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MAGNETIC RESPONSE PROPERTIES OF TWISTED BILAYER GRAPHENE

SIMON BECKER, JIHOI KIM, AND XIAOWEN ZHU

ABSTRACT. In this article, we analyse the Bistritzer–MacDonald (BM) model (also known as the continuum model) of twisted bilayer graphene (TBG) with an additional external magnetic field. We provide an explicit semiclassical asymptotic expansion of the density of states (DOS) in the limit of strong magnetic fields. The explicit expansion of the DOS enables us to study magnetic response properties such as magnetic oscillations which includes Shubnikov-de Haas and de Haas-van Alphen oscillations as well as the integer quantum Hall effect. In particular, we elucidate the role played by different types of interlayer tunnelings (AA'/BB' vs. AB'/BA') in the study of the DOS, and magnetic properties.

1. INTRODUCTION

It is arguably one of the most exciting recent discoveries in condensed matter physics that by twisting two sheets of graphene at certain *magic angles*, the material exhibits a super-conducting phase [C18]. The experimental discoveries were motivated by earlier theoretical work [LPN07, BM11] which introduced the continuum model of twisted bilayer graphene (TBG). From this model they predicted the first magic angle by observing the appearance of a relatively flat band of the Hamiltonian at a small twisting angle. To discuss our study of TBG in magnetic fields, we first briefly introduce the BM model (see §2.1, [BM11]):

The BM model is an effective 4×4 matrix-valued Hamiltonian $\begin{pmatrix} H_D^\theta & T^\theta(x) \\ (T^\theta(x))^* & H_D^{-\theta} \end{pmatrix}$, $x \in \mathbb{R}^2$, composed of two twisted-Dirac-operators $H_D^\theta, H_D^{-\theta}$ representing two isolated graphene sheets, according to the *Wallace model* [W47] respectively, and a tunneling potential term $T^\theta(x) = \begin{pmatrix} \alpha_0 V(\frac{x}{\lambda_\theta}) & \alpha_1 \overline{U}(-\frac{x}{\lambda_\theta}) \\ \alpha_1 U(\frac{x}{\lambda_\theta}) & \alpha_0 V(\frac{x}{\lambda_\theta}) \end{pmatrix}$ where the diagonal potentials and off-diagonal potentials represent two different types of interlayer tunneling potentials. In fact, when two layers of graphene are twisted at an angle θ, a macroscopic honeycomb structure of scale λ_θ, called the moiré pattern, is formed (by a purely geometrical superposition of two sheets of graphene; see Fig.1). Then the two different types of interlayer tunnelings (see Fig.1) are respectively:

(1) the chiral tunnelings $U(x/\lambda_\theta)$ and $\overline{U}(-x/\lambda_\theta)$ localized near the vertices of each moiré hexagon, with tunneling strength α_1 and a stacking similar to AB' and BA'-stacking;

(2) the anti-chiral tunneling $V(x/\lambda_\theta)$, localized near the centers of each moiré hexagon, with a tunneling strength α_0 and a stacking similar to AA'/BB'-stacking.

Here A and B label the equivalence classes of vertices on the honeycomb lattice and atoms on the lower lattice are indicated by a prime, cf. Figure 1. We refer to the BM model as
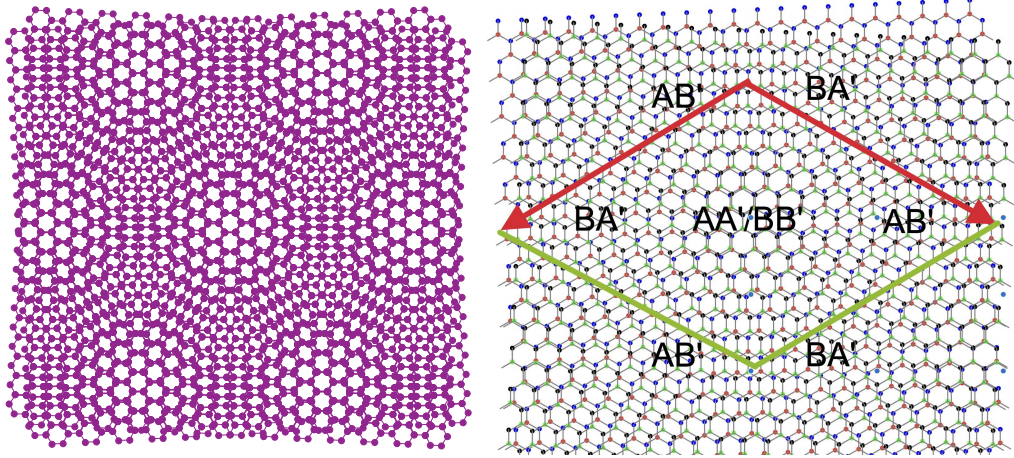
FIGURE 1. On the left: Moiré pattern at twisting angle $\theta = 5°$ with single moiré hexagon on the right, with (A=red, B=blue) and (A'=green, B'=black) denote vertices of two sheets of graphene respectively.

the *chiral* or *anti-chiral* model in the limit of purely chiral ($\alpha_0 = 0$) or anti-chiral ($\alpha_1 = 0$) tunneling interaction, respectively.

While in the full BM model, the bands close to zero appear only approximately flat, it has been shown in [TKV19, BEWZ20a, BEWZ20b, N21, NL22] that the chiral model exhibits a perfectly flat band at the magic angle [TKV19, BEWZ20a] while the anti-chiral model does not [BEWZ20b, LW21]. In our study of the magnetic response, we find that chiral and anti-chiral tunnelings exhibit intrinsically different features for the asymptotic expansion of the DOS in strong magnetic fields (see §4) which leads to different physical phenomena (see §5).

In §4, we derive the explicit asymptotic expansion of the DOS in strong magnetic fields for both models. We find that the magnetic anti-chiral model has a similar behavior as a magnetic Schrödinger operator, where Landau levels in general split under perturbations of electric potential, while the magnetic chiral model has stable Landau levels especially at energy zero. Thus, chiral tunneling enhances the peaks of the DOS at Landau levels which leads to an enhancement of physical phenomena including magnetic oscillations and the quantum hall effect, which we discuss in §5, while anti-chiral tunneling weakens them.

Our study of strong magnetic fields originates naturally from the interest in analyzing small twisting angles. In fact, as the twisting angle θ decreases to zero, the scale of the moiré hexagon $\lambda_\theta \sim (\sin \theta)^{-1}$ increases significantly. Thus, by rescaling coordinates the study of a fixed magnetic field at small twisting angle can be related to a fixed twisting angle in a strong magnetic field, see also [D21, HA21] for further physical motivation. We denote the two scaling in the following as adiabatic (see §2.3) and semiclassical (see §2.4) scalings, respectively.

In particular, this means we provide the theoretical background for the study of the dependence of Landau levels on small twisting angle that have been studied for a simplified model

in [CHK11] and numerically in [MGJ20] for a tight-binding model. Furthermore, combining with the study of chiral and anti-chiral tunnelings, we put the substantially pronounced peaks of the DOS for small twisting angles at the Landau levels in [MGJ20, Fig. 2,3] on a rigorous footing. Furthermore, our results can also be used to understand the impact of strong pseudo-magnetic fields generated by physical strain.

Finally, we summarize all our main results in an outline of the paper below:

- In Section 2, we introduce the BM model with external magnetic field for TBG.
- In Section 3, we discuss general properties of the DOS.
- In Section 4, we derive asymptotic formulae for the DOS:
    - of the chiral model: Theorem 1;
    - of the anti-chiral model: Theorem 2;
    - is termwise-differentiable with respect to B: Prop 4.9.
- In Section 5, we discuss physical applications of our semiclassical formulae.
- The article also contains two technical appendices to which some of the computations and auxiliary results for the derivation of the DOS are outsourced.

Our approach to analyze physical response properties rests on a thorough asymptotic analysis of the DOS. Here, our approach is inspired by ideas of Helffer and Sjöstrand [HS89] who studied the perturbation theory of periodic Schrödinger operators in strong magnetic fields and Wang [W95], who studied fine spectral asymptotics for random Schrödinger operators in strong magnetic fields. While Helffer and Sjöstrand stopped at studying the spectral perturbation for strong magnetic fields, the so-called Grushin problem, we obtain a full asymptotic expansion of the DOS. This has also been obtained by Helffer and Sjöstrand for weak magnetic fields [HS90] where the analysis relied on the semiclassical eigenvalue distribution close to a potential well. In our case, there is no natural well-structure and the asymptotic expansion relies on an asymptotic expansion of the parametrix with a splitting argument to overcome non-elliptic regions close to the real axis. Unlike in previous works by Helffer and Sjöstrand [HS90] and an article on single-layer graphene by the first author and Zworski [BZ19], we resolve the issue of differentiability of the asymptotic expansion with respect to the semiclassical parameter by relating the asymptotic expansion with the one of the differentiated symbol, here. This expansion is needed for the rigorous analysis of the DOS when differentiated with respect to the magnetic field which is relevant for both the de-Haas van Alphen as well as the quantum Hall effect.

2. Introduction of magnetic BM model

We start by introducing relevant notation.

Notation. *Throughout this article we identify $\mathbb{R}^2 \simeq \mathbb{C}$ by $x = (x_1, x_2) \simeq z = x_1 + ix_2$. We denote by L the Lebesgue measure on $\mathbb{R}^2 \sim \mathbb{C}$. For functions of complex variables $f(z, \bar{z})$ we often just write $f(z)$. If there exists a constant C_α such that $\|f\|_H \leq C_\alpha g$, we write $f = \mathcal{O}_\alpha(g)_H$. In particular, $f = \mathcal{O}(h^\infty)_H$ means that for any N there exists C_N such that $\|f\|_H \leq C_N h^N$. We also use the short notations $\langle x \rangle := \sqrt{1 + |x|^2}$, $B_r(x) = \{y : |y - x| \leq r\}$.*

We introduce the symbol class $S(\mathbb{R}^{2n}; \mathscr{H}) := \Big\{ p \in C^\infty(\mathbb{R}^{2n} \times \mathbb{R}_{>0}; \mathscr{H}) : \exists h_0, \text{ for all } \gamma \in \mathbb{N}^2, \exists c_\gamma > 0 \quad s.t. \text{ for all } (x, \xi) \in \mathbb{R}^{2n} \text{ for all } h \in (0, h_0) : |D_{(x,\xi)}^\gamma p(x, \xi, h)| \leq c_\gamma \Big\}$. In addition, let $S_\delta^k(\mathbb{R}^2_{x,\xi})$ denote the class of symbols $a \in C^\infty(\mathbb{R}^{2n} \times \mathbb{R}_{>0})$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi; h)| \leq C_{\alpha, \beta} h^{-k - \delta(\alpha + \beta)}, \quad \text{for all } \alpha, \beta > 0.$$

We denote standard partial derivatives in direction x_i by ∂_{x_i} and accordingly $D_{x_i} := -i\partial_{x_i}$. The principal symbol of a semiclassical operator $a(x, hD_x)$ is denoted by $\sigma_0(a(x, hD_x))$. We say a symbol a has an asymptotic expansion in S_δ^k, $a \sim \sum_{j=0}^\infty a_j$, if $a \in S_\delta^k$ and there is a sequence of $a_j \in S_\delta^{k_j}$ s.t. $k_j \to -\infty$ as $j \to \infty$ and $a - \sum_{j=0}^N a_j \in S_\delta^{k_{N+1}}$. When k or $\delta = 0$, we omit the respective sub and superscript. The spectrum of a linear operator T is denoted by $\mathrm{Spec}(T)$. We also introduce rotated Pauli matrices $\sigma_k^\theta = e^{-i\frac{\theta}{4}\sigma_3} \sigma_k e^{i\frac{\theta}{4}\sigma_3}$, for $k = 1, 2$.

2.1. **Moiré lattices and TBG.**

We recall from the introduction that by twisting two honeycomb lattices with respect to each other, the emerging moiré honeycomb pattern exhibits different scales λ_θ at different twisting angles θ. Thus it is easier to characterize such macroscopic honeycomb structures using a "unit-size honeycomb lattice" of side length $\frac{4\pi}{\sqrt{3}}$:

Let $\omega = \exp(\frac{2\pi i}{3})$, $\zeta_1 = 4\pi i\omega$, $\zeta_2 = 4\pi i\omega^2$. The "unit-size honeycomb lattice" is invariant under translations along a triangular lattice $\Gamma = \zeta_1\mathbb{Z} \oplus \zeta_2\mathbb{Z}$. We denote its unit cell, dual lattice, and the Brillouin zone of the dual lattice by $E = \mathbb{C}/\Gamma$, $\Gamma^* = \eta_1\mathbb{Z} \oplus \eta_2\mathbb{Z}$, and $E^* = \mathbb{C}/\Gamma^*$, where $\eta_1 = \frac{\omega^2}{\sqrt{3}}$ and $\eta_2 = -\frac{\omega}{\sqrt{3}}$.

2.2. **Chiral and anti-chiral tunnelings.**

The chiral and anti-chiral tunneling potentials, V and U, are smooth "unit-size" periodic functions (cf. [BM11]) satisfying for $\mathbf{a}_j = \frac{4}{3}\pi i\omega^j$ with $j = 0, 1, 2$ the following symmetries

$$V(z + \mathbf{a}_j) = \bar{\omega}V(z), \quad V(\omega z) = V(z), \quad \overline{V(z)} = V(-z), \quad V(\bar{z}) = V(-z),$$

$$U(z + \mathbf{a}_j) = \bar{\omega}U(z), \quad U(\omega z) = \omega U(z), \quad \overline{U(\bar{z})} = U(z).$$

In particular, since $\zeta_1 = 3\mathbf{a}_1$, $\zeta_2 = 3\mathbf{a}_2$, we have $V(z + \zeta_j) = V(z)$ and $U(z + \zeta_j) = U(z)$ for $j = 1, 2$. Thus $V(z)$, $U(z)$, $U_-(z) := U(-z)$ are periodic with respect to Γ. The tunneling potentials on the physical moiré scale are then $V(z/\lambda_\theta), U(z/\lambda_\theta), \overline{U_-}(z/\lambda_\theta)$.

2.3. **Magnetic BM model with Adiabatic scaling.**

To introduce the BM model with magnetic field we start with the physical or adiabatic scaling. Since we will immediately change to a semiclassical scaling, we denote all objects with a "$\sim$" in this paragraph to distinguish the two notations. Let $\tilde{A}(\tilde{z}) = (\tilde{A}_1(\tilde{z}), \tilde{A}_2(\tilde{z}), 0) \in C^\infty(\mathbb{C}; \mathbb{R}^3)$ be the magnetic vector potential of a magnetic field perpendicular to the TBG. The tunneling potentials, U and V, defined on the "unit-size honeycomb lattice" are then rescaled to the physical

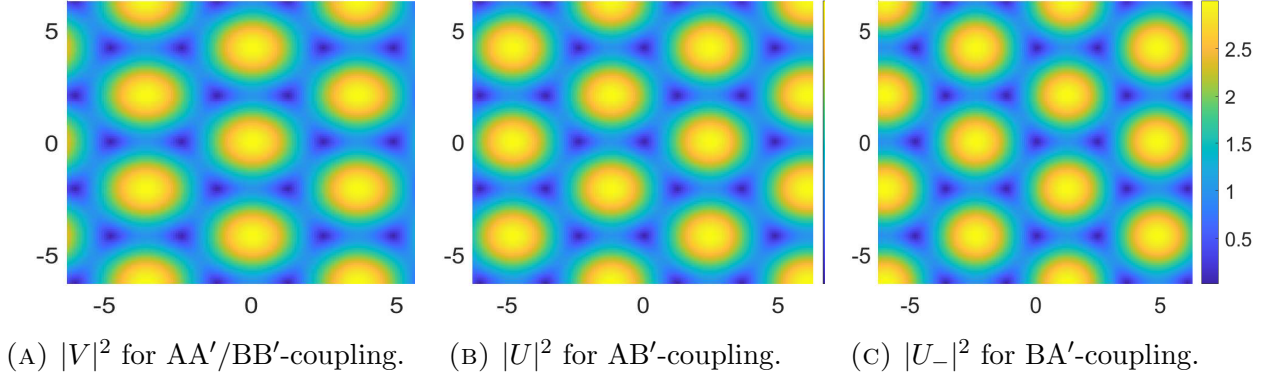(A) $|V|^2$ for AA′/BB′-coupling.   (B) $|U|^2$ for AB′-coupling.   (C) $|U_-|^2$ for BA′-coupling.

FIGURE 2. Modulus of tunneling potentials for various coupling types.

moiré-size by rescaling coordinates by λ_θ. Thus the magnetic BM model is $\tilde{\mathscr{H}}^\theta : D(\tilde{\mathscr{H}}^\theta) \subset L^2(\mathbb{C};\mathbb{C}^4) \to L^2(\mathbb{C};\mathbb{C}^4)$

$$\tilde{\mathscr{H}}^\theta := \tilde{\mathscr{H}}_0^\theta + \tilde{\mathscr{V}} := \begin{pmatrix} \tilde{H}_D^\theta & 0 \\ 0 & \tilde{H}_D^{-\theta} \end{pmatrix} + \begin{pmatrix} 0 & \tilde{T}^\theta \\ (\tilde{T}^\theta)^* & 0 \end{pmatrix}$$

with $\tilde{H}_D^\theta = \sum_{i=1}^{2} \sigma_i^\theta (D_{\tilde{x}_i} - \tilde{A}_i(\tilde{z}))$ and $\tilde{T}^\theta(\tilde{z}) = \begin{pmatrix} \tilde{\alpha}_0 V(\tilde{z}/\lambda_\theta) & \tilde{\alpha}_1 \overline{U_-}(\tilde{z}/\lambda_\theta) \\ \tilde{\alpha}_1 U(\tilde{z}/\lambda_\theta) & \tilde{\alpha}_0 V(\tilde{z}/\lambda_\theta) \end{pmatrix}$, where λ_θ, U and V are given above and $\tilde{\alpha}_i$ represent the tunneling strength, $i = 1, 2$.

2.4. Magnetic BM model with Semiclassical Scaling.

We shall now rescale the Hamiltonian in the previous paragraph to "unit-size" and multiply the Hamiltonian by λ_θ to work in another more convenient scaling called the *semiclassical scaling*: Let $z = \tilde{z}/\lambda_\theta$, $\alpha_i = \lambda_\theta \tilde{\alpha}_i$, $A_i(z) = \lambda_\theta \tilde{A}_i(\lambda_\theta z)$ (overall represented by a unitary operator U), we consider

$$\mathscr{H}^\theta(z) := \lambda_\theta (U \tilde{\mathscr{H}}^\theta U^{-1})(z) = \begin{pmatrix} H_D^\theta & 0 \\ 0 & H_D^{-\theta} \end{pmatrix} + \begin{pmatrix} 0 & T(z) \\ T(z)^* & 0 \end{pmatrix} =: \mathscr{H}_0^\theta + \mathscr{V}(z), \qquad (2.1)$$

where $H_D^\theta = \sum_{i=1}^{2} \sigma_i^\theta (D_{x_i} - A_i(z))$, or equivalently, $H_D^\theta = e^{-i\frac{\theta}{4}\sigma_3} H_D e^{i\frac{\theta}{4}\sigma_3}$ where

$$H_D = \begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix} \text{ with } \begin{cases} a = 2D_z - \overline{A(z)} \\ a^* = 2D_{\bar{z}} - A(z) \end{cases}, \quad T(x) = \begin{pmatrix} \alpha_0 V(z) & \alpha_1 \overline{U_-}(z) \\ \alpha_1 U(z) & \alpha_0 V(z) \end{pmatrix}.$$

We denote the *chiral model* by $\mathscr{H}_c^\theta = \mathscr{H}^\theta|_{\alpha_0=0}$ and the *anti-chiral model* by $\mathscr{H}_{ac}^\theta = \mathscr{H}^\theta|_{\alpha_1=0}$.

Remark 1 (Why strong magnetic fields?). *Our study of strong magnetic fields in rescaled coordinates is motivated by the observation that small twisting angles naturally correspond, for constant magnetic fields, to the limiting regimes $\alpha \gtrsim 1$ and $B \gg 1$. This provides the basis of our study of large magnetic fields which we coin the* semiclassical scaling.
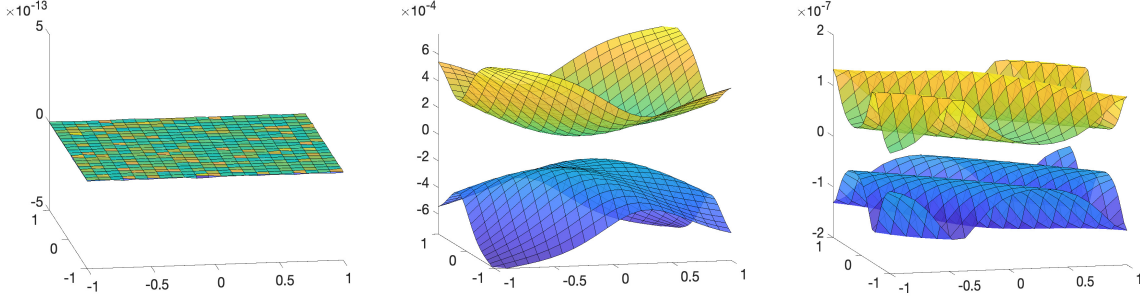
FIGURE 3. Constant magnetic field: On the left, flat bands for chiral model $(\alpha_1 = 1)$; in the middle $(\theta = 0)$ and on the right $(\theta = \pi)$ non-flat bands for anti-chiral model, $(\alpha_0 = 1)$.

2.5. **The chiral and anti-chiral model.** The chiral model is described by the Hamiltonian (2.1) for $\alpha_0 = 0$, such that upon conjugation by $\mathscr{U} = \mathrm{diag}(e^{i\theta/4}, e^{-i\theta\sigma_3/4}, e^{i\theta/4})$, $\mathscr{H}_c = \mathscr{U}\mathscr{H}^\theta\mathscr{U}$ it takes the form

$$\mathscr{H}_c = \begin{pmatrix} 0 & (\mathcal{D}_c)^* \\ \mathcal{D}_c & 0 \end{pmatrix} \text{ with } \mathcal{D}_c = \begin{pmatrix} 2D_{\bar{z}} - A_1(z) - iA_2(z) & \alpha_1 U(z) \\ \alpha_1 U_-(z) & 2D_{\bar{z}} - A_1(z) - iA_2(z) \end{pmatrix}.$$

The anti-chiral model, with $\alpha_1 = 0$, can be conjugated by a unitary $\mathcal{V}$, with $\lambda = e^{i\frac{\pi}{4}}$, to a Hamiltonian

$$\mathscr{H}_{ac}^\theta := \mathcal{V}\mathscr{H}^\theta\mathcal{V} = \begin{pmatrix} 0 & (\mathcal{D}_{ac}^\theta)^* \\ \mathcal{D}_{ac}^\theta & 0 \end{pmatrix} \text{ with } \mathcal{V} = \begin{pmatrix} \mathcal{V}_1 & \mathcal{V}_2 \\ \mathcal{V}_2 & \mathcal{V}_1 \end{pmatrix} \text{ for } \mathcal{V}_1 = \begin{pmatrix} i\lambda & 0 \\ 0 & 0 \end{pmatrix}, \mathcal{V}_2 = \begin{pmatrix} 0 & 0 \\ 0 & -\bar{\lambda} \end{pmatrix},$$

$$\mathcal{D}_{ac}^\theta = \begin{pmatrix} \alpha_0 V(z) & e^{i\theta/2}(2D_{\bar{z}} - (A_1(z) + iA_2(z))) \\ e^{i\theta/2}(2D_z - (A_1(z) - iA_2(z))) & \alpha_0 \overline{V(z)} \end{pmatrix}.$$

The off-diagonal structure implies that for both the chiral and anti-chiral model with magnetic field, the spectrum is symmetric with respect to zero. In particular, let $U := (\sigma_3 \otimes \mathrm{id}_{\mathbb{C}^2})$ then it follows that $U\mathscr{H}_c U = -\mathscr{H}_c$ and $U\mathscr{H}_{ac}^\theta U = -\mathscr{H}_{ac}^\theta$.

3. DENSITY OF STATES

In this section we study general properties of the density of states and study the possible values the density of states takes for the Hamiltonian of TBG.

3.1. **General properties.** In this subsection, we assume that the magnetic potential of the Hamiltonian is of the form $A = A_{\mathrm{per}} + A_{\mathrm{con}}$ where $A_{\mathrm{per}} \in C^\infty(E)$ and A_{con} is the vector potential of a constant magnetic field of strength B. Let $f \in C_c(\mathbb{R})$ then we define the *regularized trace*

$$\tilde{\mathrm{Tr}}(f(\mathscr{H}^\theta)) = \lim_{r \to \infty} \frac{\mathrm{Tr}(\mathbb{1}_{B_R} f(\mathscr{H}^\theta) \mathbb{1}_{B_R})}{|B_R|}$$

where $\mathbb{1}_{B_R}$ is the indicator function of the square centered at 0 of side length $2R$. By Riesz's theorem, there exists the so-called *density of states (DOS)* measure ρ satisfying

$$\tilde{\mathrm{Tr}}(f(\mathscr{H}^\theta)) = \int_{\mathbb{R}} f(t) \ d\rho(t). \tag{3.1}$$

We start by showing the existence and smoothness of the DOS.

Lemma 3.1. *For $f \in C_c^\infty(\mathbb{R})$ the regularized trace of $f(\mathscr{H}^\theta)$ exists, satisfies*

$$\tilde{\mathrm{Tr}}(f(\mathscr{H}^\theta)) = \frac{1}{|E|} \mathrm{Tr}_{L^2(E)}(f(\mathscr{H}^\theta)) = \frac{1}{|E|} \int_E f(\mathscr{H}^\theta)(x,x) \ dx,$$

and depends smoothly on $B \in \mathbb{R}$ and $\theta \in \mathbb{R} \setminus \{0\}$, with Schwartz kernel $f(\mathscr{H}^\theta)(x,y)$ of $f(\mathscr{H}^\theta)$.

Proof. Let $N_r, N_R \subset \Gamma$ be $N_r := \{\zeta \in \Gamma : \zeta + E \subset B_R\}$ and $N_R := \{\zeta \in \Gamma : \zeta + E \subset B_R \neq \emptyset\}$. Then

$$S_r := \bigcup_{\zeta \in N_r} E + \zeta \subset B_R \subset \bigcup_{\zeta \in N_R} E + \zeta =: S_R.$$

Thus for nonnegative f,

$$\frac{1}{|S_R|} \mathrm{Tr}(\mathbb{1}_{S_r} f(\mathscr{H}^\theta)) \le \frac{1}{|B_R|} \mathrm{Tr}(\mathbb{1}_{B_R} f(\mathscr{H}^\theta)) \le \frac{1}{|S_r|} \mathrm{Tr}(\mathbb{1}_{S_R} f(\mathscr{H}^\theta)). \tag{3.2}$$

Furthermore, by definition, we see that for some $C, C' > 0$, for all R,

$$\#(N_R \setminus N_r) \le CR, \quad \text{and} \quad |S_R \setminus S_r| \le C'R. \tag{3.3}$$

By standard magnetic translation $\mathbb{T}_\zeta$, which are defined e.g. in [BKZ22, Lemma 2.1] of our companion paper, satisfy $[\mathbb{T}_\zeta, \mathscr{H}^\theta] = 0$, therefore also $[\mathbb{T}_\zeta, f(\mathscr{H}^\theta)] = 0$. Furthermore, since $\mathbb{T}_\zeta \mathbb{1}_{E+\zeta} \mathbb{T}_{-\zeta} = \mathbb{1}_E$, thus $\mathrm{Tr}(\mathbb{1}_{E+\zeta} f(\mathscr{H}^\theta)) = \mathrm{Tr}(\mathbb{1}_E f(\mathscr{H}^\theta))$. Hence,

$$\mathrm{Tr}(\mathbb{1}_{S_r} f(\mathscr{H}^\theta)) = \sum_{\zeta \in N_r} \mathrm{Tr}(\mathbb{1}_{E+\zeta} f(\mathscr{H}^\theta)) = (\#N_r) \mathrm{Tr}(\mathbb{1}_E f(\mathscr{H}^\theta))$$

and similarly $\mathrm{Tr}(\mathbb{1}_{S_R} f(\mathscr{H}^\theta)) = (\#N_R) \mathrm{Tr}(\mathbb{1}_E f(\mathscr{H}^\theta))$. Inserting this into (3.2), taking $R \to \infty$ we get by using (3.3) that

$$\tilde{\mathrm{Tr}}(f(\mathscr{H}^\theta)) = \frac{1}{|E|} \mathrm{Tr}_{L^2(E)}(f(\mathscr{H}^\theta)).$$

To conclude the smooth dependence on θ and B, it suffices to adapt the arguments starting at [Sj89, p.251]. $\qquad \square$

In the next Proposition, we show that the integrated density of states of the twisted bilayer graphene Hamiltonian is stable under small perturbations of the magnetic field that do not close any spectral gaps.

Proposition 3.2. *Let the magnetic vector potential $A = A_{\mathrm{con}} + A_{\mathrm{per}}$ be the sum of a linear potential associated with a constant field B_0 and $A_{\mathrm{per}} \in C^\infty(E)$. Assuming $t_0, t_1 \notin \mathrm{Spec}(\mathscr{H}^\theta)$, there exists a neighbourhood $\mathcal{B} \subset \mathbb{R}$, open, connected, with $B_0 \in \mathcal{B}$ as well as $m = (m_1, m_2) \in \mathbb{Z}^2$ such that for any perturbation of the constant magnetic field $B \in \mathcal{B}$, $t_0, t_1 \notin \mathrm{Spec}(\mathscr{H}^\theta)$ the DOS satisfies*

$$\rho((t_0, t_1)) = \frac{1}{|E|}\left(m_1 \frac{B|E|}{2\pi} + m_2\right).$$

Proof. By density, we may assume that $B_0|E| = 2\pi\frac{p}{q} \in 2\pi\mathbb{Q}$. This implies by choosing $\lambda = q$ that $B_0|E_\lambda| \in 2\pi\mathbb{Z}$. Let $\lambda_{n,\mathbf{k}}$ be the n-th Bloch band of $\mathscr{H}_\mathbf{k}^\theta$ for $n \in \mathbb{Z}$ on $\mathbf{k} \in E_\lambda^*$. The spectrum of $\mathscr{H}^\theta$ has band structure and is given by $\mathrm{Spec}(\mathscr{H}^\theta) = \cup_n J_n$ where $J_n = \bigcup_{\mathbf{k} \in E_\lambda^*} \lambda_{n,\mathbf{k}}$.

Let $t_0,\ t_1 \notin \mathrm{Spec}(\mathscr{H}^\theta)$. We call $\mathcal{I}$ the set of bands fully contained in (t_0, t_1). In terms of $\mathbf{k} \mapsto u_{n,\mathbf{k}}$ given by the eigenvectors associated with $\lambda_{n,\mathbf{k}}$ spectral projection of $\mathscr{H}_\mathbf{k}^\theta$ is given by

$$\mathbb{1}_{(t_0,t_1)}(\mathscr{H}_\mathbf{k}^\theta)v_\mathbf{k}(x) = \int_{E_\lambda} \mathbb{1}_{(t_0,t_1)}(\mathscr{H}_\mathbf{k}^\theta)(x,y)v_\mathbf{k}(y)dy \text{ with } \mathbb{1}_{(t_0,t_1)}(\mathscr{H}_\mathbf{k}^\theta)(x,y) := \sum_{j\in\mathcal{I}} u_{j,\mathbf{k}}(x)\overline{u_{j,\mathbf{k}}(y)}.$$

So the spectral projection $\mathbb{1}_{(t_0,t_1)}(\mathscr{H}^\theta) = \mathcal{U}_{B_0}^{-1} \int_{E_\lambda^*}^\oplus \mathbb{1}_{(t_0,t_1)}(\mathscr{H}_\mathbf{k}^\theta)\frac{d\mathbf{k}}{|E_\lambda^*|}\mathcal{U}_{B_0}$ of $\mathscr{H}^\theta$ is

$$\mathbb{1}_{(t_0,t_1)}(\mathscr{H}^\theta)u(x) = \int_\mathbb{R} \mathbb{1}_{(t_0,t_1)}(\mathscr{H}^\theta)(x,y)u(y)dy \text{ with } \mathbb{1}_{(t_0,t_1)}(\mathscr{H}^\theta)(x,y) = \int_{E_\lambda^*} \mathbb{1}_{(t_0,t_1)}(\mathscr{H}_\mathbf{k}^\theta)(x,y)\frac{d\mathbf{k}}{|E_\lambda^*|}.$$

Since $t_0, t_1 \notin \mathrm{Spec}(\mathscr{H}^\theta)$ and let $N := |\mathcal{I}|$, then by Lemma 3.1

$$\rho((t_0,t_1)) := \int_{E_\lambda} \mathbb{1}_{(t_0,t_1)}(\mathscr{H}^\theta)(x,x)\frac{dx}{|E_\lambda|} = \int_{E_\lambda^*}\sum_{j\in\mathcal{I}} 1 \frac{d\mathbf{k}}{4\pi^2} = \frac{N}{|E_\lambda|} \text{ s.t. } \tilde{\mathrm{Tr}}(\mathbb{1}_{(t_0,t_1)}(\mathscr{H}^\theta)) = \frac{N}{|E_\lambda|}.$$

If $f \in C_c^\infty(\mathbb{R})$, such that $f(x) = 1$ for $x \in \mathrm{conv}\bigcup_n J_n$[1] and $f(x) = 0$ for $x \in \mathrm{Spec}(\mathscr{H}^\theta) \setminus \mathrm{conv}\bigcup_n J_n$, then

$$\rho((t_0,t_1)) = \int_\mathbb{R} f(t)\rho(dt) = \frac{N}{|E_\lambda|}.$$

Recall that $B_0|E| = \frac{B_0|E_\lambda|}{q} = 2\pi\frac{p}{q} \in 2\pi\mathbb{Q}$. We then introduce a new lattice $\tilde{\Gamma} \subset \Gamma$ generated by $\tilde{\zeta_1} = \zeta_1$ and $\tilde{\zeta_2} = q\zeta_2$. Then $B_0|\mathbb{C}/\tilde{\Gamma}| \in 2\pi\mathbb{Z}$ and $|\Gamma/\tilde{\Gamma}| = q$. As before, if $t_0, t_1 \notin \mathrm{Spec}(\mathscr{H}^\theta)$, then

$$\phi(B_0) := |E|\tilde{\mathrm{Tr}}(\mathbb{1}_{(t_0,t_1)}(\mathscr{H}^\theta)) = |E|\rho((t_0,t_1)) = |E|\int_\mathbb{R} f(t)d\rho(t) \in \frac{1}{q}\mathbb{Z} \subset \frac{B_0|E|}{2\pi}\mathbb{Z} + \mathbb{Z}$$

where the last inclusion follows since p, q are coprime i.e. there exist $c, d \in \mathbb{Z}$ such that $cp + dq = 1$. Note that if $z_0 \in \mathbb{R}\setminus\mathrm{Spec}(\mathscr{H}^\theta)$, then there exists $\varepsilon > 0$ such that $z \notin \mathrm{Spec}(\mathscr{H}^\theta)$ for all $|z - z_0|$ and small perturbations of the constant field $|B - B_0| < \varepsilon$ and $\phi(B)$ is locally

---

[1]conv is the convex hull

a smooth function of the constant field B by Lemma 3.1, so there exists $B_0 \in \mathcal{B} \subset \mathbb{R}$ open, connected and $m \in \mathbb{Z}^2$ such that for $B \in \mathcal{B}$,

$$\rho((t_0, t_1)) = \int_{\mathbb{R}} f(t) d\rho(t) = \frac{1}{|E|} \left(m_1 \frac{B|E|}{2\pi} + m_2 \right).$$

$\square$

4. Semiclassical expansion of Density of states

In this section, we provide explicit asymptotic expansions of the regularized trace in the semiclassical limit $B \gg 1$ for constant magnetic fields in the spirit of Remark 1 for the chiral and anti-chiral model respectively. We also comment on the differentiability of the DOS at the end of this section in preparation for applications in the next section.

We consider (2.1) with fixed θ and constant magnetic field B:

$$\mathscr{H}^\theta = \mathscr{H}_0^\theta + \mathscr{V}(x) = \begin{pmatrix} H_D^\theta & 0 \\ 0 & H_D^{-\theta} \end{pmatrix} + \begin{pmatrix} 0 & T(x) \\ T^*(x) & 0 \end{pmatrix}. \tag{4.1}$$

Notice that the spectrum of $\mathscr{H}_0^\theta$ is composed of *Landau levels* $\lambda_{n,B} := \operatorname{sgn}(n)\sqrt{2|n|B}$ (see Lemma 4.2) which we will perturb by the tunnelling potential $\mathscr{V}$ (see Remark 3). To simplify the notation, we therefore introduce *Landau bands* $\Lambda_{n,B,\mathscr{V}} := (\lambda_{n-1,B} + \|\mathscr{V}\|_\infty, \lambda_{n+1,B} - \|\mathscr{V}\|_\infty)$ for $n \in \mathbb{Z}$, in which the spectrum of $\mathscr{H}^\theta$ is contained around the n-th Landau level $\lambda_{n,B}$, cf. Remark 4.

We start by stating the main result of this section which is the asymptotic expansion of the DOS for the chiral model.

Theorem 1 (Chiral model). *Let $\lambda_{n,B} = \operatorname{sgn}(n)\sqrt{2|n|B}$. For a fixed $n \in \mathbb{Z}$, for $\varepsilon > 0$ small enough, for all $f \in C_c^K(\Lambda_{n,B,\mathscr{V}})$, with $K \geq \frac{6}{\varepsilon} - 2$, we have*

$$\tilde{\operatorname{Tr}}(f(\mathscr{H}_c)) = \left[\frac{B}{\pi} f(\lambda_{n,B}) + \frac{|n|}{2\pi} \operatorname{Ave}(\mathfrak{U}) f''(\lambda_{n,B}) \right] + \mathcal{O}_{n,K,f,\mathscr{V}}(B^{-\frac{1}{2}+\varepsilon}) \tag{4.2}$$

with $\mathfrak{U}(\eta) = \frac{\alpha_1^2}{8} \left[\alpha_1^2 (|U_-(\eta)|^2 - |U(\eta)|^2)^2 + 4|\partial_{\bar{\eta}} \overline{U_-(\eta)} - \partial_\eta U(\eta)|^2 \right]$, $\operatorname{Ave}(g) = \frac{1}{|E|} \int_E g(\eta) L(d\eta)$, $\eta = x_2 + i\xi_2$, and $\mathcal{O}_{n,K,f,\mathscr{V}} = \mathcal{O}_n(\|\mathscr{V}\|_\infty \|f\|_{C^K})$.

Furthermore, fix $N \in \mathbb{N}^+$ and consider $2N + 1$ Landau bands with $n \in \{-N, .., N\}$, then for all $\varepsilon > 0$ small enough, for any $f \in C_c^K([\lambda_{-(N+1),B} + \|\mathscr{V}\|_\infty, \lambda_{N+1,B} - \|\mathscr{V}\|_\infty])$, with $K \geq \frac{6}{\varepsilon} - 2$, we have

$$\tilde{\operatorname{Tr}}(f(\mathscr{H}_c)) = \sum_{n=-N}^{N} \left[\frac{B}{\pi} f(\lambda_{n,B}) + \frac{|n|}{2\pi} \operatorname{Ave}(\mathfrak{U}) f''(\lambda_{n,B}) \right] + \mathcal{O}_{(N),K,f,\mathscr{V}}(B^{-\frac{1}{2}+\varepsilon})$$

where $\mathcal{O}_{(N),K,f,\mathscr{V}} := \sum_{n=-N}^{N} \mathcal{O}_{n,K,f,\mathscr{V}}$.

Our proof also shows that all higher order terms, which in general have complicated expressions, in the expansion of $\tilde{\operatorname{Tr}}(f(\mathscr{H}_c))$ are of the form $f^{(k)}(\lambda_{n,B})$ (see (4.33)), which is different from the anti-chiral that we consider next.

For the anti-chiral Hamiltonian the sub-leading correction in the regularized trace is already of order $\sqrt{B}$. Since the dominant sub-leading correction in the anti-chiral case is one order higher than in the chiral case, we only state the correction up to order $\sqrt{B}$.

Theorem 2 (Anti-chiral model). *Under the same assumption as in Theorem 1, we have for all $\varepsilon > 0$ small enough, $f \in C_c^K(\Lambda_{n,B,\mathscr{V}})$ with $K \geq \frac{3}{\varepsilon} - 1$*

$$\tilde{\mathrm{Tr}}(f(\mathscr{H}_{ac}^\theta)) = \frac{B}{2\pi} t_{n,0}(f) - \frac{\sqrt{B}}{2\pi} t_{n,1}(f) + \mathcal{O}_{n,K,f,\mathscr{V}}(B^\varepsilon), \tag{4.3}$$

where $\mathcal{O}_{n,K,f,\mathscr{V}} = \mathcal{O}_n(\|\mathscr{V}\|_\infty \|f\|_{C^K})$ and

$$t_{n,0}(f) = \mathrm{Ave}\left(f(\lambda_{n,B} + c_n) + f(\lambda_{n,B} - c_n)\right), \; t_{n,1}(f) = \mathrm{Ave}\left(s_n^2 f'(\lambda_{n,B} + c_n) + s_n^2 f'(\lambda_{n,B} - c_n)\right),$$

$$s_n(\eta;\theta) = \begin{cases} \alpha_0 \sin(\frac{\theta}{2})|V(\eta)|, \\ \alpha_0 |V(\eta)|, \end{cases} \quad c_n(\eta;\theta) = \begin{cases} \alpha_0 \cos(\frac{\theta}{2})|V(\eta)|, & n \neq 0, \\ \alpha_0 |V(\eta)|, & n = 0. \end{cases}, \mathrm{Ave}(g) = \frac{1}{|E|} \int_E g(\eta) dL(\eta).$$

Furthermore, fix $N \in \mathbb{N}^+$ and consider $2N + 1$ Landau bands with $n \in \{-N, .., N\}$. For any $\varepsilon > 0$, $f \in C_c^K([\lambda_{-N-1,B} + \|\mathscr{V}\|_\infty, \lambda_{N+1} - \|\mathscr{V}\|_\infty])$ with $K \geq \frac{3}{\varepsilon} - 1$, we have

$$\tilde{\mathrm{Tr}}(f(\mathscr{H}_{ac}^\theta)) = \sum_{n=-N}^{N} \left[\frac{B}{2\pi} t_{n,0}(f) + \frac{\sqrt{B}}{2\pi} t_{n,1}(f)\right] + \mathcal{O}_{(N),f,K,\mathscr{V}}(B^\varepsilon)$$

where $\mathcal{O}_{(N),K,f,\mathscr{V}} := \sum_{n=-N}^{N} \mathcal{O}_{n,K,f,\mathscr{V}}.$

For the rest of this section, we shall temporarily stop using the identification $x = (x_1, x_2) \simeq z = x_1 + ix_2$. We will use the Landau gauge for the constant magnetic field, i.e. $A(z) = -iBx_1$ in 4.1. In this setup, Let $\Sigma_i^\theta = \mathrm{diag}(\sigma_i^\theta, \sigma_i^{-\theta})$. We can rewrite (4.1) as $\mathscr{H}_0^\theta = \Sigma_1^\theta D_{x_1} + \Sigma_2^\theta(D_{x_2} + Bx_1)$. We will only use $x = (x_1, x_2)$ to denote the position, while z is used in the resolvent $(\mathscr{H}^\theta - z)^{-1}$.

Quantizations. Let $x = (x_1, x_2)$, $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$. For a symbol $a(x, \xi) \in S(\mathbb{R}_{x,\xi}^4)$, we define the (h_1, h_2)-Weyl quantization $a^W(x, h_1 D_{x_1}, h_2 D_{x_2}) : L^2(\mathbb{R}_x^2) \to L^2(\mathbb{R}_x^2)$ as

$$(a^W(x, h_1 D_{x_1}, h_2 D_{x_2})u)(x) = \frac{1}{2\pi} \int e^{\frac{i}{h_1}(x_1-y_1)\xi_1 + \frac{i}{h_2}(x_2-y_2)\xi_2} a\left(\frac{x+y}{2}, \xi\right) u(y) \, dy \, d\xi. \tag{4.4}$$

In this section, we shall employ two different quantizations: in Subsections 4.1 and 4.2, we use the $(h_1, h_2) = (1, 1)$-Weyl quantization. Starting from Subsection 4.3, we use the $(x_2, h D_{x_2})$-Weyl quantization of the operator-valued symbol which is related to the $(h_1, h_2) = (1, h)$-Weyl quantization (see Subsection 4.3 for more details). Occasionally, we denote $a^W(x, h_1 D_{x_1}, h_2 D_{x_2})$ by a^W for convenience.

4.1. **First Reduction: Symplectic reduction.** In this subsection, we first apply a symplectic reduction to $\mathscr{H}^\theta$, then provide a spectral description of $\mathscr{H}_0^\theta$ and $\mathscr{H}^\theta$. In the end, we introduce the Helffer-Sjöstrand formula for our study of the regularized trace $\tilde{\mathrm{Tr}}(f(\mathscr{H}^\theta))$.

Symplectic Reduction. Let $(h_1, h_2) = (1,1)$ for this subsection. Then the operator $\mathscr{H}_0^\theta$ and $\mathscr{V}$, when viewed as a $(1,1)$-Weyl quantization, have symbols $\mathscr{H}_0^\theta(x,\xi) = \Sigma_1^\theta \xi_1 + \Sigma_2^\theta(\xi_2 + Bx_1)$ and $\mathscr{V}(x)$ respectively. The following lemma provide the symplectic reduction of $\mathscr{H}^\theta$:

Lemma 4.1. *Let $h = 1/B$. Then there is a unitary operator $\mathscr{U}$, symbols $\mathscr{G}_0^\theta(x,\xi) = \Sigma_1^\theta \xi_1 + \Sigma_2^\theta x_1$ and $\mathscr{W}(x,\xi) = \mathscr{V}(x_2 + h^{1/2}x_1, h\xi_2 - h^{1/2}\xi_1)$, s.t.*

$$\mathscr{U}\mathscr{H}_0^\theta(x, D_x)\mathscr{U}^{-1} = \sqrt{B}\mathscr{G}_0^\theta(x, D_x), \tag{4.5}$$

$$\mathscr{U}\mathscr{V}(x)\mathscr{U}^{-1} = \mathscr{W}^W(x, D_x). \tag{4.6}$$

Remark 2. *Notice that $\mathscr{G}_0^\theta(x,\xi)$ does not depend on (x_2, ξ_2), thus the $(1,1)$-Weyl-quantization is $\mathscr{G}_0^\theta(x, D_x) = (\Sigma_1^\theta D_{x_1} + \Sigma_2^\theta x_1) \otimes \mathbb{1}_{L^2(\mathbb{R}_{x_2})}$, where $\mathbb{1}_{L^2(\mathbb{R}_{x_2})}$ is the identity map on $L^2(\mathbb{R}_{x_2})$.*

Remark 3. *It follows that $\mathscr{U}\mathscr{H}^\theta\mathscr{U}^{-1} = \sqrt{B}(\mathscr{G}_0^\theta + \sqrt{h}\mathscr{W}^W)$. When $B \to \infty$, we can interpret $\mathscr{G}^\theta := \mathscr{G}_0^\theta + \sqrt{h}\mathscr{W}^W$ as a small perturbation of $\mathscr{G}_0^\theta$.*

Proof. Recall that a symplectic transformation $(y, \eta) = \kappa(x, \xi)$ applying to a symbol $a(x,\xi) = a \circ \kappa^{-1}(y, \eta) \in S(\mathbb{R}^4)$, induces a unitary operator $U_\kappa : L^2(\mathbb{R}_x^2) \to L^2(\mathbb{R}_y^2)$ s.t.

$$U_\kappa a^W(x, D_x) U_\kappa^{-1} = (a \circ \kappa^{-1})^W(y, D_y). \tag{4.7}$$

By applying the following three symplectic transformations to $\mathscr{H}^\theta(x, \xi)$:

$$\kappa_1(x,\xi) = (x_1, \xi_2, \xi_1, -x_2), \quad \kappa_2(x,\xi) = \left(x_1 + \tfrac{x_2}{B}, x_2, \xi_1, \xi_2 - \tfrac{x_1}{B}\right),$$

$$\kappa_3(x,\xi) = \left(\sqrt{B}x_1, -\tfrac{x_2}{B}, \tfrac{\xi_1}{\sqrt{B}}, -B\xi_2\right),$$

we find

$$\begin{cases} \mathscr{H}_0^\theta \circ \kappa_1^{-1} \circ \kappa_2^{-1} \circ \kappa_3^{-1}(x,\xi) = \sqrt{B}(\Sigma_1^\theta \xi_1 + \Sigma_2^\theta x_1), \\ \mathscr{V} \circ \kappa_1^{-1} \circ \kappa_2^{-1} \circ \kappa_3^{-1}(x,\xi) = \mathscr{V}(x_2 + h^{\frac{1}{2}}x_1, h\xi_2 + h^{\frac{1}{2}}\xi_1). \end{cases} \tag{4.8}$$

By (4.7) and (4.8), the unitary operator $U_\kappa := U_{\kappa_3} \circ U_{\kappa_2} \circ U_{\kappa_1}$ has then the desired properties. □

Spectral Descriptions. As mentioned in Remark 3, we study the spectral properties of $\mathscr{G}^\theta$ and $\mathscr{H}^\theta$ by viewing them as perturbations of $\mathscr{G}_0^\theta$ and $\mathscr{H}_0^\theta$. Therefore, we start with $\mathscr{G}_0^\theta$ and $\mathscr{H}_0^\theta$:

Lemma 4.2. *The spectral decompositions of $\mathscr{G}_0^\theta$ and $\mathscr{H}_0^\theta$ are given by*

$$\mathrm{Spec}(\mathscr{G}_0^\theta) = \{\lambda_n := \mathrm{sgn}(n)\sqrt{2|n|} : n \in \mathbb{Z}\} \text{ with eigenspace } N_n^\theta,$$

$$\mathrm{Spec}(\mathscr{H}_0^\theta) = \{\lambda_{n,B} := \mathrm{sgn}(n)\sqrt{2|n|B} : n \in \mathbb{Z}\} \text{ with eigenspace } \mathscr{U}N_n^\theta,$$

where

$$N_n^\theta = \mathrm{span}\left\{\begin{pmatrix} x \mapsto u_n^\theta(x_1)s_1(x_2) \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x \mapsto u_n^{-\theta}(x_1)s_2(x_2) \end{pmatrix} : \text{ For all } s_1, s_2 \in L^2(\mathbb{R}_{x_2})\right\}.$$

Here $u_n^\theta = e^{-\frac{i\theta}{4}\sigma_3} u_n e^{\frac{i\theta}{4}\sigma_3}$, $u_n = C_n \begin{pmatrix} \operatorname{sgn}(n) r_{|n|-1} \\ i r_{|n|} \end{pmatrix}$, $C_n = \begin{cases} \frac{1}{\sqrt{2}}, & n \in \mathbb{Z} \setminus \{0\} \\ 1, & n = 0 \end{cases}$, *as well as*

$r_{-1} = 0$, $r_m = C_m'(D_{x_1} + i x_1)^m e^{-\frac{x_1^2}{2}}$ *where* C_m' *is constant s.t.* $\|r_m\|_{L^2(\mathbb{R}_{x_1})} = 1$ *for* $m \in \mathbb{N}$.

Proof. The main observation here is for $G_D := \sigma_1 D_{x_1} + \sigma_2 x_1 = \begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix}$ where $a = D_{x_1} - i x_1$, we have $[a, a^*] = 2$. Thus a and a^* form a pair of annihilator and creator. By the standard argument for the ladder operators, there is a sequence of normalized $r_m(x_1) = C_m' \cdot (a^*)^m e^{-\frac{x_1^2}{2}} = C_m'(D_{x_1} + i x_1)^m e^{-\frac{x_1^2}{2}}$, for $m \geq 0$ s.t. $a r_m = \sqrt{2m} r_{m-1}$ and $a^* r_m = \sqrt{2(m+1)} r_{m+1}$. Then one can check by computation and (4.5) that u_n^θ, N_n^θ and $\mathscr{U} N_n^\theta$ defined above are eigenvectors and eigenspaces of $\mathscr{G}_0^\theta$, $\mathscr{G}_0^\theta$ and $\mathscr{H}_0^\theta$ with respect to eigenvalue λ_n, λ_n and $\lambda_{n,B}$, for all $n \in \mathbb{Z}$. $\qquad\square$

Remark 4. *Since* $\mathscr{H}^\theta = \mathscr{H}_0^\theta + \mathscr{V}$, *thus* $\operatorname{Spec}(\mathscr{H}^\theta) \subset \overline{B_{\|\mathscr{V}\|_\infty}\left(\operatorname{Spec}(\mathscr{H}_0^\theta)\right)} = \bigcup_n \overline{B_{\|\mathscr{V}\|_\infty}(\lambda_{n,B})}$. *Fix* n, *since* $\mathscr{V}$ *is bounded, when* B *is large enough,* $\left\{\overline{B_{\|\mathscr{V}\|_\infty}(\lambda_{j,B})}\right\}_{|j-n|\leq 1}$ *are disjoint. Since the DOS measure* ρ *is supported on the spectrum, by* (3.1), *the regularized trace* $\tilde{\operatorname{Tr}}(f(\mathscr{H}^\theta))$ *is not affected by modifying* f *within the spectral gap* $(\lambda_{k-1,B} + \|\mathscr{V}\|_\infty, \lambda_{k,B} - \|\mathscr{V}\|_\infty)$, *i.e.* $\tilde{\operatorname{Tr}}((\chi_{\Lambda_{k,B,\mathscr{V}}} f)(\mathscr{H}^\theta)) = \tilde{\operatorname{Tr}}((\chi_{B_{\|\mathscr{V}\|_\infty}(\lambda_{k,B})} f)(\mathscr{H}^\theta))$, *for any* $k \in \mathbb{Z}$. *Thus we will start with* f *supported on a fixed* $\Lambda_{n,B,\mathscr{V}}$ *to avoid the influence of bands nearby and then consider the general case of* f *supported on a fixed number of bands (see Theorem 1, 2 and their proofs in Subsection 4.5).*

Remark 5. *Both* $\lambda_{n,B}$ *and* λ_n *are called Landau levels of* $\mathscr{H}_0^\theta$ *and* $\mathscr{G}_0^\theta$ *respectively. To study the corresponding operators near the Landau levels, we denote* $\mathscr{H}_n^\theta := \mathscr{H}^\theta - \lambda_{n,B}$, $\mathscr{H}_{0,n}^\theta := \mathscr{H}_0^\theta - \lambda_{n,B}$, $\mathscr{G}_n^\theta := \mathscr{G}^\theta - \lambda_n$ *and* $\mathscr{G}_{0,n}^\theta := \mathscr{G}_0^\theta - \lambda_n$.

Helffer-Sjöstrand formula and regularized traces. We proceed by recalling the Helffer-Sjöstrand formula. Let $K \in \mathbb{N}$. Given $f \in C_c^{K+1}(\mathbb{R})$, we can always find $\tilde{f}$, an order-K quasi-analytic extension of f, by which we mean a function $\tilde{f} \in C_c^{K+1}(\mathbb{C})$, such that

$$\tilde{f}|_\mathbb{R} = f, \text{ and } |\partial_{\bar{z}} f| \leq C\|f\|_{C^{K+1}} |\operatorname{Im} z|^K, \text{ for some } C > 0. \tag{4.9}$$

The concrete construction can be found in [AJ06, Sec. 4.1] or [DS99, Theorem 8.1], where we can also choose $\tilde{f}$ s.t. $\operatorname{supp}(\tilde{f}) \supset \operatorname{supp}(f)$ is arbitrarily close to $\operatorname{supp}(f)$. We omit the proof which can be found in the quoted references.

Lemma 4.3 (Helffer-Sjöstrand formula). *Let* H *be a self-adjoint operator on a Hilbert space. Let* $f \in C_c^{K+1}(\mathbb{R})$ *and* $\tilde{f}$ *be its order-K quasi-analytic extension, then*

$$f(H) = \frac{1}{2\pi i} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{f}(z)(z - H)^{-1} dz \wedge d\bar{z}. \tag{4.10}$$

In particular, for $f \in C_c^{K+1}(\Lambda_{n,B,\mathscr{V}})$, define $f_0(x) = f(x + \lambda_{n,B})$ a function localized around zero. By Remark 5, (4.5) and (4.10), we have

$$
\begin{aligned}
\mathscr{U} f(\mathscr{H}^\theta) \mathscr{U}^{-1} = \mathscr{U} f_0(\mathscr{H}_n^\theta) \mathscr{U} &= -\frac{i}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{f}_0(z)(z - \mathscr{U} \mathscr{H}_n^\theta \mathscr{U})^{-1} \; dz \wedge d\bar{z} \\
&= \frac{i\sqrt{h}}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{f}_0(z)(\mathscr{G}_n^\theta - \sqrt{h}z)^{-1} \; dz \wedge d\bar{z}.
\end{aligned}
\tag{4.11}
$$

Thus to study $f(\mathscr{H}^\theta)$, it is enough to study the resolvent $(\mathscr{G}_n^\theta - \sqrt{h}z)^{-1}$.

4.2. Second reduction: Grushin problem.
In this subsection, we apply the Schur complement formula twice for operators $\mathscr{G}_{0,n}^\theta$ and $\mathscr{G}_n^\theta$ to characterize $(\mathscr{G}_n^\theta - \sqrt{h}z)^{-1}$ using the effective Hamiltonian. In our context, the Schur complement formula is also called a Grushin problem and we shall use that terminology in the sequel. See [SZ07] for more information on Grushin problem.

Unperturbed Grushin problem. To set up our Grushin problem, we introduce the space $B_{x_1}^k := B^k(\mathbb{R}_{x_1}; \mathbb{C}^4) := (1 + D_{x_1}^2 + x_1^2)^{-k/2} L^2(\mathbb{R}_{x_1}; \mathbb{C}^4)$. Then

$$
\mathscr{G}_{0,n}^\theta, \mathscr{G}_n^\theta : \; B_{x_1}^{k+1} \otimes L^2(\mathbb{R}_{x_2}; \mathbb{C}) \to B_{x_1}^k \otimes L^2(\mathbb{R}_{x_2}; \mathbb{C}) \subset L^2(\mathbb{R}_x^2; \mathbb{C}^4)
$$

are bounded. Define $R_n^+ = R_n^+(\theta) : B_{x_1}^k \otimes L^2(\mathbb{R}_{x_2}; \mathbb{C}) \to L^2(\mathbb{R}_{x_2}; \mathbb{C}^2)$ and $R_n^- = R_n^-(\theta) : L^2(\mathbb{R}_{x_2}; \mathbb{C}^2) \to B_{x_1}^k \otimes L^2(\mathbb{R}_{x_2}; \mathbb{C})$ by

$$
(R_n^+ t)(x_2) = \int_{\mathbb{R}} K_n^\theta(x_1)^* t(x_1, x_2) \; dx_1 \text{ and } R_n^-(s)(x) = K_n^\theta(x_1)s(x_2)
\tag{4.12}
$$

with

$$
K_n^\theta(x_1) = \begin{pmatrix} u_n^\theta(x_1) & 0 \\ 0 & u_n^{-\theta}(x_1) \end{pmatrix}_{4 \times 2}.
\tag{4.13}
$$

Then $(R_n^+)^* = R_n^-$.

First, we consider the Grushin problem for the unperturbed operator $\mathscr{G}_{0,n}^\theta - \sqrt{h}z$:

Lemma 4.4 (Unperturbed Grushin). *Fix $n \in \mathbb{Z}$. Let R_n^+ and R_n^- be defined as* (4.12). *Let*

$$
\mathcal{P}_{0,n} = \mathcal{P}_{0,n}(z; h, \theta) := \begin{pmatrix} \mathscr{G}_{0,n}^\theta - \sqrt{h}z & R_n^- \\ R_n^+ & 0 \end{pmatrix}.
$$

Then $\mathcal{P}_{0,n}$ is invertible iff $\sqrt{h}z \notin \{\lambda_m - \lambda_n : m \neq n\}$, and the inverse is

$$
\mathcal{E}_{0,n} := (\mathcal{P}_{0,n})^{-1} =: \begin{pmatrix} E_{0,n} & E_{0,n,+} \\ E_{0,n,-} & E_{0,n,\pm} \end{pmatrix}
\tag{4.14}
$$

where $E_{0,n,+} = R_n^-$, $E_{0,n,-} = R_n^+$, $E_{0,n,\pm}(z; h) = \sqrt{h}z \, \mathbb{1}_{\mathbb{C}^{2 \times 2}}$ and

$$
E_{0,n}^\theta(z; h) = \sum_{m \neq n} \frac{K_m^\theta (K_m^\theta)^*}{\lambda_m - \lambda_n - \sqrt{h}z} = \sum_{m \neq n} \frac{\begin{pmatrix} u_m^\theta (u_m^\theta)^* & 0 \\ 0 & u_m^{-\theta}(u_m^{-\theta})^* \end{pmatrix}}{\lambda_m - \lambda_n - \sqrt{h}z} =: \begin{pmatrix} e_{0,n}^\theta & 0 \\ 0 & e_{0,n}^{-\theta} \end{pmatrix}
\tag{4.15}
$$

with $\lambda_n = \mathrm{sgn}(n)\sqrt{2|n|}, n \in \mathbb{Z}$. Furthermore, we have

$$E_{0,n,-}(\mathscr{G}^{\theta}_{0,n} - \sqrt{h}z)E_{0,n,+} = -E_{0,n,\pm} \quad and$$

$$(\mathscr{G}^{\theta}_{0,n} - \sqrt{h}z)^{-1} = E_{0,n} - E_{0,n,+}(E_{0,n,\pm})^{-1}E_{0,n,-}.$$

Remark 6. *One can verify that $E^{\theta}_{0,n}$ maps N^{θ}_n to 0 and N^{θ}_m to $\frac{N^{\theta}_m}{\lambda_m - \lambda_n - \sqrt{h}z}$ if $m \neq n$.*

Perturbed Grushin problem. Next, we consider the perturbed Grushin problem for $\mathscr{G}^{\theta}_n - \sqrt{h}z$.

Lemma 4.5 (Perturbed Grushin). *Let $R^{\pm}_n$, $\mathscr{W}^W$ be defined as (4.12), (4.6). Let*

$$\mathcal{P}_n = \mathcal{P}_n(z; h, \theta) := \begin{pmatrix} \mathscr{G}^{\theta}_n - \sqrt{h}z & R^-_n \\ R^+_n & 0 \end{pmatrix} = \mathcal{P}_{0,n} + \sqrt{h}\mathcal{W}$$

where $\mathcal{W} := \mathrm{diag}(\mathscr{W}^W_{4\times4}, 0_{2\times2})$. Fix $n \in \mathbb{Z}$, there exist $h_0 = \min\left\{\frac{1}{2\|\mathscr{W}\|_\infty}, \frac{\lambda_{|n|+1}-\lambda_{|n|}}{4\|\mathscr{W}\|_\infty}\right\}$, s.t. for all $h \in [0, h_0)$, $\mathcal{P}_n$ is invertible with inverse

$$\mathcal{E}_n := (\mathcal{P}_n)^{-1} =: \begin{pmatrix} E_n & E_{n,+} \\ E_{n,-} & E_{n,\pm} \end{pmatrix} \tag{4.16}$$

which is analytic in $|z| \leq 2\|\mathscr{W}\|_\infty$. $E_{n,\pm}(z) : L^2(\mathbb{R}_{x_2}; \mathbb{C}^2) \to L^2(\mathbb{R}_{x_2}; \mathbb{C}^2)$ is called the effective Hamiltonian and satisfy

$$E_{n,\pm}(z) = \sqrt{h}\left(z - R^+_n\mathscr{W}^W(\mathbb{1} + \sqrt{h}E_{0,n}\mathscr{W}^W)^{-1}R^-_n\right) =: \sqrt{h}(z - Z^W). \tag{4.17}$$

In addition, we have

$$E_{n,-}(\mathscr{G}^{\theta}_n - \sqrt{h}z)E_{n,+} = -E_{n,\pm} \Rightarrow \sqrt{h}E_{n,-}E_{n,+} = \partial_z E_{n,\pm}, \tag{4.18}$$

$$(\mathscr{G}^{\theta}_n - \sqrt{h}z)^{-1} = E_n - E_{n,+}E^{-1}_{n,\pm}E_{n,-}. \tag{4.19}$$

Proof. Let h_0 be defined as above. When $h \in [0, h_0)$, $|z| < 2\|\mathscr{W}\|_\infty$, we have

$$\begin{cases} \sqrt{h}z \notin \{\lambda_m - \lambda_n : m \neq n\} & \Rightarrow \mathcal{P}_{0,n} \text{ is invertible with } \|\mathcal{P}_{0,n}\| \geq 1. \\ \sqrt{h}\|\mathcal{W}\|_\infty \leq \frac{1}{2} & \Rightarrow \mathcal{P}_n = \mathcal{P}_{0,n} + \sqrt{h}\mathcal{W} \text{ is invertible with inverse } \mathcal{E}_n. \\ |\sqrt{h}z| \leq \frac{\lambda_{|n|+1}-\lambda_{|n|}}{2} & \Rightarrow \text{ by (4.15) and (4.14), } E_{0,n}(z) \text{ and } \mathcal{E}_{0,n}(z) \text{ are analytic.} \end{cases}$$

Furthermore,

$$\mathcal{E}_n := \mathcal{P}^{-1}_n = (I + \sqrt{h}\mathcal{P}^{-1}_{0,n}\mathcal{W})^{-1}\mathcal{P}^{-1}_{0,n} = \sum^{\infty}_{j=0}(-1)^j h^{j/2}\left(\mathcal{E}_{0,n}\mathcal{W}\right)^j \mathcal{E}_{0,n}.$$

In particular, we get from the $(2,2)$-block of $\mathcal{P}^{-1}_n$ that

$$E_{n,\pm}(z) = E_{0,n,\pm}(z) + \sum^{\infty}_{j=1}(-1)^j h^{j/2}E_{0,n,-}\mathscr{W}^W(E_{0,n}\mathscr{W}^W)^{j-1}E_{0,n,+}$$

$$= \sqrt{h}z - \sqrt{h}R^+_n\mathscr{W}^W(\mathbb{1} + \sqrt{h}E_{0,n}\mathscr{W}^W)^{-1}R^-_n.$$

In fact, by direct computation, one get that $E_{0,n}$, $E_{n,+}$ and $E_{n,-}$ can all be represented by entries of $\mathcal{E}_{0,n}$ which we proved are analytic, thus $\mathcal{E}_n(z)$ is also analytic.

In the end, (4.18) and (4.19) follows from $\mathcal{E}_n \mathcal{P}_n \mathcal{E}_n = \mathcal{P}_n$ and the diagonalization on $\mathcal{P}_n$. $\square$

4.3. Properties of effective Hamiltonian.

In this subsection, we proceed with our study of $E_{n,\pm}(z)$, $E_{n,\pm}^{-1}(z)$ and $\partial_z E_{n,\pm} \circ E_{n,\pm}^{-1}$, with their symbols denoted by $E_{n,\pm}(x_2, \xi_2; z, h)$, $E_{n,\pm}^{-1}(x_2, \xi_2; z, h)$ and $r_n(x_2, \xi_2; z, h) := \partial_z E_{n,\pm} \# E_{n,\pm}^{-1}(x_2, \xi_2; z, h)$. Apart from analyzing boundedness and asymptotic expansions of symbols, we are especially interested in understanding the z-dependence and z vs. h competition of the symbols.

Before starting to analyze these properties, we introduce a key concept of this section: the operator-valued symbol and its quantization.

Operator-valued symbol. Let $b^w(x_2, \xi_2; x_1, D_{x_1}) \in S\left(\mathbb{R}^2_{x_2,\xi_2}; \mathcal{L}(B_{x_1}^{k+1}; B_{x_1}^k)\right)$, which we shall call an operator-(in-(x_1, D_{x_1}))-valued symbol (in (x_2, ξ_2)), then its (x_2, hD_{x_2})-Weyl quantization is defined as $b^W(x_2, hD_{x_2}; x_1, D_{x_1}) : L^2(\mathbb{R}_{x_2}; B_{x_1}^{k+1}) \to L^2(\mathbb{R}_{x_2}; B_{x_1}^k)$ such that

$$\left(b^W(x_2, hD_{x_2}; x_1, D_{x_1})u\right)(x) = \int e^{\frac{i(x_2-y_2)\xi_2}{h}} \left(b^w\left(\frac{x_2+y_2}{2}, \xi_2; x_1, D_{x_1}\right)u\right)(x_1; y_2) \frac{dy_2 d\xi_2}{2\pi h}.$$

In particular, if we have a symbol $a \in S(\mathbb{R}^4_{x,\xi})$, and we view (x_2, ξ_2) as parameters and consider the (x_1, D_{x_1})-Weyl quantization of it, we get $a^w(x, D_{x_1}, \xi_2)$ which is an operator-valued symbol in (x_2, ξ_2) (the superscript w represent the (x_1, D_{x_1})-Weyl quantization). If we do a further (x_2, hD_{x_2})-Weyl quantization of $a^w(x, D_{x_1}, \xi_2)$, then we get the $(1, h)$-Weyl quantization defined in (4.4).

Remark 7. *For the rest of this section, given an operator, e.g. $\mathscr{G}_0^\theta$, $E_{n,\pm}$ and $\mathscr{W}^W$ in (4.5), (4.16) and (4.6), instead of viewing them as the $(1, h)$-Weyl quantization of the scalar-valued symbol in $S(\mathbb{R}^4_{x,\xi})$, we will view them as the (x_2, hD_{x_2})-Weyl quantization of the operator-valued symbol in $S\left(\mathbb{R}^2_{x_2,\xi_2}; \mathcal{L}(B_{x_1}^{k_1}; B_{x_1}^{k_2})\right)$, for appropriate $k_1, k_2 \in \mathbb{Z}$.*

In particular, since $\mathscr{G}_0^\theta$ only depends on (x_1, D_{x_1}), $E_{n,\pm}$ only depends on (x_2, hD_{x_2}), $\mathscr{W}^W(x, D_x)$ is the $(1, h)$-Weyl quantization of the symbol $\mathscr{V}(x_2 + \sqrt{h}x_1, h\xi_2 - \sqrt{h}\xi_1)$, we see that the operator-valued symbol of $\mathscr{G}_0^\theta$, $E_{n,\pm}$ and $\mathscr{W}^W$ are respectively

$$\Sigma_1^\theta x_1 + \Sigma_2^\theta D_{x_1}, \quad E_{n,\pm}(x_2, \xi_2; z, h), \quad and \quad \tilde{\mathscr{V}}^w(x, D_{x_1}, \xi_2) := \mathscr{V}^w(x_2 + \sqrt{h}x_1, \xi_2 - \sqrt{h}D_{x_1})$$

where $\tilde{f}(x, \xi) = f(x_2 + \sqrt{h}x_1, \xi_2 - \sqrt{h}\xi_1)$. And since now

$$\mathscr{U}\mathscr{V}(x)\mathscr{U}^{-1} = \mathscr{W}^W(x, D_x) = \tilde{\mathscr{V}}^W(x, D_{x_1}, hD_{x_2}), \tag{4.20}$$

we will use $\tilde{\mathscr{V}}^W$ to replace $\mathscr{W}^W$ in Lemma 4.1 and 4.5. Finally, we mention that the proof of Lemma 4.1 implies in general

$$\mathscr{U}f(x)\mathscr{U}^{-1} = \tilde{f}^W(x, D_{x_1}, hD_{x_2}). \tag{4.21}$$

Boundedness with z dependence. We now study the boundedness of the operator-valued symbol $E_{n,\pm}$, $E_{n,\pm}^{-1}$ and r_n as well as the z dependence of them.

Notice that since $E_{n,\pm}$ only depends on (x_2, hD_{x_2}), when viewed as a (x_2, hD_{x_2})-Weyl quantization, its operator-in-(x_1, D_{x_1})-valued symbol coincides with its $\mathbb{C}_{2\times 2}$-valued symbol. For convenience, we write $S_\delta^k(\mathbb{R}^2_{x_2,\xi_2}; \mathbb{C}_{2\times 2})$ as S_δ^k and omit the "0" in δ and k.

Lemma 4.6 (Boundedness). *Let h_0, $E_{n,\pm}$ be as in Lemma 4.5. Then for all $h \in [0, h_0)$, we have the symbol of $E_{n,\pm}$, $E_{n,\pm}(x_2, \xi_2; z, h)$, belongs to $S^{-\frac{1}{2}}$ uniformly in $|z| \leq 2\|\mathscr{V}\|_\infty$, i.e. for any $\alpha, \beta > 0$, there is $C_{\alpha,\beta,n} = C_{\alpha,\beta,n}(\|\mathscr{V}\|_\infty)$, s.t.*

$$\sup_{(x_2,\xi_2)\in\mathbb{R}^2} \|\partial_{x_2}^\alpha \partial_{\xi_2}^\beta E_{n,\pm}(x_2, \xi_2; z, h)\|_{\mathbb{C}^{2\times2}} \leq C_{\alpha,\beta,n}\sqrt{h}, \quad \text{for all } |z| \leq 2\|\mathscr{V}\|_\infty.$$

Furthermore, if $|\operatorname{Im} z| \neq 0$, then we also have that for all $h \in [0, h_0)$, $|z| \leq 2\|\mathscr{V}\|_\infty$, $\alpha, \beta > 0$,

$$\|\partial_{x_2}^\alpha \partial_{\xi_2}^\beta E_{n,\pm}^{-1}(x_2, \xi_2; z, h)\|_{\mathbb{C}^{2\times2}} \leq C_{\alpha,\beta,n} \max\left(1, \frac{h^{3/2}}{|\operatorname{Im} z|^3}\right) h^{-\frac{1}{2}} |\operatorname{Im} z|^{-(|\alpha|+|\beta|)-1}, \quad (4.22)$$

$$\|\partial_{x_2}^\alpha \partial_{\xi_2}^\beta r_n(x_2, \xi_2; z, h)\|_{\mathbb{C}^{2\times2}} \leq C_{\alpha,\beta,n} \max\left(1, \frac{h^{3/2}}{|\operatorname{Im} z|^3}\right) |\operatorname{Im} z|^{-(|\alpha|+|\beta|)-1}. \quad (4.23)$$

In particular, if $0 < \delta < \frac{1}{2}$ and $|\operatorname{Im} z| \geq h^\delta$, then $E_{n,\pm}^{-1} \in S_\delta^{\frac{1}{2}+\delta}$ and $r_n \in S_\delta^\delta$.

Proof. When $h \in [0, h_0)$, $|z| \leq 2\|\mathscr{V}\|_\infty$, $E_{n,\pm}$ is a ΨDO because $\mathcal{P}_n$ is. In fact, by checking term by term, we have the operator-valued symbol $\mathcal{P}_n(x, D_{x_1}, \xi_2) \in S(\mathbb{R}^2_{x_2,\xi_2}; \mathcal{L}(B_{x_1}^{k+1} \times \mathbb{C}^2; B_{x_1}^k \times \mathbb{C}^2))$. By invertibility and Beal's lemma, $\mathcal{E}_n(x_2, \xi_2; z, h) \in S(\mathbb{R}^2_{x_2,\xi_2}; \mathcal{L}(B_{x_1}^{k+1} \times \mathbb{C}^2; B_{x_1}^k \times \mathbb{C}^2))$. In particular, we have

$$\begin{cases} R_n^+ \in S(\mathbb{R}^2_{x_2,\xi_2}; \mathcal{L}(B_{x_1}^k; \mathbb{C}^2)), \ R_n^- \in S(\mathbb{R}^2_{x_2,\xi_2}; \mathcal{L}(\mathbb{C}^2; B_{x_1}^{k+1})), \\ E_{0,n} \in S(\mathbb{R}^2_{x_2,\xi_2}; \mathcal{L}(B_{x_1}^k; B_{x_1}^{k+1})), \ \tilde{\mathscr{V}}^w \in S(\mathbb{R}^2_{x_2,\xi_2}; \mathcal{L}(B_{x_1}^{k+1}; B_{x_1}^k)). \end{cases} \quad (4.24)$$

Furthermore, by (4.15), when $|\sqrt{h}z| \leq \frac{\lambda_{|n|+1} - \lambda_{|n|}}{2}$, $E_{0,n}$ is uniformly bounded. Thus $E_{n,\pm}, \partial_z E_{n,\pm} \in S^{-\frac{1}{2}}$ uniformly.

Then we consider $E_{n,\pm}^{-1}$ and r_n. Let $l_1, l_2, \cdots$ be linear forms on $\mathbb{R}^2_{x_2,\xi_2}$. Let $L_j = l_j(x_2, hD_{x_2})$. Since $E_{n,\pm} \circ E_{n,\pm}^{-1} = I$, we get

$$\operatorname{ad}_{L_j} E_{n,\pm}^{-1} = -E_{n,\pm}^{-1} \circ \operatorname{ad}_{L_j} E_{n,\pm} \circ E_{n,\pm}^{-1},$$

where $\operatorname{ad}_{L_j} A = [L_j, A]$. Since $\operatorname{ad}_{L_j}(A \circ B) = (\operatorname{ad}_{L_j} A) \circ B + A \circ \operatorname{ad}_{L_j} B$, thus

$$\operatorname{ad}_{L_j}(\partial_z E_{n,\pm} \circ E_{n,\pm}^{-1}) = -\partial_z E_{n,\pm} \circ E_{n,\pm}^{-1} \circ \operatorname{ad}_{L_j} E_{n,\pm} \circ E_{n,\pm}^{-1} + \operatorname{ad}_{L_j} \partial_z E_{n,\pm} \circ E_{n,\pm}^{-1}.$$

By (4.19), $\|\sqrt{h}E_{n,\pm}^{-1}\|_{\mathbb{C}^{2\times2}} = \mathcal{O}(|\operatorname{Im} z|^{-1})$. Recall that $E_{n,\pm}, \partial_z E_{n,\pm} \in S^{-\frac{1}{2}}$, thus

$$\|\operatorname{ad}_{L_j}(\sqrt{h}E_{n,\pm}^{-1})\|_{\mathbb{C}^{2\times2}} = \mathcal{O}\left(\frac{h}{|\operatorname{Im} z|^2}\right) \quad \text{and} \quad \|\operatorname{ad}_{L_j}(\partial_z E_{n,\pm} \circ E_{n,\pm}^{-1})\|_{\mathbb{C}^{2\times2}} = \mathcal{O}\left(\frac{h}{|\operatorname{Im} z|^2}\right).$$

By induction,

$$\|\operatorname{ad}_{L_1} \circ \cdots \circ \operatorname{ad}_{L_N}(\sqrt{h}E_{n,\pm}^{-1})\|_{\mathbb{C}^{2\times2}} = \mathcal{O}\left(\frac{h^N}{|\operatorname{Im} z|^{N+1}}\right)$$

$$\|\operatorname{ad}_{L_1} \circ \cdots \circ \operatorname{ad}_{L_N}(\partial_z E_{n,\pm} \circ E_{n,\pm}^{-1})\|_{\mathbb{C}^{2\times2}} = \mathcal{O}\left(\frac{h^N}{|\operatorname{Im} z|^{N+1}}\right).$$

By a parametrized version of Beal's lemma, [DS99, Prop. 8.4], we get (4.22) and (4.23). $\quad\square$

Asymptotic Expansion with z dependence. We proceed by discussing the asymptotic expansion of $E_{n,\pm}$, $E_{n,\pm}^{-1}$ and r_n. Again, we are concerned with z-dependence of each term in the asymptotic expansions. In order to focus on the main points, we outsource further details concerning the asymptotic expansion of $E_{n,\pm}$ and $E_{n,\pm}^{-1}$, c.f. Prop. A.1, and its proof in the Appendix A, and present a shorter version here that only summarizes the results that we eventually need in the sequel.

Lemma 4.7 (Asymptotic expansion). *Let h_0, $E_{n,\pm}$ be as in Lemma 4.5, $0 < \delta < 1/2$. If $h \in [0, h_0)$, $|z| \leq 2\|\mathscr{V}\|_\infty$, $|\operatorname{Im} z| \geq h^\delta$, then $r_n(x_2, \xi_2; z, h) = \partial_z E_{n,\pm} \# E_{n,\pm}^{-1}$ has an asymptotic expansion in S_δ^δ:*

$$r_n(x_2, \xi_2; z, h) \sim \sum_{j=0}^{\infty} h^{\frac{j}{2}} r_{n,j}(x_2, \xi_2; z), \quad \text{with } h^{\frac{j}{2}} r_{n,j} \in S_\delta^{(j+1)\delta - \frac{j}{2}}. \tag{4.25}$$

More specifically, there are $d_{n,j,k,l}(x_2, \xi_2; z)$, $e_{n,j,k,\alpha}(x_2, \xi_2) \in S$ s.t.

$$r_{n,j} = \sum_{k=0}^{j} (z - z_{n,0})^{-1} \prod_{l=0}^{k} \left[d_{n,j,k,l}(x_2, \xi_2; z)(z - z_{n,0})^{-1} \right], \tag{4.26}$$

with $\prod_{l=0}^{k} d_{n,j,k,l}(x_2, \xi_2; z) = \sum_{\alpha=0}^{j+k-2} z^\alpha e_{n,j,k,\alpha}(x_2, \xi_2)$ and $z_{n,0}$ given in Prop. A.2. Let $R_{n,J} := r_n - \sum_{j=0}^{J-1} h^{\frac{j}{2}} r_{n,j}$, then $R_{n,J} \in S_\delta^{(J+1)\delta - \frac{J}{2}}$, i.e. for all $\alpha, \beta > 0$, there is $C'_{\alpha,\beta,n}$ s.t.

$$\sup_{(x_2, \xi_2) \in \mathbb{R}^2} |\partial_{x_2}^\alpha \partial_{\xi_2}^\beta R_{n,J}| \leq C'_{\alpha,\beta,n} h^{\frac{J}{2} - (J+1)\delta - \delta(|\alpha| + |\beta|)}. \tag{4.27}$$

Furthermore, for the expansion of $\operatorname{Tr}_{\mathbb{C}^2}(r_n)$, we have for $\eta = x_2 + i\xi_2$,

$$\begin{aligned}
\text{Chiral } \mathscr{H}_{c,n}(J=3) : &\operatorname{Tr}_{\mathbb{C}^2}(r_{c,n,0} + h^{\frac{1}{2}} r_{c,n,1} + h r_{c,n,2}) = \frac{2}{z} + 0 + \frac{\lambda_n^2}{z^3} \mathfrak{U}(\eta) h, \\
\text{Anti-Chiral } \mathscr{H}_{ac,n}^\theta(J=2) : &\operatorname{Tr}_{\mathbb{C}^2}(r_{ac,n,0} + h^{\frac{1}{2}} r_{ac,n,1}) = \frac{2z}{z^2 - c_n^2} + \frac{2s_n^2(z^2 + c_n^2)}{(z^2 - c_n^2)^2} \sqrt{h},
\end{aligned} \tag{4.28}$$

where $\mathfrak{U}(\eta) = \frac{\alpha_1^2}{8} \left[\alpha_1^2(|U_-(\eta)|^2 - |U(\eta)|^2)^2 + 4|\partial_{\bar\eta} \overline{U_-(\eta)} - \partial_\eta U(\eta)|^2 \right]$, $\partial_\eta = \frac{1}{2}(\partial_{x_2} - i\partial_{\xi_2})$, $s_n(\eta) = \begin{cases} \alpha_0 \sin(\frac{\theta}{2})|V(\eta)| & n \neq 0 \\ \alpha_0|V(\eta)| & n = 0, \end{cases}$ and $c_n(\eta) = \begin{cases} \alpha_0 \cos(\frac{\theta}{2})|V(\eta)| & n \neq 0 \\ \alpha_0|V(\eta)| & n = 0. \end{cases}$

Remark 8. *Notice by Prop. A.2, $z_{n,0} = 0$ for the chiral model. Thus we have $r_{c,n,j} = \sum_{k=0}^{2(j-1)} z^{k-j-1} f_{n,j,k}(x_2, \xi_2)$ for appropriate $f_{n,j,k} \in S$ when $j \geq 1$.*

4.4. **Trace formula.** Now we are ready to characterize $\tilde{\operatorname{Tr}} f(\mathscr{H}^\theta))$ using $E_{n,\pm}$ and still use the operator-valued symbol and (x_2, hD_{x_2})-quantization in this subsection.

Lemma 4.8. *Let $E_{n,\pm}$ be as in Lemma 4.5. Let $f \in C_c^{K+1}(\Lambda_{n,B,\mathscr{V}})$ and $f_0(x) = f(x + \lambda_{n,B})$ be as in (4.11). Then the regularized trace $\tilde{\mathrm{Tr}}(f(\mathscr{H}^\theta))$ satisfies*

$$\tilde{\mathrm{Tr}}(f(\mathscr{H}^\theta)) = -\frac{i}{4\pi^2 h|E|} \int_{\mathbb{C}} \int_E \partial_{\bar{z}} \tilde{f}_0 \, \mathrm{Tr}_{\mathbb{C}^2}(r_n(x_2, \xi_2; z, h)) \, dx_2 \, d\xi_2 \, dz \wedge d\bar{z}, \qquad (4.29)$$

Lemmas needed for the following proof are outsourced to Appendix B.

Proof. By (4.11), (4.19), and the analyticity of $E_n(z)$ when $h \in [0, h_0)$, $|z| \le 2\|\mathscr{V}\|_\infty$,

$$\mathscr{U} f(\mathscr{H}^\theta) \mathscr{U}^{-1} = -\frac{i\sqrt{h}}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{f}_0 (E_{n,+} E_{n,\pm}^{-1} E_{n,-})(z) \, dz \wedge d\bar{z}.$$

Thus we have

$$\begin{aligned}
\tilde{\mathrm{Tr}} f(\mathscr{H}^\theta) &= \lim_{R \to \infty} \frac{1}{4R^2} \mathrm{Tr}_1 \left(\mathbb{1}_R \, f(\mathscr{H}^\theta) \, \mathbb{1}_R \right) = \lim_{R \to \infty} \frac{1}{4R^2} \mathrm{Tr}_1(\tilde{\mathbb{1}}_R^W \mathscr{U} f(\mathscr{H}^\theta) \mathscr{U}^{-1} \tilde{\mathbb{1}}_R^W) \\
&\overset{2}{=} \lim_{R \to \infty} -\frac{i\sqrt{h}}{8\pi R^2} \mathrm{Tr}_1 \left(\int_{\mathbb{C}} \partial_{\bar{z}} \tilde{f}_0 (\tilde{\mathbb{1}}_R^W E_{n,+} E_{n,\pm}^{-1} E_{n,-} \tilde{\mathbb{1}}_R^W) \, dz \wedge d\bar{z} \right) \\
&\overset{3}{=} \lim_{R \to \infty} -\frac{i\sqrt{h}}{8\pi R^2} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{f}_0 \, \mathrm{Tr}_1 \left(\tilde{\mathbb{1}}_R^W E_{n,+} E_{n,\pm}^{-1} E_{n,-} \tilde{\mathbb{1}}_R^W \right) \, dz \wedge d\bar{z} \\
&\overset{4}{=} \lim_{R \to \infty} -\frac{i\sqrt{h}}{8\pi R^2} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{f}_0 \, \mathrm{Tr}_2 \left(\bar{\mathbb{1}}_R^W E_{n,-} E_{n,+} E_{n,\pm}^{-1} \bar{\mathbb{1}}_R^W \right) \, dz \wedge d\bar{z} \\
&\overset{5}{=} \lim_{R \to \infty} -\frac{i}{8\pi R^2} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{f}_0 \, \mathrm{Tr}_2 \left(\bar{\mathbb{1}}_R^W \partial_z E_{n,\pm} E_{n,\pm}^{-1} \bar{\mathbb{1}}_R^W \right) \, dz \wedge d\bar{z} \\
&\overset{6}{=} \lim_{R \to \infty} -\frac{i}{16\pi^2 h R^2} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{f}_0 \int_{\mathbb{R}^2} \mathrm{Tr}_{\mathbb{C}^2} \left(\bar{\mathbb{1}}_R \# \partial_z E_{n,\pm} \# E_{n,\pm}^{-1} \# \bar{\mathbb{1}}_R \right) \, dx_2 \, d\xi_2 \, dz \wedge d\bar{z} \\
&\overset{7}{=} -\frac{i}{4\pi^2 h|E|} \int_{\mathbb{C}} \int_E \partial_{\bar{z}} \tilde{f}_0 \, \mathrm{Tr}_{\mathbb{C}^2} \left(\partial_z E_{n,\pm} \# E_{n,\pm}^{-1} \right) \, dx_2 \, d\xi_2 \, dz \wedge d\bar{z}
\end{aligned}$$

where $\mathscr{U} \mathbb{1}_R \mathscr{U}^{-1} =: \tilde{\mathbb{1}}_R^W$ follows from (4.21). Here, $\bar{\mathbb{1}}_R^W = \bar{\mathbb{1}}_R^W(x_2, hD_{x_2})$ where $\bar{\mathbb{1}}_R(x_2, \xi_2)$ coincides with $\mathbb{1}_R(x_1, x_2)$ but is viewed as a function of phase space variables (x_2, ξ_2) rather than x. In addition, $\mathrm{Tr}_1 := \mathrm{Tr}_{L^2(\mathbb{R}_x^2; \mathbb{C}^4)}$, $\mathrm{Tr}_2 := \mathrm{Tr}_{L^2(\mathbb{R}_{x_2}; \mathbb{C}^2)}$.

The second line follows from the Helffer-Sjöstrand formula in Lemma 4.3. The third line follows from Lemma B.3, where we proved $\mathbb{1}_R^W E_{n,+} E_{n,\pm}^{-1} E_{n,-} \mathbb{1}_R^W$ is trace class. The fourth line follows directly from Lemma B.4. The fifth line follows from (4.19). The sixth line follows from

$$\mathrm{Tr}_{\mathcal{L}(L^2(\mathbb{R}_{x_2}; H_1); L^2(\mathbb{R}_{x_2}; H_2))}(a^W(x_2, hD_{x_2})) = \frac{1}{2\pi h} \int_{\mathbb{R}_{x_2, \xi_2}^2} \mathrm{Tr}_{\mathcal{L}(H_1, H_2)}(a(x_2, \xi_2)) dx_2 d\xi_2.$$

The seventh line follows from periodicity of $\mathscr{V}$ and thus periodicity of $\partial_z E_{n,\pm} \# E_{n,\pm}^{-1}$, which follows immediately from its asymptotic expansion. $\square$

4.5. **Proof of main results.** Now we can prove our main Theorems 1 and 2:

Proof of Theo. 1, 2. Let $0 < \delta < 1/2$. Assume $f \in C_c^{N+1}(\Lambda_{n,B,\mathscr{V}})$. Let $f_0(x) := f(x + \lambda_{n,B})$ which is supported on a neighbourhood of 0. Recall by Lemma 4.8, we need to compute

$$\tilde{\mathrm{Tr}}(f(\mathscr{H}_n^\theta)) = -\frac{i}{4\pi^2 h|E|} \int_E \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{f}_0 \, \mathrm{Tr}_{\mathbb{C}^2}(r_n(x_2, \xi_2; z, h)) \, dz \wedge d\bar{z} \, dx_2 \, d\xi_2. \tag{4.30}$$

We can rewrite the integral with expansion $r_n = \sum_{j=0}^{J-1} h^{j/2} r_{n,j} + R_{n,J}$ as in Lemma 4.7

$$\left[\int_{\mathbb{C}} \partial_{\bar{z}} \tilde{f}_0 \, \mathrm{Tr}_{\mathbb{C}^2}(r_n) dz \wedge d\bar{z} \right] (x_2, \xi_2; h) = \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{f}_0 \sum_{j=0}^{J-1} h^{\frac{j}{2}} \, \mathrm{Tr}_{\mathbb{C}^2}(r_{n,j}) dz \wedge d\bar{z}$$

$$+ \int_{|\mathrm{Im}\, z| \geq h^\delta} \partial_{\bar{z}} \tilde{f}_0 \, \mathrm{Tr}_{\mathbb{C}^2}(R_{n,J}) dz \wedge d\bar{z}$$

$$+ \int_{|\mathrm{Im}\, z| \leq h^\delta} \partial_{\bar{z}} \tilde{f}_0 \, \mathrm{Tr}_{\mathbb{C}^2}(R_{n,J}) dz \wedge d\bar{z} := A_1 + A_2 + A_3.$$

Notice that by Remark 4, we only need to consider f_0 supported at $|z| \leq \|\mathscr{V}\|$, for which we can pick $\tilde{f}_0$ s.t. $\tilde{f}_0$ is supported inside $|z| \leq 2\|\mathscr{V}\|_\infty$ for the integral. As in Lemma 4.7, we take $J = 3$ in the chiral case and $J = 2$ in the anti-chiral case.

First of all, we compute A_1 by (4.28) and the general version of Cauchy's integral formula, see [Ho03, (3.1.11)]: Let X be an open subset of $\mathbb{C}$. Let $g \in C_c^m(X)$, with $m \geq n$ then

$$2\pi i g^{(n)}(\zeta) = \int_X \partial_{\bar{z}} g(z) \frac{n!}{(z-\zeta)^{n+1}} dz \wedge d\bar{z}. \tag{4.31}$$

In particular, take X to be a small open neighborhood of $\mathrm{supp}(\tilde{f}_0)$. By (4.31) and the definition of f_0, we have

$$A_{1,c} = \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{f}_0 \left[\frac{2}{z} + \frac{\lambda_n^2}{z^3} \mathfrak{U}(\eta) h \right] dz \wedge d\bar{z} = 2\pi i \left[2f(\lambda_{n,B}) + \frac{\lambda_n^2}{2} \mathfrak{U}(\eta) f''(\lambda_{n,B}) h \right],$$

$$A_{1,ac} = \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{f}_0 \, \mathrm{Tr}_{\mathbb{C}^2} \left[\frac{1}{z - c_n} + \frac{1}{z + c_n} + \frac{s_n^2 \sqrt{h}}{(z - c_n)^2} + \frac{s_n^2 \sqrt{h}}{(z + c_n)^2} \right] dz \wedge d\bar{z}$$

$$= 2\pi i \left[f(\lambda_{n,B} + c_n) + f(\lambda_{n,B} - c_n) + f'(\lambda_{n,B} + c_n) s_n^2 \sqrt{h} + f'(\lambda_{n,B} - c_n) s_n^2 \sqrt{h} \right].$$

For A_2, by (4.27) and $|z| \leq 2\|\mathscr{V}\|_\infty$, when $|\mathrm{Im}\, z| \geq h^\delta$, there are $C_n, C_n' > 0$ such that

$$|A_2| \leq \int_{|\mathrm{Im}\, z| \geq h^\delta} |\partial_{\bar{z}} \tilde{f}_0| C_n h^{\frac{J}{2} - (J+1)\delta} 2L(dz) \leq C_n' \|f\|_{C^{K+1}} \|\mathscr{V}\|_\infty h^{\frac{J}{2} - (J+1)\delta}.$$

Finally, by (4.9), (4.23), (4.26), $0 < \delta < 1/2$ and $|z| \leq 2\|\mathscr{V}\|_\infty$, we have for some $C_{n,j}, C_n''$

$$
\begin{aligned}
|A_3| &\leq \int_{|\operatorname{Im} z| \leq h^\delta} |\partial_{\bar{z}} \tilde{f}_0| \left[|\operatorname{Tr}_{\mathbb{C}^2}(r_n)| + \sum_{j=0}^{J-1} \left| \operatorname{Tr}_{\mathbb{C}^2}(h^{\frac{j}{2}} r_{n,j}) \right| \right] dz \wedge d\bar{z} \\
&\leq \int_{|\operatorname{Im} z| \leq h^\delta} \|f\|_{C^{K+1}} |\operatorname{Im} z|^K \left[\max\left(\frac{1}{|\operatorname{Im} z|}, \frac{h^{\frac{3}{2}}}{|\operatorname{Im} z|^4} \right) + \sum_{j=0}^{J-1} \frac{C_{n,j} h^{\frac{j}{2}}}{|\operatorname{Im} z|^{j+1}} \right] dz \wedge d\bar{z} \\
&\leq 2 C_n'' \|f\|_{C^{K+1}} \|\mathscr{V}\|_\infty \left[\max\left(h^{(K-1)\delta}, h^{(K-4)\delta + \frac{3}{2}} \right) + \sum_{j=0}^{J-1} h^{\frac{j}{2} + (K-j-1)\delta} \right] \\
&\leq C_n'' \|f\|_{C^{K+1}} \|\mathscr{V}\|_\infty h^{(K-1)\delta}.
\end{aligned}
$$

Define $C_{n,K,f,V} = \max(C_n, C_n', C_n'') \|\mathscr{V}\|_\infty \|f\|_{C^{K+1}}$. We see

$$
|A_{2,\mathrm{c}}| \leq C_{n,K,f,\mathscr{V}} h^{\frac{3}{2} - 4\delta}, \quad |A_{2,\mathrm{ac}}| \leq C_{n,K,f,\mathscr{V}} h^{1-3\delta}, \quad |A_3| \leq C_{n,K,f,\mathscr{V}} h^{(K-1)\delta}.
$$

Combine the estimates of A_1, A_2, A_3, and plug them into (4.30), we have

$$
\begin{aligned}
\tilde{\operatorname{Tr}} f(\mathscr{H}_{\mathrm{c}}) &= \frac{1}{\pi h} f(\lambda_{n,B}) + \frac{|n|}{2\pi} f''(\lambda_{n,B}) \mathfrak{U}(\eta) + \mathcal{O}_{n,K,f,\mathscr{V}}(h^{\frac{1}{2} - 4\delta} + h^{(K-1)\delta - 1}), \\
\tilde{\operatorname{Tr}} f(\mathscr{H}_{\mathrm{ac}}^\theta) &= \frac{1}{2\pi h} t_{n,0}(f) + \frac{1}{2\pi\sqrt{h}} t_{n,1}(f) + \mathcal{O}(h^{-3\delta} + h^{(K-1)\delta - 1})
\end{aligned}
\tag{4.32}
$$

where $t_{n,0}(f) = \operatorname{Ave}[f(\lambda_{n,B} - c_n) + f(\lambda_{n,B} + c_n)]$, $t_{n,1}(f) = \operatorname{Ave}[s_n^2 f(\lambda_{n,B} - c_n) + s_n^2 f(\lambda_{n,B} + c_n)]$, and $\operatorname{Ave}(g) = \frac{1}{|E|} \int_E g(\eta) d\eta$. Thus we proved (4.2) and (4.3).

In general, fix $N \in \mathbb{N}^+$ and we consider $2N + 1$ Landau levels centered at 0. Let B be large enough such that $\left\{ \overline{B_{\|\mathscr{V}\|_\infty}(\lambda_{n,B})} \right\}_{n=-N}^N$ do not intersect. For any $f \in C_c^{K+1}(\lambda_{-(N+1),B} + \|\mathscr{V}\|_\infty, \lambda_{N+1,B} - \|\mathscr{V}\|_\infty)$, by Remark 4, values of f on the gap do not contribute to $\tilde{\operatorname{Tr}}(f(\mathscr{H}^\theta))$, thus we can apply the partition of unity of f on $\{\Lambda_{n,B,\mathscr{V}}\}_{n=-N}^N$, i.e. find f_n such that $f = \sum_{n=-N}^N f_n$ and $\operatorname{supp} f_n \subset \Lambda_{n,B,\mathscr{V}}$. Then we can apply (4.2) and (4.3) to each f_n and take the sum. That gives us the rest of the Theorem 1 and 2.

Furthermore, as mentioned in Remark 8, $z_{n,0} = 0$ in the chiral case, thus each term in the expansion of $r_{n,\mathrm{c}}$ is of the form $r_{n,j,\mathrm{c}} = \sum_{k=0}^{k-j-1} z^{k-j-1} f_{n,j,k}(x_2, \xi_2)$. Now assume f is smooth enough, then for any $J \in \mathbb{N}$, by (4.31), we can see that

$$
A_{1,\mathrm{c}} = \sum_{j=0}^{J-1} h^{j/2} \sum_{k=0}^{k-j-1} F_{n,j,k}(\eta) f^{(j-k)}(\lambda_{n,B}), \quad \text{for some } F_{n,j,k}(\eta) \in S.
\tag{4.33}
$$

Thus for the chiral case, every term in the asymptotic expansion of $\tilde{\operatorname{Tr}}(f(\mathscr{H}^\theta))$ only depends on derivatives $f^{(k)}$ at $\lambda_{n,B}$.

$\square$

4.6. **Differentiability.** Finally, we comment on the differentiability of the regularized trace with respect to the magnetic field. That $h \mapsto \tilde{\mathrm{Tr}}(f(\mathscr{H}^\theta))$ is a differentiable function follows already from Lemma 3.1. However, what does not follow from Lemma 3.1 is that the asymptotic expansion itself in Theorems 1 and 2 is differentiable. The following Proposition, which uses the same notation as Theorems 1 and 2 shows that term-wise differentiation yields the right asymptotic expansion:

Proposition 4.9 (Differentiability). *Under the same assumption of $\lambda_{n,B}$, ε, as in Theorem 1, we have that $B \mapsto \tilde{\mathrm{Tr}}(f(\mathscr{H}^\theta))$ is differentiable. For all ε, $f \in C^K(\Lambda_{n,B,\mathscr{V}})$, that $K > \frac{6}{\varepsilon} - 2$, then for $\mathcal{O}_{n,K,f,\mathscr{V}} = \mathcal{O}_n(\|\mathscr{V}\|_\infty \|f\|_{C^K})$, we have: For the chiral model $\mathscr{H}^\theta = \mathscr{H}_c$,*

$$\partial_B \tilde{\mathrm{Tr}}(f(\mathscr{H}_c)) = \frac{\sqrt{2|n|B}}{2\pi} f'(\lambda_{n,B}) + \frac{f(\lambda_{n,B})}{\pi} + \frac{(2|n|)^{\frac{3}{2}}}{8\pi\sqrt{B}} \mathrm{Ave}(\mathfrak{U}) f'''(\lambda_{n,B}) + \mathcal{O}_{n,K,f,\mathscr{V}}(B^{-1+\varepsilon})$$
(4.34)

For the anti-chiral model $\mathscr{H}^\theta = \mathscr{H}_{ac}^\theta$,

$$\partial_B \tilde{\mathrm{Tr}}(f(\mathscr{H}_{ac}^\theta)) = \frac{\sqrt{2|n|B}}{4\pi} t_{n,0}(f') + \frac{1}{4\pi}\left(2t_{n,0}(f) + \sqrt{2|n|}t_{n,1}(f')\right) + \mathcal{O}_{n,K,f,\mathscr{V}}(B^{-\frac{1}{2}+3\delta})$$
(4.35)

In particular, when $n = 0$, we get a better estimate for the chiral and anti-chiral case respectively:

$$\partial_B \tilde{\mathrm{Tr}}(f(\mathscr{H}_c)) = \frac{1}{\pi} f(0) + \mathcal{O}_{0,K,f,\mathscr{V}}(B^{-\frac{3}{2}+4\delta})$$
$$\partial_B \tilde{\mathrm{Tr}}(f(\mathscr{H}_{ac}^\theta)) = \frac{1}{2\pi} t_{0,0}(f) + \frac{3}{4\pi\sqrt{B}} t_{0,1}(f) + \mathcal{O}_{0,K,f,\mathscr{V}}(B^{-1+3\delta})$$
(4.36)

where $t_{n,0}(f)$, $t_{n,1}(f)$, $\mathfrak{U}$, s_n and c_n are the same as in Theorem 1, 2.

To prove this proposition, we will need to prove two auxiliary Lemmas 4.10 and 4.11 discussing properties of $\partial_h E_{n,\pm}$, $\partial_h E_{n,\pm}^{-1}$ and $\partial_h r_n$, which are similar to the two properties needed for $E_{n,\pm}$, $E_{n,\pm}^{-1}$ and r_n previously in 4.6 and 4.7. The rest of the proof is similar to Sec. 4.5. We start with some preparations: To discuss the differentiability of asymptotic expansions, we define $\#_h^M$ for $a(x,\xi;h), b(x,\xi;h) \in S(\mathbb{R}^2_{x,\xi})$ by

$$a\#_h^M b = \left[e^{\frac{ih}{2}\sigma(D_x,D_\xi;D_y,D_\eta)}\left(\frac{i}{2}\sigma(D_x,D_\xi;D_y,D_\eta)\right)^M \right](a(x,\xi,h)b(y,\eta,h)) \Big|_{\substack{x=y \\ \xi=\eta}}$$
$$= \sum_{|\alpha|=|\beta|=M} C_{\alpha,\beta}(\partial_{x,\xi}^\alpha a)\#(\partial_{y,\eta}^\beta b),$$
(4.37)

where $\sigma(x,\xi;y,\eta) = \langle \xi, y \rangle - \langle x, \eta \rangle$. Then we see that,

$$\partial_h^M(a\#b) = a\#_h^M b + \sum_{\substack{i+j+k=M \\ j \neq M}} C_{i,j,k} (\partial_h^i a) \#_h^j (\partial_h^k b).$$
(4.38)

The following result is derived for general $M \in \mathbb{N}$ but we will, for simplicity, only consider the $M = 1$ case later:

Lemma 4.10 (Boundedness). *Let h_0, $E_{n,\pm}$ be as in Lemma 4.5. The symbol $E_{n,\pm}(x_2, \xi_2; z, h)$ is smooth in h when $h < h_0$ and for any $M \in \mathbb{N}$, $\partial_h^M E_{n,\pm} \in S^{M-\frac{1}{2}}$ uniformly in $|z| \leq 2\|\mathscr{V}\|_\infty$, i.e. for any multi-index α, β, there is $C_{\alpha,\beta,n} = C_{\alpha,\beta,n}(\|\mathscr{V}\|_\infty)$ such that*

$$\|\partial_{x_2}^\alpha \partial_{\xi_2}^\beta \partial_h^M E_{n,\pm}(x_2, \xi_2; z, h)\|_{\mathbb{C}_{2\times 2}} \leq C_{\alpha,\beta,n}\sqrt{h}, \quad \text{for all } |z| \leq 2\|\mathscr{V}\|_\infty.$$

If $|\operatorname{Im} z| \neq 0$, $M > 0$, then $\partial_h^M E_{n,\pm}^{-1}$ and $\partial_h^M r_n$ satisfy

$$\|\partial_{x_2}^\alpha \partial_{\xi_2}^\beta \partial_h^M E_{n,\pm}^{-1}(x_2, \xi_2; z, h)\|_{\mathbb{C}_{2\times 2}} \leq C_{\alpha,\beta,n} \max\left(1, \frac{h^{\frac{3}{2}}}{|\operatorname{Im} z|^3}\right) h^{-\frac{1+2M}{2}}|\operatorname{Im} z|^{-2M-|\alpha|-|\beta|} \quad (4.39)$$

$$\|\partial_{x_2}^\alpha \partial_{\xi_2}^\beta \partial_h^M r_n(x_2, \xi_2; z, h)\|_{\mathbb{C}_{2\times 2}} \leq C_{\alpha,\beta,n} \max\left(1, \frac{h^{\frac{3}{2}}}{|\operatorname{Im} z|^3}\right) h^{-M}|\operatorname{Im} z|^{-2M-|\alpha|-|\beta|}. \quad (4.40)$$

In particular, when $0 < \delta < 1/2$ and $|\operatorname{Im} z| \geq h^\delta$, we have $\partial_h^M E_{n,\pm}^{-1} \in S_\delta^{M(2\delta+1)+\frac{1}{2}}$ and $\partial_h^M r_n \in S_\delta^{M(2\delta+1)}$.

Proof. Let $\mathcal{P}_n$ be as in Lemma 4.5, by (4.24), $\mathscr{G}^\theta - \sqrt{h}z$, $R_n^\pm \in S(\mathbb{R}^2_{x_2,\xi_2})$. Furthermore, since $\mathscr{G}^\theta = \mathscr{G}_0^\theta + \sqrt{h}\tilde{\mathscr{V}}^w$, by direct computation, we see $\partial_h^M(\mathscr{G}^\theta - \sqrt{h}z) \in S^{M-\frac{1}{2}}$ while $\partial_h^M R_n^\pm = 0$, for $M > 0$.

Then consider $\mathcal{E}_n = \mathcal{P}_n^{-1}$. First of all, by the proof of Lemma 4.6, we have $\mathcal{E}_n(x, D_{x_1}, \xi_2) \in S(\mathbb{R}^2_{x_2,\xi_2}; \mathcal{L}(B_{x_1}^k \times \mathbb{C}^2; B_{x_1}^{k+1} \times \mathbb{C}^2))$. By differentiating $\mathcal{E}_n = \mathcal{E}_n \# \mathcal{P}_n \# \mathcal{E}_n$ with respect to h and using (4.37) and (4.38), we have

$$\partial_h \mathcal{E}_n = -\mathcal{E}_n \# \partial_h \mathcal{P}_n \# \mathcal{E}_n + \sum_{|\alpha|=|\beta|=1} C_{\alpha,\beta}\left(\partial_{x_2,\xi_2}^\alpha \mathcal{E}_n \# \partial_{x_2,\xi_2}^\beta \mathcal{P}_n \# \mathcal{E}_n\right). \quad (4.41)$$

Since $\partial_h \mathcal{P}_n \in S^{\frac{1}{2}}$, thus $\partial_h \mathcal{E}_n \in S^{\frac{1}{2}}$ above. By differentiating (4.41) with respect to h and using (4.37) and (4.38), we see that $\partial_h^2 \mathcal{E}_n \in S^{\frac{3}{2}}$. An iterative argument shows that $\partial_h^M \mathcal{E}_n \in S^{M-\frac{1}{2}}$. In particular, $\partial_h^M E_{n,\pm} \in S^{M-\frac{1}{2}}$. Furthermore, by differentiating $E_{n,\pm}^{-1} = E_{n,\pm}^{-1} \# E_{n,\pm} \# E_{n,\pm}^{-1}$ with respect to h and using (4.38) and (4.37), we have

$$\partial_h E_{n,\pm}^{-1} = -E_{n,\pm}^{-1} \# \partial_h E_{n,\pm} \# E_{n,\pm}^{-1} - \sum_{|\alpha|=|\beta|=1} C_{\alpha,\beta}\partial_{x_2,\xi_2}^\alpha E_{n,\pm}^{-1} \# \partial_{x_2,\xi_2}^\beta E_{n,\pm} \# E_{n,\pm}^{-1}. \quad (4.42)$$

When $|\operatorname{Im} z| \geq h^\delta$, by (4.23) and [Zw12, Theorem 4.23(ii)], we see that

$$\|\partial_h E_{n,\pm}^{-1}\| = \mathcal{O}(h^{-\frac{3}{2}}|\operatorname{Im} z|^{-2}) + \mathcal{O}(h^{-\frac{1}{2}}|\operatorname{Im} z|^{-3}) = \mathcal{O}(h^{-\frac{3}{2}}|\operatorname{Im} z|^{-2}).$$

Furthermore, since $[D_{x_j}, A^W] = (D_{x_j}A)^W$ and $-[x_j, A^W] = (hD_{\xi_j}A)^W$, we see that

$$\|\operatorname{ad}_{L_{j_1}} \circ \cdots \operatorname{ad}_{L_{j_N}}(\partial_h E_{n,\pm}^{-1})^W\| = \mathcal{O}\left(\frac{h^{-\frac{3}{2}}}{|\operatorname{Im} z|^2} \frac{h^N}{|\operatorname{Im} z|^N}\right).$$

By [DS99, Prop. 8.4], we get

$$\|\partial_{x_2}^\alpha \partial_{\xi_2}^\beta \partial_h E_{n,\pm}^{-1}(x_2, \xi_2; z, h)\|_{\mathbb{C}_{2\times 2}} \leq C_{\alpha,\beta} \max\left(1, \frac{h^{\frac{3}{2}}}{|\operatorname{Im} z|^3}\right) h^{-\frac{3}{2}}|\operatorname{Im} z|^{-2-|\alpha|-|\beta|}. \quad (4.43)$$

Iterating this process by taking ∂_h of (4.42), expanding it and using (4.37), (4.38) and (4.43), we see that every time we differentiate, we derive an extra order of $1/(h|\operatorname{Im} z|^2)$. Thus we obtain (4.39) for $M > 0$. Then

$$\partial_h r_n = \partial_h \partial_z E_{n,\pm} \# E_{n,\pm}^{-1} + \partial_z E_{n,\pm} \# \partial_h E_{n,\pm}^{-1} + \sum_{|\alpha|=|\beta|=1} C_{\alpha,\beta}(\partial_{x_2,\xi_2}^\alpha \partial_z E_{n,\pm}) \# (\partial_{x_2,\xi_2}^\beta E_{n,\pm}^{-1}).$$

By (4.23), (4.39) and [Zw12, Theorem 4.23(ii)], we see that $\|\partial_h r_n^W\| = \mathcal{O}(h^{-1}|\operatorname{Im} z|^{-2})$. By the same argument as for $E_{n,\pm}^{-1}$, we get (4.40). $\qquad\square$

We shall now focus on $M = 1$, for simplicity, and study the asymptotic expansion of $\partial_h r_n$.

Lemma 4.11 (Asymptotic expansion). *Let $0 < \delta < 1/2$ and $|\operatorname{Im} z| \geq h^\delta$, then $\partial_h r_n$ has an asymptotic expansion in $S_\delta^{1+2\delta}$:*

$$\partial_h r_n \sim \sum_{j=1}^\infty \frac{j}{2} h^{\frac{j}{2}-1} r_{n,j} =: \sum_{j=1}^\infty h^{\frac{j}{2}-1} q_{n,j}, \quad \text{where } r_{n,j} \text{ are given in Lemma } 4.7.$$

Then $h^{\frac{j}{2}-1} q_{n,j} \in S_\delta^{(j+1)\delta+1-\frac{j}{2}}$. Let $Q_{n,J} := \partial_h r_n - \sum_{j=1}^{J-1} h^{\frac{j}{2}-1} q_{n,j} \in S_\delta^{(J+1)\delta+1-\frac{J}{2}}$, i.e., for all $\alpha, \beta > 0$, there is $C''_{\alpha,\beta,n}$ such that

$$\sup_{(x_2,\xi_2)\in\mathbb{R}^2} |\partial_{x_2}^\alpha \partial_{\xi_2}^\beta Q_{n,J}| \leq C''_{\alpha,\beta,n} h^{\frac{J}{2}-1-(J+1)\delta-\delta(|\alpha|+|\beta|)}. \tag{4.44}$$

Furthermore, for the expansion of $\operatorname{Tr}_{\mathbb{C}^2}(\partial_h r_n)$, we have for $\eta = x_2 + i\xi_2$,

$$\begin{aligned}
\text{Chiral } \mathscr{H}_{c,n}(J=3) &: \operatorname{Tr}_{\mathbb{C}^2}(h^{-\frac{1}{2}} q_{n,1} + q_{n,2}) = \frac{\lambda_n^2}{z^3} \mathfrak{U}(\eta), \\
\text{Anti-Chiral } \mathscr{H}_{ac,n}^\theta(J=2) &: \operatorname{Tr}_{\mathbb{C}^2}(h^{-\frac{1}{2}} q_{n,1}) = \frac{s_n^2(z^2 + c_n^2)}{(z^2 - c_n^2)^2 \sqrt{h}}.
\end{aligned} \tag{4.45}$$

We will prove that the termwise differentiation of the asymptotic expansion of r_n in (4.25) is indeed an asymptotic expansion of $\partial_h r_n$ in $S_\delta^{2\delta+1}$.

Proof. Let $g = \sqrt{h}$ and consider $r_n \sim \sum_{j=0}^\infty g^j r_{n,j}$. By Borel's theorem, see for instance [Zw12, Theorem 4.15] or [Ho03, Theorem 1.2.6], we see that for such $r_{n,j} \in C^\infty(\mathbb{R}^2_{x_2,\xi_2})$, there is $\tilde{r}_n \in C^\infty(\mathbb{R}_g^+ \times \mathbb{R}_{x_2,\xi_2})$ such that $\tilde{r}_n = \sum_{j=0}^\infty g^j r_{n,j}$. Thus

$$\partial_g \tilde{r}_n = \sum_{j=1}^\infty j g^{j-1} r_{n,j}. \tag{4.46}$$

On the other hand, uniqueness in Borel's theorem implies that $\tilde{r}_n - r_n = \mathcal{O}(h^\infty)$. Thus $\partial_g \tilde{r}_n - \partial_g r_n = \mathcal{O}(g^\infty)$. Thus (4.46) is also an asymptotic expansion of $\partial_g r_n$. Furthermore, since $\partial_h r_n = \frac{1}{2\sqrt{h}} \partial_g r_n$, thus we proved $\partial_h r_n$ has the following asymptotic expansion in $S_\delta^{1+2\delta}$:

$$\partial_h r_n \sim \frac{1}{2\sqrt{h}} \sum_{j=1}^\infty j h^{\frac{j-1}{2}} r_{n,j} = \sum_{j=1}^\infty \frac{j}{2} h^{\frac{j}{2}-1} r_{n,j}.$$

The rest of the Lemma follows from Lemma 4.7. $\qquad\square$

Proof of Prop. 4.9 . Recall that $f_0(z) = f(z + \sqrt{2|n|/h})$ also depends on h. By differentiating (4.29) with respect to h, we get

$$\partial_h \tilde{\mathrm{Tr}}(f(\mathscr{H}_{c,n}^\theta)) = \frac{i}{4\pi^2 h^2 |E|} \int_{\mathbb{C}} \int_E \partial_{\bar{z}} \tilde{f}_0(z) \, \mathrm{Tr}_{\mathbb{C}^2}(r_n) \, dx_2 \, d\xi_2 \, dz \wedge d\bar{z}$$

$$+ \frac{i\sqrt{2|n|/h}}{8\pi^2 h^2 |E|} \int_{\mathbb{C}} \int_E \partial_{\bar{z}} \tilde{f}'_0(z + \sqrt{2n/h}) \, \mathrm{Tr}_{\mathbb{C}^2}(r_n) \, dx_2 \, d\xi_2 \, dz \wedge d\bar{z}$$

$$- \frac{i}{4\pi^2 h |E|} \int_{\mathbb{C}} \int_E \partial_{\bar{z}} \tilde{f}_0(z) \, \mathrm{Tr}_{\mathbb{C}^2}(\partial_h r_n) \, dx_2 \, d\xi_2 \, dz \wedge d\bar{z} := -B_1 - B_2 - B_3.$$

where the asymptotic expansion of $B_1 = \frac{1}{h}\tilde{\mathrm{Tr}}(f(\mathscr{H}^\theta))$ and $B_2 = \sqrt{\frac{|n|}{2h^3}}\tilde{\mathrm{Tr}}(f'(\mathscr{H}^\theta))$ are known by (4.32). While B_3 can be computed by splitting the integral as in Subsection 4.5:

$$\left[\int_{\mathbb{C}} \partial_{\bar{z}} \tilde{f}_0 \, \mathrm{Tr}_{\mathbb{C}^2}(\partial_h r_n) dz \wedge d\bar{z} \right](x_2, \xi_2; h) = \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{f}_0 \sum_{j=1}^{J-1} h^{\frac{j}{2}-1} \mathrm{Tr}_{\mathbb{C}^2}(q_{n,j}) dz \wedge d\bar{z}$$

$$+ \int_{|\mathrm{Im}\, z| \geq h^\delta} \partial_{\bar{z}} \tilde{f}_0 \, \mathrm{Tr}_{\mathbb{C}^2}(Q_{n,J}) dz \wedge d\bar{z}$$

$$+ \int_{|\mathrm{Im}\, z| \leq h^\delta} \partial_{\bar{z}} \tilde{f}_0 \, \mathrm{Tr}_{\mathbb{C}^2}(Q_{n,J}) dz \wedge d\bar{z} := A'_1 + A'_2 + A'_3,$$

and we imitate the estimates of A_1, A_2, A_3 in the Subsection 4.5 with $\partial_h r_n$ instead of r_n, and we use Lemma 4.10 and 4.11 instead of Lemma 4.6 and 4.7. In short, we need (4.31) and (4.45) for A'_1, (4.44) for A'_2, (4.40) and Lemma 4.11 for A'_3 and we derive that

$$\begin{cases} A'_{1,c} = \pi i f''(\lambda_{n,B})\lambda_n^2 \mathfrak{U}(\eta), & A'_{1,ac} = \frac{\pi i}{\sqrt{h}} \left(s_n^2 f'(\lambda_{n,B} - c_n) + s_n^2 f'(\lambda_{n,B} + c_n) \right), \\ |A'_2| \leq C_{n,K,f,\mathscr{V}} h^{\frac{J}{2}-1-(J+1)\delta}, |A'_3| \leq C_{n,K,f,\mathscr{V}} h^{(K-2)\delta-1}, \end{cases}$$

from which we can find B_3. We summarize B_1, B_2, B_3 below:

For the chiral model where $J = 3$, we have

$$B_{1,c} = \frac{1}{\pi h^2} f(\lambda_{n,B}) + \frac{|n|}{2\pi h} \mathrm{Ave}(\mathfrak{U}) f''(\lambda_{n,B}) + \mathcal{O}_{n,K,f,\mathscr{V}}(h^{-\frac{1}{2}-4\delta} + h^{(K-1)\delta-2}),$$

$$B_{2,c} = \frac{\sqrt{2|n|}}{2\pi h^{\frac{5}{2}}} f'(\lambda_{n,B}) + \frac{(2|n|)^{\frac{3}{2}}}{8\pi h^{\frac{3}{2}}} \mathrm{Ave}(\mathfrak{U}) f'''(\lambda_{n,B}) + \mathcal{O}_{n,K,f,\mathscr{V}}(h^{-1-4\delta} + h^{(K-1)\delta-\frac{5}{2}})$$

$$B_{3,c} = \frac{|n|}{2\pi h} f''(\lambda_{n,B}) \mathrm{Ave}(\mathfrak{U}) + \mathcal{O}_{n,K,f,\mathscr{V}}(h^{-\frac{1}{2}-4\delta} + h^{(K-2)\delta-2}).$$

When $n \neq 0$ and $K > \frac{3}{2\delta} - 3$, we have

$$\partial_h \tilde{\mathrm{Tr}}(f(\mathscr{H}_{c,n})) = -\frac{\sqrt{2|n|}}{2\pi h^{\frac{5}{2}}} f'(\lambda_{n,B}) - \frac{1}{\pi h^2} f(\lambda_{n,B}) - \frac{(2|n|)^{\frac{3}{2}}}{8\pi h^{\frac{3}{2}}} \mathrm{Ave}(\mathfrak{U}) f'''(\lambda_{n,B}) + \mathcal{O}_{n,K,f,\mathscr{V}} h^{-1-4\delta}.$$

When $n = 0$ and $K > \frac{3}{2\delta} - 3$, since $B_2 = 0$, we get a better estimate:

$$\partial_h \tilde{\mathrm{Tr}}(f(\mathscr{H}_{c,0})) = -\frac{1}{\pi h^2} f(0) - O_{0,K,f,\mathscr{V}} h^{-\frac{1}{2}-4\delta}.$$
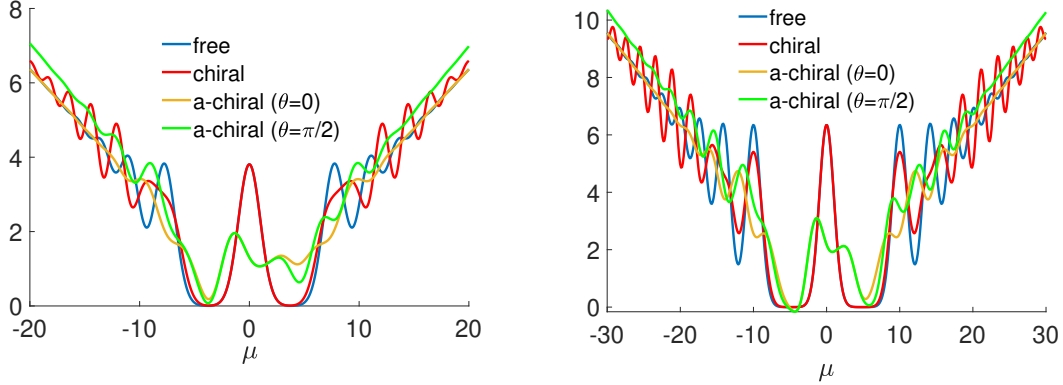
FIGURE 4. SdH oscillations: Smoothed out DOS $\rho(f_\mu)$ with $f_\mu(x) = e^{-\frac{(x-\mu)^2}{2\sigma^2}}/\sqrt{2\pi}\sigma$ illustrating the oscillatory features. On the left, $B = 30$ and on the right $B = 50$ for $\sigma = 1$.

For the anti-chiral model where $J = 2$, we have

$$B_{1,\text{ac}} = \frac{1}{2\pi h^2}t_{n,0}(f) + \frac{1}{2\pi h^{\frac{3}{2}}}t_{n,1}(f) + \mathcal{O}_{n,K,f,\mathscr{V}}(h^{-1-3\delta} + h^{(K-1)\delta-2}),$$

$$B_{2,\text{ac}} = \frac{\sqrt{2|n|}}{4\pi h^{\frac{5}{2}}}t_{n,0}(f') + \frac{\sqrt{2|n|}}{4\pi h^2}t_{n,1}(f') + \mathcal{O}_{n,K,f,\mathscr{V}}(h^{-\frac{3}{2}-3\delta} + h^{(K-1)\delta-\frac{5}{2}}),$$

$$B_{3,\text{ac}} = \frac{1}{4\pi h^{\frac{3}{2}}}t_{n,1}(f) + \mathcal{O}_{n,K,f,\mathscr{V}}(h^{-1-3\delta} + h^{(K-2)\delta-2}).$$

Thus when $n \neq 0$ and $K > \frac{1}{\delta} - 2$, we have

$$\partial_h \tilde{\text{Tr}}(f(\mathscr{H}_{\text{ac}}^\theta)) = -\frac{\sqrt{2|n|}}{4\pi h^{\frac{5}{2}}}t_{n,0}(f') - \frac{1}{4\pi h^2}\left(2t_{n,0}(f) + \sqrt{2|n|}t_{n,1}(f')\right) - \mathcal{O}_{n,K,f,\mathscr{V}}h^{-\frac{3}{2}-3\delta}.$$

If $n = 0$ and $K > \frac{1}{\delta} - 2$, since $B_2 = 0$, we get a better estimate:

$$\partial_h \tilde{\text{Tr}}(f(\mathscr{H}_{\text{ac}}^\theta)) = -\frac{1}{2\pi h^2}t_{0,0}(f) - \frac{3}{4\pi h^{\frac{3}{2}}}t_{0,1}(f) + O_{0,K,f,\mathscr{V}}h^{-1-3\delta}.$$

Recall $h = \frac{1}{B}$. By $\partial_B = -\frac{1}{B^2}\partial_h$, we get the results (4.34), (4.35) and (4.36). □

5. MAGNETIC RESPONSE QUANTITIES

This section discusses applications of the regularized trace expansions derived in the previous section, cf. Theorems 1 and 2 as well as Proposition 4.9. They form the rigorous foundation of our analysis in this section and we shall focus on qualitative features rather here, instead.

Our main contribution on magnetic response properties of TBG is a careful analysis of the oscillatory behaviour of the DOS. While this effect can be easily explained using the Poisson summation formula, we shall illustrate this phenomenon, by considering a Gaussian density $f_\mu(x) = e^{-\frac{(x-\mu)^2}{2\sigma^2}}/\sqrt{2\pi}\sigma$ and analyze the Shubnikov–de Haas (SdH) oscillations in a
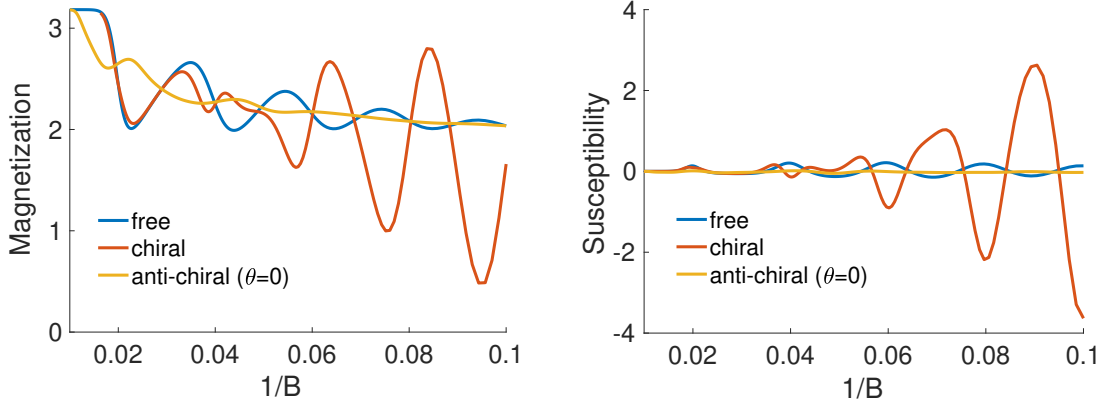
FIGURE 5. Magnetization and susceptibility for $\beta = 4$, $\alpha_i = 3/5$, and chemical potentials $\mu = 5$ (left) and $\mu = 10$ (right).

smoothed-out version of the DOS $\mu \mapsto \rho(f_\mu)$ in Figure 4 for $\sigma = 1$ using the asymptotic formulae of Theorems 1 and 2. As a general rule from our study, we find that the AB/BA interaction leads to an enhancement of this oscillatory behaviour compared to the non-interacting case, while the AA'/BB' interaction damps oscillations. The smoothing effect of the AA'/BB' interaction is due to a splitting and broadening of the highly degenerate Landau levels. This splitting has also consequences for the Quantum Hall effect, see Fig. 10. We also study the de Haas–van Alphen (dHvA) effect in TBG, see Fig. 5 and 8 for which we find a similar phenomenon.

We study magnetic response quantities by thoroughly analyzing the following cases:

- The free or non-interacting case, corresponds to two non-interacting sheets of graphene modeled by the direct sum of two magnetic Dirac operators, see also [BZ19, BHJZ21] for similar results in a quantum graph model and [SGB04] for a thorough analysis of the magnetic Dirac operator, directly.
- The chiral case, which corresponds to pure AB/BA interaction.
- The anti-chiral case, which corresponds to pure AA'/BB' interaction.

For our analysis of the de Haas-van Alphen effect, we shall employ a cut-off function $\eta_N \in C_c^\infty(\mathbb{R})$ that is one on the interval $[0, \sqrt{2BN}]$ and smoothly decays to zero outside of that interval, enclosing precisely $N + 1$ Landau levels and η_N^{sym} which is equal to one on $[-\sqrt{2BN}, \sqrt{2BN}]$. The choice of cut-off function mainly plays the role of a reference frame. In particular, for the study of magnetic oscillations it seems more natural to consider η_N instead of η_N^{sym} as the former cut-off function singles out the effect of individual Landau levels moving past a fixed chemical potential μ. We shall employ the leading order terms for the regularized trace in this section, as specified in Theorems 1 and 2 and Proposition 4.9. For this reason, we write functionals $\rho(f)$, where $f \in C^\infty(\mathbb{R})$, as $\rho(f) \sim g$, to indicate that g are the first terms in the asymptotic expansion of $\rho(f)$ and analogously for derivatives of $\rho(f)$ with respect to the magnetic field.
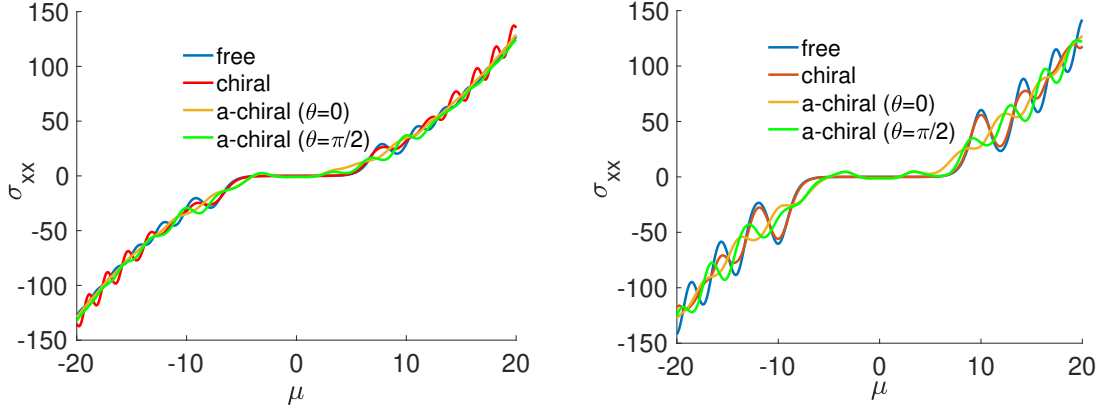
FIGURE 6. Smoothed out longitudinal conductivity $\sigma_{xx} \propto -\rho(\lambda n'_\beta(\lambda - \mu))$ with n_β, the Fermi-Dirac distribution, showing Shubnikov-de Haas oscillations. On the left, $B = 30$ and on the right $B = 50$ for $\beta = 1.5$. with $\alpha_i = \frac{3}{5}$.

5.1. **Shubnikov-de Haas oscillations.** We shall start by discussing *Shubnikov - de Haas (SdH)* oscillations in the density of states. A common method of measuring SdH oscillations is by measuring longitudinal conductivity and resistivity, see also [W11, Tan11]. In the following, let $\sigma \in \mathbb{R}^{2 \times 2}$ be the conductivity matrix, such that the current density $j = \sigma E$, where E is an external electric field, then the resistivity matrix is just $\rho = \sigma^{-1}$. Hence, we shall focus on conductivities in the sequel.

The SdH oscillations are most strongly pronounced at low temperatures in the regime of strong magnetic fields and describe oscillations in the longitudinal conductivity σ_{xx} of the material.

The expression for the longitudinal conductivity goes back to Ando et al [A70, A82] who derived the following relation, see also [FS14],

$$\sigma_{xx}(\beta, \mu, B) = -\int_0^\infty n'_\beta(\lambda - \mu)\lambda \eta_N^{\text{sym}}(\lambda) \, d\rho(\lambda),$$

where $n_\beta(x) = \frac{1}{e^{\beta x}+1}$ is the Fermi-Dirac statistics. In the free case, i.e. without any tunnelling potential, the oscillations happen precisely at the relativistic Landau levels. For the chiral model, oscillations caused by higher Landau levels are enhanced compared to the free case, whereas oscillations in the anti-chiral case are much more smoothed out.

The oscillatory behaviour of the longitudinal conductivity is visible both as a function of chemical potential, for a fixed magnetic field strength, as shown in Fig. 6 as well as function of inverse magnetic field in Fig. 7 for fixed chemical potential.

5.2. **De Haas-van Alphen oscillations.** In 1930, de Haas and van Alphen who discovered that both the magnetization and the magnetic susceptibility of metals show an oscillatory profile as a function of B^{-1}. This effect is called the de Haas-van Alphen (dHvA) effect. Even in the simpler case of graphene, both the experimental as well as theoretical foundations of that effect are not yet well-understood [L11, KH14, SGB04]. One problem in understanding
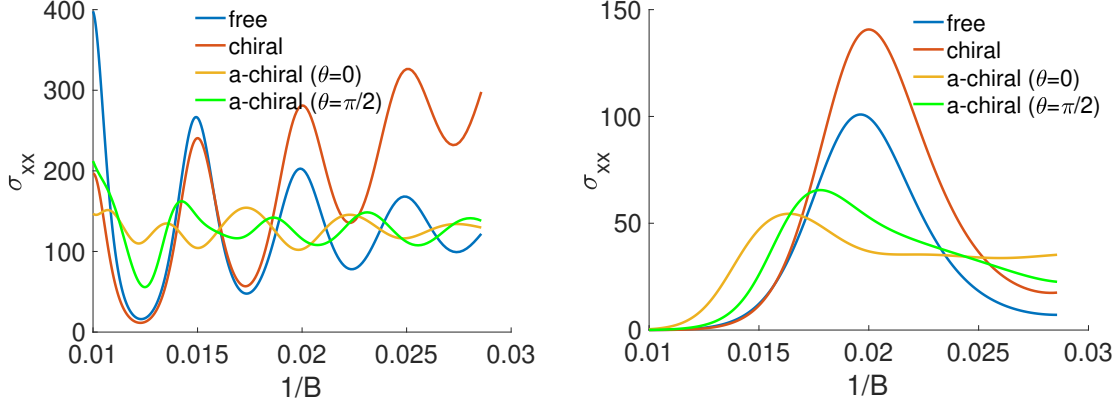
FIGURE 7. Smoothed out longitudinal conductivity $\sigma_{xx} \propto -\rho(\lambda n'_\beta(\lambda - \mu))$ with n_β, the Fermi-Dirac distribution, showing Shubnikov-de Haas oscillations. On the left, $B = 30$ and on the right $B = 50$, both for $\beta = 2.5$. with $\alpha_i = 0.35$.

the dHvA effect [SGB04], lies in the dependence of the chemical potential on the external magnetic field. To simplify mathematical analysis, it is more convenient to work in the grand-canonical ensemble, which is also discussed in [CM01, SGB04, KF17]. The comparison with the canonical ensemble is made in this subsection as well.

The grand thermodynamic potential for a DOS measure ρ, at inverse temperature β, and field-independent chemical potential μ is defined as

$$\Omega_\beta(\mu, B) := (f_\beta * (\eta_N \rho))(\mu),$$

where $f_\beta(x) := -\beta^{-1} \log(e^{\beta x} + 1)$. The magnetization M and susceptibility χ are then in the grand-canonical ensemble defined as

$$M(\beta, \mu, B) = -\frac{\partial \Omega_\beta(\mu, B)}{\partial B} \text{ and } \chi(\beta, \mu, B) = \frac{\partial M_\beta(\mu, B)}{\partial B}.$$

The susceptibility describes the response of a material to an external magnetic field. When $\chi > 0$ the material is paramagnetic, when $\chi < 0$ diamagnetic, and strongly enhanced $\chi \gg 1$ for ferromagnets.

While the approximation of computing the magnetization in the grand canonical ensemble is common, one should strictly speaking compute it in the canonical ensemble, instead.

In this case, the charge density ϱ given by the Fermi-Dirac statistics, with $n_\beta(x) := \frac{1}{e^{\beta x} + 1}$, according to

$$\varrho = -\frac{\Omega_\beta(\mu, B)}{\partial \mu} = \rho(n_\beta(\cdot - \mu))$$

is fixed and the chemical potential becomes a function of ρ and B.

To see that this uniquely defines μ as a function of ϱ and B large enough, it is sufficient to observe that

$$\mu \mapsto \sum_{n \in \mathbb{Z}} (\eta_N n_\mu)(\lambda_n \sqrt{B})$$

is a monotonically increasing function. The Helmholtz free energy is then given as
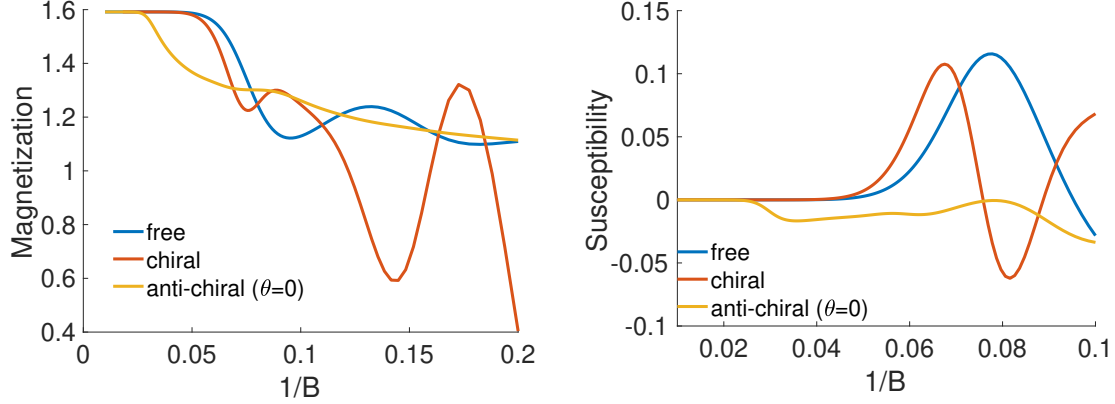
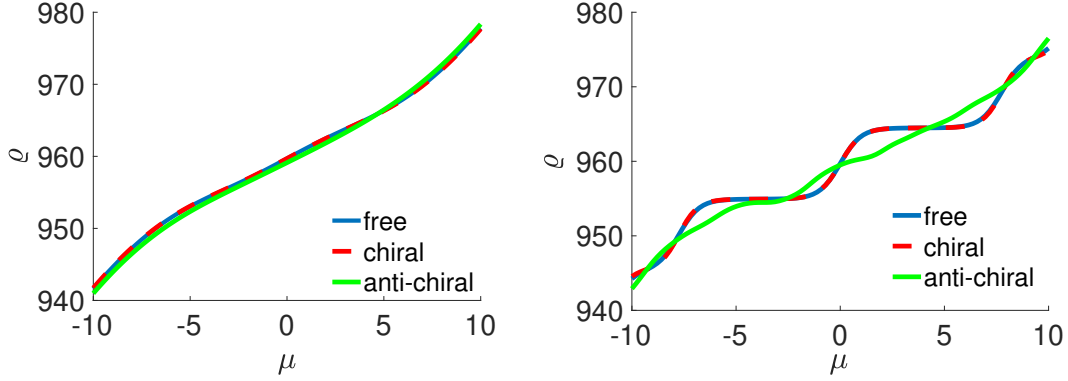FIGURE 8. Magnetization and susceptibility for $\beta = 4$, $\alpha_i = 3/5$, and chemical potential $\mu = 5$.

FIGURE 9. Charge density with respect to chemical potential. Magnetic field $B = 30$ for $\beta = 1/2$ and $\beta = 2$. We consider 100 Landau levels around zero and an anti-chiral model with $\theta = 0$.

$$F_\beta(\varrho, B) = \Omega_\beta(\mu(\varrho, B), B) + \mu(\rho, B)\varrho$$

with the magnetization given as the derivative $M(\beta, \varrho, B) = -\frac{\partial F_\beta(\varrho, B)}{\partial B}$. Hence, the magnetization in the canonical ensemble is also given by

$$M(\beta, \varrho, B) = -\frac{\partial \Omega_\beta(\mu, B)}{\partial B}\bigg|_{\mu = \mu(\varrho, B)},$$

where the difference to the grand-canonical ensemble lies in the B-dependent chemical potential. The dHvA oscillations are shown in Figures 5 and 8, with the AB'/BA' interaction leading to enhanced oscillations and the AA'/BB' interaction damping the oscillations, compared to the non-interacting case.
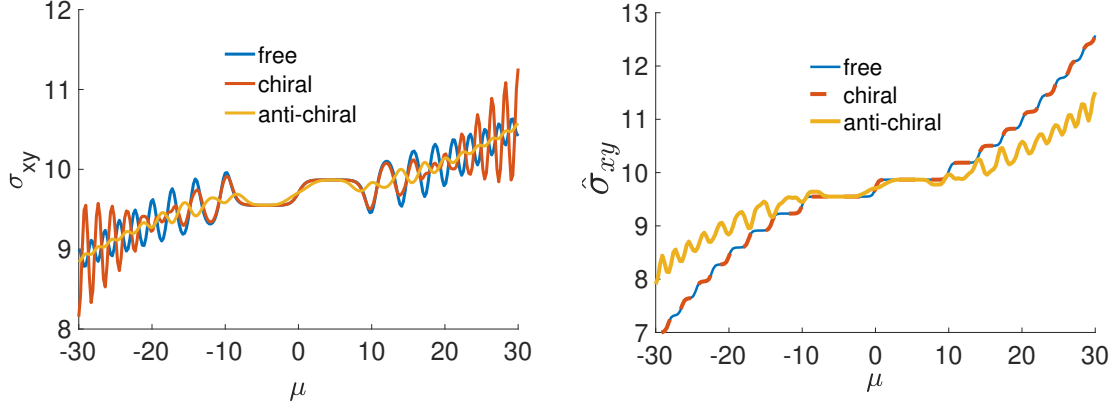
FIGURE 10. Full quantum Hall conductivity (5.1) on the left with $\beta = 2$, $B = 40$ and on the right the high temperature conductivity (5.2) with $\beta = 5, B = 50$.

5.3. Quantum Hall effect.

The (transversal) quantum Hall conductivity σ_{xy} is, by the Streda formula [MK12, (16)], for a Fermi energy μ given by

$$\sigma_{xy}(\beta, \mu, B) = \sum_{n=-N}^{N} \frac{\partial \rho(\eta_N n_\beta(\bullet - \mu))}{\partial B}. \tag{5.1}$$

In case of the chiral Hamiltonian, the Gibbs factor $\gamma_{\beta,n}(\mu) = e^{\beta(\lambda_n \sqrt{B} - \mu)}$ allows us to write

$$\sigma_{xy,c}(\beta, \mu, B) = \left(\sum_{n=-N}^{N} \frac{n_\beta(\lambda_n \sqrt{B} - \mu)}{\pi} \left(1 - \frac{\beta \lambda_n \sqrt{B}}{2} \gamma_{\beta,n}(\mu) n_\beta(\lambda_n \sqrt{B} - \mu) \right) \right.$$
$$+ \sum_{n=-N}^{N} -\frac{\lambda_n |\lambda_n|^2 \beta^3 \operatorname{Ave}(\mathfrak{U})}{4\pi \sqrt{B}} n_\beta^4(\lambda_n \sqrt{B} - \mu) \left(\gamma_{\beta,n}(\mu) - 4\gamma_{\beta,n}(\mu)^2 + \gamma_{\beta,n}(\mu)^3 \right) \right) (1 + o(1))$$

At very low temperatures, and μ well between two Landau levels, the contribution of the derivative of the Landau levels with respect to B can be discarded.

We then obtain the high-temperature limiting expression

$$\hat{\sigma}_{xy,c}(\beta, \mu, B) := \sum_{n=-N}^{N} \frac{\eta_\beta(\lambda_{n,B} - \mu)}{\pi} \xrightarrow[\beta \to \infty]{} |\{n; |\lambda_{n,B}| \leq \mu\}| \tag{5.2}$$

as $n_\beta(\lambda_{n,B} - \mu) \to 1 - H(\lambda_n \sqrt{B} - \mu)$ for $\beta \uparrow \infty$, where H is the Heaviside function.

This expression reveals the well-known staircase profile of the Hall conductivity which can already be concluded in this model in the $\beta \to \infty$ limit from Proposition 3.2.

For the AA'/BB' interaction, the situation is rather different. Due to the broadening and splitting of the Landau levels, the staircase profile is less pronounced at non-zero temperature. Setting $\hat{\sigma}_{xy,ac}(\beta, \mu, B) := t_{n,0}(n_\beta(\bullet - \mu)) - \frac{t_{n,1}(n_\beta(\bullet - \mu))}{2\sqrt{B}}$, where in the limit $\beta \to \infty$, the second term vanishes, for μ away from the spectrum as n'_β is a δ_0 approximating sequence such that also in case of the AA'/BB' interaction $\lim_{\beta \to \infty} \hat{\sigma}_{xy}(\beta, \mu, B) = |\{n; |\lambda_{n,B}| \leq \mu\}|$.

Appendix A. Asymptotic expansion

In this appendix, we shall prove Prop. A.1 which, in particular, includes the proof of Lemma 4.7. The quantization is as in Subsection 4.3.

Proposition A.1. *Let h_0, $E_{n,\pm}$ be as in Lemma 4.5. For $h \in [0, h_0)$, $|z| \leq 2\|\mathscr{W}\|_\infty$, we have*

(1) *The symbol $\frac{1}{\sqrt{h}} E_{n,\pm}$ has an asymptotic expansion in S: There are $a_{n,j,k} \in S$ such that*

$$\frac{1}{\sqrt{h}} E_{n,\pm}(x_2, \xi_2; z, h) \sim \sum_{j=0}^{\infty} h^{\frac{j}{2}} E_{n,j}(x_2, \xi_2; z) \text{ with } E_{n,j} = \sum_{k=0}^{j-1} a_{n,j,k}(x_2, \xi_2) z^k, j \geq 1. \quad \text{(A.1)}$$

In particular, $E_{n,0} = z - z_{n,0}$, $E_{n,1} = -z_{n,1}$, $E_{n,2} = -z_{n,2}$, where $z_{n,j}$ are given in Lemma A.2.

(2) *Let $0 < \delta < 1/2$, if $|\operatorname{Im} z| \geq h^\delta$, then $\sqrt{h} E_{n,\pm}^{-1}$ has an asymptotic expansions in S_δ^δ: There are $b_{n,j,k,l}, c_{n,j,k} \in S$ such that in terms of $\prod_{l=0}^{k} b_{n,j,k,l}(x_2, \xi_2; z) = \sum_{\alpha=0}^{j+k-2} z^\alpha c_{n,j,k}(x_2, \xi_2)$ the expansion of*

$$\sqrt{h} E_{n,\pm}^{-1} \sim \sum_{j=0}^{\infty} h^{\frac{j}{2}} F_{n,j}(x_2, \xi_2; z), \text{ with } F_{n,j} = \sum_{k=0}^{j}(z - z_{n,0})^{-1} \prod_{l=0}^{k} \left(b_{n,j,k,l}(x_2, \xi_2; z)(z - z_{n,0})^{-1} \right). \quad \text{(A.2)}$$

Thus $h^{\frac{j}{2}} F_{n,j} \in S^{j(\delta - \frac{1}{2}) + \delta}$. In particular, we have

$$F_{n,0} = (z - z_{n,0})^{-1}, \ F_{n,1} = F_{n,0} z_{n,1} F_{n,0}, \ F_{n,2} = F_{n,0} \left(z_{n,1} F_{n,1} + z_{n,2} F_{n,0} - \frac{\{F_{n,0}, z - z_{n,0}\}}{2i} \right), \quad \text{(A.3)}$$

where $\{\cdot, \cdot\}$ is the Poisson bracket.

(3) *Let $0 < \delta < 1/2$, if $|\operatorname{Im} z| \geq h^\delta$, then r_n has an asymptotic expansions in S_δ^δ: There are $d_{n,j,k,l}(x_2, \xi_2; z), e_{n,j,k,\alpha}(x_2, \xi_2) \in S$, such that in terms of $\prod_{l=0}^{k} d_{n,j,k,l}(x_2, \xi_2; z) = \sum_{\alpha=0}^{j+k-2} z^\alpha e_{n,j,k,\alpha}(x_2, \xi_2)$*

$$r_n(x_2, \xi_2; z, h) \sim \sum_{j=0}^{\infty} h^{\frac{j}{2}} r_{n,j}(x_2, \xi_2; z, h), \text{ with } r_{n,j} = \sum_{k=0}^{j}(z - z_{n,0})^{-1} \prod_{l=0}^{k} \left(d_{n,j,k,l}(x_2, \xi_2; z)(z - z_{n,0})^{-1} \right).$$

Thus $h^{\frac{j}{2}} r_{n,j} \in S_0^{(J+1)\delta - \frac{J}{2}}$. In particular,

$$r_{n,0} = F_{n,0}, \quad r_{n,1} = F_{n,1}, \quad r_{n,2} = F_{n,2} - (\partial_z z_{n,2}) F_{n,0}.$$

(4) *Finally, let $\eta = x_2 + i\xi_2$, then the leading terms of $\mathrm{Tr}_{\mathbb{C}^2}(r_n)$ are:*

$$\text{Chiral } \mathscr{H}_{c,n} : \mathrm{Tr}_{\mathbb{C}^2}(r_{c,n,0} + h^{\frac{1}{2}}r_{c,n,1} + hr_{c,n,2}) = \frac{2}{z} + 0 + \frac{\lambda_n^2}{z^3}\mathfrak{U}(\eta)h,$$

$$\text{Anti-Chiral } \mathscr{H}_{ac,n}^\theta : \mathrm{Tr}_{\mathbb{C}^2}(r_{ac,n,0} + h^{\frac{1}{2}}r_{ac,n,1}) = \frac{2z}{z^2 - c_n^2} + \frac{2s_n^2(z^2 + c_n^2)}{(z^2 - c_n^2)^2}\sqrt{h},$$

where $\mathfrak{U}(\eta) = \frac{\alpha_1^2}{8}\left[\alpha_1^2(|U_-(\eta)|^2 - |U(\eta)|^2)^2 + 4|\partial_{\bar\eta}\overline{U_-(\eta)} - \partial_\eta U(\eta)|^2\right]$, $\partial_\eta = \frac{1}{2}(\partial_{x_2} - i\partial_{\xi_2})$, $s_n(\eta) = \begin{cases} \alpha_0\sin(\frac{\theta}{2})|V(\eta)| & n \neq 0 \\ \alpha_0|V(\eta)| & n = 0, \end{cases}$ *and* $c_n(\eta) = \begin{cases} \alpha_0\cos(\frac{\theta}{2})|V(\eta)| & n \neq 0 \\ \alpha_0|V(\eta)| & n = 0. \end{cases}$

We will prove Proposition A.1 in the rest of this appendix in two steps: First, we compute explicitly the leading terms (three terms for the chiral model, two for anti-chiral model) in the expansion of $Z_n(x_2, \xi_2; z, h)$, the symbol of Z_n^W, where $E_{n,\pm} = \sqrt{h}(z - Z_n^W)$ by (4.17). Then, we exhibit the z dependence for each term in the expansion of $E_{n,\pm}$, from which we build up both the legitimacy of the existence of asymptotic expansions of $E_{n,\pm}^{-1}$ and r_n, and the z dependence of each term in the expansions.

Explicit leading terms. Recall that by (4.17) and (4.20), $E_{n,\pm} = \sqrt{h}(z - Z_n^W)$ with

$$
\begin{aligned}
Z_n^W(x_2, hD_{x_2}; h) &= R_n^+ \tilde{\mathscr{V}}^W(I + \sqrt{h}E_{0,n}^\theta \tilde{\mathscr{V}}^W)^{-1}R_n^- \\
&= \sum_{k=0}^\infty h^{\frac{k}{2}}(-1)^k R_n^+ \tilde{\mathscr{V}}^W(E_{0,n}^\theta \tilde{\mathscr{V}}^W)^k R_n^- =: \sum_{k=0}^\infty h^{\frac{k}{2}} Q_{n,k}^W(x_2, hD_{x_2}; h),
\end{aligned}
\tag{A.4}
$$

where $R_n^\pm$, $E_{0,n}^\theta$, $\tilde{\mathscr{V}}^W$ are given in (4.12), (4.15) and (4.20). Then we can express the asymptotic expansion of $Z_n(x_2, \xi_2)$ in terms of $Q_{n,k}(x_2, \xi_2)$:

Proposition A.2. *Let $Q_{n,k}^W(x_2, hD_{x_2}; h) = (-1)^k R_n^+ \tilde{\mathscr{V}}^W(E_{0,n}^\theta \tilde{\mathscr{V}}^W)^k R_n^-$. Then symbols $Q_{n,0}$, $Q_{n,1}$, $Q_{n,2}$ have the following asymptotic expansions*

$$Q_{n,0}(x_2, \xi_2; h) = Q_{n,0}^{(0)}(x_2, \xi_2) + \sqrt{h}Q_{n,0}^{(1)}(x_2, \xi_2) + hQ_{n,0}^{(2)}(x_2, \xi_2) + \mathcal{O}_S(h^{\frac{3}{2}}),$$

$$Q_{n,1}(x_2, \xi_2; h) = Q_{n,1}^{(0)}(x_2, \xi_2) + \sqrt{h}Q_{n,1}^{(1)}(x_2, \xi_2) + \mathcal{O}_S(h),$$

$$Q_{n,2}(x_2, \xi_2; h) = Q_{n,2}^{(0)}(x_2, \xi_2) + \mathcal{O}_S(\sqrt{h}).$$

For the chiral Hamiltonian, with $\eta = x_2 + i\xi_2$, $D_\eta = \frac{1}{2}(D_{x_2} - iD_{\xi_2})$,

$$Q_{c,n,0}^{(0)} = Q_{c,n,0}^{(2)} = Q_{c,n,2}^{(0)} = 0, \quad Q_{c,n,1}^{(0)} = -\frac{\alpha_1^2\lambda_n}{4}\left[|U|^2 - |U_-|^2\right]\sigma_3,$$

$$Q_{c,n,0}^{(1)} = \frac{\lambda_n\alpha_1}{2}\begin{pmatrix} 0 & D_\eta U - D_{\bar\eta}\overline{U_-} \\ D_\eta U_- - D_{\bar\eta}\overline{U} & 0 \end{pmatrix},$$

$$Q_{c,n,1}^{(1)} = \begin{cases} -\frac{\alpha_1^2 z}{4}\left[2|n|(|U|^2 + |U_-|^2)\,\mathbb{1}_{2\times 2} + (|U|^2 - |U_-|^2)\sigma_3\right] & n \neq 0, \\ -\frac{\alpha_1^2 z}{2}\begin{pmatrix} |U|^2 & 0 \\ 0 & |U_-|^2 \end{pmatrix} & n = 0. \end{cases}$$

Analogously, for the anti-chiral Hamiltonian, when $\mathscr{H}^\theta = \mathscr{H}_{\mathrm{ac}}$, we have, when $n = 0$,

$$Q^{(1)}_{\mathrm{ac},0,0} = Q^{(0)}_{\mathrm{ac},0,1} = Q^{(1)}_{\mathrm{ac},0,1} = Q^{(0)}_{\mathrm{ac},0,2} = 0,$$

$$Q^{(0)}_{\mathrm{ac},0,0} = \alpha_0 \begin{pmatrix} 0 & e^{-\frac{\theta}{2}i}V \\ e^{\frac{\theta}{2}}V^* & 0 \end{pmatrix}, Q^{(2)}_{\mathrm{ac},0,0} = \frac{\alpha_0}{4} \begin{pmatrix} 0 & e^{-\frac{\theta}{2}i}\Delta_{x_2,\xi_2}V \\ e^{\frac{\theta}{2}i}\Delta_{x_2,\xi_2}\bar{V} & 0 \end{pmatrix},$$

when $n \neq 0$,

$$Q^{(1)}_{\mathrm{ac},n,0} = 0, \ Q^{(0)}_{\mathrm{ac},n,1} = \frac{\alpha_0^2 |V|^2 \sin^2(\frac{\theta}{2})}{2\lambda_n} \mathbb{1}_{2\times2}, \ Q^{(1)}_{\mathrm{ac},n,1} = -\frac{z\alpha_0^2 |V|^2 \sin^2(\frac{\theta}{2})}{4\lambda_n^2} \mathbb{1}_{2\times2},$$

$$Q^{(0)}_{\mathrm{ac},n,0} = \alpha_0 \cos(\tfrac{\theta}{2}) \begin{pmatrix} 0 & V^* \\ V & 0 \end{pmatrix}, Q^{(0)}_{\mathrm{ac},n,2} = -\frac{\alpha_0^3 |V|^2 \sin^2(\frac{\theta}{2})\cos(\frac{\theta}{2})}{4\lambda_n^2} \begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix},$$

$$Q^{(2)}_{\mathrm{ac},n,0} = \frac{\alpha_0}{4} \left(2|n|\cos(\tfrac{\theta}{2}) - i\sigma_3 \sin(\tfrac{\theta}{2})\right) \begin{pmatrix} 0 & \Delta_{x_2,\xi_2}V \\ \Delta_{x_2,\xi_2}\bar{V} & 0 \end{pmatrix}.$$

In particular, Z_n has an asymptotic expansion $Z_n \sim \sum_{k=0}^{\infty} h^{\frac{k}{2}} z_{n,k}$ in S with

$$z_{n,0} = Q^{(0)}_{n,0}, \ z_{n,1} = Q^{(0)}_{n,1} + Q^{(1)}_{n,0}, \ z_{n,2} = Q^{(0)}_{n,2} + Q^{(1)}_{n,1} + Q^{(2)}_{n,0}.$$

Proof. $Q_{n,k}$ has the symbol $Q_{n,k}(x_2, \xi_2) = (-1)^k \int_{\mathbb{R}_{x_1}} (K_n^\theta(x_1))^* \tilde{\mathscr{V}}^w \#(E_{0,n}^\theta \tilde{\mathscr{V}}^w)^{\#k} K_n^\theta(x_1) dx_1$. Recall that by (2.1), (4.13), and (4.15), we have

$$K_n^\theta = \begin{pmatrix} u_n^\theta & 0 \\ 0 & u_n^{-\theta} \end{pmatrix}, \quad \mathscr{V} = \begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix}, \quad T = \begin{pmatrix} \alpha_0 V & \alpha_1 \overline{U_-} \\ \alpha_1 U & \alpha_0 V \end{pmatrix}, \quad E_{0,n}^\theta = \begin{pmatrix} e_{0,n}^\theta & 0 \\ 0 & e_{0,n}^{-\theta} \end{pmatrix}.$$

Thus, inserting the above expressions into the definition of $Q_{n,k}$, we find for its symbol

$$Q_{n,k} = \int \begin{pmatrix} u_n^{\theta*} & 0 \\ 0 & (u_n^{-\theta})^* \end{pmatrix} \begin{pmatrix} 0 & \tilde{T}^w \\ (\tilde{T}^w)^* & 0 \end{pmatrix} \left(\begin{pmatrix} e_{0,n}^\theta & 0 \\ 0 & e_{0,n}^{-\theta} \end{pmatrix} \begin{pmatrix} 0 & \tilde{T}^w \\ (\tilde{T}^w)^* & 0 \end{pmatrix} \right)^k \begin{pmatrix} u_n^{\theta*} & 0 \\ 0 & u_n^{-\theta} \end{pmatrix} \frac{dx_1}{(-1)^k},$$

where $\tilde{T}^w = T^w(x_2 + h^{\frac{1}{2}}x_1, \xi_2 - h^{\frac{1}{2}}D_{x_1})$. In particular,

$$Q_{n,0} = \begin{pmatrix} 0 & \int (u_n^\theta)^* \tilde{T}^w u_n^{-\theta} dx_1 \\ \int (u_n^{-\theta})^* (\tilde{T}^w)^* u_n^\theta dx_1 & \end{pmatrix},$$

$$Q_{n,1} = \begin{pmatrix} -\int (u_n^\theta)^* \tilde{T}^w e_{0,n}^{-\theta} (\tilde{T}^w)^* u_n^\theta dx_1 & 0 \\ 0 & -\int (u_n^{-\theta})^* (\tilde{T}^w)^* e_{0,n}^\theta \tilde{T}^w u_n^{-\theta} dx_1 \end{pmatrix}, \quad \text{and} \qquad (\text{A.5})$$

$$Q_{n,2} = \begin{pmatrix} 0 & \int (u_n^\theta)^* \tilde{T}^w e_{0,n}^{-\theta} (\tilde{T}^w)^* e_{0,n}^\theta \tilde{T}^w u_n^{-\theta} dx_1 \\ \int (u_n^{-\theta})^* (\tilde{T}^w)^* e_{0,n}^\theta \tilde{T}^w e_{0,n}^{-\theta} (\tilde{T}^w)^* u_n^\theta dx_1 & 0 \end{pmatrix}.$$

Notice that since both $\tilde{T}^w$ and $e_{0,n}^\theta$ depend on h, we need to further expand them in order to obtain asymptotic expansions of $Q_{n,k}$. Thus the proof of Proposition A.2 rests now on the following two lemmas.

Lemma A.3 (Expansion of $\tilde{T}^w$ and $e_{0,n}^\theta$).

(1) Let $T \in C_b^\infty(\mathbb{R}_x^2)$. Recall the definition $\tilde{T}(x,\xi) := T(x_2 + h^{\frac{1}{2}}x_1, \xi_2 - h^{\frac{1}{2}}\xi_1) \in S(\mathbb{R}_{x,\xi}^4)$. Then

$$\tilde{T}^w(x, D_{x_1}, \xi_2) = T(x_2, \xi_2) + \sqrt{h}\langle \nabla_{x_2,\xi_2} T(x_2,\xi_2), (x_1, -D_{x_1})\rangle$$
$$+ \frac{h}{2}\langle (x_1, -D_{x_1}), \text{Hess } T(x_2,\xi_2)(x_1, -D_{x_1})^T\rangle + \mathcal{O}_{S(\mathbb{R}_{x_2,\xi_2}^2; \mathcal{L}(B_{x_1}^3; B_{x_1}^0))}(h^{\frac{3}{2}})$$

(2) Let $e_{0,n}^\theta$ be as in (4.15). Then $e_{0,n}^\theta(x, D_{x_1}, \xi_2)$ has an asymptotic expansion $e_{0,n}^\theta \sim \sum_{k=0}^\infty h^{\frac{k}{2}} \sigma_k(e_{0,n}^\theta)$ where $\sigma_k(e_{0,n}^\theta) = \sum_{m \neq n} \frac{z^k u_m^\theta (u_m^\theta)^*}{(\lambda_m - \lambda_n)^{k+1}}$.

Lemma A.4 (Projections). Let $S_n^\theta = \text{span}\{u_n^\theta, u_{-n}^\theta\}$ with $S_n := S_n^0$ and $u_n := u_n^0$. The following properties hold:

(1) Reflection invariance with respect to θ such that $S_n^\theta = S_n^{-\theta}$, in particular $u_n^\theta = \cos\left(\frac{\theta}{2}\right) u_n^{-\theta} + i\sin\left(\frac{\theta}{2}\right) u_{-n}^{-\theta}$.

(2) Let $M = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} \in \mathbb{C}^{2\times 2}$ then $Mu_n \in S_{n-1} \cup S_{n+1}$, for any $n \geq 0$. More specifically for $\theta = 0$

$$Mu_{\pm n} = \frac{\alpha i}{2}(u_{n+1} - u_{-(n+1)}) \mp \frac{\beta i}{2}(u_{n-1} + u_{-(n-1)}), \text{ for } n \geq 2$$
$$Mu_{\pm 1} = \frac{\alpha i}{2}(u_2 - u_{-2}) \pm \frac{\beta}{\sqrt{2}}u_0, \text{ and } Mu_0 = \frac{\alpha}{\sqrt{2}}(u_1 - u_{-1}).$$

(3) We have $x_1 u_n^\theta \in S_{n-1}^\theta \cup S_{n+1}^\theta$, $D_{x_1} u_n^\theta \in S_{n-1}^\theta \cup S_{n+1}^\theta$. More specifically

$$x_1 u_{\pm n}^\theta = \frac{\sqrt{2}}{4}[u_{n-1}^\theta(\sqrt{n} \pm +\sqrt{n-1}) + u_{-(n-1)}^\theta(\sqrt{n} \mp \sqrt{n-1})$$
$$+ u_{n+1}^\theta(\sqrt{n+1} + \sqrt{n}) \pm u_{-(n+1)}^\theta(\sqrt{n+1} \mp \sqrt{n})], \text{ for } |n| \geq 2$$
$$x_1 u_{\pm 1}^\theta = \frac{i}{2}u_0^\theta + \frac{\sqrt{2}}{4}[u_2^\theta(\sqrt{2} \pm \sqrt{1}) + u_{-2}^\theta(\sqrt{2} \mp \sqrt{1})] \text{ and } x_1 u_0^\theta = \frac{\sqrt{2}i}{4}(u_1^\theta + u_{-1}^\theta).$$

Proof. We omit the proof of this Lemma here as it follows from straightforward but lengthy basis expansions and the simple observation that $\langle u_m^{-\theta}, u_n^\theta\rangle = \cos\left(\frac{\theta}{2}\right)\delta_{m,n} + i\sin\left(\frac{\theta}{2}\right)\delta_{m,-n}$. $\square$

From the preceding Lemmas A.3 and A.4, we can compute the asymptotic expansion of each term of $Q_{n,k}$ in (A.5) and therefore prove Prop.A.2.

For the $(1,2)$-entry of $Q_{n,0}$, by Lemma A.3, we have

$$\int (u_n^\theta)^* \tilde{T}^w u_n^{-\theta} dx_1 = \int (u_n^\theta)^* T u_n^{-\theta} dx_1 + \sqrt{h}\int (u_n^\theta)^* \langle \nabla_{x_2,\xi_2} T, (x_1, -D_{x_1})\rangle u_n^{-\theta} dx_1$$
$$+ \frac{h}{2}\int (u_n^\theta)^* \langle (x_1, -D_{x_1}), \text{Hess } T(x_2, \xi_2)(x_1, -D_{x_1})^T\rangle u_n^{-\theta} dx_1$$
$$=: t_{n,0}^{(0)} + \sqrt{h}t_{n,0}^{(1)} + ht_{n,0}^{(2)} + \mathcal{O}_{S(\mathbb{R}_{x_2,\xi_2}^2; \mathbb{C}_{2\times 2})}(h^{\frac{3}{2}}).$$

Specializing now to the chiral case, in which case the θ-dependence can be gauged away, we choose

$$T(x_2, \xi_2) = \begin{pmatrix} 0 & \alpha_1 \overline{U(x_2, \xi_2)} \\ \alpha_1 U_-(x_2, \xi_2) & 0 \end{pmatrix}$$

where in the chiral case, by Lemmas A.3 and A.4, we see that

$$t_{c,n,0}^{(0)} = 0, \ t_{c,n,0}^{(1)} = \frac{\lambda_n \alpha_1 i}{2}(\partial_{\bar{w}} \overline{U_-} - \partial_w U), \text{ and } t_{c,n,0}^{(2)} = 0,$$

while in the anti-chiral case, choosing $T(x_2, \xi_2) = \alpha_0 V(x_2, \xi_2) \, \mathrm{id}_{\mathbb{C}_{2\times 2}}$

$$t_{ac,n,0}^{(0)} = \begin{cases} \alpha_0 \cos(\frac{\theta}{2})V & n \neq 0, \\ \alpha_0 e^{-\frac{\theta}{2}i}V & n = 0, \end{cases}, \quad t_{ac,n,0}^{(1)} = 0, \text{ and } t_{ac,n,0}^{(2)} = \begin{cases} \frac{\alpha_0}{4}(2|n|\cos(\frac{\theta}{2}) - i\sigma_3 \sin(\frac{\theta}{2}))\Delta_{x_2,\xi_2}V, & n \neq 0 \\ \frac{\alpha_0}{4}e^{-i\frac{\theta}{2}}\Delta_{x_2,\xi_2}V & n = 0. \end{cases}$$

Due to the conjugacy relation $\int(u_n^\theta)^*(\tilde{T}^w)^* u_n^{-\theta} dx_1 = (\int (u_n^{-\theta})^* \tilde{T}^w u_n^\theta dx_1)^*$, the expansion of $Q_{n,0}^\theta$ follows by (A.5).

Similarly for the $(1,1)$-entry $Q_{n,1}^\theta$, denote

$$-\int (u_n^\theta)^* \tilde{T}^w e_{0,n}^{-\theta}(\tilde{T}^w)^* u_n^\theta dx_1 =: t_{n,1}^{(0)} + t_{n,1}^{(1)}\sqrt{h} + \mathcal{O}_{S(\mathbb{R}^2_{x_2,\xi_2};\mathbb{C}_{2\times 2})}(h)$$

where, using Lemma 1, in the chiral case,

$$t_{c,n,1}^{(0)} = -\frac{\alpha_1^2 \lambda_n}{4}(|U|^2 - |U_-|^2) \text{ and } t_{c,n,1}^{(1)} = \begin{cases} -\frac{\alpha_1^2 z}{4}[2|n|(|U|^2 + |U_-|^2) + (|U|^2 - |U_-|^2)], & n \neq 0 \\ -\frac{\alpha_1^2 z}{2}|U|^2, & n = 0 \end{cases}$$

and in the anti-chiral case

$$t_{ac,n,1}^{(0)} = \begin{cases} \frac{\alpha_0^2 |V|^2 \sin^2(\frac{\theta}{2})}{2\lambda_n}, & n \neq 0 \\ 0, & n = 0 \end{cases} \text{ and } t_{ac,n,1}^{(1)} = \begin{cases} -\frac{\alpha_0^2 |V|^2 \sin^2(\frac{\theta}{2})z}{4\lambda_n^2}, & n \neq 0 \\ 0, & n = 0. \end{cases}$$

In a similar fashion, the $(2,2)$-entry of $Q_{n,1}$, defined in (A.5), can be obtained by precisely the same computations after only replacing θ by $-\theta$ and T^* by T, i.e. U switching with U_- and using V^* instead of V. Thus the asymptotic expansion of $Q_{n,1}^\theta$ follows.

Similarly for $Q_{n,2}^\theta$ we restrict us to the $(1,2)$ entry in (A.5). Then, we denote

$$\int (u_n^\theta)^* \tilde{T}^w e_{0,n}^{-\theta}(\tilde{T}^w)^* e_{0,n}^\theta \tilde{T}^w u_n^{-\theta} dx_1 =: t_{n,2}^{(0)} + \mathcal{O}_{S(\mathbb{R}^2_{x_2,\xi_2};\mathbb{C}_{2\times 2})}(\sqrt{h}).$$

It follows then by Lemma 1, that in the chiral model, $t_{c,n,2}^{(0)} = 0$ while in the anti-chiral model, $t_{n,2}^{(0)} = -\frac{\alpha_0^3 |V|^2 \sin^2(\frac{\theta}{2})\cos(\frac{\theta}{2})}{4\lambda_n^2}V$. By the conjugacy relation

$$\int (u_n^{-\theta})^*(\tilde{T}^w)^* e_{0,n}^\theta \tilde{T}^w e_{0,n}^{-\theta}(\tilde{T}^w)^* u_n^\theta dx_1 = \left(\int (u_n^\theta)^* \tilde{T}^w e_{0,n}^{-\theta}(\tilde{T}^w)^* e_{0,n}^\theta \tilde{T}^w u_n^{-\theta} dx_1 \right)^*,$$

this also yields directly the expansion of $Q_{ac,n,2}^\theta$. $\qquad\square$

Existence, derivation and z-dependence. Now we prove the rest of Prop. A.1, which includes the existence and derivation of asymptotic expansion of $E_{n,\pm}^{-1}$ and r_n and the z dependence of each terms in the expansions of $E_{n,\pm}$, $E_{n,\pm}^{-1}$ and r_n.

Proof of Prop. A.1. By (A.4) and Prop. A.2, $E_{n,\pm} = \sqrt{h}(z - Z_n)$, and Z_n has an asymptotic expansion in S. Thus, $\frac{1}{\sqrt{h}}E_{n,\pm}$ also has an asymptotic expansion in S: $\frac{1}{\sqrt{h}}E_{n,\pm} \sim \sum_j h^{\frac{j}{2}}E_{n,j}$ with $E_{n,j} \in S$. To exhibit the z-dependence, we notice that only $E_{0,n}$ depends on z in (A.4). Thus, by (4.24), we have

$$Z_n^W = R_n^+ \tilde{\mathscr{V}}^W (\mathbb{1} + \sqrt{h}E_{0,n}\tilde{\mathscr{V}}^W)^{-1}R_n^- = R_n^+ \tilde{\mathscr{V}}^W R_n^- + \sum_{\alpha=1}^{\infty} R_n^+ \tilde{\mathscr{V}}^W (\sqrt{h}E_{0,n}\tilde{\mathscr{V}}^W)^\alpha R_n^-$$

$$= R_n^+ \tilde{\mathscr{V}}^W R_n^- + \sum_{\alpha=1}^{\infty} h^{\frac{\alpha}{2}} R_n^+ \tilde{\mathscr{V}}^W \left[\sum_{m\neq n} \frac{K_m^\theta (K_m^\theta)^*}{\lambda_m - \lambda_n} \sum_{\beta=0}^{\infty} \left(\frac{\sqrt{h}z}{\lambda_m - \lambda_n} \right)^\beta \right]^\alpha R_n^-$$

$$= R_n^+ \tilde{\mathscr{V}}^W R_n^- + \sum_{\alpha=1}^{\infty}\sum_{\gamma=0}^{\infty} h^{\frac{\alpha+\gamma}{2}} z^\gamma A_{n,\alpha,\gamma}^W(x_2, hD_{x_2}) = R_n^+ \tilde{\mathscr{V}}^W R_n^- - \sum_{j=1}^{\infty} h^{\frac{j}{2}} \left(\sum_{k=0}^{j-1} z^k a_{n,j,k}^W(x_2, hD_{x_2}) \right)$$

for some appropriate $A_{n,\alpha,\gamma}(x_2, \xi_2) \in S$ and $a_{n,j,k}(x_2, \xi_2) \in S$. Thus we proved part (1).

We can formally derive (A.2) and (A.3) for $\sqrt{h}E_{n,\pm}^{-1}$, using a formal parametrix construction by using

$$a\widetilde{\#}b \sim \sum_k \frac{1}{k!} \left(\left(\frac{ih}{2}\sigma(D_{x_2}, D_{\xi_2}; D_y, D_\eta) \right)^k (a(x_2,\xi_2)b(y,\eta)) \right) \Bigg|_{x_2=y,\ \xi_2=\eta}. \tag{A.6}$$

More specifically, there is a formal expansion of $\sqrt{h}E_{n,\pm}^{-1}$, which is denoted by $\sqrt{h}F_n \sim \sum_j h^{\frac{j}{2}}F_{n,j}$, such that $\frac{1}{\sqrt{h}}E_{n,\pm}\widetilde{\#}\sqrt{h}F_n = \mathbb{1}_{2\times 2}$. Denote $\sigma(D_{x_2}, D_{\xi_2}; D_y, D_\eta)$ in (A.6) by σ, we can solve for $F_{n,j}$ by considering

$$\mathbb{1}_{2\times 2} = E_{n,\pm}\widetilde{\#}F_n^{-1} \sim \sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty} h^{\frac{\alpha+\beta}{2}} E_{n,\alpha}\widetilde{\#}F_{n,\beta}$$

$$= \sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty} h^{\frac{\alpha+\beta}{2}} \sum_{\gamma=0}^{\infty} h^\gamma \left(\left(\frac{i\sigma}{2} \right)^\gamma (E_{n,\alpha}(x_2,\xi_2)F_{n,\beta}(y,\eta)) \right) \Bigg|_{x_2=y,\xi_2=\eta}$$

$$= \sum_{j=0}^{\infty}\sum_{\beta=0}^{j}\sum_{\alpha=0}^{j-\beta} h^{\frac{j}{2}} \left(\left(\frac{i\sigma}{2} \right)^{\frac{j-\alpha-\beta}{2}} (E_{n,\alpha}(x_2,\xi_2)F_{n,\beta}(y,\eta)) \right) \Bigg|_{x_2=y,\xi_2=\eta}.$$

Then we compare the parameter of the term of $h^{\frac{j}{2}}$ on both sides and get

$$-E_{n,0}F_{n,j} = \sum_{\beta=0}^{j-1}\sum_{\alpha=0}^{j-\beta} \left(\left(\frac{i\sigma}{2} \right)^{\frac{j-\alpha-\beta}{2}} (E_{n,\alpha}(x_2,\xi_2)F_{n,\beta}(y,\eta)) \right) \Bigg|_{x_2=y,\xi_2=\eta},$$

from which we can solve for $F_{n,j}$. Furthermore, by (A.1) and $E_{n,0} = z - z_{n,0}$, we can check inductively that for $j \geq 0$, there are $b_{n,j,k,l}$, $c_{n,j,k}$ such that

$$F_{n,j} = \sum_{k=0}^{j} (z - z_{n,0})^{-1} \prod_{l=0}^{k} \left(b_{n,j,k,l}(x_2, \xi_2; z)(z - z_{n,0})^{-1} \right),$$

with $\prod_{l=0}^{k} b_{n,j,k,l}(x_2, \xi_2; z) = \sum_{\alpha=0}^{j+k-2} z^{\alpha} c_{n,j,k}(x_2, \xi_2)$, for appropriate $c_{n,j,k} \in S$.

Notice that $\widetilde{\#}$ differs from the actual sharp product $\#$:

$$a \# b = e^{\frac{ih}{2} \sigma(D_{x_2}, D_{\xi_2}; D_y, D_\eta)} \left(a(x_2, \xi_2) b(y, \eta) \right) |_{x_2 = y, \xi_2 = \eta}. \tag{A.7}$$

Now we claim that this formal expansion for $\sqrt{h} F_n$ is legitimate as an asymptotic expansion in S_δ^δ and in fact, it is exactly the asymptotic expansion of $\sqrt{h} E_{n,\pm}$ when $|z| \leq 2\|\mathcal{V}\|_\infty$ and $|\operatorname{Im} z| \geq h^\delta$. In fact, $\sqrt{h}(E_{n,\pm}^{-1} - F_n) \in S^{-\infty}$.

In fact, since $|z|$ is bounded and $|\operatorname{Im} z| \geq h^\delta$ and $F_{n,j}$ is a rational function in z, thus $h^{\frac{j}{2}} F_{n,j} \in S_\delta^{j(\delta - \frac{1}{2}) + \delta}$. Since $j(\delta - \frac{1}{2}) + \delta \to -\infty$, (A.2) is not only a formal expansion but is indeed an asymptotic expansion of F_n in the symbol class S_δ^δ.

Furthermore, comparing (A.6) with (A.7), we see that $F_n \# E_{n,\pm} = 1 - R_n$ with $R_n \in S^{-\infty}$. By Beal's lemma, there is $\tilde{R}_n \in S^{-\infty}$ such that $(1 - R_n^W)^{-1} = 1 - \tilde{R}_n^W$. Thus $\sqrt{h} E_{n,\pm}^{-1} = F \#(1 - \tilde{R}_n^W) \in S_\delta^\delta$ and have exactly the same asymptotic expansion as F_n in (A.2) since $\tilde{R}_n \in S_\delta^{-\infty}$. Thus part (2) is proved.

It follows that $r_n := \partial_z E_{n,\pm} \# E_{n,\pm}^{-1}$ is also well-defined with an asymptotic expansion in S_δ^δ. Since

$$r_n \sim \sum_{\alpha=0}^{\infty} h^{\frac{\alpha}{2}} \partial_z E_{n,\alpha} \# \sum_{\beta=0}^{\infty} h^{\frac{\beta}{2}} F_{n,j} = \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} h^{\frac{\alpha+\beta}{2}} \sum_{\gamma=0}^{\infty} h^\gamma \left(\left(\frac{i\sigma}{2} \right)^\gamma (E_{n,\alpha}(x_2, \xi_2; z) F_{n,\beta}(y, \eta; z)) \right) \Big|_{x_2 = y, \xi_2 = \eta}$$

$$= \sum_{j=0}^{\infty} \sum_{\alpha=0}^{j} \sum_{\beta=0}^{j-\alpha} h^{\frac{j}{2}} r_{n,j,\alpha,\beta} \left(\left(\frac{i\sigma}{2} \right)^{\frac{j-\alpha-\beta}{2}} (E_{n,\alpha}(x_2, \xi_2; z) F_{n,\beta}(y, \eta; z)) \right) \Big|_{x_2 = y, \xi_2 = \eta}.$$

Combining it with part (1) and (2) and the fact that σ is linear in D_{x_2}, D_{ξ_2}, we get part (3). Part (4) follows directly from parts (1), (2), (3) with Prop. A.2. $\qquad \square$

APPENDIX B. FOR THE PROOF OF LEMMA 4.8

In this subsection, we provide several lemmas that together complete the proof of Lemma 4.8. We start with a proposition that expresses the Hilbert-Schmidt norm of the quantization in terms of its operator-valued symbol.

Proposition B.1. *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be two Hilbert spaces. Let $P : \mathbb{R}^2 \to \mathcal{L}(\mathcal{H}_1; \mathcal{H}_2)$ be an operator-valued symbol in the symbol class $S(\mathbb{R}^2_{y,\eta}; \mathcal{L}(\mathcal{H}_1; \mathcal{H}_2))$. Furthermore, let $\| \cdot \|_{\mathrm{HS}}$*

denote the Hilbert-Schmidt norm of maps $\mathscr{H}_1$ to $\mathscr{H}_2$ or $L^2(\mathbb{R}_y; \mathscr{H}_1)$ to $L^2(\mathbb{R}_y; \mathscr{H}_2)$. Then

$$\|P^W(y, hD_y)\|_{\mathrm{HS}}^2 = \frac{1}{2\pi h} \int_{\mathbb{R}^2} \|P(y, \eta)\|_{\mathrm{HS}}^2 \, dy \, d\eta.$$

In particular, if $\mathscr{H}_1 = \mathscr{H}_2 = \mathbb{R}$, for the scalar-valued symbol P, we have

$$\|P^W(y, hD_y)\|_{\mathrm{HS}}^2 = \frac{\|P(y, \eta)\|_{L^2(\mathbb{R}^2; \mathbb{R})}^2}{2\pi h}. \tag{B.1}$$

The next Lemma allows us to interchange the order of trace and integration.

Lemma B.2. *Let $E_{n,-}$, $E_{n,+}$ be as in* (4.16)*. Let $\tilde{\mathbb{1}}_R^W$, $\bar{\mathbb{1}}_R^W$ be as in the proof of Lemma* 4.7*. Then, there exists a constant $C > 0$ such that*

$$\|\bar{\mathbb{1}}_R^W E_{n,-}\|_{\mathrm{HS}(L^2(\mathbb{R}_x^2), L^2(\mathbb{R}_{x_2}))} \le C h^{-1/2} R \text{ and } \|E_{n,-}\tilde{\mathbb{1}}_R^W\|_{\mathrm{HS}(L^2(\mathbb{R}_x^2), L^2(\mathbb{R}_{x_2}))} \le C h^{-1/2} R.$$

Proof. The first equation follows from (B.1). For the second equation, we first recall that

Claim 1. *If $a \in S(\mathbb{R}^{2n}; \mathcal{L}(X, Y); m_1)$, $b \in S(\mathbb{R}^{2n}; \mathrm{HS}(Y, Z); m_2)$ and $m_1 m_2 \in L^2(\mathbb{R}_{x,\xi}^{2n})$, where m_1, m_2 are order functions, then*

$$b\#a \in S(\mathbb{R}^{2n}; \mathrm{HS}(X, Z); m_1 m_2) \text{ and } (b\#a)^W = b^W a^W \in \mathrm{HS}(L^2(\mathbb{R}_x^n; X); L^2(\mathbb{R}_x^n; Y)).$$

Similar to Lemma 1 in [W95], we can show that

Claim 2. *For any k' such that $1 < k'$, we have*

(1) $E_{n,-}(x_2, \xi_2) \in S(\mathbb{R}_{x_2,\xi_2}^2; \mathcal{L}(B_{x_1}^{-k'}; \mathbb{C}^2))$,
(2) $\tilde{\mathbb{1}}_R^w(x, D_{x_1}, \xi_2) \in S(\mathbb{R}_{x_2,\xi_2}^2; \mathrm{HS}(L_{x_1}^2; B_{x_1}^{-k'}); m)$, *where $m(x_2, \xi_2) = (1 + (|(x_2, \xi_2)| - R)_+)^{-k'}$ is the order function.*

Then it follows that, by Claim 1, we have $E_{n,-}\#\tilde{\mathbb{1}}_R^w \in S(\mathbb{R}_{x_2,\xi_2}^2; \mathrm{HS}(L_{x_1}^2); m)$, i.e.

$$\|E_{n,-}\#\tilde{\mathbb{1}}_R^w(x_2, \xi_2)\|_{\mathrm{HS}(L_{x_1}^2)} \le m(x_2, \xi_2) = (1 + (|(x_2, \xi_2)| - R)_+)^{-k'}.$$

Thus by Prop. B.1, since for all $k > 0$,

$$\int_{\mathbb{R}^2} [1 + (|(x_2, \xi_2)| - R)_+]^{-2k} dx d\xi = \pi R^2 + \mathcal{O}(R^{\max(1, -2k+2)}) = \mathcal{O}(R^2),$$

we get $\|E_{n,-}\tilde{\mathbb{1}}_R^W\|_{\mathrm{HS}(L^2(\mathbb{R}_{x_2}; L^2(\mathbb{R}_{x_1}; \mathbb{C}^4)); L^2(\mathbb{R}_{x_2}; \mathbb{C}^2))} \le C h^{-1/2} R$ and the Lemma is proved.   □

Lemma B.3. *Let $E_{n,-}, E_{n,+}, E_{n,\pm}$ be as in* (4.16)*. For $\mathrm{Im}\, z \ne 0$, both operators*

$$\tilde{\mathbb{1}}_R^W E_{n,+} E_{n,\pm}^{-1} E_{n,-} \tilde{\mathbb{1}}_R^W \text{ and } \bar{\mathbb{1}}_R^W E_{n,-} E_{n,+} E_{n,\pm}^{-1} \bar{\mathbb{1}}_R^W$$

are trace class as bounded linear operators $\mathcal{L}(L^2(\mathbb{R}_{x_2}; L^2(\mathbb{R}_{x_1}; \mathbb{C}^4)))$ and $\mathcal{L}(L^2(\mathbb{R}_{x_2}; \mathbb{C}^2))$, respectively.

Proof. By Lemma B.2, the fact that $\tilde{\mathbb{1}}_R^W E_{n,+}$ is the adjoint of $E_{n,-}\tilde{\mathbb{1}}_R^W$ and boundedness of $E_{n,\pm}$ from (4.19), we have

$$\mathrm{Tr}_1(\tilde{\mathbb{1}}_R^W E_{n,+} E_{n,\pm}^{-1} E_{n,-}\tilde{\mathbb{1}}_R^W) \leq \frac{CR^2}{h^{\frac{3}{2}}|\mathrm{Im}\,z|} \text{ and } \mathrm{Tr}_2(\bar{\mathbb{1}}_R^W E_{n,-} E_{n,+} E_{n,\pm}^{-1}\bar{\mathbb{1}}_R^W) \leq \frac{CR^2}{h^{\frac{3}{2}}|\mathrm{Im}\,z|}.$$

$\square$

The second proposition allows us to change the position of $E_{n,-}$ in the averaging and limiting process in the proof of Lemma 4.8.

Lemma B.4. *Let* $E_{n,-}, E_{n,+}, E_{n,\pm}$ *be as in* (4.16), *then*

$$\mathrm{Tr}_{L^2(\mathbb{R}_x^2;\mathbb{C}^4)}(\mathbb{1}_R^W E_{n,+} E_{n,\pm}^{-1} E_{n,-}\mathbb{1}_R^W) - \mathrm{Tr}_{L^2(\mathbb{R}_{x_2};\mathbb{C}^2)}(\bar{\mathbb{1}}_R^W E_{n,-} E_{n,+} E_{n,\pm}^{-1}\bar{\mathbb{1}}_R^W) \leq \frac{CR^{\frac{3}{2}}}{h|\mathrm{Im}\,z|}.$$

Proof. Since $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ when AB and BA are both of trace class.

$$\mathrm{Tr}_1(\tilde{\mathbb{1}}_R^W E_{n,+} E_{n,\pm}^{-1} E_{n,-}\tilde{\mathbb{1}}_R^W) - \mathrm{Tr}_2(\bar{\mathbb{1}}_R^W E_{n,-} E_{n,+} E_{n,\pm}^{-1}\bar{\mathbb{1}}_R^W)$$

$$= \mathrm{Tr}_2(E_{n,-}(\tilde{\mathbb{1}}_R^W)^2 E_{n,+} E_{n,\pm}^{-1}) - \mathrm{Tr}_2((\bar{\mathbb{1}}_R^W)^2 E_{n,-} E_{n,+} E_{n,\pm}^{-1})$$

$$= \mathrm{Tr}_2\left[(E_{n,-}\tilde{\mathbb{1}}_R^W - \bar{\mathbb{1}}_R^W E_{n,-})\tilde{\mathbb{1}}_R^W E_{n,+} E_{n,\pm}^{-1}\right] + \mathrm{Tr}_2\left[\bar{\mathbb{1}}_R^W (E_{n,-}\tilde{\mathbb{1}}_R^W - \bar{\mathbb{1}}_R^W E_{n,-}) E_{n,+} E_{n,\pm}^{-1}\right]$$

$$=: \mathrm{Tr}_2\left[[E_{n,-},\mathbb{1}_R]_w \tilde{\mathbb{1}}_R^W E_{n,+} E_{n,\pm}^{-1}\right] + \mathrm{Tr}_2\left[\bar{\mathbb{1}}_R^W [E_{n,-},\mathbb{1}_R]_w E_{n,+} E_{n,\pm}^{-1}\right]$$

$$=: \mathrm{Tr}_2(A_1) + \mathrm{Tr}_2(A_2)$$

where $[E_{n,-},\mathbb{1}_R]_W := E_{n,-}\tilde{\mathbb{1}}_R^W - \bar{\mathbb{1}}_R^W E_{n,-}$. Then the following claim completes the proof.

Claim 3. *For* $\mathrm{Im}\,z \neq 0$, A_1, A_2 *are trace class operators and there is a* $C > 0$ *such that*

$$\mathrm{Tr}_2(A_1), \mathrm{Tr}_2(A_2) \leq Ch^{-1}|\mathrm{Im}\,z|^{-3/2}R^{3/2}.$$

$\square$

Proof of Claim 3. From Lemma B.2, we already know

$$\|[E_{n,-},\mathbb{1}_R]_W\|_{\mathrm{HS}^W} \leq Ch^{-1/2}R,$$

where $\mathrm{HS}^W = \mathrm{HS}(L^2(\mathbb{R}_{x_2};L^2(\mathbb{R}_{x_1};\mathbb{C}^4));L^2(\mathbb{R}_{x_2};\mathbb{C}^2))$. We will improve the upper bound from $Ch^{-1/2}R$ to $Ch^{-1/2}R^{1/2}$.

Let $\bar{\chi}_R^c = 1 - \bar{\chi}_R$, $\tilde{\mathbb{1}}_R^c = 1 - \tilde{\mathbb{1}}_R$. First notice that from the proof of Lemma B.2, and replacing $\bar{\chi}_R$ by $\bar{\chi}_R^c$, we have

$$\|[E_{n,-},\mathbb{1}_R]_w(x_2,\xi_2)\|_{\mathrm{HS}} \leq \frac{C_k}{[1+(R-|(x_2,\xi_2)|)_+]^k} \text{ and } \|[E_{n,-},\mathbb{1}_R^c]_w(x_2,\xi_2)\|_{\mathrm{HS}} \leq \frac{C_k}{[1+(|(x_2,\xi_2)|-R)_+]^k}$$

where $[E_{n,-},\mathbb{1}_R]_w(x_2,\xi_2) = E_{n,-}\#\tilde{\mathbb{1}}_R^w - \bar{\mathbb{1}}_R\#E_{n,-}$ is the symbol in (x_2,ξ_2) of $[E_{n,-},\mathbb{1}_R]_W$ and $\mathrm{HS} = \mathrm{HS}(L^2(\mathbb{R}_{x_1};\mathbb{C}^4);\mathbb{C}^2)$. Since $[E_{n,-},\mathbb{1}_R]_w = -[E_{n,-},\mathbb{1}_R^c]_w$, we have

$$\|[E_{n,-},\mathbb{1}_R]_w(x_2,\xi_2)\|_{\mathrm{HS}} \leq C_k[1+||(x_2,\xi_2)|-R|]^{-k}.$$

Thus by Prop. B.1 and a straightforward computation of the following integral

$$\int_{\mathbb{R}^2_{x_2,\xi_2}} [1 + ||(x_2,\xi_2)| - R|]^{-2k} dx_2 d\xi_2 = \frac{1}{(2k-2)(2k-1)} + \frac{R}{2k-1} = \mathcal{O}(R),$$

we find that $\|[E_{n,-}, \mathbb{1}_R]_W\|_{\mathrm{HS}}w \leq Ch^{-1/2}R^{1/2}$. Since $\tilde{\mathbb{1}}_R^W E_{n,+}$ is the adjoint of $E_{n,-}\tilde{\mathbb{1}}_R^W$, this yields that

$$\mathrm{Tr}(A_1) \leq Ch^{-3/2}R^{3/2}, \quad \mathrm{Tr}(A_2) \leq Ch^{-3/2}R^{3/2}.$$

$\square$

In next Lemma, we state the averaging property of periodic symbols to reduce the regularized trace to a fundamental cell.

Lemma B.5. *Let $E_{n,-}, E_{n,+}, E_{n,\pm}, \bar{\mathbb{1}}_R$ be as in* (4.16). *Then*

$$\lim_{R\to\infty} \frac{1}{4R^2} \int_{\mathbb{R}^2} \mathrm{Tr}_{\mathbb{C}^2}(\bar{\mathbb{1}}_R \# \partial_z E_{n,\pm} \# E_{n,\pm}^{-1} \# \bar{\mathbb{1}}_R) \, dx_2 \, d\xi_2 = \frac{1}{|E|} \int_E \partial_{\bar{z}}\tilde{f} \, \mathrm{Tr}_{\mathbb{C}^2}(\partial_z E_{n,\pm} \# E_{n,\pm}^{-1}) \, dx_2 \, d\xi_2.$$

The proof of this Lemma can be found in [W95, Prop.3].

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