Sparse High Dimensional Expanders via Local Lifts

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Abstract

High dimensional expanders (HDXs) are a hypergraph generalization of expander graphs. They are extensively studied in the math and TCS communities due to their many applications. Like expander graphs, HDXs are especially interesting for applications when they are bounded degree, namely, if the number of edges adjacent to every vertex is bounded. However, only a handful of constructions are known to have this property, all of which rely on algebraic techniques. In particular, no random or combinatorial construction of bounded degree high dimensional expanders is known. As a result, our understanding of these objects is limited.

The degree of an i-face in an HDX is the number of $(i+1)$-faces that contain it. In this work we construct complexes whose higher dimensional faces have bounded degree. This is done by giving an elementary and deterministic algorithm that takes as input a regular k-dimensional HDX X and outputs another regular k-dimensional HDX $\widehat{X}$ with twice as many vertices. While the degree of vertices in $\widehat{X}$ grows, the degree of the $(k-1)$-faces in $\widehat{X}$ stays the same. As a result, we obtain a new "algebra-free" construction of HDXs whose $(k-1)$-face degree is bounded.

Our construction algorithm is based on a simple and natural generalization of the expander graph construction by Bilu and Linial [12], which build expander graphs using lifts coming from edge signings. Our construction is based on *local lifts* of high dimensional expanders, where a local lift is a new complex whose top-level links are lifts of the links of the original complex. We demonstrate that a local lift of an HDX is also an HDX in many cases.

In addition, combining local lifts with existing bounded degree constructions creates new families of bounded degree HDXs with significantly different links than before. For every large enough D, we use this technique to construct families of bounded degree HDXs with links that have diameter $\geq D$.

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1 Introduction

Expander graphs are graphs that are well connected. These objects are studied extensively in computer science and mathematics [40], and since their discovery they have found numerous applications in complexity [25, 62], coding theory [64, 66, 26], derandomization [40, 33] and more. Most of these applications rely on families of expander graphs that have a bounded degree. It is well known that random regular graphs are expanders, and many explicit bounded degree constructions are also in hand [55, 51, 63, 12, 54].

Recently, the study of *high dimensional expanders* (HDXs) emerged (see surveys [49, 37]). These are hypergraph analogues of expander graphs. While the full potential of high dimensional expanders is yet to be discovered, they are already important objects of study. High dimensional expanders, and especially bounded degree high dimensional expanders[1] have already yielded exciting applications in various areas such as locally testable codes [26, 61, 28], quantum complexity [7], sampling and Markov chains [27, 43], agreement testing [27, 24, 8], high dimensional geometry and topology [38, 30], pseudorandomness [19] and random (hyper)graph theory [46, 56].

The specific family of high dimensional expanders used in many of the aforementioned applications is tailor-made to satisfy other desired properties, in addition to high dimensional expansion. For example, the local neighborhoods in the high dimensional expanders used in [26, 61, 28] are tailored so that one can define a small locally testable code on them; the high dimensional expanders in [38, 24, 8] also have a vanishing 1-cohomology over certain group coefficients.

However, constructing bounded degree high dimensional expanders (for arbitrarily small spectral expansion of the links) is still a serious challenge. No random model for bounded degree high dimensional expanders is known, and all deterministic constructions known use non-trivial algebraic techniques. The fact that we have only a handful of bounded degree constructions to choose from, makes these objects difficult to understand and to work with. We believe that many further applications await us once we learn how to diversify these constructions, in the same way that many of the above-mentioned applications of expander graphs grew out of more varied expander constructions that were discovered.

Nowadays, all known constructions of HDXs rely on algebraic techniques, including quotients of the Bruhat Tits buildings [10, 16, 45, 52, 22, 24, 8] and coset complexes [42, 31, 59] (see also [39] for a more elementary analysis of some of these HDXs). There have been attempts at constructing bounded degree HDXs with combinatorial tools, but all these constructions fall short either in bounded degreeness [35, 47] or in their local spectral expansion [20, 21, 17, 48, 34].

In particular, it is an important open question whether an algorithm à la Zig-Zag product [63] exists for bounded degree high dimensional expanders. That is, an algorithm that given a bounded degree high dimensional expander as input, outputs another high dimensional expander with more vertices and the same bound on the degree and spectral expansion.

As an intermediate result, in this work we develop an algorithm that takes a high dimensional expander as input, and outputs another high dimensional expander with more vertices, the same bound on spectral expansion and the same bound on *the degree of high dimensional faces* (but not on the degree vertices). This algorithm is entirely combinatorial, relying only on the theory of graph covers initiated by [5, 12]. While families of complexes

[1] A family of HDXs is called *bounded degree* if there is some $M > 0$ so that all vertices in every HDX in the family have degree at most M.

constructed via such an algorithm are not sufficient for applications that require the vertex degree to be bounded, we view this as a stepping stone towards an "algebra free" construction of bounded degree HDXs. One exception to this is the recent work by [3], which analyzes a variant of the well known Glauber dynamics (or up-down) random walk on HDXs. The walk that [3] analyzes has bounded degree if and only if the underlying HDX is $(k-1)$-bounded degree.

1.1 Preliminaries on High Dimensional Expanders

To better understand our results, let us introduce some standard definitions and notation on simplicial complexes (see Section 2.3 for a more elaborated definitions). A simplicial complex is a hypergraph that is downwards closed to containment. A simplicial complex is k-dimensional if the largest hyperedge in the complex is of size $(k+1)$. We denote by $X(\ell)$ the sets (aka faces) of size $\ell+1$. Let X be a k-dimensional simplicial complex.

The degree of a face $\sigma \in X(\ell)$, is $d(\sigma) := |\{\tau \in X(\ell+1) \mid \tau \supseteq \sigma\}|$. A family of complexes $\{X_i\}_{i=0}^{\infty}$ is *ℓ-bounded degree* if there exists an $M > 0$ that bounds the degrees of *all ℓ-faces* across *all the complexes simultaneously*. That is, for every i and any $\sigma \in X_i(\ell)$, $d(\sigma) \leq M$. We say that a family of complexes are bounded degree if they are 0-bounded degree. We say that a complex is $(d_0, d_1, \ldots, d_{k-1})$-regular if for every $\sigma \in X(\ell)$, $d(\sigma) = d_\ell$.

In this paper we are mainly interested in the local spectral expansion definition of high dimensional expanders (see [49] for a survey on other definitions). For this we need to define "links", the generalization of vertex neighborhoods in graphs. For a face $\sigma \in X$, the link of σ is the simplicial complex $X_\sigma = \{\tau \setminus \sigma \mid \sigma \subseteq \tau \in X\}$. We will be interested in the graph structure underlying the complex and its links. The 1-skeleton of X is the graph whose vertices are $X(0)$ and whose edges are $X(1)$.

▶ **Definition 1** (High dimensional expander). *For $\lambda > 0$ we say that X is a λ-two sided (one sided) high dimensional expander if for every $\ell \leq k-2$ and $\sigma \in X(\ell)$, the 1-skeleton of X_σ is a λ-two sided (one sided) spectral expander.*

1.2 Our Results

Our results are based on the notion of a graph lifts. We say a graph $\widehat{G} = (\widehat{V}, \widehat{E})$ is a lift of a graph $G = (V, E)$ if there exists a graph homomorphism $\phi : \widehat{V} \to V$ such that for every $\hat{v} \in \widehat{V}$, the mapping ϕ is a bijection on the neighborhood of $\hat{v}$. Intuitively, a lift of a graph G is a large graph $\widehat{G}$ that locally looks the same as G. Graph lifts are essential in many constructions of expander graphs [5, 12, 54], and our construction builds on the beautiful work of [12]. We elaborate more on this below.

Our main result is a construction algorithm that maintains both expansion and the degree of the $(k-1)$-faces of a regular complex. This algorithm takes as input a $(d_0, d_1, \ldots, d_{k-1})$-regular λ-high dimensional expander X, and outputs a $(2d_0, 2d_1, \ldots, 2d_{k-2}, d_{k-1})$-regular $\widehat{X}$ with twice as many vertices that is also a λ-HDX (even though the number of intermediate faces grows like $|\widehat{X}(i)| = 2^i|X(i)|$ for $i \leq k-1$). This algorithm uses the notion of random lifts [12], and in particular, it requires no algebraic machinery for the construction nor the analysis. More formally, this is the theorem we prove.

▶ **Theorem 2** (See Theorem 28 for a more precise statement). *There exists a randomized algorithm $\mathcal{A}$ that takes as input a k-dimensional complex X_0 and an integer $i \geq 1$, runs in expected time $\mathrm{poly}((2^i|X_0(0)|)^k)$ at most, and outputs a k-dimensional complex X_i with $2^i|X_0(0)|$ vertices. The algorithm has the following guarantees.*

1. *If* X_0 *is a* $(d_0, d_1, \dots, d_{k-1})$*-regular* λ*-two sided high dimensional expander, then* X_i *is a* $(2^i d_0, \dots, 2^i d_{k-2}, d_{k-1})$*-regular* λ'*-two sided high dimensional expander where* $\lambda' = O\left(\max\left\{\lambda\left(1 + \log\frac{1}{\lambda}\right), \sqrt{\frac{\log^3 d_{k-1}}{d_{k-1}}}\right\}\right)$.
2. *For every* $\hat{\sigma} \in X_i(k-2)$*, the link* $(X_i)_{\hat{\sigma}}$ *is a lift of* $(X_{i-1})_\sigma$ *for some* $\sigma \in X_{i-1}(k-2)$*.*

For every $j \leq k-2$*,* $|X_i(j)| = 2^{(j+1)i}|X_0(j)|$*.*

There are various complexes in hand that one can use as the input to this algorithm. These include the complete complex, the complexes from [50], and even complexes from bounded degree families that are regular, such as those constructed by [31].

We give two proofs to Theorem 2, building on the techniques developed by [12] to analyze lifts in expander graphs, and extend them to high dimensional expanders.

While most of the work in [12] regards random lifts of graphs, they also show how to deterministically find expander graphs using lifts. Building on their method, we also give a deterministic algorithm for finding the complexes in Theorem 2, albeit under some more assumptions on the input X_0. This provides a *deterministic, polynomial time and elementary* construction of a family of $(k-1)$-bounded k-dimensional high dimensional expanders.

▶ **Theorem 3** (See Theorem 35). *There exists a deterministic algorithm* $\mathcal{B}$ *that takes as input a* k*-dimensional complex* X_0 *and an integer* $i \geq 1$*, runs in time* $\mathrm{poly}((2^i|X_0(0)|)^k)$ *at most, and outputs a* k*-dimensional complex* X_i *with* $2^i|X_0(0)|$ *vertices. The algorithm has the following guarantees.*

1. *If* X_0 *is a* $(d_0, d_1 \dots, d_{k-1})$*-regular* λ*-two sided high dimensional expander, such that* $d_{k-1} > 2^{10k}$ *and* $|X_0(k-2)| \leq (d_{k-2})^{10k}$*, then* X_i *is a* $(2^i d_0, \dots, 2^i d_{k-2}, d_{k-1})$*-regular* λ'*-two sided high dimensional expander where* $\lambda' = O\left(2^{5k}\max\left\{\lambda\left(1 + \log\frac{1}{\lambda}\right), \sqrt{\frac{\log^3 d_{k-1}}{d_{k-1}}}\right\}\right)$.
2. *For every* $\hat{\sigma} \in X_i(k-2)$*, the link* $(X_i)_{\hat{\sigma}}$ *is a lift of* $(X_{i-1})_\sigma$ *for some* $\sigma \in X_{i-1}(k-2)$*.*

For every $j \leq k-2$*,* $|X_i(j)| = 2^{(j+1)i}|X_0(j)|$*.*

Not only is our construction deterministic, but it is also simple and versatile; one can apply it to various kinds of high dimensional expanders, and the family of HDXs obtained by doing so is changes according to the initiating HDX given at the beginning of the process.

1.3 Comparing to Random Constructions of HDXs

In the graph case the configuration model yields regular and bounded degree expanders. In contrast, there is no immediate generalization of this model to higher dimensions, that leads to bounded degree HDXs, even if one only wishes to bound the degrees of higher dimensional faces. If one settles for logarithmic degree, then one could use the [46] random model to construct random HDXs. The degree of the top-level faces of these complexes is $O(\log n)$, where n is the number of vertices, and the degree of the lower dimensional faces is polynomial in n. For 2-dimensional complexes, the random geometric model in [47] offers an improvement to the vertex degree that one gets from [46], but it is still polynomial.

It is tempting to try and adapt the [46] model for constructing $(k-1)$-bounded degree HDXs, but doing so in a straightforward manner falls short of achieving that. The work by [50] found an appropriate generalization of the random model that gives $(k-1)$-bounded degree HDXs, utilizing the breakthrough work of [44] on Steiner systems. In their model, one takes a complete $(k-1)$-skeleton and samples k-faces by sampling random Steiner systems on this complex.

Our construction sidesteps this difficulty by taking a different approach; it uses random *local lifts* of HDXs (presented in the following subsection) instead of trying to construct random ones from scratch. In this setting, the high dimensional case behaves more similar to the 1-dimensional one - our work shows that random local lifts of HDXs are HDXs with high probability. Of course, this requires an appropriate modification of the lift notion to local lifts.

1.4 The Construction

We now dive into the heart of Theorem 2. Our construction builds upon the work of random lifts of graphs studied in [12]. Random lifts of expanders have been subject to extensive research (e.g., [6, 12, 2, 54, 14]). However, in this work, we do not try to lift the complex itself. Instead, we construct another complex where the $(k-2)$-*links* are lifts of links in the original complex. We call such a complex a *local lift*.

Let us first explain the idea behind the work of [12]. Their work suggests a construction of bounded degree family of expander graphs $\{G_i\}_{i=0}^{\infty}$, where for every i, G_{i+1} is a lift of G_i. The fact that the maximal degree of G_{i+1} is equal to the maximal degree of G_i promises that the sequence is bounded degree. Therefore, one only needs to worry about expansion.

The work [12] studies random lifts sampled using signings on the edges of a graph $G = (V, E)$, that is, functions $f : E \to \{\pm 1\}$. Given such a signing f, one can construct the following lift $\widehat{G} = (\widehat{V}, \widehat{E})$ by setting $\widehat{V} = V \times \{\pm 1\}$ and $\{(v, i), (u, j)\} \in \widehat{E}$ if $\{v, u\} \in E$ and $i \cdot j = f(\{u, v\})$.

The work of [12] analyzes when a lift $\widehat{G}$ obtained by random signing f is an expander graph. They give a proof (based on the Lovász Local Lemma) that every expander G has such a “good” signing. They also provide a deterministic algorithm to construct such a lift using the conditional probabilities method [4]. Our construction generalizes this idea, only instead of lifts coming from edge signings, we define *local lifts* coming from *face*-signings.

Let X be a k-dimensional simplicial complex and let $f : X(k) \to \{\pm 1\}$. Define the k-dimensional complex $\widehat{X}$ (where f is implicit in the notation) as a complex whose vertices are $\widehat{X}(0) = X(0) \times \{\pm 1\}$, and whose k-faces are

$$\widehat{X}(k) = \left\{ \left\{ v_0^{j_0}, v_1^{j_1}, \dots, v_k^{j_k} \right\} \middle| \; \{v_0, v_1, \dots, v_k\} \in X(k) \text{ and } \prod_{i=0}^{k} j_i = f(\{v_0, v_1, \dots, v_k\}) \right\}.$$

For $1 \le \ell \le k-1$ the ℓ-faces are independent of the second coordinate, that is,

$$\widehat{X}(\ell) = \left\{ \left\{ v_0^{j_0}, v_1^{j_1}, \dots, v_\ell^{j_\ell} \right\} \middle| \; \{v_0, v_1, \dots, v_\ell\} \in X(\ell) \right\}.$$

Obviously, the underlying graph of $\widehat{X}$ is *not* a lift of the underlying graph of X. Indeed, the degree of each vertex is doubled. However, for every $\hat{\sigma} \in \widehat{X}(k-2)$, we show that $\widehat{X}_{\hat{\sigma}}$ is isomorphic to a lift of X_σ (where $\sigma = \left\{ v \middle| \; v^j \in \hat{\sigma} \right\}$).

Indeed, let us assume for simplicity that $k = 2$. Consider the link of a vertex $v^j \in \widehat{X}(0)$ and define the function $g : X_v(1) \to \{\pm 1\}$ by $g(uw) = j \cdot f(uw)$. We claim that $\widehat{X}_{v^j}$ is the cover g induces on X_v. It is easy to see that its vertices are $\widehat{X}_{v^j}(0) = X_v(0) \times \{\pm 1\}$, since the vertices in $\widehat{X}_{v^j}$ correspond to edges in $\widehat{X}(1)$. These are precisely all u^1, u^{-1} where $u \in X_v(0)$.

The edges are more delicate. Edges $\left\{ u^{j'}, w^{j''} \right\} \in \widehat{X}_{v^j}(1)$ correspond to triangles $\left\{ v^j, u^{j'}, w^{j''} \right\} \in \widehat{X}$. Indeed, such a triangle is in $\widehat{X}$ if and only $\{u, w\} \in X_v(1)$ and $j \cdot j' \cdot j'' = f(\{v, u, w\})$. The second condition occurs if and only if $j \cdot j' \cdot j'' = g(uw)$. Hence, $\widehat{X}_{v^j}$ is the cover g induces on X_v.

As mentioned above, we give two proofs that signings f so that $\widehat{X}$ is a high dimensional expander exist. The first proof is based on the Lovász Local Lemma and follows the argument in [12, Lemma 3.3], and generalizes it so that multiple links may be taken into account simultaneously. The second proof is based on a different way to use the Lovász Local Lemma (together with other results from [12]) which we find simpler, to deduce high dimensional expansion. This proof, while more restrictive on the link sizes, can be combined with the algorithmic version of the Lovász Local Lemma [57], to prove Theorem 2. Afterwards, we show that if the links of the complex X are already dense, then the derandomization technique in [12] works for high dimensional expanders, and we can obtain a deterministic construction for $(k-1)$-bounded k-dimensional high dimensional expanders, proving Theorem 3.

1.5 Understanding Vertex vs. Edge Degree in Bounded-Degree Constructions

We can use Theorem 2 to diversifying links in other existing bounded degree constructions, and thus gain more understanding on how possible high dimensional expanders may look like. For simplicity, let us stick to the 2-dimensional case, and consider the question *how small could d_1 be given d_0 in a (d_0, d_1)-regular high dimensional expander?*

Let us consider the behavior of d_0 and d_1 in the known bounded degree constructions [10, 16, 45, 52, 42, 31, 22, 24, 8]. In all the above, d_0 grows to infinity as λ goes to 0, and $d_1 = \text{poly}(d_0)$[2]. In other words, the links themselves are "locally" dense. A natural question to ask is whether the lower bound of $d_1 \geq d_0^{\Omega(1)}$ is necessary for *bounded degree* constructions. In expander *graphs* it is well known that one can increase the size of the graph without increasing the bound on the degree, but this is not the behavior in the known bounded-degree HDX constructions.

We note that if one allows d_0 to tend to infinity with n, rather than staying constant, then works such as [50] (and also infinite families of complexes constructed by iteratively applying Theorem 3) show that this is false. But this question is more interesting when its bounded degree.

Theorem 3 gives a negative answer to this question in the 2-dimensional case, by proving the following.

▶ **Theorem 4.** *For every $\lambda > 0$ and any sufficiently large $M > 0$, there exists an infinite family of 2-dimensional λ-two sided high dimensional expanders that are $(d_0, \exp(\text{poly}(\frac{1}{\lambda})))$-regular, for $M \leq d_0 \leq 2M$.*

In particular, for every large enough $D > 0$, there exists an infinite family of 2-dimensional λ-two sided HDXs such that the diameter in every link X_v, is at least D.

We stress that $d_1 = \exp(\text{poly}(\frac{1}{\lambda}))$ depends only on the spectral expansion and not on the number of vertices or d_0.

The proof of Theorem 4 appears in the full version of this paper [11].

1.6 Related Work

Bounded degree HDX

As discussed above, all known constructions of bounded degree high dimensional expanders use algebraic techniques. The first bounded degree high dimensional expanders for arbitrarily small $\lambda > 0$ was by [10]. This was followed by many other works that aimed to construct the high dimensional equivalent to Ramanujan graphs [16, 45, 53, 52]. All these constructions

[2] Technically most of the constructions above are not *regular*, only bounded degree, so d_0 and d_1 should be average values, but we ignore this point for the sake of presentation.

are quotients of $\tilde{A}$-type Bruhat Tits buildings. The work by [22] used this building together with complex lifts to construct other high dimensional expanders. Recently, high dimensional expanders that come from $\tilde{C}$ Bruhat Tits buildings were also constructed and studied [18, 24, 8]. A second type of constructions come from coset complexes, first studied by [42]. More complexes of this type were constructed by [31, 59]. We mention that the work by [39] simplified the analysis of these coset complexes, and gave a description of the complexes in [42] in relatively elementary means (albeit still relying on algebraic methods).

Interestingly, [50] give a randomized construction of a $(k-1)$-bounded degree family of λ-HDXs for arbitrarily small $\lambda > 0$. This construction is based on random Steiner systems and given in the breakthrough result of [44]. The underlying $(k-1)$-skeletons of the complexes in that family are complete.

There are other bounded degree constructions [20, 21, 17, 48, 34]. These constructions have various mixing properties, but none of them are λ-HDXs for $\lambda < \frac{1}{2}$ (where λ is normalized between 0 and 1). There are other constructions of λ-HDXs for $\lambda < \frac{1}{2}$, which are not bounded degree, but are still non-trivially sparse. These include [35] - based on Grassmann posets, and [47] - based on random geometric graphs of the sphere.

Finally, we comment that previous works also considered the possible degrees (d_0, d_1) possible in a high dimensional expanders. The work by [31] used irregular algebraic constructions of bounded degree λ-HDXs and "regularized" them, thus showing that there exists bounded degree HDXs that are regular for arbitrarily small λ. The work by [17] gives a lower bound on the expansion of the underlying graph of the complex in terms of (d_0, d_1), but this lower bound does not rule out (or construct) such HDXs with $d_1 \ll d_0$.

Graph and HDX lifts

The study of random graph lifting was initiated in [5]. Random lifts from signings of expanders were studied in [12] where it was proven that with high probability they are also expanders. This was extended to larger lifts as well [60, 1]. Friedman showed that random lifts of Ramanujan graphs are nearly Ramanujan [32] (see also [14]). In the seminal paper by [54], Ramanujan bipartite graphs were constructed by using graph lifts. One can also define lifts of simplicial complexes. Most known bounded degree high dimensional expanders are constructed using a dual notion of lifts - that is, taking quotients of an infinite object [10, 16, 45, 53, 52, 42], in a way such that the infinite object is a lift of the complex that is constructed. [22] studied taking random lifts of simplicial complexes as in [12], but the construction there needed the use of algebraic techniques as well.

1.7 Open Questions

Combinatorial constructions

As we mention earlier, there is no construction of bounded degree high dimensional expanders that does not rely on non-trivial algebraic techniques. As an intermediate step towards such a construction, can one give a construction of k-dimensional simplicial complexes that are $(k-2)$-bounded degree (or i-bounded degree for any $i < k-1$)?

Links with other properties

Fix a vertex set $[n]$ and graphs $\{G_i\}_{i=1}^{n}$ (one graph for every vertex $i \in [n]$). It is interesting to understand whether there exists a graph whose vertex set is $V = [n]$, and such that the neighborhood of every vertex $i \in V$ is (isomorphic to) G_i. The structure of such graphs is an extensive topic of study, especially in the case where all G_i's are equal (see, e.g., [15, 13, 58]). One of the major components in the works [26, 61] that constructed asymptotically good locally testable codes and quantum codes, is a construction of graphs that locally look like a neighborhood of a graph product, but globally have improved expansion properties.

In this work, we propose a technique that addresses a related problem. Given a graph G (which is the one skeleton of a regular 2-dimensional complex X), we find a graph $\widehat{G}$ where every neighborhood in $\widehat{G}$ is a random (or deterministic) 2-lift of a corresponding neighborhood in G. Is there a technique that allows us to do so for *any* set of 2-lifts of the respective vertex neighborhoods?

Other notions of expansion

In this paper we mainly deal with local spectral expansion, but other definitions of high dimensional expansion also exist. Most notable is the notion of coboundary expansion defined independently in [46] and [38]. This notion is important for many applications of high dimensional expanders such as code construction [26], topological expansion [38] and property testing [41, 36, 23, 9]. Does a local lift maintain coboundary expansion? If not, is it maintained in interesting special cases?

Better local spectral expansion

Works following [12] such as [54, 14] improved the bounds on the spectrum of lifts of regular graphs. Can one construct local lifts of regular high dimensional expanders that are also *Ramanujan*?

1.7.1 Organization of This Paper

The necessary preliminaries are given in Section 2. We describe local lifts in Section 3 and describe some of their basic properties. In Section 4 we show existence of good local lifts by modifying a Lovász Local Lemma argument by [12]. In Section 5 we prove Theorem 2 using the algorithmic Lovász Local Lemma [57] and derive Theorem 4. In Section 6 we show that the method of derandomization in [12] could be generalized to our case as well and prove Theorem 3.

2 Preliminaries

Unless explicitly stated, all logarithms are with base 2. The ln function is a logarithm with base e. We write $A \sqcup B$ to denote a disjoint union of sets A, B. The For $n \geq 0$ we write $[n] = \{0, 1, \dots, n\}$. For a square matrix (or equivalently, a linear operator on a finite vector space), we write $\|A\|$ to denote the operator norm.

2.1 Graphs

Let $G = (V, E)$ be a graph. For $u, v \in V$ we write $\Gamma(v)$ for the set of v's neighbors in G and $u \sim v$ if u and v are neighbors. The *indicator vector* of a set $S \subseteq V$, denoted by $\mathbf{1}_S$, is $\mathbf{1}_S : V \to \{0,1\}$ with $\mathbf{1}_S(v) = 1 \iff v \in V$. For two sets $S, T \subseteq V$ we write $E_G(S,T)$ for the set of edges in G between S and T. The *graph induced by S and T* is $G' = (S \cup T, E_G(S,T))$. For a d-regular graph we denote $\ell_2(V) = \{f : V \to \mathbb{R}\}$ endowed with the inner product $\langle f, g \rangle = \sum_{v \in V} f(v)g(v)$.

2.1.1 Expander Graphs

Expander graphs are graphs with good connectivity properties. There are many equivalent ways to define expanders [40]. In this manuscript we focus on *spectral expansion*.

Let $G = (V, E)$ be a d-regular graph. The *random walk matrix* of G is a matrix $A \in \mathbb{R}^{V \times V}$ defined by $A(u, v) = \frac{1}{d}$ if $u \in \Gamma(v)$ and otherwise 0. Equivalently, it corresponds to the random walk operator $A : \ell_2(V) \to \ell_2(V)$ with $Af(v) = \frac{1}{d}\sum_{u\in\Gamma(v)} f(u)$. We abuse notation and use A for both the matrix and the random walk operator it represents. We sometimes denote this operator by A_G when G is unclear from the context.

The operator A is self adjoint with respect to the inner product. Therefore, it has an orthonormal basis of real-valued eigenvectors, where the eigenvalues are denoted by $1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. We elaborate and write $\lambda_i(G)$ when the graph in question is unclear from the context. The *spectrum of* G is the spectrum of its random walk matrix and is denoted by $\mathsf{Spec}(G)$.

▶ **Definition 5** (spectral expander). *For $\lambda \in [0, 1]$ we say that G is λ-*two sided *(resp.* one sided*)* spectral expander *(or* expander *for short) if $\lambda \geq \max\{\lambda_2, |\lambda_n|\}$ (resp. $\lambda \geq \lambda_2$).*

2.1.2 Tensor Product

Let G, H be any graphs. The *tensor product* of G and H, denoted by $G \otimes H$, is the graph with vertices $V(G) \times V(H)$, and edges $E(G \otimes H) = \{(a, b)(a', b') \mid \{a, a'\} \in E(G) \text{ and } \{b, b'\} \in E(H)\}$. The following fact is well known.

▶ **Fact 6.** *Let G, H be graphs. If H, G are λ, λ'-two sided spectral expanders respectively, then $G \otimes H$ is a $\max\{\lambda, \lambda'\}$-two sided spectral expander. Moreover, if H is a λ-two sided spectral expander and G is a λ'-*one sided *spectral expander, then $G \otimes H$ is a $\max\{\lambda, \lambda'\}$-one sided spectral expander.*

2.2 Graph Lifts

Graph lifts are an important notion, studied first by [5, 6] (although the notion of lifts themselves is a classical notion in algebraic topology known for about a century).

▶ **Definition 7** (lift). *For finite, connected and simple graphs G and $\widehat{G}$, a* lift *(also known as a covering map) $\phi : \widehat{G} \to G$ is a graph homomorphism with the property that for all $\hat{v} \in V(\widehat{G})$, ϕ maps the neighborhood of $\hat{v}$ in $\widehat{G}$ onto the neighborhood of $\phi(\hat{v})$ in G. Finally, we say that $\widehat{G}$ is an ℓ-*lift of G *if there exists an ℓ-to-1 covering map $\phi : \widehat{G} \to G$.*

One way to construct a 2-lift is to use a signing function on the edges as follows.

▶ **Definition 8** (Function induced lift). *Let $G = (V, E)$ be a graph and let $f : E \to \{\pm 1\}$ be a signing. The f-*induced lift *$\widehat{G} = \widehat{G}^f$ is the graph whose vertices are $\widehat{V} = V \times \{\pm 1\} = \left\{v^j \;\middle|\; v \in V, j \in \{\pm 1\}\right\}$ and whose edges are $\widehat{E} = \left\{\left\{v^j, u^i\right\} \;\middle|\; \{v, u\} \in E, ij = f(\{u, v\})\right\}$. The lift map is $\phi(v^j) = v$.*

It is elementary to prove this construction is indeed a lift, so we omit this proof. It is also easy to show that any 2-lift is an induced lift for some signing $f : V(G) \to \{\pm 1\}$. See [65] for a more general statement and proof.

In Section 3 we generalize the notion of graph lifts to *local lifts* of simplicial complexes.

2.2.1 Signing Functions and Lift Expansion

Fix a graph G and a signing $f : E(G) \to \{\pm 1\}$. In this subsection, we characterize the eigenvalues of an f-induced lift. For this, we need to define the f-signing of an adjacency operator. For a d-regular graph the f-signing of the adjacency operator is the matrix $A^f(u,v) = f(u,v) \cdot A(u,v)$ for $\{u,v\} \in E$ and $A^f(u,v) = 0$ if $\{u,v\} \notin E$.

This signing matrix is closely related to the random walk operator of the lift. In particular, the following is by now classical.

▶ **Lemma 9.** *Let G be a d-regular graph and let $\widehat{G}$ be an f-induced 2-lift. Then the eigenvalues of $A_{\widehat{G}}$ are the union (with multiplicities) of the eigenvalues of A and those of A^f. In particular, if $\|A^f\| \leq \lambda$ and G is a λ'-two sided (resp. one sided) spectral expander, then $\widehat{G}$ is a $\max\{\lambda, \lambda'\}$-two sided (resp. one sided) spectral expander.*

Using this lemma, [12] gave a criterion for the expansion of the lift graph.

▶ **Lemma 10** ([12, Lemma 3.3])**.** *Let G, f and A^f be as above and assume that G is a λ-two sided (resp. one sided) spectral expander with no self loops. Assume that for any pair of disjoint $S, T \subseteq V(G)$ it holds that $\left|\left\langle \mathbf{1}_S, A^f \mathbf{1}_T \right\rangle\right| \leq \alpha\sqrt{|S||T|}$, then $\widehat{G}^f$ is a λ'-two sided (resp. one-sided) spectral expander where $\lambda' = \max\left\{\lambda, O\left(\alpha\left(1+\log\frac{1}{\alpha}\right)\right)\right\}$.*

We note that there is a nice formula for this inner product, which is $\left\langle \mathbf{1}_S, A^f \mathbf{1}_T \right\rangle = \frac{1}{d}\sum_{(v,u):\{v,u\}\in E(G)} f(u,v)\mathbf{1}_S(v)\mathbf{1}_T(u)$.

2.3 High Dimensional Expanders

▶ **Definition 11** (simplicial complex)**.** *A* k-dimensional simplicial complex *is a finite hypergraph X that is downwards closed to containment. That is, if $\tau \in X$ and $\sigma \subseteq \tau$ then $\sigma \in X$.*

We write $X = X(-1) \sqcup X(0) \sqcup X(1) \sqcup \cdots \sqcup X(k)$, where $X(\ell) = \{\sigma \in X \mid |\sigma| = \ell + 1\}$ (here $X(-1) = \{\emptyset\}$ is mainly a formality) and the maximal size of a set $\sigma \in X$ is $k+1$. We call elements $\sigma \in X(\ell)$ *ℓ-faces*, and in this case we say that X is k-dimensional. In this paper we will always assume the simplicial complex in question is *pure*, that is, that every $\sigma \in X(\ell)$ contained in some $\tau \in X(k)$. In addition, we assume it has no self-loops or multifaces. Namely, every vertex appears at most once in each face and any face appears at most once in X.

The degree of a face $\sigma \in X(i)$ is $d(\sigma) = |\{\tau \in X(i+1) \mid \tau \supseteq \sigma\}|$. We say that a family of simplicial complexes $\{X_i\}_{i=0}^{\infty}$ is *j-bounded degree* if there is an integer $M > 0$ so that for all X_i and all $\sigma \in X_i(j)$, $d(\sigma) \leq M$. If the family is 0-bounded degree, we sometimes just say bounded degree (without the zero).

▶ **Definition 12** (hyper-regularity)**.** *Let $d_0 \geq d_1 \geq \cdots \geq d_{k-1}$ be positive integers. A k-dimensional simplicial complex X is* $(d_0, d_1, \ldots, d_{k-1})$-regular *if for any $i \in \{0, \ldots, k-1\}$ and any i-face σ, $d(\sigma) = d_i$.*

We say that X is regular if there exists such a tuple so that X is $(d_0, d_1, \ldots, d_{k-1})$-regular. In this case we denote by $d_i(X) = d_i$.

The *j-skeleton* of a simplicial complex X is the simplicial complex obtained by taking all the i-faces of X, for all $i \leq j$. The 1-skeleton of a complex is also called an *underlying graph.*

A link is a generalization of a graph neighborhood.

▶ **Definition 13** (link)**.** *For a k-dimensional simplicial complex X and a face $\sigma \in X$, the* link of σ *is the $(k-1-|\sigma|)$-dimensional simplicial complex $X_\sigma = \{\tau \setminus \sigma \mid \tau \in X, \tau \supset \sigma\}$.*

For $\ell \leq k-2$ and $\sigma \in X(\ell)$ we denote by A_σ the random walk operator of the 1-skeleton of X_σ. We often abuse of notation and for a face $\sigma = \{v_0, \ldots, v_\ell\} \in X(\ell)$ write $\sigma = v_0 \ldots v_\ell$.

Natural analogues of expander graphs to higher dimensions are simplicial complexes where the neighborhoods of the faces are themselves expander graphs. See Section 1 for more context on this important definition.

▶ **Definition 14** (λ-high dimensional expander). *Let $\lambda \in [0,1]$. A k-dimensional simplicial complex X is a* λ-two sided (resp. one sided) high dimensional expander *if for all $i \leq k-2$ and all $\sigma \in X(i)$, the 1-skeleton of X_σ is a λ-two sided (resp. one sided) spectral expander.*

2.4 The Lovász Local Lemma

The Lovász Local Lemma is a classical result in the probabilistic method.

▶ **Lemma 15** (Lovász Local Lemma [29]). *Let $\mathfrak{B} = \{B_1, \dots, B_n\}$ be a finite set of events in some arbitrary probability space. The* dependency graph of $\mathfrak{B}$ *is a digraph $G_{\mathfrak{B}} = (\mathfrak{B}, E)$ so that any event $B_i \in \mathfrak{B}$ is mutually independent of all the events $\mathfrak{B} \setminus \Gamma(B_i)$, where $\Gamma(B_i)$ is the neighborhood of B_i in $G_{\mathfrak{B}}$.*

If there exists a real function $\rho : \mathfrak{B} \to [0,1)$ so that

$$\mathbb{P}[B_i] \leq \rho(B_i) \prod_{B_j \sim B_i} (1 - \rho(B_j)) \tag{2.1}$$

for any $B_i \in \mathfrak{B}$, then with strictly positive probability, none of the events B_i occur.

This lemma also has an algorithmic version, first given in the seminal work of [57]. We give below a slightly less general version than the one in [57].

▶ **Lemma 16** ([57]). *Let Ω be a finite set and let $\mathfrak{P} = (P_1, P_2, \dots, P_m)$ be a tuple of independent random variables supported on Ω^m. Let $\mathfrak{B} = \{B_1, B_2, \dots, B_n\}$ be a finite set of events in the sigma algebra of $\mathfrak{P}$. Let the dependency graph and the assignment $\rho : \mathfrak{B} \to [0,1)$ be as in Lemma 15. Then there exists a randomized algorithm that finds an assignment $p \in \Omega^m$ such that $p \notin \bigcup_{i=1}^n B_i$. If one can verify whether B_i holds in time t, then the randomized algorithm runs in $tn \sum_{i=1}^n \frac{\rho(B_i)}{1-\rho(B_i)}$ expected time.*

The algorithm described in this lemma is simple. The algorithm starts with randomly sampling some $p \in \Omega^m$. While there exists some B_i such that $p \in B_i$, the algorithm takes an arbitrary such B_i, and resamples all the coordinates P_j that B_i depends upon. Of course, if the algorithm halts, then $p \notin \bigcup_{i=1}^n B_i$. The paper [57] shows that the expected number of times an event B_i is resampled is at most $\frac{\rho(B_i)}{1-\rho(B_i)}$ which explains the runtime of this algorithm.

3 Local lifts

This section presents our basic construction, the *local lift* of a complex. We will define this construction formally and describe some of its properties.

▶ **Construction 17** (Local Lift). Let X be a k-dimensional simplicial complex and let $f : X(k) \to \{\pm 1\}$. The *f-local lift* of X denoted by $\widehat{X} = \widehat{X}^f$, is the following k-dimensional simplicial complex:

- $\widehat{X}(0) = X(0) \times \{\pm 1\}$ and we denote the vertices by $\widehat{X}(0) = \left\{ v^j \;\middle|\; v \in X(0), j \in \{\pm 1\} \right\}$.
- For any $1 \leq \ell \leq k-1$,

$$\widehat{X}(\ell) = \left\{ \{v_0^{j_0}, v_1^{j_1}, \dots, v_\ell^{j_\ell}\} \;\middle|\; \{v_0, v_1 \dots, v_\ell\} \in X(\ell),\ j_0, \dots, j_\ell \in \{\pm 1\} \right\}.$$

- Finally, $\widehat{X}(k)$ is the set of all faces $\sigma = \left\{v_0^{j_0}, v_1^{j_1}, \ldots, v_d^{j_k}\right\}$ so that the face without the signs is $\{v_0, v_1, \ldots, v_k\} \in X(k)$, and the product of the j_i's are equal to $f(\sigma)$. Namely,

$$\widehat{X}(k) = \left\{ \left\{v_0^{j_0}, v_1^{j_1}, \ldots, v_k^{j_k}\right\} \,\middle|\, \{v_0, v_1, \ldots, v_k\} \in X(k), \;\; f(\{v_0, v_1, \ldots, v_k\}) = \prod_{i=0}^{k} j_i \right\}.$$

One can already see that the $(k-1)$-skeleton of $\widehat{X}$ doesn't depend on f and is just some inflation of the original complex. The dependence on f is only in the top-level faces. Thus, in particular, $\widehat{X}$ is not a lift of (the underlying graph) of X, except when $k = 1$. However, the links of $(k-2)$-faces in $\widehat{X}$ are lifts of links in X, which is why we named this complex a local lift. We will see this in the next subsection.

3.1 Local Properties of Local Lifts

For the rest of this subsection, we fix X to be a k-dimensional pure simplicial complex, $f : X(k) \to \{\pm 1\}$ to be a signing function, and $\widehat{X}$ to be the f-local lift of X. We also need the following three pieces of notation:

1. Let $\pi : \widehat{X}(0) \to X(0)$ be the projection map $\pi(v^j) = v$, and we extend it to higher dimensional faces as well by $\pi(\left\{v_0^{j_0}, v_1^{j_1}, \ldots, v_i^{j_i}\right\}) = \{v_0, v_1, \ldots, v_i\}$.
2. Let $\mathrm{sign} : \widehat{X} \to \{\pm 1\}$ be $\mathrm{sign}(\hat{\sigma}) \to \prod_{v^j \in \hat{\sigma}} j$.
3. For any $\hat{\sigma} \in \widehat{X}(k-2)$ with $\sigma = \pi(\hat{\sigma})$ we denote by $f_\sigma : X_\sigma(1) \to \{\pm 1\}$ the function $f_\sigma(e) = f(\sigma \sqcup e)$ and by $f_{\hat{\sigma}} : X_\sigma(1) \to \{\pm 1\}$ the function $f_{\hat{\sigma}}(e) = \mathrm{sign}(\hat{\sigma}) \cdot f_\sigma(e)$.

The first observation is that the degrees of $\widehat{X}$ are twice the degrees of X, except for d_{k-1}, which stays the same.

▶ **Observation 18.** *If X is $(d_0, \ldots, d_{k-1})$-regular then $\widehat{X}$ is $(2d_0, 2d_1 \ldots, 2d_{k-2}, d_{k-1})$-regular.* ⌟

It is a direct calculation, so we have omitted its proof. We just comment that the reason that d_{k-1} remains the same is that for every $\hat{\sigma} \in \widehat{X}(k-1)$ and $v \in X_{\pi(\hat{\sigma})}(0)$ there is exactly one $j \in \{\pm 1\}$ such that $\hat{\sigma} \cup \left\{v^j\right\} \in \widehat{X}(k)$. Therefore, $d_{k-1}(X) = |X_{\pi(\hat{\sigma})}(0)| = |\widehat{X}_{\hat{\sigma}}(0)| = d_{k-1}(\widehat{X})$.

The next lemma gives a complete description of the links of $\widehat{X}$.

▶ **Lemma 19** (on the structure of the links)**.** *Let $\hat{\sigma} \in \widehat{X}$ and denote by $\sigma = \pi(\hat{\sigma})$.*

1. *If $\dim(\hat{\sigma}) < k-2$, then the 1-skeleton of $\widehat{X}_{\hat{\sigma}}$ (the $\hat{\sigma}$-link of $\widehat{X}$), is isomorphic to the 1-skeleton of X_σ tensored with the complete graph on two vertices with self loops[3].*
2. *If $\dim(\hat{\sigma}) = k-2$, then $\widehat{X}_{\hat{\sigma}}$ is isomorphic to a lift of X_σ induced by $f_{\hat{\sigma}}$.*

Proof. The first item directly follows from the definition of a tensor product. For the second item, suppose $\dim(\hat{\sigma}) = k-2$. Both the vertices of $\widehat{X}_{\hat{\sigma}}$ and of the $f_{\hat{\sigma}}$-induced lift of X_σ are $X_\sigma(0) \times \{\pm 1\}$. As for the edges, $\left\{u^i, v^j\right\} \in \widehat{X}_{\hat{\sigma}}(1)$ if and only if $\{u, v\} \in X_\sigma(1)$ and $ij \cdot \mathrm{sign}(\hat{\sigma}) = f(\sigma \sqcup \{u, v\})$ (or equivalently $ij = f_{\hat{\sigma}}(\{u, v\})$). This is precisely the relation that defines edges in the $f_{\hat{\sigma}}$-induced lift of X_σ. ◀

[3] Note that this is also true for the link of $\sigma = \emptyset$, i.e. $\widehat{X}$ itself.

The following corollary that bounds the spectrum of the links is direct.

▶ **Corollary 20.** *Let $\hat{\sigma} \in \widehat{X}$ and denote $\sigma = \pi(\hat{\sigma})$.*

1. *If $\dim(\hat{\sigma}) < k-2$ then $\lambda(\widehat{X}_{\hat{\sigma}}) = \lambda(X_\sigma)$.*
2. *If $\dim(\hat{\sigma}) = k-2$ then $\mathsf{Spec}(\widehat{X}_{\hat{\sigma}}) = \mathsf{Spec}(A_\sigma) \cup \mathsf{Spec}(A_\sigma^{f_{\hat{\sigma}}})$, where A_σ is the normalized adjacency matrices of X_σ and $A_\sigma^{f_{\hat{\sigma}}}$ is its signed normalized adjacency matrix with respect to $f_{\hat{\sigma}}$.*

Proof. The first item follows from the first item in Lemma 19 that shows the link of $\widehat{X}_{\widehat{\sigma}}$ is isomorphic to the link of X_σ tensored with a complete graph, and Fact 6 that bounds the expansion of such a graph. The second item follows from Lemma 9 and the fact that the link is the $f_{\hat{\sigma}}$-induced lift of X_σ as we saw in Lemma 19. ◀

4 Families of HDXs via Random Local Lifts

This section is dedicated to existential proofs of high dimensional expanders based on our local lifts from Construction 17. We start by stating the main theorem of this section which asserts that given an arbitrary HDX X, there exists a family of HDXs with parameters comparable to those of X so that any member of the family is a local lift of the former. Formally,

▶ **Theorem 21.** *Let X_0 be a $(d_0, d_1, \ldots, d_{k-1})$-regular λ-two sided (resp. one sided) HDX over n vertices, for $\lambda \in [0,1]$. Then there exists a family of $\max\left\{\lambda, O\left(\sqrt{\frac{k^2\log^3 d_{k-1}}{d_{k-1}}}\right)\right\}$-two sided (resp. one sided) high dimensional expanders $\{X_i\}_{i=0}^{\infty}$ so that X_i is a $(2^i d_0, \ldots, 2^i d_{k-2}, d_{k-1})$-regular complex over $2^i n$ vertices and X_{i+1} is a local lift of X_i.*

The proof of Theorem 21 is based on proving the single-step version of it, given in Theorem 22, and applying it iteratively.

▶ **Theorem 22.** *Let $\lambda \in [0,1]$. For any k-dimensional, $(d_0, \ldots, d_{k-1})$-regular, λ-two sided (resp. one sided) high dimensional expander X over n vertices, there exists a signing $f : X(k) \to \{\pm 1\}$ so that $\widehat{X}$ is a $\max\left\{\lambda, O\left(\sqrt{\frac{k^2\log^3 d_{k-1}}{d_{k-1}}}\right)\right\}$-two sided (resp. one sided) high dimensional expander with regularity $(2d_0, \ldots, 2d_{k-2}, d_{k-1})$ and $2n$ vertices.*

We start by proving Theorem 21 given Theorem 22. The proof of Theorem 22 is more involved and is provided in the remainder of this section.

Proof of Theorem 21 assuming Theorem 22. Let X_0 as specified in Theorem 21 and denote $\lambda' := \max\left\{\lambda, O\left(\sqrt{\frac{k^2\log^3 d_{k-1}}{d_{k-1}}}\right)\right\}$. The proof is by induction on i. Clearly X_0 holds the requirements.

For the induction step, let X_i be a $(2^i d_0, \ldots, 2^i d_{k-2}, d_{k-1})$-regular λ'-two sided (resp. one sided) HDX with $2^i n$ vertices received in the i-th step of the process. By Theorem 22, there exists a singing function $f_i : X_i(k) \to \{\pm 1\}$ so that the f_i-local lift of X_i (denoted by $\widehat{X_i}$) is a $(2^{i+1}d_0, \ldots, 2^{i+1}d_{k-2}, d_{k-1})$-regular λ'-two sided (resp. one sided) HDX over $2^{i+1}n$ vertices. Setting $X_{i+1} := \widehat{X_i}$ concludes the proof. ◀

4.1 Proof Outline of Theorem 22

The proof of Theorem 22 is based on Lovász Local Lemma [29] and Lemma 10, and closely follows the lines of the existential proof in [12].

Recall that one approach for proving a given k-dimensional simplicial complex is λ-HDX, is considering all of its ℓ-links for $\ell \leq k-2$ and bound the spectrum of each of their 1-skeletons by λ. By Corollary 20, the only links one should be concerned with are those obtained by $(k-2)$-faces, as the links of all other faces inherent the expansion from the initial HDX. In addition, by the same corollary, it's enough to analyze the spectra of the signed random walk matrices of the $(k-2)$-links of X, with respect to the signing induced on them as defined in Lemma 19. Indeed, doing so is the most technical part of the proof and follows by the next lemma combined with Lemma 10:

▶ **Lemma 23.** *For any k-dimensional pure $(d_0, \ldots, d_{k-1})$-regular simplicial complex X over n vertices, there exists a signing function $f : X(k) \to \{\pm 1\}$ such that for any $(k-2)$-face $\hat{\sigma} \in \widehat{X}$ and any disjoint subsets of vertices $S, T \subseteq X_\sigma(0)$ for $\sigma = \pi(\hat{\sigma})$,*

$$|\langle \mathbf{1}_S, A_\sigma^{f_{\hat{\sigma}}} \mathbf{1}_T \rangle| \leq 10 \sqrt{\frac{k^2 \log d_{k-1}}{d_{k-1}} |S||T|} \tag{4.1}$$

where $f_{\hat{\sigma}}$ is the signing on X_σ's edges induced by f as defined in Section 3.1.

By the $(d_0, \ldots, d_{k-1})$-regularity of X, X_σ is a d_{k-1}-regular graph over d_{k-2}-vertices. Furthermore, since any signing over the k-faces induces a signing function on the edges of any $(k-2)$-link, our goal is to find a single signing function f such that these lifts of all the links of the $(k-2)$-dimensional faces expand.

The proof of Lemma 23 is by the Lovász Local Lemma Lemma 15. We define the set of "bad" events $\mathfrak{B} = \{B_\sigma^{S,T}\}$. The event $B_\sigma^{S,T}$ is that (4.1) doesn't hold for a fixed $\sigma \in X(k-2)$ and fixed disjoint sets $S, T \subseteq X_\sigma(0)$. In [12], similar bad events were considered, but only the sets S, T needed to be specified. The main difference between our proof and theirs is that we need to take care of the dependencies between events corresponding to different $(k-2)$-faces σ, σ'.

To apply the lemma and deduce Lemma 23, one needs to understand and analyze the dependency relation of the events in $\mathfrak{B}$.

On the dependency of bad events in $\mathfrak{B}$

Fix $\hat{\sigma} \in \widehat{X}(k-2)$ and disjoint $S, T \subseteq X_\sigma(0)$ for $\sigma = \pi(\hat{\sigma})$, and define $F(\sigma, S, T) \subseteq X(k)$ to be the set of all k-faces of X so that $\sigma \subseteq \tau$ and $\tau \setminus \sigma$ is an edge in the graph induces by $S \sqcup T$ on X_σ. Recall that $\mathrm{sign} : \widehat{X} \to \{\pm 1\}$ is defined by $\mathrm{sign}(\hat{\sigma}) = \prod_{v^j \in \hat{\sigma}} j$, and note that

$$d_{k-1} \langle \mathbf{1}_S, A_\sigma^{f_{\hat{\sigma}}} \mathbf{1}_T \rangle = \sum_{\substack{uv \in X_\sigma(1) \\ \text{s.t. } u \in S, v \in T}} f_{\hat{\sigma}}(uv) = \mathrm{sign}(\hat{\sigma}) \sum_{\substack{uv \in X_\sigma(1) \\ \text{s.t. } u \in S, v \in T}} f(\sigma \sqcup uv) = \mathrm{sign}(\hat{\sigma}) \sum_{\tau \in F(\sigma, S, T)} f(\tau).$$

Since the signs f assigns to the k-faces are chosen independently, if the event $B_\sigma^{S,T}$ is *not* mutually independent of a subset $\mathfrak{A} \subseteq \mathfrak{B}$, there must exists some event $B_{\sigma'}^{S',T'} \in \mathfrak{A}$ for which $F(\sigma, S, T) \cap F(\sigma', S', T')$ is not empty.

Towards using the Lovász Local Lemma, we need to bound the probability of the event $B_\sigma^{S,T}$ as well as the number of neighbors it has in the dependency graph considered in the Local Lemma. The bound on the probability follows directly from the arguments in [12], but bounding the number of neighbors each event is more involved. In contrast to the expander graph case considered in [12] (where the "bad" events only depend on S and T), in simplicial complexes events corresponding to different faces σ, σ' (and therefore to different links) can depend on one another as long as they have a common k-face in $F(\sigma, S, T) \cap F(\sigma', S', T')$.

A naive count of the number of such k-faces won't suffice, and would lead the bound in Equation (4.1) to scale with d_{k-2}. Therefore, we need to carefully characterize when exactly $F(\sigma, S, T)$ and $F(\sigma', S', T')$ intersect.

The first case where dependency can occur is when $\sigma = \sigma'$. In this case, we are in the same setting as in [12], which observed that there must be an edge in the subgraph induced by $S \cup T$ as well as in the one induced by $S' \cup T'$ for a dependency to happen.

In the second case, which is new to our proof, $\sigma \neq \sigma'$ meaning that each of the events considers a different graph. In this case, we observe that this implies both that there is a k-face τ containing both σ, σ' and that either there exist vertices $v \in \sigma \cap (S' \sqcup T')$ and $s \in S \sqcup T$ so that $\tau \setminus \sigma' = \{v, s\}$, or that $\tau \setminus \sigma' \subseteq \sigma$. As we show below, this characterization is useful to bound the number of possible events that are dependent on a certain $B_\sigma^{S,T}$. The following claim gives this characterization formally.

▷ Claim 24. Let an event $B_\sigma^{S,T} \in \mathfrak{B}$ and some subset $\mathfrak{A} \subseteq \mathfrak{B}$. If

- for any $B_{\sigma'}^{S',T'} \in \mathfrak{A}$ with $\sigma = \sigma'$ there is no edge lying in both $E_{X_\sigma}(S,T)$ and $E_{X_\sigma}(S',T')$, and
- for any $B_{\sigma'}^{S',T'} \in \mathfrak{A}$ with $\sigma \neq \sigma'$ there is no k-face $\tau \in X$ containing both σ and σ', so that $\tau \setminus \sigma$ and $\tau \setminus \sigma'$ are edges in $E_{X_\sigma}(S,T)$ and $E_{X_{\sigma'}}(S',T')$ respectively,

then $B_\sigma^{S,T}$ is mutually independent of $\mathfrak{A}$.

All left to conclude the proof of Lemma 23 is carefully counting the number of events that fulfill one of the claim conditions and provide a real function that bounds the probability of each event as in Equation (2.1). We leave the details for the formal proof, which is given in the next section.

4.2 Proving Lemma 23

This subsection is dedicated to the proof of Lemma 23 together with the subclaims it requires. We start by setting notations and highlighting features that will be needed for the proof.

Notations

We say that sets $S, T \subseteq X_\sigma(0)$ *induce a connected subgraph* if the subgraph obtained by projecting X_σ on $S \cup T$ is connected. In addition, we write $E_{X_\sigma}(S,T)$ to indicate the set of edges between S and T in X_σ. For $\hat{\sigma} \in \widehat{X}(k-2)$, we denote $\sigma = \pi(\hat{\sigma})$. When the face $\hat{\sigma}$ being considered is clear from the context we abbreviate and write f for $f_{\hat{\sigma}}$.

In addition, we rely on the following observation:

▶ **Observation 25.** *If a signing $f : X(k) \to \{\pm 1\}$ independently assigns a uniform sign to each k-face, then for any $\hat{\sigma} \in \widehat{X}(k-2)$, $f_{\hat{\sigma}}$ independently assigns a uniform sign to each edge in X_σ.*

Proof. Let $\hat{\sigma} \in \widehat{X}(k-2)$ and let $e \neq e' \in X_\sigma(1)$. Then for any $j, j' \in \{\pm 1\}$

$$\begin{aligned}\mathbb{P}[f_{\hat{\sigma}}(e) = j \wedge f_{\hat{\sigma}}(e') = j'] &= \mathbb{P}[\operatorname{sign}(\hat{\sigma}) \cdot f(\sigma \sqcup e) = j \wedge \operatorname{sign}(\hat{\sigma}) \cdot f(\sigma \sqcup e') = j'] \\ &= \mathbb{P}[\operatorname{sign}(\hat{\sigma}) \cdot f(s \sqcup e) = j] \cdot \mathbb{P}[\operatorname{sign}(\hat{\sigma}) \cdot f(\sigma \sqcup e') = j'] = \frac{1}{4}. \quad ◀\end{aligned}$$

In addition, as in [12], we can restrict the proof to consider only a pair of sets inducing connected subgraphs and deduce the result to any pair of sets. In addition, we can assume that d is as large as 996 as it is always the case that $\langle \mathbf{1}_S, A_\sigma^f \mathbf{1}_T \rangle \leq \sqrt{|S||T|}$ and $1 \leq 10\sqrt{\frac{k^2 \log d_{k-1}}{d_{k-1}}}$ for $d_{k-1} \in [1, 996]$.

We are now ready to prove Lemma 23.

Proof of Lemma 23. We set f to be a randomized signing of $X(k)$ by setting a uniform and independent sign from $\{\pm 1\}$ to any k-face. Fix some face $\hat{\sigma} \in \widehat{X}(k-2)$ with $\pi(\hat{\sigma}) = \sigma$, and disjoint sets $S, T \subseteq X_\sigma(0)$. Denote the "bad" event in which the claim does not hold for our fixed face and sets by $B_\sigma^{S,T}$. That is, $\mathbb{P}\big[B_\sigma^{S,T}\big] = \mathbb{P}\Big[|\langle \mathbf{1}_S, A_\sigma^f \mathbf{1}_T\rangle| > 10\sqrt{\frac{k^2 \log d_{k-1}}{d_{k-1}}|S||T|}\Big]$.

Fix for a moment some edge $uv \in X_\sigma(1)$, and consider the (u,v)-th entry of A_σ^f. By Definition 8, $A_\sigma^f(u,v) = \frac{1}{d_{k-1}} f_{\hat{\sigma}}(uv)$, which, per Observation 25, distributed uniformly in $\{\pm 1\}$ and is independent of all other edges signs. In addition, since

$$\langle \mathbf{1}_S, A_\sigma^f \mathbf{1}_T\rangle = \frac{1}{d_{k-1}} \sum_{uv \in E_{X_\sigma}(S,T)} f_{\hat{\sigma}}(uv),$$

$\langle \mathbf{1}_S, A_\sigma^f \mathbf{1}_T\rangle$ is a sum of independent uniform random variables over $\{\pm 1\}$, implying that

$$\mathop{\mathbb{E}}_f \left[\langle \mathbf{1}_S, A_\sigma^f \mathbf{1}_T\rangle\right] = \frac{1}{d_{k-1}} \sum_{uv \in E_{X_\sigma}(S,T)} \mathop{\mathbb{E}}_f \left[f_{\hat{\sigma}}(uv)\right] = 0.$$

Hence, by Hoeffding's inequality,

$$\begin{aligned}\mathbb{P}\big[B_\sigma^{S,T}\big] &= \mathbb{P}\left[\left|\langle \mathbf{1}_S, A_\sigma^f \mathbf{1}_T\rangle\right| > 10\sqrt{\frac{k^2 \log d_{k-1}}{d_{k-1}}|S||T|}\right] \\ &\leq 2\exp\left(-\frac{2\cdot 100\frac{k^2 \ln d_{k-1}}{d_{k-1}}|S||T|}{\sum\limits_{uv \in E_{X_\sigma}(S,T)} \left(\frac{1}{d_{k-1}} - (-\frac{1}{d_{k-1}})\right)^2}\right). \end{aligned} \tag{4.2}$$

Assuming w.l.o.g. that $|S| \geq |T|$ we get

$$\begin{aligned}\frac{200\frac{k^2 \ln d_{k-1}}{d_{k-1}}|S||T|}{\sum\limits_{uv \in E_{X_\sigma}(S,T)} \left(\frac{1}{d_{k-1}} - (-\frac{1}{d_{k-1}})\right)^2} &= \frac{200k^2 d_{k-1}(\ln d_{k-1})|S||T|}{4|E_{X_\sigma}(S,T)|} \geq \frac{50k^2 d_{k-1}(\ln d_{k-1})|S||T|}{d_{k-1}|T|} \\ &\geq 25k^2 \ln d_{k-1}|S \sqcup T|.\end{aligned}$$

Hence, denoting $c = |S \sqcup T|$,

$$\textit{Equation } (4.2) \leq 2\exp\left(-25ck^2 \ln d_{k-1}\right) \leq d_{k-1}^{-10ck^2}. \tag{4.3}$$

We turn to analyze the dependency graph of the "bad" events:[4] Recall that $\mathfrak{B}$ is the set of all events $B_\sigma^{S,T}$ for $\sigma \in X(k-2)$ and disjoint subsets $S, T \subseteq X_\sigma(0)$. Using the characterization of correlated events in $\mathfrak{B}$ given in Claim 24, we get the following bound on the neighborhood size of the events in the dependency graph:

▷ Claim 26. Let $B_\sigma^{S,T} \in \mathfrak{B}$ and denote $c = |S \sqcup T|$. Then $B_\sigma^{S,T}$ has at most $3k^2 c d^{c'-1}$ neighbors $B_{\sigma'}^{S',T'}$ with $|S' \sqcup T'| = c'$.

Now, to apply Lovász Local Lemma, one needs to define a function $\rho : \mathfrak{B} \to [0,1)$ such that $\mathbb{P}\big[B_\sigma^{S,T}\big] \leq \rho(B_\sigma^{S,T}) \prod_{B_{\sigma'}^{S',T'} \sim B_\sigma^{x,y}} \left(1 - \rho(B_{\sigma'}^{S',T'})\right)$. Set $\rho(B_\sigma^{S,T}) = d_{k-1}^{-3ck^2}$. Indeed

[4] Recall that the dependency graph of a set of events $\mathfrak{B}$ is a digraph with a vertex for each event $B \in \mathfrak{B}$ and any event B is mutually independent of $\mathfrak{B} \setminus \Gamma(B)$.

$$
\begin{aligned}
\rho(B_\sigma^{S,T}) \prod_{B_{\sigma'}^{S',T'} \in \sim B_\sigma^{S,T}} \left(1-\rho(B_{\sigma'}^{S',T'})\right) &= d_{k-1}^{-3ck^2} \prod_{B_{\sigma'}^{S',T'} \sim B_\sigma^{S,T}} \left(1-d_{k-1}^{-3|S'\cup T'|k^2}\right) \\
&= d_{k-1}^{-3ck^2} \prod_{c'\in[n]} \left(1-d_{k-1}^{-3c'k^2}\right)^{2^{c'}3ck^2 d_{k-1}^{c'-1}} && (4.4) \\
&\geq d_{k-1}^{-3ck^2} \exp\left(-6ck^2 \sum_{c'\in[n]} 2^{c'} d_{k-1}^{c'-1} d_{k-1}^{-3c'k^2}\right) && (4.5) \\
&\geq d_{k-1}^{-3ck^2} e^{-7ck^2} \\
&\geq d_{k-1}^{-10ck^2} \geq \mathbb{P}\left[B_\sigma^{S,T}\right] && (4.6)
\end{aligned}
$$

where Equation (4.4) is since for any $U \subseteq X_\sigma(0)$ of cardinality c', there are at most $2^{c'}$ pairs of disjoint sets S', T' with $S' \sqcup T' = U$, Equation (4.5) is by $e^{-x} \leq 1 - \frac{x}{2}$ for any $x \in [0, 1.59]$ and Equation (4.6) is by taking $d_{k-1} \geq 3$. Together with Equation (4.3), this concludes the proof. ◀

The formal proofs of Claim 24 and Claim 26, and the proof of Theorem 22 given Lemma 23 appear in the full version of this paper [11].

4.3 Concluding Theorem 22

Proof of Theorem 22. Let X be a $(d_0, \ldots, d_{k-1})$-regular, $\lambda(X)$-two sided (resp. one sided) HDX over n vertices, fix f to be the signing provided by Lemma 23, and set $\widehat{X}$ to be the f-local lift of X.

By Observation 18, $\widehat{X}$ is a $(2d_0, \ldots, 2d_{k-2}, d_{k-1})$-regular graph over $2n$ vertices. We need to prove that for any $\hat{\sigma} \in \widehat{X}$ of dimension $\leq k-2$, the 1-skeleton of $\widehat{X}_{\hat{\sigma}}$ is a $\max\left\{\lambda(X), O\left(\sqrt{\frac{k^2 \log^3 d_{k-1}}{d_{k-1}}}\right)\right\}$-two sided (resp. one sided) expander.

By Corollary 20, the spectra of all links $\widehat{X}_{\hat{\sigma}}$ with $\hat{\sigma}$ of dimension $< k-2$ are bounded by $\lambda(X)$. In addition, by Lemma 23, for any $\hat{\sigma} \in \widehat{X}(k-2)$ and any disjoint sets $S, T \subseteq X_\sigma(0)$ for $\sigma = \pi(\hat{\sigma})$, we have that $|\langle \mathbf{1}_S, A_\sigma^f \mathbf{1}_T \rangle| \leq O\left(\sqrt{\frac{k^2 \log d_{k-1}}{d_{k-1}} |S||T|}\right)$ where A_σ^f is the $f_{\hat{\sigma}}$-signed random walk matrix of X_σ. Together with Lemma 10 this implies

$$
\begin{aligned}
\lambda(X_\sigma) &= O\left(\sqrt{\frac{k^2 \log d_{k-1}}{d_{k-1}}}\left(1 + \log\left(\sqrt{\frac{d_{k-1}}{k^2 \log d_{k-1}}}\right)\right)\right) \\
&\leq O\left(\sqrt{\frac{k^2 \log d_{k-1}}{d_{k-1}}} \cdot \log \sqrt{d_{k-1}}\right) = O\left(\sqrt{\frac{k^2 \log^3 d_{k-1}}{d_{k-1}}}\right)
\end{aligned}
$$

hence, by Lemma 9, $\lambda(X_\sigma) = \max\left\{\lambda(X), O\left(\sqrt{\frac{k^2 \log^3 d_{k-1}}{d_{k-1}}}\right)\right\}$ and by Corollary 20, this is also the case for $\lambda(\widehat{X}_{\hat{\sigma}})$. ◀

5 An Algorithmic Version of Theorem 21

In this subsection, we prove that there is a randomized algorithm that finds a family of local lifts as in Theorem 21 when X is a high dimensional expander under mild assumptions on the degree sequence which we encapsulate in the following definition.

▶ **Definition 27** (Nice complex). *Let X be a k-dimensional simplicial complex. We say that X is* nice *if X is regular, and*

$$d_{k-2}^{1-4\log d_{k-1}} < \frac{2}{e(k+1)kd_{k-1}+1}. \tag{5.1}$$

We prove the following.

▶ **Theorem 28.** *There exists a randomized algorithm $\mathcal{A}$ that takes as input a k-dimensional complex X_0 and an integer $i \geq 1$, runs in expected time $\mathrm{poly}((2^i|X_0(0)|)^k)$ at most, and outputs a k-dimensional complex X_i with $2^i|X_0(0)|$ vertices. The algorithm has the following guarantee.*

If X_0 is a nice $(d_0, \ldots, d_{k-1})$-regular λ-two sided high dimensional expander, then X_i is a $(2^i d_0, \ldots, 2^i d_{k-2}, d_{k-1})$-regular λ'-two sided high dimensional expander where $\lambda' = O\left(\max\left\{\lambda\left(1+\log\frac{1}{\lambda}\right), \sqrt{\frac{\log^3 d_{k-1}}{d_{k-1}}}\right\}\right)$ [5].

Moreover, one can modify the algorithm so that it outputs a sequence $X_1, X_2, \ldots, X_i$ of complexes all satisfying the same guarantees (instead of just the last one), so that for every $j = 0, 1, \ldots, i-1$, X_{j+1} is a local lift of X_j.

Loosely speaking, in order to prove Theorem 28, we need to prove that there is an algorithm $\mathcal{A}$ that finds a single local lift for X in polynomial time (just as Theorem 21 was proved by the "one-step theorem" Theorem 22) with good enough spectral expansion. Then we just iteratively use $\mathcal{A}$ on its own output, setting $X_{j+1} = \mathcal{A}(X_j)$, until reaching $j = i-1$. For this to work, we also need to address the issue that $\lambda' \geq \lambda$ so naively the expansion deteriorates as we reiterate. We differ the proof of Theorem 28 for the full version of this paper [11].

▶ Remark 29 (A non-uniform algorithm for any HDX). Theorem 28 requires that X_0 be a nice complex, i.e. that (5.1) holds. However, in any family $\{X_i\}_{i=0}^{\infty}$ where X_{i+1} is a local lift of X_i, the degree $d_{k-2}(X_i)$ tends to infinity with i while the other side of both inequalities stays fixed. Thus, the inequalities will eventually hold for any family of consecutive local lifts. In fact, they should hold for any $i \geq C\log(k + d_{k-1}(X_0))$ for some large enough constant $C > 0$. Thus we can modify the algorithm to work even if X_0 is not nice (albeit with the spectral expansion bound guaranteed in Theorem 21, which is slightly worse than the expansion in Theorem 28). This is done by allowing the algorithm to do a brute-force search for the first few steps, to produce a nice X_i, and then continuing as the original algorithm does. The first few steps will eventually stop because Theorem 21 promises the existence of such an X_j. This process takes $\mathrm{poly}(|X_i|^k) + \exp(O(|X_0|^k))$ time.

Towards the proof of Theorem 28, we need the following definition and lemma from [12].

▶ **Definition 30.** *A graph G with adjacency operator A is said to be (β, t)-sparse if for every $S, T \subseteq V(G)$ such that $|S \cup T| \leq t$, $\langle \mathbf{1}_S, A\mathbf{1}_T\rangle \leq \beta\sqrt{|S||T|}$. For a k-dimensional hyper-regular complex X, we say that it is (β, t)-sparse if for every $\sigma \in X(k-2)$, the graph X_σ is (β, t)-sparse.*

▶ Remark 31. While the definition here regards any S, T with $|S \cup T| \leq t$, it is in fact equivalent to regarding only S, T with $|S \cup T| \leq t$ *such that the graph induced on $S \cup T$ is connected*. We also remark that if X is (β, t)-sparse then it is also (β', t)-sparse for any $\beta' \geq \beta$.

[5] We will not calculate the constants in the big O notation explicitly.

The reason we need this definition of sparseness is that in a random local lift, sparseness does not deteriorate with high probability. More formally, the following lemma was proven in [12].

▶ **Lemma 32** ([12, Lemma 3.4])**.** *Let $G = (V, E)$ be a d-regular graph with n vertices that is $(\beta, \log n)$-sparse for $\beta \geq 10\sqrt{\frac{\log d}{d}}$. Then with probability $\geq 1 - n^{-4\log d}$ over $f : E \to \{\pm 1\}$:*
- *For every $S, T \subseteq V$, $\left|\langle \mathbf{1}_S, A^f \mathbf{1}_T\rangle\right| \leq \beta\sqrt{|S||T|}$ and,*
- *$\widehat{G}^f$ is $(\beta, \log n + 1)$-sparse,*

We comment that [12, Lemma 3.4] does not explicitly calculate the probability $1 - n^{-4\log d}$; rather, they only say the events happen with high probability. This is the probability that is implicit in their proof. They also prove this theorem for $\beta = 10\sqrt{\frac{\log d}{d}}$ but the same proof extends to $\beta \geq 10\sqrt{\frac{\log d}{d}}$ with no additional changes.

This next claim easily follows from the definition of expansion and says that a spectral expander is sparse.

▷ Claim 33. Let G be a d-regular λ-two sided spectral expander over n vertices such that $\lambda > \frac{1}{\sqrt{d}}$ and $d \geq 3$. Then G is $(2\lambda, \log n)$-sparse.

Proof. Fix S, T such that with $|S \cup T| \leq \log n$. By the λ-expansion and the expander mixing lemma (see e.g. [40]), $\langle \mathbf{1}_S, A\mathbf{1}_T\rangle \leq \frac{|S||T|}{n} + \lambda\sqrt{|S||T|}$. We bound this term by $\left(\lambda + \frac{\log n}{n}\right)\sqrt{|S||T|}$. As $\frac{\log n}{n} \leq \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{d}} \leq \lambda$ the claim follows. ◁

We are ready to state our one-step theorem.

▶ **Theorem 34.** *There exists a randomized algorithm $\mathcal{A}$ with the following guarantees. Let X be a k-dimensional $\bar{d}$-regular λ-two sided (resp. one sided) high dimensional expander over n vertices, where $\bar{d} = (d_0, \ldots, d_{k-1})$. Let $\beta \geq 10\sqrt{\frac{\log d_{k-1}}{d_{k-1}}}$ and denote by $\lambda' = \max\left\{\lambda, O(\beta(1 + \log\frac{1}{\beta}))\right\}$. Assume that X is $(\beta, \log d_{k-2})$-sparse, and suppose that d_{k-2} and d_{k-1} satisfy $d_{k-2} > d_{k-1}^2$ and (5.1). Then $\mathcal{A}(X) = \widehat{X}$ is a local lift of X such that:*
1. *The complex $\widehat{X}$ is a λ'-two sided (resp. one sided) high dimensional expander.*
2. *The complex $\widehat{X}$ is $(\beta, \log 2d_{k-2})$-sparse.*

Upon input X satisfying the above, the algorithm runs in time $\mathrm{poly}(|X(0)|^k)$.

Proof of Theorem 34. We intend to use Lemma 16. For this, we fix the following "bad" events $\mathfrak{C} = \{C_\sigma \mid \ \sigma \in X(k-2)\}$ where $C_\sigma \subseteq \{f : X(k) \to \{\pm 1\}\}$ is the event where :
1. Either $\widehat{X_\sigma}^{\pm f_\sigma}$ is *not* a λ'-two sided spectral expander, or
2. $\widehat{X_\sigma}^{\pm f_\sigma}$ is *not* $(\beta, \log 2d_{k-2})$-sparse.

By Lemma 32 (and Lemma 10 that relates the first item in Lemma 32 to spectral expansion), $\mathbb{P}_f\left[C_\sigma\right] \leq 2d_{k-2}^{-4\log d_{k-1}}$. Moreover, because every link of a $\hat{\sigma} \in \widehat{X}(k-2)$ is a lift of X_σ with respect to $f_{\hat{\sigma}}$, then if none of the events C_σ occur, then $\widehat{X}$ satisfies both items in Theorem 34. We will use Lemma 16 to find such an assignment.

We now construct a dependency graph for $\mathfrak{C}$. Let $\sigma \in X(k-2)$ and $U \subseteq X(k-2)$. The event C_σ only depends on f_σ, so it only depends on k-faces $\tau \supseteq \sigma$. Therefore, if C_σ and $\{C_{\sigma'} \mid \ \sigma' \in U\}$ are *not* mutually independent, then in particular there is a k-face $\tau \in X$ and $\sigma' \in U$ such that $\tau \supseteq \sigma, \sigma'$. Hence, in our dependency graph we connect $C_\sigma \sim C_{\sigma'}$ if there exists such a k-face containing both σ and σ'. Let us upper bound the neighborhood size of an event C_σ. The number of neighbors that C_σ has is upper bounded by the number of k-faces containing σ times $\binom{k+1}{k-1}$ (the number of ways to choose $\sigma' \subseteq \tau$). Therefore, the number of neighbors is bounded by

$$D := \binom{k+1}{k-1} \cdot |\{\tau \in X(k) \mid \tau \supseteq \sigma\}| = \binom{k+1}{k-1}|X_\sigma(1)| = \frac{(k+1)k}{4}d_{k-2}d_{k-1}.$$

By setting $\rho : \mathfrak{C} \to [0,1)$ to be the constant function $\rho(C_\sigma) = \frac{1}{D+1}$ we have that $\mathbb{P}[C_\sigma] \leq \rho(C_\sigma)\prod_{\sigma' \sim \sigma}(1-\rho(C_\sigma))$, because $\rho(C_\sigma)\prod_{\sigma' \sim \sigma}(1-\rho(C_\sigma)) \geq \frac{1}{D+1}\left(1-\frac{1}{D+1}\right)^D \geq \frac{1}{e(D+1)}$ and $\mathbb{P}[C_\sigma] \leq 2d_{k-2}^{-4\log d_{k-1}} \leq \frac{1}{e(D+1)}$ by (5.1).

Let us now verify that the algorithm in Lemma 16 runs in polynomial time. We note that there the number of events in $\mathfrak{C}$ is $\mathrm{poly}(|X(0)|^k)$, and checking whether C_σ occurs could be done in $\mathrm{poly}(|X(0)|)$-time because it amounts to:

1. Find the spectrum of a signed adjacency operator of a d_{k-2}-sized graph.
2. Going over all connected sets $U \subseteq \widehat{X}_\sigma^{\pm f_\sigma}$ of size $\leq \log 2d_{k-2}$ for every $\sigma \in X(k-2)$, finding S, T such that $S \cup T = U$, and counting the number of edges between S and T to check if the pair S, T violates sparseness. There is a $\mathrm{poly}(|X(0)|)$ such U, S, T at most.

Therefore, the randomized algorithm in Lemma 16 will find a signing in $\mathrm{poly}(|X(0)|^k) \cdot \sum_{\sigma \in X(k-2)} \frac{1}{D} = \mathrm{poly}(|X(0)|^k)$-time. ◀

6 Derandomizing the Construction

In this section we provide a deterministic construction of $(k-1)$-bounded families of high dimensional expanders, as referred to in Theorem 3. For the rest of this section, we denote $\alpha_k(d) = 10\sqrt{\frac{k^2 \log d}{d}}$ (when k is clear from context, we will write $\alpha(d)$).

We will prove the following theorem.

▶ **Theorem 35** (Restatement of Theorem 3). *There exists a deterministic algorithm $\mathcal{B}$ that takes as input a k-dimensional complex X_0 and an integer $i \geq 1$, runs in time $\mathrm{poly}((2^i|X_0(0)|)^k)$, and outputs a k-dimensional complex X_i with $2^i|X_0(0)|$ vertices. The algorithm has the following guarantee: If X_0 is a $(d_0,\dots,d_{k-1})$-regular λ-two sided high dimensional expander, with $\lambda > \alpha_k(d_{k-1})$, $d_{k-1} > 2^{10k}$ and $|X_0(k-2)| \leq d_{k-2}^{10k}$, then X_i is a $(2^i d_0,\dots,2^i d_{k-2}, d_{k-1})$-regular λ'-two sided high dimensional expander where $\lambda' = O\left(2^{5k}\lambda\left(1+\log\frac{1}{\lambda}\right)\right)$[6]. In particular, for every $n \in \mathbb{N}$, choosing $i = \log n$ yields a complex with at least n-vertices.*

This explicit construction generalizes the explicit construction for expanders given in [12], which is based on the conditional probabilities method [4, Chapter 16].

We first observe that under the assumption that the base complex is sparse (as in Definition 30) and that $|X(k-2)|$ is not too large, then a random local lift of X is also sparse and is a high dimensional expander with high probability. Then, we explain how we can find such a lift deterministically by greedily selecting the values $f(\tau)$ one k-face at a time.

▶ **Lemma 36.** *Let X be a k-dimensional, $(d_0,\dots,d_{k-1})$-regular and $(\beta, \log d_{k-2})$-sparse simplicial complex so that $\beta \geq \alpha(d_{k-1})$ and $|X(k-2)| \leq d_{k-2}^{\log d_{k-1}}$.*

Then, for $f : X(k) \to \{\pm 1\}$ drawn uniformly at random, with probability at least $1 - d_{k-2}^{-3\log d_{k-1}}$:

1. *For every $\sigma \in X(k-2)$ and every $S, T \subseteq X_\sigma(0)$: $\left|\left\langle \mathbf{1}_S, A_\sigma^f \mathbf{1}_T \right\rangle\right| \leq \beta\sqrt{|S||T|}$.*
2. *The local lift $\widehat{X} = \widehat{X}^f$ is $(\beta, \log d_{k-2}+1)$-sparse.* ⌟

[6] We will not calculate the constants in the big O notation explicitly.

We comment that the condition $|X(k-2)| \leq d_{k-2}^{\log d_{k-1}}$ may seem odd at first glance. However, similar to Remark 29, this is eventually satisfied by every sequence $\{X_i\}_{i=0}^{\infty}$ where X_{i+1} is a local lift of X_i. Thus, we do not lose too much generality by assuming it.

The proof of Lemma 36 follows by applying Lemma 32 to every link and taking a union bound over the links. We omit the proof since it is a direct calculation.

The deterministic construction mentioned at the beginning of this section is composed of iterative applications of the local lift, where each application is according to the algorithm described in the following lemma.

▶ **Lemma 37.** *Let X be a k-dimensional $(d_0, \dots, d_{k-1})$-regular $(\beta, \log d_{k-2})$-sparse simplicial complex with $d_{k-1} > 2^{10k}$, $\beta \geq \alpha(d_{k-1})$ and such that $|X(k-2)| \leq d_{k-2}^{10k}$.*

Then, there is a deterministic $\operatorname{poly}(|X(0)|^k)$ *time algorithm for finding a function $f : X(k) \to \{\pm 1\}$ such that:*

1. *For every $\sigma \in X(k-2)$, $\|A_\sigma^f\| = O\left(2^{5k}\beta\left(1 + \log\frac{1}{\beta}\right)\right)$.*
2. *$\widehat{X}^f$ is $(\beta, \log d_{k-2} + 1)$-sparse.*

The proof of Lemma 37 appears in the full version of this paper [11]. We give here a short discussion of the techniques used there. The proof uses the method of conditional probabilities. The main idea is that, given the conditions on the input complex, we can define random variables denoted $Z^{(\sigma)}$, which serve as "error" indicators, where these errors occur with very small probability. By defining another set of random variables $Y^{(\sigma)}$ which correlate with the links' expansions, and amplifying the impact of each error, we are able to choose $f(\tau)$ k-face by k-face, while tracking the expected value of the sum of those variables efficiently and making sure no error occurs. We are now ready to prove our main result in this section.

Proof of Theorem 35. Let $\bar{d} = (d_0, d_1, \dots, d_{k-1} = d)$ and let X_0 be a $\bar{d}$-regular λ-two sided high dimensional expander for $\lambda > \alpha_k(d)$, such that $|X_0(k-2)| \leq d_{k-2}^{10k}$. By Claim 33, it is also $(2\lambda, \log d_{k-2})$-sparse.

Denote by $\mathcal{B}'$ the algorithm suggested by Lemma 37, and let $X_1, X_2, \dots, X_i$ be such that $X_j = \mathcal{B}'(X_{j-1})$ for $j \in [i]$. We set X_i to be $\mathcal{B}$'s output.

Let us show that, X_i meets the guarantees of Theorem 35. By Observation 18, for every $j \in [i]$, X_j is $(2^j d_0, 2^j d_1, \dots, 2^j d_{k-2}, d)$-regular and $|X_j(0)| = 2^j |X_0(0)|$.

In addition, one can verify by a direct calculation that, for any $j \in [i]$, $|X_j(k-2)| = 2^{k-1}|X_{j-1}(k-2)|$, so if $|X_{j-1}(k-2)| \leq d_{k-2}(X_{j-1})^{10k}$ then $|X_j(k-2)| = 2^{k-1}|X_{j-1}(k-2)| \leq d_{k-2}(X_{j-1})^{10k} \cdot 2^{k-1} \leq d_{k-2}(X_j)^{10k}$. Thus, by induction and the fact that this inequality holds for X_0, this holds for every j.

Finally, by Lemma 37 one inductively obtains that for any j:

1. X_j is an $O\left(2^{5k}\lambda\left(1 + \log\frac{1}{\lambda}\right)\right)$-high dimensional expander.
2. X_j is $(2\lambda, \log d_{k-2}(X_i))$-sparse.
3. X_j computed in time $\operatorname{poly}(|X_{j-1}(0)|^k) = \operatorname{poly}(2^{j-1}|X_0(0)|^k)$.

as required. ◀

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