The asymmetry of Thurston's earthquake flow

Francisco Arana-Herrera

Alex Wright

The asymmetry of Thurston's earthquake flow

FRANCISCO ARANA-HERRERA

ALEX WRIGHT

We show that Thurston's earthquake flow is strongly asymmetric in the sense that its normalizer is as small as possible inside the group of orbifold automorphisms of the bundle of measured geodesic laminations over moduli space. (At the level of Teichmüller space, such automorphisms correspond to homeomorphisms that are equivariant with respect to an automorphism of the mapping class group.) It follows that the earthquake flow does not extend to an $SL(2, \mathbb{R})$–action of orbifold automorphisms and does not admit continuous renormalization self-symmetries. In particular, it is not conjugate to the Teichmüller horocycle flow via an orbifold map. This contrasts with a number of previous results, most notably Mirzakhani's theorem that the earthquake and Teichmüller horocycle flows are measurably conjugate.

30F60; 32G15

1 Introduction

Context The bundle $\mathcal{P}^1 \mathcal{M}_g$ of unit-length measured geodesic laminations over the moduli space $\mathcal{M}_g$ of hyperbolic or Riemann surfaces of genus g is most naturally seen as a construction of hyperbolic geometry, whereas the bundle $\mathcal{Q}^1 \mathcal{M}_g$ of unit-area quadratic differentials over $\mathcal{M}_g$ is most naturally seen from the perspective of either complex analysis or flat geometry. The bundle $\mathcal{P}^1 \mathcal{M}_g$ supports Thurston's rather mysterious earthquake flow, which is most concisely defined as a Hamiltonian flow using the Weil–Petersson symplectic form, whereas the bundle $\mathcal{Q}^1 \mathcal{M}_g$ supports the Teichmüller horocycle flow, easily defined as part of the much-studied $SL(2, \mathbb{R})$–action. Mirzakhani showed that, despite their different origins, these flows are measurably isomorphic.

Theorem 1.1 [Mirzakhani 2008] *There is a measurable conjugacy $\mathcal{P}^1 \mathcal{M}_g \to \mathcal{Q}^1 \mathcal{M}_g$ between the earthquake flow and the Teichmüller horocycle flow.*

In addition to being of fundamental interest as a bridge between different perspectives on the geometry of surfaces and their moduli spaces, this theorem has powered many applications concerning equidistribution, counting and the study of random surfaces [Mirzakhani 2007a; 2016; Arana-Herrera 2021; 2022; Liu 2022; Lu and Su 2022].

Mirzakhani's conjugacy is only defined on a full-measure subset of $\mathcal{P}^1 \mathcal{M}_g$, and, as remarked by Mirzakhani herself [2008, Section 6], this conjugacy cannot be extended to a continuous map on all of $\mathcal{P}^1 \mathcal{M}_g$. Despite

this, Calderon and Farre [2024] extended Mirzakhani's conjugacy to a bijection which, although not continuous, is geometrically natural and has exciting new applications.

One reason Theorem 1.1 is plausible is that there are many conceptual similarities between the earthquake flow and the Teichmüller horocycle flow, such as the following:

(1)   Both arise from some notion of shearing.

(2)   Both have been understood by analogy to unipotent flows on homogeneous spaces.

(3)   Both are Hamiltonian with respect to related symplectic structures [Masur 1995; Sözen and Bonahon 2001].

(4)   Both are associated to natural complex disks in Teichmüller space, namely Teichmüller discs for the Teichmüller horocycle flow and complex earthquake discs for the earthquake flow [McMullen 1998].

(5)   Both have quantitative nondivergence properties [Minsky and Weiss 2002].

No continuous conjugacy   In light of all these similarities and the work of Mirzakhani, Calderon and Farre, one might wonder if a result stronger than Theorem 1.1 holds: perhaps the earthquake and Teichmüller horocycle flows are isomorphic from the point of view of continuous dynamics, ie perhaps there is a different conjugacy between these flows that is also a homeomorphism. This question was advertised by Wright [2020, Problem 12.3; 2022, Remark 5.6]. Our main result on asymmetry, which we will state shortly as Theorem 1.4, implies a negative solution to this problem.

Theorem 1.2   *There does not exist an orbifold conjugacy $\mathcal{P}^1 \mathcal{M}_g \to \mathcal{Q}^1 \mathcal{M}_g$ between the earthquake flow and the Teichmüller horocycle flow.*

The technical restriction in Theorem 1.2 that the conjugacy respects the orbifold structure of these spaces is natural since both spaces have the same orbifold structure [Hubbard and Masur 1979].

The existence of an orbifold conjugacy $\mathcal{P}^1 \mathcal{M}_g \to \mathcal{Q}^1 \mathcal{M}_g$ as in Theorem 1.2 is equivalent to the existence of a topological conjugacy $\mathcal{P}^1 \mathcal{T}_g \to \mathcal{Q}^1 \mathcal{T}_g$ of the lifts to Teichmüller space of the earthquake and Teichmüller horocycle flows that intertwines an automorphism $\rho \colon \mathrm{Mod}_g \to \mathrm{Mod}_g$ of the mapping class group. For detailed discussions on the theory of orbifolds, see [Thurston 1979, Chapter 13; Erlandsson and Souto 2022, Section 2]. In particular, the following corollary holds:

Corollary 1.3   *There does not exist a mapping class group equivariant topological conjugacy $\mathcal{P}^1 \mathcal{T}_g \to \mathcal{Q}^1 \mathcal{T}_g$ between the earthquake flow and the Teichmüller horocycle flow.*

Strong asymmetry   A flow $E = \{E_t \colon \mathcal{X} \to \mathcal{X}\}_{t \in \mathbb{R}}$ on a space $\mathcal{X}$ can be interpreted as a group homomorphism $E \colon \mathbb{R} \to \mathrm{Aut}(\mathcal{X})$ mapping $t \in \mathbb{R}$ to $E_t \in \mathrm{Aut}(\mathcal{X})$, where the automorphism group $\mathrm{Aut}(\mathcal{X})$ is defined in whatever category (smooth, continuous, measurable, etc) is under consideration.

The centralizer of the flow E is defined as

$$C(E) = \{F \in \mathrm{Aut}(\mathcal{X}) : (\forall t \in \mathbb{R}) \ E_t \circ F = F \circ E_t\}.$$

The centralizer corresponds to the most narrow concept of the set of symmetries of a flow one can consider, consisting only of the automorphisms that commute with it. A slightly broader notion is the extended centralizer of a flow, defined here as

$$C_\pm(E) = \{F \in \mathrm{Aut}(\mathcal{X}) : (\exists \varepsilon \in \{1, -1\})(\forall t \in \mathbb{R})\ E_t \circ F = F \circ E_{\varepsilon t}\}.$$

The extended centralizer includes time-reversing symmetries of a flow.

Even more broadly, one can consider the normalizer of a flow, defined as

$$N(E) = \{F \in \mathrm{Aut}(\mathcal{X}) : (\exists \varepsilon \in \{1, -1\}, s \in \mathbb{R})(\forall t \in \mathbb{R})\ E_t \circ F = F \circ E_{\varepsilon e^{2s} t}\}.$$

The normalizer includes symmetries that scale time, ie which conjugate the flow to a constant-speed reparametrization of itself. If $F \in N(E)$ is as above, we call F a *normalizer* of the flow, or an *s–normalizer* if we wish to specify the time dilation factor e^{2s}.

The smallest $N(E)$ can be is the flow itself, namely $N(E) = \{E_t\}_{t \in \mathbb{R}}$. When this is the case, we say that the flow E is *strongly asymmetric*. Our main result establishes this strong asymmetry property for the earthquake flow.

Theorem 1.4 *The normalizer of the earthquake flow inside the group of orbifold automorphisms of $\mathcal{P}^1\mathcal{M}_g$ is the flow itself.*

Theorem 1.2 follows immediately from Theorem 1.4, since the Teichmüller horocycle flow is normalized by the Teichmüller geodesic flow.

A few remarks Before discussing the proof of Theorem 1.4, let us make a couple of remarks.

Remark 1.5 In testing the plausibility of Theorem 1.4, it is natural to consider both Thurston's stretch map flow, defined in [Thurston 1998], and grafting, so we discuss both in turn.

The stretch map flow already makes a natural appearance in any discussion regarding Mirzakhani's conjugacy. Indeed, Mirzakhani's conjugacy shows that the earthquake flow is part of a measurable $SL(2, \mathbb{R})$–action in which the diagonal subgroup acts via the stretch map flow. The stretch map flow does normalize the earthquake flow, but, since it is only defined on a full-measure subset of $\mathcal{P}^1\mathcal{M}_g$, this does not contradict Theorem 1.4.

Grafting plays a central role in the definition of complex earthquake discs. If one compares Teichmüller discs to complex earthquake discs, the Teichmüller geodesic flow corresponds to grafting. Grafting is continuous, but, since it does not normalize the earthquake flow, this does not contradict Theorem 1.4.

In the next two remarks, it is implicit that we are working in the category of topological orbifolds (so in particular all conjugacies are continuous).

Remark 1.6 Theorem 1.4 implies that the earthquake flow is not conjugate to its own inverse. (The inverse of a flow $t \mapsto E_t$ is the flow $t \mapsto E_{-t}$.)

Remark 1.7 Theorem 1.4 implies that the earthquake flow is not the restriction of any $\mathrm{SL}(2,\mathbb{R})$–action to any one-parameter subgroup. (One way to see this is to note that every noncompact one-parameter subgroup of $\mathrm{SL}(2,\mathbb{R})$ has nontrivial normalizer, since the horocycle flow is normalized by the geodesic flow and the geodesic flow is normalized by an involution.)

Outline of the proof Every normalizer can and should be considered as a conjugacy between the earthquake flow and a (possibly trivial) linear time change of itself. Given an s–normalizer $F: \mathscr{P}^1 \mathcal{M}_g \to \mathscr{P}^1 \mathcal{M}_g$, we constrain its behavior until we are eventually able to show it is an element of the flow. This involves four main steps, each occupying a different section of this paper. Throughout we assume $(X, \lambda) \in \mathscr{P}^1 \mathcal{M}_g$ and $F(X, \lambda) = (Y, \mu)$.

(1) By studying minimal sets, we show in Proposition 2.1 that μ is a multicurve if and only if λ is, and, moreover, that the number of components of μ is equal to the number of components of λ. This is strongly related to [Minsky and Weiss 2002; Smillie and Weiss 2004].

(2) Leveraging the rigidity of the curve complex, we show in Proposition 3.1 that μ is a multiple of λ. This relies on [Ivanov 1997] and applies to all $(X, \lambda) \in \mathscr{P}^1 \mathcal{M}_g$.

(3) By carefully analyzing the periods of specific closed orbits, we determine in Lemma 4.2 what the multiple is, showing $\mu = e^s \cdot \lambda$. We moreover show in Lemma 4.3 that, often, many curves shrink by at least a factor of e^{-s} in the passage of X to Y. This gives a contradiction unless $s = 0$, showing that the normalizer is equal to the extended centralizer, a conclusion we record as Proposition 4.1.

(4) In Proposition 5.1, we show that the extended centralizer of the earthquake flow is trivial, by showing that many and hence all orbits are preserved, and using ergodicity. We use the generalized McShane identity of [Mirzakhani 2007b] as a technical tool.

Open problems Many interesting questions related to Mirzakhani's conjugacy remain open. We highlight a few of them here.

To our knowledge, the only previously established dynamical difference between the earthquake and Teichmüller horocycle flows concerns cusp excursions in the specific case of once-punctured tori [Fu 2019]. Previous to this, it was known that certain orbits of the two flows do not stay finite distance apart in one-dimensional Teichmüller spaces [Minsky and Weiss 2002, Proposition 8.1].

Theorem 1.4 is a dynamical difference, since it relates to renormalization, but it would be illuminating to find less subtle differences.

Problem 1.8 Find a dynamical, non-group-theoretic property that is invariant under topological conjugacies and which holds for exactly one of the earthquake flow and the Teichmüller horocycle flow.

It is easy to construct topological joinings between the earthquake flow and the Teichmüller horocycle flow. For example, consider the set of pairs

$$((X, \lambda), q) \in \mathscr{P}^1 \mathcal{M}_g \times \mathscr{Q}^1 \mathcal{M}_g$$

such that the horizontal foliation of q is equal to λ. This construction of a topological joining admits many different variations.

Problem 1.9 Classify all the topological joinings between the earthquake flow and the Teichmüller horocycle flow.

More generally, our dynamical understanding of the earthquake flow remains incomplete, leaving questions such as the following open.

Question 1.10 Is the earthquake flow polynomially mixing?

In comparison, it is known that the Teichmüller horocycle flow is polynomially mixing [Avila et al. 2006; Avila and Resende 2012; Avila and Gouëzel 2013; Ratner 1987].

There are also interesting open questions related to strong asymmetry, including the following deliberately vague question:

Question 1.11 How common is strong asymmetry in smooth dynamics?

The most interesting setting for this question may be flows that share some properties with the earthquake flow, such as volume-preserving flows with zero entropy and having closed orbits of all periods.

Centralizers of flows (and diffeomorphisms) have been studied, for example, in [Obata 2021; Bakker and Fisher 2014; Bonomo and Varandas 2019]. Actions of Baumslag–Solitar groups and other discrete solvable groups have been studied, for example, in [Bonatti et al. 2017; Guelman and Liousse 2011; 2013; Wilkinson and Xue 2020; Burslem and Wilkinson 2004; McCarthy 2010]. Actions of solvable Lie groups have been studied, for example, in [Ghys 1985; Ghys and Verjovsky 1994]. See [Wilkinson 2010; Navas 2018] for some open questions and additional context. See [Navas 2011] for more on the one-dimensional case.

In [Frączek et al. 2014], a continuous flow on the torus is constructed that (in particular) has a measurable s–normalizer for every $s \in \mathbb{R}$ but has no continuous s–normalizers for $s \neq 0$. In light of Theorem 1.4 and the work of Mirzakhani, this is analogous to the situation for the earthquake flow. In [Frączek and Lemańczyk 2009], the symmetries of certain flows are studies in the measurable category. In [Berk et al. 2020], time-reversing translation flows are studied.

2 A dimension argument using minimal sets

In this section we analyze minimal sets to obtain the following:

Proposition 2.1 *Let $F: \mathcal{P}^1 \mathcal{M}_g \to \mathcal{P}^1 \mathcal{M}_g$ be a normalizer of the earthquake flow, and suppose $(X, \lambda) \in \mathcal{P}^1 \mathcal{M}_g$ and $F(X, \lambda) = (Y, \mu)$. Then, for any $k \in \mathbb{N}$, λ is a simple closed multicurve with k components if and only if μ is a simple closed multicurve with k components.*

We begin by showing that every normalizer must preserve the locus of points $(X, \lambda) \in \mathcal{P}^1 \mathcal{M}_g$ with λ a simple closed multicurve. We do this using the minimal sets of the earthquake flow.

A minimal set of the earthquake flow is a closed, earthquake flow–invariant subset of $\mathcal{P}^1 \mathcal{M}_g$ that does not contain any proper, nonempty, closed, earthquake flow–invariant subsets.

We will be interested in compact minimal sets. Minsky and Weiss [2002] showed that all minimal sets for the earthquake flow are compact, but we will not require such a strong statement. The result we will need is the following:

Theorem 2.2 *A point $(X, \lambda) \in \mathcal{P}^1 \mathcal{M}_g$ is contained in a compact minimal set if and only if λ is a simple closed multicurve.*

Smillie and Weiss [2004] prove the analogous statement for the Teichmüller horocycle flow and state that it should be possible to similarly obtain a result for the earthquake flow. However, as far as we know, even the statement of Theorem 2.2 has not previously appeared in the literature. For the convenience of the reader we sketch a proof in the appendix.

Since normalizers preserve minimal sets, we deduce the following corollary:

Corollary 2.3 *Let $F: \mathcal{P}^1 \mathcal{M}_g \to \mathcal{P}^1 \mathcal{M}_g$ be a normalizer of the earthquake flow, and suppose $(X, \lambda) \in \mathcal{P}^1 \mathcal{M}_g$ and $F(X, \lambda) = (Y, \mu)$. Then λ is a simple closed multicurve if and only if μ is a simple closed multicurve.*

To get a grasp on the number of components of a simple closed multicurve, we study the local topology of the lift to $\mathcal{P}^1 \mathcal{T}_g$ of the union of the compact minimal sets of the earthquake flow on $\mathcal{P}^1 \mathcal{M}_g$. The following result is crucial to our approach:

Lemma 2.4 *Let $\gamma \in \mathcal{PML}_g$ be the projective class of a simple closed multicurve with $k \in \mathbb{N}$ components, $U \subseteq \mathcal{PML}_g$ be a small open neighborhood of γ in $\mathcal{PML}_g$, and W be the path-connected component containing γ of the intersection of U with the subset of $\mathcal{PML}_g$ of projective classes of simple closed multicurves. Then, if U is sufficiently small, $U \cap \overline{W}$ is locally homeomorphic to $\mathbb{R}^{6g-7-k}$.*

Proof Denote $\gamma := \sum_{i=1}^{k} a_i \gamma_i \in \mathcal{P}\mathcal{M}\mathcal{L}_g$. Then, if U is sufficiently small, W consists of projective classes of simple closed multicurves of the form

$$\gamma' := \sum_{i=1}^{k} (a_i + \epsilon_i) \gamma_i + \sum_{j=1}^{k'} \delta_j \gamma_j',$$

where $\epsilon := (\epsilon_i)_{i=1}^{k} \in \mathbb{R}^k$ is a small vector, $k' \geq 0$ is a nonnegative integer, $(\gamma_j')_{j=1}^{k'}$ are pairwise nonhomotopic and nonintersecting simple closed curves that are not homotopic and do not intersect any of the components of γ, and $\delta := (\delta_j)_{j=1}^{k'} \in \mathbb{R}_+^{k'}$ is a small vector with positive entries. This fact can be readily verified using Dehn–Thurston coordinates [Penner and Harer 1992, Section 1.2]. Indeed, if U is sufficiently small, projective classes in W correspond to simple closed multicurves whose geometric intersection number with any of the components of γ is zero.

Furthermore, the closure of W in U is given by the connected component containing γ of the intersection of U with the projectivization of

$$\mathcal{L}_g(\gamma) := \{\lambda \in \mathcal{M}\mathcal{L}_g : i(\gamma, \lambda) = 0\}.$$

Notice that $\mathcal{L}_g(\gamma)$ is homeomorphic to $\mathbb{R}^k \times \mathbb{R}^{6g-6-2k}$, where the first term of this product corresponds to changing the weights of the components of γ and the second term corresponds to choosing a measured geodesic lamination on S_g supported away from γ. In particular, $U \cap \overline{W}$ is locally homeomorphic to $\mathbb{R}^{6g-7-k}$. $\square$

Suppose $(X, \gamma) \in \mathcal{P}^1 \mathcal{T}_g$, where γ is a simple closed multicurve with $k \in \mathbb{N}$ components. Consider a small open neighborhood $U \subseteq \mathcal{P}^1 \mathcal{T}_g$ of (X, γ). Denote by W the path-connected component containing (X, γ) of the intersection of U with the subset of points of $\mathcal{P}^1 \mathcal{T}_g$ where the lamination is a simple closed multicurve. Directly from Lemma 2.4, we see that, if U is sufficiently small, $U \cap \overline{W}$ is locally homeomorphic to $\mathbb{R}^{12g-13-k}$; the $6g - 6$ increase in dimension with respect to Lemma 2.4 comes from the dimension of Teichmüller space. In particular, we can recover the number of components of γ from the dimension of this intersection.

As the number of components of γ can be recovered from information depending exclusively on the minimal sets of $\mathcal{P}^1 \mathcal{M}_g$, this quantity is preserved by any earthquake flow normalizer. This concludes the proof of Proposition 2.1.

3 An automorphism of the curve complex

In this section we use the rigidity of the curve complex to obtain the following:

Proposition 3.1 *Every normalizer* $F : \mathcal{P}^1 \mathcal{M}_g \to \mathcal{P}^1 \mathcal{M}_g$ *of the earthquake flow admits a* Mod_g*–equivariant lift* $\widehat{F} : \mathcal{P}^1 \mathcal{T}_g \to \mathcal{P}^1 \mathcal{T}_g$ *such that, for every* $(X, \lambda) \in \mathcal{P}^1 \mathcal{T}_g$*, if* $\widehat{F}(X, \lambda) = (Y, \mu)$*, then* μ *belongs to the projective class of* $\lambda \in \mathcal{M}\mathcal{L}_g$*.*

Because we assume F is an orbifold map, there exists a lift $\hat{F} \colon \mathscr{P}^1 \mathscr{T}_g \to \mathscr{P}^1 \mathscr{T}_g$ that is equivariant with respect to some automorphism of Mod_g. We start with this lift and show how to modify it to get the desired lift $\hat{F}$.

Denote by $\mathscr{S}_g$ the discrete set of free homotopy classes of unoriented simple closed curves on the marking surface S_g. By Proposition 2.1, every $X \in \mathscr{T}_g$ induces a map $\Psi_X \colon \mathscr{S}_g \to \mathscr{S}_g$ in the following way: given $\gamma \in \mathscr{S}_g$, let $\Psi_X(\gamma) \in \mathscr{S}_g$ be the free homotopy class of the simple closed curves γ' given by

$$(Y, \gamma'/\ell_{\gamma'}(Y)) := \hat{F}(X, \gamma/\ell_\gamma(X)).$$

As $\mathscr{T}_g$ is connected and as $\mathscr{S}_g$ is discrete, the map $\Psi_X \colon \mathscr{S}_g \to \mathscr{S}_g$ is independent of $X \in \mathscr{T}_g$. From now on we denote this map simply by $\Psi \colon \mathscr{S}_g \to \mathscr{S}_g$.

We claim that Ψ induces an automorphism of the curve complex of S_g, meaning that it is bijective and that any pair of simple closed curves can be realized disjointly if and only if their images under Ψ can be realized disjointly.

Lemma 3.2 *The map $\Psi \colon \mathscr{S}_g \to \mathscr{S}_g$ defined above induces an automorphism of the curve complex of S_g.*

Proof An inverse of $\Psi \colon \mathscr{S}_g \to \mathscr{S}_g$ can be constructed using the inverse of $\hat{F}$. It follows that Ψ is bijective.

Notice that a pair $\alpha, \beta \in \mathscr{S}_g$ of simple closed curves can be realized disjointly if and only if there exists a path

$$[0, 1] \to \mathscr{P}^1 \mathscr{T}_g, \quad t \mapsto (X_t, \gamma_t),$$

such that γ_t is a simple closed multicurve on S_g for every $t \in [0, 1]$, $\gamma_0 = \alpha/\ell_{X_0}(\alpha)$, $\gamma_1 = \beta/\ell_{X_1}(\beta)$ and γ_t has exactly two components for every $t \in (0, 1)$. It follows from Proposition 2.1 that $\hat{F}$ preserves these types of paths. In particular, for every pair of simple closed curves $\alpha, \beta \in \mathscr{S}_g$, their images $\Psi(\alpha), \Psi(\beta) \in \mathscr{S}_g$ are nonintersecting if and only if α and β are nonintersecting. $\square$

A well-known result of Ivanov [1997] shows that every automorphism of the curve complex of a closed, connected, oriented surface S_g of genus $g \geq 2$ is induced by the isotopy class of a diffeomorphism of S_g. Thus there exists a diffeomorphism $\psi \colon S_g \to S_g$ such that the map $\Psi \colon \mathscr{S}_g \to \mathscr{S}_g$ defined above is given by $\Psi(\gamma) = \psi(\gamma)$ for every $\gamma \in \mathscr{S}_g$. The diffeomorphism ψ acts on $\mathscr{P}^1 \mathscr{T}_g$ by changing the markings even if it does not preserve the orientation of S_g. It also acts naturally on the mapping class group Mod_g by conjugation.

Since $\hat{F} \colon \mathscr{P}^1 \mathscr{T}_g \to \mathscr{P}^1 \mathscr{T}_g$ is the lift of an orbifold map, there exists an automorphism $\rho \colon \mathrm{Mod}_g \to \mathrm{Mod}_g$ such that

$$\hat{F}(\phi.(X, \lambda)) = \rho(\phi).\hat{F}(X, \lambda)$$

for every $\phi \in \mathrm{Mod}_g$ and every $(X, \lambda) \in \mathscr{P}^1 \mathscr{T}_g$. Consider the lift $\hat{F}' \colon \mathscr{P}^1 \mathscr{T}_g \to \mathscr{P}^1 \mathscr{T}_g$ of F defined by

$$\hat{F}'(X, \lambda) := \psi^{-1}.\hat{F}(X, \lambda).$$

This lift intertwines the automorphism $\rho' \colon \mathrm{Mod}_g \to \mathrm{Mod}_g$ given by

$$\rho'(\phi) := \psi^{-1} \circ \rho(\phi) \circ \psi$$

for every $\phi \in \mathrm{Mod}$. Thus, by replacing $\widehat{F}$ with $\widehat{F}'$, we can assume without loss of generality that the map $\Psi \colon \mathscr{S}_g \to \mathscr{S}_g$ defined above is the identity.

As $\widehat{F}$ intertwines the automorphism $\rho \colon \mathrm{Mod}_g \to \mathrm{Mod}_g$, the map $\Psi \colon \mathscr{S}_g \to \mathscr{S}_g$ defined above, which we are assuming is the identity, also intertwines this automorphism. It follows that $\rho(\phi).\gamma = \phi.\gamma$ for every $\phi \in \mathrm{Mod}_g$ and every $\gamma \in \mathscr{S}_g$. As the kernels of the Mod_g–actions on $\mathscr{S}_g$ and $\mathscr{T}_g$ are equal, $\rho(\phi).X = X$ for every $\phi \in \mathrm{Mod}_g$ and every $X \in \mathscr{T}_g$. It follows that, without loss of generality, we can assume that the automorphism $\rho \colon \mathrm{Mod}_g \to \mathrm{Mod}_g$ is the identity.

The discussion above shows that the lift $\widehat{F}$ satisfies the following property: for every $X \in \mathscr{T}_g$ and every simple closed curve $\gamma \in \mathscr{S}_g$, if $(Y, \mu) := \widehat{F}(X, \gamma/\ell_\gamma(X)) \in \mathscr{P}^1\mathscr{T}_g$, then μ belongs to the projective class of $\gamma \in \mathcal{ML}_g$. As simple closed curves are dense in $\mathcal{PML}_g$, the same property holds for arbitrary measured geodesic laminations. This concludes the proof of Proposition 3.1.

4 Inspecting the periods of closed orbits

In this section we show that the normalizer of the earthquake flow is equal to its extended centralizer.

Proposition 4.1
$$N(E) = C_\pm(E).$$

In other words, given an s–normalizer F as above, we show that $s = 0$. We begin by strengthening Proposition 3.1 to control the scaling between λ and μ.

Lemma 4.2 *Let $\widehat{F}$ be the lift produced by Proposition 3.1 of an s–normalizer F. Then, for every $(X, \lambda) \in \mathscr{P}^1\mathscr{T}_g$, if $(Y, \mu) := \widehat{F}(X, \lambda)$, then $\mu = e^s \cdot \lambda$.*

Proof Since for every $(X, \lambda) \in \mathscr{P}^1\mathscr{T}_g$ the measured geodesic lamination $\mu := \mu(X, \lambda)$ given by $(Y, \mu) := \widehat{F}(X, \lambda)$ belongs to the projective class of $\lambda \in \mathcal{ML}_g$, we can consider the continuous function $c \colon \mathscr{P}^1\mathscr{T}_g \to \mathbb{R}^+$ which to every $(X, \lambda) \in \mathscr{P}^1\mathscr{T}_g$ assigns the unique scaling factor $c(X, \lambda) > 0$ such that

$$(4\text{-}1) \qquad\qquad \mu(X, \lambda) = c(X, \lambda) \cdot \lambda.$$

Our goal is to show that $c \colon \mathscr{P}^1\mathscr{T}_g \to \mathbb{R}^+$ is identically equal to e^s.

Denote by $T_\gamma \in \mathrm{Mod}_g$ the Dehn twist of S_g along a simple closed curve γ. One can check that, for every $(X, a \cdot \gamma) \in \mathscr{P}^1\mathscr{T}_g$ with $a > 0$ and γ a simple closed curve on S_g, the period of the earthquake flow orbit of

$$(X, a \cdot \gamma) \in \mathscr{P}^1\mathscr{T}_g/\langle T_\gamma \rangle$$

is exactly $\ell_\gamma(X)/a$.

Now consider $\lambda = \gamma/\ell_\gamma(X)$ with γ a simple closed curve on S_g. Note that (X, λ) has period $\ell_\gamma(X)^2$ in $\mathscr{P}^1 \mathscr{T}_g / \langle T_\gamma \rangle$ and $\widehat{F}(X, \lambda) = (Y, c(X, \lambda)\lambda)$ has period

$$\frac{\ell_\gamma(Y)\ell_\gamma(X)}{c(X, \lambda)} = \frac{\ell_\gamma(X)^2}{c(X, \lambda)^2},$$

where the last equality uses the fact that $c(X, \lambda)\lambda$ must have length 1 on Y. As $\widehat{F}$ is Mod_g–equivariant and as s–normalizers multiply periods by e^{-2s}, it follows that

$$\frac{\ell_\gamma(X)^2}{c(X, \lambda)^2} = e^{-2s}\ell_\gamma(X)^2.$$

Hence, $c(X, \gamma/\ell_\gamma(X)) = e^s$. As $c \colon \mathscr{P}^1 \mathscr{T}_g \to \mathbb{R}^+$ is continuous and as points of the form $(X, \gamma/\ell_\gamma(X)) \in \mathscr{P}^1 \mathscr{T}_g$ with γ a simple closed curve on S_g are dense in $\mathscr{P}^1 \mathscr{T}_g$, this finishes the proof. $\qquad\square$

We now prove a loop-shrinking property for lifts $\widehat{F}$ of s–normalizers of the earthquake flow. This property will play a crucial role in the proof of Theorem 1.4.

Lemma 4.3 *Let $\widehat{F}$ be the lift produced by Proposition 3.1 of an s–normalizer F of the earthquake flow. Then, for every $X \in \mathscr{T}_g$ and every simple closed curve $\alpha \in \mathscr{S}_g$, if $(Y, \mu) := \widehat{F}(X, \alpha/\ell_\alpha(X))$, then*

$$\ell_\beta(Y) \le e^{-s}\ell_\beta(X)$$

for every simple closed curve $\beta \in \mathscr{S}_g$ that can be realized disjointly from α, with equality if $\beta = \alpha$.

Proof By Lemma 4.2, $\mu = e^s \cdot \alpha/\ell_\alpha(X)$. It follows that

$$1 = \ell_\mu(Y) = e^s \cdot \ell_\alpha(X)^{-1} \cdot \ell_\alpha(Y).$$

Reorganizing the terms in this equality, we deduce

$$\ell_\alpha(Y) = e^{-s} \cdot \ell_\alpha(X).$$

Let $\beta \in \mathscr{S}_g$ be a simple closed curve that can be realized disjointly from α and is not equal to α. We average α and β with appropriate weights to obtain simple closed multicurves on S_g converging to $\alpha/\ell_\alpha(X)$, with unit length with respect to X, and whose corresponding earthquake flow orbits are periodic with explicit periods. Indeed, for every $k \in \mathbb{N}$, consider the positive weights

$$a_k = a_k(X, \alpha, \beta) := (\ell_\alpha(X) + k^{-1} \cdot \ell_\alpha(X)^{-1} \cdot \ell_\beta(X)^2)^{-1},$$
$$b_k = b_k(X, \alpha, \beta) := (\ell_\beta(X) + k \cdot \ell_\alpha(X)^2 \cdot \ell_\beta(X)^{-1})^{-1}.$$

These choices guarantee that, for every $k \in \mathbb{N}$,

(4-2) $$\ell_\beta(X)/b_k = k \cdot \ell_\alpha(X)/a_k.$$

For every $k \in \mathbb{N}$, consider the simple closed multicurve on S_g given by

$$\gamma_k = \gamma_k(X, \alpha, \beta) := a_k(X, \alpha, \beta) \cdot \alpha + b_k(X, \alpha, \beta) \cdot \beta.$$

Direct computations show that $\ell_{\gamma_k}(X) = 1$ for every $k \in \mathbb{N}$. Directly from the definitions, one can also check that

$$\lim_{k \to \infty} \gamma_k = \alpha/\ell_\alpha(X).$$

For every $k \in \mathbb{N}$, consider $(Y_k, \mu_k) := \widehat{F}(X, \gamma_k)$. By Lemma 4.2, $\mu_k = e^s \cdot \gamma_k$ for every $k \in \mathbb{N}$. As $\widehat{F}$ is continuous,

$$(4\text{-}3) \qquad\qquad Y = \lim_{k \to \infty} Y_k.$$

Fix $k \in \mathbb{N}$. Denote by $T_\alpha, T_\beta \in \mathrm{Mod}_g$ the Dehn twists of S_g along α and β. A direct computation using (4-2) shows that the earthquake flow orbit of the image of (X, γ_k) in $\mathcal{P}^1 \mathcal{T}_g / \langle T_\alpha, T_\beta \rangle$ is periodic with period given by the least common multiple

$$(4\text{-}4) \qquad\qquad \mathrm{lcm}(\ell_\alpha(X)/a_k, \ell_\beta(X)/b_k) = \ell_\beta(X)/b_k.$$

Analogously, the earthquake flow orbit of the image of (Y_k, μ_k) in $\mathcal{P}^1 \mathcal{T}_g / \langle T_\alpha, T_\beta \rangle$ is periodic if and only if the following least common multiple is finite, in which case it is exactly the period of the orbit:

$$(4\text{-}5) \qquad\qquad \mathrm{lcm}(\ell_\alpha(Y_k)/(e^s \cdot a_k), \ell_\beta(Y_k)/(e^s \cdot b_k)).$$

Since s–normalizers multiply periods by e^{-2s}, for the periods in (4-4) and (4-5) to agree, it is necessary that

$$\ell_\beta(Y_k) \le e^{-s} \cdot \ell_\beta(X).$$

Taking limits as $k \to \infty$ and using (4-3), we conclude

$$\ell_\beta(Y) \le e^{-s} \cdot \ell_\beta(X). \qquad\qquad \square$$

We can now conclude the proof of Proposition 4.1 as follows:

Proof of Proposition 4.1 Suppose by contradiction that $s \ne 0$. By working with the inverse of F if $s < 0$, we can assume without loss of generality that $s > 0$. Denote by $\widehat{F}$ the Mod_g–equivariant lift provided by Proposition 3.1. Let $\alpha, \beta, \gamma \in \mathcal{S}_g$ be simple closed curves such that α can be realized disjointly from β and γ, and such that β and γ have positive geometric intersection number. Fix $X \in \mathcal{T}_g$ and let

$$(X_n, \lambda_n) := \widehat{F}^n(X, \alpha/\ell_\alpha(X))$$

for every $n \in \mathbb{N}$. By Lemma 4.3, there exists $N \in \mathbb{N}$ such that $\ell_\beta(X_N)$ and $\ell_\gamma(X_N)$ are arbitrarily small, contradicting the collar lemma for hyperbolic surfaces. $\qquad\qquad \square$

5 The centralizer of the earthquake flow

In this section we show that the extended centralizer of the earthquake flow is trivial.

Proposition 5.1 $$C_\pm(E) = E.$$

We proceed in several steps, starting with the following geometric result:

Lemma 5.2 *Let X and Y be a pair of compact, connected and orientable diffeomorphic hyperbolic surfaces with at least one totally geodesic boundary component. Suppose that, for some pair of markings on X and Y, the lengths of the boundary components of X agree with those of Y, and, for every simple closed curve, the length of its geodesic representative on Y is at most the length of its geodesic representative on X. Then X and Y are isometric.*

An analogous statement for closed surfaces is well known [Thurston 1998, Theorem 3.1]. We do not know if the exact statement of Lemma 5.2 has appeared before in the literature, but, in any case, a short proof is possible from known results.

Proof The monotonicity of the summands in Mirzakhani's generalized McShane's identity [2007b, Theorem 1.3] guarantees that, if X and Y satisfy the assumptions, then, for every simple closed curve, the lengths of its geodesic representatives on X and Y are equal. As the isometry class of a marked hyperbolic structure with totally geodesic boundary components on a compact, connected, orientable surface is determined by its marked length spectrum,[1] we conclude that X and Y are isometric.   □

The following result shows that centralizers of the earthquake flow map points of the form $(X, \alpha/\ell_\alpha(X)) \in \mathcal{P}^1 \mathcal{M}_g$ into their own earthquake flow orbit.

Lemma 5.3 *Suppose $F \in C_\pm(E)$ and let $\widehat{F}$ be the lift provided by Proposition 3.1. Then, for every $X \in \mathcal{T}_g$ and every simple closed curve $\alpha \in \mathcal{S}_g$, there exists a unique $t \in \mathbb{R}$ satisfying*

$$\widehat{F}(X, \alpha/\ell_\alpha(X)) = E_t(X, \alpha/\ell_\alpha(X)).$$

For the proof, it is helpful to recall that an element of the extended centralizer is nothing other than an s–normalizer with $s = 0$.

Proof Let $(Y, \mu) := \widehat{F}(X, \alpha/\ell_\alpha(X)) \in \mathcal{P}^1 \mathcal{T}_g$. Lemmas 4.2 and 4.3 ensure that $\mu = \alpha/\ell_\alpha(X) \in \mathcal{ML}_g$, $\ell_\alpha(Y) = \ell_\alpha(X)$, and $\ell_\beta(Y) \leq \ell_\beta(X)$ for every simple closed curve $\beta \in \mathcal{S}_g$ that can be realized disjointly from α.

Cutting X and Y along the corresponding geodesic representatives of α on each surface yields a pair of (possibly disconnected) hyperbolic surfaces with totally geodesic boundary components of matching lengths. Lemma 5.2 guarantees these surfaces are isometric. As X and Y can be recovered from isometric pieces by gluing along the boundary components corresponding to α, we deduce that X and Y only differ by a Fenchel–Nielsen twist along α. In other words,

$$\widehat{F}(X, \alpha/\ell_\alpha(X)) = (Y, \mu) = E_t(X, \alpha/\ell_\alpha(X)).   □$$

The following result extends the conclusion of Lemma 5.3 to arbitrary points $(X, \lambda) \in \mathcal{P}^1 \mathcal{T}_g$:

---

[1] A proof can be obtained by adapting the arguments in [Farb and Margalit 2012, Proof of Theorem 10.7].

Lemma 5.4 *Suppose $F \in C_{\pm}(E)$ and let $\widehat{F}$ be the lift provided by [Proposition 3.1](). Then there exists a continuous, Mod_g–invariant function $t \colon \mathcal{P}^1 \mathcal{T}_g \to \mathbb{R}$ such that, for every $(X, \lambda) \in \mathcal{P}^1 \mathcal{T}_g$, $t = t(X, \lambda)$ satisfies*

$$\widehat{F}(X, \lambda) = E_t(X, \lambda)$$

and is the unique real number satisfying this equation.

Furthermore, if $F \in C(E)$, then t is earthquake flow–invariant, and, if $F \in C_{\pm}(E) \setminus C(E)$, then T is "twisted-equivariant", in the sense that

$$t(E_s(X, \lambda)) = t(X, \lambda) - 2s.$$

Proof Fix $(X, \lambda) \in \mathcal{P}^1 \mathcal{T}_g$. As weighted simple closed curves are dense in $\mathcal{ML}_g$, one can find a sequence $(\lambda_n)_{n \in \mathbb{N}}$ of length 1 weighted simple closed curves such that $\lambda_n \to \lambda$ in $\mathcal{ML}_g$ as $n \to \infty$. By [Lemma 5.3](), for every $n \in \mathbb{N}$, there exists $t_n \in \mathbb{R}$ such that

(5-1)
$$\widehat{F}(X, \lambda_n) = E_{t_n}(X, \lambda_n).$$

Claim 5.5 *The sequence $(t_n)_{n \in \mathbb{N}}$ is bounded.*

Proof Suppose by contradiction this was not the case. Assume t_n diverges to $+\infty$ along a subsequence; the case when t_n diverges to $-\infty$ along a subsequence can be treated in an analogous way. Rename this subsequence as $(t_n)_{n \in \mathbb{N}}$ and assume without loss of generality that all of its terms are positive. Let $\mu \in \mathcal{ML}_g$ be a measured geodesic lamination such that

(5-2)
$$\iint_X \cos \theta \, d\lambda \, d\mu > 0,$$

where θ is the angle measured counterclockwise from μ to λ at each intersection between μ and λ. The existence of such a measured geodesic lamination $\mu \in \mathcal{ML}_g$ can be argued as follows. By the infinitesimal version of Thurston's earthquake theorem (see for instance [Kerckhoff 1983, Appendix, Theorem 2]), every tangent vector at $X \in \mathcal{T}_g$ can be realized by an infinitesimal earthquake. In particular, by Kerckhoff's derivative formula [loc. cit., Corollary 3.4], the only way μ could not exist is if the function $Y \mapsto \ell_\lambda(Y) > 0$ for $Y \in \mathcal{T}_g$ had a critical point, and, by convexity of length functions [loc. cit., Section 3, Theorem 1], a minimum at X. This is not possible, as can be seen, for instance, using shear coordinates and reverse stretch lines.

By [loc. cit., Corollary 3.4], the integral in (5-2) is equal to the derivative at $t = 0$ of the convex function $t \mapsto \ell_\mu(E_t(X, \lambda))$. By continuity, there exists $c > 0$ and $N \in \mathbb{N}$ such that, for every $n \geq N$,

$$\iint_X \cos \theta \, d\lambda_n \, d\mu > c.$$

Kerckhoff's work guarantees that, for every $n \geq N$,

(5-3)
$$\ell_\mu(E_{t_n}(X, \lambda_n)) \geq \ell_\mu(X) + t_n \cdot c.$$

Denote by $\pi\colon \mathcal{P}^1\mathcal{T}_g \to \mathcal{T}_g$ the natural projection defined by $\pi(X,\lambda) = X$. By definition,

$$E_{t_n}(X,\lambda_n) = \widehat{F}(X,\lambda_n) \in \widehat{F}(\pi^{-1}(X)).$$

As $\widehat{F}$ is continuous, the set $\widehat{F}(\pi^{-1}(X)) \subseteq \mathcal{P}^1\mathcal{T}_g$ is compact. Thus, the sequence $\big(\ell_\mu(E_{t_n}(X,\lambda_n))\big)_{n\in\mathbb{N}}$ must be bounded. Taking limits as $n \to \infty$ in (5-3) yields a contradiction, concluding the proof of the claim. □

As $(t_n)_{n\in\mathbb{N}}$ is bounded, it admits a subsequence converging to some $t \in \mathbb{R}$. Taking limits in (5-1) along this subsequence, we deduce

$$(5\text{-}4) \qquad\qquad\qquad \widehat{F}(X,\lambda) = E_t(X,\lambda).$$

The uniqueness of $t \in \mathbb{R}$ satisfying this condition follows directly from the fact that earthquake flow orbits in $\mathcal{P}^1\mathcal{T}_g$ are embedded. The continuity of the corresponding function $t\colon \mathcal{P}^1\mathcal{T}_g \to \mathbb{R}$ follows from (5-4) and uniqueness. The Mod_g–invariance of t can be verified using (5-4) and the fact that $\widehat{F}$ is Mod_g–equivariant. The earthquake flow–invariance or twisted-equivariance of t can be verified directly from (5-4) and the fact that $\widehat{F}$ is in the extended centralizer of the earthquake flow. □

We are now ready to conclude.

Proof of Proposition 5.1 Consider the function $t\colon \mathcal{P}^1\mathcal{T}_g \to \mathbb{R}$ above. Since it is Mod_g–equivariant, it induces a function $t\colon \mathcal{P}^1\mathcal{M}_g \to \mathbb{R}$.

If $F \in C(E)$, the function t is earthquake flow–invariant. As the earthquake flow on $\mathcal{P}^1\mathcal{M}_g$ is ergodic with respect to a measure of full support, t is equal to a constant $t_0 \in \mathbb{R}$ on a dense set of $\mathcal{P}^1\mathcal{M}_g$. Applying continuity and density, we conclude $F = E_{t_0}$, as desired.

Suppose $F \in C_\pm(E) \setminus C(E)$. There exists c such that the set $t^{-1}((c, c+2))$ has positive measure. The twisted-equivariance gives that, for all k, E_k maps $t^{-1}((c, c+2))$ into $t^{-1}((c-2k, c+2-2k))$. For different k integral, the sets $t^{-1}((c-2k, c+2-2k))$ are disjoint, and, since earthquake flow is measure-preserving, they all have the same measure. So considering all k integral contradicts the fact that the space has finite measure, showing that such an F cannot exist. □

We are now ready to prove that the earthquake flow is strongly asymmetric.

Proof of Theorem 1.4 Proposition 4.1 shows that $N(E) = C_\pm(E)$ and Proposition 5.1 shows that $C_\pm(E) = E$. □

Appendix   Minimal sets

In this appendix we sketch, for the convenience of the reader, a proof of Theorem 2.2. The corresponding result in the case of the Teichmüller horocycle flow is discussed in detail by Smillie and Weiss [2004], who remark there that "an analogous result for the earthquake flow may be proved by a similar argument". Our starting point is the following observation, the details of whose proof are left to the reader:

Lemma A.1 *If $K \subset \mathcal{P}^1 \mathcal{M}_g$ is a minimal set for the earthquake flow, and (X, λ) and (X', λ') are in K, then $X - \lambda$ is isometric to $X' - \lambda'$.*

Sketch of proof For any fixed $(X, \lambda) \in K$, consider the set $K' \subseteq K$ of all $(X', \lambda') \in K$ for which there exists an isometric embedding

$$X - \lambda \hookrightarrow X' - \lambda'$$

of complementary regions. Since the complementary regions are not changed by the earthquake flow, K' is invariant. A limit argument shows that K' is closed, so the definition of minimality guarantees $K' = K$.

Thus, for every $(X, \lambda), (X', \lambda') \in K$, each complementary region embeds isometrically into the other. Hence $X - \lambda = X' - \lambda'$. $\qquad\qquad\square$

We also need the following nontrivial result:

Proposition A.2 *If λ is not a multicurve and the orbit of (X, λ) is bounded in $\mathcal{P}^1 \mathcal{M}_g$, then the orbit accumulates on some (X', λ') with $X - \lambda \neq X' - \lambda'$.*

In fact, experts believe the following stronger statement is true (and accessible):

Problem A.3 Prove that, if λ is not a multicurve, then the earthquake flow orbit of (X, λ) is not bounded.

We will not consider this problem here since it is certainly harder than what we require. The analogous problem for the Teichmüller horocycle flow is item (IV) in the list of problems at the end of [Smillie and Weiss 2004] and has been considered in unpublished work of those authors.

Before addressing Proposition A.2, we note it implies Theorem 2.2.

Proof of Theorem 2.2 assuming Proposition A.2 If K is a compact minimal set and $(X, \lambda) \in K$, then Lemma A.1 implies that any (X', λ') in the orbit closure of (X, λ) has $X - \lambda = X' - \lambda'$, and so Proposition A.2 implies λ is a multicurve.

The converse implication — that if λ is a multicurve then the orbit closure of (X, λ) is a minimal set — is well known. Indeed, if $T \subseteq \mathcal{P}^1 \mathcal{M}_g$ is the subset obtained by starting at (X, λ) and independently twisting at each component of λ, then T is an invariant torus and the earthquake flow is continuously conjugate to a straight-line flow on T. The converse implication follows from the fact that, for straight-line flows on tori, every orbit closure is a minimal set. $\qquad\qquad\square$

We conclude by briefly sketching how the ideas of Smillie and Weiss apply to Proposition A.2. Most of the work is divided into two lemmas.

Lemma A.4 *Suppose λ is a measured geodesic lamination on X that is not a multicurve. Then there exists some $\delta > 0$ such that, for all $\epsilon > 0$, we can find segments γ_1 and γ_2 of leaves of λ that stay within distance 1 of each other and are such that all leaves of λ that come within δ of the starting point p_1 of γ_1 do so on the side of γ_1 containing γ_2, and all leaves that come within δ of the endpoint p_2 of γ_2 do so*
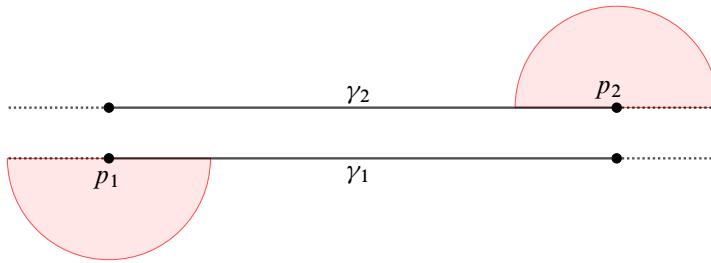
Figure 1: The γ_i. The shaded half balls of radius δ do not intersect λ.

on the side of γ_2 containing γ_1, and such that the transverse measure of a segment from γ_1 to γ_2 is less than ϵ. Moreover, γ_1 and γ_2 can be taken to lie on nonisolated leaves of λ.

In particular, it follows that both γ_i are segments of leaves of λ adjacent to regions of $X - \lambda$. See Figure 1.

The proof will use the concept of the thick part of a surface with boundary, which can be defined by embedding the surface in its double and taking the thick part there; see for example [Lipnowski and Wright 2024, Section 2.1] for details.

Sketch of proof Without loss of generality assume γ has no closed leaves. Start with p_1 on the boundary of the thick part of $X - \lambda$, on a leaf α of λ. Pick a point q that is very close to p_1 and on a leaf β of λ. Follow both leaves α and β in the same direction until they are distance $\frac{1}{10}$ apart. The region R between these segments of α and β, illustrated in Figure 2, has definite area.

The area of the thin part of $X - \lambda$ is small, so the thick part must intersect R. (Here the thick part should be defined appropriately using δ, and δ should be taken small enough.)

We then pick p_2 to be on the boundary of the thick part of $X - \lambda$ intersected with R. (One should pick p_2 so that the thick part and α are on different sides of the leaf through p_2.) We define γ_1 to be the segment of α from p_1 to the projection of p_2 onto α, and similarly define γ_2 using the leaf through p_2. ☐

Lemma A.5 *There exists a universal constant $C > 0$ such that the following holds. Consider any measured geodesic lamination on $\mathbb{H}$, any segments γ_1 and γ_2 of nonatomic leaves of λ that stay within distance 1 of each other, and any $p_1 \in \gamma_1$ and $p_2 \in \gamma_2$. Assume there are leaves of λ that go between p_1 and p_2. Let $\lambda_{\max}$ be a maximal geodesic lamination containing λ. Assume the p_i lie on the boundary of $\mathbb{H} - \lambda_{\max}$. Then there is a unique $t \in \mathbb{R}$ such that the image of p_1 and p_2 under the time t earthquake*
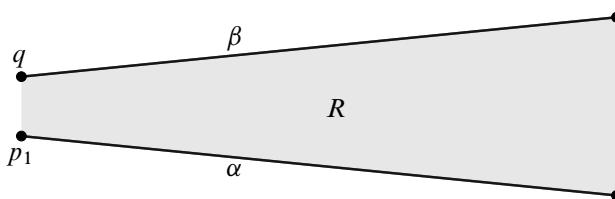
Figure 2: The region R.

of λ can be joined by a segment s of a leaf of the horocyclic foliation of $\lambda_{\max}$ and this segment has length at most C.

In applications, often λ is already maximal, so $\lambda_{\max} = \lambda$. The main conclusion here is that p_1 and p_2 become bounded distance from each other; the use of the horocyclic foliation (and $\lambda_{\max}$) is merely a convenient technical tool to obtain this.

One should of course think of $\mathbb{H}$ as the universal cover of a closed surface X; we use the universal cover only so that we do not have to specify a homotopy class for the arc s.

Sketch of proof   The first claim is related to the fact that shears change linearly under earthquakes; see for example the survey [Wright 2022, Section 4].

If one considers a rectangle R bounded by γ_1 and γ_2, then λ divides this rectangle up into countably many small rectangles bounded by leaves of λ. The preimage of s on (X, λ) consists of one horocyclic arc in each small rectangle; compare to a Cantor staircase.

For each small rectangle, one can define its maximum height to be the maximum length of a horocyclic arc crossing that rectangle. A standard estimate shows that the sum of the maximum heights is at most some constant C; see [Thurston 1998, page 16]. This uses the fact that the γ_i remain within distance 1 of each other.

The length of s is the sum of the lengths of the horocyclic arcs of $E_{-t}(s)$, which is at most C. This gives the result. $\qquad\square$

Sketch of proof of Proposition A.2   Consider a sequence $\epsilon_n \to 0$ and, for each $n \in \mathbb{N}$, let $\gamma_{1,n}$, $\gamma_{2,n}$, $p_{1,n}$ and $p_{2,n}$ be as provided by Lemma A.4 with $\epsilon = \epsilon_n$.

The output of Lemma A.5 is a sequence of points $(X_n, \lambda_n) \in \mathcal{P}^1 \mathcal{M}_g$ on the earthquake flow orbit of (X, λ) such that two points on the boundary of the thick part of $X_n - \lambda_n = X - \lambda$ are joined by a path on X_n of hyperbolic length at most C and transverse measure going to 0 as $n \to \infty$. By extending these paths into the thick part and taking geodesic representatives, we obtain geodesic paths σ_n on X_n, of lengths bounded above and below, which are uniformly transverse to λ_n and which have the same transverse measure as the original paths of length at most C.

Passing to a subsequence if necessary, we can assume (X_n, λ_n) converges to some $(X_\infty, \lambda_\infty) \in \mathcal{P}^1 \mathcal{M}_g$. For convenience, we can also assume that the supports of the λ_n converge to a geodesic lamination $\hat{\lambda}_\infty$ which contains the support of λ_∞.

Since the complementary regions $X_n - \lambda_n$ are constant, it follows that $X - \lambda = X_\infty - \hat{\lambda}_\infty$. Thus, to show that $X - \lambda \neq X_\infty - \lambda_\infty$, it suffices to show that some leaves of the geodesic lamination $\hat{\lambda}_\infty$ are not contained in the support of λ_∞.

This is verified by considering a limit σ of the geodesic segments σ_n; the limit σ has length bounded above and below, is transverse to $\hat{\lambda}_\infty$, and has 0 transverse measure with respect to λ_∞. $\qquad\square$

References

[Arana-Herrera 2021]   **F Arana-Herrera**, *Equidistribution of families of expanding horospheres on moduli spaces of hyperbolic surfaces*, Geom. Dedicata 210 (2021) 65–102   MR Zbl

[Arana-Herrera 2022]   **F Arana-Herrera**, *Counting hyperbolic multigeodesics with respect to the lengths of individual components and asymptotics of Weil–Petersson volumes*, Geom. Topol. 26 (2022) 1291–1347

[Avila and Gouëzel 2013]   **A Avila**, **S Gouëzel**, *Small eigenvalues of the Laplacian for algebraic measures in moduli space, and mixing properties of the Teichmüller flow*, Ann. of Math. 178 (2013) 385–442   MR Zbl

[Avila and Resende 2012]   **A Avila**, **M J Resende**, *Exponential mixing for the Teichmüller flow in the space of quadratic differentials*, Comment. Math. Helv. 87 (2012) 589–638   MR Zbl

[Avila et al. 2006]   **A Avila**, **S Gouëzel**, **J-C Yoccoz**, *Exponential mixing for the Teichmüller flow*, Publ. Math. Inst. Hautes Études Sci. 104 (2006) 143–211   MR Zbl

[Bakker and Fisher 2014]   **L Bakker**, **T Fisher**, *Open sets of diffeomorphisms with trivial centralizer in the C^1 topology*, Nonlinearity 27 (2014) 2869–2885   MR Zbl

[Berk et al. 2020]   **P Berk**, **K Frączek**, **T de la Rue**, *On typicality of translation flows which are disjoint with their inverse*, J. Inst. Math. Jussieu 19 (2020) 1677–1737   MR Zbl

[Bonatti et al. 2017]   **C Bonatti**, **I Monteverde**, **A Navas**, **C Rivas**, *Rigidity for C^1 actions on the interval arising from hyperbolicity, I: Solvable groups*, Math. Z. 286 (2017) 919–949   MR Zbl

[Bonomo and Varandas 2019]   **W Bonomo**, **P Varandas**, *A criterion for the triviality of the centralizer for vector fields and applications*, J. Differential Equations 267 (2019) 1748–1766   MR Zbl

[Burslem and Wilkinson 2004]   **L Burslem**, **A Wilkinson**, *Global rigidity of solvable group actions on S^1*, Geom. Topol. 8 (2004) 877–924   MR Zbl

[Calderon and Farre 2024]   **A Calderon**, **J Farre**, *Shear-shape cocycles for measured laminations and ergodic theory of the earthquake flow*, Geom. Topol. 28 (2024) 1995–2124

[Erlandsson and Souto 2022]   **V Erlandsson**, **J Souto**, *Counting curves on orbifolds*, Trans. Lond. Math. Soc. 9 (2022) 56–85   MR Zbl

[Farb and Margalit 2012]   **B Farb**, **D Margalit**, *A primer on mapping class groups*, Princeton Math. Ser. 49, Princeton Univ. Press (2012)   MR Zbl

[Frączek and Lemańczyk 2009]   **K Frączek**, **M Lemańczyk**, *On the self-similarity problem for ergodic flows*, Proc. Lond. Math. Soc. 99 (2009) 658–696   MR Zbl

[Frączek et al. 2014]   **K Frączek**, **J Kułaga-Przymus**, **M Lemańczyk**, *Non-reversibility and self-joinings of higher orders for ergodic flows*, J. Anal. Math. 122 (2014) 163–227   MR Zbl

[Fu 2019]   **S-W Fu**, *Cusp excursions for the earthquake flow on the once-punctured torus*, Conform. Geom. Dyn. 23 (2019) 251–261   MR Zbl

[Ghys 1985]   **É Ghys**, *Actions localement libres du groupe affine*, Invent. Math. 82 (1985) 479–526   MR Zbl

[Ghys and Verjovsky 1994]   **É Ghys**, **A Verjovsky**, *Locally free holomorphic actions of the complex affine group*, from "Geometric study of foliations" (T Mizutani, K Masuda, S Matsumoto, T Inaba, T Tsuboi, Y Mitsumatsu, editors), World Sci., River Edge, NJ (1994) 201–217   MR Zbl

[Guelman and Liousse 2011]   **N Guelman**, **I Liousse**, *C^1–actions of Baumslag–Solitar groups on S^1*, Algebr. Geom. Topol. 11 (2011) 1701–1707   MR Zbl

[Guelman and Liousse 2013]   **N Guelman**, **I Liousse**, *Actions of Baumslag–Solitar groups on surfaces*, Discrete Contin. Dyn. Syst. 33 (2013) 1945–1964   MR Zbl

[Hubbard and Masur 1979]   **J Hubbard**, **H Masur**, *Quadratic differentials and foliations*, Acta Math. 142 (1979) 221–274   MR Zbl

[Ivanov 1997]   **N V Ivanov**, *Automorphism of complexes of curves and of Teichmüller spaces*, Int. Math. Res. Not. 1997 (1997) 651–666   MR Zbl

[Kerckhoff 1983]   **S P Kerckhoff**, *The Nielsen realization problem*, Ann. of Math. 117 (1983) 235–265   MR Zbl

[Lipnowski and Wright 2024]   **M Lipnowski**, **A Wright**, *Towards optimal spectral gaps in large genus*, Ann. Probab. 52 (2024) 545–575   MR Zbl

[Liu 2022]   **M Liu**, *Length statistics of random multicurves on closed hyperbolic surfaces*, Groups Geom. Dyn. 16 (2022) 437–459   MR Zbl

[Lu and Su 2022]   **S Lu**, **W Su**, *Counting mapping class group orbits under shearing coordinates*, Geom. Dedicata 216 (2022) art. id. 16   MR Zbl

[Masur 1995]   **H Masur**, *The Teichmüller flow is Hamiltonian*, Proc. Amer. Math. Soc. 123 (1995) 3739–3747 MR Zbl

[McCarthy 2010]   **A E McCarthy**, *Rigidity of trivial actions of abelian-by-cyclic groups*, Proc. Amer. Math. Soc. 138 (2010) 1395–1403   MR Zbl

[McMullen 1998]   **C T McMullen**, *Complex earthquakes and Teichmüller theory*, J. Amer. Math. Soc. 11 (1998) 283–320   MR Zbl

[Minsky and Weiss 2002]   **Y Minsky**, **B Weiss**, *Nondivergence of horocyclic flows on moduli space*, J. Reine Angew. Math. 552 (2002) 131–177   MR Zbl

[Mirzakhani 2007a]   **M Mirzakhani**, *Random hyperbolic surfaces and measured laminations*, from "In the tradition of Ahlfors–Bers, IV" (D Canary, J Gilman, J Heinonen, H Masur, editors), Contemp. Math. 432, Amer. Math. Soc., Providence, RI (2007) 179–198   MR Zbl

[Mirzakhani 2007b]   **M Mirzakhani**, *Simple geodesics and Weil–Petersson volumes of moduli spaces of bordered Riemann surfaces*, Invent. Math. 167 (2007) 179–222   MR Zbl

[Mirzakhani 2008]   **M Mirzakhani**, *Ergodic theory of the earthquake flow*, Int. Math. Res. Not. 2008 (2008) art. id. rnm116   MR Zbl

[Mirzakhani 2016]   **M Mirzakhani**, *Counting mapping class group orbits on hyperbolic surfaces*, preprint (2016) arXiv 1601.03342

[Navas 2011]   **A Navas**, *Groups of circle diffeomorphisms*, Univ. of Chicago Press (2011)   MR Zbl

[Navas 2018]   **A Navas**, *Group actions on 1–manifolds: a list of very concrete open questions*, from "Proceedings of the International Congress of Mathematicians, III" (B Sirakov, P N de Souza, M Viana, editors), World Sci., Hackensack, NJ (2018) 2035–2062   MR Zbl

[Obata 2021]   **D Obata**, *Symmetries of vector fields: the diffeomorphism centralizer*, Discrete Contin. Dyn. Syst. 41 (2021) 4943–4957   MR Zbl

[Penner and Harer 1992]   **R C Penner**, **J L Harer**, *Combinatorics of train tracks*, Ann. of Math. Stud. 125, Princeton Univ. Press (1992)   MR Zbl

[Ratner 1987]   **M Ratner**, *The rate of mixing for geodesic and horocycle flows*, Ergodic Theory Dynam. Systems 7 (1987) 267–288   MR Zbl

[Smillie and Weiss 2004]   **J Smillie**, **B Weiss**, *Minimal sets for flows on moduli space*, Israel J. Math. 142 (2004) 249–260   MR   Zbl

[Sözen and Bonahon 2001]   **Y Sözen**, **F Bonahon**, *The Weil–Petersson and Thurston symplectic forms*, Duke Math. J. 108 (2001) 581–597   MR   Zbl

[Thurston 1979]   **W P Thurston**, *The geometry and topology of three-manifolds*, lecture notes, Princeton University (1979) Available at https://url.msp.org/gt3m

[Thurston 1998]   **W P Thurston**, *Minimal stretch maps between hyperbolic surfaces*, preprint (1998)   arXiv math/9801039   Reprinted in his "Collected works with commentary, I: Foliations, surfaces and differential geometry" (B Farb, D Gabai, S P Kerckhoff, editors), Amer. Math. Soc., Providence, RI (2022) 533–585

[Wilkinson 2010]   **A Wilkinson**, *Conservative partially hyperbolic dynamics*, from "Proceedings of the International Congress of Mathematicians, III" (R Bhatia, A Pal, G Rangarajan, V Srinivas, M Vanninathan, P Gastesi, editors), Hindustan, New Delhi (2010) 1816–1836   MR   Zbl

[Wilkinson and Xue 2020]   **A Wilkinson**, **J Xue**, *Rigidity of some abelian-by-cyclic solvable group actions on $\mathbb{T}^N$*, Comm. Math. Phys. 376 (2020) 1223–1259   MR   Zbl

[Wright 2020]   **A Wright**, *A tour through Mirzakhani's work on moduli spaces of Riemann surfaces*, Bull. Amer. Math. Soc. 57 (2020) 359–408   MR   Zbl

[Wright 2022]   **A Wright**, *Mirzakhani's work on earthquake flow*, from "Teichmüller theory and dynamics" (P Dehornoy, E Lanneau, editors), Panor. Synth. 58, Soc. Math. France, Paris (2022) 101–134   MR   Zbl

Department of Mathematics, University of Maryland
College Park, MD, United States

Department of Mathematics, University of Michigan
Ann Arbor, MI, United States

farana@umd.edu,   alexmw@umich.edu

GEOMETRY & TOPOLOGY