HONDA-HUANG'S WORK ON CONTACT CONVEXITY REVISITED

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Abstract. Following the overall strategy of the paper [14] by Ko Honda and Yang Huang on contact convexity in high dimensions, we present a simplified proof of their main result.

1. Introduction

A hypersurface in a contact manifold is said to be convex if it admits a transverse contact vector field (see Section 3.3 below for precise definitions). The central result of the article *"Convex hypersurfaces in contact topology"* by Ko Honda and Yang Huang is the following:

Theorem 1.1 (Ko Honda and Yang Huang, [14])**.** *Let (M, ξ_M) be a manifold with a co-orientable contact structure and $\Sigma \subset M$ a co-oriented hypersurface. Then there exists a C^0–small isotopy sending Σ to a convex hypersurface $\widetilde{\Sigma}$.*

If $\dim M = 2$ then Theorem 1.1 holds, according to a classical result of Emmanuel Giroux, [9], in a stronger form, with a C^∞-small isotopy instead of a C^0-small isotopy. The purpose of this article is to provide a more accessible proof of Theorem 1.1. While the proof follows the overall strategy of [14] it is significantly different in its implementation. In particular, we do not use any contact open book techniques. Besides Theorem 1.1 we do not discuss in this paper any other results formulated in [14].

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2. Dynamics of vector fields

This section and Section 3 contains some background material which is mostly well-known.

2.1. **Lyapunov functions.** An isolated zero p of a vector field X on an m-dimensional manifold Σ is called *non-degenerate* if d_pX is non-degenerate, and it is called an *embryo* or *death-birth* singularity if the corank of its linearization d_pX is equal to 1 and the quadratic differential $d_p^2X : \operatorname{Ker} d_pX \to \operatorname{Coker} d_pX$, which is defined up to scaling by a non-zero coefficient, does not vanish. We will call a non-degenerate or death-birth zero *hyperbolic* if d_pX has no pure imaginary (non-zero) eigenvalues.

Let X be a vector field on a compact manifold Σ. Let us endow Σ with a Riemannian metric. A function $f : \Sigma \to \mathbb{R}$ is called *Lyapunov* for X if $df(X) \geq C(||X||^2 + ||df||^2)$ for a positive constant C. Equivalently, one says that X is a *gradient like* vector field for f.

It is a standard fact that isolated hyperbolic zeroes, non-degenerate or embryos, admit local Lyapunov function, e.g. see [1]. The *stable manifold* of a non-degenerate zero is diffeomorphic to $\mathbb{R}^k$ for some $k = 0, \dots, m$ in the non-degenerate case, and to $\mathbb{R}^k_+$ in the case of an embryo. The dimension k of the stable manifold of a non-degenerate hyperbolic zero O is called its *index* and denoted by $\operatorname{ind}(O)$. For an embryo the index is usually defined to be equal to $k - \frac{1}{2}$.

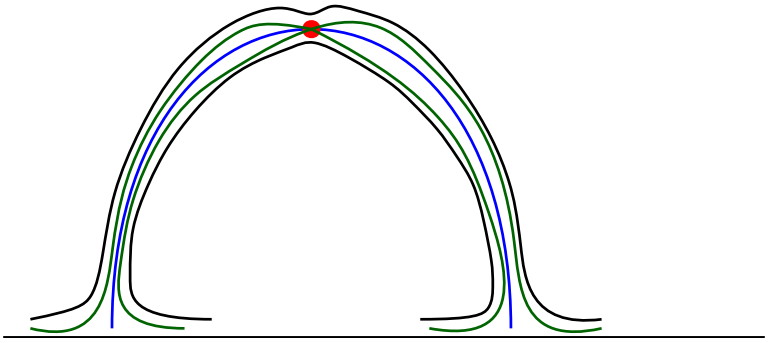

Figure 1

Lemma 2.1. *Let X be a vector field with isolated hyperbolic zeroes which are non-degenerate or of embryo type on a closed m-dimensional manifold Σ. Then X admits a Lyapunov function if and only if the following conditions are satisfied:*

(L1) *every trajectory of X originates and terminates at a zero of X;*
(L2) *there exists an ordering $O_1, \dots, O_N$ of zeroes such that there are no trajectories of X which originate at O_i and terminate in O_j if $i > j$.*

Proof. If X admits a Lyapunov function then both conditions (L1) and (L2) are straightforward. Suppose that these conditions are satisfied. We construct a Lyapunov function $f : \Sigma \to \mathbb{R}$ by extending it inductively to neighborhoods of stable manifolds of zeroes O_j.

We start with a local Lyapunov function near O_1 (which has to be of index 0) and set $f(O_1) = 1$. We assume that $\Sigma_1 := \{f \leq \frac{3}{2}\}$ is a small ball surrounding O_1, with boundary transverse to X.

Suppose that we already constructed f on a domain $\Sigma_{k-1} := \{f \leq k-\frac{1}{2}\}$, $2 \leq k \leq N$, such that zeroes $O_1, \dots, O_{k-1}$ and their stable manifolds are contained in $\operatorname{Int} \Sigma_{k-1}$ and $f(O_{k-1}) = k-1$. The stable manifold P_k of O_k transversely intersects $\partial\Sigma_{k-1}$. Denote $\widetilde{P}_k := P_k \setminus \operatorname{Int} \Sigma_{k-1}$. Then $\widetilde{P}_k$ is an embedded disk (or a half-disc, if O_k is an embryo) of dimension $\operatorname{ind}(O_k)$ with boundary transverse to $\partial\Sigma_{k-1}$. Extend f to a neighborhood $U_k \supset \Sigma_{k-1} \cup \widetilde{P}_k$ as a Lyapunov function for X such that $f(O_k) = k$ and the regular level set $\{f = k + \frac{1}{2}\}$ is compact and is contained in U_k, see Figure 1 and [4] for details. Denote $\Sigma_k = \{f \leq k + \frac{1}{2}\}$. For $k = N$ we have $\Sigma_N = \{f \leq N\} = \Sigma$, and this completes the construction. □

Note that condition (L2) is guaranteed by the *Morse-Smale property*, i.e. transversality of stable and unstable manifolds for any pair of zeros. While the Morse-Smale property can be arranged by a C^∞-small perturbation, it is not clear to us whether this perturbation can be always done without destroying property (L1).

Lemma 2.1 can be extended to 1-parametric families.

Lemma 2.2. *Any family of vector fields X_s, $s \in [0,1]$, which satisfy conditions (L1) and (L2) admits a family of Lyapunov functions.*

Proof. The space of Lyapunov functions for a given vector field X is contractible, because a convex linear combination of two Lyapunov functions for X is again a Lyapunov function for X. Also note that a Lyapunov function f_{s_0} for X_{s_0} can always be included into a family f_s of Lyapunov functions for X_s for s close to s_0. Hence, the projection of the space of pairs ((L1)+(L2) field, Lyapunov function) to the space of (L1)+(L2) fields is a micro-fibration with a (non-empty!) contractible fiber, and hence, it is a Serre fibration, see [11, 18]. □

Let us also formulate a version of Lemma 2.1 for a (trivial) cobordism. Let W be an $(m-1)$-dimensional manifold with boundary and $\Sigma := W \times [0,1]$. Denote by y the coordinate which corresponds to the second factor. Let X be a vector field on Σ which coincides with $\frac{\partial}{\partial y}$ near $\partial\Sigma$.

Lemma 2.3. *Suppose that*

- (L1′) *every trajectory of X originates and terminates at a zero of X or at a point of $\partial\Sigma$;*
- (L2) *there exists an ordering $O_1, \dots, O_N$ of zeroes such that there are no trajectories of X which originate at O_i and terminate in O_j if $i > j$.*

Then X admits a Lyapunov function which is equal to y near $\partial\Sigma$.

Proof. We construct f by the process described in the proof of Lemma 2.1 with $W \times 0$ and $W \times 1$ playing the role of the first and last zeroes, O_0 and O_{N+1}. We then adjust f near $\partial W \times [0,1]$, by making it linear with respect to y and then scaling it to make equal to 1 on $W \times 1$. □

2.2. **Blocking collections.** The material of this section is fairly standard and its various versions appear in many places (e.g. see [12, 19]). In particular, Lemma 2.5 is a corollary of [19, Lemma 2].

A non-vanishing vector field X in a neighborhood of a hypersurface V in an m-dimensional manifold Σ is called *in general position* with respect to V if it has Thom-Boardman-Morin tangency singularities of type $\Sigma^{1,\dots,1}$, see [17, 3]. Let us fix a Riemannian metric on Σ.

Arguing by induction over strata of tangency singularity, it is straightforward to prove the following statement (e.g. it is a corollary of Morin's normal forms [15] for $\Sigma^{1,\dots,1}$-singularities).

Lemma 2.4. *Suppose that X is in general position with respect to V. Then there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ there exists $\delta > 0$ such that any connected trajectory arc of length ϵ contains a connected sub-arc of length $C(m)\epsilon$ which does not intersect the δ-neighborhood of V. Here $C(m)$ denotes a constant which depends only on the dimension m*

Proof. We can assume that the vector field X on a neighborhood of V has a unit length. We will be measuring below arcs γ of X-trajectories by the flow-parameter. This measurement, which we call *length* is equivalent to the diameter of γ for sufficiently short arcs.

For any point $p \in V$ and $\epsilon > 0$ denote $\gamma_\epsilon(p) = \bigcup_{u\in[-\epsilon,\epsilon]} X^u(p)$. There exists $\epsilon_0 > 0$ such that for each point $p \in V$ the arc $\gamma_{3\epsilon_0}(p)$ intersects V at no more than m points, and moreover $V \cap \gamma_{3\epsilon_0}(p) \subset \gamma_{\epsilon_0}(p)$. Given $\delta > 0$ denote by $N_\delta(V)$ the δ-tubular neighborhood of V. If For any $\epsilon < \epsilon_0$ there exists $\delta > 0$ such that for every $p \in V$ the intersection $\gamma_{3\epsilon_0}(p) \cap N_\delta(V)$ consists of no more than m components of length $< \frac{\epsilon}{2m+2}$. Any trajectory arc σ of length ϵ which intersects $N_\delta(V)$ is contained in $\gamma_{3\epsilon_0}(p)$ for some $p \in V$. Hence $N_\delta(V) \cap \sigma$ consists of no more than m arcs of length $< \frac{\epsilon}{2m+2}$. Thus, the complement $\sigma \setminus N_\delta(V)$ contains an arc of length $> \frac{\epsilon}{m+1} - \frac{\epsilon}{2m+2} = \frac{\epsilon}{2m+2}$. □

Let X be a vector field on a compact m-dimensional manifold Σ, possibly with boundary. Given $\epsilon > 0$, a finite collection $\{D_j\}_{1\le j\le K}$ of transverse to X embedded into $\operatorname{Int}\Sigma$ codimension one discs of diameter $< \epsilon$ is called *ϵ-blocking* if any connected trajectory arc of diameter $> \epsilon$ intersects $\bigcup_1^K \operatorname{Int} D_j$.

Lemma 2.5. *Let X be a vector field on a compact m-dimensional manifold Σ, possibly with boundary. Suppose that all zeros of X are in $\operatorname{Int}\Sigma$, isolated and hyperbolic (non-degenerate or embryos). Suppose that X is in general position with respect to $\partial\Sigma$. Then for any $\epsilon > 0$ the field X admits an ϵ-blocking collection.*

Proof. Part I. Suppose first that the vector field $X|_\Sigma$ admits a Lyapunov function $f : \widetilde{\Sigma} \to \mathbb{R}$. without critical points. Suppose that $\min f = 0, \max f = 1$ and choose N large enough to guarantee that any connected arc of an X-trajectory in $\{\frac{j}{N} \le f \le \frac{j+1}{N}\}$ has its diameter $< \frac{\epsilon}{2}$, $j = 0, \dots, N-1$. Suppose that ϵ is chosen $< \epsilon_0$ from Lemma 2.4 and δ is chosen so small that any connected trajectory arc of length ϵ contains a connected sub-arc of length $\frac{\epsilon}{C(m)}$ which does not intersect the δ-neighborhood of V. Here $C(m)$ is the constant from Lemma 2.4. Choose an interior tubular collar $\partial\Sigma \times [-1, 0] \subset \Sigma$ such that $\partial\Sigma = \partial\Sigma \times 0$ and $\partial\Sigma \times (-1)$ is at a distance δ from $\partial\Sigma$. Denote $\Sigma_0 := \Sigma \setminus (\partial\Sigma \times (-1, 0])$. For each $j = 1, \dots, N-1$ choose finitely many closed discs of radius ϵ in $\operatorname{Int}\{f = \frac{j}{N}\}$ whose interiors cover $\{f = \frac{j}{N}\} \cap \Sigma_0$. By shifting these discs to disjoint level sets $\{f = t_{j,k}\}$, $t_{j,k} \in (\frac{2j-1}{2N}, \frac{2j+1}{2N})$, we get the required ϵ-blocking collection.

Part II. In the general case let us choose any smooth function $f : \Sigma \to \mathbb{R}$. Let us surround zeroes of X by the union B of disjoint closed ϵ-balls. We can assume that X is in general position with respect to ∂B. Denote $\widetilde{\Sigma} := \Sigma \setminus \operatorname{Int} B$.

Denote

$$\widetilde{\Sigma}_+ := \{df(X) \geq 0\},\ \widetilde{\Sigma}_- := \{df(X) \leq 0\},\ V := \{df(X) = 0\} = \widetilde{\Sigma}_+ \cap \widetilde{\Sigma}_-.$$

By C^∞-perturbing f, if necessary, we can arrange that V is a codimension 1 submanifold, and X is in general position with respect to V. Let us assume that $\delta > 0$ is chosen in such a way that any connected trajectory arc of length ϵ contains a connected sub-arc of length $\frac{\epsilon}{C(m)}$ which does not intersect the δ-neighborhood of V. Consider a δ-tubular neighborhood $N \supset V, N \subset \operatorname{Int}\Sigma$. Denote $\widehat{\Sigma}_\pm := \widetilde{\Sigma}_\pm \setminus \operatorname{Int} N$. We can assume that $\partial\widehat{\Sigma}_\pm$ is in general position with respect to X. By applying Part 1 we can construct $\frac{\epsilon}{2}$-blocking collections for $\widehat{\Sigma}_+$ and $\widehat{\Sigma}_+$. The union of these collections is the required ϵ-blocking collection for X on Σ.

□

Note that a compact arc γ of a non-constant trajectory X has a *flow-box* neighborhood $U = D \times [0, c]$ such that $D \times 0$ is an embedded transverse disc, and $x \times [0, c], x \in D$ are trajectories of X. Denote by $\pi_U : U \to D$ the projection of the flow-box neighborhood to the first factor. We will call an ϵ-blocking collection $\{D_j\}$ generic, if for any flow-box U projections $\pi_U|_{\partial D_j \cap U} \to D$ are transverse to each other. Any ϵ-blocking collection can be made generic by a C^∞-perturbation.

2.3. **Plugs.** Given an ϵ-blocking collection $\{D_j\}$, let us thicken discs D_j to disjoint flow-boxes $Q_j = D_j \times [0, a]$ such that intervals $x \times [0, a]$, $x \in D_j$ are time a trajectories of X originated at $x \in D_j = D_j \times 0$. We will assume that a is chosen small enough to guarantee that flow-boxes Q_j have diameter $< 2\epsilon$.

Let D be an $(m-1)$-dimensional disc. A vector field Y on $D \times [0, a]$ is called a *σ-plug* if the following conditions are satisfied:

- P1. Y coincides with $\frac{\partial}{\partial y}$ on ∂Q, where y is the coordinate on $D \times [0, a]$ corresponding to the second factor;
- P2. Y satisfies the Morse-Smale condition and admits a Morse Lyapunov function;
- P3. for any point $p \in D$ with $\operatorname{dist}(p, \partial D) > \sigma$ the trajectory of Y through $p \times 0$ converges to a critical point of Y;
- P4. given any point $p \in D$ the trajectory of Y through $p \times 0$ either converges to a critical point of Y, or exit Q at a point $p' \times a$ where $\operatorname{dist}(p', p) < \sigma$.

Lemma 2.6. *Let Σ be a closed manifold of dimension m, and X a vector field on Σ with non-degenerate hyperbolic zeroes. Let $\{D_j\}$ be a generic ϵ-blocking collection, and $\{Q_j\}$ a collection of their disjoint flow-boxes of diameter $< 2\epsilon$. Then there exists $\sigma > 0$ such that by replacing for each j the vector field $X|_{Q_j}$ by a σ-plug Y one gets a vector field $\widehat{X}$ which satisfies condition (L1) and such that all its trajectories has diameter $< 3\epsilon$. Moreover, property (L1) survives a sufficiently small C^1-perturbation X away from flow-boxes and neighborhoods of zeroes of X.*

Proof. By assumption any point of p belongs to a connected arc γ of an X-trajectory of diameter $< \epsilon$, which has either both ends at $\operatorname{Int} D_i$ and $\operatorname{Int} D_j$ for some i, j, or it limits at one of the end to a zero and the other end is at $\operatorname{Int} D_j$, or both ends it limits to zeroes. By compactness argument we can find smaller closed discs $D_{j,-} \subset \operatorname{Int} D_j$ such that in the above condition one can replace $\operatorname{Int} D_j$ by $D_{j,-}$.

The genericity property for the blocking collection D_j implies that if an arc γ of an X-trajectory does not intersect $\bigcup_1^K \operatorname{Int} D_j$ then it cannot have more than $m-1$ intersection points with ∂D_j. Hence, we can choose the discs $D_{j,-}$ so close to D_j that any arc γ which does not intersect $\bigcup_1^K D_{i,-}$ intersects no more than $m-1$ annuli $A_j := D_j \setminus \operatorname{Int} D_{j,-}$. Choose $\sigma < \frac{1}{m+1} \min_j(\operatorname{dist}(\partial D_{j,-}.\partial D_j)$. We claim that the vector field $\widehat{X}$ obtained by replacing $X|_{Q_j}$ by σ-plugs has the required properties. Indeed, consider any trajectory $\widehat{\gamma}$ of $\widehat{X}$ which enters a flow-box Q_i through a point $p \in D_{i,-} = D_{i,-} \times 0$. Let γ be an X-trajectory through p which intersect $D_{j,-}$ at a point p', and does not contain any other points from $\bigcup_1^K D_{k,-}$. Then if $\widehat{\gamma}$ is not locked in Q_i, or any other of $< m$ plugs Q_k for discs D_k which intersect γ, then it enters Q_j through a point p'' with $\operatorname{dist}(p', p'') < m\sigma$, and thus $\operatorname{dist}(p'', \partial D_i) > \sigma$. But this means that the trajectory $\widehat{\gamma}$ converges to a zero of $\widehat{X}$ in Q_j. Analysis of trajectories with limiting at one of the ends to a zero of X is similar. Moreover, the trajectories of the vector field $\widehat{X}$ have their diameter bounded by 3ϵ. It remains to observe that the above analysis remains valid if X is perturbed by a sufficiently C^1-small homotopy outside flow-boxes.

□

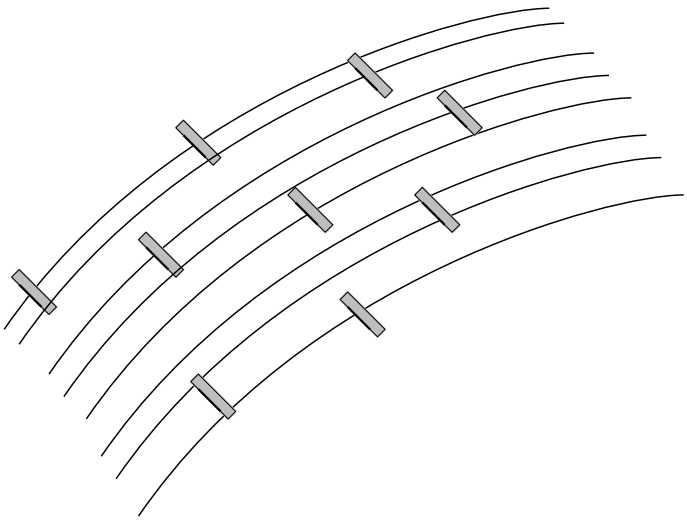

FIGURE 2. Blocking discs with their flow-boxes.

3. Contact convexity

3.1. **Characteristic foliation.** Let M be a contact manifold of dimension $2n+1$ with a co-oriented contact structure $\xi = \{\alpha = 0\}$. The volume form $\mu := \alpha \wedge (d\alpha)^n$ defines an orientation of M, and $(d\alpha)^n|_\xi$ defines an orientation of ξ. If $n = 2k+1$ then the former orientation, and if $n = 2k$ than the latter orientation, depends only on ξ.

Let $\Sigma \subset M$ be a co-oriented hypersurface. If ν is a vector field defining its co-orientation then the orientation of Σ is given by the $2n$-form $\iota(\nu)\mu$. At any point $p \in \Sigma$ where $\xi_p \pitchfork T_p\Sigma$, there is defined a *characteristic line* $\ell_p := \operatorname{Ker}\{d\alpha|_{\xi_p \cap T_p\Sigma}\} \subset \xi_p \cap T_p\Sigma$. Note that the co-orientation of Σ defines a co-orientation of $\xi_p \cap T_p\Sigma$ in ξ_p. We orient ℓ_p by a vector $X_p \in \ell_p$ such that the 1-form $\iota(X_p)d\alpha|_{\xi_p}$ defines that co-orientation.

The line field ℓ, which is defined in the complement of the tangency locus T between ξ and Σ, integrates to a singular foliation on Σ with singularities at the points of T. We will keep the notation ℓ for this foliation, and write ℓ_Σ, ℓ_ξ or $\ell_{\xi,\Sigma}$ when it is important to stress the dependence of ℓ on Σ, ξ, or both.

The singular locus T splits as a union of disjoint closed subsets, $T = T_+ \cup T_-$, where T_+ (resp. T_-) consists of *positive* (resp. *negative*) points, where the orientations of ξ_p and $T_p(\Sigma)$ coincide (resp. opposite). On neighborhoods $U_\pm \supset T_\pm$, $U_\pm \subset \Sigma$, the form $d\beta$, $\beta = \alpha|_\Sigma$, is symplectic. We define a vector field X on Σ which directs ℓ as equal to the Liouville field $d\beta$-dual to β on U_+, and as a vector field $d\beta$-dual to $-\beta$ on U_-, and extend it to the rest of Σ as any non-vanishing vector field. The following lemma is due to E. Giroux in [9] and was pointed out to us by D. Salamon. It provides an equivalent characterization of a vector field directing the characteristic foliation. Choose a positive volume form ρ on Σ equal to $(d\beta)^n$ on U_+ and to $-(d\beta)^n$ on U_-.

Lemma 3.1. *The vector field X defined by the equation*

$$\iota(X)\rho = n\beta \wedge (d\beta)^{n-1}. \tag{1}$$

directs the characteristic foliation ℓ.

Proof. On $U_\pm$ equation (1) is equivalent to $\iota(X)d\beta = \pm\beta$, i.e. X coincides with the Liouville field dual to β on U_+ and to the $d\beta$-dual to $-\beta$ vector field on U_-. Elsewhere, $X \neq 0$ and $\iota(X)(\beta \wedge (d\beta)^{n-1}) = 0$. If $n = 1$ this implies that $\beta(X) = 0$ and hence, $X \in \ell$. For $n \geq 2$ we have

$$0 = \iota(X)(\beta \wedge (d\beta)^{n-1}) = \beta(X)(d\beta)^{n-1} + (n-1)\,(\iota(X)d\beta) \wedge \beta \wedge d\beta^{n-2}.$$

By restricting to $\operatorname{Ker}\beta$ we conclude that $\beta(X) = 0$, and hence, $(\iota(X)d\beta) \wedge \beta \wedge d\beta^{n-2} = 0$. But then $(\iota(X)d\beta)\,|_{\operatorname{Ker}\beta} = 0$. Indeed, the form $d\beta$ descends as a symplectic form to the $(2n-2)$-dimensional quotient space $Q_p := (\xi_p \cap T_p\Sigma)/T\ell_p$ as a symplectic form. Hence, the multiplication by $d\beta^{n-2}$ defines an isomorphism between 1- and $(2n-3)$-forms on Q_p, and the claim follows.

□

Let us recall that the contact structure on a neighborhood of a hypersurface Σ is determined by its restriction to the hypersurface.

Proposition 3.2 (A. Givental, [2])**.** *Let $\xi = \operatorname{Ker}\alpha, \xi' = \operatorname{Ker}\alpha'$ be two contact structures defined on a neighborhood of $\Sigma = \Sigma \times 0 \subset \Sigma \times \mathbb{R}$. Suppose that $\alpha|_\Sigma = h\alpha'|_{\Sigma'}$ for a positive function $h : \Sigma \to \mathbb{R}$. Then there exists a diffeomorphism $g : \mathcal{O}p\,\Sigma \to \mathcal{O}p\,\Sigma$ which is fixed on Σ and such that $dg(\xi) = \xi'$.*

In fact, the statement formulated in [2] is slightly weaker. We thank D. Salamon for providing the details of the proof of the above result.

All singularities of a vector field X directing the characteristic foliation ℓ can be made non-degenerate and hyperbolic by a C^∞-small perturbation of Σ, see e.g. [4]. For a generic 1-parametric family of characteristic foliations, the directing vector field X_s can also have (hyperbolic) embryo singularities for isolated values of the parameter s.

We will need the following version of Lemma 2.5 for a vector field X directing a characteristic foliation. Define the *standard contact* $(2n-1)$*-disc* $(D = D^{2n-1}, \xi_{\text{st}})$ as contactomorphic to the hemisphere $D = S^{2n-1}_+ := S^{2n-1} \cap \{y_n \geq 0\}$ endowed with the contact structure $\xi_{\text{st}} := \left\{ \sum_1^n (x_j dy_j - y_j dx_j)|_{S^{2n-1}} = 0 \right\}$.

Lemma 3.3. *Let X be a vector field directing a characteristic foliation on a closed hypersurface Σ in a contact manifold of dimension $2n+1$. Suppose that all zeros of X are isolated and hyperbolic (non-degenerate or embryos). Then for any $\epsilon > 0$ the field X admits an ϵ-blocking collection $\{D_j\}$ which consists of standard contact $(2n-1)$-dimensional discs.*

Proof. We only need to ensure that discs forming the blocking collections can be chosen contactomorphic to the standard contact disc. We recall that in the proof of Lemma 2.5 discs D_j arise as elements of a covering of a transverse contact hypersurface. But the covering can always be chosen to be formed by standard small Darboux balls. □

3.2. **Lyapunov functions for characteristic foliations.** For a vector field X directing a characteristic foliation ℓ on a hypersurface Σ stable manifolds of positive zeroes are isotropic with respect to $d\beta$, while unstable are coisotropic, see [4]. Near negative zeroes the field $-X$ is Liouville, and thus stable manifolds of negative zeroes are co-isotropic while unstable are isotropic, In particular, a local Lyapunov function on $\mathcal{O}p\,(T_+ \cup T_-)$, have critical points of index $\leq n$ at the positive points, and of index $\geq n$ in the negative ones. It is important to note that critical points of index n can be either negative or positive. The stable manifold of a positive (resp. negative) embryo is an isotropic (resp. co-isotropic) half-space.

We call a Lyapunov function $f : \Sigma \to \mathbb{R}$ for X *good* if there exists a regular value c such that all positive zeroes of X are in $\{f > c\}$ and all negative ones are in $\{f < c\}$. Sometimes we will call f a Lyapunov function for ℓ, rather than X.

Following Giroux, we call a trajectory γ of X a *retrograde connection* if it originates at a negative point of X and terminates at a positive one. Lemma 2.1 implies

Corollary 3.4. *Suppose that a vector field X satisfies conditions (L1) and (L2) from Lemma 2.1. Then it admits a good Lyapunov function if and only if it has no retrograde connections. In particular, any X which satisfies (L1) and the Morse-Smale condition admits a good Lyapunov function.*

Proof. If there are no retrograde connections, then one can always order zeroes in such a way that positive zeroes go first, and hence the construction in 2.1 yields a good Lyapunov function. The necessity of the absence of retrograde connections for existence of good Lyapunov function is straightforward. □

Similarly, using Lemma 2.2 we get a parametric version of this statement.

Corollary 3.5. *Any family X_s, $s \in [0,1]$, which satisfy (L1) and (L2) and have no retrograde connections admits a family of good Lyapunov functions.*

Proof. An additional observation which is needed for the proof, in addition to the argument in 2.2, is that the space of good Lyapunov functions is contractible. Indeed, if we normalize Lyapunov functions by the condition that the 0 level is separating positive and negative points, then their convex linear combination is again a good Lyapunov function. □

3.3. **Flavors of contact convexity.** The notion of contact convexity was first defined in [7], and then explored by Emmanuel Giroux, see [9], Ko Honda, see [13], and others.

Definition 3.6.
(1) A hypersurface $\Sigma \subset (M, \xi)$ is called *convex* if it admits a transverse contact vector field Υ.
(2) A hypersurface Σ is called *Weinstein convex* if its characteristic foliation ℓ admits a good Lyapunov function.

As we will see below in Lemma 3.10, Weinstein convexity is a stronger condition which implies convexity.

E. Giroux proved in [9] that for 2-dimensional surfaces contact convexity can be achieved by a C^∞-perturbation.

Using Corollary 3.4 we can equivalently characterize Weinstein convexity by conditions (L1) and (L2) (or, equivalently, existence of *any* Lyapunov function) for X and absence of retrograde connections. As it was pointed out above, condition (L2) is implied by the Morse-Smale condition, which is generic for individual hypersurfaces.

Following Giroux, the set $S := \{x \in \Sigma; \Upsilon(x) \in \xi_x\}$ is called the *dividing set* of Σ.

Lemma 3.7 (E. Giroux, [9])**.** *Suppose X is a contact vector field transverse to a hypersurface Σ and S the corresponding dividing set. Let t be the flow coordinate such that $\Sigma = \{t = 0\}$ and $X = \frac{\partial}{\partial t}$. Then ξ on $\mathcal{O}p\,\Sigma$ can be defined by a contact 1-form $f(x)dt+\beta$, where $f : \Sigma \to \mathbb{R}$ is a function transversely changing sign across S.*

Note that the contact condition implies that $df \neq 0$ along S, and $\alpha|_S$ is a contact form. In particular, the characteristic foliation ℓ_Σ transverse to S.

Hence, we have the following:

Lemma 3.8 (E. Giroux, [9])**.** *Dividing set S is a smooth submanifold, which is transverse to the characteristic foliation, and independent of the choice of a contact vector field transverse to Σ, up to an isotopy transverse to the characteristic foliation.*

Indeed, the space of contact vector fields transverse to Σ is convex subset of the vector space of all contact vector fields, and hence, contractible.

The dividing hypersurface $S \subset \Sigma$ divides Σ into $\Sigma_+ := \{f > 0\}$, $\Sigma_- := \{f < 0\}$. The form $\alpha = (f(x)dt + \beta)|_{\Sigma \setminus S}$ can be divided by f,

$$\frac{\alpha}{f} = dt + \frac{\beta}{f}.$$

Denote $\lambda_\pm := \frac{\beta}{f}|_{\Sigma_\pm}$. The contact condition then is equivalent to $(d\lambda_\pm)^n \neq 0$. In other words, $\lambda_\pm$ are Liouville forms on $\Sigma_\pm$. Note that the corresponding Liouville fields $Z_\pm$ directs the characteristic foliation on Σ. Indeed, $\lambda_\pm \wedge \iota(Z_\pm)d\lambda_\pm = \lambda_\pm \wedge \lambda_\pm = 0$.

Lemma 3.9. *Let $\Sigma \subset (M, \xi = \operatorname{Ker}\alpha)$ be a co-oriented hypersurface. Denote $\beta := \alpha|_\Sigma$. Then Σ is convex if and only if there exists a function $f : \Sigma \to \mathbb{R}$ such that the form $\beta + fdt$ is contact on $\Sigma \times \mathbb{R}$.*

Proof. The necessity is a reformulation of Lemma 3.7. To see the sufficiency we observe that $\Sigma = \Sigma \times 0 \subset (\Sigma \times \mathbb{R}, \operatorname{Ker}(\beta + fdt))$ is convex because the field $\Upsilon := \frac{\partial}{\partial t}$ is manifestly contact. On the other hand, by Proposition 3.2 neighborhoods of Σ in (M, ξ) and $\Sigma \times 0$ in $(\Sigma \times \mathbb{R}, \operatorname{Ker}(\beta + fdt))$ are contactomorphic. □

Lemma 3.10. *Any Weinstein convex hypersurface is convex.*

Proof. According to Lemma 3.9 it is sufficient to find a function $f : \Sigma \to \mathbb{R}$ such that the form $\widetilde{\alpha} := \beta + fdt$ is contact. We claim that in turn this condition is equivalent to the inequality

$$f(d\beta)^n + n\beta \wedge (d\beta)^{n-1} \wedge df > 0. \tag{2}$$

Indeed,

$$\begin{aligned}\widetilde{\alpha} \wedge (d\widetilde{\alpha})^n &= (\beta + fdt) \wedge (d\beta + df \wedge dt)^n = fdt \wedge (d\beta)^n + n\beta \wedge (d\beta)^{n-1} \wedge df \wedge dt \\ &= dt \wedge \left(f(d\beta)^n + n\beta \wedge (d\beta)^{n-1} \wedge df\right),\end{aligned}$$

and hence, the inequality $\widetilde{\alpha} \wedge (d\widetilde{\alpha})^n > 0$ is equivalent to (2).

Suppose ρ is a volume form on Σ chosen as in Lemma 3.1, and the vector field X directing ℓ satisfies equation (1). Let $h : \Sigma \to \mathbb{R}$ be a good Lyapunov function on Σ. We will assume that $S := \{h = 0\}$ is a regular level set of h separating values in negative and positive zeroes of X. In particular, $h|_{U_+} < 0, h|_{U_-} > 0$ for neighborhoods $U_\pm \supset T_\pm$ of the singular point loci.

Define a function $g : \Sigma \to \mathbb{R}$ by the equation $(d\beta)^n = g\rho$. We have $g|_{U_+} > 0$ and $g|_{U_-} < 0$. Hence, $gh < 0$ on $U := U_+ \cup U_-$. We have $dh(X) > 0$ on $\Sigma \setminus ((T := T_+ \cup T_-) \cup S)$. Furthermore, $dh(X) - hg > 0$ on a neighborhood of $S = \{h = 0\}$, and for a sufficiently large constant $C > 0$ we have

$$dh(X) - hg + Ch^2 dh(X) > 0$$

everywhere on Σ. The function $f : \Sigma \to \mathbb{R}$, which satisfies (2) can now be defined by the formula $f := -e^{\frac{Ch^2}{2}}h$. Indeed, we have $df = -e^{\frac{Ch^2}{2}}(1 + Ch^2)dh$ and therefore,

$$fd\beta^n + n\beta \wedge (d\beta)^{n-1} \wedge df = fg\rho + n(\iota(X)\rho) \wedge df.$$

But for any 1-form γ we have $\iota(X)\rho\wedge\gamma = -\iota(X)(\rho\wedge\gamma) - \gamma(X)\rho = \gamma(X)\rho$, and hence

$$fg\rho + n(\iota(X)\rho)\wedge df = (fg - df(X))\rho = -e^{\frac{Ch^2}{2}}\left(hg - (1+Ch^2)dh(X)\right)\rho$$
$$= e^{\frac{Ch^2}{2}}(dh(X) - hg + Ch^2 dh(X)) > 0.$$

□

Corollary 3.4 and Lemma 3.10 reduce Theorem 1.1 (in a stronger form replacing convexity by Weinstein convexity) to

Theorem 3.11 (Honda-Huang,[14])**.** *Let $\Sigma \subset (M,\xi)$ be a co-orientable hypersurface in a contact manifold with a co-orientable contact structure. Then there exists a C^0-small isotopy deforming Σ into $\widetilde{\Sigma} \subset (M,\xi)$ such that the characteristic foliation induced on $\widetilde{\Sigma}$ satisfies condition (L1) and the Morse-Smale property.*

For the case $\dim M = 2$ this result can be deduced from E. Giroux's theorem about C^∞-genericity of contact convexity in 3-dimensional contact manifolds, [9].

4. Construction of plugs

4.1. **Main proposition.** In view of Lemmas 2.6 and 3.3, the proof of Theorem 3.11 will be completed if for any $\sigma > 0$ one can create a σ-plug by a C^0-small isotopy of the flow box of a standard, transverse to the flow contact disc. The next proposition asserts that this is possible. As the statement will be proven by induction, we need more properties of a σ-plug in order for the induction to go through.

Let (D^{2n-1}, α_{st}) be the standard contact disc. Choose $c, b > 0$ and consider $U^b := D \times T^*[0,b]$ endowed with a contact form $\alpha_{\mathrm{st}} + xdy$. Denote

$$Q^b := \{x = 0\} \subset U = D \times [0,b],\ U^b_c := \{|x| \le c\} \subset U^b.$$

We will omit the superscript b when $b = 1$.

Proposition 4.1. *For any positive ϵ and $\sigma \ll \epsilon$ there exists an isotopy $h_s : Q \to U_\epsilon$, $s \in [0,1]$, which is fixed on $\mathcal{O}p\,\partial Q$, begins with the inclusion $h_0 : Q \hookrightarrow U_\epsilon$ and has the following properties:*

a) *(Q, X_1) is a σ-plug, where we denoted by X_s the vector field directing the characteristic foliation ℓ_s induced by $h_s^*(\alpha_{\mathrm{st}} + xdy)$;*
b) *for any $\sigma_1 \ll \sigma$ there exists a family of compact manifolds with boundary $C_s^+ \subset \operatorname{Int} Q$ and $C_s^- \subset \operatorname{Int} Q$, and an extension of the isotopy h_s to a 2-parametric isotopy $h_{s,t}$, $s,t \in [0,1]$, such that*
   (i) *$h_{s,0} = h_s$, $h_{0,t} = h_0$ for all $s,t \in [0,1]$;*
   – *the foliation $\ell_{s,1}$ induced on $h_{s,1}(Q)$ has no singular points;*
   (ii) *for any fixed $s \in [0,1]$ the isotopy $h_{s,t}$, $t \in [0,1]$, is σ_1-small in the C^0-sense and supported in a σ_1-neighborhood of C_s;*
   (iii) *For each $s \in [0,1]$ the submanifold C_s^+ (resp. C_s^-) contains all positive (resp. negative) singularities of X_s, $C_s^+ \cap C_S^- = \emptyset$ and C_s^+ (resp. C_s^-) is invariant with respect to the backward (resp. forward) flow of X_s;*

(iv) *there exists a family of generalized Morse Lyapunov functions* $\psi_{s,t} : Q \to \mathbb{R}$ *for* $X_{s,t}$ *such that* $\psi_{s,t}|_{\mathcal{O}p\,\partial Q} = y$*;*

(v) *there exists a stratified* $(n-1)$*-dimensional subset* $E \subset Q \cap \{y = 0\}$ *which contains all the intersection points of* $Q \cap \{y = 0\}$ *with stable manifolds of positive singular point of* $X_{s,t}$ *for all* $s, t \in [0, 1]$*.*

We will refer to the statement of Proposition 4.1 as an *installation of a* σ*-plug of height* ϵ *over* $Q = D^{2n-1} \times [0, 1]$, where $(D^{2n-1}, \alpha_{\mathrm{st}})$ is the standard contact disc. The same statement with Q, U_c are replaced by Q^b and U_c^b will be referred as an installation over Q^b. Note that the contactomorphism $U_a^b \to U_{ab}^1$ induced by the linear map $(x, y) \mapsto (bx, \frac{y}{b})$ of the second factor always allows us to reduce the installation to the case $b = 1$.

If we further replace $(D, \alpha_{\mathrm{st}})$ in the statement of 4.1 by any compact $(2n-1)$-dimensional manifold V manifold with boundary (and possibly with corners) and with a fixed contact form α, we will say that we are *installing a* σ*-plug of height* ϵ *over* $V \times [0, b]$*.*

4.2. **Plan of the proof of Proposition 4.1.** We begin the proof in Section 5 by showing that Proposition 4.1 can be deduced from a weaker Proposition 5.1, where the required isotopy h_s is constructed in U_K for a large K which may depend on σ, rather than U_ϵ for an arbitrary small ϵ. This is done by a scaling argument. One of the subtleties here is that contact scalings are better adjusted to Carnot-Caratheodory type metrics, rather than Riemannian ones. Thus, we have to analyze separately an effect of the scaling on measurements in directions tangent and transverse to contact planes.

The continuation of the proof is by induction on dimension $2n-1$. Lemma 6.4 in Section 6.2 serves as the base of the induction for $n = 2$, as well as an important ingredient in the proof of the induction step. The Giroux-Fuchs creation-elimination construction, which we recall in Section 6.1, is an essential ingredient to the proof of Lemma 6.4.

By taking a product of the two-dimensional plug constructed in Lemma 6.4 with a $(2n-2)$-dimensional Weinstein domain (W, λ) and appropriately adjusting the product over $\mathcal{O}p\,\partial W$ we construct in Section 7.1 a $2n$-dimensional *preliminary plug* over $((W \times [0, 1]) \times [0, 1]$. We call the contact domain $(W \times [0, 1], \lambda + dz)$ a *Weinstein cylinder*. Similar to a σ-plug, a preliminary plug blocks all trajectories entering $W \times [0, 1] \times 0$ at a distance $\geq \sigma$ from the boundary of the Weinstein cylinder. However, one has a much weaker control of the dynamics of the trajectories entering near the boundary. Constructions of 2-dimensional and preliminary plugs are variations of similar constructions in [14].

Next, we show in Section 7.4 that by a special arrangement of Weinstein cylinders ($V_1 = W_1 \times [0, 1], \dots, V_k = W_k \times [0, 1]$), $k \geq 2$, see the definition of a *good position* in Section 7.3, and by composing preliminary plugs over $V_j \times [\frac{j-1}{k}, \frac{j}{k}]$, $j = 1, \dots, k$, we create, see Lemma 7.8, a σ*-quasi-plug* over $(\widehat{V} := \bigcup_1^k V_j) \times [0, 1]$ which blocks trajectories entering at a distance $\geq \sigma$ from $\partial\widehat{V} \times 0$, while a non-blocked trajectory which enters at a point $(p_0, 0) \in \widehat{V} \times 0$ with $\mathrm{dist}(p_0, \partial\widehat{V}) < \sigma$ exits at a point $p_1 \in \widehat{V} \times 1$ which satisfies the following condition: there exist points $p_0', p_1' \in \partial\widehat{W}$ such that $\mathrm{dist}(p_0, p_0'), \mathrm{dist}(p_1, p_1') < \sigma$ and p_1' belongs to the forward trajectory of p_0' for a vector field directing the characteristic foliation $\ell_{\partial\widehat{V}}$ on $\partial\widehat{V}$. We note

that if the characteristic foliation $\ell_{\partial\widehat{V}}$ is a σ-short, then any σ-quasi-plug is automatically a 3σ-plug.

Crucial Proposition 7.6 asserts that if a contact domain V with a Weinstein convex boundary ∂V and a dividing set $S \subset \partial V$ can be C^0-approximated by standard contact balls which coincide with V in $\mathcal{O}p\, S$ then V can be approximated by 3 Weinstein cylinders in a good position.

In Section 8.2 we use the induction hypothesis to show that the standard contact ball D^{2n-1} can be be deformed by a σ-small in the C^0-sense isotopy to a ball $\widetilde{D}$ with Weinstein convex boundary and a dividing set $S \subset \partial\widetilde{D}$ such that the characteristic foliations $\ell_{\partial\widetilde{D}}$ is σ-short and $\widetilde{D}$ can be C^0-approximated by standard contact balls which coincide with $\widetilde{D}$ on a neighborhood of S.

Together with Proposition 7.6 and Lemma 7.8 this leads in Section 8.3 to a proof of Proposition 5.1, and with it, of all main results of the paper.

5. Reducing the height of a σ-plug

The goal of this section is to reduce Proposition 4.1 to the following weaker statement.

Proposition 5.1. *For any $\sigma > 0$ there exists $K = K(\sigma)$ and an isotopy $h_s : Q \to U_K$, $s \in [0,1]$, which satisfy properties a) and b) from Proposition 4.1. In other words, one can install a σ-plug over $Q = D^{2n-1} \times [0,1]$ of height K which may depend on σ.*

5.1. **Changing the base.** Let $(V_1, \operatorname{Ker}\alpha_1), (V_2, \operatorname{Ker}\alpha_2)$ be two contact manifolds with boundary with corners endowed with contact forms and Riemannian metrics. Any contactomorphism $f : V_1 \to V_2$ can be extended to a contactomorphism

$$F : (V_1 \times T^*\mathbb{R}, \operatorname{Ker}(\alpha_1 + xdy)) \to (V_2 \times T^*\mathbb{R}, \operatorname{Ker}(\alpha_2 + xdy)$$

by the formula $F(v,x,y) = (f(v), g(v)x, y),\ v \in V_1,\ x,y \in \mathbb{R}$, where the function g is defined by the equation $f^*\alpha_2 = g\alpha_1$.

Lemma 5.2. *Let $h_{s,t} : V_1 \times [0,1] \to V_1 \times T^*[0,1]$ be an isotopy installing a σ-plug of height ϵ over $V_1 \times [0,1]$. Denote $C_1 := \max\limits_{v \in V_1} ||d_v f||$, $C_2 := \max\limits_{V_1} g$. Then the isotopy $\widehat{h}_s := F \circ h_s \circ F^{-1} : V_2 \times [0,1] \to V_2 \times T^*[0,1]$ is installing a $C_1\sigma$-plug of height $C_2\epsilon$ over $V_2 \times [0,1]$.*

Proof. First, we note that $\widehat{h}_s(V_2 \times [0,1]) \subset \{|x| \le C_2\epsilon\}$ because $h_s(V_1 \times [0,1]) \subset \{|x| \le \epsilon\}$ by assumption. If X_1^1 is the vector field on $V_1 \times [0,1]$ directing the characteristic foliation of the form $h_s^*(\alpha_1 + xdy)$ then the vector field $X_1^2 := df(X_1^1)$ on $V_2 \times [0,1]$ directs the characteristic foliation defined by the form $h_s^*(\alpha_2 + xdy)$. Let N_a^1 and N_a^2 denote metric a-neighborhoods of ∂V_1 in V_1 and ∂V_2 in V_2, respectively. Then $f(N_\sigma^1) \subset N_{C_1\sigma}^2$. Hence, all trajectories of X_1^2 originated in $V_2 \setminus N_{C_1\sigma}^2$ are blocked. On the other hand, the non-blocked trajectories originated in $N_{C_1\sigma}^2$ exit with a distortion for no more than $C_1\sigma$. All other properties of a $C_1\sigma$-plug installation isotopy listed in Proposition 4.1 are straightforward. □

Consider a class $\mathcal{D}$ of $(2n-1)$-dimensional compact manifolds with boundary (and possibly with corners) which are contactomorphic to a domain in the standard contact $(\mathbb{R}^{2n-1}, dz + \lambda_{\text{st}})$

with boundary transverse to the contact vector field $\Upsilon = 2\frac{\partial}{\partial z} + \sum_1^{n-1} x_j \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j}$. For instance, the standard $(2n-1)$-dimensional contact ball belongs to $\mathcal{D}$.

Lemma 5.3. *If there exists a domain $V \in \mathcal{D}$ such that for any $\sigma > 0$ one can install a plug of height $K = K(\sigma, V)$ over $V \times [0,1]$, then for any domain $V' \in \mathcal{D}$ and any $\sigma > 0$ one can install a plug of height $K' := K(\sigma, V'))$ over $V' \times [0,1]$. If there exists a domain $V \in \mathcal{D}$ such that for any $\sigma > 0$ and any $\epsilon > 0$ one can install a plug of height ϵ over $V \times [0,1]$, then the same is true for any domain $V' \in \mathcal{D}$.*

Proof. For any domain $(V, \operatorname{Ker}\alpha) \in \mathcal{D}$ its interior $\operatorname{Int} V$ is contactomorphic to the standard contact $\mathbb{R}^{2n-1}$, see [6]. Besides, the boundary ∂V is convex, and hence $\operatorname{Int} V = \bigcup V_j$, where V_j is contactomorphic to V, Hence, for any $V, V' \in \mathcal{D}$ $\operatorname{Int} V' = \bigcup V_j$, where V_j is contactomorphic to V. Hence, the statement follows from Lemma 5.2. □

5.2. **Scaling.** For $a, b > 0$ denote $R_{a,b} := \{|x_j|, |y_j| \le a, |z| \le b,\ j = 1, \dots, n-1\} \subset (\mathbb{R}^{2n-1}, dz + \lambda_{\mathrm{st}})$. Note that $R_{a,b} \in \mathcal{D}$.

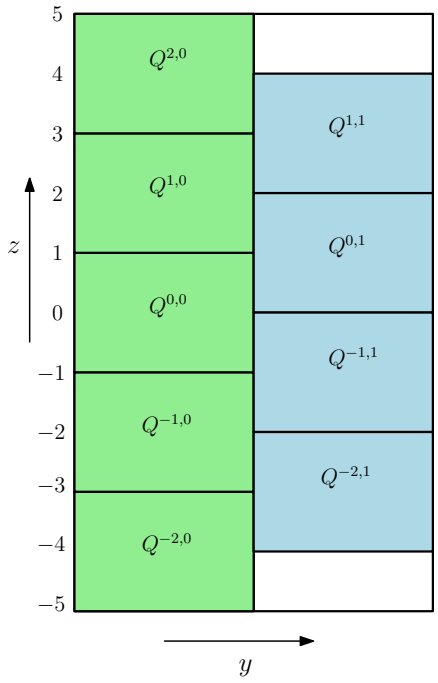

FIGURE 3. The arrangement of blocks $Q^{i,0}$, $i = 0, \pm 1, \pm 2$, and $Q^{i,1}$, $i = -2, \dots, 1$, for the case $N = 2$.

Lemma 5.4. *Choose $\sigma > 0$. Suppose that one can install a σ-plug of height K over $R_{1,1} \times [0,1]$. Then for any integer $N \ge 0$ one can install a σ-plug of height K over $R_{1,2N+1} \times [0,2]$.*

Proof. Denote $\widetilde{Q} := R_{1,2N+1} \times [0,2]$, $\widehat{U} := R_{1,2N+1} \times T^*[0,2]$. We assume $\widehat{U}$ is endowed with the contact form $dz + \lambda_{\mathrm{st}} + x dy$. Furthermore, for $i = 0, \pm 1, \dots, \pm N$ denote $Q^{i,0} := \{z \in [2i-1, 2i+1], y \in [0,1]\} \subset \widehat{Q}$, $U^{i,0} := \{z \in [2i-1, 2i+1], y \in [0,1]\} \subset \widehat{U}$, and for $i = -N, \dots, N-1$

denote $Q^{i,1} := \{z \in [2i, 2i+2], y \in [1,2]\} \subset \widehat{Q}$, $U^{i,0} := \{z \in [2i, 2i+2], y \in [1,2]\} \subset \widehat{U}$, see Fig. 3. Note that $Q^{0,0} = R_{1,1} \times [0,1]$ and $U^{0,0} = R_{1,1} \times T^*[0,1]$. The diffeomorphisms $(z,y) \overset{\Pi^{i,0}}{\mapsto} (z+2i, y)$ and $(z,y) \overset{\Pi^{i,1}}{\mapsto} (z+1+2i, y+1)$ preserves the contact form $dz+\lambda_{\mathrm{st}}+xdy$ and identify $(U := R_{1,1} \times T^*[0,1], Q := R_{1,1} \times [0,1])$ with $(U^{i,0}, Q^{i,0})$ and $(U^{i,1}, Q^{i,1})$, respectively. Let $h_s : Q \to U$ be an isotopy installing a plug of height K over $Q = R_{1,1} \times [0,1]$. Then the isotopy $g_s : \widehat{Q} \to \widehat{U}$ which is equal to $\Pi^{i,0} \circ g_s \circ (\Pi^{i,0})^{-1}$ on $Q^{i,0}$, $i = 0, \dots, \pm N$, and to $\Pi^{i,1} \circ g_s \circ (\Pi^{i,1})^{-1}$ on $Q^{i,1}$, $i = -N, \dots, N-1$ is installing the required σ-plug of height K over $R_{1,2N+1} \times [0,2]$. □

Lemma 5.5. *If one can install a σ-plug of height K over $R_{1,1} \times [0,1]$ then for any integer $N \geq 0$ one can install a $\frac{\sigma}{2N+1}$-plug of height $\frac{2K}{(2N+1)^2}$ over $R_{\frac{1}{2N+1},1} \times [0,1]$.*

Proof. By applying Lemma 5.4 we install a σ-plug of height K over $R_{1,(2N+1)^2} \times [0,2]$. This is equivalent to a σ-plug of height $2K$ over $R_{1,(2N+1)^2} \times [0,1]$. Let h_s be an isotopy which installs this plug. Consider a contactomorphism

$$\begin{aligned}&(x_1, y_1, \dots, x_{n-1}, y_{n-1}, x, y, z)\\ &\overset{f}{\mapsto} \left(\frac{x_1}{2N+1}, \frac{y_1}{2N+1}, \dots, \frac{x_{n-1}}{2N+1}, \frac{y_{n-1}}{2N+1}, \frac{x}{(2N+1)^2}, y, \frac{z}{(2N+1)^2}\right).\end{aligned}$$

Then the isotopy $f \circ h_s \circ (f)^{-1}$ is installing the required $\frac{\sigma}{2N+1}$-plug of height $\frac{2K}{(2N+1)^2}$ over $R_{\frac{1}{2n+1},1} \times [0,1]$. □

Lemma 5.5 and Proposition 5.1 imply

Corollary 5.6. *For any $\sigma, \epsilon > 0$ and $p \in D^{2n-2}$ there exists N_0 such that for any $N \geq N_0$ one can install a $\frac{\delta}{N}$-plug of height ϵ over $R_{\frac{1}{N},1} \times [0,1]$.*

For a point $p = (a_1, b_1, \dots, a_{n-1}, b_{n-1}) \subset \mathbb{R}^{2n-2}$ consider a map

$$\tau_p : \mathbb{R}^{2n-1} = \mathbb{R}^{2n-2} \times \mathbb{R} \to \mathbb{R}^{2n-2} \times \mathbb{R}$$

given by the formula

$$\begin{aligned}&\tau_p(x_1, y_1, \dots, x_{n-1}, y_{n-1}, z)\\ &= \left(x_1 + a_1, y_1 + b_1, \dots, x_{n-1} + a_{n-1}, y_{n-1} + b_{n-1}, z - \sum_1^{n-1} a_j y_j - b_j x_j\right).\end{aligned}$$

Note the τ_p preserves the contact form $\lambda_{\mathrm{st}} + dz$:

$$\tau_p^*(\lambda_{\mathrm{st}} + dz) = \lambda_{\mathrm{st}} + dz.$$

Fix an integer $N > 1$. Given an integer vector

$$I := (i_1, j_1, \dots, i_{n-1}, j_{n-1}) \in [1-N, N-1]^{2n-2}$$

denote $p_I = \frac{I}{N}$, $\delta_N := \frac{2n-2}{N}$ and

$$P_I := \tau_{p_I}\left(R_{\frac{1}{N},1}\right),\ P_{I,-} := \tau_{p_I}\left(R_{\frac{2}{3N},1-\delta_N}\right),\ \text{see Fig. 4.}$$

Note that we have

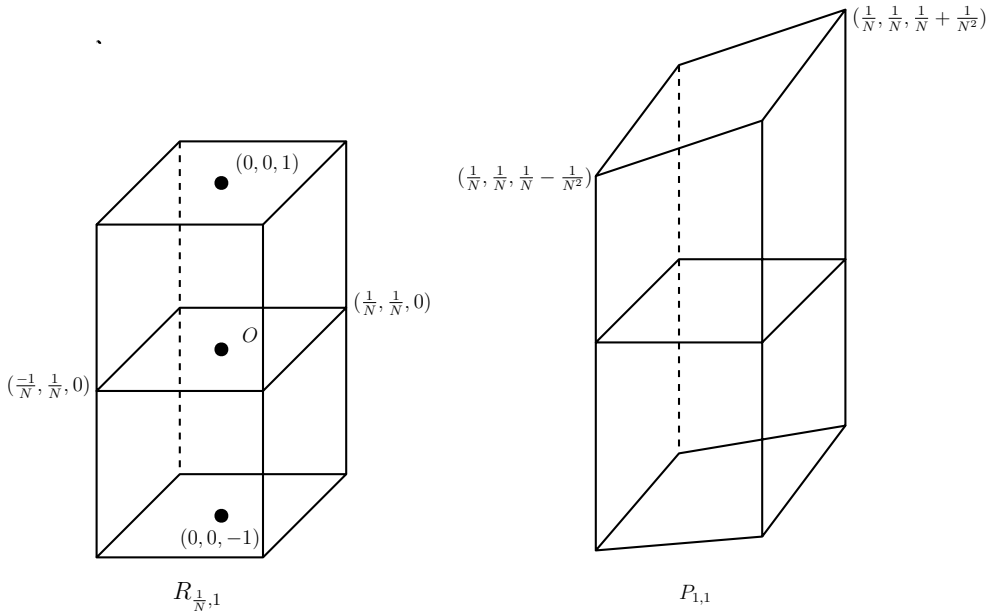

FIGURE 4. Domain $R_{\frac{1}{N},1}$ and its image $P_{1,1}$ under the shear $\tau_{p_{11}}$.

Lemma 5.7.

$$\text{Int}\left(R_{1-\frac{1}{3N},1-\frac{2n-2}{N}}\right) \subset \bigcup_{I\in[1-N,N-1]^{2n-2}} \text{Int}\,(P_{I,-})) \subset \text{Int}\,(R_{1,1})$$

and the multiplicity of the covering $\bigcup_{I\in[1-N,N-1]^{2n-2}} \text{Int}\,(P_{I,-})) \supset \text{Int}\left(R_{1-\frac{1}{3N},1-\frac{2n-2}{N}}\right)$ *is equal to* 2^{2n-2}.

Reduction of Proposition 4.1 to Proposition 5.1. Suppose that for any $\sigma > 0$ one can install a σ-plug of some height $K = K(\sigma)$ over $D^{n-1} \times [0, 1]$. As the domain $R_{1,1}$ belongs to the class $\mathcal{D}$ it follows that for any $\sigma > 0$ one can install a σ-plug of some height $K' = K'(\sigma)$ over $R_{1,1} \times [0, 1]$.

We can assume $\sigma < \frac{1}{4}$. Set $\widehat{\sigma} := \frac{\sigma}{2^{2n-2}\sqrt{2n}}, \widehat{\epsilon} := \frac{\epsilon}{2^{2n-2}}$. Let N be the integer provided by Corollary 5.6 for the pair $(\widehat{\sigma}, \widehat{\epsilon})$. In other words, one can install a $\frac{\widehat{\sigma}}{N}$-plug of height $\widehat{\epsilon}$ over $R_{\frac{1}{N},1} \times [0, 1]$. Choosing N large enough we will ensure that $\frac{2n}{N} < \sigma$ which implies that $\text{dist}(\partial R_{1-\frac{1}{3N},1-\frac{2n-2}{N}}, \partial R_{1,1}) < \sigma$. Note that $||d\tau_p|| \leq \sqrt{2n}$ for any $p = (a_1, b_1, \ldots, a_{n-1}, b_{n-1}) \in [-1, 1]^{2n-2}$. Hence, we can apply Lemma 5.2 to install a $\frac{\sqrt{2n}\widehat{\sigma}}{N}$-plug of height $\widehat{\epsilon}$ over $P_I \times [0, 1]$ for any I.

Let us partition the set $\mathcal{I} = [1 - N, N - 1]^{2n-2}$ of all indices into 2^{2n-2} subsets $\mathcal{I}_A$ indexed by subsets $A \subset \{1, \ldots, 2n - 2\}$: the subset $\mathcal{I}_A$ consists of $I = (i_1, j_1, \ldots, i_{n-1}, j_{n-1})$ which have odd entries at positions of the subset A, and even at other places. For instance, for $A = \varnothing$ the set $\mathcal{I}_\varnothing$ consists of $I = (i_1, j_1, \ldots, i_{n-1}, j_{n-1})$, where all i_k, j_k are even. Note that for $I, I' \in \mathcal{I}_A$, $I \neq I'$ we have $\text{Int}\, P_I \cap \text{Int}\, P_{I'} = \varnothing$. We enumerate all subsets $A \subset \{1, \ldots, 2n-2\}$ as $A_1, \ldots, A_{2^{2n-2}}$, and write $\mathcal{I}_j$ instead of $\mathcal{I}_{A_j}$.

We claim that by installing for each $I \in \mathcal{I}_j$, $j = 1, \dots, 2^{2n-2}$ a $\frac{\sqrt{2n}\widehat{\sigma}}{N}$-plug of height $2^{2n-2}\widehat{\epsilon} = \epsilon$ over $P_I \times [\frac{j-1}{2^{2n-2}}, \frac{j}{2^{2n-2}}]$ we construct the required σ-plug of height ϵ over $R_{1,1} \times [0,1]$. Indeed, let $h_s : R_{1,1} \times [0,1] \to R_{1,1} \times T^*[0,1]$ be the resulting isotopy. First, note that $h_s(R_{1,1} \times [0,1]) \subset \{|x| \leq 2^{2n-2}\widehat{\epsilon} = \epsilon$. Let us verify that the vector field X_1 directing the characteristic foliation ℓ_1 induced by $h_1^*(\alpha_{\text{st}} + xdy)$ is a σ-plug. As there are 2^{2n-2} layers of plugs, each trajectory γ of X_1 beginning at $(p,0) \in R_{1,1} \times 0$ intersects no more than 2^{2n-2} plugs $P_I \times [\frac{j-1}{2^{2n-2}}, \frac{j}{2^{2n-2}}]$. Each of these plugs either blocks γ, or displaces it for no more than $\frac{\sqrt{2n}\widehat{\sigma}}{N}$. Hence if γ exits through a point $(p',1) \in R_{1,1} \times 1$ then $\text{dist}(p,p') < 2^{2n-2}\sqrt{2n}\frac{\widehat{\sigma}}{N} = \frac{\sigma}{N}$. On the other hand, if $\text{dist}(p, \partial R_{1,1}) \geq \sigma > \frac{2n}{N}$ then $p \in P_{I,-}$ for a multi-index $I \in \mathcal{I}_j$ for some $j = 1, \dots, 2^{2n-2}$. If the trajectory γ originates at $(p,0)$ and it is not blocked by any of the plugs on the layers $[\frac{i-1}{2^{2n-2}}, \frac{i}{2^{2n-2}}]$ for $i < j$ then by the above argument it enters the plug $P_{I,-} \times [\frac{j-1}{2^{2n-2}}, \frac{j-1}{2^{2n-2}}]$ through a point $(p', \frac{j-1}{2^{2n-2}})$ with $\text{dist}(p,p') < \frac{\sigma}{N}$. Hence, $\text{dist}(p', \partial P_I) > \text{dist}(\partial P_I, \partial P_{I,-}) - \frac{\sigma}{N} > \frac{1}{3N} - \frac{1}{4N} > \frac{\sigma}{3N} > \frac{\sqrt{2n}\widehat{\sigma}}{N}$. But $(P_{I,-} \times [\frac{j-1}{2^{2n-2}}, \frac{j-1}{2^{2n-2}}], X_1)$ is a $\frac{\sqrt{2n}\widehat{\sigma}}{N}$-plug, and therefore, the trajectory γ is blocked inside $P_{I,-} \times [\frac{j-1}{2^{2n-2}}, \frac{j-1}{2^{2n-2}}]$. This verifies the property a) of Proposition 4.1. Property b) follows from the fact that it holds for plugs $(P_{I,-} \times [\frac{j-1}{2^{2n-2}}, \frac{j-1}{2^{2n-2}}, X_1)]$ for each $I \in \mathcal{I}$ and transversality arguments.

Finally, we again apply Lemma 5.3 to conclude that the installation for any σ, ϵ of a σ-plug of height ϵ over $R_{1,1}$ is equivalent to the installation for any σ, ϵ of a σ-plug of height ϵ over $D^{2n-1} \times [0,1]$, because both domains D^{2n-1} and $R_{1,1}$ belong to the class $\mathcal{D}$. □

Remark 5.8. Note that the above proof of the height-reduction for σ-plugs significantly simplifies for $n = 1$, i.e. when the plug is 2-dimensional. Indeed, in this case $D^{2n-1} = R_{1,1} = R_{\frac{1}{N},1} = [-1,1]$, and the claim follows directly from Corollary 5.6.

6. The 2-dimensional case

We will prove in this section Proposition 5.1 (and hence, Proposition 4.1) in the 2-dimensional case. In fact, we will establish a stronger statement, Lemma 6.4, which will enable us to continue the construction by induction on dimension of the plug.

6.1. **Creation and elimination of singularities of a 2-dimensional characteristic foliation.**

The following statement is a slight modification of the Giroux-Fuchs elimination lemma, see [9].

Lemma 6.1. *Let Σ be a 2-dimensional surface in a contact 3-manifold $(M, \xi = \text{Ker}\,\alpha)$. Let $p \in \Sigma$ be a non-singular point of the characteristic foliation $\ell = \ell_{\Sigma,\xi}$. Let $\gamma \ni p$ be an arc of the leaf of ℓ through p. Suppose that $(d\alpha)_p > 0$ Then for any positive ϵ and $\sigma \ll \epsilon$ there exists an ϵ-small 2-parametric isotopy $\phi_{t,s} : \Sigma \to (M, \xi)$ supported in an ϵ-neighborhood of $p \in \Sigma$ with the following properties. Denote $\beta_{s,t} := \phi_{s,t}^*\alpha$ and let $\ell_{s,t}$ be the characteristic foliation defined by $\beta_{s,t}$, and $X_{s,t}$ the vector field directing $\ell_{s,t}$.*

- *$\phi_{0,t} = \phi_{0,0}$ is the inclusion $\Sigma \hookrightarrow M$;*
- *the 1-form $\beta_{s,1}$ has no zeros for all $s \in [0,1]$;*
- *the 1-form $\beta_{1,0}$ has exactly two zeros, one positive elliptic and one hyperbolic on the arc γ;*
- *the arc γ is tangent to $X_{s,t}$ for all $s, t \in [0,1]$.*

- *$d\beta_{s,t} = d\beta_{0,0}$ for all $s, t \in [0,1]$;*
- *If ℓ admits a Lyapunov function $f : \Sigma \to \mathbb{R}$ then for all $s, t \in [0,1]$ the characteristic foliation $\ell_{s,t}$ admits a Lyapunov function $f_{s,t}$ which coincides with f outside an ϵ-neighborhood of p.*

We split the proof into two parts.

Lemma 6.2. *Consider the form $\delta = dz + dx + xdy$ in $\mathbb{R}^3$. Under assumptions of Lemma 6.1 there is a neighborhood $U \ni p$ in M, a neighborhood U' of 0 in $\mathbb{R}^3$ and a contactomorphism $h : (U', \delta) \to (U, \alpha)$ such that $h^{-1}(\Sigma \cap U) = \mathbb{R}^2 \cap U'$.*

Proof. We first find a diffeomorphism $h_1 : \mathcal{O}p_{\,\mathbb{R}^2}(0) \to \mathcal{O}p_{\,\Sigma}(p)$ such that $h_1^* d\alpha = dx \wedge dy$, and then compose it with a symplecomorphism to equate the pull-back of α with $dx + xdy$. Finally, we evoke Proposition 3.2 to conclude that two contact structures with the same restriction to a surface are contactomorphic via a contactomorphism fixed on the surface. □

Lemma 6.3. *For any $\epsilon \gg \sigma > 0$ there exists a 2-parametric family of C^∞-functions $G_{s,t} : \mathbb{R}^2 \to \mathbb{R}$, $s, t \in [0,1]$ which are supported in $\{|x|, |y| < \epsilon\}$ which have the following properties. Denote $\alpha_{s,t} := xdy + dx + dG_{s,t}$.*

- *$G_{0,0} = 0$;*
- *$G_{s,t} - G_{s,0}$ is supported in $\{x < \sigma, y < \epsilon\}$.*
- *the 1-form $\alpha_{s,1}$ has no zeros for all $s \in [0,1]$;*
- *the 1-form $\alpha_{1,0}$ has exactly two positive zeroes, one elliptic and one hyperbolic, on the line $\{x = 0\}$.*
- *for all $s, t \in [0,1]$ the vector field $Y_{s,t}$ directing the characteristic foliation on $\{x = 0\}$ generated by $\alpha_{s,t}$ admits a Lyapunov function which is equal to y outside of a compact set.*

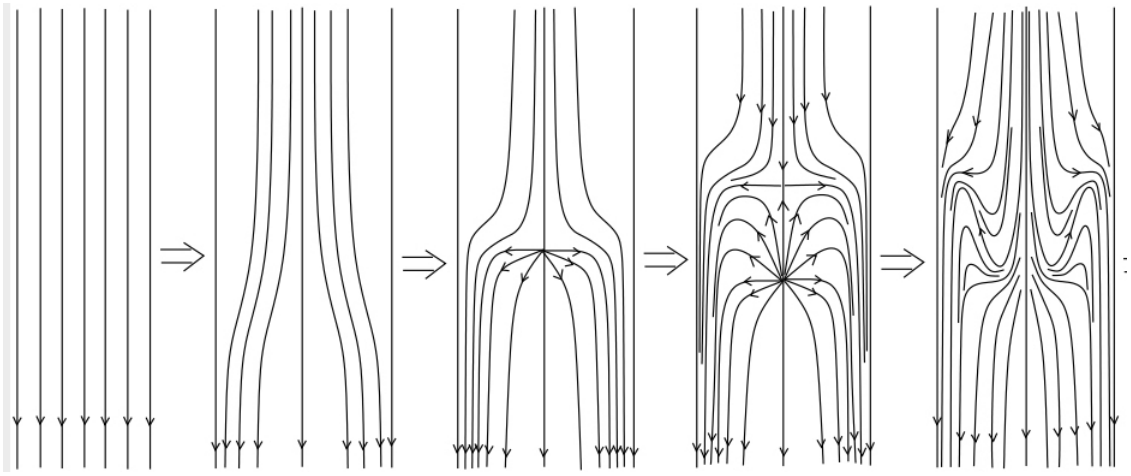

FIGURE 5. Creating and eliminating zeroes.

Proof. Consider an even function $\theta : \mathbb{R} \to \mathbb{R}_+$, $\theta(u) = \theta(-u)$, and for any $c > 0$ consider an odd functions $\eta_c : \mathbb{R} \to \mathbb{R}$, $\eta_c(u) = -\eta_c(u)$ such that the following properties are satisfied:

- $\theta(u) = u^2 - 1 - \frac{\epsilon^2}{9}$ on $u \in [-\epsilon/2, \epsilon/2]$, $\theta(u) = 0$ for $|u| \geq \epsilon$ and $0 \leq \theta'(u)$ for $u \geq 0$;
- $\eta_c(u) = u$ for $u \in [-c\sigma, c\sigma]$, $\eta_c(u) = 0$ for $|u| \geq c\epsilon$, $|\eta_c(u)| \leq |u|$ and $-\frac{2\sigma}{\epsilon} \leq \eta'(u) \leq 1$ for $u \geq 0$.

Consider a family of functions $G_s,\ s \in [0,1]$ by the formula

$$G_s(x,y) = s\theta(y)\eta_1(x)$$

and a family of 1-forms

$$\alpha_s = dx + xdy + dG_s = (1 + s\theta(y)\eta_1'(x))dx + (x + s\theta'(y)\eta_1(x))dy = f_s(x,y)dx + g_s(x,y)dy.$$

Let us check that the form α_1 have a hyperbolic 0 at the points $(x = 0, y = -\frac{\epsilon}{3})$, an elliptic zero at $(x = 0, y = \frac{\epsilon}{3})$ and no other zeroes. We have

$$\begin{aligned} &|g_1(x,y)| = |x + 2y\eta_1(x)| \geq |x|(1 - 2|y|) \geq (1-\epsilon)|x| \ \text{ for } \ |y| \leq \frac{\epsilon}{2}; \\ &f_1(x,y) = 1 + \left(y^2 - 1 - \frac{\epsilon^2}{9}\right)\eta_1'(x) \geq \frac{5\epsilon^2}{36} > 0, \ \ |y| \geq \frac{\epsilon}{2}. \end{aligned} \tag{3}$$

Hence α_1 has only zeros along the interval $\{x = 0, -\frac{\epsilon}{2} < y < \frac{\epsilon}{2}\}$. In the neighborhood of this interval we have $\alpha_1 = (y^2 - \frac{\epsilon^2}{9})dx + (1 + 2y)xdy$, which has 2 zeroes, elliptic and hyperbolic, respectively at the points $(x = 0, y = \frac{\epsilon}{3})$ and $(x = 0, y = -\frac{\epsilon}{3})$.

Let us now extend the family G_s to the 2-parametric family of functions $G_{s,t} : \mathbb{R}^2 \to \mathbb{R}$ by setting

$$G_{s,t} = H_s(x,y) - st\theta(y)\eta_\sigma(x) = f_{s,t}(x,y)dx + f_{s,t}(x,y)dy.$$

Let us verify that the form $\alpha_{s,1} = dx + xdy + dG_{s,1}$ has no zeros. First, note that for $|x| \leq \frac{\sigma^2}{\epsilon}$, we have $f_{s,1} = 1$ and $g_{s,1}(x,y) = x$.

Similarly to the above estimates (3) for f_s and g_s we conclude that $f_{s,1} \neq 0$ for $|y| \geq \frac{\epsilon}{2}$ and $x \neq 0$ and $g_{s,1} \neq 0$ $|y| \leq \frac{\epsilon}{2}$.

It remains to show existence of a family of Lyapunov functions for the family of vector fields $Y_{s,t}$. According to Corollary 3.4 it is sufficient to verify for $Y_{s,t}$ the property (L1) and the Morse-Smale condition. Because it is the 2-dimensional case then by Poincaré-Bendixson's theorem it is sufficient to show that there are no periodic orbits. But any periodic orbit in R bounds a disc and the sum of indices of singular points in this disc should be equal to 1. On the other hand, the only 2 singular points of $Y_{s,t}$ are connected by a separatrix trajectory, and hence the disc bounded by a periodic orbit must enclose both singular points, whose sum of indices is equal to 0. □

6.2. **Special 2-dimensional plug.** We construct in this section a special 2-dimensional plug. In the contact space $(\mathbb{R}^3, \mathrm{Ker}\,\{dz + xdy\})$ consider

$$O := \{0 \leq y, z \leq 1, -4 \leq x \leq 0\}, \ \ R := O \cap \{x = 0\}.$$

For a sufficiently small ϵ let us choose non-decreasing C^∞-functions $\psi, \theta : [0,1] \to \mathbb{R}$ such that

- $\theta(y) = \epsilon$ for $y \in [0, \frac{1}{3}]$; $\theta(y) = 1 - \epsilon$ for $y \in [\frac{2}{3}, 1]$ and $0 < \theta'(y) < 4$ for $y \in (\frac{1}{3}, \frac{2}{3})$.
- $\psi(0) = 0$, $\psi_1(1) < \frac{\epsilon}{2}$;

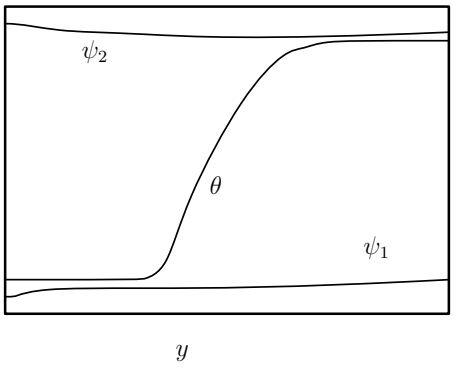

FIGURE 6. Graphs of functions Θ, Ψ_1, and Ψ_2.

- ψ has vanishing derivatives of all orders at the points 0 and 1, and $0 < \psi'(y) < \epsilon$ for $y \in (0,1)$.

Denote

$$\psi_1 := \psi + \frac{\epsilon}{2},\ \psi_2 := -\psi + (1 - \psi(1) - \frac{\epsilon}{2}).$$

and consider the graphs Θ, Ψ_1, Ψ_2 of the functions θ, ψ_1 and ψ_2:

$$\Theta := \{(y, \theta(y));\ y \in [0,1]\},\ \Psi_1 := \{(y, \psi_1(y));\ y \in [0,1]\},$$
$$\Psi_2 := \{(y, \psi_2(y));\ y \in [0,1]\} \subset R.$$

Denote

$$R_+ := \{(y,z) \in R, z \le \theta(y)\},\ R_- := \{(y,z) \in R, z \ge \theta(y)\}.$$

Lemma 6.4. *For any $\epsilon > 0$ there exists an isotopy $h_s : R \to O$, $s \in [0,1]$, which is fixed together with its ∞-jet along ∂R, constant for $s \in [0, \frac{1}{8}] \cup [\frac{7}{8}, 1]$, and such that the following properties a)–i) are satisfied. Denote $\beta_s := h_s^*(dz + xdy)$. Let Y_s be the vector field directing the characteristic foliation ℓ_s of β_s.*

a) *$d\beta$ restricted to the interior $\operatorname{Int} R_+$ of R_+ is positive, and $d\beta$ restricted to the interior $\operatorname{Int} R_-$ of R_- is negative for all $s \in (\frac{1}{8}, 1]$;*

b) *β_s has*
   - *no zeros for $s < \frac{3}{4}$,*
   - *a positive and negative embryos $o_+ := (\frac{1}{6}, \psi_1(\frac{1}{6})), o_- = (\frac{5}{6}, \psi_2(\frac{5}{6}))$ for $s = \frac{3}{4}$,*
   - *a pair*

$$e_+(s) = (e_+^1(s), \psi_1(e_+^1(s))), \hbar_+(s) = (\hbar_+^1(s), \psi_1(\hbar_+^1(s))),\ 0 < e_+^1(s) < \hbar_+^1(s) < \frac{1}{3}$$

   of positive elliptic and hyperbolic points, and a pair

$$e_-(s) = (e_-^1(s), \psi_1(e_-^1(s))), \hbar_-(s) = (\hbar_-^1(s), \psi_1(\hbar_-^1(s))),\ 1 > e_+^1(s) > \hbar_+^1(s) > \frac{2}{3}$$

   of negative elliptic and hyperbolic points for $s > \frac{3}{4}$.

c) *the incoming separatrices of $\hbar_+(s)$ and o_+ for Y_s, are contained in Ψ_1, and outgoing separatrices of $\hbar_-(s)$ and o_- for Y_s are contained in Ψ_2, $s \ge \frac{3}{4}$;*

d) *there exists* $\epsilon_1 \in (0,\epsilon)$ *such that the outgoing separatrices of* $\hbar_+(s)$ *for* Y_1 *terminate at* e_- *and* $(1,\epsilon_1)$*, and the incoming separatrices of* $\hbar_-(s)$ *for* Y_1 *originate at* e_+ *and* $(0,1-\epsilon_1)$*;*

f) Y_s *for* $s \in [\frac{1}{2},1]$ *is outward transverse to the graph* Θ*, viewed as a part of the boundary of the domain* R_+*;*

g) Y_s *admits a family of good Lyapunov function* $\psi_s : R \to \mathbb{R}$ *such that* $\psi_s|_{\mathcal{O}p\,\partial R} = y$*;*

h) $\beta_s(\frac{\partial}{\partial z}) > 0$ *everywhere in* R *for* $s \in [\frac{1}{2},1]$*;*

i) *for any* $\sigma > 0$ *the isotopy* h_s*,* $s \in [0,1]$*, can be extended to a 2-parametric isotopy* $h_{s,t}, 0 \le s,t \le 1$*, such that*
   - $h_{s,0} = h_s$*,* $h_{0,t} = h_0$ *for all* $s,t \in [0,1]$*;*
   - $h_{s,t} = h_s$ *for* $s \le \frac{3}{4} - \sigma, t \in [0,1]$*;*
   - *for each* $s \in (\frac{3}{4},1]$ *the isotopy* $h_{s,t}, t \in [0,1]$*, is supported in a* σ*-neighborhood of the separatrices connecting* $\hbar_\pm(s)$ *with* $e_\pm(s)$*; for each* $s \in [\frac{3}{4}-\sigma,1]$ $h_{s,t}$ *is supported in a* σ*-neighborhood of* $o_\pm$*;*
   - $h_{s,t}$ *is* σ*-close in the* C^0*-sense to* $h_{s,0}$ *for all* $s,t \in [0,1]$*;*
   - *the family of vector fields* $Y_{s,t}$ *directing the characteristic foliations* $\ell_{s,t}$ *of* $\beta_{s,t} := h_{s,t}^*\alpha$ *admits a family good Lyapunov functions* $\psi_{s,t} : R \to \mathbb{R}$ *such that* $h_{s,t}|_{\mathcal{O}p\,\partial R} = y$*;*
   - $Y_{s,t}$ *has a pair of positive elliptic and hyperbolic zeroes at* $e_\pm(s(1-2t)+\frac{3t}{2})$ *and* $\hbar_\pm(s(1-2t)+\frac{3t}{2})$ *for* $s > \frac{3}{4}, t < \frac{1}{2}$*, pairs of embryos at* $o_\pm$ *for* $s = \frac{3}{4}, t = \frac{1}{2}$ *and no zeroes otherwise;*
   - $h_{s,t} = h_0$ *for* $s \in [0,\frac{1}{8}], t \in [0,1]$*, and* $h_{s,t} = h_{1,t}$ *for* $s \in [\frac{7}{8},1]$ $t \in [0,1]$*;*
   - $X_{s,t}$ *is outwardly transverse to* Θ *for all* $s \in [\frac{1}{2},1],\ t \in [0,1]$.

Proof. Choose a function $H : R \to \mathbb{R}$ such that

(C1) H vanishes on ∂R together with all its derivatives;
(C2) $0 \ge H(y,z) \ge -4$, $y,z \in [0,1]$;
(C3) $H(y,\psi_1(y)) = H(y,\psi_2(y)) = -\psi'(y)$.
(C4) $\frac{\partial H}{\partial z}(y,z) = \begin{cases} < 0, & (y,z) \in R_+ \\ > 0, & (y,z) \in R_-; \end{cases}$
(C5) $H(y,\theta(y)) \le -\theta'(y)$.

An additional property (C6) will be imposed later.

Define an isotopy $h_s : R \to \mathbb{R}$, $s \in [0,1]$, as follows. For $s \in [0,1/2]$ we define

$$h_s(y,z) := (y,z,2sH(y,z)),\ \ (y,z) \in R.$$

Let ℓ_s be the characteristic foliation defined by $h_s^*\widehat{\mu}$ on R. Leaves of the characteristic foliation on ℓ_s are graphs of solutions of the equation

$$\frac{dz}{dy} = -2sH(y,z). \tag{4}$$

For $\zeta \in [0,1]$ we denote by ℓ_ζ and ℓ^ζ the solutions of (4) with the initial data $\ell_\zeta(0) = \zeta$ and $\ell^\zeta(3) = \zeta$, respectively. Condition (C3) ensures that $\ell_{\frac{\epsilon}{2}} = \psi_1$, $\ell^{\frac{\epsilon}{2}} = \psi_2$.

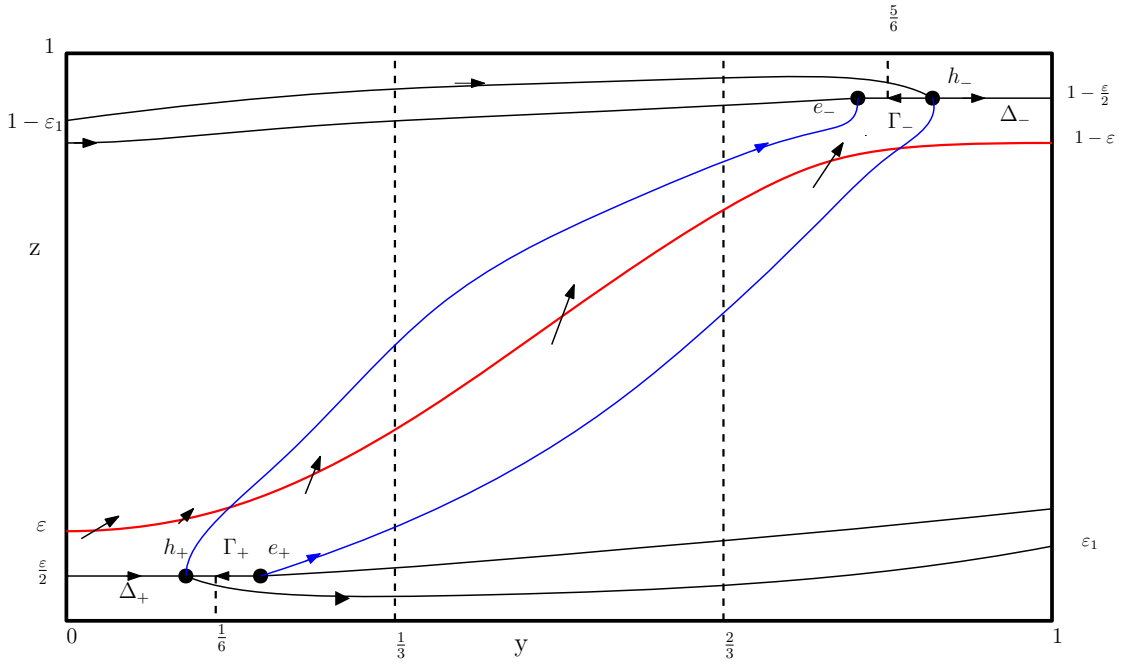

FIGURE 7. The characteristic foliation on $h_{\frac{4}{5}}(R)$. The red curve is the dividing set $\Gamma = \{d\mu = 0\}$ while blue curves depict separatrices connecting $e^{\pm}$ to $h^{\mp}$.

For $s \in [1/2, 1]$ we use Lemma 6.1 to create for $s > \frac{3}{4}$ pairs of elliptic-hyperbolic positive and negative points at $e_{\pm}(s), \hbar_{\pm}(s)$ through embryos at $o_{\pm}$ for $s = \frac{3}{4}$. The isotopy can be constructed arbitrary C^0-small and supported in a neighborhood of separatrices connecting $e_{\pm}$ and $\hbar_{\pm}$. It can also be arranged that the isotopy also fixes the leaves Ψ_1 and Ψ_2 of the foliation $\ell_{\frac{1}{2}}$, so that these leaves become broken leaves of the characteristic foliation ℓ_s for $s \in [\frac{3}{4}, 1]$. In particular, these curves contain, respectively, the incoming separatrix of $\hbar_+(s)$ and outgoing separatrix of $\hbar_-(s)$. It then follows that one of the outgoing separatrices of $\hbar_+(s)$ terminates at $(1, \epsilon_1)$ for $\epsilon_1 < \epsilon$, and it could be arranged that one of the incoming separatrices of $\hbar_-$ originates at $(1, 1-\epsilon_1)$.

Suppose that the second outgoing separatrix of $\hbar_+(1)$ intersects the line $y = \frac{1}{3}$ at a point $(\frac{1}{3}, a)$, $a \in (\psi_1(\frac{1}{3}), \theta(\frac{1}{3}))$, while the second incoming separatrix of $\hbar_-(1)$ intersects the line $y = \frac{2}{3}$ at a point $(\frac{2}{3}, b)$, $b \in (\theta(\frac{2}{3}), \psi_2(\frac{2}{3}))$. We now impose the remaining condition on the function H:

(C6) $H(y, z) < -3$ for $y \in [1, 2], z \in [a, b]$;

This guarantees that one of the outgoing separatrices of $\hbar_+(1)$ terminates at $e_-(1)$, and one of the incoming separatrices of $\hbar_-(1)$ originates at $e_+(1)$.

Using the extension to the 2-parametric isotopy in Lemma 6.1 we extend the isotopy h_s to a 2-parametric isotopy $h_{s,t}$ for $s, t \in [0, 1]$ with the required properties. □

Let us denote by $\Gamma_{\pm}(s)$, $s \in [\frac{3}{4}, 1]$ the (closure of the) trajectory of Y_s connecting $e_{\pm}(s)$ and $\hbar_{\pm}(s)$ and by $\Delta_+(s)$ (resp. $\Delta_-(s)$) the (closure of the) incoming (resp. outgoing) separatrix

of $\hbar_+(s)$ (resp. $\hbar_-(s)$). For $s = \frac{3}{4}$ we assume that $e_\pm(\frac{3}{4}) = \hbar_\pm(\frac{3}{4}) = o_\pm$. We extend the definition of $\Gamma_\pm(s)$ and $\Delta_\pm(s)$ to all $s \in [0,1]$ by setting $\Gamma_\pm(s) = \Delta_\pm(s) = \varnothing$ for $s < \frac{3}{4}$.

7. Preliminary and quasi-plugs

7.1. **Preliminary plug.** Let (W, λ) be a Weinstein domain. We denote by Z the Liouville field dual to λ. Consider an interior boundary collar $C := \partial W \times [1-\epsilon, 1]$ such that $\partial W \times 1 = \partial W$ and $\lambda|_C = \tau\gamma$, $\tau \in [1-\epsilon, 1]$, for a contact form $\gamma = \lambda|_{\partial W}$. Denote $W_0 := W \setminus C$. Furthermore, denote by $\mathrm{Skel}(W, Z)$ the skeleton of W, i.e. the union of stable manifolds of zeroes of Z. Alternatively, $\mathrm{Skel}(W, Z) = \bigcap\limits_{s\in[0,\infty)} Z^{-s}(W)$. Here we denote by Z^{-s} the flow of $-Z$, which is defined for all $s \geq 0$.

Consider a contact manifold $(V := W \times [0,1], \mathrm{Ker}\,(\lambda + dz))$, and in $(V \times T^*[0,1], \mathrm{Ker}\,(\lambda + dz + xdy))$ take the domain $U = \{-4 \leq x \leq 0\}$ and a hypersurface $Q = \{x = 0\}$. Note that $Q = V \times [0,1]$, and we can naturally identify $V \times T^*[0,1]$ and Q with $W \times O$ and $W \times R$ respectively, where we use the notation O, R introduced above in Section 6.2.

Let $h_{s,t} : R \to O$ be the isotopy constructed in Lemma 6.4. Define an isotopy

$$g_{s,t} : Q = W \times R \to U = W \times O$$

by the formula

$$g_{s,t}(w,q) = \begin{cases} (w, h_{s,t}(q)), & w \in W_0, q \in R \\ (w, h_{s\overline{\tau},t}(q)), & w = (v,\tau) \in C = \partial W \times [1-\epsilon.\epsilon],\ \overline{\tau} = \frac{1-\tau}{\epsilon} \end{cases}$$

Denote $\widehat{\beta}_{s,t} := g_{s,t}^*(\lambda + dz + xdy)$ and let $X_{s,t}$ be the vector field directing the characteristic foliation defined by the form $\widehat{\beta}_{s,t}$. Set $\widehat{\beta}_s := \widehat{\beta}_{s,0}$ and $X_s := X_{s,0}$. It follows from the corresponding properties of the isotopy $h_{s,t}$ that for each fixed $s \in [0,1]$ the isotopy $g_{s,t}$, $t \in [0,1]$, is σ-close to g_s in the C^0-sense.

Denote

$$\begin{aligned} &{}^{in}V := V \times 0 \subset Q,\ \ {}^{out}V := V \times 1 \subset Q, \\ &{}^{in}P := W_0 \times [\epsilon, 1 - \frac{\epsilon}{2}] \times 0 \subset {}^{in}V,\ \ {}^{out}P := W_0 \times [\epsilon_1, 1 - \frac{\epsilon}{2}] \times 1 \subset {}^{out}V, \\ &{}^{in}T := W_0 \times (1 - \frac{\epsilon}{2}, 1] \times 0 \subset {}^{in}V,\ \ {}^{out}T := W_0 \times [0, \epsilon_1) \times 1 \subset {}^{out}V. \end{aligned}$$

Furthermore, denote

$$\widehat{\Gamma}_\pm := W_0 \times \Gamma_\pm(1) \cup \bigcup_{u\in\partial W,\ 1-\epsilon\leq\tau\leq 1} (u,\tau) \times \Gamma_\pm(\overline{\tau}) \subset Q,\ \overline{\tau} = \frac{1-\tau}{\epsilon}.$$

The following proposition lists the necessary for our application properties of the family of vector field $X_{s,t}$.

Proposition 7.1. *The vector field $X_{s,t}$ on Q, $s,t \in [0,1]$, has the following properties.*

(1) *every trajectory of X_1*
   (a) *which starts at ${}^{in}P$ converges to a zero of X_1;*

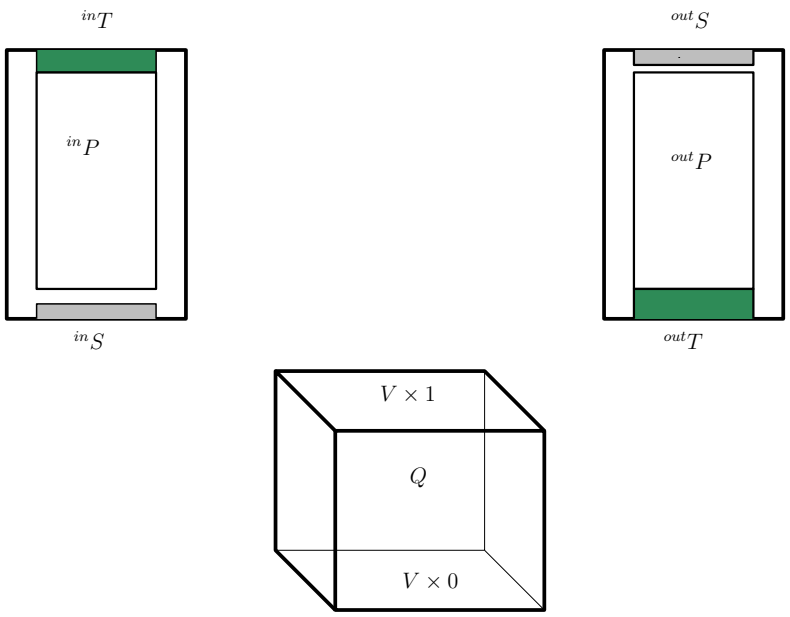

FIGURE 8. Regions in the preliminary block with controlled dynamics.

(b) *which starts at a point* $(u, z, 0) \in {}^{in}T$ *exits at a point* $(u', z', 1) \in V$*, then* $u' = Z^{-a}(u), a > 0$ *and* $z' \in (1 - \frac{\epsilon}{2}, 1]$*;*
(c) *which ends* ${}^{out}P$ *originates at a zero of* X_1*;*
(d) *which terminates at a point* $(u, z, 1) \in {}^{out}T$ *begins at a point* $(u', z', 0)$ *such that* $u' = Z^{-a}(u), a > 0$*, and* $z' \in [0, \frac{\epsilon}{2})$*;*

(2) *vector fields* $X_{s,1}$*,* $s \in [0, 1]$*, have no zeroes;*
(3) $\widehat{\Gamma}_+$ *is invariant with respect to the negative flow of* X_1*, and* $\widehat{\Gamma}_-$ *is invariant with respect to the positive one;*
(4) *All trajectories of* X_s *which converge to positive singularities either do not intersect* ∂Q *and are contained in* $\widehat{\Gamma}_+$*, or intersect it at points of* $\mathrm{Skel}(W) \times \frac{\epsilon}{2} \subset {}^{in}V$*; all trajectories of* X_s *which originate at negative singularities either do not intersect* ∂Q *and are contained in* $\widehat{\Gamma}_-$*, or intersect it at points of* $\mathrm{Skel}(W) \times (1 - \frac{\epsilon}{2}) \subset {}^{out}V$*;*
(5) *the family of vector fields* $X_{s,t}$ *admits a family of good Lyapunov functions* $\psi_{s,t}$*, equal to* y *on* ∂Q*;*

We begin the proof with an explicit computation of the vector field $X_{s,t}$. Let Z denote the Liouville field on W corresponding to the Liouville form λ. Recall that $\lambda|_C = \tau\gamma$, $\tau \in [1 - \epsilon, 1]$, where γ is a contact form on ∂W. We have $Z|_C = \tau\frac{\partial}{\partial\tau}$. Let R be the Reeb vector field on ∂W lifted to $C = \partial W \times [1 - \epsilon, \epsilon]$ via the projection to the first factor.

Let us view $Q = W \times R$ as the fiber bundle over W. The form $\widehat{\beta}_{s,t}$ restricts to the fiber $w \times Q$, $w \in W$ as $\beta_{s,t}$ if $w \in W_0$ and for $w = (u, \tau) \in C = \partial W \times [1 - \epsilon, 1]$ as $\beta_{s\overline{\tau},t}$, where $\overline{\tau} = \frac{1-\tau}{\epsilon}$. To simplify the notation we will write $\widehat{\mu}$ instead of $\widehat{\beta}_{s,t}$, X instead of $X_{s,t}$, μ instead of $\beta_{s,t}$ and μ_τ instead of $\beta_{s\overline{\tau},t}$. Denote $\dot{\mu} := \frac{d\mu_\tau}{d\tau}$.

For each $\tau \in [1-\epsilon, 1]$ consider a form $\kappa_\tau := \mu_\tau - \tau\dot{\mu}_\tau$ on Q and choose a vector field Y_τ directing Ker κ_τ.

Consider a tangent to fibers $w \times Q, w \in W$, vector field Y which is equal to $X_{s,t}$ on $w \times Q$ for $w \in W_0$ and to Y_τ on $w \times Q$ for $w = (u, \tau) \in C = \partial W \times [1-\epsilon, 1]$.

Lemma 7.2. *We have*

1. *$X = Y + aZ$ over $W_0 \times Q$, where the function $a : Q \to \mathbb{R}$ is determined by the equation*

$$a\mu = \iota(Y)d\mu. \tag{5}$$

2. *$X = Y + aZ + bR$ over $C \times Q$, where the functions $a, b : Q \to \mathbb{R}$ are determined by the equations*

$$\begin{aligned} a\kappa_\tau &= \iota(Y_\tau)d\mu_\tau, \\ b\tau + \mu_\tau(Y_\tau) &= 0. \end{aligned} \tag{6}$$

Remark 7.3. Note that for any vector v and a symplectic form ω we have $\iota(v)\omega(v) = 0$. Hence, the equation (5) and the first equation (6) are always solvable for some function $a : Q \to \mathbb{R}$.

Proof. 1. According to Remark 7.3 we can solve equation (5) with respect to a function a. We have

$$\iota(aZ + Y)d(\mu + \lambda) = \iota(Y)d\mu + a\iota(Z)d\lambda = a(\mu + \lambda).$$

But this means that $aZ + Y$ is tangent to the characteristic foliation defined by the form $\mu + \lambda = \widehat{\mu}$.

2. Note that $d\widehat{\mu} = d\mu_\tau + d(\tau\gamma) = d\tau \wedge \dot{\mu_\tau} + d\mu_\tau + d\tau \wedge \gamma + \tau d\gamma$. Hence, we get that

$$\iota(Y + aZ + bR)d\widehat{\mu} = -(\dot{\mu}(Y_\tau) + b)d\tau + a\mu + a\tau\gamma = a(\mu + \tau\gamma),$$

where we used the second equation (6) to conclude that $\dot{\mu}(Y_\tau) + b = 0$. But this implies that

$$\widehat{\mu} \wedge (\iota(Y + aZ + bR)d\widehat{\mu}) = 0,$$

which means that $Y + aZ + bR$ generates the characteristic foliation on $C \times Q$ defined by the form $\widehat{\mu}$, as required.

□

Proof of Proposition 7.1. (1a) When a trajectory of X_1 which enters at a point of ${}^{in}P$ it is in the region where $X_1 = Y_1 + aZ$ with $a < 0$. Since a is negative the trajectory continues to remain in the region where the plug is given by $W_0 \times R$, and hence projects onto trajectories of $-Z$ and Y, when projected to the corresponding factors. Moreover, the projection to R remains in R_-, according to Property f) of Lemma 6.4, and therefore remains in the region where the coefficient a is negative. But in R_- any trajectory of Y_1 entering at a point in $[\epsilon, 1-\epsilon_1) \times 0$ converges to a negative zero of Y_1, while every trajectory of $-Z$ converges to a zero of Z. Hence, any trajectory of Y_1 entering through ${}^{in}P$ converges to a 0 of the vector field X_1.

(1b) If a trajectory enters at a $(u, z) \in {}^{in}T$ then similarly to 1a) we have $X_1 = Y_1 + aZ$ for a negative coefficient a, and therefore, it projects onto trajectories of $-Z$ and Y_1 in the

factors W and R. But every trajectory of Y_1 which enters through $(1-\epsilon_1, 1]\times 0$ exits R at a point of $[1-\frac{\epsilon}{2}, 1]$, and hence the corresponding trajectory of X_1 exits ${}^{out}V$ at a point $(u' = Z^{-c}(u), z')$ for some positive $c > 0$ and $z' \in (-\frac{\epsilon}{2}.1]$.

(1c) and (1d) follows from the same arguments as, respectively, (1a) and (1b) applied to the vector field $-X_1$.

Properties (2)-(4) are straightforward from the corresponding properties in Lemma 6.4 and Lemma 7.2.

(5) According to Corollary 3.4 it is sufficient to verify that

- each trajectory of $X_{s,t}$ either originates at $V\times 0$, or at a critical point of $X_{s,t}$;
- each trajectories of $X_{s,t}$ either terminates at $V\times 1$, or at a critical point of $X_{s,t}$;
- $X_{s,t}$ has no retrograde connections.

In addition to the Weinstein subdomain $W_0 = W\setminus \partial W\times(1-\epsilon, 1]$ consider also a larger subdomain $W_1 = W\setminus \partial W\times(1-\frac{\epsilon}{2}, 1]$. Denote $Q_\pm := W_1\times R_\pm, Q_b := (W\setminus W_1)\times R \subset Q$. We have $Q = Q_+\cup Q_-\cup Q_b$. Let us first analyze the forward trajectory $X^u_{s,t}(p) = (w(u), r(u))$, $u\in\mathbb{R}_+$ of a point $p = (w, r)\in Q$.

If $p\in Q_-$ and $r\in W_0$ then $w(u)$ belongs to the negative Liouville trajectory $\bigcup\limits_{\tau\geq 0} Z^{-\tau}(w)$ as long as $r(u)\in R_-$. Similarly, if $w\in W_1\setminus W_0 = \partial W\times[1-\epsilon, 1-\frac{\epsilon}{2}]$ the second coordinate of w decreases as long as $w(u)\in R_-$. But the Q-component $Y_{s,t}$ of $X_{s,t}$ is by construction inwardly transverse to the boundary of R_-, and hence, remains in R_- for all $u\geq 0$. Therefore, the trajectory either converges to a singular point of $X_{s,t}$, or exits through $V\times 1$.

Suppose $p\in Q_b$. Recall that the Q-component $Y_{s,t}$ of $X_{s,t}$ has in Q_b a positive projection to the y-direction. Hence, $X^u_{s,t}(p)$ either exits through $V\times 1$, or enters $Q_+\cup Q_-$. But in that case it only can enter Q_-, and therefore, the analysis of the previous case does apply.

Suppose now that $p\in Q_+$. If $w\in W_0$ then $w(u)$ moves along a positive Liouville trajectory of w, and if $w\in\partial W\times[1-\epsilon, 1-\frac{\epsilon}{2}]$, then the second coordinate of $w(u)$ increases as long as $r(u)\in R_+$. Hence, the trajectory either exits through $V\times 1$, or enters either Q_b or Q_-, and therefore, the previous analysis applies.

The incoming trajectories could be analyzed similarly, with exchanging Q_+ and Q_- cases. The absence of retrograde connections follow from (4).

□

7.2. **Approximating balls by Weinstein cylinders.** A hypersurface $\Sigma\subset(M,\xi)$ is said to have an *admissible corner* along a smooth hypersurface $S\subset\Sigma$ if

- S is a codimension 2 contact submanifold of (M,ξ);
- $\Sigma\cap\mathcal{O}p\, S = \Sigma_1\cup\Sigma_2$, where Σ_1 and Σ_2 are two manifolds with common boundary S which transversely intersect along S.

We will call the hypersurface $S\subset\Sigma$ the *corner locus* of Σ and denoted by $\mathrm{Corner}(\Sigma)$.

Suppose $\widetilde{\Sigma}$ is a smooth hypersurface, let us choose its tubular ϵ-neighborhood N and denote by $\widehat{\tau}$ the hyperplane field on N orthogonal to the fibers of the projection $N\to\widetilde{\Sigma}$. We say that a hypersurface with admissible corners $\Sigma\subset N$ is C^1 *ϵ-close to* $\widetilde{\Sigma}$ if all its tangent planes do not deviate for more than ϵ from $\widehat{\tau}$.

Given a Weinstein domain (W, λ), a domain U in a contact manifold (M, ξ) is called a *Weinstein cylinder* if there is given a contactomorphism $\phi : (W \times [0, a], \mathrm{Ker}\,(\lambda + dz)) \to (U, \xi|_U)$. We already encountered Weinstein cylinders in Proposition 7.1.

Note that the boundary of a Weinstein cylinder $\phi(W \times [0, a])$ is a hypersurface with admissible corners along $\mathrm{Corner}(U) = \partial W \times 0 \cup \partial W \times a$. We denote $\partial_- U := \phi(W \times 0)$, $\partial_+ U := \phi(W \times a)$.

For a general W the contact topology of the Weinstein cylinder $W \times [0, a]$ is very sensitive to the value of the parameter a. However, there is one exception (see e.g. [8]):

Lemma 7.4. *Let $D = D^{2n-2}$ be the unit ball on $\mathbb{R}^{2n-2}$ endowed with the Liouville form $\lambda_{\mathrm{st}} := \sum_1^{n-1}(x_j dy_j - y_j dx_j)$. Then for any $a > 0$ there is a contactomorphism*

$$\Delta_a := (D \times [0, a], \mathrm{Ker}\,(\lambda + dz)) \to \Delta_1 := (D \times [0, 1], \mathrm{Ker}\,(\lambda + dz)).$$

Hence, we will use the notation Δ for any Weinstein cylinder of the type $D \times [0, a]$.

Lemma 7.5. *Let D be the standard contact ball and $p_\pm \in \partial D$ its poles. Then for any $\epsilon > 0$ there exists a contact embedding $h : \Delta \to D$ such that $h(\partial_+\Delta) \subset \partial D$, $\partial D \setminus h(\partial_+\Delta)$ is contained in an ϵ-neighborhood of the pole p_- and $D \setminus \Delta$ is contained in the ϵ-neighborhood of ∂D.*

Proof. Consider a $(2n-2)$-dimensional open disc $B_- \subset \partial D$ of radius ϵ centered at p_- with boundary ∂B_- transverse to the characteristic foliation $\ell_{\partial D}$. Denote $D_+ := \partial D \setminus B_-$ By scaling the contact form along ∂D we can arrange that D_+ with the resulted form is the standard Liouville ball, and flowing for some time δ with the corresponding Reeb field Υ, inwardly transverse to D_+, we construct a Weinstein cylinder $\Delta = \bigcup_{t \in [0,\delta]} \Upsilon^t(D_+) \subset D$. Let $D', D'' \subset D$ be smaller standard contact balls such that $D' \subset \mathrm{Int}\,\Delta$, $D' \subset \mathrm{Int}\,D''$ and $D \setminus D''$ is in an ϵ-neighborhood of ∂D. Note that the space of contact embeddings of a standard contact ball into any connected contact manifold is connected. Hence, there exists a contact diffeotopy $h_t : D \to D$, $t \in [0, 1]$, which is fixed on $\mathcal{O}p\,\partial D$, and such that $h_1(D') = D''$. Then the Weinstein cylinder $h_1(\Delta)$ has the required properties. □

7.3. **Weinstein cylinders in a good position.** We say that Weinstein cylinders $V_1, \dots, V_k$ are in a *good position*, see Fig. 9, if

- $V_j \cap V_{j+2} = \varnothing$ for all $j = 1, \dots, k-2$;
- $\partial V_1 \setminus V_2 \subset \mathrm{Int}\;\partial_- V_1$;
- for each $j = 2, \dots, k$ we have $\partial V_j \setminus V_{j-1} \subset \mathrm{Int}\;\partial_+ U_j$;
- for each $j = 2, \dots, k-q$ we have $\partial V_j \setminus V_{j+1} \subset \mathrm{Int}\;\partial_+ V_j$;
- ∂V_j and ∂V_{j+1}, $j = 1, \dots, k-1$, intersect transversely along a codimension 2 contact submanifold S_j, and the orientations induced on S_j from $\partial V_{j+1} \setminus \mathrm{Int}\,V_j$ and from $\partial V_j \setminus \mathrm{Int}\,V_{j+1}$ are opposite.

Note that if $V_1, \dots, V_k$ are in a good position then $\partial\left(\bigcup_1^k V_i\right)$ is a piecewise smooth hypersurface with admissible corners, and it can be made Weinstein convex by a C^∞-small perturbation.

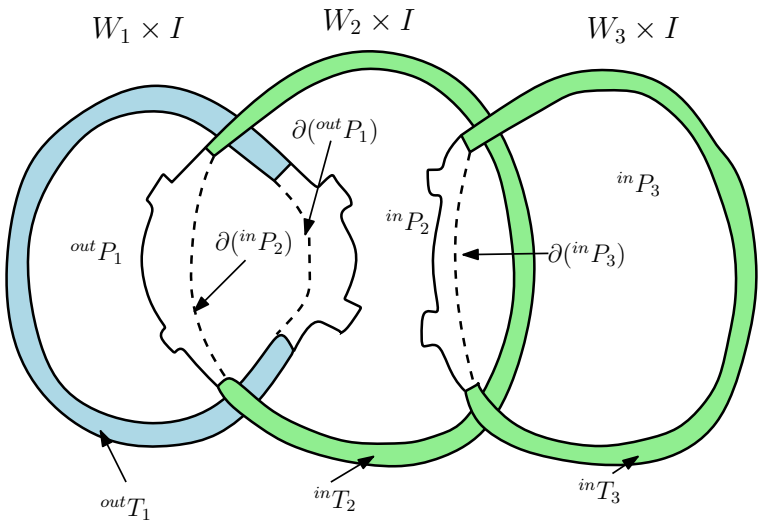

FIGURE 9. Three Weinstein cylinders $V_1 = W_1 \times I$, $V_2 = W_2 \times I$, and $V_3 = W_3 \times I$ in a good position. The blue region is ${}^{out}T_1$, while the green ones are ${}^{out}T_2$ and ${}^{out}T_3$.

Let $V \subset (M, \xi)$ be a domain diffeomorphic to a closed ball with a Weinstein convex boundary ∂V. Let $S \subset \partial V$ be a dividing set. We say that (V, S) *can be approximated by standard contact balls* if there exists a neighborhoood N of S in M such that for every $\sigma > 0$ there is a (iso-)contact embedding $g : D \to (M, \xi)$ of the standard contact ball such that

- $g(\partial D)$ is contained in a σ-neighborhood of ∂V;
- $g(\partial D) \cap N = \partial V \cap N$.

Proposition 7.6. *Let $V \subset (M, \xi)$ be a domain with a Weinstein convex boundary ∂V and $S \subset \partial V$ be a dividing set. Suppose that (V, S) can be approximated by standard contact balls. Then for any $\epsilon > 0$ there exist three Weinstein cylinders $V_1, V_2, V_3 \subset V$ in a good position such that a piecewise smooth hypersurface $\partial(V_1 \cup V_2 \cup V_3)$ is C^1 ϵ-close to ∂V, see Fig. 10.*

Proof. Choose an inwardly pointing transverse to $\Sigma := \partial V$ contact vector field and consider an interior collar $C := \partial\Sigma \times [0, 1] \subset V$ such that $\Sigma \times 0 = \Sigma$, $x \times [0, 1], x \in \Sigma$ are trajectories of Υ and $\Upsilon^t(\Sigma) = \Sigma \times t$. By scaling X we can assume that C is contained in an ϵ-small neighborhood of Σ. Choose an $\frac{\epsilon}{4}$-approximation of $(X^{\frac{1}{2}}(V), X^{\frac{1}{2}}(S))$ by a standard ball $h(D)$. By assumption the standard sphere $\widetilde{\Sigma} := h(\partial D)$ coincides with $\Sigma_{\frac{1}{2}} := \Sigma \times \frac{1}{2}$ along a tubular neighborhood $N = S \times [-1, 1] \subset \Sigma_{\frac{1}{2}}$. We can assume that $S \times t \subset N$ is transverse to the characteristic foliation on $\Sigma_{\frac{1}{2}}$ for all $t \in [-1, 1]$. Take a C^∞-function $\theta : [0, 1] \to \mathbb{R}$ which is equal to $1 - u$ for $u \in [\frac{1}{4}, \frac{3}{4}]$, equal to 0 near 1 and has non-positive derivative everywhere. Let $\Theta : \Sigma \to \mathbb{R}$ be a function supported in N and defined on N by the formula $\Theta(x, u) = \theta(|u|),\ (x, u) \in N = S \times [-1, 1]$. For $\sigma \in (0, \frac{1}{2})$ define an isotopy $g_s : \Sigma \to V$:

$$g_s(w) = \Upsilon^{-s\Theta(u)}(u),\ u \in \Sigma, s \in [0, \sigma].$$

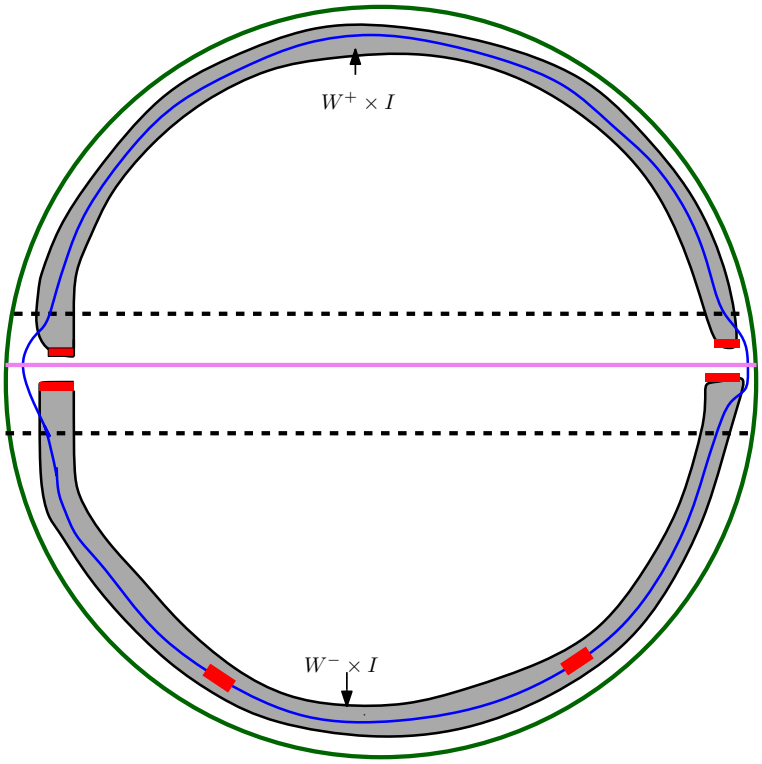

FIGURE 10. Three Weinstein cylinders in good position approximating a domain which admits an approximation by standard balls.

If σ is chosen sufficiently small then the spheres $g_s(\widetilde{\Sigma})$, $s \in [0, \sigma]$, are almost standard, and hence, by Lemma 8.2 a C^0-small adjustment near one of the poles makes them standard. Assuming this is done we can extend the isotopy to a global compactly supported contact diffeotopy $G_s : V \to V, s \in [0, \sigma]$.

The dividing set $S \subset \Sigma_{\frac{1}{2}}$ divides Σ into domains $\Sigma_{\frac{1}{2},\pm}$ with the common boundary S. Denote

$$\widehat{\Sigma}_+ := \Sigma_{\frac{1}{2},+} \setminus \left(S \times \left(-\frac{1}{2}, 0 \right] \right);$$
$$\widehat{\Sigma}_- := \Sigma_{\frac{1}{2},-} \setminus \left(S \times \left[0, \frac{1}{2} \right) \right).$$

Consider two Weinstein cylinders: $\widehat{W}_\pm = \widehat{\Sigma}_\pm \times [-\frac{\sigma}{2}, \frac{\sigma}{2}] = \bigcup_{|s| \le \frac{\sigma}{2}} \Upsilon^s(\widehat{\Sigma}_\pm)$, see Fig. 10. We have $\partial_\pm \widehat{W}_+ = \Upsilon^{\mp\frac{\sigma}{2}}(\widehat{\Sigma}_+)$ and $\partial_\pm \widehat{W}_- = \Upsilon^{\pm\frac{\sigma}{2}}(\widehat{\Sigma}_-)$. Consider the standard ball $\widehat{D} := G_\sigma(h(D)) \subset V$. The standard sphere $\partial\widehat{D}$ transversely intersects $\partial\widehat{W}_+$ along $\Upsilon^{\frac{\sigma}{2}}(S \times (-\frac{1}{2})) \in \partial_+\widehat{W}_+$, and transversely intersects $\partial\widehat{W}_-$ along $\Upsilon^{-\frac{\sigma}{2}}(S \times \frac{1}{2}) \in \partial_-\widehat{W}_-$. Note that the South pole of the sphere $\partial\widehat{D}$ is contained in $\operatorname{Int}\widehat{W}_-$. Hence, using Lemma 7.5 we can C^1-approximate $\widehat{D}$ by a contact embedding $f : \Delta = D \times I \to \widehat{D}$ such that $f(\operatorname{Corner}(\partial\Delta))$ is contained in $\operatorname{Int}\widehat{W}_-$. It

then follows that the Weinstein cylinders $\widehat{W}_+$, $f(\Delta)$ and $\widehat{W}_-$ are in good position, while the boundary $\widehat{W}_+ \cup f(\Delta) \cup \widehat{W}_-$ C^1-approximates Σ. □

7.4. **Quasi-plugs.** Let $V \subset (M, \xi)$ be a domain whose boundary ∂U is a hypersurface with admissible corners. Denote $Q := V \times [0, a]$ and let y be the coordinate corresponding to the second factor. Given sufficiently small $\sigma > 0$ and a vector field Y on Q, we call (Q, Y) a *σ-quasi-plug* if the following conditions are satisfied:

- QP1. Y coincides with $\frac{\partial}{\partial y}$ on $\mathcal{O}p\, \partial Q$;
- QP2. Y admits a Morse Lyapunov function which is equal to y on $\mathcal{O}p\, \partial Q$;
- QP3. for any point $p \in V$ with $\operatorname{dist}(p, \partial V) > \sigma$ the trajectory of Y through $p \times 0$ converges to a critical point of Y;
- QP4. given any point $p \in V$ with $\operatorname{dist}(p, \partial V) \leq \sigma$, there exists a point $p' \in \partial V$ and a positive $u(p)$ such that the trajectory of Y through a point $p \times 0$ either converges to a critical point of Y, or exit Q at a point $p'' \times a$ with $\operatorname{dist}(p'', Y^{u(p)} p') < \sigma$.

We will use quasi-plugs in combination with the following simple observation.

Lemma 7.7. *Suppose that the characteristic foliation on ∂V is σ-short. Then any σ-quasi-plug is a 3σ-plug.*

Lemma 7.8. *Let $V_1, \dots, V_k$, $k \geq 2$ be k Weinstein cylinders in a good position. Denote $\widehat{V} = \bigcup_1^k V_j$, $\widehat{Q} := \widehat{V} \times [0, k]$, $Q_j := V_j \times [j-1, j]$, $j = 1, \dots, k$. Denote by Y_j the vector field X_1 constructed in Proposition 7.1 and implanted to Q_j, $j = 1, \dots, k$. Let Y be the resulted field on $\widehat{Q}$. Then $(\widehat{Q}, Y)$ is a $C\sigma$-quasi-plug for some constant C.*

Proof. For each $j = 1, \dots, k$ we denote ${}^{in}P^j$, ${}^{out}P^j$, ${}^{in}U^j$, ${}^{out}U^j$, ${}^{in}T^j$ and ${}^{out}T^j$ the corresponding domains defined in Proposition 7.1 for Q_j. Denote $\widehat{V}_{\leq i} = \bigcup_1^i V_j$, $\widehat{Q}_{\leq i} := \widehat{V}_{<k} \times [0, i]$. We will prove the following more precise statement by induction in i:

$(\widehat{Q}_{\leq i}, Y|_{\widehat{Q}_{\leq i}})$ is a σ-quasi-plug for any $i \geq 2$. Moreover, any non-blocked trajectory entering $\widehat{V}_{<k} \times 0$ exit through ${}^{out}U_1 \cup \bigcup_2^i T_j$.

Suppose $i = 2$. For a sufficiently small σ we have ${}^{out}V_1 \setminus {}^{out}R_1 \subset {}^{in}P_2$ and ${}^{in}V_2 \setminus {}^{in}R_2 \subset {}^{out}P_1$. This implies that not blocked trajectories which enter ${}^{in}V^1 \times 0$ either exit through ${}^{out}U^1 \times 2$, or through ${}^{out}T^2 \times 2$. In the former case the exit point moves, possibly with a σ-error, in the positive direction of the Liouville flow of W_1, and in the latter one in the negative direction of the Liouville flow of W_2. which means that in both cases they are moved forward along the characteristic foliation on $\partial\widehat{V}$. Similarly, not blocked trajectories entering $V^2 \setminus V^1$ must enter through ${}^{in}U^2 \times 0$ and exit through ${}^{out}T_2 \times 2$, and hence, also are moved forward by the characteristic flow on $\partial\widehat{V}$.

Suppose now that the statement holds for $< i$ blocks. By the induction assumption all the non-blocked trajectories which enter through $V_{\leq i-1} \times 0$ exit through $({}^{out}U_1 \cup \bigcup_2^{i-1} T_j) \times (i-1)$.

On the other hand, $V_i \cap (\bigcup_2^{i-1} {}^{out}T_j) \subset {}^{out}T_{i-1} \cap ({}^{in}U_i \cup {}^{in}P_i)$. But all trajectories entering Q_i through ${}^{in}P_i \times (i-1)$ are blocked and those entering through ${}^{in}U_i \times (i-1)$ exit through ${}^{out}T_i$ and are pulled in the negative direction by the flow of Y_i. Finally, trajectory entering $(V_i \setminus V_{\leq i-1}) \times 0$ which are not blocked similarly exit through ${}^{out}T_i$ and are pulled in the negative direction by the flow of Y_i.

□

8. From a quasi-plug to a σ-plug

As we already mentioned above, we prove Proposition 4.1 by induction on dimension. Lemma 6.4 (together with the height reduction argument from Section 5, see also Remark 5.8) serves as the base of the induction for $2n = 2$. Suppose that Proposition 4.1 is already proven in dimension $< 2n$.

8.1. **Standard and almost standard spheres in a contact manifold.** Recall that we defined the standard contact $(2n+1)$-ball as the upper hemisphere $D := S^{2n+1}_+ = \{y_{n+1} \geq 0\}$ in the unit sphere $S^{2n+1} = \left\{ \sum_1^{n+1} x_j^2 + y_j^2 = 1 \right\} \subset \mathbb{R}^{2n+2}$ endowed with the standard contact structure $\xi_{\text{st}} = \{\sum_1^{n+1} x_j dy_j - y_j dx_j = 0\}$.

We call a germ of a contact structure on $S^{2n} = \partial D$ *standard* if it contactomorphic to the germ of of ξ_{st} along ∂D and coincides with ξ_{st} on a neighborhood of poles

$$p_\pm := \{x_{n+1} = \pm 1,\ x_j = 0 \text{ for } j = 1, \dots, n,\ y_j = 0 \text{ for } j = 1, \dots, n+1\}.$$

A germ ξ along a sphere is called *almost standard* if it coincides with ξ_{st} on a neighborhood of poles $p_\pm$ and its is characteristic foliation on S^{2n} admits a Lyapunov Morse function with exactly 2 critical points. Recall (see Proposition 3.2) that a germ of a contact structure along a hypersurface Σ is determined by its restriction to Σ up to a diffeomorphism fixed on Σ. Hence, we will not distinguish below between the germs and their restrictions.

Recall that any linear (conformal) symplectic structure ω on a vector space E defines a canonical contact structure ζ_ω on its sphere at infinity $S(E)$, i.e. the space of oriented lines through the origin. The group $Sp(E, \omega)$ of linear symplectic transformations acts by linear projective contactomorphisms on $(S(E), \zeta_\omega)$ and this representation is faithful. Hence, we can view $Sp(E, \omega)$ as a subgroup of the group of contactomorphisms of $(S(E), \zeta_\omega)$. In particular, given a hypersurface Σ in a contact manifold (M, ξ) the contact structure ξ defines a conformal symplectic structure on $T_p\Sigma$ for each singularity p of the characteristic foliation $\ell_{\Sigma,\xi}$. We denote the corresponding contact structure on the sphere $S_p = S(T_p\Sigma)$ by $\zeta_{p,\xi}$.

For the contact structure ξ_{st} along $S^{2n} = \partial D$ the holonomy along the leaves of the characteristic foliation $\ell_{\xi_{\text{st}}}$ allows us to identify the contact spheres at infinity $(S_\pm, \zeta_\pm) := \left(S\left(T_{p_\pm}(S^{2n})\right), \zeta_{p_\pm}\right)$. In turn, the contact sphere (S_+, ζ_+) can be canonically identified with the standard contact sphere $(S^{2n-1}, \xi_{\text{st}})$ Hence, for any almost standard germ ξ along S^{2n} the holonomy along the leaves of ℓ_ξ can be viewed as a contactomorphism $h_\xi : (S^{2n-1}, \xi_{\text{st}}) \to (S^{2n-1}, \xi_{\text{st}})$. We will call h_ξ the *clutching contactomorphism* of an almost standard germ ξ.

Let $AlSt$ be the space of almost standard contact germs on S^{2n} and $\mathcal{D}$ the group of contactomorphisms of the standard contact sphere $(S^{2n-1}, \xi_{\rm st})$. The image of the map $\pi : AlSt \to \mathcal{D}$ is the subgroup $\mathcal{D}_0 \subset \mathcal{D}$ which consists of contactomorphisms which are pseudo-isotopic to the identity. Denote by St the subspace of $AlSt$ which consists of standard contact germs. For $\xi \in St$ we have $h_\xi \in \mathcal{D}_1 := Sp(\mathbb{R}^{2n}, \omega_{\rm st}) \subset \mathcal{D}_0$. The following lemma is straightforward.

Lemma 8.1. *The projection $\pi : AlSt \to \mathcal{D}_0$ is a Serre fibration. If $h_\xi \in Sp(\mathbb{R}^{2n}, \omega_{\rm st})$ then ξ is standard, i.e. $St = \pi^{-1}(\mathcal{D}_1)$.*

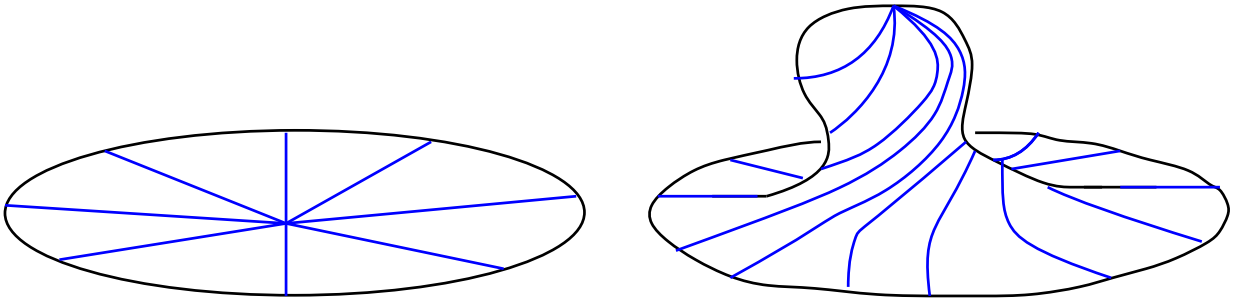

FIGURE 11. Adjusting a characteristic foliation near a pole.

Lemma 8.2. *Let ξ_t be a family of contact structures on $\mathcal{O}p\,\partial D \subset (S^{2n+1}, \xi_{\rm st})$ such that their germs along ∂D are almost standard. Suppose that for $t \in [0, \frac{1}{8}]$ the germ ξ_t is standard. Then there exists a diffeotopy $g_t : \mathcal{O}p\,\partial D \to \mathcal{O}p\,\partial D$ supported in an arbitrarily small neighborhood of p_+ such that the germs $g_t^* \xi_t$ along ∂D are standard and g_0 is the identity. See Fig. 11.*

Proof. Recall that by assumption $\xi_t = \xi_0 = \xi_{\rm st}$ in a neighborhood $U \supset \{p_+\}, U \in \mathcal{O}p\,\partial D$. There exists a smaller neighborhood $U' \subset U$ such that the pair $(U', \partial D \cap U'; \xi_{\rm st})$ is contactomorphic to $(D^{2n} \times (-\epsilon, \epsilon), D^{2n} \times 0; \mathrm{Ker}\,(\gamma := \sum_1^n x_i dy_i - y_i dx_i + dz))$. Denote $u := \sum_1^n (x_j^2 + y_j^2)$, and consider the splitting $D^{2n} \setminus 0 = S^{2n-1} \times (0, 1]$, given by the radial projection to the unit sphere and the u-coordinate. With respect to this splitting the form γ can be written as $dz + u\alpha_{\rm st}$, where $\alpha_{\rm st}$ is the standard contact form of the standard contact $(2n-1)$-sphere.

Choose a positive $\sigma \ll \epsilon$, and consider a C^∞-function $\theta : [0, \infty) \to [-3\sigma, 0]$, see 12 which is supported in $[0, 6\sigma]$, and such that $\theta(u) = 0$ for $u < \sigma$, $\theta(u) = -3\sigma$ for $u \in [4\sigma, 5\sigma]$, $\theta(u) = \sigma - u$ for $u \in [2\sigma, 3\sigma]$ and $\theta'(u) \le 0$ for $u \in [0, 5\sigma]$. Let us view $u \in [0, 1], z \in (-\epsilon, \epsilon)$ and $w \in S^{2n-1}$ as coordinates in the neighborhood U', so that the equation $z = 0$ defines $\partial D \cap U'$. Consider the family of hypersurfaces $\Theta_s \subset \mathcal{O}p\,\partial D$ which coincides with graphs $\{z = s\theta(u)\}$ in U' and equal to ∂D elsewhere. Take a neighborhood $U'' := \{u < 7\sigma, |z| < \sigma\} \Subset U'$. There exists a supported in U'' diffeotopy $\phi_s : \mathcal{O}p\,\partial D \to \mathcal{O}p\,\partial D$ such that $\phi_0 = \mathrm{Id}$, $\phi_s(\partial D) = \Theta_s$. Define a diffeotopy $\psi_t : \mathcal{O}p\,\partial D \to \mathcal{O}p\,\partial D$ as $\psi_t := \phi_{8t}$ for $t \in [0, \frac{1}{8}]$, $\psi_t = \phi_1$ for $t \in [\frac{1}{8}, \frac{7}{8}]$ and $\psi_t = \phi_{8-8t}$ for $t \in [\frac{7}{8}, 1]$. Note that the germs of contact structures $\widetilde{\xi}_t := \psi_t^* \xi_t$ along ∂D are almost standard, and moreover, for each $t \in [0, 1]$ the clutching diffeomorphism $\widetilde{h}_t := h_{\widetilde{\xi}_t}$ differs from h_{ξ_t} by a unitary rotation $w \mapsto e^{C(t) i \phi} w$ of the sphere $S^{2n-1} \subset \mathbb{C}^n$. Consider the domain $\widehat{U} := \{u \le 3\sigma,\ -2\sigma \le z \le -\sigma\} \subset U''$. Note that for $t \in [\frac{1}{8}, \frac{7}{8}]$ we

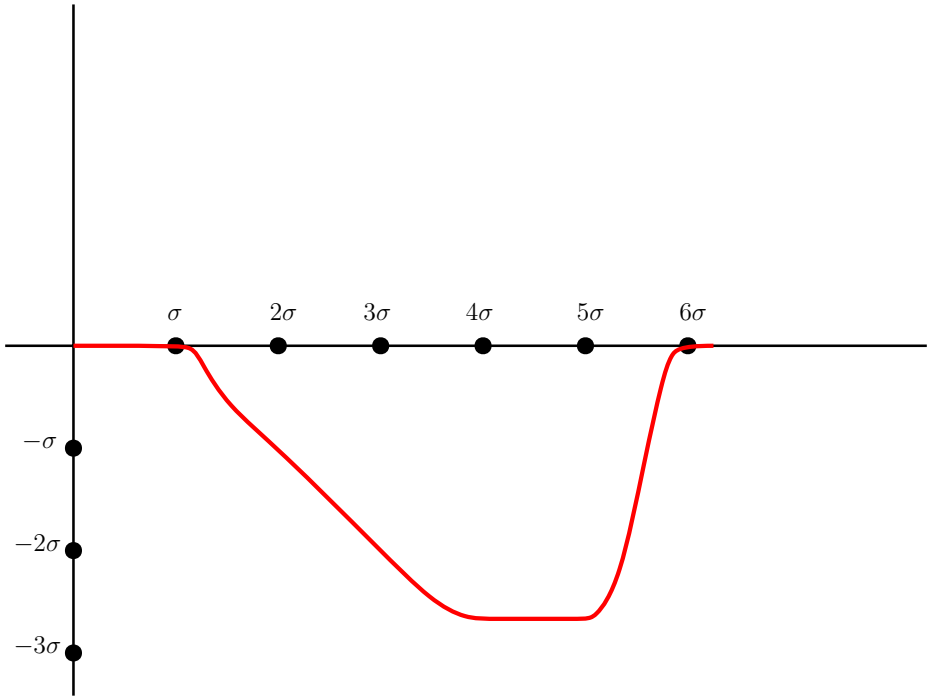

FIGURE 12. The function $\theta(u)$.

have $\psi_t(\partial D) \cap \widehat{U} = \{u = \sigma - z,\ -2\sigma \leq z \leq -\sigma\}$. Consider the space $\mathcal{K}$ of functions $K : [-2\sigma, -\sigma] \times S^{2n-1} \to (0, 3\sigma]$ which are equal to $\sigma - z$ near the boundary. Given $K \in \mathcal{K}$ consider its graph $\Gamma_K := \{u = K(z, w);\ (z, w) \in [-2\sigma, -\sigma] \times S^{2n-1}\} \subset \widehat{U}$. The contact structures ξ_t in $\widehat{U}$ is given by the form $\frac{1}{u}dz + \alpha_{\text{st}}$ and hence, the holonomy along the leaves of the characteristic foliation ℓ_{Γ_K} is equal to the time σ map of the contact flow of the contact Hamiltonian $\frac{1}{K(z,w)}$. We view here $z \in [-2\sigma, -\sigma]$ as the time parameter.

Define a contact isotopy $g_t := \widetilde{h}_t^{-1} = (h_{\widetilde{\xi}_t})^{-1} : S^{2n-1} \to S^{2n-1}$. While its contact Hamiltonian $G_t : S^{2n-1} \to S^{2n-1}$ is not necessarily positive, it can be made positive and even arbitrarily large by composing g_t with appropriate unitary rotations $w \mapsto e^{\widetilde{C}(t)i\phi}$ of the sphere $S^{2n-1} \subset \mathbb{C}^n$. We will keep the notation g_t for the modified isotopy. Hence, there exists a family of functions $K_t \in \mathcal{K}$, $t \in [\frac{1}{8}, \frac{7}{8}]$ such that $K_t = \sigma - z$ for $t = \frac{1}{8}, \frac{7}{8}$, and such that the holonomy along the leaves of the characteristic foliation $\ell_{\Gamma_{K_t}}$ coincide with g_t up to a unitary rotation of the sphere S^{2n-1}. Let us modify the diffeotopy ψ_t for $t \in [\frac{1}{8}, \frac{7}{8}]$, keeping it supported in $\widehat{U}$, so that $\psi_t(\partial D \cap \widehat{U}) = \Gamma_{K_t}$. Denote $\widehat{\xi}_t := \psi_t^* \xi_t$. By construction the clutching diffeomorphisms $h_{\widehat{\xi}_t} : S^{2n-1} \to S^{2n-1}$ are unitary rotations, and hence, the germs of contact structures $\widehat{\xi}_t$ along ∂D are standard.

□

8.2. **Making the characteristic foliation short.**

Proposition 8.3. *Suppose that Proposition 4.1 holds for plug installation over $D^{2n-3} \times [0, 1]$. Let Σ be the standard $(2n-2)$-dimensional sphere in $(M^{2n-1}, \xi = \operatorname{Ker} \alpha)$. Then for any*

$\sigma > 0$ there exists an σ-small in the C^0-sense isotopy $f_s : \Sigma \to M$ starting with the inclusion $f_0 : \Sigma \hookrightarrow M$ such that

a) *f_s is fixed on a neighborhood of poles of the characteristic foliation ℓ of Σ;*
b) *the family of characteristic foliations ℓ_s, $s \in [0,1]$, induced on Σ by $f_s^*\alpha$; admits a family of good Lyapunov functions $F_s : \Sigma \to \mathbb{R}$;*
c) *the characteristic foliation ℓ_1 is σ-short;*
d) *for any $\sigma > 0$ the isotopy f_s can be included into a 2-parametric isotopy $f_{s,t}$, $s,t \in [0,1]$, such that*
   (i) *$f_{s,0} = f_s$, $f_{0,t} = f_0$ for all $s,t \in [0,1]$;*
   (ii) *the spheres $f_{s,1}(\Sigma)$ are almost standard for all $s \in [0,1]$;*
   (iii) *$f_{s,t}$ is σ-close to $f_{s,0}$ for all $(s,t) \in [0,1]$;*
   (iv) *the isotopy $f_{1,t}, t \in [0,1]$, is fixed on a neighborhood of a dividing set S_1 of $\Sigma_1 = f_1(\Sigma)$.*

Proof. Using Corollary 3.4 we can find an σ-blocking system $\{D_j\}$, $j = 1, \dots, N$, of transverse standard contact discs. We can assume that the Lyapunov function F for X on Σ is constant on each D_j. Denote $c_j = F|_{D_j}$ and assume that $c_1 < c_2 < \dots < c_N$. Let $Q_j \subset \Sigma$ be disjoint flow-boxes of D_j of diameter $< 2\sigma$. Let $h_{s,t} : Q := D \times [0,1] \to D \times T^*[0,1]$ be a C^0 small isotopy constructed in Proposition 4.1. We define the required isotopy $f_{s,t}$ by successively deforming flow-boxes Q_j. Denote $\Delta_j := [\frac{j-1}{N}, \frac{j}{N}] \subset [0,1]$, $1 \le j \le N$. For $s \in \Delta_j, t \in [0,1]$ define $f_{s,t} := \Phi_j \circ h_{Ns-j+1,t} \circ \Phi_j^{-1}$ on Q_j, $f_{s,t} = \Phi_i \circ h_{1,t} \circ \Phi_i^{-1}$ on Q_i for $i < j$, and fixed elsewhere on Σ. Then Proposition 4.1 guarantees all the required properties of the isotopy $f_{s,t}$ except b), d)(ii) and d)(iv). More precisely, for b) we automatically get a family of Lyapunov functions for X_s, but not necessarily good ones. Moreover, we get a family of Lyapunov functions for the whole 2-parametric family $X_{s,t}$. In particular, Lyapunov functions for $X_{s,1}$ have no critical points except the maximum and the minimum of the original Lyapunov function F. This implies that the sphere $f_{s,1}(\Sigma)$ are almost standard. Recall that according to Corollary 3.5 it is sufficient to ensure properties (L1) and (L2) and absence of retrograde connections. According to property c)(v) of Proposition 4.1 trajectories of $X_{s,t}$ converging to positive zeroes enter each plug through the same $(n-2)$-dimensional stratified subset $E_j \subset D_j \subset \partial\widehat{Q}_j$. According to our staged construction of the isotopy, the isotopy $f_{s,0}$ for $s \in [\frac{j-1}{N}, \frac{j}{N}]$, $j = 1, \dots, N$, which creates a plug in $\widehat{Q}_j$ does not change trajectories of $X_s = X_{s,0}$ in $F \le c_j$. Hence, by a C^∞-small adjustment of embeddings ϕ_j before each step of the isotopy we can arrange that the closure $\overline{G_j}$ of the negative tail $\bigcup_{u \ge 0} X_{\frac{j-1}{N}}(E_j)$ of the set E_j does not contain any negative zeroes of $X_{\frac{j-1}{N}}$, and hence, the same property holds for all $s \ge \frac{j-1}{N}$. The deformation $f_{s,t}$ for a fixed $s \in \Delta_j$ changes the field $X_{s,t}$ only in an arbitrarily small neighborhood of $\overline{G}_j$, and hence thanks to compactness of the set of zeroes, one can arrange that $X_{s,t}$ have no retrograde connections. Corollary 3.4 then guaranteed that $f_{s,t}(\Sigma)$ are Weinstein convex. It remains to satisfy property d)(iv). Consider the dividing set S for X_1. Using Property c)(iii) of Proposition 4.1 we can find an isotopy of Σ preserving leaves of ℓ_1 with disjoins S with compact subsets $C_1^\pm$, and hence with their neighborhoods $U^\pm \supset C_1^\pm$. According to c)(ii) we can arrange the isotopy $f_{1,t}$ to be supported in $U^\pm$, which implies property d)(iv).

□

Proposition 8.4. *Suppose that Proposition 4.1 holds for plug installation over $D^{2n-3}\times[0,1]$. Let $D = (D^{2n-1}, \operatorname{Ker}\alpha_{\mathrm{st}})$ be the standard contact disc. Then for any $\sigma > 0$ there exists a σ-small in the C^0-sense isotopy $h_s : D \to D$ such that $h_s(D) \subset \operatorname{Int} D$ for all $s > 0$ and*

(i) *the ball $\widetilde{D} := h_1(D)$ has a Weinstein convex boundary $\partial\widetilde{D}$ with the dividing set $S \subset \partial\widetilde{D}$;*
(ii) *the characteristic foliation $\ell_{\partial\widetilde{D}}$ is σ-short;*
(iii) *$(\widetilde{D}, S)$ can be approximated by standard contact balls.*

Remark 8.5. In the case $n = 2$ property (iii) is automatic from the classification of tight contact structures on the 3-ball, see [5].

Proof. Let us first shrink $D \to \operatorname{Int} D$ by a C^∞-small contracting contact isotopy and then apply to the image D' Proposition 8.3. Let us extend the constructed there isotopy $f_{s,t} : \partial D' \to D$, $s, t \in [0,1]$, to an isotopy $D' \to D$. We will continue using the notation $f_{s,t}$ for the extension. We claim that that the isotopy obtained by concatenating the shrinking isotopy with the isotopy $f_{s,0}$ has the required properties. Indeed, the balls $D'_s := f_{s,0}(D')$ have Weinstein convex boundaries with dividing sets $S_s \subset \partial D'_s$ and the characteristic foliation on $\partial D'_1$ is σ-short and the family of spheres $\Sigma_s := f_{s,1}(\partial D')$, $s \in [0,1]$ are almost standard and hence, can be made standard by an arbitrarily C^0-small adjustment away from the poles and dividing sets by applying Lemma 8.2. Th sphere $\Sigma_0 = \partial D'$ bounds the standard ball, and hence, the same holds, by continuation of the contact isotopy argument, for Σ_1. But by construction Σ_1 coincides with $\partial D'_1$ on a neighborhood of the dividing set $S_1 \subset \partial D'_1$ and can be made arbitrarily C^0-close to $\partial D'_1$, i.e. (D'_1, S_1) can be approximated by standard contact balls. □

8.3. **Proofs on main results.**

Proof of Proposition 5.1. Applying the induction hypothesis and Proposition 8.4 we find a disc $\widetilde{D} \subset D$ such that

- $\widetilde{D}$ has a Weinstein convex boundary $\partial\widehat{D}$ with the dividing set $S \subset \partial\widehat{D}$;
- the characteristic foliation $\ell_{\partial\widetilde{D}}$ is σ-short;
- $(\widetilde{D}, S)$ can be approximated by standard contact balls;
- $D \setminus f(\partial D)$ is contained in σ-small neighborhood of ∂D,

Next, we apply Proposition 7.6 and find 3 Weinstein cylinders $V_1, V_2, V_3 \subset f(D)$ such that a piecewise smooth hypersurface $\partial(V_1 \cup V_2 \cup V_3)$ is C^1 σ-close to $\partial f(D)$. Applying now Lemma 7.8 we install into $D \times [0,1]$ a σ-quasi-plug in $(V_1 \cup V_2 \cup V_3) \times [0,1] \subset f(D) \times [0,1]$. But the characteristic foliation on $f(\partial D)$ is σ-short, and hence the constructed plug is a genuine $C\sigma$-plug for $D \times [0,1]$ for some universal constant $C > 0$.

This concludes the proof of Proposition 5.1, and hence, of Proposition 4.1. □

Proof of Theorem 3.11. First, we adjust Σ by a C^∞-isotopy to make all singularities of its characteristic foliation ℓ_Σ non-degenerate and hyperbolic. Next we apply Lemma 3.3 to find a blocking collection of transverse standard contact discs $D_j \subset \Sigma$. According to Lemma 2.6 there exists $\sigma > 0$ such that by installing σ-plugs instead of flow boxes one can arrange

the resulting flow to satisfy condition (L1). Proposition 4.1 asserts that such plugs can be installed by deforming flow-boxes via an arbitrarily small in the C^0-sense isotopy. By an additional C^∞-perturbation of the hypersurface outside plugs we can satisfy the Morse-Smale property, while still preserving condition (L1), see Lemma 2.6, and hence, by Corollary 3.4 the resulting Σ is Weinstein convex. This concludes the proof of Theorem 3.11, and in combination with Lemma 3.10 of Theorem 1.1. □

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