfy any kind of convexity properties in general.

Given these similarities, it is natural to wonder if Anosov representations could also be characterised geometrically in terms of some suitable notion of convex cocompactness. We will see below that this is indeed the case. This will give more geometric intuition about Anosov representations, and yield new examples constructed geometrically.

Two attempts

Our starting point is the following special case of Fact 4.5.7.

Fact 4.5.18 *Let Γ_0 be an infinite group and $\rho : \Gamma_0 \rightarrow \text{PO}(n, 1) = \text{Isom}(\mathbb{H}^n)$ a representation. Then ρ is convex cocompact (Definition 4.4.1) if and only if Γ_0 is Gromov hyperbolic and $\rho : \Gamma_0 \rightarrow \text{PO}(n, 1) \hookrightarrow \text{PGL}(n + 1, \mathbb{R})$ is P_1 -Anosov.*

We would like to generalise this equivalence to higher-rank semisimple Lie groups G .

A natural first attempt would be to replace \mathbb{H}^n by the Riemannian symmetric space of G . However, this turns out to be rather restrictive: Kleiner–Leeb [106] and Quint [121] proved that if G is a real simple Lie group of real rank ≥ 2 , with Riemannian symmetric space $X = G/K$, then any Zariski-dense discrete subgroup of G , acting with compact quotient on some nonempty convex subset of X , is a cocompact lattice in G ; in particular, Γ is *not* Gromov hyperbolic and ρ is *not* Anosov. Thus this approach does not provide a generalisation of Fact 4.5.18.

Instead, we make a second attempt by viewing \mathbb{H}^n as a properly convex open set in projective space as in (4.1); we can then try to generalise Fact 4.5.18 by replacing \mathbb{H}^n with any properly convex open subset Ω of $\mathbb{P}(\mathbb{R}^{n+1})$.

Recall from the proof of Theorem 4.3.7 that Ω being *properly convex* means that it is convex and bounded in some affine chart of $\mathbb{P}(\mathbb{R}^{n+1})$. In this setting Ω carries a natural proper metric d_Ω , the *Hilbert metric*, which is invariant under $\text{Aut}(\Omega) := \{g \in \text{PGL}(n+1, \mathbb{R}) \mid g \cdot \Omega = \Omega\}$ (see Figure 4.11). In particular, any discrete subgroup of $\text{Aut}(\Omega)$ acts properly discontinuously on Ω .

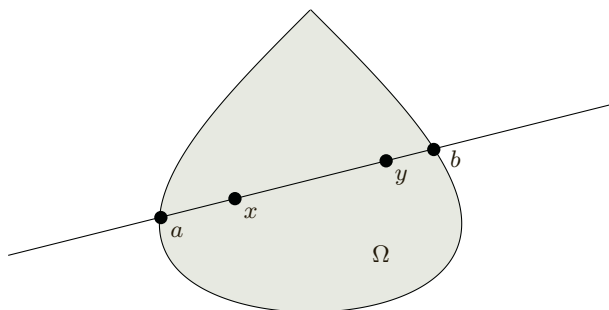


Figure 4.11 In a properly convex open subset Ω of $\mathbb{P}(\mathbb{R}^d)$, the *Hilbert distance* between two distinct points $x, y \in \Omega$ is given by $d_\Omega(x, y) := \frac{1}{2} \log [a, x, y, b]$, where $[\cdot, \cdot, \cdot, \cdot]$ is the cross-ratio on $\mathbb{P}^1(\mathbb{R})$, normalised so that $[0, 1, y, \infty] = y$, and a, b are the intersection points of $\partial\Omega$ with the projective line through x and y , with a, x, y, b in this order. The Hilbert metric d_Ω coincides with the hyperbolic metric when $\Omega = \mathbb{H}^n$ as in (4.1), but in general d_Ω is not Riemannian, only Finsler.

Note that \mathbb{H}^n , viewed as a properly convex open subset of $\mathbb{P}(\mathbb{R}^{n+1})$, does not contain any nontrivial projective segments in its boundary. For properly convex open sets Ω with this property (also known as *strictly convex* open sets), we consider the following analogue of Definition 4.4.1.

Definition 4.5.19 Let Ω be a properly convex open subset of $\mathbb{P}(\mathbb{R}^d)$, whose boundary $\partial\Omega$ does not contain any nontrivial projective segments. Let Γ_0 be a group and $\rho : \Gamma_0 \rightarrow \text{Aut}(\Omega) \subset \text{PGL}(d, \mathbb{R})$ a representation. We say that the action of Γ_0 on Ω via ρ is *convex cocompact* if it is properly discontinuous and if there exists a nonempty $\rho(\Gamma_0)$ -invariant convex subset \mathcal{C} of Ω such that $\rho(\Gamma_0) \backslash \mathcal{C}$ is compact.

In that case, the representation ρ has finite kernel and discrete image and, as in Remark 4.4.2, the group Γ_0 is finitely generated.

Similarly to Fact 4.4.5, we can rephrase convex cocompactness in terms of some specific convex set in Ω . Namely, define the *orbital limit set* $\Lambda_{\rho(\Gamma_0)}^{\text{orb}}(\Omega)$ of $\rho(\Gamma_0)$ in Ω to be the set of accumulation points in $\partial\Omega$ of some $\rho(\Gamma_0)$ -orbit of Ω ; one easily checks that $\Lambda_{\rho(\Gamma_0)}^{\text{orb}}(\Omega)$ does not depend

on the choice of $\rho(\Gamma_0)$ -orbit, because $\partial\Omega$ does not contain any nontrivial segments. Define the *convex core* $\mathcal{C}_{\rho(\Gamma_0)}^{\text{cor}}(\Omega) \subset \Omega$ of $\rho(\Gamma_0)$ to be the convex hull of $\Lambda_{\rho(\Gamma_0)}^{\text{orb}}(\Omega)$ in Ω (i.e. the smallest closed convex subset of Ω whose closure in $\mathbb{P}(\mathbb{R}^d)$ contains $\Lambda_{\rho(\Gamma_0)}^{\text{orb}}(\Omega)$). Similarly to Fact 4.4.5, for infinite Γ_0 , the action of Γ_0 on Ω via ρ is then convex cocompact if and only if it is properly discontinuous and $\rho(\Gamma_0) \backslash \mathcal{C}_{\rho(\Gamma_0)}^{\text{cor}}(\Omega)$ is compact and nonempty.

The following result is a generalisation of Fact 4.5.18 in this setting. It was first proved in [53] for representations ρ with values in $\text{PO}(p, q)$, and then in general in [54] and independently (in a slightly different form and under some irreducibility assumption) in [141]. See also [10, 16, 52, 114] for related earlier results.

Theorem 4.5.20 *Let Γ_0 be an infinite group and $\rho : \Gamma_0 \rightarrow \text{PGL}(d, \mathbb{R})$ a representation. Suppose that $\rho(\Gamma_0)$ preserves a nonempty properly convex open subset of $\mathbb{P}(\mathbb{R}^d)$. Then the following are equivalent:*

- (1) Γ_0 is Gromov hyperbolic and $\rho : \Gamma_0 \rightarrow \text{PGL}(d, \mathbb{R})$ is P_1 -Anosov;
- (2) ρ is strongly convex cocompact in $\mathbb{P}(\mathbb{R}^d)$: the group Γ_0 acts convex cocompactly (Definition 4.5.19) via ρ on some properly convex open subset Ω of $\mathbb{P}(\mathbb{R}^d)$ such that $\partial\Omega$ is C^1 and contains no segments.

Here we say that $\partial\Omega$ is C^1 if every point of $\partial\Omega$ has a unique supporting hyperplane. The phrase *strongly convex cocompact* is meant to reflect the strong regularity imposed on $\partial\Omega$ (namely, C^1 and no segments).

A few comments on Theorem 4.5.20

In certain situations, the assumption in Theorem 4.5.20 that $\rho(\Gamma_0)$ preserve a properly convex open subset of $\mathbb{P}(\mathbb{R}^d)$ is automatically satisfied for P_1 -Anosov representations ρ . For instance, this is the case when $\partial_\infty\Gamma_0$ is connected and ρ takes values in $\text{PO}(p, q) \subset \text{PGL}(d, \mathbb{R})$ for some $p, q \geq 1$ with $p + q = d$, by [53]. In this case, the $\rho(\Gamma_0)$ -invariant properly convex open set Ω given by Theorem 4.5.20.(2) can be taken in

$$\{[v] \in \mathbb{P}(\mathbb{R}^{p+q}) \mid \langle v, v \rangle_{p,q} < 0\} = \mathbb{H}^{p,q-1}$$

(we then say that ρ is $\mathbb{H}^{p,q-1}$ -convex cocompact) or in

$$\{[v] \in \mathbb{P}(\mathbb{R}^{p+q}) \mid -\langle v, v \rangle_{p,q} < 0\} \simeq \mathbb{H}^{q,p-1}$$

(we then say that ρ is $\mathbb{H}^{q,p-1}$ -convex cocompact), where $\langle \cdot, \cdot \rangle_{p,q}$ is the symmetric bilinear form of signature (p, q) on \mathbb{R}^{p+q} defining $\text{PO}(p, q)$.

On the other hand, there exist P_1 -Anosov representations that do

not preserve any properly convex open subset of $\mathbb{P}(\mathbb{R}^d)$: e.g. Hitchin representations (see Sections 4.3.3 and 4.5.1) into $\mathrm{PSL}(d, \mathbb{R})$ for even d .

However, one can always reduce to preserving a properly convex open set by considering a larger projective space. Indeed, consider the natural action of $\mathrm{GL}(d, \mathbb{R})$ on the vector space $\mathrm{Sym}_d(\mathbb{R})$ of symmetric $(d \times d)$ real matrices by $g \cdot A = gAg^T$. It induces a representation $\tau : \mathrm{PGL}(d, \mathbb{R}) \rightarrow \mathrm{PGL}(\mathrm{Sym}_d(\mathbb{R}))$, which preserves the open subset Ω_{sym} of $\mathbb{P}(\mathrm{Sym}_d(\mathbb{R}))$ corresponding to positive definite symmetric matrices. The set Ω_{sym} is properly convex. One can check (see [84], or use one of the characterisations of Theorem 4.5.13) that a representation $\rho : \Gamma_0 \rightarrow \mathrm{PGL}(d, \mathbb{R})$ is P_1 -Anosov if and only if $\tau \circ \rho : \Gamma_0 \rightarrow \mathrm{PGL}(\mathrm{Sym}_d(\mathbb{R}))$ is P_1 -Anosov. Theorem 4.5.20 then implies the following.

Corollary 4.5.21 *For any infinite group Γ_0 and any representation $\rho : \Gamma_0 \rightarrow \mathrm{PGL}(d, \mathbb{R})$, the following are equivalent:*

- (1) Γ_0 is Gromov hyperbolic and $\rho : \Gamma_0 \rightarrow \mathrm{PGL}(d, \mathbb{R})$ is P_1 -Anosov;
- (2) $\tau \circ \rho$ is strongly convex cocompact in $\mathbb{P}(\mathrm{Sym}_d(\mathbb{R}))$.

This actually yields a characterisation of P -Anosov representations into G for any proper parabolic subgroup P of any noncompact semisimple Lie group G , by considering an appropriate representation of G to some large projective linear group. For instance, for $G = \mathrm{PGL}(d, \mathbb{R})$ and $P = P_i$ with $1 \leq i \leq d-1$ as in Section 4.5.1, we can consider the natural representation $\tau_i : \mathrm{PGL}(d, \mathbb{R}) \rightarrow \mathrm{PGL}(S^2(\Lambda^i \mathbb{R}^d))$ where $S^2(\Lambda^i \mathbb{R}^d)$ is the second symmetric power of the i -th exterior power of the standard representation of $\mathrm{GL}(d, \mathbb{R})$ on \mathbb{R}^d . (For $i = 1$, this identifies with $\tau : \mathrm{PGL}(d, \mathbb{R}) \rightarrow \mathrm{PGL}(\mathrm{Sym}_d(\mathbb{R}))$ above.) Again, one can check that $\rho : \Gamma_0 \rightarrow \mathrm{PGL}(d, \mathbb{R})$ is P_i -Anosov if and only if $\tau_i \circ \rho : \Gamma_0 \rightarrow \mathrm{PGL}(S^2(\Lambda^i \mathbb{R}^d))$ is P_1 -Anosov. Theorem 4.5.20 then implies the following.

Corollary 4.5.22 *For any infinite group Γ_0 , any representation $\rho : \Gamma_0 \rightarrow \mathrm{PGL}(d, \mathbb{R})$, and any $1 \leq i \leq d-1$, the following are equivalent:*

- (1) Γ_0 is Gromov hyperbolic and $\rho : \Gamma_0 \rightarrow \mathrm{PGL}(d, \mathbb{R})$ is P_i -Anosov;
- (2) $\tau_i \circ \rho$ is strongly convex cocompact in $\mathbb{P}(S^2(\Lambda^i \mathbb{R}^d))$.

Sketch of proof of Theorem 4.5.20

Proof of (1) \Rightarrow (2): Suppose that Γ_0 is Gromov hyperbolic, that ρ is P_1 -Anosov with boundary maps $\xi_1 : \partial_\infty \Gamma_0 \rightarrow \mathrm{Gr}_1(\mathbb{R}^d) = \mathbb{P}(\mathbb{R}^d)$ and $\xi_{d-1} : \partial_\infty \Gamma_0 \rightarrow \mathrm{Gr}_{d-1}(\mathbb{R}^d) = \mathbb{P}((\mathbb{R}^d)^*)$, and that $\rho(\Gamma_0)$ preserves a nonempty properly convex open subset Ω of $\mathbb{P}(\mathbb{R}^d)$. Since Ω was chosen

without care, it is possible that $\partial\Omega$ contains segments or that the action of Γ_0 on Ω via ρ is not convex cocompact. Therefore, we do not work with Ω itself, but consider instead the connected component Ω_{\max} of $\mathbb{P}(\mathbb{R}^d) \setminus \bigcup_{w \in \partial_\infty \Gamma_0} \xi_{d-1}(w)$ containing Ω (where we view each $\xi_{d-1}(w)$ as a projective hyperplane in $\mathbb{P}(\mathbb{R}^d)$); it is $\rho(\Gamma_0)$ -invariant, open, and convex (not necessarily bounded) in some affine chart of $\mathbb{P}(\mathbb{R}^d)$. Using Lemma 4.5.16, one can show that the action of Γ_0 on Ω_{\max} via ρ is properly discontinuous, and that the set of accumulation points of any $\rho(\Gamma_0)$ -orbit of Ω_{\max} is $\xi_1(\partial_\infty \Gamma_0)$.

Consider the convex hull \mathcal{C} of $\xi_1(\partial_\infty \Gamma_0)$ in Ω_{\max} . One easily checks, using the transversality of ξ_1 and ξ_{d-1} , that $\xi_1(\partial_\infty \Gamma_0)$ is not contained in a single supporting hyperplane to Ω_{\max} in $\mathbb{P}(\mathbb{R}^d)$, and therefore that \mathcal{C} is nonempty. Using the expansion property (6) of Theorem 4.5.13 for Anosov representations, a similar reasoning to the proof of (6) \Rightarrow (1) in Section 4.4.4 then shows that $\rho(\Gamma_0) \backslash \mathcal{C}$ is compact: see [54, § 8].

By transversality of ξ_1 and ξ_{d-1} , there are no nontrivial segments in $\partial\Omega_{\max}$ between points of $\xi_1(\partial_\infty \Gamma_0)$. This makes it possible to “smooth out” Ω_{\max} to obtain a $\rho(\Gamma_0)$ -invariant properly convex open subset $\Omega' \subset \Omega_{\max}$ containing \mathcal{C} such that $\partial\Omega'$ is C^1 and contains no segments: see [54, § 9]. The action of Γ_0 on Ω' via ρ is convex cocompact as desired. \square

Proof of (2) \Rightarrow (1): Suppose that Γ_0 acts convex cocompactly via ρ on some properly convex open subset Ω of $\mathbb{P}(\mathbb{R}^d)$ such that $\partial\Omega$ is C^1 and contains no segments. Because $\partial\Omega$ contains no segments, the geodesic rays of Ω for the Hilbert metric d_Ω (Figure 4.11) are exactly the projective segments between a point of Ω and a point of $\partial\Omega$, and two such rays remain at bounded Hausdorff distance for d_Ω if and only if their endpoints in $\partial\Omega$ are the same. Therefore the convex core $\mathcal{C}_{\rho(\Gamma_0)}^{\text{cor}}$, endowed with the restriction of d_Ω , is a geodesic metric space whose visual boundary $\partial_\infty \mathcal{C}_{\rho(\Gamma_0)}^{\text{cor}}$ identifies with its ideal boundary $\overline{\mathcal{C}_{\rho(\Gamma_0)}^{\text{cor}}} \cap \partial\Omega$ in $\partial\Omega$.

Using the fact that $\partial\Omega$ contains no segments, one can check by a limiting argument that all triangles in $\mathcal{C}_{\rho(\Gamma_0)}^{\text{cor}}$ must be uniformly thin, i.e. the metric space $(\mathcal{C}_{\rho(\Gamma_0)}^{\text{cor}}, d_\Omega)$ is Gromov hyperbolic: see [54, Lem. 6.3]. Since the action of Γ_0 on $(\mathcal{C}_{\rho(\Gamma_0)}^{\text{cor}}, d_\Omega)$ via ρ is properly discontinuous, by isometries, with compact quotient, we deduce that Γ_0 is Gromov hyperbolic and (as in the proof of (1) \Rightarrow (5) in Section 4.4.4) that any orbital map $\Gamma_0 \rightarrow \mathcal{C}_{\rho(\Gamma_0)}^{\text{cor}}$ extends to a continuous ρ -equivariant boundary map $\xi_1 : \partial_\infty \Gamma_0 \rightarrow \partial_\infty \mathcal{C}_{\rho(\Gamma_0)}^{\text{cor}} \subset \mathbb{P}(\mathbb{R}^d)$.

Consider the dual $\Omega^* = \{H \in \mathbb{P}((\mathbb{R}^d)^*) \mid H \cap \overline{\Omega} = \emptyset\}$ of Ω (where we view $\mathbb{P}((\mathbb{R}^d)^*)$ as the set of projective hyperplanes in $\mathbb{P}(\mathbb{R}^d)$). It is a

properly convex open subset of $\mathbb{P}((\mathbb{R}^d)^*)$. The boundary $\partial\Omega^*$ of Ω^* is C^1 (because $\partial\Omega$ contains no segments), and it contains no segments (because $\partial\Omega$ is C^1). One can show that the dual action of Γ_0 on Ω^* via ρ is still convex cocompact: see [54, § 5]. Then the same reasoning as above yields a continuous ρ -equivariant boundary map $\xi_{d-1} : \partial_\infty\Gamma_0 \rightarrow \mathbb{P}((\mathbb{R}^d)^*)$.

By construction, ξ_1 and ξ_{d-1} are transverse: indeed, $\xi_{d-1}(w)$ is a supporting hyperplane to Ω at $\xi_1(w)$ for any w , and $\partial\Omega$ contains no segments. One checks that ξ_1 and ξ_{d-1} are dynamics-preserving and (using Lemma 4.5.16) that $(\mu_1 - \mu_2)(\rho(\gamma)) \rightarrow +\infty$ as $\mathbf{d}_{\text{Cay}(\Gamma_0)}(e, \gamma) \rightarrow +\infty$: see [54, § 7]. We then apply the implication (4) \Rightarrow (1) of Theorem 4.5.13. \square

Applications

Theorem 4.5.20 and Corollaries 4.5.21–4.5.22 give geometric interpretations for Anosov representations.

Example 4.5.23 For odd d , any Hitchin representation $\rho : \pi_1(S) \rightarrow \text{PSL}(d, \mathbb{R})$ as in Section 4.3.3 preserves a nonempty properly convex open subset of $\mathbb{P}(\mathbb{R}^d)$ (see [53, 54, 141]). Therefore these representations are strongly convex cocompact in $\mathbb{P}(\mathbb{R}^d)$ by Theorem 4.5.20. This extends the case $d = 3$ due to Choi and Goldman (see the proof of Theorem 4.3.7).

Example 4.5.24 For $n \geq 2$, any maximal representation $\rho : \pi_1(S) \rightarrow \text{SO}(2, n)$ as in Section 4.3.3 preserves a nonempty properly convex open subset of $\mathbb{P}(\mathbb{R}^{n+2})$, contained in $\mathbb{H}^{2, n-1} = \{[v] \in \mathbb{P}(\mathbb{R}^{n+2}) \mid \langle v, v \rangle_{2, n} < 0\}$ (see [47, 53]). Therefore these representations are strongly convex cocompact in $\mathbb{P}(\mathbb{R}^{n+2})$ by Theorem 4.5.20, and in fact $\mathbb{H}^{2, n-1}$ -convex cocompact as in Section 4.3.4 (see the comments after Theorem 4.5.20).

Theorem 4.5.20 can also be used to construct new examples of Anosov representations. One source of examples comes from representations of Coxeter groups as linear reflection groups. Recall that a Coxeter group is a group with a presentation by generators and relations of the form

$$W = \langle s_1, \dots, s_N \mid (s_i s_j)^{m_{i,j}} = e \quad \forall 1 \leq i, j \leq N \rangle \quad (4.7)$$

where $m_{i,i} = 1$ (i.e. s_i is an involution) and $m_{i,j} \in \{2, 3, 4, \dots\} \cup \{\infty\}$ for all $i \neq j$. (By convention, $(s_i s_j)^\infty = e$ means that $s_i s_j$ has infinite order in the group W .) Vinberg [130] developed a theory of representations of W as a *reflection group* in a finite-dimensional real vector space V : these are by definition representations $\rho : W \rightarrow \text{GL}(V)$ such that each $\rho(s_i)$ is a linear reflection in a hyperplane of V and the configuration of these reflections is such that ρ is injective, discrete, and

the associated fundamental polytope has nonempty interior. These representations may preserve a nondegenerate quadratic form on V (e.g. the image of ρ could be a discrete subgroup of $O(n, 1)$ generated by orthogonal reflections in the faces of a right-angled polyhedron of \mathbb{H}^n as in Section 4.4.3), but in general they need not preserve any nonzero quadratic form. Representations of W as a reflection group constitute a subset $\text{Hom}_{\text{refl}}(W, \text{GL}(V))$ of $\text{Hom}(W, \text{GL}(V))$ which is semialgebraic (defined by finitely many equalities and inequalities).

Example 4.5.25 ([53, 56, 110]) Let W be a Coxeter group in N generators as in (4.7). Suppose W is infinite and Gromov hyperbolic. Then for any $d \geq N$ there exist representations $\rho : W \rightarrow \text{SL}^\pm(d, \mathbb{R})$ of W as a reflection group which are strongly convex cocompact in $\mathbb{P}(\mathbb{R}^d)$; for $d \geq 2N - 2$, they constitute the full interior of $\text{Hom}_{\text{refl}}(W, \text{GL}(d, \mathbb{R}))$. By Theorem 4.5.20, these representations are P_1 -Anosov.

By [64], a conclusion similar to that of Example 4.5.25 holds if W is an infinite Gromov hyperbolic group which is not necessarily a Coxeter group, but which embeds into a right-angled Coxeter group as a so-called *quasiconvex subgroup*. Using celebrated work of Agol and Haglund–Wise, this provides Anosov representations for a large class of infinite Gromov hyperbolic groups, namely all those which admit a properly discontinuous and cocompact action on a CAT(0) cube complex.

One can also use the geometric interpretation of Anosov representations from Theorem 4.5.20 to prove that free products $\Gamma_1 * \Gamma_2$ of Anosov subgroups Γ_1, Γ_2 are Anosov [55], using a generalisation of the ping pong arguments of Sections 4.3.1–4.3.2. For instance, for $1 \leq i \leq d - 1$, let $\tau_i : \text{SL}(d, \mathbb{R}) \rightarrow \text{SL}(S^2(\Lambda^i \mathbb{R}^d))$ be the second symmetric power of the i -th exterior power of the standard representation as in Corollary 4.5.22, let $V'_i := S^2(\Lambda^i \mathbb{R}^d) \oplus \mathbb{R}$, and let $\tau'_i : \text{SL}(d, \mathbb{R}) \rightarrow \text{SL}(V'_i)$ be the direct sum of τ_i and of the trivial representation. Then the following holds.

Example 4.5.26 ([55]) Let $1 \leq i \leq d - 1$ and let Γ_1, Γ_2 be any discrete subgroups of $\text{SL}(d, \mathbb{R})$. Then there exists $g \in \text{SL}(V'_i)$ such that the representation $\rho : \Gamma_1 * g\Gamma_2g^{-1} \rightarrow \text{SL}(V'_i)$ induced by the restrictions of τ'_i to Γ_1 and $g\Gamma_2g^{-1}$ has finite kernel and discrete image. If moreover Γ_1 and Γ_2 are P_i -Anosov, then we can choose g so that ρ is P_1 -Anosov.

(Note that beyond Anosov representations, this construction can be used to prove that the free product of two \mathbb{Z} -linear groups is \mathbb{Z} -linear, and that there exist Zariski-dense discrete subgroups of $\text{SL}(V'_i)$ which are not lattices but contain cocompact lattices of $\tau'_i(\text{SL}(d - 1, \mathbb{R}))$: see [55].)

We refer to [61, 62] for other combination theorems for Anosov representations which do not use Theorem 4.5.20.

Finally we note that, although we have seen many constructions of Anosov representations above, it is expected that not every linear Grovov hyperbolic group admits an Anosov representation into some non-compact semisimple Lie group; a concrete example remains to be found.

4.5.3 Generalisations of Anosov subgroups

In the past few years, several fruitful generalisations of Anosov subgroups have appeared, which are currently being actively investigated. These generalisations exploit both the dynamical definition of Anosov subgroups from Section 4.5.1 and their geometric characterisation from Section 4.5.2. Let us briefly mention three of these generalisations.

More general convex cocompact subgroups

We just saw in Theorem 4.5.20 and Corollaries 4.5.21 and 4.5.22 that Anosov representations can be characterised geometrically by a strong convex cocompactness condition in projective space. Here *strong* refers to the regularity imposed on the properly convex open set Ω (its boundary $\partial\Omega$ should be C^1 and contain no segments).

It is natural to try to generalise Anosov representations by relaxing this strong regularity requirement. Removing it altogether in Definition 4.5.19 leads to a notion which is not stable under small deformations (see [54, 55]). Instead, we impose the following mild condition, which relies on the notions of *full orbital limit set* and *convex core*.

Definition 4.5.27 ([54]) Let Ω be a properly convex open subset of $\mathbb{P}(\mathbb{R}^d)$. Let Γ_0 be a group and $\rho : \Gamma_0 \rightarrow \text{Aut}(\Omega) \subset \text{PGL}(d, \mathbb{R})$ a representation.

- The *full orbital limit set* $\Lambda_{\rho(\Gamma_0)}^{\text{orb}}(\Omega)$ of $\rho(\Gamma_0)$ in Ω is the set of all accumulation points in $\partial\Omega$ of all possible $\rho(\Gamma_0)$ -orbits of Ω .
- The *convex core* $\mathcal{C}_{\rho(\Gamma_0)}^{\text{cor}}(\Omega) \subset \Omega$ of $\rho(\Gamma_0)$ is the convex hull of $\Lambda_{\rho(\Gamma_0)}^{\text{orb}}(\Omega)$ in Ω .
- The action of Γ_0 on Ω via ρ is *convex cocompact* if it is properly discontinuous and if there exists a nonempty $\rho(\Gamma_0)$ -invariant convex subset \mathcal{C} of Ω such that $\rho(\Gamma_0) \backslash \mathcal{C}$ is compact and \mathcal{C} is “large enough” in the sense that it contains the convex core $\mathcal{C}_{\rho(\Gamma_0)}^{\text{cor}}(\Omega)$.

Note that Definition 4.5.27 coincides with Definition 4.5.19 when $\partial\Omega$

does not contain any nontrivial projective segments. Indeed, in that case the full orbital limit set $\Lambda_{\rho(\Gamma_0)}^{\text{orb}}(\Omega)$ is the set of accumulation points of any single $\rho(\Gamma_0)$ -orbit of Ω , hence any nonempty $\rho(\Gamma_0)$ -invariant convex subset \mathcal{C} of Ω contains the convex core $\mathcal{C}_{\rho(\Gamma_0)}^{\text{cor}}(\Omega)$ (see the comments after Definition 4.5.19).

Definition 4.5.28 Given a group Γ_0 , we say that a representation $\rho : \Gamma_0 \rightarrow \text{PGL}(d, \mathbb{R})$ is *convex cocompact* in $\mathbb{P}(\mathbb{R}^d)$ if Γ_0 acts convex cocompactly via ρ on some properly convex open subset Ω of $\mathbb{P}(\mathbb{R}^d)$. In that case, we also say that the image $\rho(\Gamma_0)$ is *convex cocompact* in $\mathbb{P}(\mathbb{R}^d)$.

As above, if ρ is convex cocompact in $\mathbb{P}(\mathbb{R}^d)$, then it has finite kernel and discrete image, and the group Γ_0 is finitely generated.

This notion turns out to be quite fruitful: by [54], the set of convex cocompact representations is open in $\text{Hom}(\Gamma_0, \text{PGL}(d, \mathbb{R}))$, and it is stable under duality and under embedding into a larger projective space; moreover, a representation $\rho : \Gamma_0 \rightarrow \text{PGL}(d, \mathbb{R})$ is strongly convex cocompact in $\mathbb{P}(\mathbb{R}^d)$ in the sense of Theorem 4.5.20.(2) if and only if it is convex cocompact in $\mathbb{P}(\mathbb{R}^d)$ and Γ_0 is Gromov hyperbolic. Theorem 4.5.20 then shows that convex cocompact representations are generalisations of P_1 -Anosov representations, for finitely generated infinite groups Γ_0 that are not necessarily Gromov hyperbolic, and that may therefore contain subgroups isomorphic to \mathbb{Z}^2 (see Remark 4.4.11).

In fact, Weisman [131] has recently given a dynamical characterisation of convex cocompact representations of Γ_0 that extends the characterisation of Anosov representations of Theorem 4.5.13.(6). The expansion now takes place in various Grassmannians (not only projective space): namely, at each face of the full orbital limit set in $\partial\Omega$, there is expansion in the Grassmannian of i -planes of \mathbb{R}^d where $i - 1$ is the dimension of the face.

We conclude this section by mentioning a few examples of convex cocompact groups that are not necessarily Gromov hyperbolic (i.e. that are not necessarily Anosov subgroups).

Example 4.5.29 Let Γ be a discrete subgroup of $\text{PGL}(d, \mathbb{R})$ *dividing* (i.e. acting properly discontinuously with compact quotient on) some properly convex open subset Ω of $\mathbb{P}(\mathbb{R}^d)$. Then $\Lambda_{\rho(\Gamma_0)}^{\text{orb}}(\Omega) = \partial\Omega$ and the action of Γ on Ω is convex cocompact. By [16], the group Γ is Gromov hyperbolic if and only if $\partial\Omega$ contains no segments. Examples where $\partial\Omega$ contains segments include the symmetric divisible convex sets $\Omega_{\text{sym}} \subset \mathbb{P}(\text{Sym}_{d'}(\mathbb{R})) \simeq \mathbb{P}(\mathbb{R}^d)$ with $d = d'(d' + 1)/2 \geq 6$ discussed before

Corollary 4.5.21. The first nonsymmetric irreducible examples were constructed in small dimensions ($4 \leq d \leq 7$) by Benoist [18]; examples in all dimensions $d \geq 4$ were recently constructed by Blayac and Viaggi [23].

Example 4.5.30 For Γ dividing Ω as in Example 4.5.29, we can lift Γ to a subgroup $\hat{\Gamma}$ of $\mathrm{SL}^\pm(d, \mathbb{R})$ preserving a properly convex cone of \mathbb{R}^d lifting Ω , and then embed $\hat{\Gamma}$ into $\mathrm{PGL}(D, \mathbb{R})$ for some $D \geq d$. By the result of [54] mentioned above, the discrete subgroup of $\mathrm{PGL}(D, \mathbb{R})$ obtained in this way will be convex cocompact in $\mathbb{P}(\mathbb{R}^D)$; moreover, it will remain convex cocompact in $\mathbb{P}(\mathbb{R}^D)$ after any small deformation in $\mathrm{PGL}(D, \mathbb{R})$.

Recall that, given a Coxeter group W as in (4.7), a subgroup of W is called *standard* if it is generated by a subset of the generating set $\{s_1, \dots, s_N\}$. The Coxeter group W is called *affine* if it is irreducible (i.e. it cannot be written as a direct product of two nontrivial Coxeter groups) and if it is virtually (i.e. it admits a finite-index subgroup which is) isomorphic to \mathbb{Z}^k for some $k \geq 1$. Affine Coxeter groups have been completely classified; they include the Coxeter groups of type \tilde{A}_k (which are virtually isomorphic to \mathbb{Z}^k), where we say that W is of type \tilde{A}_1 if $N = 2$ and $m_{1,2} = \infty$, and W is of type \tilde{A}_{N-1} for $N \geq 3$ if $m_{i,j} = 3$ for all $i \neq j$ with $|i - j| = 1 \bmod N$ and $m_{i,j} = 2$ for all other $i \neq j$.

Example 4.5.31 ([56]) As a generalisation of Example 4.5.25, let W be a Coxeter group in N generators as in (4.7). Suppose W is infinite. Then there exists a representation $\rho \in \mathrm{Hom}_{\mathrm{refl}}(W, \mathrm{GL}(d, \mathbb{R}))$ which is convex cocompact in $\mathbb{P}(\mathbb{R}^d)$ for some d if and only if any affine standard subgroup of W is of type \tilde{A}_k for some $k \geq 1$ and W does not contain a direct product of two infinite standard subgroups. If this holds, then we can take any $d \geq N$ and the convex cocompact representations then constitute a large open subset of $\mathrm{Hom}_{\mathrm{refl}}(W, \mathrm{GL}(d, \mathbb{R}))$: see [56, § 1.5].

Examples 4.5.29, 4.5.30, and 4.5.31 provide many convex cocompact groups which are not Gromov hyperbolic. (In Example 4.5.31, the group W is nonhyperbolic as soon as it contains an affine standard subgroup of type \tilde{A}_k with $k \geq 2$, see Remark 4.4.11.)

Some of these groups are still relatively hyperbolic: e.g. in Example 4.5.31, the group W is relatively hyperbolic with respect to a collection of virtually abelian subgroups of rank ≥ 2 (see [56, Cor. 1.7]). We refer to [93] for general results about the structure of relatively hyperbolic groups which are convex cocompact in $\mathbb{P}(\mathbb{R}^d)$ and about the geometry of the associated convex sets. On the other hand, Example 4.5.29 includes,

for $d = d'(d' + 1)/2 \geq 6$, discrete subgroups of $\mathrm{PGL}(d, \mathbb{R})$ which divide a symmetric properly convex open set $\Omega_{\mathrm{sym}} \subset \mathbb{P}(\mathbb{R}^d) \simeq \mathbb{P}(\mathrm{Sym}_{d'}(\mathbb{R}))$ and which are isomorphic to cocompact lattices of $\mathrm{PGL}(d', \mathbb{R})$, hence *not* relatively hyperbolic (see Section 4.2.3). Further examples of convex cocompact groups which are not relatively hyperbolic can be constructed e.g. using free products inside larger projective spaces: see [55].

Relatively Anosov subgroups

Kapovich–Leeb [97] and Zhu [137, 138] have developed notions of a *relatively Anosov representation* of a relatively hyperbolic group into a noncompact semisimple Lie group G , which generalise the notion of an Anosov representation of a hyperbolic group into G from Section 4.5.1. They obtain various characterisations similar to those of Theorem 4.5.13. The original definition of Anosov representations using flows (Definition 4.5.1 and Condition 4.5.4) is recovered in this more general setting by recent work of Zhu and Zimmer [139].

Extending Fact 4.5.6, if Γ_0 is relatively hyperbolic with respect to a collection of subgroups (called *peripheral subgroups*), then relatively Anosov representations of Γ_0 into a given G are stable under small deformations that preserve the conjugacy class of the image of each peripheral subgroup [97, 139].

Any relatively Anosov representation $\rho : \Gamma_0 \rightarrow G$ has finite kernel and discrete image $\rho(\Gamma_0)$, called a *relatively Anosov subgroup* of G . There are many examples of relatively Anosov subgroups (see [97, 140]), including:

- geometrically finite subgroups of G for $\mathrm{rank}_{\mathbb{R}}(G) = 1$ (Definition 4.4.1),
- some of the Schottky groups of Section 4.3.2,
- the images of certain compositions $\tau \circ \sigma_0 : \Gamma_0 \rightarrow G$ where $\sigma_0 : \Gamma_0 \rightarrow G'$ is a geometrically finite representation into a semisimple Lie group G' with $\mathrm{rank}_{\mathbb{R}}(G') = 1$ and $\tau : G' \rightarrow G$ is a representation with compact kernel (e.g. Fact 4.5.7 generalises to the relative setting);
- similarly to Section 4.3.3, small deformations in G of such $\tau \circ \sigma_0(\Gamma_0)$, preserving the conjugacy class of the image of each peripheral subgroup;
- certain representations of $\mathrm{PSL}(2, \mathbb{Z})$ into $\mathrm{PGL}(3, \mathbb{R})$ constructed by Schwartz [123] by iterating Pappus's theorem (see [140, § 13]);
- for a finite-volume hyperbolic surface S , the images of positive (in the sense of Fock–Goncharov [70]) type-preserving representations of $\Gamma_0 =$

$\pi_1(S)$ into a real split simple Lie group G , see [43] (for closed S , these coincide with the Hitchin representations of Sections 4.3.3–4.3.4);

- discrete subgroups of $\mathrm{PGL}(d, \mathbb{R})$ preserving a properly convex open subset Ω of $\mathbb{P}(\mathbb{R}^d)$ with strong regularity ($\partial\Omega$ is C^1 with no segments), and whose action on Ω is geometrically finite in the sense of [52].

It would be interesting to determine whether relatively Anosov representations of relatively hyperbolic groups can also be fully characterised geometrically similarly to Theorem 4.5.20 and Corollaries 4.5.21–4.5.22.

Extended geometrically finite subgroups

Recently, Weisman [132, 133] has introduced a notion of *extended geometrically finite* (or *EGF* for short) representation of a relatively hyperbolic group. This is a dynamical notion, which extends a dynamical characterisation of Anosov representations in terms of multicones [26]. EGF representations include all Anosov or relatively Anosov representations, all representations of relatively hyperbolic groups which are convex cocompact in the sense of Definition 4.5.28, as well as other examples (see [133, Th. 1.5–1.7] and [23, Prop. 6.5 & Rem. 6.2]). They are stable under certain small deformations, called *peripherally stable*, for which the dynamics of the peripheral subgroups does not degenerate too much.

On the other hand, it would be interesting to define a general notion of geometric finiteness in convex projective geometry (involving properly convex open subsets Ω of $\mathbb{P}(\mathbb{R}^d)$ where $\partial\Omega$ may contain segments or not be C^1), and to make the link with Weisman's EGF representations. A good notion of geometric finiteness should contain as a particular case the notion of convex cocompactness from Definition 4.5.27. More precisely, a convex projective manifold $M = \Gamma \backslash \Omega$ should be geometrically finite if its convex core $\Gamma \backslash \mathcal{C}_\Gamma^{\mathrm{cor}}(\Omega)$ (see Definition 4.5.27) is covered by a compact piece and finitely many ends of M , called *cusps*, with a controlled geometry. It is not completely clear what the right definition of a cusp should be. Following Cooper, Long, and Tillmann [48], one could define a (full) cusp to be the image in M of some convex open subset of Ω whose stabiliser in Γ is infinite and does not contain any hyperbolic element (i.e. any element of this stabiliser has all its complex eigenvalues of the same modulus); in that case, the cusp is diffeomorphic to the direct product of \mathbb{R} with an affine $(d - 2)$ -dimensional manifold called the *cusp cross-section*, and the stabiliser of the cusp is virtually nilpotent [48, Th. 5.3]. The cusp is said to have *maximal rank* if the cross-section is compact. A more general notion of cusp of maximal rank, where the

stabiliser may contain hyperbolic elements but is still assumed to be virtually nilpotent, was studied in [5]. A notion of geometric finiteness involving only such generalised cusps of maximal rank was introduced and studied in [136], where it was characterised in dynamical terms. Examples (both of finite and infinite volume) were constructed in [6, 24] as small deformations of finite-volume real hyperbolic manifolds, using a stability result from [49]; the corresponding representations are EGF by [133]. On the other hand, the study of convex projective cusps of nonmaximal rank, possibly allowing for hyperbolic elements, is still at its infancy, and a good general notion of geometric finiteness in this setting still remains to be found, together with appropriate dynamical characterisations.

Acknowledgements. These notes grew out of a minicourse given at Groups St Andrews 2022 in Newcastle. I would like to warmly thank the organisers (Colin Campbell, Martyn Quick, Edmund Robertson, Colva Roney-Dougal, and David Stewart) for a very stimulating and enjoyable meeting.

I am grateful to Pierre-Louis Blayac, Jean-Philippe Burelle, Jeffrey Danciger, Sami Douba, Balthazar Fléchettes, Ilia Smilga, Jérémy Toulisse, and the referee for many useful comments on a preliminary version of this text, and to Florian Stecker for providing Figure 4.9.

This project received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (ERC starting grant DiGGeS, grant agreement No. 715982). The final proofreading was done at the Institute for Advanced Study in Princeton, supported by the National Science Foundation under Grant No. DMS-1926686.

References

- [1] I. Agol, “Systoles of hyperbolic 4-manifolds”, arXiv:0612290.
- [2] G. Arzhantseva, A. Minasyan, D. Osin, “The SQ-universality and residual properties of relatively hyperbolic groups”, *J. Algebra* **315** (2007), 165–177.
- [3] U. Bader, P.-E. Caprace, A. Furman, A. Sisto, “Hyperbolic actions of higher-rank lattices come from rank-one factors”, arXiv:2206.06431.
- [4] U. Bader, A. Furman, T. Gelander, N. Monod, “Property (T) and rigidity for actions on Banach spaces”, *Acta Math.* **198** (2007), 57–105.

- [5] S. Ballas, D. Cooper, A. Leitner, “Generalized cusps on convex projective manifolds: Classification”, *J. Topol.* **13** (2020), 1455–1496.
- [6] S. Ballas, L. Marquis, “Convex projective bendings of hyperbolic manifolds”, *Groups Geom. Dyn.* **14** (2020), 653–688.
- [7] E. P. van den Ban, M. Flensted-Jensen, H. Schlichtkrull, “Harmonic analysis on semisimple symmetric spaces: A survey of some general results”, in “Representation theory and automorphic forms”, Proceedings of an instructional conference (Edinburgh, UK, 1996), 191–217, Proceedings of Symposia in Pure Mathematics, vol. 61, American Mathematical Society, Providence, RI, 1997.
- [8] T. Barbot, “Flag structures on Seifert manifolds”, *Geom. Topol.* **5** (2001), 227–266.
- [9] T. Barbot, “Deformations of Fuchsian AdS representations are quasi-Fuchsian”, *J. Differential Geom.* **101** (2015), 1–46.
- [10] T. Barbot, Q. Mérigot, “Anosov AdS representations are quasi-Fuchsian”, *Groups Geom. Dyn.* **6** (2012), 441–483.
- [11] H. Bass, J. W. Milnor, J.-P. Serre, “Solution of the congruence subgroup problem for SL_n ($n \geq 3$) and Sp_{2n} ($n \geq 2$)”, *Pub. Math. Inst. Hautes Études Sci.* **33** (1967), 59–137.
- [12] J. Behrstock, C. Druţu, L. Mosher, “Thick metric spaces, relative hyperbolicity, and quasi-isometric rigidity”, *Math. Ann.* **344** (2009), 543–595.
- [13] M. V. Belolipetsky, S. A. Thomson, “Systoles of hyperbolic manifolds”, *Algebr. Geom. Topol.* **11** (2010).
- [14] N. Benakli, I. Kapovich, “Boundaries of hyperbolic groups”, in “Combinatorial and geometric group theory (New York, 2000/Hoboken, NJ, 2001)”, 39–93, Contemporary Mathematics, vol. 296, American Mathematical Society, Providence, RI, 2002.
- [15] Y. Benoist, “Propriétés asymptotiques des groupes linéaires”, *Geom. Funct. Anal.* **7** (1997), 1–47.
- [16] Y. Benoist, “Convexes divisibles I”, in “Algebraic groups and arithmetic”, Tata Inst. Fund. Res. Stud. Math. **17** (2004), 339–374.
- [17] Y. Benoist, “Convex divisibles III”, *Ann. Sci. Éc. Norm. Supér.* **38** (2005), 793–832.
- [18] Y. Benoist, “Convexes divisibles IV”, *Invent. Math.* **164** (2006), 249–278.
- [19] Y. Benoist, “Five lectures on lattices in semisimple Lie groups”, in “Géométries à courbure négative ou nulle, groupes discrets et rigidités”, 117–176, Séminaires et Congrès, vol. 18, Société Mathématique de France, Paris, 2009.
- [20] N. Bergeron, F. Haglund, D. T. Wise, “Hyperplane sections in arithmetic hyperbolic manifolds”, *J. Lond. Math. Soc.* **83** (2011), 431–448.
- [21] J. Beyrer, F. Kassell, “ $\mathbb{H}^{p,q}$ -convex cocompactness and higher higher Teichmüller spaces”, arXiv:2305.15031.
- [22] J. Beyrer, M. B. Pozzetti, “Positive surface group representations in $PO(p,q)$ ”, arXiv:2106.14725.
- [23] P.-L. Blayac, G. Viaggi, “Divisible convex sets with properly embedded cones”, arXiv:2302.07177.

- [24] M. D. Bobb, “Convex projective manifolds with a cusp of any non-diagonalizable type”, *J. Lond. Math. Soc.* **100** (2019), 183–202.
- [25] J. Bochi, N. Gourmelon, “Some characterizations of domination”, *Math. Z.* **263** (2009), 221–231.
- [26] J. Bochi, R. Potrie, A. Sambarino, “Anosov representations and dominated splittings”, *J. Eur. Math. Soc.* **21** (2019), 3343–3414.
- [27] F. Bonahon, G. Dreyer, “Parameterizing Hitchin components”, *Duke Math. J.* **163** (2014), 2935–2975.
- [28] A. Borel, “Density properties for certain subgroups of semi-simple groups without compact components”, *Ann. of Math.* **72** (1960), 179–188.
- [29] A. Borel, “Compact Clifford–Klein forms of symmetric spaces”, *Topology* **2** (1963), 111–122.
- [30] A. Borel, Harish-Chandra, “Arithmetic subgroups of algebraic groups”, *Ann. of Math.* **75** (1962), 485–535.
- [31] B. H. Bowditch, “Geometrical finiteness with variable negative curvature”, *Duke Math. J.* **77** (1995), 229–274.
- [32] S. Bradlow, “Global properties of Higgs bundle moduli spaces”, arXiv:2312.00762, to appear in a volume in honor of Peter Newstead on the occasion of his 80th birthday.
- [33] S. Bradlow, B. Collier, O. García-Prada, P. B. Gothen, A. Oliveira, “A general Cayley correspondence and higher Teichmüller spaces”, to appear in *Ann. Math.*
- [34] S. B. Bradlow, O. García-Prada, P. B. Gothen, “Maximal surface group representations in isometry groups of classical Hermitian symmetric spaces”, *Geom. Dedicata* **122** (2006), 185–213.
- [35] M. Bridgeman, R. D. Canary, F. Labourie, A. Sambarino, “The pressure metric for Anosov representations”, *Geom. Funct. Anal.* **25** (2015), 1089–1179.
- [36] J.-P. Burelle, F. Kassel, “Crooked surfaces and symplectic Schottky groups”, in preparation.
- [37] M. Burger, A. Iozzi, F. Labourie, A. Wienhard, “Maximal representations of surface groups: Symplectic Anosov structures”, *Pure Appl. Math. Q.* **1** (2005), special issue in memory of Armand Borel, 543–590.
- [38] M. Burger, A. Iozzi, A. Wienhard, “Surface group representations with maximal Toledo invariant”, *Ann. of Math.* **172** (2010), 517–566.
- [39] M. Burger, A. Iozzi, A. Wienhard, “Higher Teichmüller spaces: from $SL(2, \mathbb{R})$ to other Lie groups”, in “Handbook of Teichmüller theory IV”, 539–618, IRMA Lectures in Mathematics and Theoretical Physics, vol. 19, EMS Publishing House, Zürich, 2014.
- [40] M. Burger, M. B. Pozzetti, “Maximal representations, non Archimedean Siegel spaces, and buildings”, *Geom. Topol.* **21** (2017), 3539–3599.
- [41] R. D. Canary, “Hitchin representations of Fuchsian groups”, *EMS Surv. Math. Sci.* **9** (2022), 355–388, special issue in honor of Dennis Sullivan on the occasion of his 80th birthday.

- [42] R. D. Canary, M. Lee, A. Sambarino, M. Stover, “Projective Anosov Schottky groups and strongly amalgam Anosov representations”, appendix to “Amalgam Anosov representations” by Canary, Lee and Stover, *Geom. Topol.* **21** (2017), 240–248.
- [43] R. D. Canary, T. Zhang, A. Zimmer, “Cusped Hitchin representations and Anosov representations of geometrically finite Fuchsian groups”, *Adv. Math.* **404** (2022), 1–67.
- [44] S. Cantat, “Progrès récents concernant le programme de Zimmer [d’après A. Brown, D. Fisher et S. Hurtado]”, *Séminaire Bourbaki, Exposé 1136, Astérisque* **414** (2019), 1–48.
- [45] C. Chevalley, “Théorie des groupes de Lie : II. Groupes algébriques”, *Actualités Scientifiques et Industrielles*, vol. 1152, Hermann & Cie, Paris, 1951.
- [46] S. Choi, W. M. Goldman, “Convex real projective structures on closed surfaces are closed”, *Proc. Amer. Math. Soc.* **118** (1993), 657–661.
- [47] B. Collier, N. Tholozan, J. Toulisse, “The geometry of maximal representations of surface groups into $SO_0(2, n)$ ”, *Duke Math. J.* **168** (2019), 2873–2949.
- [48] D. Cooper, D. D. Long, S. Tillmann, “On convex projective manifolds and cusps”, *Adv. Math.* **277** (2015), 181–251.
- [49] D. Cooper, D. Long, S. Tillmann, “Deforming convex projective manifolds”, *Geom. Topol.* **22** (2018), 1349–1404.
- [50] D. Cooper, S. Tillmann, “The space of properly-convex structures”, arXiv:2009.06568.
- [51] M. Coornaert, T. Delzant, A. Papadopoulos, “Géométrie et théorie des groupes. Les groupes hyperboliques de Gromov”, *Lecture Notes in Mathematics*, vol. 1441, Springer-Verlag, Berlin, 1990.
- [52] M. Crampon, L. Marquis, “Finitude géométrique en géométrie de Hilbert”, with an appendix by C. Vernicos, *Ann. Inst. Fourier* **64** (2014), 2299–2377.
- [53] J. Danciger, F. Guéritaud, F. Kassel, “Convex cocompactness in pseudo-Riemannian hyperbolic spaces”, *Geom. Dedicata* **192** (2018), 87–126, special issue “Geometries: A Celebration of Bill Goldman’s 60th Birthday”.
- [54] J. Danciger, F. Guéritaud, F. Kassel, “Convex cocompact actions in real projective geometry”, *Ann. Sci. Éc. Norm. Supér.* **57** (2024), 1751–1841.
- [55] J. Danciger, F. Guéritaud, F. Kassel, “Combination theorems in convex projective geometry”, arXiv:2407.09439.
- [56] J. Danciger, F. Guéritaud, F. Kassel, G.-S. Lee, L. Marquis, “Convex cocompactness for Coxeter groups”, to appear in *J. Eur. Math. Soc.*
- [57] P. Deligne, G. D. Mostow, “Monodromy of hypergeometric functions and non-lattice integral monodromy”, *Publ. Math. Inst. Hautes Études Sci.* **63** (1986), 5–89.
- [58] M. Deraux, “A new non-arithmetic lattice in $PU(3,1)$ ”, *Algebr. Geom. Topol.* **20** (2020), 925–963.
- [59] M. Deraux, J. R. Parker, J. Paupert, “New non-arithmetic complex hyperbolic lattices”, *Invent. Math.* **203** (2016), 681–771.

- [60] M. Desgroseilliers, F. Haglund, “On some convex cocompact groups in real hyperbolic space”, *Geom. Topol.* **17** (2013), 2431–2484.
- [61] S. Dey, M. Kapovich, “Klein–Maskit combination theorem for Anosov subgroups: Free products”, *Math. Z.* **305** (2023), article number 35.
- [62] S. Dey, M. Kapovich, “Klein–Maskit combination theorem for Anosov subgroups: Amalgams”, *arXiv:2301.02345*.
- [63] S. Douba, “Systoles of hyperbolic hybrids”, *arXiv:2309.16051*.
- [64] S. Douba, B. Fléchéllles, T. Weisman, F. Zhu, “Cubulated hyperbolic groups admit Anosov representations”, *arXiv:2309.03695*.
- [65] C. Druţu, M. Kapovich, “Geometric group theory”, American Mathematical Society Colloquium Publications, vol. 63, American Mathematical Society, Providence, RI, 2018.
- [66] P. B. Eberlein, “Geometry of nonpositively curved manifolds”, *Chicago Lectures in Mathematics*, The University of Chicago Press, Chicago, IL, 1996.
- [67] D. Fisher, “Superrigidity, arithmeticity, normal subgroups: results, ramifications and directions”, in “Dynamics, geometry, number theory — The impact of Margulis on modern mathematics”, 9–46, Chicago University Press, Chicago, IL 2022.
- [68] D. Fisher, “Rigidity, lattices and invariant measures beyond homogeneous dynamics”, *Proceedings of the International Congress of Mathematicians 2022 (ICM 2022)*, 3484–3507, EMS Publishing House, Berlin, 2023.
- [69] D. Fisher, G. A. Margulis, “Local rigidity for cocycles”, in “Surveys in differential geometry, vol. VIII” (Boston, MA, 2002), 191–234, *Surveys in Differential Geometry*, vol. 8, International Press, Somerville, MA, 2003.
- [70] V. V. Fock, A. B. Goncharov, “Moduli spaces of local systems and higher Teichmüller theory”, *Publ. Math. Inst. Hautes Études Sci.* **103** (2006), 1–211.
- [71] M. Fraczyk, T. Gelander, “Infinite volume and infinite injectivity radius”, *Ann. of Math.* **197** (2023), 389–421.
- [72] C. Frances, “The conformal boundary of Margulis space-times”, *C. R. Acad. Sci. Paris* **336** (2003), 751–756.
- [73] A. Furman, “A survey of measured group theory”, in “Geometry, rigidity, and group actions”, 296–374, *Chicago Lectures in Mathematics*, University of Chicago Press, Chicago, IL, 2011.
- [74] O. García-Prada, “Higgs bundles and higher Teichmüller spaces”, *Handbook of Teichmüller theory VII*, 239–285, *IRMA Lectures in Mathematics and Theoretical Physics*, vol. 30, EMS Publishing House, Zürich, 2020.
- [75] O. García-Prada, P. B. Gothen, I. Mundet i Riera, “The Hitchin–Kobayashi correspondence, Higgs pairs and surface group representations”, *arXiv:0909.4487*.
- [76] G. Gardam, D. Kielak, A. D. Logan, “Algebraically hyperbolic groups”, *arXiv:2112.01331*.
- [77] W. M. Goldman, “Discontinuous groups and the Euler class”, PhD thesis, University of California, Berkeley, 1980, see <https://www.math.umd.edu/~wmg/PhDthesis.pdf>

- [78] W. M. Goldman, “Geometric structures on manifolds”, Graduate Studies in Mathematics, vol. 227, American Mathematical Society, Providence, RI, 2022.
- [79] M. Gromov, “Hyperbolic groups”, in “Essays in group theory”, 75–263, Mathematical Sciences Research Institute Publications, vol. 8, Springer, New York, 1987.
- [80] M. Gromov, I. Piatetski-Shapiro, “Nonarithmetic groups in Lobachevsky spaces”, *Publ. Math. Inst. Hautes Études Sci.* **66** (1988), 93–103.
- [81] F. Guéritaud, O. Guichard, F. Kassel, A. Wienhard, “Anosov representations and proper actions”, *Geom. Topol.* **21** (2017), 485–584.
- [82] F. Guéritaud, F. Kassel, “Maximally stretched laminations on geometrically finite hyperbolic manifolds”, *Geom. Topol.* **21** (2017), 693–840.
- [83] O. Guichard, F. Labourie, A. Wienhard, “Positivity and representations of surface groups”, [arXiv:2106.14584](https://arxiv.org/abs/2106.14584).
- [84] O. Guichard, A. Wienhard, “Anosov representations : Domains of discontinuity and applications”, *Invent. Math.* **190** (2012), 357–438.
- [85] O. Guichard, A. Wienhard, “Positivity and higher Teichmüller theory”, *Proceedings of the European Congress of Mathematics (Zürich 2016)*, 289–310, European Mathematical Society, 2018.
- [86] O. Guichard, A. Wienhard, “Generalizing Lusztig’s total positivity”, [arXiv:2208.10114](https://arxiv.org/abs/2208.10114).
- [87] T. Haettel, “Hyperbolic rigidity of higher rank lattices”, with an appendix by C. Horbez and V. Guirardel, *Ann. Sci. Éc. Norm. Supér.* **53** (2020), 437–468.
- [88] E. Hamilton, “Geometrical finiteness for hyperbolic orbifolds”, *Topology* **37** (1998), 635–657.
- [89] B. B. Healy, G. C. Hruska, “Cusped spaces and quasi-isometries of relatively hyperbolic groups”, [arXiv:2010.09876](https://arxiv.org/abs/2010.09876).
- [90] S. Helgason, “Differential geometry, Lie groups, and symmetric spaces”, corrected reprint of the 1978 original, Graduate Studies in Mathematics, vol. 34, American Mathematical Society, Providence, RI, 2001.
- [91] G. Hinojosa, A. Verjovsky, “Actions of discrete groups on spheres and real projective spaces”, *Bull. Braz. Math. Soc. (N. S.)* **39** (2008), 157–171.
- [92] N. J. Hitchin, “Lie groups and Teichmüller space”, *Topology* **31** (1992), 339–365.
- [93] M. Islam, A. Zimmer, “Convex co-compact actions of relatively hyperbolic groups”, *Geom. Topol.* **27** (2023), 417–511.
- [94] D. Johnson, J. J. Millson, “Deformation spaces associated to compact hyperbolic manifolds”, in “Discrete groups in geometry and analysis”, 48–106, Progress in Mathematics, vol. 67, Birkhäuser, Boston, MA, 1987.
- [95] M. Kapovich, “Kleinian groups in higher dimensions”, in “Geometry and dynamics of groups and spaces”, 485–562, Progress in Mathematics, vol. 265, Birkhäuser Verlag, Basel, 2007.
- [96] M. Kapovich, B. Leeb, “Discrete isometry groups of symmetric spaces”, in “Handbook of group actions IV”, 191–290, Advanced Lectures in Mathematics, vol. 41, International Press, Boston, MA, 2018.

- [97] M. Kapovich, B. Leeb, “Relativizing characterizations of Anosov subgroups, I”, *Groups Geom. Dyn.* **17** (2023), 1005–1071.
- [98] M. Kapovich, B. Leeb, J. Porti, “Morse actions of discrete groups on symmetric spaces”, arXiv:1403.7671.
- [99] M. Kapovich, B. Leeb, J. Porti, “Some recent results on Anosov representations”, *Transform. Groups* **21** (2016), 1105–1121.
- [100] M. Kapovich, B. Leeb, J. Porti, “Dynamics on flag manifolds: domains of proper discontinuity and cocompactness”, *Geom. Topol.* **22** (2018), 157–234.
- [101] M. Kapovich, B. Leeb, J. Porti, “A Morse Lemma for quasigeodesics in symmetric spaces and euclidean buildings”, *Geom. Topol.* **22** (2018), 3827–3923.
- [102] F. Kassel, “Geometric structures and representations of discrete groups”, *Proceedings of the International Congress of Mathematicians 2018 (ICM 2018)*, 1113–1150, World Scientific, 2019.
- [103] F. Kassel, M. B. Pozzetti, A. Sambarino, A. Wienhard (editors), “Arbeitsgemeinschaft: Higher rank Teichmüller theory”, *Oberwolfach Rep.* **19** (2022), 2687–2740.
- [104] F. Kassel, R. Potrie, “Eigenvalue gaps for hyperbolic groups and semi-groups”, *J. Mod. Dyn.* **18** (2022), 161–208.
- [105] B. Kleiner, B. Leeb, “Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings”, *Publ. Math. Inst. Hautes Études Sci.* **86** (1997), 115–197.
- [106] B. Kleiner, B. Leeb, “Rigidity of invariant convex sets in symmetric spaces”, *Invent. Math.* **163** (2006), 657–676.
- [107] A. Kontorovich, D. D. Long, A. Lubotzky, A. W. Reid, “What is... a thin group?”, *Notices Amer. Math. Soc.* **66** (2019), 905–910.
- [108] C. Kourouniotis, “Deformations of hyperbolic structures on manifolds of several dimensions”, *Math. Proc. Cambridge Philos. Soc.* **98** (1985), 247–261.
- [109] F. Labourie, “Anosov flows, surface groups and curves in projective space”, *Invent. Math.* **165** (2006), 51–114.
- [110] G.-S. Lee, L. Marquis, “Anti-de Sitter strictly GHC-regular groups which are not lattices”, *Trans. Amer. Math. Soc.* **372** (2019), 153–186.
- [111] G. Lusztig, “Total positivity in reductive groups”, in “Lie theory and geometry”, 531–568, *Progress in Mathematics*, vol. 123, Birkhäuser, Boston, MA, 1994.
- [112] G. A. Margulis, “Discrete subgroups of semisimple Lie groups”, *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*, vol. 17, Springer-Verlag, Berlin, 1991.
- [113] B. Maskit, “Kleinian groups”, *Grundlehren der mathematischen Wissenschaften*, vol. 287, Springer-Verlag, Berlin, 1988.
- [114] G. Mess, “Lorentz spacetimes of constant curvature” (1990), *Geom. Dedicata* **126** (2007), 3–45.
- [115] D. Monclair, J.-M. Schlenker, N. Tholozan, “Gromov–Thurston manifolds and anti-de Sitter geometry”, arXiv:2310.12003.

- [116] G. D. Mostow, “On a remarkable class of polyhedra in complex hyperbolic space”, *Pacific J. Math.* **86** (1980), 171–276.
- [117] D. Mumford, C. Series, D. Wright, “*Indra’s pearls. The vision of Felix Klein*”, Cambridge University Press, 2002.
- [118] D. Osin, “Small cancellations over relatively hyperbolic groups and embedding theorems”, *Ann. of Math.* **172** (2010), 1–39.
- [119] P. Pansu, “Sous-groupes discrets des groupes de Lie : rigidité, arithmétique”, *Séminaire Bourbaki, Exposé 778, Astérisque* **227** (1995), 69–105.
- [120] M. B. Pozzetti, “Higher rank Teichmüller theories”, *Séminaire Bourbaki, Exposé 1159, Astérisque* **422** (2019), 327–354.
- [121] J.-F. Quint, “Groupes convexes cocompacts en rang supérieur”, *Geom. Dedicata* **113** (2005), 1–19.
- [122] H. P. de Saint-Gervais, “Uniformization of Riemann surfaces. Revisiting a hundred-year old theorem”, *Heritage of European Mathematics, European Mathematical Society, Zürich*, 2016.
- [123] R. Schwartz, “Pappus’s theorem and the modular group”, *Publ. Math. Inst. Hautes Études Sci.* **78** (1993), 187–206.
- [124] J. Seade, A. Verjovsky, “Complex Schottky groups”, *Astérisque* **287** (2003), 251–272.
- [125] A. Selberg, “On discontinuous groups in higher-dimensional symmetric spaces” (1960), in “*Collected papers*”, vol. 1, 475–492, Springer-Verlag, Berlin, 1989.
- [126] C. Series, “A crash course on Kleinian groups”, *Rend. Istit. Mat. Univ. Trieste* **37** (2005), 1–38.
- [127] B. Sury, “The congruence subgroup problem — An elementary approach aimed at applications”, *Texts and Readings in Mathematics*, vol. 24, Hindustan Book Agency, New Delhi, 2003.
- [128] J. Tits, “Free subgroups in linear groups”, *J. Algebra* **20** (1972), 250–270.
- [129] E. B. Vinberg, “Discrete groups generated by reflections in Lobachevskii spaces”, *Math. USSR Sb.* **1** (1968), 429–444.
- [130] E. B. Vinberg, “Discrete linear groups generated by reflections”, *Math. USSR Izv.* **5** (1971), 1083–1119.
- [131] T. Weisman, “Dynamical properties of convex cocompact actions in projective space”, *J. Topol.* **16** (2023), 990–1047.
- [132] T. Weisman, “An extended definition of Anosov representation for relatively hyperbolic groups”, *arXiv:2205.07183*.
- [133] T. Weisman, “Examples of extended geometrically finite representations”, *arXiv:2311.18653*.
- [134] A. Wienhard, “An invitation to higher Teichmüller theory”, *Proceedings of the International Congress of Mathematicians 2018 (ICM 2018)*, 1007–1034, World Scientific, 2019.
- [135] D. Witte Morris, “*Introduction to Arithmetic Groups*”, Deductive Press, 2015.
- [136] A. Wolf, “Convex projective geometrically finite structures”, PhD thesis, Stanford University, 2020.

- [137] F. Zhu, “Relatively dominated representations”, *Ann. Inst. Fourier* **71** (2021), 2169–2235.
- [138] F. Zhu, “Relatively dominated representations from eigenvalue gaps and limit maps”, *Geom. Dedicata* **217** (2023), article number 39.
- [139] F. Zhu, A. Zimmer, “Relatively Anosov representations via flows I: theory”, *arXiv:2207.14737*.
- [140] F. Zhu, A. Zimmer, “Relatively Anosov representations via flows II: examples”, *arXiv:2207.14738*.
- [141] A. Zimmer, “Projective Anosov representations, convex cocompact actions, and rigidity”, *J. Differential Geom.* **119** (2021), 513–586.