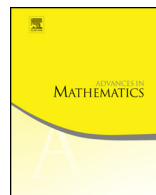




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Advances in Mathematics

journal homepage: www.elsevier.com/locate/aim

Global well-posedness of the energy subcritical nonlinear wave equation with initial data in a critical space



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ARTICLE INFO

Article history:

Received 6 August 2021

Received in revised form 31 August 2023

Accepted 11 June 2024

Available online 26 June 2024

Communicated by C. Fefferman

MSC:

35L05

Keywords:

Wave equation

Scattering

ABSTRACT

In this paper we prove global well-posedness and scattering for the defocussing, energy-subcritical, nonlinear wave equation on $\mathbb{R}^{1+3}$ with initial data in a critical Besov space. No radial symmetry assumption is needed.

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1. Introduction

In this paper, we continue the study of the defocussing, energy subcritical nonlinear wave equation,

$$u_{tt} - \Delta u + |u|^{p-1}u = 0, \quad u(0, x) = u_0, \quad u_t(0, x) = u_1, \quad 3 \leq p < 5, \quad (1.1)$$

with initial data in a critical space. A critical space is a space that is invariant under the scaling symmetry. Observe that (1.1) is invariant under the scaling symmetry

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<https://doi.org/10.1016/j.aim.2024.109811>

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$$u(t, x) \mapsto \lambda^{\frac{2}{p-1}} u(\lambda t, \lambda x), \quad \lambda > 0. \quad (1.2)$$

Under the above scaling symmetry, the size of the initial data changes by a factor of

$$\begin{aligned} \|\lambda^{\frac{2}{p-1}} u_0(\lambda x)\|_{\dot{H}^s(\mathbb{R}^3)} &= \lambda^{s+\frac{2}{p-1}-\frac{3}{2}} \|u_0\|_{\dot{H}^s(\mathbb{R}^3)}, \\ \|\lambda^{\frac{p+1}{p-1}} u_1(\lambda x)\|_{\dot{H}^{s-1}(\mathbb{R}^3)} &= \lambda^{s+\frac{2}{p-1}-\frac{3}{2}} \|u_1\|_{\dot{H}^{s-1}(\mathbb{R}^3)}. \end{aligned} \quad (1.3)$$

Thus, (1.1) is called $\dot{H}^{s_c} \times \dot{H}^{s_c-1}$ -critical when

$$s_c = \frac{3}{2} - \frac{2}{p-1}, \quad (1.4)$$

because this norm is invariant under (1.2).

The scaling symmetry (1.2) completely determines the local well-posedness theory for (1.1), where local well-posedness is defined in the usual way.

Definition 1 (*Locally well-posed*). The initial value problem (1.1) is said to be locally well-posed if there exists an open interval $I \subset \mathbb{R}$ containing 0 such that:

- (1) A unique solution $u \in L_t^\infty \dot{H}^{s_c}(I \times \mathbb{R}^3) \cap L_{t,loc}^{2(p-1)} L_x^{2(p-1)}(I \times \mathbb{R}^3)$, $u_t \in L_t^\infty \dot{H}^{s_c-1}(I \times \mathbb{R}^3)$ exists.
- (2) The solution u is continuous in time, $u \in C(I; \dot{H}^{s_c}(\mathbb{R}^3))$, $u_t \in C(I; \dot{H}^{s_c-1}(\mathbb{R}^3))$.
- (3) The solution u depends continuously on the initial data in the topology of item one.

Theorem 1. Equation (1.1) is locally well-posed for initial data in $(u_0, u_1) \in \dot{H}^{s_c}(\mathbb{R}^3) \times \dot{H}^{s_c-1}(\mathbb{R}^3)$ on some interval $[-T(u_0, u_1), T(u_0, u_1)]$. The time of well-posedness $T(u_0, u_1)$ depends on the profile of the initial data (u_0, u_1) , not just its size. For data sufficiently small in $\dot{H}^{s_c} \times \dot{H}^{s_c-1}$, global well-posedness and scattering hold.

Additional regularity is enough to give a lower bound on the time of well-posedness. Therefore, there exists some $T(\|u_0\|_{\dot{H}^s}, \|u_1\|_{\dot{H}^{s-1}}) > 0$ for any $s_c < s < \frac{3}{2}$.

Negatively, equation (1.1) is ill-posed for $u_0 \in \dot{H}^s(\mathbb{R}^3)$ and $u_1 \in \dot{H}^{s-1}(\mathbb{R}^3)$ when $s < s_c$.

Proof. See [14]. $\square$

Definition 2 (*Scattering*). A solution to (1.1) with initial data (u_0, u_1) is said to be scattering in some $\dot{H}^s(\mathbb{R}^3) \times \dot{H}^{s-1}(\mathbb{R}^3)$ if there exist $(u_0^+, u_1^+), (u_0^-, u_1^-) \in \dot{H}^s \times \dot{H}^{s-1}$ such that

$$\lim_{t \rightarrow +\infty} \|(u(t), u_t(t)) - S(t)(u_0^+, u_1^+)\|_{\dot{H}^s \times \dot{H}^{s-1}} = 0, \quad (1.5)$$

and

$$\lim_{t \rightarrow -\infty} \|(u(t), u_t(t)) - S(t)(u_0^-, u_1^-)\|_{\dot{H}^s \times \dot{H}^{s-1}} = 0, \quad (1.6)$$

where u is the solution to (1.1) with initial data (u_0, u_1) and $S(t)(f, g)$ is the solution operator to the linear wave equation. That is, if $(u(t), u_t(t)) = S(t)(f, g)$, then

$$u_{tt} - \Delta u = 0, \quad u(0, x) = f, \quad u_t(0, x) = g. \quad (1.7)$$

Equation (1.1) is called scattering for data in a certain subset X if the solution to (1.1) with initial data in X is globally well-posed, the solution scatters both forward and backward in time, and the scattering states (u_0^+, u_1^+) and (u_0^-, u_1^-) depend continuously on the initial data.

For large data, global well-posedness and scattering is known to hold for (1.1) for sufficiently regular, sufficiently rapidly decaying initial data.

Theorem 2. *Equation (1.1) is globally well-posed and scattering for initial data (u_0, u_1) satisfying*

$$\|\langle x \rangle \nabla u_0\|_{L^2} + \|u_0\|_{L^2} + \|\langle x \rangle u_1\|_{L^2} < \infty. \quad (1.8)$$

Proof. See [20]. $\square$

For any (u_0, u_1) satisfying (1.8), $(u_0, u_1) \in \dot{H}^{s_c} \times \dot{H}^{s_c-1}$. However, the question of the long time behavior of initial data in the critical Sobolev space that does not belong to (1.8) remains open. In the radially symmetric case, this question was settled in [4] and [3].

Theorem 3. *For $3 \leq p < 5$, the initial value problem (1.1) is globally well-posed and scattering for radial initial data $(u_0, u_1) \in \dot{H}^{s_c}(\mathbb{R}^3) \times \dot{H}^{s_c-1}(\mathbb{R}^3)$. Moreover, there exists a function $f : [3, 5) \times [0, \infty) \rightarrow [0, \infty)$ such that if u solves (1.1) with initial data $(u_0, u_1) \in \dot{H}^{s_c} \times \dot{H}^{s_c-1}$, then*

$$\|u\|_{L_{t,x}^{2(p-1)}(\mathbb{R} \times \mathbb{R}^3)} \leq f(p, \|u_0\|_{\dot{H}^{s_c}(\mathbb{R}^3)} + \|u_1\|_{\dot{H}^{s_c-1}(\mathbb{R}^3)}), \quad (1.9)$$

and thus the solution scatters both forward and backward in time.

Proof. This was proved in [4] when $p = 3$ and in [3] when $3 < p < 5$. The proof used the computations on hyperbolic coordinates in [16] and [17]. $\square$

Remark 1. The argument in [14] may be used to show that (1.9) is equivalent to scattering in the critical Sobolev norm.

An important stepping stone in the proof of Theorem 3 was the result of [2] for radially symmetric initial data in a critical Besov space.

Theorem 4. *The defocussing, cubic nonlinear wave equation ((1.1) when $p = 3$) is globally well-posed and scattering for radially symmetric initial data $u_0 \in B_{1,1}^2$ and $u_1 \in B_{1,1}^1$. $B_{p,q}^s$ is the Besov space defined by the norm*

$$\|u\|_{B_{p,q}^s(\mathbb{R}^3)} = \left(\sum_j 2^{jsp} \|P_j u\|_{L^q}^p \right)^{1/p}. \quad (1.10)$$

The operator P_j is the usual Littlewood–Paley projection operator.

In this paper we generalize Theorem 4 to the case when $3 < p \leq 4$ with nonradial initial data.

Theorem 5. *Equation (1.1) is globally well-posed and scattering when $3 < p \leq 4$ initial data $u_0 \in B_{1,1}^{s_c + \frac{3}{2}}$ and $u_1 \in B_{1,1}^{s_c + \frac{1}{2}}$. Furthermore, there exists $f : (3, 4] \times [0, \infty) \rightarrow [0, \infty)$ such that*

$$\|u\|_{L_{t,x}^{2(p-1)}(\mathbb{R} \times \mathbb{R}^3)} \leq f(p, \|u_0\|_{B_{1,1}^{s_c + \frac{3}{2}}(\mathbb{R}^3)} + \|u_1\|_{B_{1,1}^{s_c + \frac{1}{2}}(\mathbb{R}^3)}). \quad (1.11)$$

Global well-posedness alone holds for $3 \leq p < 5$.

The $B_{1,1}^{s_c + \frac{3}{2}} \times B_{1,1}^{s_c + \frac{1}{2}}$ norm is invariant under the scaling symmetry (1.2). By the Sobolev embedding theorem, $B_{1,1}^{s_c + \frac{3}{2}} \subset \dot{H}^{s_c}$ and $B_{1,1}^{s_c + \frac{1}{2}} \subset \dot{H}^{s_c - 1}$. The main advantage that $B_{1,1}^{s_c + \frac{3}{2}} \times B_{1,1}^{s_c + \frac{1}{2}}$ provides is the dispersive estimate for the wave equation

$$\|S(t)(u_0, u_1)\|_{L^\infty} \lesssim \frac{1}{t} \|(\nabla^2 u_0, \nabla u_1)\|_{L^1 \times L^1}, \quad (1.12)$$

which implies good behavior for the solution to the linear wave equation with initial data (u_0, u_1) for $t \neq 0$. Therefore, a helpful heuristic in thinking about Theorem 4 is that blowup of a solution to (1.1) with initial data in $B_{1,1}^2 \times B_{1,1}^1$ must occur when $t = 0$ if it occurs at all. Radial symmetry further implies that the blowup must occur at the origin in space and time.

The results in Theorem 3 addressed initial data that was merely radially symmetric, but not in $B_{1,1}^2 \times B_{1,1}^1$, so the blowup could occur at any time, but only at the origin in space, $x = 0$. Theorem 5 approaches this problem from the other direction. The fact that $(u_0, u_1) \in B_{1,1}^{s_c + \frac{3}{2}} \times B_{1,1}^{s_c + \frac{1}{2}}$ means that, heuristically, the blowup may occur anywhere in $\mathbb{R}^3$, but only at time $t = 0$, if it occurs at all.

1.1. Outline of the proof

The only obstacle to proving Theorem 5 is that the $\dot{H}^{s_c} \times \dot{H}^{s_c - 1}$ norm of (u, u_t) may blow up either forward or backward in time.

Theorem 6. Suppose $(u_0, u_1) \in \dot{H}^{s_c}(\mathbb{R}^3) \times \dot{H}^{s_c-1}(\mathbb{R}^3)$ and u solves (1.1) on a maximal interval $0 \in I \subset \mathbb{R}$, with $3 < p < 5$ and

$$\sup_{t \in I} \|u(t)\|_{\dot{H}^{s_c}(\mathbb{R}^3)} + \|u_t(t)\|_{\dot{H}^{s_c-1}(\mathbb{R}^3)} < \infty. \quad (1.13)$$

Then $I = \mathbb{R}$ and the solution u scatters both forward and backward in time.

Proof. This theorem was proved in [1]. The proof uses the concentration compactness method. This method has been applicable to many scattering results for nonlinear dispersive equations. Two important results in this area are scattering for the cubic nonlinear Schrödinger equation for solutions with bounded $\dot{H}^{1/2}$ norm, [12] and scattering/blowup phenomena for the focusing nonlinear wave equation [11]. $\square$

While there is no known conserved quantity that controls the $\dot{H}^{s_c} \times \dot{H}^{s_c-1}$ norm of $(u(t), u_t(t))$ for a solution to (1.1) with generic initial data $(u_0, u_1) \in \dot{H}^{s_c} \times \dot{H}^{s_c-1}$, a solution to (1.1) does have the conserved energy

$$E(u(t)) = \frac{1}{2} \int |\nabla u(t, x)|^2 dx + \frac{1}{2} \int u_t(t, x)^2 dx + \frac{1}{p+1} \int |u(t, x)|^{p+1} dx = E(u(0)). \quad (1.14)$$

For $u_0 \in \dot{H}^1 \cap \dot{H}^{s_c}$ and $u_1 \in L^2 \cap \dot{H}^{s_c-1}$, the Sobolev embedding theorem implies

$$\|u(0)\|_{L^{p+1}(\mathbb{R}^3)}^{p+1} \lesssim \|u_0\|_{\dot{H}^{s_c}(\mathbb{R}^3)}^{p-1} \|u_0\|_{\dot{H}^1(\mathbb{R}^3)}^2, \quad (1.15)$$

so

$$E(u(0)) \lesssim_{\|u_0\|_{\dot{H}^{s_c}}} \|u_0\|_{\dot{H}^1}^2 + \|u_1\|_{L^2}^2. \quad (1.16)$$

Conservation of energy then implies a uniform bound on the $\|(u(t), u_t(t))\|_{\dot{H}^1 \times L^2}$ norm for the entire time of existence of u , which by Theorem 1 implies that the solution to (1.1) with initial data $u_0 \in \dot{H}^1 \cap \dot{H}^{s_c}$ and $u_1 \in L^2 \cap \dot{H}^{s_c-1}$ is global.

For generic initial data $u_0 \in B_{1,1}^{s_c+\frac{3}{2}}$ and $u_1 \in B_{1,1}^{s_c+\frac{1}{2}}$, there is no reason to think that the initial data lies in $\dot{H}^1 \times L^2$. However, using the dispersive estimate (1.12), we can split a solution $u(t)$ into a piece lying in $\dot{H}^1 \times L^2$ and a piece with good decay estimates as t becomes large. A similar computation was used in [2] to prove Theorem 4.

The local well-posedness result of Theorem 1 implies that there exists an open neighborhood I of 0 for which (1.1) has a solution, and

$$\|u\|_{L_{t,x}^{2(p-1)}(I \times \mathbb{R}^3)} \leq \epsilon, \quad (1.17)$$

for some $\epsilon > 0$ small. Rescaling by (1.2),

$$\|u\|_{L_{t,x}^{2(p-1)}([-1,1] \times \mathbb{R}^3)} \leq \epsilon. \quad (1.18)$$

This solution satisfies Duhamel's principle

$$u(t) = S(t)(u_0, u_1) - \int_0^t S(t - \tau)(0, |u|^{p-1}u) d\tau. \quad (1.19)$$

Next, combining the dispersive estimate (1.12) and local well-posedness theory, it is possible to prove that

$$t^{\frac{2-s_c}{p}} \|u(t)\|_{L^{2p}}, \quad (1.20)$$

is uniformly bounded for all $t \in [-1, 1]$. Therefore, by standard energy estimates,

$$\left\| \int_{1/2}^1 S(1 - \tau)(0, |u|^{p-1}u) d\tau \right\|_{\dot{H}^1 \times L^2} \lesssim 1, \quad (1.21)$$

with implicit constant bounded by the norm of the initial data in $B_{1,1}^{s_c + \frac{3}{2}} \times B_{1,1}^{s_c + \frac{1}{2}}$.

Let

$$v(1) = \int_{1/2}^1 S(1 - \tau)(0, |u|^{p-1}u) d\tau, \quad v_t(1) = \partial_t \int_{1/2}^t S(t - \tau)(0, |u|^{p-1}u) d\tau|_{t=1}, \quad (1.22)$$

and let

$$w(1) = u(1) - v(1), \quad w_t(1) = u_t(1) - v_t(1). \quad (1.23)$$

It follows from (1.18) and Theorem 1 that (1.1) has a local solution on $[1, T)$ for some $T > 1$. Decompose this solution $u = v + w$, which solve

$$\begin{aligned} w_{tt} - \Delta w &= 0, & w(1, x) &= w(1), & w_t(1, x) &= w_t(1), \\ v_{tt} - \Delta v + u^3 &= 0, & v(1, x) &= v(1), & v_t(1, x) &= v_t(1). \end{aligned} \quad (1.24)$$

To prove that T may be extended to $T = \infty$, it is enough to prove that $E(v(t))$, where E is given by (1.14), is uniformly bounded on any compact subset of $[1, \infty)$. To see why, first note that $w_{tt} - \Delta w = 0$ has a global solution. Next, the rescaling used to obtain (1.18) will be used to show that for any $T \geq 0$,

$$\|w\|_{L_{t,x}^{2(p-1)}([T, T+1] \times \mathbb{R}^3)} \leq \frac{\epsilon}{2}. \quad (1.25)$$

Therefore, using standard perturbative arguments,

$$v_{tt} - \Delta v + |u|^{p-1}u = 0, \quad (1.26)$$

may be treated as a perturbation of

$$v_{tt} - \Delta v + |v|^{p-1}v = 0, \quad (1.27)$$

on short time intervals. Therefore, if $E(v(t_0)) < \infty$, (1.1) is locally well-posed on the interval $[t_0, t_0 + \frac{1}{E(v(t_0))}]$, so it is enough to prove that $E(v(t))$ is uniformly bounded on any compact subset of $[1, \infty)$.

To prove the uniform bound, standard calculations imply

$$\frac{d}{dt}E(v(t)) = -\langle v_t, |u|^{p-1}u - |v|^{p-1}v \rangle. \quad (1.28)$$

The most difficult component of (1.28) is a term of the form

$$-\langle v_t, v^{p-1}w \rangle \lesssim \| |\nabla|^{s_c - \frac{1}{2}} w \|_{L^\infty} E(v(t)). \quad (1.29)$$

Using the dispersive estimate (1.12) it is possible to prove $\| |\nabla|^{s_c - \frac{1}{2}} w \|_{L^\infty} \lesssim \frac{1}{t}$. Plugging this estimate into (1.29) and using Gronwall's inequality then proves a uniform bound on $E(v(t))$ on any compact subset, completing the proof of global well-posedness.

The above computations are not enough to prove scattering. In fact, even if one assumed initial data $u_0 \in \dot{H}^1 \cap \dot{H}^{s_c}$ and $u_1 \in L^2 \cap \dot{H}^{s_c-1}$, conservation of energy would not guarantee a uniform bound on $\|u(t), u_t(t)\|_{\dot{H}^{s_c} \times \dot{H}^{s_c-1}}$. Indeed, recall that [20] assumed sufficient decay on the initial data.

However, the Lebesgue dominated convergence theorem implies that outside a compact set, the initial data has small $\dot{H}^{s_c} \times \dot{H}^{s_c-1}$ norm. By finite propagation speed, this implies scattering outside a light cone. Inside the light cone, we follow and use the conformal energy, as in [20], [5], and [7].

We obtain the bound (1.11) using the profile decomposition argument in [15].

Acknowledgments

A prior version of this paper asserted global well-posedness and scattering for $3 < p < 5$, using hyperbolic coordinates, following [16] and [17]. Such coordinates only hold for the radially symmetric equation, requiring the result to be proved in a different way. The author is extremely grateful to Changxing Miao and Ruipeng Shen for pointing out to him this mistake in the earlier draft.

The author is also grateful to several anonymous referees for their many helpful changes and corrections.

During the time of writing this paper, the author was partially supported by NSF Grants DMS-1764358 and DMS-2153750. The author is also grateful to Andrew Lawrie and Walter Strauss for some helpful conversations at MIT on subcritical nonlinear wave equations.

2. Local behavior of the solution to (1.1)

As was mentioned in the introduction, the main singularity for initial data in a Besov space occurs when $t = 0$, from (1.12). To combat this, we study the local behavior of the solution in greater detail. Using (1.2), it is possible to rescale equation (1.1) so that (1.1) is locally well-posed on $[-1, 1]$ and the solution satisfies

$$\|u\|_{L_{t,x}^{2(p-1)}([-1,1] \times \mathbb{R}^3)} \leq \epsilon. \quad (2.1)$$

Proof of (2.1). Recall the Strichartz estimates for the wave equation.

Theorem 7. *Let I be a time interval and let $u : I \times \mathbb{R}^3 \rightarrow \mathbb{R}$ be a Schwartz solution to the wave equation*

$$u_{tt} - \Delta u = F, \quad u(0) = u_0, \quad \partial_t u(0) = u_1, \quad (2.2)$$

where $0 \in I$. Then we have the estimates,

$$\begin{aligned} & \|u\|_{L_t^q L_x^r(I \times \mathbb{R}^3)} + \|u\|_{C_t^0 \dot{H}_x^s(I \times \mathbb{R}^3)} + \|\partial_t u\|_{C_t^0 \dot{H}_x^s(I \times \mathbb{R}^3)} \\ & \lesssim_{q,r,s} (\|u_0\|_{\dot{H}^s(\mathbb{R}^3)} + \|u_1\|_{\dot{H}^{s-1}(\mathbb{R}^3)} + \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(I \times \mathbb{R}^3)}), \end{aligned} \quad (2.3)$$

for any $s \geq 0$, $2 < q, \tilde{q} \leq \infty$, and $2 \leq r, \tilde{r} < \infty$ obey the scaling condition,

$$\frac{1}{q} + \frac{3}{r} = \frac{3}{2} - s = \frac{1}{\tilde{q}'} + \frac{3}{\tilde{r}'} - 2, \quad (2.4)$$

and satisfy the wave admissibility conditions

$$\frac{1}{q} + \frac{1}{r}, \quad \frac{1}{\tilde{q}} + \frac{1}{\tilde{r}} \leq \frac{1}{2}. \quad (2.5)$$

Proof. This theorem is copied from [23]. See [21], [10], [8], [9], [14], [18], [19], [13] for the proof of this theorem. $\square$

By Theorem 7, if u solves (2.1), then

$$\|u\|_{L_t^q L_x^r \cap L_{t,x}^{2(p-1)}([-1,1] \times \mathbb{R}^3)} \lesssim_p \|u_0\|_{\dot{H}^{s_c}} + \|u_1\|_{\dot{H}^{s_c-1}} + \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}([-1,1] \times \mathbb{R}^3)}, \quad (2.6)$$

where

$$\begin{aligned} \frac{1}{q} &= \frac{1}{2} s_c, & \frac{1}{r} &= \frac{1}{2} - \frac{1}{2} s_c, & s_c &= \frac{3}{2} - \frac{2}{p-1}, \\ \frac{1}{\tilde{q}'} &= \frac{1}{q} + \frac{1}{2}, & \frac{1}{\tilde{r}'} &= \frac{1}{r} + \frac{1}{2}. \end{aligned} \quad (2.7)$$

When $3 \leq p < 5$, (q, r) is an admissible pair that satisfies (2.5), and $\tilde{q}'$ and $\tilde{r}'$ satisfies (2.4).

Since $(u_0, u_1) \in B_{1,1}^{s_c+\frac{3}{2}} \times B_{1,1}^{s_c+\frac{1}{2}}$, there exists some $j_0 \in \mathbb{Z}$ such that

$$\sum_{j \geq j_0} 2^{j(s_c+\frac{3}{2})} \|P_j u_0\|_{L^1} + 2^{j(s_c+\frac{1}{2})} \|P_j u_1\|_{L^1} \leq c\epsilon, \quad (2.8)$$

for some $c > 0$ that is determined by the implicit constant in (2.3). Using (1.2) with

$$\lambda \sim 2^{-j_0} (\|u_0\|_{B_{1,1}^{s_c+3/2}} + \|u_1\|_{B_{1,1}^{s_c+1/2}})^{-\frac{1}{1-s_c}},$$

rescale. After the rescaling, we have a new j_0 that satisfies (2.8) and furthermore,

$$2^{j_0(1-s_c)} \cdot \|(u_0, u_1)\|_{B_{1,1}^{s_c+\frac{3}{2}} \times B_{1,1}^{s_c+\frac{1}{2}}} \leq c\epsilon. \quad (2.9)$$

Theorem 7 and (2.8) imply

$$\|S(t)(P_{\geq j_0} u_0, P_{\geq j_0} u_1)\|_{L_t^q L_x^r \cap L_{t,x}^{2(p-1)}(\mathbb{R} \times \mathbb{R}^3)} \leq \frac{\epsilon}{4}. \quad (2.10)$$

Also, by the Sobolev embedding theorem, (2.8), and the fact that $S(t)$ is a unitary operator on $\dot{H}^s \times \dot{H}^{s-1}$,

$$\|S(t)(P_{\leq j_0} u_0, P_{\leq j_0} u_1)\|_{L_t^\infty L^{2(p-1)}(\mathbb{R} \times \mathbb{R}^3)} \leq \frac{\epsilon}{4}, \quad (2.11)$$

so by Hölder's inequality,

$$\|S(t)(u_0, u_1)\|_{L_{t,x}^{2(p-1)}([-1,1] \times \mathbb{R}^3)} \leq \frac{3\epsilon}{4}. \quad (2.12)$$

A similar calculation also implies

$$\|S(t)(u_0, u_1)\|_{L_t^q L_x^r([-1,1] \times \mathbb{R}^3)} \leq \frac{3\epsilon}{4}. \quad (2.13)$$

Plugging (2.12) into (1.19) and using (2.3) and Picard iteration implies that for $\epsilon > 0$ sufficiently small, (1.1) is locally well-posed on $[-1, 1]$, and the solution satisfies

$$\|u\|_{L_{t,x}^{2(p-1)}([-1,1] \times \mathbb{R}^3)} \leq \epsilon. \quad (2.14)$$

This proof is a modification of the proof of scattering in [14] for small data and local well-posedness on a time interval depending on the profile of the initial data. $\square$

The constant $\epsilon > 0$ will eventually be chosen to depend on $\|u_0\|_{B_{1,1}^{s_c+\frac{3}{2}}} + \|u_1\|_{B_{1,1}^{s_c+\frac{1}{2}}}$. Under (2.1), the behavior of u on the interval $[-1, 1]$ is approximately linear.

Proposition 1. *If u is a solution to (1.1) on $[-1, 1]$ with $\|u\|_{L_{t,x}^{2(p-1)}([-1,1] \times \mathbb{R}^3)} \leq \epsilon(A)$, where $(u_0, u_1) \in B_{1,1}^{3/2+s_c} \times B_{1,1}^{1/2+s_c}$ with $A = \|u_0\|_{B_{1,1}^{3/2+s_c}} + \|u_1\|_{B_{1,1}^{1/2+s_c}}$, then*

$$\sum_j 2^{js_c} \|P_j u\|_{L_t^\infty L_x^2([-1,1] \times \mathbb{R}^3)} \lesssim A. \quad (2.15)$$

Proof. Using the Strichartz estimates in Theorem 7, if (q, r) and $(\tilde{q}, \tilde{r})$ are given by (2.7),

$$\begin{aligned} & 2^{js_c} \|P_j u\|_{L_t^\infty L_x^2([-1,1] \times \mathbb{R}^3)} + \|P_j u\|_{L_{t,x}^{2(p-1)} \cap L_t^q L_x^r([-1,1] \times \mathbb{R}^3)} + 2^{j(s_c - \frac{1}{2})} \|P_j u\|_{L_{t,x}^4([-1,1] \times \mathbb{R}^3)} \\ & + 2^{-j(1-s_c)/2} \|P_j u\|_{L_t^{\frac{4q}{2+q}} L_x^{2r}([-1,1] \times \mathbb{R}^3)} \lesssim 2^{js_c} \|P_j u_0\|_{L^2} + 2^{j(s_c-1)} \|P_j u_1\|_{L^2} \\ & + 2^{-j(1-s_c)/2} \|P_j F_1\|_{L_t^{\frac{4q}{3q+2}} L_x^{\frac{2r}{r+1}}([-1,1] \times \mathbb{R}^3)} + 2^{j(s_c - \frac{1}{4})} \|P_j F_2\|_{L_t^{8/5} L_x^{8/7}([-1,1] \times \mathbb{R}^3)}, \end{aligned} \quad (2.16)$$

where $P_j F_1 + P_j F_2 = P_j(|u|^{p-1}u)$ is a decomposition of the nonlinearity. Using Taylor's theorem, decompose

$$F_1 = |P_{\leq j} u|^{p-1} (P_{\leq j} u), \quad F_2 = |u|^{p-1} u - |P_{\leq j} u|^{p-1} (P_{\leq j} u) = O(|u|^{p-1} |P_{\geq j} u|). \quad (2.17)$$

Proposition 1 follows directly from (2.16) and $u_0 \in B_{1,1}^{s_c+3/2}$, $u_1 \in B_{1,1}^{s_c+1/2}$. Indeed,

$$\begin{aligned} & \|F_1\|_{L_t^{\frac{4q}{3q+2}} L_x^{\frac{2r}{r+1}}([-1,1] \times \mathbb{R}^3)} \\ & \lesssim \|P_{\leq j} u\|_{L_{t,x}^{2(p-1)}([-1,1] \times \mathbb{R}^3)}^{p-1} \|P_{\leq j} u\|_{L_t^{\frac{4q}{q+2}} L_x^{2r}([-1,1] \times \mathbb{R}^3)}, \end{aligned} \quad (2.18)$$

and

$$\|F_2\|_{L_t^{8/5} L_x^{8/7}([-1,1] \times \mathbb{R}^3)} \lesssim \|P_{\geq j} u\|_{L_t^8 L_x^{8/3}([-1,1] \times \mathbb{R}^3)} \|u\|_{L_{t,x}^{2(p-1)}([-1,1] \times \mathbb{R}^3)}^{p-1}, \quad (2.19)$$

so by Young's inequality and (2.16), the proof of Proposition 1 is complete. Indeed, letting X_j denote the left hand side of (2.16),

$$\begin{aligned} X_j & \lesssim 2^{js_c} \|P_j u_0\|_{L^2} + 2^{j(s_c-1)} \|P_j u_1\|_{L^2} \\ & + \epsilon^{p-1} \sum_{k \geq j} 2^{(j-k)(s_c - \frac{1}{4})} X_k + \epsilon^{p-1} \sum_{k \leq j} 2^{(k-j) \frac{(1-s_c)}{2}} X_k, \end{aligned} \quad (2.20)$$

which implies (2.15). $\square$

The dispersive estimates (1.12) also give additional $L_t^q L_x^r$ bounds on the solution u in $[-1, 1]$ that lie outside the admissible pairs in Theorem 7. These estimates fail at the endpoint $p = 3$, which is the main technical obstacle to proving scattering in the cubic case.

Theorem 8. For $3 < p < 5$, if $\frac{1}{q} = \frac{3}{2} - s_c = \frac{2}{p-1}$,

$$\|u\|_{L_t^q L_x^\infty([-1,1] \times \mathbb{R}^3)} + \sum_j 2^{j(s_c - \frac{1}{2})} \sup_{t \in [-1,1]} t \|P_j u\|_{L^\infty} \lesssim \epsilon. \quad (2.21)$$

Proof. Using the dispersive estimate

$$\|S(t)(u_0, u_1)\|_{L^\infty} \lesssim \frac{1}{t} \|(u_0, u_1)\|_{B_{1,1}^2 \times B_{1,1}^1}, \quad (2.22)$$

for any $j \in \mathbb{Z}$,

$$\|S(t)(P_j u_0, P_j u_1)\|_{L^\infty} \lesssim \frac{1}{t} 2^{-j(s_c - \frac{1}{2})} [2^{j(\frac{3}{2} + s_c)} \|P_j u_0\|_{L^1} + 2^{j(\frac{1}{2} + s_c)} \|P_j u_1\|_{L^1}]. \quad (2.23)$$

Interpolating (2.23) with

$$\|S(t)(P_j u_0, P_j u_1)\|_{L^\infty} \lesssim 2^{j(\frac{3}{2} - s_c)} \|(P_j u_0, P_j u_1)\|_{\dot{H}^{s_c} \times \dot{H}^{s_c-1}}, \quad (2.24)$$

and making use of (2.8) and (2.10), we have proved

$$\begin{aligned} \sum_j \sup_{t \in [-1,1]} t^{\frac{3}{2} - s_c} \|S(t)(P_j u_0, P_j u_1)\|_{L^\infty} + \sum_j \|S(t)(P_j u_0, P_j u_1)\|_{L_t^q L_x^\infty(\mathbb{R} \times \mathbb{R}^3)} \\ + \sum_j 2^{j(s_c - \frac{1}{2})} \sup_{t \in [-1,1]} t \|S(t)(P_j u_0, P_j u_1)\|_{L^\infty} \lesssim \epsilon. \end{aligned} \quad (2.25)$$

Turning to the second term in (1.19) and using the formula for the solution to the linear wave equation in $\mathbb{R}^3$, see for example [18], for any $x \in \mathbb{R}^3$,

$$|S(t - \tau)(0, |u|^{p-1}u)(x)| \lesssim \frac{1}{|t - \tau|} \int_{\partial B(x, t-\tau)} |u(y, \tau)|^p d\sigma(y). \quad (2.26)$$

Once again, split

$$P_j(|u|^{p-1}u) = P_j F_1 + P_j F_2, \quad F_1 = |P_{\leq j} u|^{p-1} (P_{\leq j} u), \quad F_2 = O(|P_{\geq j} u| |u|^{p-1}). \quad (2.27)$$

Plugging F_2 into (2.26), for any $t \in [-1, 1]$, $x \in \mathbb{R}^3$,

$$\begin{aligned} \left| \int_0^{\frac{t}{2}} S(t - \tau)(0, P_j F_2)(t, x) d\tau \right| \lesssim \frac{1}{t} \| |u|^{\frac{p-1}{2}} \|_{L_\tau^1 L_x^\infty([0, \frac{t}{2}] \times \mathbb{R}^3)} \\ \cdot \sup_{\tau \in [0, \frac{t}{2}]} \left(\int_{\partial B(x, t-\tau)} |u(\tau, y)|^{p-1} d\sigma(y) \right)^{1/2} \cdot \sup_{\tau \in [0, \frac{t}{2}]} \left(\int_{\partial B(x, t-\tau)} |P_{\geq j} u(\tau, y)|^2 dy \right)^{1/2}. \end{aligned} \quad (2.28)$$

By an argument similar to the Sobolev embedding theorem, for any $k \in \mathbb{Z}$,

$$\int_{\partial B(x, t-\tau)} |P_k u(y, \tau)|^{p-1} d\sigma(y) \lesssim 2^k \|P_k u\|_{L^{p-1}}^{p-1}. \quad (2.29)$$

Remark 2. To see why this is so, recall that the Littlewood–Paley kernel for P_k may be approximated by 2^{3k} multiplied by the characteristic function of a ball of radius 2^{-k} . Then consider the cases when $2^{-k} \leq |t - \tau|$ and $2^{-k} > |t - \tau|$ separately. Indeed, for $|t - \tau| \lesssim 2^{-k}$, there exists some C such that

$$\begin{aligned} & \int_{\partial B(x, t-\tau)} |P_k u(y, \tau)|^{p-1} d\sigma(y) \\ & \lesssim 2^{3k} |t - \tau|^2 \int_{B(x, C2^{-k})} |P_k u(\tau, y)|^{p-1} dy \lesssim 2^k \|P_k u(\tau)\|_{L^{p-1}}^{p-1}. \end{aligned} \quad (2.30)$$

Meanwhile, for $|t - \tau| \gg 2^{-k}$,

$$\begin{aligned} & \int_{\partial B(x, t-\tau)} |P_k u(y, \tau)|^{p-1} d\sigma(y) \\ & \lesssim 2^k \int_{\text{dist}(B(x, t-\tau), y) \leq 2^{-k}} |P_k u(\tau, y)|^{p-1} dy \lesssim 2^k \|P_k u(\tau)\|_{L^{p-1}}^{p-1}. \end{aligned} \quad (2.31)$$

Now, then, since the Littlewood–Paley kernel obeys the bounds

$$\mathcal{F}(P_k(y)) \lesssim_N 2^{3k} (1 + 2^k |y|)^{-N}, \quad (2.32)$$

for any N , calculations similar to (2.30) and (2.31) imply (2.29). Indeed, for any $l \in \mathbb{Z}$, $l \geq 0$, (2.30) and (2.31) imply,

$$2^{-Nl} 2^k \int_{\text{dist}(B(x, t-\tau)) \leq 2^{l-k}} |P_k u(\tau, y)|^{p-1} dy \lesssim 2^{-Nl} 2^k \|P_k u(\tau)\|_{L^{p-1}}^{p-1},$$

which can easily be summed in l .

Plugging (2.29) into (2.28), by Young's inequality and (2.15),

$$\begin{aligned}
 & \sum_j 2^{j(s_c - \frac{1}{2})} \sup_{t \in [-1, 1]} t \left\| \int_0^{\frac{t}{2}} S(t - \tau)(0, P_j F_2) d\tau \right\|_{L^\infty} \\
 & \lesssim \|u\|_{L_t^{\frac{p-1}{2}} L_x^\infty([-1, 1] \times \mathbb{R}^3)}^{\frac{p-1}{2}} \sum_j 2^{j(s_c - \frac{1}{2})} \cdot \sup_{\tau \in [-1, 1]} \left(\int_{\partial B(x, t-\tau)} |u(\tau, y)|^{p-1} d\sigma(y) \right)^{1/2} \\
 & \quad \cdot \sup_{\tau \in [-1, 1]} \left(\int_{\partial B(x, t-\tau)} |P_{\geq j} u(\tau, y)|^2 dy \right)^{1/2} \tag{2.33} \\
 & \lesssim \|u\|_{L_t^{\frac{p-1}{2}} L_x^\infty([-1, 1] \times \mathbb{R}^3)}^{\frac{p-1}{2}} \sum_j 2^{j(s_c - \frac{1}{2})} \sup_{t \in [-1, 1]} \left(\sum_l 2^l \|P_l u(t)\|_{L_x^{p-1}}^{p-1} \right)^{1/2} \left(\sum_{l \geq j} 2^l \|P_l u(t)\|_{L^2}^2 \right)^{1/2} \\
 & \lesssim \|u\|_{L_t^{\frac{p-1}{2}} L_x^\infty([-1, 1] \times \mathbb{R}^3)}^{\frac{p-1}{2}} A^{\frac{p+1}{2}}.
 \end{aligned}$$

Meanwhile, since by Bernstein's inequality,

$$P_j(F_1) \sim 2^{-j} \nabla P_j F_1 \sim 2^{-j} |P_{\leq j} u|^{p-1} |\nabla P_{\leq j} u|, \tag{2.34}$$

$$\begin{aligned}
 & \left| \int_0^{\frac{t}{2}} S(t - \tau)(0, P_j F_1)(t, x) d\tau \right| \lesssim \frac{2^{-j}}{t} \|u\|_{L_\tau^1 L_x^\infty([0, \frac{t}{2}] \times \mathbb{R}^3)}^{\frac{p-1}{2}} \\
 & \quad \cdot \sup_{\tau \in [0, \frac{t}{2}]} \left(\int_{B(x, t-\tau)} |u(\tau, y)|^{p-1} d\sigma(y) \right)^{1/2} \cdot \sup_{\tau \in [0, \frac{t}{2}]} \left(\int_{B(x, t-\tau)} |\nabla P_{\leq j} u(\tau, y)|^2 dy \right)^{1/2}, \tag{2.35}
 \end{aligned}$$

and therefore,

$$\sum_j 2^{j(s_c - \frac{1}{2})} \sup_{t \in [-1, 1]} t \left\| \int_0^{\frac{t}{2}} S(t - \tau)(0, P_j |u|^{p-1} u) d\tau \right\|_{L^\infty} \lesssim \|u\|_{L_t^{\frac{p-1}{2}} L_x^\infty([-1, 1] \times \mathbb{R}^3)}^{\frac{p-1}{2}} A^{\frac{p+1}{2}}. \tag{2.36}$$

For $\tau \in [\frac{t}{2}, t]$, energy estimates and the Sobolev embedding theorem imply,

$$\begin{aligned}
 \|S(t - \tau)(0, P_j(|u|^{p-1} u))\|_{L^6} & \lesssim \frac{1}{t^2} \left(\sup_{\tau \in [\frac{t}{2}, t]} \tau \cdot \| |u(\tau)|^{\frac{p-1}{2}} \|_{L_x^\infty(\mathbb{R}^3)}^2 \sup_{\tau \in [\frac{t}{2}, t]} \|P_{\geq j} u\|_{L^2} \right. \\
 & \quad \left. + \frac{2^{-j}}{t^2} \left(\sup_{\tau \in [\frac{t}{2}, t]} \tau \cdot \| |u(\tau)|^{\frac{p-1}{2}} \|_{L_x^\infty(\mathbb{R}^3)}^2 \|P_{\leq j} \nabla u\|_{L^2} \right) \right). \tag{2.37}
 \end{aligned}$$

Therefore, by Young's inequality, the Sobolev embedding theorem, and Proposition 1,

$$\sum_j 2^{j(s_c - 1/2)} \sup_{t \in [-1, 1]} t \left\| \int_{t/2}^t S(t - \tau) P_j(0, |u|^{p-1} u) d\tau \right\|_{L^\infty}$$

$$\lesssim \left(\sup_{t \in [-1,1]} t^{\frac{3}{2}-s_c} \|u(t)\|_{L^\infty} \right)^{p-1} A. \quad (2.38)$$

Combining (2.9), (2.23), (2.36), and (2.38),

$$\begin{aligned} & \sum_j 2^{j(s_c - \frac{1}{2})} \sup_{t \in [-1,1]} t \|P_j u(t)\|_{L^\infty} \lesssim \sum_j 2^{j(s_c - \frac{1}{2})} \sup_{t \in [-1,1]} t \|P_j S(t)(u_0, u_1)\|_{L^\infty} \\ & \quad + \|u\|_{L_t^{\frac{p-1}{2}} L_x^\infty([-1,1] \times \mathbb{R}^3)}^{\frac{p+1}{2}} + \left(\sup_{t \in [-1,1]} t^{\frac{3}{2}-s_c} \|u(t)\|_{L^\infty} \right)^{p-1} A \\ & + \sum_{j \geq 0} (2^{j(s_c + \frac{3}{2})} \|P_j u_0\|_{L^1} + 2^{j(s_c + \frac{1}{2})} \|P_j u_1\|_{L^1}) + \sum_{j \geq 0} (2^{3j} \|P_j u_0\|_{L^1} + 2^{2j} \|P_j u_1\|_{L^1}) \\ & \quad + \|u\|_{L_t^{\frac{p-1}{2}} L_x^\infty([-1,1] \times \mathbb{R}^3)}^{\frac{p+1}{2}} + \left(\sup_{t \in [-1,1]} t^{\frac{3}{2}-s_c} \|u(t)\|_{L^\infty} \right)^{p-1} A \\ & \lesssim \epsilon + \|u\|_{L_t^{\frac{p-1}{2}} L_x^\infty([-1,1] \times \mathbb{R}^3)}^{\frac{p+1}{2}} + \left(\sup_{t \in [-1,1]} t^{\frac{3}{2}-s_c} \|u(t)\|_{L^\infty} \right)^{p-1} A. \end{aligned} \quad (2.39)$$

Now then, for any $3 < p < 5$, Proposition 1, (2.10), (2.11), and the Sobolev embedding theorem imply

$$\begin{aligned} & \sum_j \|P_j u\|_{L_t^{\frac{p-1}{2}} L_x^\infty([-1,1] \times \mathbb{R}^3)} + \sum_j \sup_{t \in [-1,1]} t^{\frac{3}{2}-s_c} \|P_j u\|_{L_x^\infty(\mathbb{R}^3)} \\ & \lesssim \epsilon^{\frac{p-3}{p-1}} \left(\sum_j 2^{j(s_c - \frac{1}{2})} \sup_{t \in [-1,1]} t \|P_j u(t)\|_{L^\infty} \right)^{\frac{2}{p-1}}. \end{aligned} \quad (2.40)$$

Combining (2.39) with (2.40) proves the Theorem. $\square$

Theorem 8 implies finite energy for a piece of the Duhamel term.

Corollary 1. *For any $t \in [-1, 1]$,*

$$\int_{t/2}^t \|u^p(\tau)\|_{L^2} d\tau \lesssim \frac{A^p}{t^{1-s_c}}. \quad (2.41)$$

Proof. Use the energy estimate in (2.37). $\square$

In the cubic case, Theorem 8 is out of reach, but we are still able to prove some decay estimates.

Proposition 2. *For all $t \in [-1, 1]$,*

$$\sup_{t \in [-1,1]} t^{1/2} \|u(t)\|_{L^6} \lesssim A. \quad (2.42)$$

Proof. Interpolating the dispersive estimate

$$\|S(t)(u_0, u_1)\|_{L^\infty} \lesssim \frac{1}{t} \|(u_0, u_1)\|_{B_{1,1}^2 \times B_{1,1}^1} \lesssim \frac{A}{t}, \quad (2.43)$$

with the trivial bound

$$\|S(t)(u_0, u_1)\|_{L^3} \lesssim \|(u_0, u_1)\|_{\dot{H}^{1/2} \times \dot{H}^{1/2}} \lesssim A, \quad (2.44)$$

gives the dispersive estimate

$$\|S(t)(u_0, u_1)\|_{L^6} \lesssim \frac{A}{t^{1/2}}. \quad (2.45)$$

Next, by dispersive estimates,

$$\left\| \int_0^t S(t-\tau)(0, u^3) d\tau \right\|_{L^6} \lesssim \int_0^t \frac{1}{|t-\tau|^{2/3}} \| |\nabla|^{1/3} u^3 \|_{L^{6/5}} d\tau. \quad (2.46)$$

Then by the fractional product rule,

$$\left\| \int_0^t S(t-\tau)(0, u^3) d\tau \right\|_{L^6} \lesssim \int_0^t \frac{1}{|t-\tau|^{2/3}} \| |\nabla|^{1/2} u \|_{L^2}^{2/3} \| u \|_{L^4}^{2/3} \| u \|_{L^6}^{4/3} \| u \|_{L^3}^{1/3} d\tau. \quad (2.47)$$

Plugging (2.9) into (2.47),

$$\begin{aligned} \left\| \int_0^t S(t-\tau)(0, u^3) d\tau \right\|_{L^6} &\lesssim A \int_0^t \frac{1}{|t-\tau|^{2/3}} \| u(\tau) \|_{L^4}^{2/3} \| u(\tau) \|_{L^6}^{4/3} d\tau \\ &\lesssim_A \epsilon^{1/6} \left(\sup_{t \in [-1, 1]} t^{1/2} \| u(\tau) \|_{L^6} \right)^{4/3}. \end{aligned} \quad (2.48)$$

In the last line, we used the fact that $\|u\|_{L_{t,x}^4([-1, 1] \times \mathbb{R}^3)} \lesssim_A \epsilon$. This follows from (2.8) and (2.9). By (2.8) and Strichartz estimates,

$$\|S(t)(P_{\geq j_0} u_0, P_{\geq j_0} u_1)\|_{L_{t,x}^4} \lesssim \epsilon. \quad (2.49)$$

Meanwhile, by the Sobolev embedding theorem and (2.9),

$$\|S(t)(P_{\leq j_0} u_0, P_{\leq j_0} u_1)\|_{L_t^\infty L_x^6} \lesssim 2^{j_0/2} (\|P_{\leq j_0} u_0\|_{\dot{H}^{1/2}} + \|P_{\leq j_0} u_1\|_{\dot{H}^{-1/2}}) \lesssim \epsilon. \quad (2.50)$$

Therefore,

$$\|S(t)(u_0, u_1)\|_{L_{t,x}^4([-1, 1] \times \mathbb{R}^3)} \lesssim_A \epsilon, \quad (2.51)$$

which by small data arguments implies $\|u\|_{L^4_{t,x}([-1,1] \times \mathbb{R}^3)} \lesssim_A \epsilon$, for $\epsilon(A) > 0$ sufficiently small. Choosing $\epsilon(A) > 0$ sufficiently small, combined with (2.45), gives (2.42). $\square$

Then by direct computation,

Corollary 2. *For any $t \in [-1, 1]$,*

$$\int_{t/2}^t \|u^3(\tau)\|_{L^2} d\tau \lesssim \frac{A^3}{t^{1/2}}. \quad (2.52)$$

An argument similar to Proposition 2 also proves $u \in L^2_t L^6_x([-1, 1] \times \mathbb{R}^3)$ in the cubic case.

Proposition 3.

$$\sum_j \|P_j u\|_{L^2_t L^6_x([-1,1] \times \mathbb{R}^3)} \lesssim A^3. \quad (2.53)$$

Proof. Interpolating

$$\|S(t)P_k(u_0, u_1)\|_{L^\infty} \lesssim \frac{1}{t} (2^{2k} \|P_k u_0\|_{L^1} + 2^k \|P_k u_1\|_{L^1}), \quad (2.54)$$

with

$$\begin{aligned} \|S(t)P_k(u_0, u_1)\|_{L^2} &\lesssim 2^{-k/2} (\|P_k u_0\|_{\dot{H}^{1/2}} + \|P_k u_1\|_{\dot{H}^{-1/2}}) \\ &\lesssim 2^{-k/2} (2^{2k} \|P_k u_0\|_{L^1} + 2^k \|P_k u_1\|_{L^1}), \end{aligned} \quad (2.55)$$

implies that

$$\|S(t)P_k(u_0, u_1)\|_{L^6} \lesssim \frac{2^{-k/6}}{t^{2/3}} (2^{2k} \|P_k u_0\|_{L^1} + 2^k \|P_k u_1\|_{L^1}). \quad (2.56)$$

Since the Sobolev embedding theorem implies that

$$\begin{aligned} \|S(t)P_k(u_0, u_1)\|_{L^6} &\lesssim 2^{k/2} \|P_k(u_0, u_1)\|_{\dot{H}^{1/2} \times \dot{H}^{-1/2}} \\ &\lesssim 2^{k/2} (2^{2k} \|P_k u_0\|_{L^1} + 2^k \|P_k u_1\|_{L^1}). \end{aligned} \quad (2.57)$$

Therefore,

$$\sum_k \|S(t)P_k(u_0, u_1)\|_{L^2_t L^6_x} \lesssim A. \quad (2.58)$$

Now then, plugging in Propositions 1 and 2,

$$\begin{aligned} \left\| \int_0^t S(t-\tau)(0, P_k(u^3)) d\tau \right\|_{L^6} &\lesssim 2^{k/3} \int_0^t \frac{1}{|t-\tau|^{2/3}} \|P_{\geq k-3} u\|_{L^{36/17}} \|u\|_{L^6}^{11/6} \|u\|_{L^3}^{1/6} d\tau \\ &\lesssim_A 2^{-k/12} \left(\sum_{j \geq k-3} 2^{\frac{5}{12}(k-j)} 2^{j/2} \|P_j u\|_{L_t^\infty L_x^2([-1,1] \times \mathbb{R}^3)} \right) \int_0^t \frac{1}{|t-\tau|^{2/3}} \cdot \frac{1}{|\tau|^{11/12}} d\tau \\ &\lesssim_A \frac{2^{-k/12}}{t^{1/12}} \frac{1}{t^{1/2}} \left(\sum_{j \geq k-3} 2^{\frac{5}{12}(k-j)} 2^{j/2} \|P_j u\|_{L_t^\infty L_x^2([-1,1] \times \mathbb{R}^3)} \right). \end{aligned} \quad (2.59)$$

Also, by the Sobolev embedding theorem and the proof of Proposition 2,

$$\left\| \int_0^t S(t-\tau)(0, P_k(u^3)) d\tau \right\|_{L_x^6([-1,1])} \lesssim_A 2^{k/2} \|P_k u\|_{L_t^\infty \dot{H}^{1/2}([-1,1] \times \mathbb{R}^3)}. \quad (2.60)$$

Summing up (2.58), (2.59), and (2.60) using Young's inequality,

$$\sum_k \|P_j u\|_{L_t^2 L_x^6([-1,1] \times \mathbb{R}^3)} \lesssim_A 1. \quad \square \quad (2.61)$$

Proposition 4.

$$\sum_j 2^{j/3} \sup_{t \in [-1,1]} t^{1/3} \|P_j u\|_{L^3} \lesssim_A 1. \quad (2.62)$$

Proof. Again using dispersive estimates and the Sobolev embedding theorem,

$$2^{k/3} \left\| \int_0^t S(t-\tau)(0, P_k(u^3)) d\tau \right\|_{L^3} \lesssim \int_0^t \frac{2^{k/2}}{|t-\tau|^{1/3}} \|P_{\geq k-3} u(t)\|_{L^2} \|u\|_{L^6}^2 d\tau. \quad (2.63)$$

Splitting this integral into two pieces, by Proposition 3,

$$\begin{aligned} &\int_0^{t/2} \frac{2^{k/2}}{|t-\tau|^{1/3}} \|P_{\geq k-3} u(t)\|_{L^2} \|u\|_{L^6}^2 d\tau \\ &\lesssim t^{-1/3} \left(\sum_{j \geq k-3} 2^{j/2} 2^{\frac{k-j}{2}} \|P_j u\|_{L_t^\infty L_x^2} \|u\|_{L_t^2 L_x^6([-1,1] \times \mathbb{R}^3)}^2 \right) \\ &\lesssim_A t^{-1/3} \left(\sum_{j \geq k-3} 2^{j/2} 2^{\frac{k-j}{2}} \|P_j u\|_{L_t^\infty L_x^2} \right). \end{aligned} \quad (2.64)$$

Meanwhile, by Proposition 2,

$$\begin{aligned}
\int_{t/2}^t \frac{2^{k/2}}{|t-\tau|^{1/3}} \|P_{\geq k-3} u(t)\|_{L^2} \|u\|_{L^6}^2 d\tau &\lesssim_A \left(\sum_{j \geq k-3} 2^{j/2} 2^{\frac{k-j}{2}} \|P_j u\|_{L_t^\infty L_x^2} \right) \int_{t/2}^t \frac{1}{\tau} \frac{1}{|t-\tau|^{1/3}} d\tau \\
&\lesssim_A t^{-1/3} \left(\sum_{j \geq k-3} 2^{j/2} 2^{\frac{k-j}{2}} \|P_j u\|_{L_t^\infty L_x^2} \right).
\end{aligned} \tag{2.65}$$

Therefore, by Young's inequality and Proposition 1, the proof is complete. $\square$

Remark 3. For the rest of the paper, we will wish to study long time behavior for the nonlinear wave equation. All of the above estimates also hold for small data solutions to (1.1), as well as free solutions of the form

$$\int_0^{1/2} S(t-\tau)(0, |u|^{p-1}u) d\tau, \tag{2.66}$$

even when $t > 1$. We will frequently use this fact in the rest of the paper.

3. Proof of global well-posedness for $3 < p < 5$

By time reversal symmetry and local well-posedness on the interval $[-1, 1]$, to prove global well-posedness, it suffices to prove global well-posedness in the positive time direction, $t > 1$ for (1.1) with initial data $(u(1, x), u_t(1, x))$. The local well-posedness arguments used to prove Theorem 1 imply that (1.1) has a solution on some open interval $[0, T)$ for some $T > 1$, so to prove global well-posedness it suffices to show that T can be taken to go to infinity.

Split

$$\begin{aligned}
\begin{pmatrix} u(1, x) \\ u_t(1, x) \end{pmatrix} &= S(1)(u_0, u_1) \\
&+ \int_0^{1/2} S(1-\tau)(0, |u|^{p-1}u) d\tau + \int_{1/2}^1 S(1-\tau)(0, |u|^{p-1}u) d\tau.
\end{aligned} \tag{3.1}$$

By Corollary 1, the second Duhamel term has finite energy.

$$\left\| \begin{pmatrix} v(1, x) \\ v_t(1, x) \end{pmatrix} \right\|_{\dot{H}^1 \times L^2} = \left\| \int_{1/2}^1 S(1-\tau)(0, |u|^{p-1}u) d\tau \right\|_{\dot{H}^1 \times L^2} \lesssim_A 1. \tag{3.2}$$

Now let u be the solution to (1.1) on $[1, T)$. Split $u = v + w$, where v solves

$$v_{tt} - \Delta v + |u|^{p-1}u = 0, \tag{3.3}$$

on $[1, T)$ with initial data given by (3.2), and

$$w_{tt} - \Delta w = 0, \quad w(1, x) = u(1, x) - v(1, x), \quad w_t(1, x) = u_t(1, x) - v_t(1, x). \quad (3.4)$$

Set

$$E(v) = \int \left[\frac{1}{2} |v_t|^2 + \frac{1}{2} |\nabla v|^2 + \frac{1}{p+1} |v|^{p+1} \right] dx, \quad (3.5)$$

and compute

$$\frac{d}{dt} E(v) = \langle v_t, -|u|^{p-1}u + |v|^{p-1}v \rangle. \quad (3.6)$$

By Taylor's theorem,

$$|u|^{p-1}u - |v|^{p-1}v = p|v|^{p-1}w + O(|w|^2|v|^{p-2}) + O(|w|^p). \quad (3.7)$$

By Hölder's inequality,

$$\langle O(|w|^2|v|^{p-2}), v_t \rangle \lesssim \|v_t\|_{L^2} \|v\|_{L^{p+1}}^{p-2} \|w\|_{L^\infty}^{\frac{p-1}{2}} \|w\|_{L^{p+1}}^{\frac{5-p}{2}} \lesssim E(v(t))^{\frac{1}{2} + \frac{p-2}{p+1}} \|w\|_{L^\infty}^{\frac{p-1}{2}} \|w\|_{L^{p+1}}^{\frac{5-p}{2}}. \quad (3.8)$$

Interpolating (2.21) with $\|w\|_{L^{\frac{3(p-1)}{2}}} \lesssim \|w\|_{\dot{H}^{s_c}} \lesssim_A 1$, proves $\|w\|_{L^{p+1}} \lesssim_A 1$. Also,

$$\langle |w|^p, v_t \rangle \lesssim \|v_t\|_{L^2} \|w\|_{L^\infty}^{\frac{p-1}{2}} \|w\|_{L^{p+1}}^{\frac{p+1}{2}} \lesssim E(v(t))^{1/2} \|w\|_{L^\infty}^{\frac{p-1}{2}} \|w\|_{L^{p+1}}^{\frac{p+1}{2}}. \quad (3.9)$$

If we could ignore the term

$$\langle v_t, p|v|^{p-1}w \rangle, \quad (3.10)$$

then $E(v(t))$ would be uniformly bounded on $\mathbb{R}$ by Gronwall's inequality. Indeed, by (2.21),

$$\begin{aligned} & \int_1^T E(v(t))^{\frac{1}{2} + \frac{p-2}{p+1}} \|w\|_{L^\infty}^{\frac{p-1}{2}} \|w\|_{L^{p+1}}^{\frac{5-p}{2}} dt + \int E(v(t))^{1/2} \|w\|_{L^\infty}^{\frac{p-1}{2}} \|w\|_{L^{p+1}}^{\frac{p+1}{2}} dt \\ & \lesssim \sup_{t \in [1, T)} \epsilon E(v(t))^{\frac{1}{2} + \frac{p-2}{p+1}} + \sup_{t \in [1, T)} \epsilon E(v(t))^{1/2}, \end{aligned} \quad (3.11)$$

which implies a uniform bound on $E(v(t))$.

To deal with the contribution of (3.10), take the modified energy

$$\mathcal{E}(v(t)) = E(v(t)) + \langle |v|^{p-1}v, w \rangle. \quad (3.12)$$

Then (3.6) and (3.7) imply

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}(v(t)) &= \langle v_t, -|u|^{p-1}u + |v|^{p-1}v \rangle + \langle p|v|^{p-1}w, v_t \rangle + \langle |v|^{p-1}v, w_t \rangle \\
&= \langle |v|^{p-1}v, w_t \rangle + O(E(v(t))^{\frac{1}{2} + \frac{p-2}{p+1}} \|w\|_{L^\infty}^{\frac{2}{p-1}} \|w\|_{L^{p+1}}^{\frac{5-p}{2}}) + O(E(v(t))^{1/2} \|w\|_{L^\infty}^{\frac{p-1}{2}} \|w\|_{L^{p+1}}^{\frac{p+1}{2}}).
\end{aligned} \tag{3.13}$$

Remark 4. In the last line we use the fact that

$$-|u|^{p-1}u + |v|^{p-1}v + p|v|^{p-1}w = O(|w|^2|v|^{p-2}) + O(|w|^p).$$

Also,

$$\langle |v|^{p-1}v, w \rangle \lesssim \|v\|_{L^{p+1}}^p \|w\|_{L^{p+1}} \lesssim E(v(t))^{\frac{p}{p+1}}, \tag{3.14}$$

so when $E(v(t))$ is large,

$$E(v(t)) \sim \mathcal{E}(v(t)), \tag{3.15}$$

and

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}(v(t)) &= \langle |v|^{p-1}v, w_t \rangle + O(\mathcal{E}(v(t))^{\frac{1}{2} + \frac{p-2}{p+1}} \|w\|_{L^\infty}^{\frac{2}{p-1}} \|w\|_{L^{p+1}}^{\frac{5-p}{2}}) \\
&\quad + O(\mathcal{E}(v(t))^{1/2} \|w\|_{L^\infty}^{\frac{p-1}{2}} \|w\|_{L^{p+1}}^{\frac{p+1}{2}}).
\end{aligned} \tag{3.16}$$

Splitting $w_t = \sum_j P_j w_t$,

$$\langle |v|^{p-1}v, w_t \rangle = \sum_j \langle P_j(|v|^{p-1}v), P_j w_t \rangle. \tag{3.17}$$

Now by Bernstein's inequality and (2.21),

$$\begin{aligned}
&\sum_j \langle P_j(|v|^{p-1}v - |P_{\leq j}v|^{p-1}(P_{\leq j}v)), P_j w_t \rangle \\
&\lesssim \sum_j \|P_j w_t\|_{L^\infty} \|P_{\geq j}v\|_{L^{\frac{p+1}{2}}}^{p-1} \|v\|_{L^{p+1}}^{p-1} \lesssim \frac{\epsilon}{t} E(v(t)).
\end{aligned} \tag{3.18}$$

Indeed, by (2.21) and the fact that since w is a solution to the linear wave equation, when $t > 1$, and thus, $\|P_j w_t\|_{L^\infty} \sim 2^j \|P_j w\|_{L^\infty}$,

$$\sum_j 2^{j(s_c - \frac{1}{2})} 2^{-j} \|P_j w_t\|_{L^\infty} \lesssim \frac{\epsilon}{t}. \tag{3.19}$$

Meanwhile, by Bernstein's inequality, for any fixed $j \in \mathbb{Z}$,

$$\begin{aligned}
\frac{\epsilon}{t} 2^j 2^{-j(s_c - \frac{1}{2})} \|P_{\geq j} v\|_{L^{\frac{p+1}{2}}} \|v\|_{L^{p+1}}^{p-1} &\lesssim \frac{\epsilon}{t} 2^{j(\frac{3}{2} - s_c)} \|P_{\geq j} v\|_{L^2}^{\frac{2}{p-1}} \|v\|_{L^{p+1}}^{\frac{p-3}{p-1} + p-1} \\
&\lesssim \frac{\epsilon}{t} \|\nabla v\|_{L^2}^{\frac{2}{p-1}} \|v\|_{L^{p+1}}^{\frac{(p-2)(p+1)}{p-1}} \lesssim \frac{\epsilon}{t} E(v(t)). \quad (3.20)
\end{aligned}$$

Also, by Bernstein's inequality,

$$\begin{aligned}
&\sum_j \langle P_j (|P_{\leq j} v|^{p-1} (P_{\leq j} v), P_j w_t) \rangle \\
&\lesssim \sum_j 2^{-j(\frac{2}{p-1})} \|\nabla v\|_{L^2}^{\frac{2}{p-1}} \|P_j w_t\|_{L^\infty} \|v\|_{L^{p+1}}^{(p+1)\frac{p-2}{p-1}} \lesssim \frac{\epsilon}{t} E(v(t)). \quad (3.21)
\end{aligned}$$

Therefore, by Gronwall's inequality,

$$\mathcal{E}(v(t)) < \infty, \quad \text{and} \quad E(v(t)) < \infty, \quad (3.22)$$

for any $t \in \mathbb{R}$. This proves global well-posedness.

4. Proof of global well-posedness in the cubic case

The argument for global well-posedness for the cubic wave equation

$$u_{tt} - \Delta u + u^3 = 0, \quad u(0, x) = u_0, \quad u_t(0, x) = u_1, \quad (4.1)$$

is similar.

Theorem 9. Equation (4.1) is globally well-posed for initial data $u_0 \in B_{1,1}^2$ and $u_1 \in B_{1,1}^1$.

Proof of Theorem 9. Again by time reversal symmetry and local well-posedness on the interval $[-1, 1]$, to prove Theorem 9, it suffices to prove global well-posedness in the positive time direction, $t > 1$ for (1.1) with initial data $(u(1, x), u_t(1, x))$.

Again, split

$$\begin{pmatrix} u(1, x) \\ u_t(1, x) \end{pmatrix} = S(1)(u_0, u_1) + \int_0^{1/2} S(1-\tau)(0, u^3) d\tau + \int_{1/2}^1 S(1-\tau)(0, u^3) d\tau. \quad (4.2)$$

By Proposition 2, the second Duhamel term has finite energy. Indeed,

$$\left\| \begin{pmatrix} v(1, x) \\ v_t(1, x) \end{pmatrix} \right\|_{\dot{H}^1 \times L^2} = \left\| \int_{1/2}^1 S(1-\tau)(0, u^3) d\tau \right\|_{\dot{H}^1 \times L^2} \lesssim \int_{1/2}^1 \|u(\tau)\|_{L^6}^3 d\tau \lesssim_A 1. \quad (4.3)$$

Now suppose v solves

$$v_{tt} - \Delta v + v^3 = 0, \quad (4.4)$$

on $[1, T]$ with initial data given by (4.2), and $u = v + w$, where

$$w_{tt} - \Delta w = 0, \quad w(1, x) = u(1, x) - v(1, x), \quad w_t(1, x) = u_t(1, x) - v_t(1, x). \quad (4.5)$$

Computing,

$$\frac{d}{dt}E(v) = \langle v_t, -u^3 + v^3 \rangle \lesssim \|v_t\|_{L^2} \|v^2 w\|_{L^2} + \|v_t\|_{L^2} \|v w^2\|_{L^2} + \|v_t\|_{L^2} \|w^3\|_{L^2}. \quad (4.6)$$

By Proposition 2,

$$\|w^3\|_{L^2} \lesssim \|w\|_{L^6}^3 \lesssim_A \frac{1}{t^{3/2}}, \quad (4.7)$$

so

$$\|v_t\|_{L^2} \|w^3\|_{L^2} \lesssim_A \frac{1}{t^{3/2}} E(v(t))^{1/2}. \quad (4.8)$$

Also, by Proposition 2,

$$\|v w^2\|_{L^2} \lesssim \|v\|_{L^6} \|w\|_{L^6}^2 \lesssim \frac{1}{t} E(v)^{1/2}, \quad (4.9)$$

so

$$\|v_t\|_{L^2} \|v w^2\|_{L^2} \lesssim \frac{1}{t} E(v(t)). \quad (4.10)$$

Next, split

$$S(t-1)S(1)(u_0, u_1) + \int_0^{1/2} S(t-\tau)(0, u^3) d\tau = w_1 + w_2. \quad (4.11)$$

By the dispersive estimate (1.12),

$$\|S(t-1)S(1)(u_0, u_1)\|_{L^\infty} \lesssim_A \frac{1}{t}, \quad (4.12)$$

so

$$\|v^2 w_1\|_{L^2} \lesssim \|v\|_{L^4}^2 \|w_1\|_{L^\infty} \lesssim_A \frac{1}{t} E(v(t))^{1/2}. \quad (4.13)$$

To control the term $\|w_2 v^2\|_{L^2}$, split $w_2 = w_{21} + w_{22}$,

$$w_{21} = \int_0^{E(t)^{-\kappa}} S(t-\tau)(0, u^3) d\tau, \quad w_{22} = \int_{E(t)^{-\kappa}}^{1/2} S(t-\tau)(0, u^3) d\tau, \quad (4.14)$$

for some $\kappa > 0$ to be specified later. Following (2.47),

$$\|w_{21}\|_{L^6} \lesssim \int_0^{E(t)^{-\kappa}} \frac{1}{|t-\tau|^{2/3}} \| |\nabla|^{1/2} u \|_{L^2}^{2/3} \|u\|_{L^4}^{4/3} \|u\|_{L^6} d\tau \lesssim_A \frac{1}{t^{2/3}} E(t)^{-\kappa/6}. \quad (4.15)$$

Taking $\kappa = 3$,

$$\|w_{21} \cdot v^2\|_{L^2} \lesssim \|w_{21}\|_{L^6} \|v\|_{L^6}^2 \lesssim_A \frac{1}{t^{2/3}} E(t)^{1/2}. \quad (4.16)$$

Following (2.26)–(2.32),

$$|S(t-\tau)(0, u^3)(x)| \lesssim \frac{1}{|t-\tau|} \int_{\partial B(x, t-\tau)} |u(y, \tau)|^3 d\sigma(y), \quad (4.17)$$

and

$$\int_{\partial B(x, t-\tau)} |P_k u(y, \tau)|^3 d\sigma(y) \lesssim 2^k \|P_k u\|_{L^3}^3. \quad (4.18)$$

Therefore, by (2.62),

$$\begin{aligned} \int_{\partial B(x, t-\tau)} |u(\tau, y)|^3 d\sigma(y) &\lesssim \left(\sum_k \left(\int_{\partial B(x, t-\tau)} |P_k u|^3 d\sigma(y) \right)^{1/3} \right)^3 \\ &\lesssim \left(\sum_k 2^{k/3} \|P_k u\|_{L^3} \right)^3 \lesssim_A \frac{1}{\tau}, \end{aligned}$$

so

$$\|w_{22}(t)\|_{L^\infty} \lesssim_A \int_{E(t)^{-\kappa}}^{1/2} \frac{1}{\tau} \frac{1}{|t-\tau|} d\tau \lesssim \frac{1}{t} \ln(E), \quad (4.19)$$

and

$$\|w_{22} v^2\|_{L^2} \lesssim \frac{1}{t} \ln(E(v(t))) \|v\|_{L^4}^2 \lesssim \frac{1}{t} \ln(E(v(t))) E(v(t)). \quad (4.20)$$

Therefore, since we are only concerned with E large,

$$\frac{d}{dt} E \lesssim \frac{1}{t} E + \frac{\ln(t)}{t} \ln(E) \cdot E + \frac{1}{t^2} \lesssim \frac{\ln(t)}{t} \ln(E) E + \frac{1}{t^2}. \quad (4.21)$$

Doing some algebra,

$$\frac{1}{E} \frac{d}{dt} E = \frac{d}{dt} \ln(E) \lesssim \frac{\ln(t)}{t} \ln(E) + \frac{1}{E} \frac{1}{t^2}. \quad (4.22)$$

$$\frac{1}{\ln(E)} \frac{d}{dt} \ln(E) = \frac{d}{dt} \ln(\ln(E)) \lesssim \frac{\ln(t)}{t} + \frac{1}{E \ln(E)} \cdot \frac{1}{t^2}. \quad (4.23)$$

Therefore,

$$\ln(\ln(E(T))) \lesssim \ln(T)^2 + C. \quad (4.24)$$

Therefore,

$$E(T) \lesssim \exp(\exp(C + C \ln(T)^2)). \quad (4.25)$$

This proves Theorem 9. $\square$

5. Scattering

To prove scattering, we make use of the conformal energy of a wave equation in [20] and [22]. See also [23]. Indeed, if v solves

$$v_{tt} - \Delta v + |v|^{p-1}v = 0, \quad (5.1)$$

then define the conformal energy,

$$\begin{aligned} \mathcal{E}(t) = & \frac{1}{4} \langle (t + |x|)Lv + 2v, (t + |x|)Lv + 2v \rangle + \frac{1}{4} \langle (t - |x|)\underline{L}v + 2v, (t - |x|)\underline{L}v + 2v \rangle \\ & + \frac{1}{2} \int (t^2 + |x|^2) |\nabla v|^2 dx + \frac{1}{p+1} \int (t^2 + |x|^2) |v|^{p+1} dx, \end{aligned} \quad (5.2)$$

where $L = \partial_t + \frac{x}{|x|} \cdot \nabla$ and $\underline{L} = \partial_t - \frac{x}{|x|} \cdot \nabla$.

By direct computation,

$$\frac{d}{dt} \mathcal{E}(t) = \frac{8}{p+1} t \int |v|^{p+1} dx - 2t \int |v|^{p+1} dx. \quad (5.3)$$

Therefore, when $p \geq 3$, $0 \leq \mathcal{E}(t) \leq \mathcal{E}(0)$, which implies

$$\|u\|_{L_{t,x}^{p+1}(\mathbb{R} \times \mathbb{R}^3)} < \infty, \quad (5.4)$$

when $\mathcal{E}(0) < \infty$, since $\|v(t, \cdot)\|_{L^{p+1}} \lesssim \mathcal{E}(0)^{\frac{1}{p+1}} t^{-\frac{2}{p+1}}$. Combining (5.4) with the usual conservation of energy implies scattering.

Usually, if $(u_0, u_1) \in B_{1,1}^{s_c + \frac{3}{2}} \times B_{1,1}^{s_c + \frac{1}{2}}$, it is not true that $\mathcal{E}(0) < \infty$. Therefore, we make an argument similar to the argument in [5] and [7] to prove scattering.

Theorem 10. *Let r satisfy*

$$\frac{1}{r} = \frac{2}{p-1} - \frac{3}{p+1}. \quad (5.5)$$

Then,

$$\|v\|_{L_t^r L_x^{p+1}(\mathbb{R} \times \mathbb{R}^3)} < \infty. \quad (5.6)$$

Remark 5. Notice that when $p = 3$, which is the conformal case, $r = p + 1$, and when $p = 5$, the energy-critical case, $r = \infty$.

Proof. It is clear from Theorem 8 that (5.6) holds for the interval $[-1, 1]$. Therefore, by time reversal symmetry, it suffices to show that (5.6) holds on the interval $[1, \infty)$. By interpolation, (5.6) follows from

$$\int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \|v(t)\|_{L^{p+1}}^{p+1} dt < \infty, \quad (5.7)$$

and

$$\sup_{t \geq 1} t^{2(1-s_c)} \|v(t)\|_{L^{p+1}}^{p+1} < \infty. \quad (5.8)$$

As in the proof of global well-posedness, decompose the initial data

$$u|_{t=1} = v_0 + w_0, \quad \partial_t u(t, \cdot)|_{t=1} = v_1 + w_1, \quad (5.9)$$

where this time,

$$v_0 = \chi\left(\frac{x}{R}\right) P_{\leq j_0} u|_{t=1}, \quad v_1 = \chi\left(\frac{x}{R}\right) P_{\leq j_0} \partial_t u(t, \cdot)|_{t=1}. \quad (5.10)$$

In this case, j_0 satisfies (2.9) and (2.10). Here $\chi \in C_0^\infty(\mathbb{R}^3)$ is a smooth cutoff function, $\chi(x) = 1$ for $|x| \leq 1$, and $\chi(x) = 0$ for $|x| > 2$. Also, R is chosen so that

$$\|w_0\|_{\dot{H}^{s_c}(\mathbb{R}^3)} + \|w_1\|_{\dot{H}^{s_c-1}(\mathbb{R}^3)} \leq \epsilon \ll 1. \quad (5.11)$$

Now, let $u = v + w$, where v and w solve,

$$\begin{aligned} v_{tt} - \Delta v + |u|^{p-1}u - |w|^{p-1}w &= 0, & v(0, x) &= v_0, & v_t(0, x) &= v_1, \\ w_{tt} - \Delta w + |w|^{p-1}w &= 0, & w(0, x) &= w_0, & w_t(0, x) &= w_1, \end{aligned} \quad (5.12)$$

starting from $t = 1$. Then by (5.2) and (5.3),

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}(t) &= \frac{8t}{p+1} \int |v|^{p+1} dx - 2t \int |v|^{p+1} dx \\
&\quad - \frac{1}{2} \langle (t+|x|)Lv + 2v, (t+|x|)\{|u|^{p-1}u - |v|^{p-1}v - |w|^{p-1}w\} \rangle \\
&\quad - \frac{1}{2} \langle (t-|x|)\underline{L}v + 2v, (t-|x|)\{|u|^{p-1}u - |v|^{p-1}v - |w|^{p-1}w\} \rangle.
\end{aligned} \tag{5.13}$$

Again by Taylor's theorem, as in (3.7),

$$|u|^{p-1}u - |v|^{p-1}v - |w|^{p-1}w = p|v|^{p-1}w + O(|w|^2|v|^{p-2}) + O(|v||w|^{p-1}). \tag{5.14}$$

Furthermore, by finite propagation speed, v is supported on the set $\{x : |x| \leq t + R\}$.

Remark 6. This is the main reason for the difference between (5.12) and (3.3). Here, we seek to take advantage of finite propagation speed when computing (5.13).

As in section three,

$$\begin{aligned}
&\langle (t+|x|)Lv + 2v, (t+|x|)|w|^2|v|^{p-2} \rangle \\
&\lesssim (t+R)\|(t+|x|)Lv + 2v\|_{L^2}\|v\|_{L^{p+1}}^{p-2}\|w\|_{L^{p+1}}^{\frac{5-p}{2}}\|w\|_{L^\infty}^{\frac{p-1}{2}}.
\end{aligned} \tag{5.15}$$

Modifying the proof of Theorem 8 to a small data result,

$$\|w\|_{L^\infty}^{\frac{p-1}{2}} \lesssim \frac{\epsilon}{t}. \tag{5.16}$$

Remark 7. This is why we cannot use the methods in this paper to prove scattering in the $p = 3$ case, since in that case we would have $\frac{\ln(t)}{t}$ decay (as in (4.21)), which is not enough.

Therefore, for $t \geq 1$,

$$\begin{aligned}
&(t+R)\|(t+|x|)Lv + 2v\|_{L^2}\|v\|_{L^{p+1}}^{p-2}\|w\|_{L^{p+1}}^{\frac{5-p}{2}}\|w\|_{L^\infty}^{\frac{p-1}{2}} \\
&\lesssim_R \mathcal{E}(t)^{1/2}\|v\|_{L^{p+1}}^{p-2}\|w\|_{L^{p+1}}^{\frac{5-p}{2}} \lesssim \frac{\mathcal{E}(t)^{\frac{1}{2} + \frac{p-2}{p+1}}}{t^{\frac{2(p-2)}{p+1}}}\|w\|_{L^{p+1}}^{\frac{5-p}{2}}.
\end{aligned} \tag{5.17}$$

By Proposition 1 and Theorem 8,

$$\|w\|_{L_t^r L_x^{p+1}(\mathbb{R} \times \mathbb{R}^3)} \lesssim \epsilon, \tag{5.18}$$

where r satisfies (5.5). Furthermore,

Lemma 1. If w solves (5.12), and (w_0, w_1) has $B_{1,1}^{s_c+3/2} \times B_{1,1}^{s_c+1/2}$ norm of size $\epsilon \ll 1$, and $\frac{1}{2} \leq s_c < 1$,

$$\int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \|w(t)\|_{L^{p+1}}^{p+1} dt \lesssim \epsilon^{p+1}. \quad (5.19)$$

Proof. By direct computation, using Theorem 8,

$$\begin{aligned} & \int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \|P_j w\|_{L^{p+1}}^{p+1} dt \lesssim \int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \|P_j w\|_{L^2}^2 \|P_j w\|_{L^\infty}^{p-1} dt \\ & \lesssim a_j^{p+1} \int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} 2^{-2js_c} \inf\{(2^{j(\frac{3}{2} - s_c)})^{p-1}, (\frac{2^{-j(s_c - \frac{1}{2})}}{t})^{p-1}\} dt \lesssim a_j^{p+1}, \end{aligned} \quad (5.20)$$

where

$$a_j = 2^{j(s_c - \frac{1}{2})} \sup_t |t| \|P_j w\|_{L^\infty} + 2^{js_c} \|P_j w\|_{L_x^\infty L_x^2}, \quad \sum a_j \lesssim \epsilon. \quad \square \quad (5.21)$$

Following the computations in [7],

$$\begin{aligned} & \int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \frac{1}{t^2} \int_1^t \frac{\mathcal{E}(\tau)^{\frac{1}{2} + \frac{p-2}{p+1}}}{\tau^{\frac{2(p-2)}{p+1}}} \|w\|_{L^{p+1}}^{\frac{5-p}{2}} d\tau dt \lesssim \int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \frac{\mathcal{E}(t)^{\frac{1}{2} + \frac{p-2}{p+1}}}{t^{1 + \frac{2(p-2)}{p+1}}} (\|w(t)\|_{L_x^{p+1}}^{p+1})^{\frac{5-p}{2(p+1)}} dt \\ & \lesssim \left(\int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \frac{\mathcal{E}(t)}{t^2} dt \right)^{\frac{1}{2} + \frac{p-2}{p+1}} \left(\int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \|w(t)\|_{L^{p+1}}^{p+1} dt \right)^{\frac{5-p}{2(p+1)}}. \end{aligned} \quad (5.22)$$

Similarly,

$$\begin{aligned} & \langle (t + |x|)Lv + 2v, (t + |x|)|w|^{p-1}|v\rangle \\ & \lesssim (t + R) \|(t + |x|)Lv + 2v\|_{L^2} \|v\|_{L^{p+1}} \|w\|_{L^{p+1}}^{\frac{p-1}{2}} \|w\|_{L^\infty}^{\frac{p-1}{2}} \end{aligned} \quad (5.23)$$

$$\lesssim_R \frac{\mathcal{E}(t)^{\frac{1}{2} + \frac{1}{p+1}}}{t^{\frac{2}{p+1}}} \|w\|_{L^{p+1}}^{\frac{p-1}{2}}. \quad (5.24)$$

Again by Lemma 1,

$$\begin{aligned} & \int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \frac{1}{t^2} \int_1^t \frac{\mathcal{E}(\tau)^{\frac{1}{2} + \frac{1}{p+1}}}{\tau^{\frac{2}{p+1}}} \|w\|_{L^{p+1}}^{\frac{p-1}{2}} d\tau dt \\ & \lesssim \left(\int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \frac{\mathcal{E}(t)}{t^2} dt \right)^{\frac{1}{2} + \frac{1}{p+1}} \left(\int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \|w(t)\|_{L_x^{p+1}}^{p+1} dt \right)^{\frac{p-1}{2(p+1)}}. \end{aligned} \quad (5.25)$$

By (5.22) and (5.25), it only remains to compute

$$p \int \{(t + |x|)Lv + 2v\}(t + |x|)|v|^{p-1}w dx dt, \quad (5.26)$$

and

$$p \int \{(t - |x|)\underline{L}v + 2v\}(t - |x|)|v|^{p-1}w dx dt. \quad (5.27)$$

Using the computations in the proof of Theorem 8, we see that w is summable in Littlewood–Paley. Therefore, it suffices to compute (5.26) and (5.27) for one specific frequency, $P_j w$, provided our final estimates can be summed.

First compute

$$\iint_1^t (\tau + |x|)|v|^{p-1}v(P_j w) dx d\tau \lesssim \int_1^t (\tau + R) \frac{\mathcal{E}(\tau)^{\frac{p}{p+1}}}{\tau^{\frac{2p}{p+1}}} \|P_j w\|_{L^{p+1}} d\tau. \quad (5.28)$$

Therefore, as in (5.22)

$$\begin{aligned} & \int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \int_1^t (\tau + R) \frac{\mathcal{E}(\tau)^{\frac{p}{p+1}}}{\tau^{\frac{2p}{p+1}}} \|P_j w\|_{L^{p+1}} d\tau dt \\ & \lesssim \left(\int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \frac{\mathcal{E}(t)}{t^2} dt \right)^{\frac{p}{p+1}} \left(\int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \|P_j w\|_{L_x^{p+1}}^{p+1} dt \right)^{\frac{1}{p+1}}. \end{aligned} \quad (5.29)$$

Next, for (5.26) with L and (5.27) with $\underline{L}$, we consider three areas separately, inside the light cone, near the boundary of the light cone, and outside the light cone. Let $\chi \in C_0^\infty(\mathbb{R}^3)$ be a radially symmetric function such that $\chi(x) = 1$ for $|x| \leq \frac{9}{10}$ and $\chi(x) = 0$ for $|x| > \frac{19}{20}$. Then, we decompose

$$(\tau + |x|)Lv = \chi\left(\frac{x}{\tau}\right)(\tau + |x|)Lv + \chi\left(\frac{\tau x}{|x|^2}\right)(\tau + |x|)Lv + \left(1 - \chi\left(\frac{x}{\tau}\right) - \chi\left(\frac{\tau x}{|x|^2}\right)\right)(\tau + |x|)Lv. \quad (5.30)$$

Remark 8. Observe that $\chi(\frac{\tau x}{|x|^2})$ is supported on $|x| \geq \frac{20}{19}\tau$.

By the product rule,

$$\begin{aligned} & p \iint_1^t \chi\left(\frac{x}{\tau}\right) \{(\tau + |x|)Lv\} (\tau + |x|)|v|^{p-1}(P_j w) dx d\tau \\ & = \iint_1^t \chi\left(\frac{x}{4\tau}\right) (P_j w) (\tau + |x|)^2 L(|v|^{p-1}v) dx d\tau. \end{aligned} \quad (5.31)$$

If $2^{-j} \gtrsim \tau$, then the computations are rather easy. If we integrate by parts with L , then we pick up either a $\frac{1}{\tau}$ multiplier or a $\frac{1}{|x|}$ multiplier, and we then proceed by the analysis shown below in (5.34). There is no need to split $|v|^{p-1}v$.

When $2^j \gg \tau$, we do decompose $|v|^{p-1}v$. By Theorem 8, as in (3.18),

$$\begin{aligned} & \iint_1^t \chi\left(\frac{x}{\tau}\right) (P_j w)(\tau + |x|)^2 L(|P_{\leq j} v|^{p-1}(P_{\leq j} v)) dx d\tau \\ & \lesssim \int_1^t \frac{2^{j(s_c-1/2)} \|\tau P_j w\|_{L^\infty}}{\tau} \|(t + |x|) L v\|_{L^2}^{\frac{2}{p-1}} \|(t^2 + |x|^2)^{\frac{1}{p+1}} v\|_{L^{p+1}}^{p-\frac{2}{p-1}} d\tau \\ & \lesssim \int_1^t \frac{2^{j(s_c-1/2)} \|\tau P_j w\|_{L^\infty}}{\tau} \mathcal{E}(\tau) d\tau + \int_1^t (\tau + R) \frac{\mathcal{E}(\tau)^{\frac{p}{p+1}}}{\tau^{\frac{2p}{p+1}}} \|P_j w\|_{L^{p+1}} d\tau. \end{aligned} \quad (5.32)$$

By straightforward computation,

$$\begin{aligned} & \int_1^\infty \frac{1}{t^{2(s_c-\frac{1}{2})}} \frac{1}{t^2} \int_1^t \frac{\|\tau P_j w\|_{L^\infty}}{\tau} \mathcal{E}(\tau) d\tau dt \\ & \lesssim \sup_{t \in [1, \infty)} 2^{j(s_c-1/2)} \|t P_j w\|_{L^\infty} \cdot \int_1^\infty \frac{1}{t^{2(s_c-\frac{1}{2})}} \frac{\mathcal{E}(t)}{t^2} dt. \end{aligned} \quad (5.33)$$

Next, integrating by parts,

$$\begin{aligned} & \iint_1^t \chi\left(\frac{x}{\tau}\right) (P_j w)(\tau + |x|)^2 L(|v|^{p-1}v - |P_{\leq j} v|^{p-1}(P_{\leq j} v)) dx d\tau \\ & = - \iint_1^t \chi\left(\frac{x}{\tau}\right) L(P_j w)(\tau + |x|)^2 \{|v|^{p-1}v - |P_{\leq j} v|^{p-1}(P_{\leq j} v)\} dx d\tau \\ & \quad - 2 \iint_1^t \chi\left(\frac{x}{\tau}\right) (P_j w)(\tau + |x|) \{|v|^{p-1}v - |P_{\leq j} v|^{p-1}(P_{\leq j} v)\} dx d\tau \\ & \quad - \iint_1^t L(\chi\left(\frac{x}{\tau}\right)) (P_j w)(\tau + |x|)^2 \{|v|^{p-1}v - |P_{\leq j} v|^{p-1}(P_{\leq j} v)\} dx d\tau \\ & \quad - \iint_1^t \chi\left(\frac{x}{\tau}\right) (P_j w)(\tau + |x|)^2 \frac{1}{|x|} \{|v|^{p-1}v - |P_{\leq j} v|^{p-1}(P_{\leq j} v)\} dx d\tau \\ & \quad + \int \chi\left(\frac{x}{\tau}\right) (P_j w)(\tau + |x|)^2 \{|v|^{p-1}v - |P_{\leq j} v|^{p-1}(P_{\leq j} v)\} dx \Big|_{\tau=1}^{\tau=t}. \end{aligned} \quad (5.34)$$

Following (5.29),

$$\begin{aligned} & \int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \frac{1}{t^2} \{2 \iint_1^t \chi(\frac{x}{\tau})(P_j w)(\tau + |x|) \{|v|^{p-1}v - |P_{\leq j}v|^{p-1}(P_{\leq j}v)\} dx d\tau\} dt \\ & \lesssim \left(\int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \frac{\mathcal{E}(t)}{t^2} dt \right)^{\frac{p}{p+1}} \left(\int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \|P_j w\|_{L_x^{p+1}}^{p+1} dt \right)^{\frac{1}{p+1}}. \end{aligned} \quad (5.35)$$

Similarly, since $\frac{(\tau+|x|)^2}{\tau} \lesssim (\tau + R)$,

$$\begin{aligned} & \int \frac{1}{t^{2(s_c - \frac{1}{2})}} \frac{1}{t^2} \{ \iint_1^t L(\chi(\frac{x}{\tau}))(P_j w)(\tau + |x|)^2 \{|v|^{p-1}v - |P_{\leq j}v|^{p-1}(P_{\leq j}v)\} dx d\tau \} dt \\ & \lesssim \left(\int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \frac{\mathcal{E}(t)}{t^2} dt \right)^{\frac{p}{p+1}} \left(\int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \|P_j w\|_{L_x^{p+1}}^{p+1} dt \right)^{\frac{1}{p+1}}. \end{aligned} \quad (5.36)$$

Split $\frac{(\tau+|x|)^2}{|x|} = \frac{\tau^2}{|x|} + O(\tau + R)$. The contribution of the $O(\tau + R)$ term is estimated identically to (5.36). We set aside the contribution of

$$- \int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \frac{1}{t^2} \{2 \iint_1^t \chi(\frac{x}{\tau})(P_j w) \frac{\tau^2}{|x|} \{|v|^{p-1}v - |P_{\leq j}v|^{p-1}(P_{\leq j}v)\} dx d\tau\} dt, \quad (5.37)$$

for now.

Next, it is straightforward to compute

$$\begin{aligned} & \int \frac{1}{t^{2(s_c - \frac{1}{2})}} \frac{1}{t^2} \int \chi(\frac{x}{\tau})(P_j w)(\tau + |x|)^2 \{|v|^{p-1}v - |P_{\leq j}v|^{p-1}(P_{\leq j}v)\} dx \Big|_{\tau=1}^\tau dt \\ & \lesssim \left(\int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \frac{\mathcal{E}(t)}{t^2} dt \right)^{\frac{p}{p+1}} \left(\int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \|P_j w\|_{L_x^{p+1}}^{p+1} dt \right)^{\frac{1}{p+1}} + \|P_j w(1)\|_{L^{p+1}} \|v(1)\|_{L^{p+1}}^p \int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \frac{1}{t^2} dt \\ & \lesssim \left(\int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \frac{\mathcal{E}(t)}{t^2} dt \right)^{\frac{p}{p+1}} \left(\int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \|P_j w\|_{L_x^{p+1}}^{p+1} dt \right)^{\frac{1}{p+1}} + \|P_j w(1)\|_{L^{p+1}} \mathcal{E}(1)^{\frac{p}{p+1}}. \end{aligned} \quad (5.38)$$

Finally, by Bernstein's inequality, examining the kernel of the Littlewood–Paley projection, for any $N < \infty$, for $2^{-j} \ll \tau$,

$$\begin{aligned} & \|\chi(\frac{x}{\tau})P_{\geq j}v\|_{L^2} \lesssim 2^{-j} \|\chi(\frac{x}{\tau})\nabla v\|_{L^2} + \frac{2^{-j}}{\tau} \|\chi'(\frac{x}{\tau})v\|_{L^2} + \| [P_{\geq j}, \chi(\frac{x}{\tau})]v \|_{L^2} \\ & \lesssim 2^{-j} \|\chi(\frac{x}{\tau})\nabla v\|_{L^2} + \frac{2^{-j}}{\tau} \|v\|_{L^2(|x| \leq \tau)} + C_N \frac{2^{-j}}{\tau} \left\| \frac{1}{\langle 2^j |x| \rangle^N} v \right\|_{L^2(|x| \geq \tau)}. \end{aligned} \quad (5.39)$$

Now then, since

$$\begin{aligned} \|(\tau^2 + |x|^2)^{1/2} \chi(\frac{x}{\tau}) \nabla_{t,x} v\|_{L^2} &\sim \|(\tau + |x|) \chi(\frac{x}{\tau}) L v\|_{L^2} + \|(\tau - |x|) \chi(\frac{x}{\tau}) \underline{L} v\|_{L^2} \\ &\quad + \|(\tau^2 + |x|^2)^{1/2} \nabla v\|_{L^2}, \end{aligned} \quad (5.40)$$

$$2^{-j} \|\chi(\frac{x}{\tau}) \nabla v\|_{L^2} \lesssim \frac{2^{-j}}{\tau} \mathcal{E}(\tau)^{1/2}. \quad (5.41)$$

Therefore, as in (5.33), using the proof of Theorem 8, from which we can see that the contribution of ∂_t can be analyzed in a manner identical to the contribution of ∂_r , $\|\partial_t P_j w\|_{L^\infty} + \|\partial_r P_j w\|_{L^\infty} \lesssim 2^j \|P_j w\|_{L^\infty}$,

$$\begin{aligned} &\int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \frac{1}{t^2} \iint_1^t \chi(\frac{x}{\tau}) L(P_j w) (\tau + |x|)^2 \{|v|^{p-1} v - |P_{\leq j} v|^{p-1} (P_{\leq j} v)\} dx d\tau dt \\ &\lesssim \int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \frac{1}{t^2} \int_1^t \frac{2^{j(s_c - 1/2)} \|\tau P_j w\|_{L^\infty}}{\tau} \mathcal{E}(\tau) d\tau dt \\ &\quad + \left(\int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \frac{\mathcal{E}(t)}{t^2} dt \right)^{\frac{1}{p+1}} \left(\int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \|P_j w\|_{L_x^{p+1}}^{p+1} dt \right)^{\frac{1}{p+1}}. \end{aligned} \quad (5.42)$$

The computations involving the $\underline{L}$ are virtually identical, the only difference being that since $\underline{L} = \partial_t - \partial_r$, we have a term that is the negative of (5.37), canceling it out. Thus, the contribution of the $\chi(\frac{x}{\tau})$ term is complete.

For $\chi(\frac{\tau x}{|x|^2})$, (5.40) also holds, so we can use the same arguments as in the $\chi(\frac{x}{\tau})$ case.

For the case when $|x|$ is close to t , it is useful to work in the diagonal coordinates $t - |x| = c_1$ and $t + |x| = c_2$. It is also useful to let $\psi(\frac{x}{\tau}) = 1 - \chi(\frac{x}{\tau}) - \chi(\frac{\tau x}{|x|^2})$. For any T , by a change of variables,

$$\begin{aligned} &p \iint_{\frac{T}{2}}^T \psi(\frac{x}{t}) \{(t + |x|) L v\} (t + |x|) |v|^{p-1} (P_j w) dx dt \\ &= \int_{c \lesssim T} \int_{t - |x| = c: \frac{T}{2} \leq t \leq T} \psi(\frac{x}{t}) (t + |x|)^2 (P_j w) L(|v|^{p-1} v) dx dc. \end{aligned} \quad (5.43)$$

As before, we would like to integrate by parts and move part of the L onto $(P_j w)$ while keeping the rest of the L on v . To do this successfully, we will need to make a Littlewood–Paley decomposition along the coordinates $t - |x| = c$. Indeed, let

$$1_{\frac{T}{2} \leq t \leq T} (c + |x|) v(c + |x|, x) = f_{c,T}(x), \quad (5.44)$$

and then let $\tilde{P}_{\leq j}v$ and $\tilde{P}_{\geq j}v$ denote the standard Littlewood–Paley projections of $f_{c,T}(x)$. Now, compute

$$\begin{aligned} & \int_{t-|x|=c: \frac{T}{2} \leq t \leq T} \psi\left(\frac{x}{t}\right)(t+|x|)^2(P_j w)L(|\tilde{P}_j v|^{p-1}\tilde{P}_j v)dx \\ & \lesssim \frac{2^{j(s_c-\frac{1}{2})}}{\tau} \sup_{\tau \in [1, \infty)} \|\tau P_j w\|_{L^\infty} \|(t+|x|)Lv\|_{L^2}^{\frac{2}{p-1}} \|(t+|x|)^{\frac{2}{p+1}}v\|_{L^{p+1}}^{\frac{(p-2)(p+1)}{p-1}}, \end{aligned} \quad (5.45)$$

where L^r is the norm of $f(c+|x|, x)$ on the set $\frac{T}{2} \leq t \leq T$.

Remark 9. In the above computation, we are using the fact that $f_{c,T}$ is supported on the annulus of integration in (5.45). Indeed, if Ω is an annulus,

$$\|1_\Omega L\tilde{P}_j f_{c,T}\|_{L^{p+1}}^{p+1} \leq \|L\tilde{P}_j f_{c,T}\|_{L^{p+1}}^{p+1} \lesssim 2^j \|f_{c,T}\|_{L^{p+1}}^{p+1} = 2^j \|1_\Omega f_{c,T}\|_{L^{p+1}}^{p+1}.$$

Therefore,

$$\begin{aligned} & \int_{c \lesssim T} \int_{t-|x|=c: \frac{T}{2} \leq t \leq T} \psi\left(\frac{x}{t}\right)(t+|x|)^2(P_j w)L(|\tilde{P}_j v|^{p-1}\tilde{P}_j v)dx dc \\ & \lesssim \int_{\frac{T}{2}}^T \frac{2^{j(s_c-1/2)} \|\tau P_j w\|_{L^\infty}}{\tau} \mathcal{E}(\tau) d\tau. \end{aligned} \quad (5.46)$$

Meanwhile, for

$$\int_{c \lesssim T} \int_{t-|x|=c: \frac{T}{2} \leq t \leq T} \psi\left(\frac{x}{t}\right)(t+|x|)^2(P_j w)L(|v|^{p-1}v - |\tilde{P}_j v|^{p-1}\tilde{P}_j v)dx dc, \quad (5.47)$$

we can integrate by parts, obtaining terms in the form of (5.34). The computations are similar for the $\underline{L}$ terms. The only potential difficulty lies in the fact that when we integrate by parts

$$\int_{c \lesssim T} \int_{t-|x|=c: \frac{T}{2} \leq t \leq T} \psi\left(\frac{x}{t}\right)(t-|x|)^2(P_j w)\underline{L}(|v|^{p-1}v - |\tilde{P}_j v|^{p-1}\tilde{P}_j v)dx dc, \quad (5.48)$$

$\underline{L}$ may hit $(t-|x|)^2$, since in this case it is not true that $\underline{L}(t-|x|)^2 \sim \frac{(t-|x|)^2}{t}$. However, this is no problem, since following (5.35),

$$\begin{aligned}
& \int_{c \lesssim T} \int_{t-|x|=c: \frac{T}{2} \leq t \leq T} \psi\left(\frac{x}{t}\right)(t-|x|)(P_j w)(|v|^{p-1}v - |\tilde{P}_j v|^{p-1}\tilde{P}_j v) dx dc \\
& \lesssim T \|P_j w\|_{L_{t,x}^{p+1}([\frac{T}{2}, T] \times \mathbb{R}^3)} \|v\|_{L_{t,x}^{p+1}([\frac{T}{2}, T] \times \mathbb{R}^3)}^p \\
& \lesssim \left(\int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \frac{\mathcal{E}(t)}{t^2} dt \right)^{\frac{p}{p+1}} \left(\int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \|P_j w\|_{L_x^{p+1}}^{p+1} dt \right)^{\frac{1}{p+1}}.
\end{aligned} \tag{5.49}$$

Finally, it should be noted that (5.47) only includes an integral over $\frac{T}{2} \leq t \leq T$, whereas before, our integrals were always on the interval $1 \leq \tau \leq t$. This means that we will pick up several boundary terms. However, the boundary terms are easily summed. Indeed,

$$\begin{aligned}
& \frac{1}{t^2} \sum_k \int \psi\left(\frac{x}{\tau}\right)(\tau + |x|)^2 (P_j w)(|v|^{p-1}v) dx|_{\tau=2^{-k}t} \\
& \lesssim \frac{1}{t^2} \sum_k (2^{-k}t)^{\frac{2}{p+1}} \mathcal{E}(2^{-k}t)^{\frac{p}{p+1}} \|P_j w(2^{-k}t)\|_{L^{p+1}}.
\end{aligned} \tag{5.50}$$

By a straightforward change of variables, using Proposition 1 and Theorem 8,

$$\begin{aligned}
& \sum_{j,k} \int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \frac{1}{t^2} (2^{-k}t)^{\frac{2}{p+1}} \mathcal{E}(2^{-k}t)^{\frac{p}{p+1}} \|P_j w(2^{-k}t)\|_{L^{p+1}} dt \\
& \lesssim \sum_{j,k} \left(\int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \frac{\mathcal{E}(2^{-k}t)}{t^2} dt \right)^{\frac{p}{p+1}} \left(\int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \|P_j w\|_{L_x^{p+1}}^{p+1} dt \right)^{\frac{1}{p+1}} \\
& \lesssim \sum_{j,k} 2^{-k \cdot \frac{p}{p+1}} 2^{-2k(s_c - \frac{1}{2})} 2^{k \cdot \frac{1}{p+1}} \left(\int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \frac{1}{t^2} \mathcal{E}(t) dt \right)^{\frac{p}{p+1}} \left(\int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \|P_j w(t)\|_{L^{p+1}}^{p+1} dt \right)^{\frac{1}{p+1}} \\
& \lesssim \epsilon \left(\int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \frac{1}{t^2} \mathcal{E}(t) dt \right)^{\frac{p}{p+1}}.
\end{aligned} \tag{5.51}$$

In conclusion, we have proved

$$\begin{aligned}
& \int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \frac{\mathcal{E}(t)}{t^2} dt \lesssim \epsilon \int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \frac{\mathcal{E}(t)}{t^2} dt \\
& + c(R) \left(\int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \frac{\mathcal{E}(t)}{t^2} dt \right)^{\frac{1}{2} + \frac{p-2}{p+1}} \left(\int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \|w\|_{L_x^{p+1}}^{p+1} dt \right)^{\frac{5-p}{2(p+1)}}
\end{aligned}$$

$$+ c(R) \left(\int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \frac{\mathcal{E}(t)}{t^2} dt \right)^{\frac{p}{p+1}} \left(\int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \|w\|_{L_x^{p+1}}^{p+1} dt \right)^{\frac{1}{p+1}} + \mathcal{E}(1). \quad (5.52)$$

Then by standard bootstrap arguments, we have proved

$$\int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \frac{\mathcal{E}(t)}{t^2} dt < \infty, \quad (5.53)$$

which implies (5.7).

Plugging (5.53) back into (5.22)–(5.52), but taking $\sup_t$ rather than integrating over $[1, \infty)$ implies

$$\sup_{t \geq 1} t^{-2s_c} \mathcal{E}(t) < \infty, \quad (5.54)$$

which implies (5.8). $\square$

Remark 10. We have actually proved something more. By (5.13), we have actually proved

$$\int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \frac{\mathcal{E}(t)}{t^2} dt + \frac{p-3}{p+1} \int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \frac{1}{t^2} \int_1^t \tau \|v(\tau)\|_{L^{p+1}}^{p+1} d\tau dt < \infty, \quad (5.55)$$

which will be useful in the proof of scattering.

Unlike in the case of the nonlinear Schrödinger equation, see [6], (5.6) is not enough to prove scattering, even though we have a uniform bound on a norm of v , and thus $v + w$, which can be partitioned into finitely many small intervals. To prove scattering, we will utilize integrals on cones.

Theorem 11. For $3 < p \leq 4$,

$$\|v\|_{L_{t,x}^{2(p-1)}(\mathbb{R} \times \mathbb{R}^3)} < \infty. \quad (5.56)$$

Proof. Again by (5.40),

$$\|(\tau^2 + |x|^2)^{1/2} \chi(\frac{x}{\tau}) \nabla v\|_{L^2}^2 \lesssim \mathcal{E}(\tau) + \|\chi(\frac{x}{\tau}) v\|_{L^2}^2. \quad (5.57)$$

Therefore, by the Sobolev embedding theorem,

$$\begin{aligned} \|(\tau^2 + |x|^2)^{1/2} \chi(\frac{x}{\tau}) v\|_{L^6}^2 &\lesssim \|(\tau^2 + |x|^2)^{1/2} \chi(\frac{x}{\tau}) \nabla v\|_{L^2}^2 \\ &+ \|(\tau^2 + |x|^2)^{1/2} \frac{1}{\tau} \chi'(\frac{x}{\tau}) v\|_{L^2}^2 + \|\chi(\frac{x}{\tau}) v\|_{L^2}^2 \end{aligned}$$

$$\lesssim \mathcal{E}(\tau) + \tau^{\frac{3(p-1)}{p+1}} \frac{\mathcal{E}(\tau)^{\frac{2}{p+1}}}{\tau^{\frac{4}{p+1}}}. \quad (5.58)$$

Therefore, by (5.8),

$$\|(\tau^2 + |x|^2)^{1/2} \chi(\frac{x}{\tau}) v\|_{L^6}^2 \lesssim \tau^{\frac{3(p-1)}{p+1}} \frac{\mathcal{E}(\tau)^{\frac{2}{p+1}}}{\tau^{\frac{4}{p+1}}} \quad (5.59)$$

Combining (5.59),

$$\|\chi(\frac{x}{\tau}) v\|_{L^{p+1}} \lesssim \frac{\mathcal{E}(\tau)^{\frac{1}{p+1}}}{\tau^{\frac{2}{p+1}}}, \quad (5.60)$$

and (5.53) implies

$$\|\chi(\frac{x}{\tau}) v\|_{L_{t,x}^{2(p-1)}(\mathbb{R} \times \mathbb{R}^3)} < \infty. \quad (5.61)$$

Remark 11. It is here that it is necessary to use the restriction $3 < p \leq 4$, since it is only here that $p+1 < 2(p-1) \leq 6$.

The same exact computations, combined with the support of v implies

$$\|\chi(\frac{\tau x}{|x|^2}) v\|_{L_{t,x}^{2(p-1)}(\mathbb{R} \times \mathbb{R}^3)} < \infty. \quad (5.62)$$

For r close to t , let $\tilde{E}$ denote the conformal energy inside a cone,

$$\begin{aligned} \tilde{E}(t) = & \frac{1}{4} \int_{r \leq t+c} ((t+r)Lv + 2v)^2 dx + \frac{1}{4} \int_{r \leq t+c} ((t-r)Lv + 2v)^2 dx \\ & + \frac{1}{2} \int_{r \leq t+c} (t^2 + r^2) |\nabla v|^2 dx + \frac{1}{p+1} \int_{r \leq t+c} (t^2 + r^2) |v|^{p+1} dx. \end{aligned} \quad (5.63)$$

If v solves (5.1),

$$\begin{aligned} \frac{d}{dt} \tilde{E}(t) = & \frac{1}{4} \int_{r=t+c} ((t+r)Lv + 2v)^2 d\sigma - \frac{1}{2(p+1)} \int_{r=t+c} (t+r)^2 |v|^{p+1} d\sigma \\ & - \frac{1}{4} \int_{r=t+c} ((t-r)\underline{L}v + 2v)^2 + \frac{1}{2(p+1)} \int_{r=t+c} (t-r)^2 |v|^{p+1} d\sigma \\ & + \frac{1}{4} \int_{r=t+c} ((t+r)Lv + 2v)^2 d\sigma + \frac{1}{4} \int_{r=t+c} ((t-r)\underline{L}v + 2v)^2 d\sigma \\ & + \frac{1}{p+1} \int_{r=t+c} (t^2 + r^2) |v|^{p+1} d\sigma \end{aligned}$$

$$+ \frac{1}{2} \int_{r=t+c} (t^2 + r^2) |\nabla v|^2 d\sigma + t \left(\frac{8}{p+1} - 2 \right) \int_{r \leq t+c} |v|^{p+1} dx, \quad (5.64)$$

where $d\sigma$ is the surface measure on the cone $r = t + c$. Summing up,

$$\begin{aligned} \frac{d}{dt} \tilde{E}(t) &= \frac{1}{2} \int_{r=t+c} ((t+r)Lv + 2v)^2 d\sigma + \frac{1}{p+1} \int_{r=t+c} (t+r)^2 |v|^{p+1} d\sigma \\ &\quad + \frac{1}{2} \int_{r=t+c} (t^2 + r^2) |\nabla v|^2 d\sigma + t \left(\frac{8}{p+1} - 2 \right) \int_{r \leq t+c} |v|^{p+1} dx. \end{aligned} \quad (5.65)$$

Computing

$$\tilde{E}(T) - \tilde{E}\left(\frac{T}{2}\right), \quad (5.66)$$

when v solves (5.12) using the fundamental theorem of calculus implies, by (5.53), (5.54), (5.55), and the proof of Theorem 10,

$$\begin{aligned} &\int_{c \sim T, \frac{T}{2} \leq t \leq T} \frac{1}{t^2} \cdot \frac{1}{t^{2(s_c - \frac{1}{2})}} \int_{r=t+c} ((t+r)Lv + 2v)^2 d\sigma dc \\ &\quad + \int_{c \sim T, \frac{T}{2} \leq t \leq T} \frac{1}{t^2} \cdot \frac{1}{t^{2(s_c - \frac{1}{2})}} \int_{r=t+c} (t+r)^2 |v|^{p+1} d\sigma dc \\ &\quad + \int_{c \sim T, \frac{T}{2} \leq t \leq T} \frac{1}{t^2} \cdot \frac{1}{t^{2(s_c - \frac{1}{2})}} \int_{r=t+c} (t^2 + r^2) |\nabla v|^2 d\sigma dc \lesssim \int_T^{2T} \frac{1}{t^{2(s_c - \frac{1}{2})}} \frac{\mathcal{E}(t)}{t^2} dt < \infty. \end{aligned} \quad (5.67)$$

Therefore, following the proof of (5.61), only integrating along cones instead of in dx ,

$$\int_1^\infty \frac{1}{t^{2(s_c - \frac{1}{2})}} \int \psi\left(\frac{x}{\tau}\right) |v|^{2(p-1)} dx dt < \infty, \quad (5.68)$$

which proves the Theorem. $\square$

6. Profile decomposition argument

Having obtained a scattering result for any $u_0 \in B_{1,1}^{\frac{3}{2}+s_c}$, $u_1 \in B_{1,1}^{\frac{1}{2}+s_c}$, it only remains to show that this bound is uniform over all (u_0, u_1) satisfying

$$\|(u_0, u_1)\|_{B_{1,1}^{\frac{3}{2}+s_c} \times B_{1,1}^{\frac{1}{2}+s_c}} \leq A, \quad (6.1)$$

for some $A < \infty$. The proof argument is exactly parallel to the arguments in [4], [3], and especially in [2]. Here we are in the nonradial setting, however, we are aided by the fact that the nonlinearity is not the Lorentz invariant nonlinearity.

Let (u_0^n, u_1^n) be a bounded sequence in $B_{1,1}^{\frac{3}{2}+s_c} \times B_{1,1}^{\frac{1}{2}+s_c}$. Since this sequence is bounded in $\dot{H}^{s_c} \times \dot{H}^{s_c-1}$, then by Theorem 3.1 in [15], we may make the profile decomposition

$$S(t)(u_{0,n}, u_{1,n}) = \sum_{j=1}^N \Gamma_j^n S(t)(\phi_0^j, \phi_1^j) + S(t)(R_{0,n}^N, R_{1,n}^N), \quad (6.2)$$

where

$$\lim_{N \rightarrow \infty} \limsup_{n \rightarrow \infty} \|S(t)(R_{0,n}^N, R_{1,n}^N)\|_{L_{t,x}^{2(p-1)}(\mathbb{R} \times \mathbb{R}^3)} = 0. \quad (6.3)$$

The group Γ_j^n is the group of operators generated by translation in space and in time, and also by the scaling symmetry. That is, there exist $x_j^n \in \mathbb{R}^3$, $t_j^n \in \mathbb{R}$, and $\lambda_j^n \in (0, \infty)$ such that

$$\Gamma_j^n v(t, x) = (\lambda_j^n)^{\frac{2}{p-1}} v(\lambda_j^n(t - t_j^n), \lambda_j^n(x - x_j^n)). \quad (6.4)$$

Furthermore, the Γ_j^n 's have the asymptotic orthogonality property that when $j \neq k$,

$$\lim_{n \rightarrow \infty} |\ln(\frac{\lambda_j^n}{\lambda_k^n})| + (\lambda_j^n)^{1/2} (\lambda_k^n)^{1/2} (|x_j^n - x_k^n| + |t_j^n - t_k^n|) = \infty. \quad (6.5)$$

Using the dispersive estimate in (1.12), $\frac{|t_j^n|}{\lambda_j^n}$ is uniformly bounded for any j .

Lemma 2. *If $\frac{|t_j^n|}{\lambda_j^n} \rightarrow \infty$ then $\phi_0^j = 0$ and $\phi_1^j = 0$.*

Proof. Indeed, from [15], for any fixed j ,

$$\lim_{n \rightarrow \infty} (\Gamma_j^n)^{-1} S(t)(u_0^n, u_1^n) \rightharpoonup S(t)(\phi_0^j, \phi_1^j) \quad (6.6)$$

weakly in $L_{t,x}^{2(p-1)}$. Rewriting $(\Gamma_j^n)^{-1}$,

$$(\Gamma_j^n)^{-1} S(t)(u_0^n, u_1^n) = S(t + \frac{t_j^n}{\lambda_j^n})((\lambda_j^n)^{-\frac{2}{p-1}} u_0^n(\frac{x + x_j^n}{\lambda_j^n}), (\lambda_j^n)^{-\frac{p+1}{p-1}} u_1^n(\frac{x + x_j^n}{\lambda_j^n})), \quad (6.7)$$

and then by the dispersive estimate (1.12), for any fixed Littlewood–Paley projection, if $\frac{t_j^n}{\lambda_j^n} \rightarrow \pm\infty$,

$$S(t + \frac{t_j^n}{\lambda_j^n})((\lambda_j^n)^{-\frac{2}{p-1}} u_0^n(\frac{x + x_j^n}{\lambda_j^n}), (\lambda_j^n)^{-\frac{p+1}{p-1}} u_1^n(\frac{x + x_j^n}{\lambda_j^n})) \rightharpoonup 0, \quad (6.8)$$

weakly in $L_{t,x}^{2(p-1)}$, which proves the lemma. $\square$

Since t_j^n is bounded for any j , after passing to a subsequence, $t_j^n \rightarrow t_j$. Absorbing the remainder into R_N , we may rewrite (6.2) with Γ_j^n having no translation in time, that is,

$$\Gamma_j^n v(t, x) = (\lambda_j^n)^{\frac{2}{p-1}} v(\lambda_j^n t, \lambda_j^n(x + x_j^n)). \quad (6.9)$$

Furthermore, since

$$(\lambda_j^n)^{\frac{2}{p-1}} u_0(\lambda_j^n x) \rightharpoonup \phi_0^j, \quad \text{and} \quad (\lambda_j^n)^{\frac{2}{p-1}+1} u_1(\lambda_j^n x) \rightharpoonup \phi_1^j, \quad (6.10)$$

we have the bounds

$$\|\phi_0^j\|_{B_{1,1}^{\frac{3}{2}+s_c}} + \|\phi_1^j\|_{B_{1,1}^{\frac{1}{2}+s_c}} \leq A. \quad (6.11)$$

Therefore, the solution to (1.1) with initial data equal to (ϕ_0^j, ϕ_1^j) has a finite $L_{t,x}^{2(p-1)}$ norm. Furthermore,

$$\lim_{N \rightarrow \infty} \sum_{j=1}^N \|(\phi_0^j, \phi_1^j)\|_{\dot{H}^{s_c} \times \dot{H}^{s_c-1}}^2 \leq \limsup_{n \rightarrow \infty} \|(u_{0,n}, u_{1,n})\|_{\dot{H}^{s_c} \times \dot{H}^{s_c-1}}^2, \quad (6.12)$$

so for only finitely many j , $\|(\phi_0^j, \phi_1^j)\|_{\dot{H}^{s_c} \times \dot{H}^{s_c-1}} \geq \epsilon$. If $\|(\phi_0^j, \phi_1^j)\|_{\dot{H}^{s_c} \times \dot{H}^{s_c-1}} \leq \epsilon$, then the solution to (1.1) with initial data (ϕ_0^j, ϕ_1^j) has the bound

$$\|u\|_{L_{t,x}^{2(p-1)}} \lesssim \|(\phi_0^j, \phi_1^j)\|_{\dot{H}^{s_c} \times \dot{H}^{s_c-1}}. \quad (6.13)$$

Therefore, by standard perturbative arguments combined with the asymptotic orthogonality in (6.5), if u_n is the solution to (1.1) with initial data $(u_{0,n}, u_{1,n})$,

$$\lim_{n \rightarrow \infty} \|u_n\|_{L_{t,x}^{2(p-1)}} < \infty. \quad (6.14)$$

Thus, there must exist a uniform upper bound on the $L_{t,x}^{2(p-1)}$ norm of a solution u to (1.1) whose initial data has bounded Besov norm.

References

- [1] B. Dodson, A. Lawrie, D. Mendelson, J. Murphy, Scattering for defocusing energy subcritical nonlinear wave equations, *Anal. PDE* 13 (7) (2020) 1995–2090.
- [2] B. Dodson, Global well-posedness and scattering for the radial, defocusing, cubic wave equation with initial data in a critical Besov space, *Anal. PDE* 12 (4) (2018) 1023–1048.
- [3] B. Dodson, Global well-posedness for the radial, defocusing, nonlinear wave equation for $3 < p < 5$, *Am. J. Math.* 146 (1) (2024) 1–46, <https://doi.org/10.1353/ajm.2024.a917538>, Project MUSE.
- [4] B. Dodson, Global well-posedness and scattering for the radial, defocusing, cubic nonlinear wave equation, *Duke Math. J.* 170 (15) (2021) 3267–3321.

- [5] B. Dodson, Global well-posedness of the radial conformal nonlinear wave equation with initial data in a critical space, preprint, arXiv:2206.13586, 2022.
- [6] B. Dodson, Scattering for the defocusing, nonlinear Schrödinger equation with initial data in a critical space, *Int. Math. Res. Not.* (2022).
- [7] B. Dodson, Sharp global well-posedness and scattering for the radial conformal nonlinear wave equation, preprint, arXiv:2305.15633, 2023.
- [8] J. Ginibre, G. Velo, Generalized Strichartz inequalities for the wave equation, *J. Funct. Anal.* 133 (1) (1995) 50–68.
- [9] L.V. Kapitanski, Some generalizations of the Strichartz–Brenner inequality, *Algebra Anal.* 1 (3) (1989) 127–159.
- [10] T. Kato, An $L^{q,r}$ -theory for nonlinear Schrödinger equations, in: *Spectral and Scattering Theory and Applications*, Mathematical Society of Japan, 1994, pp. 223–238.
- [11] C. Kenig, F. Merle, Global well-posedness, scattering and blow-up for the energy critical focusing non-linear wave equation, *Acta Math.* 201 (2) (2008) 147–212.
- [12] C. Kenig, F. Merle, Scattering for $H^{1/2}$ bounded solutions to the cubic, defocusing NLS in 3 dimensions, *Trans. Am. Math. Soc.* 362 (4) (2010) 1937–1962.
- [13] M. Keel, T. Tao, Endpoint Strichartz estimates, *Am. J. Math.* 120 (5) (1998) 955–980.
- [14] H. Lindblad, C.D. Sogge, On existence and scattering with minimal regularity for semilinear wave equations, *J. Funct. Anal.* 130 (2) (1995) 357–426.
- [15] J. Ramos, A refinement of the Strichartz inequality for the wave equation with applications, *Adv. Math.* 230 (2) (2012) 649–698.
- [16] R. Shen, On the energy subcritical, nonlinear wave equation in $\mathbb{R}^3$ with radial data, *Anal. PDE* 6 (8) (2014) 1929–1987.
- [17] R. Shen, Scattering of solutions to the defocusing energy subcritical semi-linear wave equation in 3D, *Commun. Partial Differ. Equ.* 42 (4) (2017) 495–518.
- [18] C.D. Sogge, *Lectures on Non-linear Wave Equations*, vol. 2, International Press, Boston, MA, 1995.
- [19] J.M. Ihsan Shatah, M. Struwe, *Geometric Wave Equations*, vol. 2, American Mathematical Soc., 2000.
- [20] W.A. Strauss, Decay and asymptotics for $\square u = f(u)$, *J. Funct. Anal.* 2 (4) (1968) 409–457.
- [21] R.S. Strichartz, Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations, *Duke Math. J.* 44 (3) (1977) 705–714.
- [22] W.A. Strauss, Nonlinear scattering theory at low energy, *J. Funct. Anal.* 41 (1) (1981) 110–133.
- [23] T. Tao, *Nonlinear Dispersive Equations: Local and Global Analysis*. Number 106, American Mathematical Soc., 2006.