Control-based Conditions for Graph Distinguishability *

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Abstract

The graph distinguishability problem investigates whether a graph can be uniquely identified by the spectrum of its adjacency matrix, specifically determining if two graphs with the same spectrum are isomorphic. This issue is central to spectral graph theory and has significant implications for graph machine learning. In this paper, we explore the intricate connections between graph distinguishability and graph controllability–an essential concept in the control of networked systems. Focusing on oriented graphs and their skew-adjacency matrices, we establish controllability-based conditions that ensure their distinguishability. Notably, our conditions are less restrictive than existing methods, enabling a broader class of graphs to satisfy the distinguishability criteria. We illustrate the effectiveness of our results with several examples. Our findings highlight the applications of network control methods in tackling this crucial problem in algebraic graph theory, with implications for machine learning and network design.

1 Introduction

Graphs are fundamental mathematical structures widely used to model complex systems across science, engineering, and social sciences. One of the key challenges in graph theory is *graph distinguishability*, which aims to determine whether a graph can be uniquely identified by its spectral properties, such as the adjacency spectrum [1]. This problem has significant implications for graph isomorphism, graph classification, and representation learning. The *distinguishability conjecture* posits that *almost all graphs are determined (up to isomorphism) by their spectrum (**DS**)*. Though certain graph families, such as trees and strongly regular graphs, are known to be indistinguishable from their adjacency spectra, a comprehensive characterization of distinguishable and non-distinguishable graphs remains an open problem and an active area of research [2–4].

One approach to enhancing graph distinguishability involves using the *generalized spectrum*, which considers both the graph's spectrum and that of its complement, enabling improved criteria to distinguish non-isomorphic cospectral graphs [5, 6]. Other works have explored graph modifications that preserve spectral properties while altering structural properties, leading to non-distinguishable graphs, i.e., non-isomorphic graphs with identical spectra. A notable example is GM-switching, a technique used to construct cospectral, non-isomorphic graphs [7]. Another major direction, and arguably the most challenging yet most practically relevant, is establishing criteria that guarantee distinguishability, effectively identifying families of graphs uniquely determined by their spectra. *Our work in this paper falls into this category as we develop new conditions under which graphs can be distinguished based on their spectral properties.*

We leverage the recent work of Wang [6, 8, 9], which establishes intriguing connections between *graph distinguishability and network controllability*, a fundamental concept in network control. Network or graph controllability refers to the ability to steer a network (or graph) to desired states through external inputs applied to a subset of its nodes [10–13]. A key object in determining controllability is the *controllability matrix* (or *walk matrix*). Wang utilized the determinant of the walk matrix to establish distinguishability conditions, integrating control theory with spectral graph theory. This perspective refines spectral methods for graph distinction by incorporating control-theoretic insights.

In this paper, we extend this connection to oriented (directed) graphs, deriving general conditions for their distinguishability based on the spectrum of their skew-adjacency matrices, which are commonly associated with directed graphs and further detailed in Section 2.2. Our approach addresses the limitations of existing methods, which primarily focus on *self-converse* graphs [14], a restricted class of oriented graphs that are isomorphic to their converse, as further discussed in Section 2.2. Ad-

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ditionally, we broaden the applicability of controllability-based techniques to a wider class of graphs by introducing a flexible control input framework. The main contributions are summarized as:

- Expanding the class of distinguishable oriented graphs by deriving general conditions based on skew-adjacency spectra.
- Establishing new controllability-based criteria for graph distinguishability by generalizing existing methods and eliminating restrictive assumptions such as self-converseness.
- Introducing a flexible control input framework to enhance applicability.

The paper is organized as follows: Section 2 introduces the notations, the graph distinguishability problem and the relevant background. It also highlights the role of Network Control Theory. Section 3 presents control-based criteria (Theorem 3.2) for distinguishing oriented graphs, with proofs. Section 4 provides illustrative examples. Finally, Section 5 concludes the paper.

2 Problem Description and Background

We first examine the problem of determining graphs by their spectrum (DS) in the context of undirected graphs, introducing key notations, definitions, and illustrative examples. This choice is motivated by the fact that most of the literature focuses on undirected graphs, making them a natural starting point for understanding the problem. However, our work extends to oriented (directed) graphs, which we discuss in Section 3 after establishing the necessary groundwork in Section 2.2.

2.1 Undirected Graphs A network of interconnected entities is typically modeled as an undirected graph, denoted by $G = (V, E)$, where $V = \{v_1, v_2, \ldots, v_n\}$ represents the set of vertices (or nodes), and $E \subseteq V \times V$ represents the edges defining relationships between them. The terms 'vertex,' 'node,' and 'agent' are used interchangeably. The *adjacency matrix* $A(G)$ is an $n \times n$ matrix where $A_{ij} = A_{ji} = 1$ if an edge exists between vertices v_i and v_j, and 0 otherwise. The *degree matrix* $D(G)$ is diagonal, with entries D_{ii} corresponding to the degree of v_i, and the *Laplacian matrix* is given by $L(G) = D(G) - A(G)$. Given a simple, undirected graph G, its *complement*, denoted $\overline{G}$, is a graph with the same vertex set V, where two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

Definition *(Graph Isomorphism)* Two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ are *isomorphic* if there exists a bijection $f : V_G \to V_H$ such that for all $v_i, v_j \in V_G$, $(v_i, v_j) \in E_G \iff (f(v_i), f(v_j)) \in E_H$. The function f is called an *isomorphism* between the graphs G and H, and we write $G \cong H$ if such an isomorphism exists.

The *spectrum* of an undirected graph G, denoted $\lambda(G)$, is the multiset of eigenvalues of $A(G)$. These eigenvalues satisfy:

$$A(G)\mathbf{v}_i = \lambda_i \mathbf{v}_i, \quad i = 1, 2, \ldots, n,$$

where $\mathbf{v}_i$ is the corresponding eigenvector. The *generalized spectrum* of a graph G is the spectrum of G together with the spectrum of its complement $\overline{G}$. Two graphs are *cospectral* if their adjacency matrices share the same spectrum. More generally, for $t \in \mathbb{R}$, two graphs G and H are *t-cospectral* if the matrices $tJ - A(G)$ and $tJ - A(H)$, where J is the all-ones matrix, have the same spectrum. A graph pair is $\mathbb{R}$-cospectral if they are t-cospectral for all $t \in \mathbb{R}$. Johnson and Newman [15] established that if two graphs are t-cospectral for two distinct t-values, they are also $\mathbb{R}$-cospectral.

The *walk matrix* of a graph G, denoted $W(G)$, is given by:

$$W(G) = [e,\ Ae,\ A^2e,\ \ldots,\ A^{n-1}e], \tag{2.1}$$

where e is the all-ones vector. The determinant $\det(W)$ plays a crucial role in determining whether G is *determined by its generalized spectrum* (DGS), a concept discussed later.

For matrix operations, we use $X^\top$ to denote the transpose of X, $\mathbf{0}_n$ for an n-dimensional zero vector, and e_n for an all-ones vector. A matrix A is *integral* if $a_{ij} \in \mathbb{Z}$, *rational* if $a_{ij} \in \mathbb{Q}$, and *orthogonal* if $Q^\top Q = I$. A rational orthogonal matrix Q is *regular* if $Qe = e$, meaning each row sums to one. The notation $a \mid b$ denotes that a divides b.

The *graph distinguishability problem*, also known as the *Determined by the Spectrum* (**DS**) problem, seeks to determine whether a graph can be uniquely identified by its spectrum. A graph G is DS if no non-isomorphic graph shares its spectrum:

$$\lambda(G) = \lambda(H) \implies G \cong H.$$

However, cospectral graphs—distinct graphs with the same spectrum—complicate this problem. A graph is *Determined by the Generalized Spectrum* (**DGS**) if it is uniquely identified by both its spectrum and the spectrum of its complement.

Figure 1 illustrates two non-isomorphic cospectral graphs G_1 and H_1 that also have cospectral complements. Their spectra are:

$$\lambda(G_1) = \lambda(H_1) = \{-2, -1.78, -1, 0, 0, 1.29, 3.49\},$$
$$\lambda(\overline{G}_1) = \lambda(\overline{H}_1) = \{-2.46, -1.38, -1, 0, 0.77, 1, 3.07\}.$$

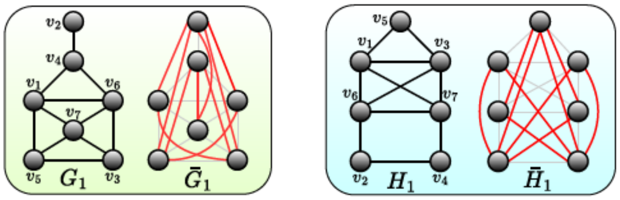


Figure 1: Two non-isomorphic cospectral graphs G_1 and H_1 with cospectral complements $\overline{G}_1$ and $\overline{H}_1$.

Since these graphs remain indistinguishable even when considering their generalized spectrum, the DS classification problem remains unresolved. Haemers conjectured that almost all graphs are DS [2]. While empirical evidence supports this conjecture for small graphs (e.g., $n \leq 12$ [16, 17]), the problem remains open for larger graphs and for oriented graphs.

Recent work by Wang et al. [9, 14, 17] connects control theory with spectral graph theory, offering new tools for DS and DGS classification. In the next subsection, we explore this connection and introduce a broader control-theoretic perspective on Wang's approach to graph distinguishability.

2.1.1 Potential of Network Control Theory for Graph DS

Network control theory provides a powerful framework for addressing the graph distinguishability (DS) and generalized spectrum (DGS) problems. Building on the work of Wang et al. [9], control theory-centric techniques have demonstrated significant potential in identifying controllable and DGS graphs. A central tool in this analysis is the *walk matrix*, which is closely tied to graph controllability.

In dynamical systems, the walk matrix, defined in (2.1), corresponds to the *controllability matrix* of a system evolving over a graph, governed by

$$\dot{x}(t) = Ax(t) + \mathcal{H}(G)u(t), \tag{2.2}$$

where $x(t)$ is the state vector, $u(t)$ is the control input, and $\mathcal{H}(G)$ specifies the input configuration. More generally, the system dynamics can be described as:

$$\dot{x}(t) = \mathcal{M}(G)x(t) + \mathcal{H}(G)u(t), \tag{2.3}$$

where $\mathcal{M}(G)$ encodes the network topology. Wang [8, 17] studied the specific case where $\mathcal{M}(G) = A(G)$ and $\mathcal{H}(G) = e$, meaning all nodes receive the same input. Extending this framework—by incorporating the graph Laplacian or signless Laplacian for $\mathcal{M}(G)$ and varying $\mathcal{H}(G)$—yields a more flexible controllability matrix:

$$\mathcal{C} = \left[\mathcal{H}(G)\ \mathcal{M}(G)\mathcal{H}(G)\ \ldots\ (\mathcal{M}(G))^{n-1}\mathcal{H}(G)\right]. \tag{2.4}$$

This generalization provides deeper insights into the graph distinguishability problem beyond the walk matrix alone.

A fundamental result from O'Rourke and Behrouz [18] states:

THEOREM 2.1. *([18]) Almost all graphs are controllable.*

Furthermore, numerical analysis by Wang et al. [17] suggests that approximately 25% of controllable graphs are DGS, leading to the following conclusion:

REMARK 2.1. *Approximately 25% of all graphs are determined by their generalized spectra (DGS).*

The relationship between the determinant of the walk matrix $\det(W)$ and a graph's generalized spectrum is a key insight from [6, 17]. Integrating control theory with spectral graph theory provides a novel approach to distinguishing cospectral graphs by analyzing how external control inputs propagate through the network. This fusion of disciplines opens new avenues for addressing the DS problem, particularly for large and complex graph families.

In the next section, we introduce the necessary notation and terminology for oriented graphs before presenting our main results, which leverage control-theoretic tools, in Section 3.

2.2 Oriented Graphs

An *oriented graph* G^σ is derived from a simple[1] undirected graph G by assigning a direction to each edge. The undirected graph G is called the *underlying graph* of G^σ. For the remainder of this paper, we focus exclusively on oriented graphs.

An oriented graph is *self-converse* if it is isomorphic to its *converse* $(G^\sigma)^\top$, which is obtained by reversing all directed edges. The *skew-adjacency matrix*, introduced by Tutte [19], is an $n \times n$ matrix $S(G^\sigma) = (s_{ij})$, where:

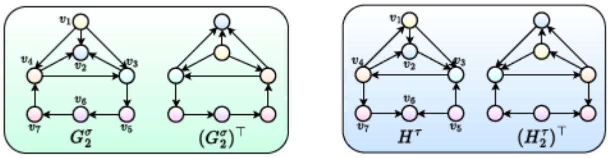
$$s_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \text{ is an arc,} \\ -1 & \text{if } (v_j, v_i) \text{ is an arc,} \\ 0 & \text{otherwise.} \end{cases}$$

The *generalized skew-adjacency spectrum* of G^σ is defined as the pair $(\lambda(S(G^\sigma)), \lambda(tJ - S(G^\sigma)))$, where J is the all-ones matrix and $t \in \mathbb{R}$. Two oriented graphs, H^τ and G^σ, are $\mathbb{R}$-cospectral if, for every $t \in \mathbb{R}$, the matrices $tJ - S(G^\sigma)$ and $tJ - S(H^\tau)$ share the same spectrum. The *skew-walk matrix* of an oriented graph G^σ is defined as:

$$W(G^\sigma) = [e,\ S(G^\sigma)e,\ S(G^\sigma)^2 e, \ldots,\ S(G^\sigma)^{n-1}e],$$

where e is the all-ones vector.

[1] A graph without multiple edges or self-loops.



(a) (b)

Figure 2: (a) A graph G_2^σ and its isomorphic converse $(G_2^\sigma)^\top$. (b) A graph H_2^τ and its non-isomorphic converse $(H_2^\tau)^\top$. All four graphs are cospectral.

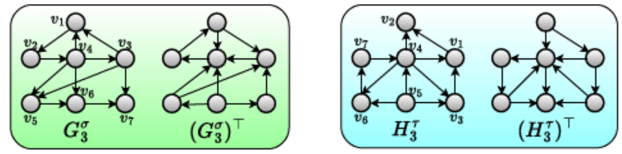


Figure 3: A graph G_3^σ and its non-isomorphic converse $(G_3^\sigma)^\top$ and a graph H_3^τ and its non-isomorphic converse $(H_3^\tau)^\top$. All four graphs are cospectral but non-isomorphic to each other.

An oriented graph G^σ is *Determined by the Generalized Skew Spectrum* (**DGSS**) if any graph H^τ sharing the same generalized skew spectrum with G^σ is isomorphic to G^σ and its converse $(G^\sigma)^\top$. For a graph to be DGSS, it must be *self-converse*, meaning its converse is isomorphic to itself. This strict condition limits its applicability [14].

In this work, we address the *Weakly Determined by the Generalized Skew Spectrum* (**WDGSS**) problem, which provides a more flexible alternative. A graph G^σ is *WDGSS* if any oriented graph H^τ with the same generalized skew spectrum as G^σ is either isomorphic to G^σ or its converse $(G^\sigma)^\top$, since *all oriented graphs are cospectral with their converses, even if they are not self-converse [14].* Unlike DGSS, WDGSS does not require a graph to be self-converse. This distinction makes the two conditions mutually exclusive: a DGSS graph has an isomorphic converse, whereas a WDGSS graph is distinguishable from all graphs except its converse.

Thus, WDGSS broadens the applicability of spectral methods by enabling the classification of graphs whose converses are not isomorphic to themselves. This extension refines spectral graph analysis, offering new insights into the structure of oriented graphs. Below, we present a few examples of graphs that are neither DGSS nor WDGSS for illustration purposes.

Figure 2 shows non-DGSS oriented graphs. The skew adjacency matrices of $S(G_2^\sigma)$ and $S(H_2^\tau)$ are:

$$S(G_2^\sigma) = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & -1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}, S(H_2^\tau) = \begin{pmatrix} 0 & 1 & 1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}.$$

The skew adjacency spectrum of all four graphs are identical i.e., $\lambda(S(G_2^\sigma)) = \lambda(S((G_2^\sigma)^\top)) = \lambda(S(H_2^\tau)) = \lambda(S((H_2^\tau)^\top))$:

$$\{-2.68j,\ -1.34j,\ -1j,\ 0j,\ 1j,\ 1.34j,\ 2.68j\}.$$

In this example, the graph G_2^σ is self-converse, i.e., it is isomorphic to its converse $(G_2^\sigma)^\top$, which is obtained by reversing each directed edge in G_2^σ. However, there exists a graph H_2^τ that is cospectral to G_2^σ, but is neither isomorphic to G_2^σ nor its converse $(G_2^\sigma)^\top$. Therefore, the graphs in Figure 2 cannot be determined by their generalized skew-spectrum (not DGSS).

Now, consider the graph $S(G_3^\sigma)$ in Figure 3. The skew adjacency spectrum of all four graphs are identical i.e., $\lambda(S(G_3^\sigma)) = \lambda(S((G_3^\sigma)^\top)) = \lambda(S(H_3^\tau)) = \lambda(S((H_3^\tau)^\top))$:

$$\{-2.82j,\ -1.54j,\ -0.83j,\ 0j,\ 0.83j,\ 1.54j,\ 2.82j\}.$$

The graph G_3^σ is not self-converse and has a non-isomorphic cospectral mate, $S(H_3^\tau)$, which is also not self-converse. This means that both G_3^σ and H_3^τ are neither DGSS nor WDGSS. These graphs share the same spectrum, and their generalized skew-spectrum does not provide enough information to distinguish them.

In the next section, we introduce novel criteria to identify a class of oriented graphs that can be weakly distinguished by their generalized skew spectrum, i.e., WDGSS graphs. Recent works have leveraged the walk matrix to address the DS problem in undirected graphs. Extending this, the skew-walk matrix has been applied to construct DGSS families for oriented graphs [14]. Building on these ideas, we introduce a new family of oriented graphs that can be classified as WDGSS using properties of the skew-walk matrix and network control-based criteria.

3 A New Family of WDGSS Graphs

In this section, we introduce a new class of *Weakly Determined by Generalized Skew Spectrum* (WDGSS) graphs, extending prior work on *Determined by Generalized Skew Spectrum* (DGSS) graphs [14]. While previous studies have established conditions for DGSS graphs, these results apply only to graphs whose converses are isomorphic to themselves. We provide the first systematic criterion for identifying WDGSS graphs, which allows us to distinguish graphs whose converses are not isomorphic to themselves based on their spectral properties.

Our approach builds on the arithmetic properties of the *skew-walk matrix* $W(G^\sigma)$, which has been used in prior work to characterize DGSS graphs. We now generalize this framework to include graphs whose converses are not isomorphic to themselves. Qiu et al. [14] demonstrated that for any oriented graph G^σ,

$$\lambda(tJ - S(G^\sigma)) = \lambda(tJ - S((G^\sigma)^\top)) \quad \text{for every } t \in \mathbb{R},$$

implying that an oriented graph and its converse always share the same generalized skew spectrum. If the converse of an oriented graph is isomorphic to itself, then it belongs to the DGSS category. However, if its converse is not isomorphic to itself, the graph instead falls into the WDGSS category. The following theorem provides an arithmetic criterion for determining whether a graph belongs to the DGSS category.

THEOREM 3.1. *([14]) Let G^σ be an oriented graph of order n whose converse is isomorphic to itself. If $2^{-\lfloor \frac{n}{2} \rfloor} \det W(G^\sigma)$ (which is always an integer) is odd and square-free, then G^σ is DGSS.*

Building on this result, we extend the applicability of the skew-walk matrix to identify graphs whose converses are not isomorphic to themselves using spectral properties. We now propose a more general criterion that allows for the classification of WDGSS graphs, which are distinguishable from all other graphs except their converses. Specifically, we show that certain graphs satisfy a weaker but still robust distinguishability condition, leading to the following theorem:

THEOREM 3.2. *Let G^σ be an oriented graph of order n whose converse is not isomorphic to itself. If $2^{-\lfloor \frac{n}{2} \rfloor} \det W(G^\sigma) = p$, where p is an* odd prime, *then G^σ is weakly determined by generalized skew spectrum (WDGSS).*

This result provides the first formal classification of WDGSS graphs, addressing a gap in the literature where only DGSS graphs have been systematically studied. Unlike DGSS graphs, which belong to the category where the graph and its converse are isomorphic, WDGSS graphs belong to the category where the converse is not isomorphic to the graph. This extension enhances the utility of spectral graph methods and provides new insights into how the skew-walk matrix encodes structural information in oriented graphs.

3.1 Proof of Theorem 3.2

In this section, we present the proof of Theorem 3.2. Before proceeding, we first introduce several key lemmas and theorems, some of which are adapted from the literature, alongside new theorems that we establish in this paper.

We define the family of graphs that satisfy Theorem 3.2 as:

$$\mathcal{F}_n^p = \{\text{n-vertex graphs } G : 2^{-\lfloor \frac{n}{2} \rfloor} \det W(G^\sigma) = p\}, \tag{3.5}$$

where p is an odd prime, and G^σ is not self-converse. We exclude self-converse graphs since these are already covered by Theorem 3.1 [14].

Since we work exclusively with oriented graphs, we omit the explicit notation G^σ when clear from context, and write $S = S(G^\sigma)$ and $W = W(G^\sigma)$. We use $\mathbb{F}_p$ for the finite field of prime order p, and $\text{rank}_p(M)$ for the rank of a matrix M over $\mathbb{F}_p$, computed by reducing each entry m_{ij} mod p. We write $M \equiv 0 \pmod p$ if all entries of M are divisible by p.

THEOREM 3.3. *([20]) Let G^σ be an oriented graph with $\det(W(G^\sigma)) \neq 0$. There exists an oriented graph H^τ such that G^σ and H^τ share the same generalized skew spectrum if and only if a unique regular rational orthogonal matrix Q exists, satisfying:*

$$Q^\top S(G^\sigma) Q = S(H^\tau), \quad Qe = e.$$

Define the set $\Gamma(G^\sigma)$ as:

$$\Gamma(G^\sigma) = \{Q \in O_n(\mathbb{Q}) \mid Q^\top S(G^\sigma) Q = S(H^\tau),\, Qe = e\},$$

where $O_n(\mathbb{Q})$ is the set of orthogonal matrices with rational entries.

Definition (*Level of a Matrix*) The *level* of an orthogonal matrix Q with rational entries, denoted by $\ell(Q)$, is the smallest positive integer k such that kQ becomes an integral matrix.

Note that a rational orthogonal matrix Q satisfying $Qe = e$ is a permutation matrix if and only if its level $\ell(Q)$ equals 1. An integral matrix V of order n is said to be *unimodular* if its determinant is ± 1. The Smith Normal Form (SNF) is a valuable tool for analyzing integral matrices.

THEOREM 3.4. *For any full-rank integral matrix M, there exist unimodular matrices V_1 and V_2 such that $M = V_1 F V_2$, where $F = \text{diag}(d_1, d_2, \ldots, d_n)$ is a diagonal matrix satisfying $d_i \mid d_{i+1}$ (i.e., d_{i+1} divides d_i) for $i = 1, 2, \ldots, n-1$.*

The matrix F is referred to as the SNF of M, with its diagonal entries d_i known as the *elementary divisors* or *invariant factors*. Since the matrices V_1 and V_2 are unimodular, it follows that $\text{rank}_p(F) = \text{rank}_p(M)$ for any prime p. Moreover, the determinant of M can be expressed as:

$$\det M = \pm \prod_{i=1}^{n} d_i \tag{3.6}$$

The last divisor d_n is crucial: $d_n Q$ is integral for every matrix $Q \in \Gamma(G^\sigma)$ [8], implying that $\ell(Q)$ must divide d_n. The following lemma characterizes the SNF of walk matrices.

LEMMA 3.1. *([6]) If $2^{-\lfloor \frac{n}{2} \rfloor} \det W = b$, where b is an odd, square-free integer, then the Smith Normal Form of $W(G^\sigma)$ is:*

$$[\underbrace{1, 1, \ldots, 1}_{\lceil \frac{n}{2} \rceil}, \underbrace{2, 2, \ldots, 2, 2b}_{\lfloor \frac{n}{2} \rfloor}].$$

Therefore, for graphs belonging to $\mathcal{F}_n^p$, using the Equation 3.6, the last invariant factor of W is given by $d_n = 2p$. Moreover, the authors of [8] proved that for every matrix $Q \in \Gamma(G^\sigma)$, the level $\ell(Q) \mid d_n$.

LEMMA 3.2. *[14] Let $G^\sigma \in \mathcal{F}_n^p$ be an oriented graph of order n, and let $Q \in \Gamma(G^\sigma)$. Then the $\ell(Q)$ is odd.*

Although in [14], the authors proved Lemma 3.2 for self-converse graphs only, the same proof applies to graphs that are not self-converse. Hence, for every matrix $Q \in \Gamma(G^\sigma)$, the level $\ell(Q)$ can either be 1 or p.

The following lemmas further explore the properties of matrices in $\Gamma(G^\sigma)$ with level p.

LEMMA 3.3. *Let $G^\sigma \in \mathcal{F}_n^p$ and $Q \in \Gamma(G^\sigma)$. If $\ell(Q) = p$, then $\text{rank}_p(\hat{Q}) = 1$, where $\hat{Q} = pQ$.*

Proof. Consider the oriented graph H^τ such that $Q^\top S(G^\sigma)Q = S(H^\tau)$. Using the facts that Q is orthogonal and $Qe = e$, it is easy to see that:

$$(Q^\top S(G^\sigma)Q)^k e = Q^\top S(G^\sigma)^k e = S(H^\tau)^k e$$

It follows that $Q^\top W(G^\sigma) = W(H^\tau)$, or equivalently, $W^\top(G^\sigma)Q = W^\top(H^\tau)$. Therefore, we have $W^\top(G^\sigma)\hat{Q} = pW^\top(H^\tau) \equiv 0 \pmod p$.

Since $p \mid d_n$ while $p^2 \nmid d_n$, using SNF, we have that $\text{rank}_p(W(G^\sigma)) = n-1$, and thus $\text{rank}_p(W^\top(G^\sigma)) = n-1$. This means the solution space of $W^\top(G^\sigma)v \equiv 0 \pmod p$ is one-dimensional. Consequently, $\text{rank}_p(\hat{Q}) \le 1$.

Since $\ell(Q) = p$, the minimality of $\ell(Q)$ ensures that $\hat{Q}$ has at least one nonzero entry over $\mathbb{F}_p$ implying that $\text{rank}_p(\hat{Q}) \ge 1$. Hence, we conclude that $\text{rank}_p(\hat{Q}) = 1$, as required. ∎

REMARK 3.1. Let $Q_1, Q_2 \in \Gamma(G^\sigma)$ such that $\ell(Q_1) = \ell(Q_2) = p$, and let $\hat{Q}_1 = pQ_1$ and $\hat{Q}_2 = pQ_2$. Since $W^\top\hat{Q}_1 = W^\top\hat{Q}_2 = 0 \pmod p$, the columns of matrices $\hat{Q}_1$ and $\hat{Q}_2$ span the same one-dimensional subspace over $\mathbb{F}_p$.

LEMMA 3.4. *Let $Q \in \Gamma(G^\sigma)$ such that $\ell(Q) = p$. If Q contains at least one non-integral entry, then by performing appropriate row and column permutations, Q can be transformed into a quasi-diagonal form $\text{diag}[Q', I]$, where Q' is a regular rational orthogonal matrix consisting of non-integral entries with $\ell(Q') = p$ and $\text{rank}_p(pQ') = 1$, and I is an identity matrix of suitable size.*

Proof. It is clear that for any regular rational orthogonal matrix Q, the integral entries can only be 0 or 1. We claim that Q contains a 0 if and only if it contains a 1. The "if" part is straightforward: since each row (and column) of Q has a norm of one in $\mathbb{R}^n$, the presence of a one implies the presence of a zero. Let q_{ij} be the (i,j)-entry of Q, and suppose $q_{ij} = 0$. Now, consider $\hat{Q} = pQ$. Either the i-th row or the j-th column of $\hat{Q}$ must be the zero vector over $\mathbb{F}_p$, because otherwise, there would be a 2×2 invertible submatrix in $\hat{Q}$, contradicting $\text{rank}_p(\hat{Q}) = 1$. In either case, $\hat{Q}$ contains p as an entry, meaning that Q must also contain a 1.

Now, assume Q has exactly k entries equal to one. These k entries must lie in distinct rows and columns, with all other entries in those rows and columns being zero. By applying row and column permutations, Q can be transformed into a quasi-diagonal form $\text{diag}[Q', I_k]$, where Q' satisfies the aforementioned properties and I_k is an identity matrix of size k. Since Q' contains no 1's, it also contains no 0's, meaning that Q' consists of non-integral entries. ∎

The following lemma is a key component of our work.

LEMMA 3.5. *([9]) Let u and v be two n-dimensional integral column vectors, each with nonzero entries modulo p. Suppose that: (i) u and v are linearly dependent over $\mathbb{F}_p$; (ii) $u \neq \pm v$; and (iii) $u^\top u = v^\top v = p^2$. Then, it follows that $u^\top v = 0$.*

The following lemma explore the relationships between matrices in $\Gamma(G^\sigma)$ having the same level.

LEMMA 3.6. *Let $Q_1, Q_2 \in \Gamma(G^\sigma)$ such that $\ell(Q_1) = \ell(Q_2) = p$. Then, Q_2 can be obtained from Q_1 through a series of column permutations.*

Proof. Let $\hat{Q}_1 = pQ_1$ and $\hat{Q}_2 = pQ_2$. Denote the i-th and j-th columns of $\hat{Q}_1$ and $\hat{Q}_2$ as α_i and β_j, respectively, for $i, j = 1, 2, \ldots, n$. By Remark 3.1, every column in $\hat{Q}_1$ and $\hat{Q}_2$ is a multiple of a vector v over $\mathbb{F}_p$. First, consider the case where every entry of v is nonzero modulo p.

We claim that both Q_1 and Q_2 contain no integral entries. Suppose, for contradiction, that Q_1 contains an integral entry, say $q_{ij} = 1$.

Now, let α_k be a column of $\hat{Q}_1$ such that $\alpha_k \not\equiv 0 \pmod p$. This implies $\alpha_k \equiv cv \pmod p$ for some integer c. Since α_k has at least one nonzero entry modulo p, it follows that $c \not\equiv 0 \pmod p$. Consequently, as we are assuming that every entry in v is nonzero modulo p, every entry of α_k must be nonzero modulo p, which means each entry of α_k/p is not an integer. However, since $q_{ij} = 1$, the i-th row of Q_1 is a standard unit vector, so at least one entry of α_k/p must indeed be an integer.

This contradiction confirms our claim that neither Q_1 nor Q_2 contains integral entries.

Next, we assert that for any pair of indices i and j, either $\alpha_i = \beta_j$ or $\alpha_i^\top \beta_j = 0$.

We may assume that $\alpha_i \neq \beta_j$. As $e^\top \alpha_i = e^\top \beta_j = p$, it implies that $\alpha_i \neq -\beta_j$ and hence $\alpha_i \neq \pm\beta_j$. Since both columns are nonzero multiples of v over $\mathbb{F}_p$, they must be linearly dependent with each entry not equal to zero modulo p. Applying Lemma 3.5 (where $u = \alpha_i$ and $v = \beta_j$), we conclude $\alpha_i^\top \beta_j = 0$.

If we fix β_j and consider all α_i's, the set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ forms a basis of $\mathbb{R}^n$. Since β_j is a nonzero vector in $\mathbb{R}^n$, $\alpha_i^\top \beta_j = 0$ cannot hold for all i. Therefore, there must exist some i such that $\beta_j = \alpha_i$. This implies that the j-th column of $\hat{Q}_2$ matches a column of $\hat{Q}_1$. As all columns of $\hat{Q}_2$ are distinct, we conclude that $\hat{Q}_2$ can be obtained from $\hat{Q}_1$ by column permutations, and hence, Q_2 can be obtained from Q_1 through column permutations.

Now, consider the case where v has at least one zero entry modulo p. Assume, for simplicity, that the first k entries of v are nonzero. Then we can express v as: $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, where v_1 is a k-dimensional vector with nonzero entries, and v_2 is an $(n-k)$-dimensional zero vector. This case corresponds to row permutations, as in Lemma 3.4.

Instead of using row permutations, we can apply column permutations only to transform Q_1 and Q_2 into quasi-diagonal forms. Both Q_1 and Q_2 can be written as $\mathrm{diag}[Q_1', I]$ and $\mathrm{diag}[Q_2', I]$, where Q_1' and Q_2' are $k \times k$ matrices with each column being a multiple of v_1 over $\mathbb{F}_p$.

By our previous argument, Q_2' can be obtained from Q_1' through column permutations. Applying the same column permutations to $\mathrm{diag}[Q_1', I]$ yields $\mathrm{diag}[Q_2', I]$. Therefore, Q_2 can be obtained from Q_1 through column permutations, completing the proof. □

With the groundwork laid by the preceding lemmas and theorems, we now present a proof of Theorem 3.2.

Proof. (Proof of Theorem 3.2) Let $G^\sigma \in \mathcal{F}_n^p$. Since G^σ is assumed not to be self-converse, $(G^\sigma)^\top$ is not isomorphic to G^σ, yet both share the same generalized skew spectrum. Therefore, G^σ is not DGSS. Now, we need to show that any two generalized cospectral mates of G^σ, say H^τ and $(G^\sigma)^\top$, are isomorphic.

We denote the rational orthogonal matrices corresponding to H^τ and $(G^\sigma)^\top$ as Q_1 and Q_2, respectively, where $Q_1, Q_2 \in \Gamma(G^\sigma)$. Without loss of generality, assume that H^τ is not isomorphic to G^σ, which implies that $\ell(Q_1) \neq 1$. Since G^σ is not self-converse, we also have that $\ell(Q_2) \neq 1$. Hence, it follows that $\ell(Q_1) = \ell(Q_2) = p$.

Furthermore, these matrices satisfy the following relationships:

$$Q_1^\top S(G^\sigma) Q_1 = S(H^\tau) \quad \text{and} \quad Q_2^\top S(G^\sigma) Q_2 = S((G^\sigma)^\top).$$

According to Lemma 3.6, we have that $Q_2 = Q_1 P$ for some permutation matrix P.

We can now express the relationship between the matrices $S((G^\sigma)^\top)$ and $S(H^\tau)$:

$$\begin{aligned} S((G^\sigma)^\top) &= Q_2^\top S(G^\sigma) Q_2 \\ &= Q_2^\top Q_1 S(H^\tau) Q_1^\top Q_2 \\ &= P^\top S(H^\tau) P. \end{aligned}$$

This equation shows that H^τ and $(G^\sigma)^\top$ are isomorphic. Therefore, the proof of Theorem 3.2 is complete. □

In the next section, we provide two illustrative examples of oriented graphs that are DGSS and WDGSS according to the Theorem 3.1 and 3.2 respectively.

4 Example Illustrations

In this section, we present examples to illustrate our theoretical results on DGSS and WDGSS graphs. Specifically, we analyze two types of oriented graphs: a self-converse graph satisfying the DGSS condition from Theorem 3.1, and a non-self-converse graph satisfying the WDGSS condition from Theorem 3.2. These graphs are selected from a dataset containing all oriented graphs of size 7. We present the eigenvalue spectrum for each graph and the invariant factors $d_i \ \forall i = 1, \ldots, n$ of the skew-walk matrix.

DGSS Graphs Satisfying Theorem 3.1 We begin with the self-converse graph G_4^σ, which satisfies the DGSS criterion. Figure 4(a) shows G_4^σ and its isomorphic converse $(G_4^\sigma)^\top$.

The eigenvalue spectrum of $S(G_4^\sigma)$ is:

$$\begin{aligned} \lambda(S(G_4^\sigma)) &= \lambda(S((G_4^\sigma)^\top)) \\ &= \{-2.20j,\ -1.62j,\ -1.21j,\ 0.j,\ 1.21j,\ 1.62j,\ 2.20j\}. \end{aligned}$$

The invariant factors of $W(G_4^\sigma)$ are $[1,1,1,1,2,2,14]$, satisfying Qiu et al. [14]'s DGSS condition. Since $2^{-\lfloor \frac{n}{2} \rfloor} \det(W(G_4^\sigma))$ is odd and square-free (i.e., $\frac{d_n}{2} = 7$), G_4^σ is DGSS.

WDGSS Graphs Satisfying Theorem 3.2 Next, we consider the non-self-converse graph G_5^σ, as shown in Figure 4(b). The eigenvalue spectrum of $S(G_5^\sigma)$ is:

$$\begin{aligned} \lambda(S(G_5^\sigma)) &= \lambda(S((G_5^\sigma)^\top)) \\ &= \{-2.58j,\ -1.45j,\ -0.46j,\ 0.j,\ 0.46j,\ 1.45j,\ 2.58j\}. \end{aligned}$$

The invariant factors of $W(G_5^\sigma)$ are $[1,1,1,1,2,2,1466]$. Since $b = \frac{d_n}{2} = 733$ is an odd prime, G_5^σ is WDGSS.

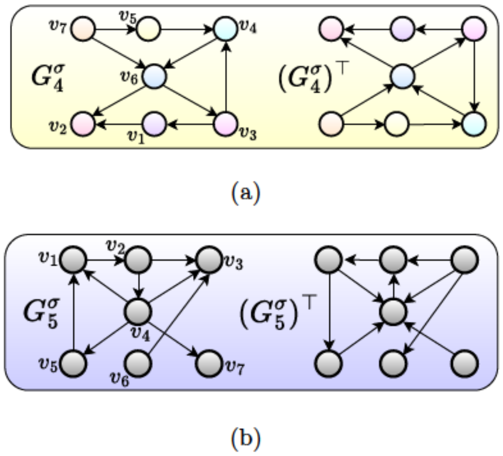


Figure 4: (a) A graph G_4^σ with its isomorphic converse $(G_4^\sigma)^\top$ and (b) a graph G_5^σ with its non-isomorphic converse $(G_5^\sigma)^\top$.

These examples illustrate how skew-adjacency matrices and walk matrices are instrumental in determining the spectral properties of DGSS and WDGSS graphs.

5 Conclusion

In this paper, we expanded the scope of network controllability as a promising tool for graph distinguishability, applying it to oriented graphs through their skew-adjacency spectrum. We developed a controllability-based criterion, leveraging the walk matrix, to determine the weak distinguishability of non-self-converse oriented graphs by their generalized skew spectrum. This result significantly broadens the class of distinguishable oriented graphs compared to existing methods, which are typically limited to self-converse graphs and rely on more restrictive control setups. Our findings demonstrate that network controllability-based criteria, rooted in the walk matrix, offer a powerful approach to resolving the graph distinguishability problem.

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