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THE ANISOTROPIC CALDERÓN PROBLEM FOR HIGH FIXED FREQUENCY*

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Abstract. We consider Schrödinger operators at a fixed high frequency on simply connected compact Riemannian manifolds with nonpositive sectional curvatures and smooth strictly convex boundaries. We prove that the Dirichlet-to-Neumann map uniquely determines the potential.

Key words. anisotropic Calderón problem, high frequency, geodesic ray transform

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1. Introduction. Let (M,g) be a simply connected compact Riemannian manifold of dimension $n=3$ with smooth strictly convex boundary ∂M. Let Δ_g be the positive Laplace–Beltrami operator on (M,g). For $V\in C_0^\infty(M)$ and $\lambda>0$, we consider the Dirichlet problem

$$(1)\qquad \begin{aligned}(\Delta_g+V-\lambda^2)u&=0 \text{ in } M,\\ u&=f \text{ at } \partial M.\end{aligned}$$

Suppose that λ^2 is not an eigenvalue of Δ_g+V. Let u be the unique $C^\infty(M)$ solution for $f\in C^\infty(\partial M)$. We consider the Dirichlet-to-Neumann map Λ defined by

$$(2)\qquad \Lambda f\doteq \partial_\nu u=|g|^{\frac12}\sum_{i,j=1}^{3} g^{ij}(\partial_i u)\nu_j,$$

where ν is the unit outward normal to ∂M. The problem we study is the determination of V from Λ for a large but fixed λ. Our main result is the following theorem.

THEOREM 1.1. *Let (M,g) be a simply connected compact Riemannian manifold of dimension three with smooth boundary ∂M which is strictly convex. Assume that the sectional curvatures are nonpositive. Let $V,\widetilde V\in C_0^\infty(M)$ be two potentials which are supported away from ∂M, and let $\Lambda,\widetilde\Lambda$ be the corresponding Dirichlet-to-Neumann map for* (1). *Suppose that λ is sufficiently large (depending on $V,\widetilde V$) and λ^2 is not an eigenvalue for* (1) *for both V and $\widetilde V$. Then $\Lambda=\widetilde\Lambda$ implies $V=\widetilde V$.*

We remark that one can find some conditions on the potential stated in Remark 10.1 so that Theorem 1.1 holds for λ large depending on those conditions.

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It is also possible to obtain a Hölder type stability estimate for those potentials. Finally, our analysis leads to an approximate reconstruction method for high frequencies by inverting a geodesic ray transform on (M, g); see the end of section 10.

When $\lambda = 0$ the problem we study is related to the anisotropic Calderón problem. In fact the Calderón problem in the isotropic case can be reduced to studying an inverse boundary problem for the Schrödinger equation at zero energy. It is well known that in dimension three or larger the anisotropic Calderón problem can be reduced to studying the Dirichlet-to-Neumann (DtN) map for the Laplace–Beltrami operator of a Riemannian metric uniquely determined by the conductivity; see [12]. The case we are considering here corresponds to determining a metric in the same conformal class. For the Euclidean metric the problem we are considering here at any fixed energy was solved in dimension three or larger by Sylvester and Uhlmann [21] and in two dimensions by Bukhgeim [1]. For the anisotropic case on conformally transversally anisotropic manifolds this problem was solved under some conditions on the transversal manifold; see [2], [3]. In two dimensions using Bukhgeim's method the problem was solved on Riemann surfaces in [6]. The problem of recovering the metric itself has only been solved in dimensions three or larger in the real-analytic category; see [11] and [10]. We plan to consider the problem of recovering the Riemannian metric at large fixed frequency in a subsequent article. For further references and a review of Calderón's problem, see [24].

We will outline the proof of Theorem 1.1 in section 2, but we want to point out that some assumptions in the statement of the theorem are made to simplify the analysis in order to give a clear demonstration of our method. They can potentially be removed. First, the parametrix construction we use in the proof is simpler in dimension three with the nonpositive curvature assumptions, but it can be done for higher dimensions (and even without the curvature assumptions). Second, we will use stability estimates for geodesic ray transforms on (M, g) which are simple to state with the nonpositive sectional curvature assumptions. It is known that such estimates hold under weaker curvature conditions or foliation conditions; see [17] and [25]. Third, we assumed that $V, \widetilde{V}$ are supported away from ∂M to save us from some technical discussions related to the singularities of the distance function. Also, it should be possible to relax the regularity of potential to finite smoothness. Finally, our results have been stated for the DtN map Λ, but similar arguments would probably hold for the set of Cauchy data defined as $\{(f, h) \in C^\infty(\partial M) \times C^\infty(\partial M) :$ there is $v \in C^\infty(\overline{M})$ such that $(\Delta_g + V - \lambda^2)v = 0$ in M and $v|_{\partial M} = f, \partial_\nu v|_{\partial M} = h\}$.

2. The strategy. Since we consider high frequencies, we can assume $\lambda >> 0$ and take $h = 1/\lambda$ as a semiclassical parameter. We consider the semiclassical problem

$$\begin{aligned} h^2(\Delta_g + V)u - \sigma^2 u = 0 \text{ in } M, \\ u = f \text{ at } \partial M \end{aligned} \tag{3}$$

for $h \in (0, 1]$. Here, we allow σ in a compact set of $\mathbb{C}\backslash 0$, but σ will be fixed in later arguments. We will use Green's representation formula which connects Λ to the fundamental solution of the equation in (3).

Since (M, g) has nonpositive sectional curvature and M is simply connected, without loss of generality we take M as a subset of $\mathbb{R}^3$. Let $\mathcal{R}(\sigma, h)$ be the fundamental solution such that

$$(h^2(\Delta_g + V) - \sigma^2)\mathcal{R}(\sigma, h) = \text{Id} \text{ in } \mathbb{R}^3. \tag{4}$$

Here, we can extend V to be zero outside M so V is a C_0^∞ potential on $\mathbb{R}^3$. The construction of the fundamental solution is known; see, e.g., [18]. In particular, for fixed h, σ and modulo a smoothing operator, $\mathcal{R}(\sigma, h)$ is a pseudodifferential operator of order -2 on $\mathbb{R}^3$.

Next, we follow the argument in [22, page 80]. Let u be a C^∞ solution of (3) on $\overline{M}$. Let χ_M be the characteristic function for M in $\mathbb{R}^3$. Then we have in the sense of distribution

$$\Delta_g(\chi_M u) = \chi_M \Delta_g u + 2\sum_{i,j=1}^{3} g^{ij}\frac{\partial u}{\partial z^i}\frac{\partial \chi_M}{\partial z^j} + u\Delta_g \chi_M.$$

Thus,

$$h^2(\Delta_g + V)(\chi_M u) - \sigma^2 \chi_M u = 2h^2 \sum_{i,j=1}^{3} g^{ij}\frac{\partial u}{\partial z^i}\frac{\partial \chi_M}{\partial z^j} + uh^2(\Delta_g \chi_M).$$

Using the fundamental solution $\mathcal{R}(\sigma, h)$, we obtain for $z \in M$ that

$$\chi_M u = 2h^2 \mathcal{R}(\sigma, h)\left(\sum_{i,j=1}^{3} g^{ij}\frac{\partial u}{\partial z^i}\frac{\partial \chi_M}{\partial z^j}\right) + \mathcal{R}(\sigma, h)(u(h^2\Delta_g \chi_M)). \tag{5}$$

Following the calculation in [22, page 80], we get that

$$u(z) = -h^2 \int_{\partial M} \mathcal{R}(z, z', \sigma, h)\frac{\partial u}{\partial \nu}(z')dz' + h^2 \int_{\partial M} u(z')\frac{\partial}{\partial \nu}\mathcal{R}(z, z', \sigma, h)dz'. \tag{6}$$

In this work, we will use the same notation $\mathcal{A}$ for both the operator $\mathcal{A} : \mathcal{E}'(M) \to \mathcal{D}'(M)$ and its Schwartz kernel. Formula (6) holds in the interior of M. But we will use this formula for $z \in \partial M$ which will be justified in section 4.

Now suppose we have two potentials $V, \widetilde{V}$ on M. Let $\Lambda, \widetilde{\Lambda}$ be the corresponding DtN maps. Let $\mathcal{R}, \widetilde{\mathcal{R}}$ be the corresponding fundamental solutions. For $(z, z') \in \partial M \times \partial M$ away from $z = z'$, we use (6) to get

$$\begin{aligned} &\partial_\nu \widetilde{\mathcal{R}}(z, z', \sigma, h) - \partial_\nu \mathcal{R}(z, z', \sigma, h) \\ &\quad = \int_{\partial M} \widetilde{\mathcal{R}}(z, z'', \sigma, h)\widetilde{\Lambda}(z'', z')dz'' - \int_{\partial M} \mathcal{R}(z, z'', \sigma, h)\Lambda(z'', z')dz''. \end{aligned} \tag{7}$$

The justification of the formula will be given in Remark 4.4 after we establish some properties of the fundamental solution. Our proof is based on the investigation of this formula. We list the key steps and describe the structure of the paper.

Step 1: We construct an approximation of the kernel of $\mathcal{R}(\sigma, h)$ with an explicit leading order term as $h \to 0$. This is done in sections 3 and 4. Roughly speaking, $\mathcal{R}(\sigma, h) = \mathcal{G}(\sigma, h) + O(h^{-1})$ and we have

$$\widetilde{\mathcal{R}}(\sigma, h) - \mathcal{R}(\sigma, h) = \widetilde{\mathcal{G}}(\sigma, h) - \mathcal{G}(\sigma, h) + \mathcal{F}_{para},$$

where $\mathcal{F}_{para}$ denotes the error term. Here, $\mathcal{G}$ is of the form

$$h^{-2}e^{-i\frac{\sigma}{h}r(z,z')}r(z, z')^{-1}\mathcal{A}(z, z'), \quad z, z' \in M, \tag{8}$$

where $\mathcal{A}$ is smooth and r denotes the distance function on M. Because there are no conjugate points on M, the distance function is smooth on $M \times M$ away from the diagonal $\{(z, z') \in M \times M : z = z'\}$.

Step 2: We show that the leading order term in $\widetilde{\mathcal{G}}(\sigma,h)-\mathcal{G}(\sigma,h)$ as $h\to 0$ is a weighted geodesic ray transform of the difference $W=\widetilde{V}-V$, denoted by $\mathcal{X}^w W$, so

$$\widetilde{\mathcal{G}}(\sigma,h)-\mathcal{G}(\sigma,h)=\mathcal{X}^w W+\mathcal{F}_{lead,1},$$

where $\mathcal{F}_{lead,1}$ denotes the error term. This is obtained from some perturbation argument in section 5 and a stationary phase argument by considering the oscillatory behavior in (8) done in section 6. The error term comes from the application of stationary phase argument. It is smaller in terms of h, but involves higher order derivatives of W.

Step 3: For $\mathcal{X}^w W$, we have some stability estimates in Theorem 6.1. After careful estimates of the error terms which is done in section 7 and semiclassical analysis of the DtN map in sections 8 and 9, we derive that

$$\|W\|_{L^2(M)}\le Ch|W|_{C^2(M)}, \tag{9}$$

where C is a generic constant. Here, for $u\in C^m(M)$, we denote the seminorm $|u|_{C^m(M)}=\sup_{x\in M}\sum_{|\alpha|\le m}|\partial^\alpha u|$. For given $V,\widetilde{V}$, there exists a constant C_1 such that $|W|_{C^2(M)}\le C_1\|W\|_{L^2(M)}$. We thus obtain that $W=0$ for h sufficiently small (depending on $V,\widetilde{V}$). This is done in section 10.

3. The semiclassical parametrix. Consider the semiclassical operator $\mathcal{P}\doteq h^2(\Delta_g+V)-\sigma^2$ on $\mathbb{R}^3$ where $V\in C_0^\infty(\mathbb{R}^3)$. Denote the resolvent by

$$\mathcal{R}(\sigma,h)=(h^2(\Delta_g+V)-\sigma^2)^{-1}.$$

We follow the discussion in [14, page 24] to construct an approximation of $\mathcal{R}(\sigma,h)$.

In polar coordinates (r,θ) based at a point $z'\in M$, the metric

$$g=dr^2+H(r,\theta,d\theta),$$

where H is a smooth 1-parameter family of metrics on $\mathbb{S}^2$. This is the Gauss lemma; see [5, page 91]. The Laplacian in this coordinate reads

$$\Delta_g=-\partial_r^2-A\partial_r+\Delta_H,\quad A=|g|^{-\frac{1}{2}}\partial_r(|g|^{\frac{1}{2}}),$$

where $|g|^{\frac{1}{2}}$ is the volume element and Δ_H is the positive Laplacian on $\mathbb{S}^2$ with respect to $H(r,\theta,d\theta)$. As $r\to 0$, we have

$$|g|^{\frac{1}{2}}(r,\theta)=r^2(1+r^2g_1(r,\theta)),$$

where g_1 is smooth at $r=0$; see [5, page 144]. Thus

$$\Delta_g=-\partial_r^2-(2/r+rA(r,\theta))\partial_r+\Delta_H$$

in which $A(r,\theta)$ is smooth up to $r=0$.

Now we look for an approximation of the resolvent whose Schwartz kernel is of the form

$$\mathcal{G}(\sigma,h,z,z')=e^{-i\frac{\sigma}{h}r(z,z')}(h^{-2}\mathcal{U}_0(z,z')+h^{-1}\mathcal{U}_1(z,z')).$$

We formally compute that

$$\begin{aligned}(h^2(\Delta_g+V)-\sigma^2)\mathcal{G}(\sigma,h)&=e^{-i\frac{\sigma}{h}r}(2i\sigma h^{-1}|g|^{-1/4}\partial_r(|g|^{\frac{1}{4}}\mathcal{U}_0)\\&+2i\sigma|g|^{-1/4}\partial_r(|g|^{1/4}\mathcal{U}_1)+(\Delta_g+V)\mathcal{U}_0+h(\Delta_g+V)\mathcal{U}_1).\end{aligned} \tag{10}$$

First, the $O(h^{-1})$ terms have to vanish and this gives

$$|g|^{-1/4}\partial_r(|g|^{\frac{1}{4}}\mathcal{U}_0) = 0, \quad r > 0.$$

Notice that $|g|^{\frac{1}{2}}$ is the density factor on $M \times M$ so $|g|^{1/4}$ is indeed the half-density factor and the above equation is the Lie derivative of the principal symbols; see [14]. We obtain that

$$\mathcal{U}_0 = \frac{1}{4\pi}|g|^{-1/4}.$$

Near $r = 0$, we get

$$\mathcal{U}_0 = \frac{1}{4\pi}(r^{-1} + rB),$$

where B is smooth in r, θ up to $r = 0$. This implies that

$$\Delta_g \mathcal{U}_0 = \delta(z, z') + \frac{1}{4\pi}Ar^{-1} + \frac{1}{4\pi}\Delta_g(rB).$$

So we removed the singularities at the diagonal to the leading order. This only happens in dimension three.

Next, the $O(h^0)$ terms in (10) have to vanish and we obtain

$$\begin{gathered} 2i\sigma|g|^{-1/4}\partial_r(|g|^{1/4}\mathcal{U}_1) + (\Delta_g + V)\mathcal{U}_0 = 0, \quad r > 0, \\ \mathcal{U}_1 = 0 \text{ at } r = 0. \end{gathered}$$

So we get

$$\mathcal{U}_1 = -\frac{1}{2i\sigma}|g|^{-1/4}\int_0^r |g|^{\frac{1}{4}}(\Delta_g + V)\mathcal{U}_0 ds.$$

In particular, $|g|^{1/4}$ is smooth up to $r = 0$ and vanishes at $r = 0$. Thus the integrand in $\mathcal{U}_1$ is smooth at $r = 0$. This implies that $\mathcal{U}_1$ is smooth up to $r = 0$.

Consider the remainder term. We have

$$\begin{aligned} (h^2(\Delta_g + V) - \sigma^2)\mathcal{G}(\sigma, h) &= \delta(z, z') + he^{-i\frac{\sigma}{h}r}(\Delta_g + V)\mathcal{U}_1 \\ &= \mathrm{Id} + h\mathcal{E}(\sigma, h). \end{aligned}$$

We observe that the remainder term $\mathcal{E}$ has a $1/r$ type singularity at $r = 0$. Thus, in this approach, we removed the semiclassical error to order h but in the classical sense, we only removed the leading order singularity at $r = 0$.

We are done with the construction and we summarize the result. Hereafter, we use $r(z, z')$ for the distance function on (M, g), where $z, z' \in \mathcal{M}$.

Proposition 3.1. *For $h \in (0, 1], \sigma \in \mathbb{C}\backslash 0$, there exist operators $\mathcal{G}(\sigma, h)$ and $\mathcal{E}(\sigma, h)$ such that*

$$(h^2(\Delta_g + V) - \sigma^2)\mathcal{G}(\sigma, h) = \mathrm{Id} + h\mathcal{E}(\sigma, h),$$

where the Schwartz kernel

$$\mathcal{G}(\sigma, h, z, z') = e^{-i\frac{\sigma}{h}r(z, z')}(h^{-2}\mathcal{U}_0(z, z') + h^{-1}\mathcal{U}_1(z, z'))$$

with

$$\mathcal{U}_0 = \frac{1}{4\pi}|g|^{-1/4}, \quad \mathcal{U}_1 = -\frac{1}{2\pi i\sigma}|g|^{-1/4}\int_0^r |g|^{\frac{1}{4}}(\Delta_g + V)\mathcal{U}_0 ds,$$

and

$$\mathcal{E}(\sigma, h, z, z') = e^{-i\frac{\sigma}{h}r(z,z')}(\Delta_g + V)\mathcal{U}_1(z, z').$$

Remark 3.2. For higher dimensions $n \geq 4$, a similar but more involved construction was also done in [14] for small metric perturbations of hyperbolic spaces, and it was further developed in [16] for nontrapping asymptotically hyperbolic manifolds. For our purpose, the asymptotic behavior near infinity does not matter. The construction in [14] and [16] away from infinity does not rely on the hyperbolic structure and can be modified to obtain a parametrix for simply connected manifolds.

4. The resolvent kernel and estimates. Hereafter, we fix $\sigma \in \mathbb{C}\backslash 0$. We will drop σ in the notation and write, e.g., $\mathcal{R}(h) = \mathcal{R}(\sigma, h)$. We consider the resolvent $\mathcal{R}(h)$ and use the parametrix to find the resolvent kernel. The construction and estimate for the Green's function are known for elliptic problems in general; see, for example, [18, Chapter VI, section 4]. Here, the point is the dependency on h.

We start from

$$(h^2(\Delta_g + V) - \sigma^2)\mathcal{G}(h) = \mathrm{Id} + h\mathcal{E}(h). \tag{11}$$

Using the resolvent $\mathcal{R}(h)$ in (11), we get

$$\mathcal{G}(h) = \mathcal{R}(h)(\mathrm{Id} + h\mathcal{E}(h)).$$

We first have the following lemma.

LEMMA 4.1. *Let* $\sigma \in \mathbb{C}\backslash 0$ *and* $\operatorname{Im}\sigma \leq 0$. *There is* $h_0 > 0$ *such that for* $0 < h < h_0$, *the operator* $\mathrm{Id} + h\mathcal{E}(h)$ *is invertible on* $L^2(M)$.

Proof. We recall that

$$r\Delta_g = -r\partial_r^2 - (2 + r^2 A(r, \theta))\partial_r + r\Delta_H$$

is a second order differential operator with smooth coefficients up to $r = 0$. From the formula of $\mathcal{E}(z, z', h)$ in Proposition 3.1, we see that $r(z, z')\mathcal{E}(z, z', h)$ is smooth up to $r = 0$. We use Schur's lemma to estimate the L^2 norm of $\mathcal{E}(h)$. First, we have

$$\int_M |\mathcal{E}(h, z, z')|dz' \leq C\frac{1}{|\sigma|}\int_M \frac{1}{r(z, z')}dz' \leq C\frac{1}{|\sigma|},$$

where the constant C depends on M but not on h, σ. Here, we use $\operatorname{Im}\sigma \leq 0$. Using Schur's lemma, we get

$$\|h\mathcal{E}(h)\|_{L^2(M)\to L^2(M)} \leq Ch|\sigma|^{-1}.$$

Then we can find $h_0 > 0$ such that for $h < h_0|\sigma|^{-1}$, we have $Ch|\sigma|^{-1} < 1$ and thus $\mathrm{Id} + h\mathcal{E}(h)$ is invertible on $L^2(M)$. □

The proof implies that for $h < h_0$, the inverse can be written as a Neumman series. We write

$$\begin{aligned}\mathcal{R}(h) &= \mathcal{G}(h)(\mathrm{Id} + h\mathcal{E}(h))^{-1} = \mathcal{G}(h) - \mathcal{G}(h)h\mathcal{E}(h) + \cdots \\ &= \mathcal{G}(h) + \mathcal{F}(h), \text{ where } \mathcal{F}(h) = \sum_{j=1}^{\infty}\mathcal{G}(h)(-h\mathcal{E}(h))^j.\end{aligned} \tag{12}$$

Next we find the Schwartz kernel of $\mathcal{F}(h)$. We start with some estimates of the terms in the parametrix.

LEMMA 4.2. *There exists $C>0$ depending on $g, |V|_{C^0(M)}$ such that for fixed $\sigma \in \mathbb{C}\backslash 0$ and $\operatorname{Im}\sigma \le 0$, we have*

(1) $|r(z,z')\mathcal{G}(h,z,z')|_{C^0(M\times M)} \le Ch^{-2}$,
(2) $|r^2(z,z')\partial_\nu \mathcal{G}(h,z,z')|_{C^0(M\times M)} \le Ch^{-3}$,
(3) $|r(z,z')\mathcal{E}(h,z,z')|_{C^0(M\times M)} \le C$.

Proof. This is straightforward. For (1), the expression of $\mathcal{G}$ is

$$\mathcal{G}(h,z,z') = e^{-i\frac{\sigma}{h}r(z,z')}(h^{-2}\mathcal{U}_0(z,z') + h^{-1}\mathcal{U}_1(z,z')).$$

From the expression of $\mathcal{U}_0, \mathcal{U}_1$, we know that $r\mathcal{U}_0, r\mathcal{U}_1$ are both smooth up to $r=0$. Therefore, we get that

$$|r(z,z')\mathcal{G}(z,z',h)| \le Ch^{-2} + Ch^{-1}|\sigma|^{-1}$$

for some constant dependent only on $g, |V|_{C^0(M)}$. For (2), we have

$$\begin{aligned}\partial_\nu \mathcal{G}(z,z',h) = -i\frac{\sigma}{h}\partial_\nu r(z,z')e^{-i\frac{\sigma r(z,z')}{h}}(h^{-2}\mathcal{U}_0 + h^{-1}\mathcal{U}_1)\\ + e^{-i\frac{\sigma r(z,z')}{h}}\partial_\nu(h^{-2}\mathcal{U}_0 + h^{-1}\mathcal{U}_1).\end{aligned}$$

Therefore,

$$|r^2(z,z')\partial_\nu \mathcal{G}(z,z',h)| \le C|\sigma|h^{-3} + Ch^{-2} + Ch^{-1}|\sigma|^{-1}.$$

The proof of (3) follows from the same argument. ∎

LEMMA 4.3. *Let $\sigma \in \mathbb{C}\backslash 0$ and $\operatorname{Im}\sigma \le 0$. There exist $h_0>0$, $C>0$ depending on $g, |V|_{C^0(M)}$ and σ such that for $h<h_0$, we have*

(1) $|r(z,z')\mathcal{F}(h,z,z')|_{C^0(M\times M)} \le Ch^{-1}$,
(2) $|r^2(z,z')\partial_\nu \mathcal{F}(h,z,z')|_{C^0(M\times M)} \le Ch^{-2}$.

Proof. We estimate the Schwartz kernel of $h^j\mathcal{G}(h)\mathcal{E}^j(h), j\ge 1$. The kernel can be written as

$$\begin{aligned}h^j\mathcal{G}(h)\mathcal{E}^j(h)(z_0,z_j) = h^j\int_{M\times M\times\cdots\times M}\mathcal{G}(h,z_0,z_1)\mathcal{E}(h,z_1,z_2)\mathcal{E}(h,z_2,z_3)\\ \cdots\mathcal{E}(h,z_{j-1},z_j)dz_2\cdots dz_{j-1},\end{aligned}$$

where $z_0, z_j \in M$. Since $r(z,z')\mathcal{E}(h,z,z')$ are smooth up to $r=0$, we can estimate

$$\begin{aligned}|h^j\mathcal{G}(h)\mathcal{E}^j(h)(z_0,z_j)| \le C^jh^j|\sigma|^{-j}\int_{M\times M\times\cdots\times M}|\mathcal{G}(h,z_0,z_1)|\frac{1}{r(z_1,z_2)}\frac{1}{r(z_2,z_3)}\\ \cdots\frac{1}{r(z_{j-1},z_j)}dz_2\cdots dz_{j-1}.\end{aligned}$$

The kernel is integrable. Moreover, we observe that

$$\int_M \frac{1}{r(x,y)}\frac{1}{r(y,z)}dy \le \frac{1}{r(x,z)}C\int_M\frac{1}{r(x,y)}\frac{1}{r(y,z)}dy \le C\frac{1}{r(x,z)},$$

where C only depends on (M,g). So we get

$$|h^jr(z,z')\mathcal{G}(h)\mathcal{E}^j(h)(z,z')| \le (C_0h/|\sigma|)^jCh^{-2}$$

for some $C_0 > 0$. For $h/|\sigma| < 1$, we can sum the terms from $j = 1$ and get

$$|r(z,z')\mathcal{F}(h,z,z')|_{C^0(M\times M)} \leq \frac{C_0 h/|\sigma|}{1 - C_0 h/|\sigma|} C h^{-2} \leq C h^{-1} |\sigma|^{-1},$$

where we used $h/|\sigma| < 1/2$. The derivative estimate is similar. □

Remark 4.4. Now we justify the formula (6) for $z \in \partial M$. First of all, the first term on the right of (6) is continuous at ∂M because the kernel $\mathcal{R}(z,z',\sigma,h)$ is locally integrable. For the last term in (6), we notice that the kernel $\partial_\nu \mathcal{R}(z,z',\sigma,h)$ is not locally integrable, but the kernel is only singular at $z = z'$. We will stay away from the diagonal as follows. For any $p \in \partial M$, let χ_p be a compactly supported smooth cut-off function supported near p. For any $f \in C^\infty(\partial M)$, we consider Dirichlet data $f\chi_p$. Then the last term is smooth when we consider z outside of support of χ_p. This means that if we stay away from the diagonal, all the terms in (6) can be extended continuously to ∂M. Then we derive (7) by taking f to be test functions on ∂M.

5. The perturbation argument. Let $V, \widetilde{V}$ be two potentials on (M,g). Let $\mathcal{R}(h), \widetilde{\mathcal{R}}(h)$ be the corresponding resolvent of $h^2(\Delta_g + V) - \sigma^2, h^2(\Delta_g + \widetilde{V}) - \sigma^2$. We are interested in the difference $\widetilde{\mathcal{R}}(h) - \mathcal{R}(h)$. Let $W = \widetilde{V} - V$. From the resolvent formula, we get

$$\widetilde{\mathcal{R}}(h) - \mathcal{R}(h) = \widetilde{\mathcal{R}}(h) h^2 W \mathcal{R}(h)$$

so that

$$\widetilde{\mathcal{R}}(h) = \mathcal{R}(h)(\mathrm{Id} + h^2 W \mathcal{R}(h))^{-1}. \tag{13}$$

Here, because W is compactly supported in M, the invertibility of $\mathrm{Id} + h^2 W\mathcal{R}(h)$ on $L^2(M)$ follows from the analytic Fredholm theory; see pages 19–20 of [13]. If we apply $h^2(\Delta_g + V) - \sigma^2$ to (13), we get

$$(h^2(\Delta_g + V) - \sigma^2)\widetilde{\mathcal{R}}(h) = (\mathrm{Id} + h^2 W \mathcal{R}(h))^{-1}.$$

Then it follows from the structure of $\widetilde{\mathcal{R}}$ in section 4 that $(\mathrm{Id} + h^2 W\mathcal{R}(h))^{-1}$ has an integrable kernel and the L^1 norm of the kernel is bounded as $h \to 0$.

Now we describe the approximation of (7) that we use later. From (13), we write

$$\widetilde{\mathcal{R}}(h) - \mathcal{R}(h) = -\mathcal{R}(h) h^2 W \mathcal{R}(h) + \mathcal{F}_{res},$$

where

$$\mathcal{F}_{res} = \mathcal{R}(h) h^2 W \mathcal{R}(h) h^2 W \mathcal{R}(h)(\mathrm{Id} + h^2 W \mathcal{R}(h))^{-1}. \tag{14}$$

Here, $\mathcal{F}_{res}$ accounts for the error in the potential perturbation. We use the parametrix $\mathcal{R}(h) = \mathcal{G}_0 + \mathcal{G}_1 + \mathcal{F}$ to get

$$\begin{aligned} -h^2\mathcal{R}(h) W \mathcal{R}(h) &= \mathcal{F}_{lead} + \mathcal{F}_{para}, \\ \mathcal{F}_{lead} &= -\mathcal{G}_0 h^2 W \mathcal{G}_0, \\ \mathcal{F}_{para} &= -\mathcal{G}_0 h^2 W(\mathcal{G}_1 + \mathcal{F}) - (\mathcal{G}_1 + \mathcal{F}) h^2 W \mathcal{G}_0 - (\mathcal{G}_1 + \mathcal{F}) h^2 W (\mathcal{G}_1 + \mathcal{F}). \end{aligned} \tag{15}$$

The term $\mathcal{F}_{lead}$ is what we use to get a geodesic ray transform of W. The term $\mathcal{F}_{para}$ accounts for the error in parametrix construction. To summarize, we get

$$\mathcal{R}(z,z',h) - \widetilde{\mathcal{R}}(z,z',h) = \mathcal{F}_{lead} + \mathcal{F}_{para} + \mathcal{F}_{res}. \tag{16}$$

In the next section, we analyze $\mathcal{F}_{lead}$. Then we estimate $\mathcal{F}_{para}, \mathcal{F}_{res}$.

6. Geodesic ray transform. By the assumption that (M,g) has nonpositive sectional curvatures, we know that for every $z, z' \in \partial M, z \neq z'$, there is a unique distance minimizing geodesic $\gamma_{z,z'}$ between them. Let $r(z,z')$ be the distance between $z, z' \in M$. Let $\gamma_{z,z'}(s) : [0, r(z,z')] \to M$ be the unit speed geodesic from z to z'. It satisfies the geodesic equation

$$\nabla_{\dot\gamma_{z,z'}(s)}\dot\gamma_{z,z'}(s) = 0,$$
$$\gamma_{z,z'}(0) = z, \quad \dot\gamma_{z,z'}(0) = \partial_z r(z,z').$$

For $z, z' \in \partial M$, we consider the geodesic ray transform on scalar functions

$$\mathcal{X} f(z,z') = \int_0^{r(z,z')} f(\gamma_{z,z'}(s))ds. \tag{17}$$

Note that we parametrize the geodesics using $z, z' \in \partial M$. Usually, the geodesic transform is parametrized by using inward pointing unit tangent bundle at ∂M,

$$\Omega_- M = \{(z,\xi) \in TM | z \in \partial M, -\langle \xi, \nu\rangle \geq 0, |\xi|_g = 1\};$$

see, for example, [17]. For (M,g) that we consider, there are no conjugate points. There is a diffeomorphism between $(z,z') \in \partial M \times \partial M$ away from the diagonal $z = z'$ and $(z,\xi) \in \Omega_- M$ away from $\xi = 0$ via $\xi = \partial_z r(z,z')$. For our purpose, the function f in (17) is supported away from ∂M; thus it suffices to consider geodesics corresponding to $(z,z') \in \partial M \times \partial M$ away from the diagonal. For this reason, we can use $\mathscr{C} = \partial M \times \partial M$ with a measure which away from the diagonal is the one induced from $\Omega_- M$. We use $\mathscr{C}$ as the set for parametrizing geodesics.

Later, we shall consider a weighted geodesic ray transform

$$\mathcal{X}^w f(z,z') = \int_0^{r(z,z')} \mathcal{W}(z,z',\gamma_{z,z'}(s)) f(\gamma_{z,z'}(s))ds, \tag{18}$$

where $\mathcal{W}$ is a smooth nonvanishing function on $\partial M \times \partial M \times M$. In particular, we can find $C_1, C_2 > 0$ such that $C_1 \leq |\mathcal{W}|_{C^0} \leq C_2$ on M. We need an invertibility result.

THEOREM 6.1. *Let (M,g) be a simply connected compact Riemannian manifold with strictly convex boundary ∂M and nonpositive sectional curvatures. Suppose $f \in L^2(M)$ and f is supported away from ∂M. Then f is uniquely determined by the ray transform $\mathcal{X}^w f$ and the stability estimate holds:*

$$\|f\|_{L^2(M)} \leq C \|\mathcal{X}^w f\|_{H^1(\mathscr{C})},$$

where C is a constant independent of f.

Proof. First, $\mathcal{X}^w$ is injective. This follows from Remark 4.3 of [25] that the weighted X-ray transform is locally invertible and that the foliation condition is satisfied because of the nonpositive curvature assumption. Next, we use the argument in section 7 of [20]. For nonvanishing weight, a similar calculation shows that the normal operator $\mathcal{N} = \mathcal{X}^{w,*}\mathcal{X}^w$ is an elliptic pseudodifferential operator of order -1. Together with the injectivity, we get the estimate.

$$\|f\|_{L^2(M)} \leq C \|\mathcal{N} f\|_{H^1(M)}. \tag{19}$$

Note that we assumed f is supported away from ∂M. This is why we can take the H^1 space on M. Finally, we can use a simple estimate that $\mathcal{X}^{w,*} : H^1 \to H^1$ is bounded to finish the proof. ∎

Now we analyze $\mathcal{F}_{lead}$. With $\mathcal{G}_0(h,z,z') = (4\pi h)^{-2}e^{-i(\sigma/h)r(z,z')}|g(z')|^{-1/4}$, we have

$$\begin{aligned}\mathcal{F}_{lead} &= h^2\mathcal{G}_0(h)W\mathcal{G}_0(h)\\ &= (4\pi)^{-4}h^{-2}\int_M e^{-i\sigma r(z,z')/h}|g|^{-1/4}(z,z')W(z')e^{-i\sigma r(z',z'')/h}|g|^{-1/4}(z',z'')dz'\\ &= (4\pi)^{-4}h^{-2}\int_M e^{-i\frac{\sigma}{h}(r(z,z')+r(z',z''))}|g|^{-1/4}(z,z')|g|^{-1/4}(z',z'')W(z')dz'.\end{aligned}$$

At this point, we will apply a stationary phase argument for a nonhomogeneous phase function

$$\Phi(z,z',z'') = r(z,z') + r(z',z''), \quad z'\in M,$$

and we consider $z,z''\in\partial M$. To find critical points, we see from

$$\partial_{z'}\Phi(z,z',z'') = \partial_{z'}r(z,z') + \partial_{z'}r(z',z'') = 0$$

that $\partial_{z'}r(z,z') = -\partial_{z'}r(z',z'')$. This happens if and only if z' is on $\gamma_{z,z''}$, the unique distance minimizing geodesic between z,z'. To see the rank of the Hessian $\partial^2_{z'}\Phi$, it is helpful to look at the Carleson–Sjölin condition in the estimates of oscillatory integrals; see [15, 19].

Consider an oscillatory integral of the form

$$S_\lambda f(x) = \int_{\mathbb{R}^n} e^{i\lambda\phi(x,y)}a(x,y)f(y)dy,$$

where $a\in C_0^\infty(\mathbb{R}^n\times\mathbb{R}^n)$. For our problem, $\phi(x,y) = r(x,y)$ which is smooth away from $x=y$. The real valued smooth function ϕ satisfies the Carleson–Sjölin condition in this case if $\nabla_x\phi, \nabla_y\phi$ never vanish and

$$\text{rank}\,\phi''_{xy} = n-1, \tag{20}$$

and condition that there is a neighborhood $\mathcal{U}$ of supp a so that the immersed hypersurfaces $\Sigma_{x_0} = \{\phi'_x(x_0,y) : (x_0,y)\in\mathcal{U}\}$ have everywhere nonvanishing Gaussian curvature. If $\phi = r(x,y)$, then

$$\Sigma_{x_0} = \left\{\xi\in\mathbb{R}^n : \sum_{j,k=1}^n g^{jk}(x_0)\xi_j\xi_k = 1\right\}.$$

The curvature condition implies that

$$\text{rank}\left(\frac{\partial^2}{\partial y_j\partial y_k}\langle\phi'_x,\theta\rangle\right) = n-1, \tag{21}$$

where $\pm\theta\in\mathbb{S}^{n-1}$ are the directions for which $\nabla_{y'}\langle\phi'_x,\theta\rangle = 0$. In fact, θ is orthogonal to Σ_{x_0} at ξ. For the distance function r on (M,g), the Carleson–Sjölin condition holds.

Let $z,z''\in\partial M$ and let $z'\in K$ a compact set of M. We observe that

$$\partial_{z'}\Phi(z,z',z'') = \partial_{z'}r(z,z') + \partial_{z'}r(z',z'') = \partial_{z'}r(z,z') - \partial_{z'}r(z'',z').$$

We let $\widetilde{z},\widetilde{z}''\in M$ be points on a neighborhood of z' such that

$$\partial_{z'}r(z,z') = \partial_{z'}r(\widetilde{z},z'), \quad \partial_{z'}r(z'',z') = \partial_{z'}r(\widetilde{z}'',z').$$

These are unit tangent vectors. Actually, if we let $\gamma_{z,z'}(s)$ be the geodesic with $\gamma_{z,z'}(0)=z', \dot\gamma_{z,z'}(0)=\partial_{z'}r(z,z')$, then we can take $\widetilde z=\gamma_{z,z'}(s_0)$ for some s_0 small. We can find $\widetilde z''$ similarly on $\gamma_{z,z'}(s)$. Now we find that

$$\partial^2_{z'}\Phi(z,z',z'')=\partial_{z'}(\partial_{z'}r(\widetilde z,z')-\partial_{z'}r(\widetilde z'',z'))\approx\frac{\partial^2}{\partial z'\partial z'}\partial_z r(\widetilde z,z')(\widetilde z-\widetilde z'')$$

for $\widetilde z,\widetilde z''$ close to z' and $|\widetilde z-\widetilde z''|$ small. Then the Carleson–Sjölin condition tells us that the Hessian has rank $n-1=2$. Now one can choose local coordinates $z=(x,y), x\in\mathbb{R}, y=(y_1,y_2)\in\mathbb{R}^2$ such that the rank in (21) in y variable is 2. We can perform the method of stationary phase in y variable and obtain the asymptotic expansion of the oscillatory integral using $h/|\sigma|$ as the small parameter.

We recall the standard stationary phase expansion; see [4, Proposition 1.2.4]. Let Q be a nonnegative and symmetric matrix on $\mathbb{R}^2$ depending continuously on parameter $a\in\mathbb{R}^m$. Then

$$\begin{aligned}\int e^{it\langle Q(a)y,y\rangle/2}g(y,a,t)dy &\sim\left(\frac{2\pi}{t}\right)|\det Q(a)|^{-1/2}e^{\pi i\operatorname{sgn}Q(a)/4}\\ &\cdot\sum_{k=0}^{\infty}\frac{1}{k!}(i\langle Q(a)^{-1}\partial_y,\partial_y\rangle/2)^k g(0,a,t)t^{-k}\end{aligned}$$

for $t\to\infty$ uniformly in a. Applying this result, we get

$$\begin{aligned}\mathcal{F}_{lead}(z,z'')=h^{-2}e^{i\sigma r(z,z'')/h}\left(\frac{h}{\sigma}\right)\int_0^{r(z,z'')}&|g|^{-1/4}(z,\gamma_{z,z''}(s))|g|^{-1/4}(\gamma_{z,z''}(s),z'')\\ &\cdot J(z,z'',\gamma_{z,z''}(s))W(\gamma_{z,z''}(s))ds+\mathcal{F}_{lead,1},\end{aligned}$$

where J is a nonvanishing function independent of W which comes from the Hessian of Φ and change of variables. Note that the first term on the right side can be written as $h^{-1}e^{i\sigma r(z,z'')/h}\mathcal{X}^wW(z,z')$, where $\mathcal{X}^wW$ is a weighted geodesic ray transform with nonvanishing weight $\mathcal{W}$. For the remainder term, we have

$$\begin{aligned}&|\mathcal{F}_{lead,1}|_{C^0(\partial M\times\partial M)}\le C\left(\frac{h}{\sigma}\right)^2h^{-2}|W|_{C^2(M)},\\ &|\mathcal{F}_{lead,1}|_{C^1(\partial M\times\partial M)}\le C\left(\frac{h}{\sigma}\right)h^{-2}|W|_{C^2(M)}.\end{aligned}\tag{22}$$

Finally, we need the ∂_ν derivative and we find that

$$\begin{aligned}\partial_\nu\mathcal{F}_{lead}&=h^2\partial_\nu\mathcal{G}_0(h)W\mathcal{G}_0(h)\\ &=Ch^{-2}\partial_\nu r(z,z'')e^{i\sigma r(z,z'')/h}\mathcal{X}^w(W)+Ch^{-2}\frac{h}{\sigma}e^{i\sigma r(z,z'')/h}\partial_\nu\mathcal{X}^w(W)+\partial_\nu\mathcal{F}_{lead,1},\end{aligned}\tag{23}$$

where $\partial_\nu\mathcal{F}_{lead,1}$ satisfy (22) as well.

7. Estimates of the remainder terms. First, we consider $\mathcal{F}_{para}$ and $\partial_\nu\mathcal{F}_{para}$.

LEMMA 7.1. *Let $\sigma\in\mathbb{C}\backslash 0$ with $\operatorname{Im} z\le 0$. For $h<h_0$ small depending on g and $|V|_{C^2(M)}$, the Schwartz kernel for $z,z'\in\partial M$ satisfies*

(1) $|r(z,z')\mathcal{F}_{para}(h,z,z')|_{C^0(\partial M\times\partial M)}\le C|W|_{C^0(M)}$,

(2) $|r^2(z,z')\partial_\nu\mathcal{F}_{para}(h,z,z')|_{C^0(\partial M\times\partial M)}\le Ch^{-1}|W|_{C^0(M)}$.

Proof. We recall that

$$\mathcal{F}_{para} = -\mathcal{G}_0 h^2 W(\mathcal{G}_1 + \mathcal{F}) - (\mathcal{G}_1 + \mathcal{F}) h^2 W \mathcal{G}_0 - (\mathcal{G}_1 + \mathcal{F}) h^2 W(\mathcal{G}_1 + \mathcal{F}).$$

For terms $\mathcal{G}_0 W \mathcal{G}_1$ and $\mathcal{G}_1 W \mathcal{G}_1$, we can apply the stationary phase argument as in section 6 to get the conclusion. For $\mathcal{G}_0 h^2 W \mathcal{F}$ and $\mathcal{G}_1 h^2 W \mathcal{F}$, one can estimate the integral directly as $\mathcal{F} = O(h^{-1})$ instead of $O(h^{-2})$. ∎

Next, we consider $\mathcal{F}_{res}, \partial_\nu \mathcal{F}_{res}$.

LEMMA 7.2. *Let $\sigma \in \mathbb{C}\backslash 0$ with $\operatorname{Im} z \leq 0$. For $h < h_0$ small depending on $g, |V|_{C^2(M)}$, the Schwartz kernel for $z, z' \in \partial M$ satisfies*

(1) $|r(z,z')\mathcal{F}_{res}(h,z,z')|_{C^0(\partial M \times \partial M)} \leq C|W|^2_{C^0(M)}$,

(2) $|r^2(z,z')\partial_\nu \mathcal{F}_{res}(h,z,z')|_{C^0(\partial M \times \partial M)} \leq Ch^{-1}|W|^2_{C^0(M)}$.

Proof. We start with the formula

$$\mathcal{F}_{res} = \mathcal{R}(h) h^2 W \mathcal{R}(h) h^2 W \mathcal{R}(h) (\mathrm{Id} + h^2 W \mathcal{R}(h))^{-1}.$$

We will study the kernel of $\mathcal{F}_{res,1} = \mathcal{R}(h) h^2 W \mathcal{R}(h) h^2 W \mathcal{R}(h)$ because we know that the Schwartz kernel of $(\mathrm{Id} + h^2 W \mathcal{R}(h))^{-1}$ is integrable and the L^1 norm is bounded in h as $h \to 0$. We use that $\mathcal{R} = \mathcal{G} + \mathcal{F}$ and see that the Schwartz kernel is of the form

$$\begin{aligned}\mathcal{F}_{res,1}(z,z''') = Ch^{-6} \int_{M\times M} & e^{-i\frac{\sigma}{h} r(z,z')} h^2 W(z') e^{-i\frac{\sigma}{h} r(z',z'')} h^2 W(z'') e^{-i\frac{\sigma}{h} r(z'',z'''')} \\ & \cdot \frac{A(z,z',z'',z''')}{r(z',z'')} dz' dz'',\end{aligned}$$

where $z \in \partial M$ and $z', z'', z''' \in M$ and the amplitude A is smooth. We would like to apply the stationary phase argument, but the distance function is not smooth at the diagonal. Let's consider the phase function $\Phi(z,z',z'') = r(z,z') + r(z',z'')$ with integration in $z' \in M$. Here, $z \in \partial M, z'' \in M$. Since W is supported away from ∂M, we just need to consider when z' is close to z''. For fixed $z'' \in M$, we let $B_\epsilon(z'')$ be the ball of radius ϵ centered at z'' and we split the integral (for fixed z'', z'''):

$$\begin{aligned}&\int_M e^{-i\frac{\sigma}{h} r(z,z')} h^2 W(z') e^{-i\frac{\sigma}{h} r(z',z'')} \cdot \frac{A(z,z',z'',z''')}{r(z',z'')} dz' \\ &= \int_{M\backslash B_\epsilon(z'')} e^{-i\frac{\sigma}{h} r(z,z')} h^2 W(z') e^{-i\frac{\sigma}{h} r(z',z'')} \cdot \frac{A(z,z',z'',z''')}{r(z',z'')} dz' \\ &\quad + \int_{B_\epsilon(z'')} e^{-i\frac{\sigma}{h} r(z,z')} h^2 W(z') e^{-i\frac{\sigma}{h} r(z',z'')} \cdot \frac{A(z,z',z'',z''')}{r(z',z'')} dz' = I_1 + I_2.\end{aligned}$$

For integral I_1, we can apply stationary phase argument as before to conclude that $|I_1(z,z'')| \leq Ch^3 |W|_{C^0}$. For I_2, we change the integral to polar coordinate $z' = z'' + \rho w, \rho \in (0,\epsilon), w \in \mathbb{S}^1$, and get

$$\begin{aligned} I_2 &= \int_{\mathbb{S}^1} \int_0^\epsilon e^{-i\frac{\sigma}{h} r(z,z''+\rho w)} h^2 W(z''+\rho w) e^{-i\frac{\sigma}{h}\rho} \cdot \frac{A(z,z''+\rho w,z'',z''')}{\rho} \rho^2 d\rho dw \\ &= \int_{\mathbb{S}^1} \int_0^1 e^{-i\frac{\sigma}{h} r(z,z''+shw)} h^2 W(z''+shw) e^{-i\sigma s} A(z,z''+shw,z'',z''') shh ds dw,\end{aligned}$$

where we changed variables $s = \rho/h$ in the second line. Thus $|I_2(z,z'')| \leq Ch^4 |W|_{C^0}$. To summarize, we get

$$\mathcal{F}_{res,1}(z,z''') = Ch^{-3} \int_M e^{-i\frac{\sigma}{h} r(z,z'')} \mathcal{A}_0(z,z'') h^2 W(z'') e^{-i\frac{\sigma}{h} r(z'',z'''')} \cdot A(z,z',z'',z''') dz'',$$

where $|\mathcal{A}_0|_{C^0} \le C|W|_{C^0}$ and is smooth in z, z''. Now we can apply the stationary phase argument again to get

$$|\mathcal{F}_{res,1}(z, z''')|_{C^0(\partial M\times\partial M)} \le C|W|^2_{C^0(M)}.$$

Similarly, we get

$$|\partial_\nu \mathcal{F}_{res,1}(z, z''')|_{C^0(\partial M\times\partial M)} \le Ch^{-1}|W|^2_{C^0(M)}.$$

This completes the proof of the lemma. ∎

8. The semiclassical DtN map. In this section, we look for the dependency of the DtN map on h. For the classical Calderón problem, it is known (see, for example, [12]) that the DtN map is a pseudodifferential operator of order 1 on ∂M. The approach there is to decompose the elliptic operators in boundary normal coordinates. The method implicitly relies on standard elliptic regularity results which can be studied in this approach; see [23]. For the semiclassical problem, we re-examine the approach and pay attention to the dependency on h. We carry out the construction for dimensions $n\ge 2$ as the argument is the same.

We recall the decomposition of Δ_g from [12, section 2]. We consider the Laplace–Beltrami operator Δ_g in boundary normal coordinates $(x_1, \ldots, x_n)$ in which $\partial M = \{x_n = 0\}$, and we denote $x' = (x_1, \ldots, x_{n-1})$. The metric is of the form

$$g = \sum_{\alpha,\beta=1}^{n-1} g_{\alpha\beta}dx_\alpha dx_\beta + (dx_n)^2.$$

We recall that

$$\Delta_g = -\sum_{i,j=1}^{n} \delta^{-\frac12}\partial_{x_i}(\delta^{\frac12} g^{ij}\partial_{x_j}),$$

where $\delta = |\det g_{ij}|$. We write $D_{x_n} = -i\partial_{x_n}$. In boundary normal coordinates, we have

$$\Delta_g = D^2_{x_n} + iE(x)D_{x_n} + Q(x, D_{x'}),$$

where

$$E(x) = -\frac12 \sum_{\alpha,\beta=1}^{n-1} g^{\alpha\beta}(x)\partial_{x_n} g_{\alpha\beta}(x),$$

$$Q(x, D_{x'}) = \sum_{\alpha,\beta=1}^{n-1} g^{\alpha\beta} D_{x_\alpha}D_{x_\beta} - i\sum_{\alpha,\beta=1}^{n-1}\left(\frac12 g^{\alpha\beta}(x)\partial_{x_\alpha}\log\delta(x) + \partial_{x_\alpha} g^{\alpha\beta})\right) D_{x_\beta};$$

see [12, page 1101]. It is proved in [12, Proposition 1.1] that there exists a pseudodifferential operator $A(x, D_{x'})$ of order 1 in x' depending smoothly on x_n such that

$$\Delta_g = (D_{x_n} + iE(x) - iA(x, D_{x'}))(D_{x_n} + iA(x, D_{x'})) + B,$$

where B denotes a smoothing operator. Now we consider the problem

$$(\Delta_g + V)u - \lambda^2 u = 0.$$

We use the decomposition to get

$$(D_{x_n} + iE(x) - iA(x, D_{x'}))(D_{x_n} + iA(x, D_{x'}))u + Bu - \lambda^2 u = 0.$$

Dividing by λ^2 and setting $h = 1/\lambda$, we get

$$(hD_{x_n} + ihE(x) - ihA(x, D_{x'}))(hD_{x_n} + ihA(x, D_{x'}))u + (h^2 B - 1)u = 0.$$

Now we convert the elliptic problem to a system

$$\begin{aligned} &(hD_{x_n} + ihA(x, D_{x'}))u = v, \\ &(hD_{x_n} + ihE(x) - ihA(x, D_{x'}))v + (h^2 B - 1)u = 0. \end{aligned} \tag{24}$$

The boundary condition $u = f$ at ∂M is converted to $u = f$ at $x_n = 0$. Notice that $hA(x, D_{x'})$ is a semiclassical pseudodifferential operator of order 1. Later, we will denote it by $A(x, hD_{x'})$ to signify the semiclassical nature.

Let's see what we need to do to find the DtN map. We let $U(x_n, h)$ be a semiclassical parametrix such that

$$\begin{aligned} (hD_{x_n} + iA(x, hD_{x'}))U(x_n, h) &= 0 \bmod O(h^\infty) \text{ for } x_n > 0, \\ U(0, h) &= \mathrm{Id}. \end{aligned}$$

Then we can write using Duhamel's principle

$$u(x_n, x') = U(x_n, h)f(x') + \int_0^{x_n} U(x_n - s, h)v(s, x')ds \tag{25}$$

modulo an $O(h^\infty)$ term. To find v, we consider the backward heat equation from $t = T > 0$. Let $W(x_n - T, h)$ be the parametrix of

$$\begin{aligned} (hD_{x_n} + ihE - iA(x, hD_{x'}))W(x_n - T, h) &= 0 \bmod O(h^\infty), \quad x_n < T, \\ W(0, h) &= \mathrm{Id}. \end{aligned}$$

We can write

$$v(x_n, x') = W(x_n - T, h)v(T, x') - \int_{x_n}^{T} W(x_n - s, h)(h^2 B + 1)u(s, x')ds \tag{26}$$

modulo a $O(h^\infty)$ term. The important property of the parametrix we need to establish (in section 9) is that for $t > 0$, $U(t, h)$ and $W(-t, h)$ are smoothing operators of order h^∞. Suppose this is done. We know a priori that the solution u to the Dirichlet problem is smooth and the H^1 norm is of order h^{-2}, which can be seen from the variational form. Using (24), we see that $v(T, x')$ is smooth and the L^2 norm is of order h^{-1}. We conclude from (26) that v is smooth and is of order h^∞ for $x_n \in [0, T)$. Finally, we can use (25) to conclude that $u(x_n, x')$ is smooth for $x_n \in [0, T)$. Also, the second term on the right-hand side of (25) is of order h^∞ for $x_n \in [0, T)$. Now we can derive the DtN map from the first equation of (24) as

$$D_{x_n} u = h^{-1} iA(x, hD_{x'})u = h^{-1} A(x, hD_{x'})f \text{ at } x_n = 0$$

up to an $O(h^\infty)$ smooth term.

Using this description of the DtN map, we will prove that the following proposition holds.

PROPOSITION 8.1. *Let (M,g) be as in Theorem 1.1 except that the dimension of M is $n \geq 2$. Consider the semiclassical Dirichlet problem*

$$h^2(\Delta_g + V) - u = 0 \text{ in } M,$$
$$u = f \text{ at } \partial M.$$

For the DtN map Λ defined in (2), the Schwartz kernel $\Lambda(z,z'), z, z' \in \partial M$, is such that $|z-z'|\Lambda(z,z')$ is continuous in z,z' and bounded for h small. Here, $|\cdot|$ denotes the Euclidean norm.

The key in this approach is to construct a parametrix of the semiclassical heat equations (24) which we study next.

9. Parametrix of semiclassical heat equations. We briefly review the basics of semiclassical quantization from [26]. For $h \in [0,1)$, consider $a(h,x,\xi) \in C^\infty([0,1); S^m(\mathbb{R}^n_x; \mathbb{R}^n_\xi))$. Here, $S^m(\mathbb{R}^n; \mathbb{R}^n)$ is the standard symbol class, which is the set of C^∞ functions on $\mathbb{R}^n_x \times \mathbb{R}^n_\xi$ satisfying

$$|D_x^\alpha D_\xi^\beta a(h,x,\xi)| \leq C_{\alpha\beta}\langle\xi\rangle^{m-|\beta|}$$

for all $\alpha, \beta \in \mathbb{N}^n$. The estimate is uniform on a compact set of $\mathbb{R}^n_x$. When the context is clear, we also abbreviate the notation as S^m. The semiclassical operator with symbol a is defined as

$$A(x,hD)u(x) = \mathrm{Op}_\hbar(a)u(x) = (2\pi h)^{-n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi/h}a(h,x,\xi)u(y)dyd\xi.$$

Here, we only use the standard quantization. The semiclassical principal symbol is $\sigma_{\hbar,m}(A) = a|_{h=0} \in S^m$.

For $h \in [0,1), t \geq 0, x', \xi' \in \mathbb{R}^{n-1}$, we consider symbols $a(h,t,x',\xi') \in C^\infty([0,1)_h \times [0,\infty); S^m(\mathbb{R}^{n-1}_{x'}, \mathbb{R}^{n-1}_{\xi'}))$. We assume that

$$a(h,t,x',\xi') \sim \sum_{j=0}^\infty h^j a_j(t,x',\xi'), \quad a_j \in C^\infty([0,1)_h; S^m).$$

Moreover, the semiclassical principal symbol a_0 is elliptic, namely

$$|a_0(t,x',\xi')| \geq \gamma|\xi'|^m \text{ for } \gamma > 0 \text{ and all } (t,x').$$

Let $A(t,x',hD)$ be the semiclassical quantization of $a(h,t,x',\xi')$. We consider semiclassical heat equations

$$(hD_t + A(t,x',hD_{x'}))u = 0 \text{ for } t > 0, x' \in \mathbb{R}^{n-1},$$
$$u = f, \quad t = 0. \tag{27}$$

The argument below follows closely section 1.1 of Chapter III of [23]. We aim to find a semiclassical parametrix of the form

$$U(t,x',hD_{x'}) = \sum_{j=0}^\infty h^j U_j(t,x',hD_{x'}),$$

where U_j are semiclassical quantizations of $u_j(t,x',\xi') \in S^{-m}$, that is,

$$U_j(t)f(x') = (2\pi h)^{-(n-1)}\int e^{i(x'-y')\xi'/h}u_j(t,x',\xi')f(y')d\xi'dy'. \tag{28}$$

We write $x=(t,x')\in\mathbb{R}^n$ below. Formally, we have

$$\begin{aligned}&(hD_t+A(x,hD_{x'}))U(x,hD_{x'})\\ &\quad=\sum_{j=0}^{\infty}h^jh\partial_tU_j(x,hD_{x'})+\sum_{j=0}^{\infty}h^jA(x,hD_{x'})U_j(x,hD_{x'}).\end{aligned}\tag{29}$$

Let $C_j(x,hD_{x'})=A(x,hD_{x'})U_j(x,hD_{x'})$ be the composition. We use semiclassical calculus for standard quantization to conclude that C_j are semiclassical pseudodifferential operators with full symbols

$$c_j(h,x,\xi')=\sum_{k=0}^{N}\frac{h^k}{k!}(i\langle D_{\xi'},D_{y'}\rangle)^k(a(x',\xi')u_j(y',\eta'))|_{y'=x',\eta'=\xi'}+O(h^{N+1})$$

as $h\to 0$. Here, $c_j\in C^\infty([0,1)_h;S^0)$. See Theorems 4.14 and 4.18 of [26]. Using the asymptotic expansion of $a(x,\xi)$, we find that the symbol expansion of $A(x,hD_{x'})$ $U(x,hD_{x'})$ is $\sum_{j=0}^{\infty}h^jd_j(t,x',\xi')$, where

$$d_j(t,x',\xi')=\sum_{k=0}^{j}\sum_{l=0}^{k}\frac{1}{l!}(i\langle D_{\xi'},D_{y'}\rangle)^l(a_{k-l}(x',\xi')u_{j-k}(y',\eta'))|_{y'=x',\eta'=\xi'}.$$

Using this formula, we get from (29) that

$$\begin{aligned}&(hD_t+A(x,hD_{x'}))U(x,hD_{x'})=(2\pi h)^{-(n-1)}\\ &\quad\cdot\sum_{j=0}^{\infty}h^j\int_{\mathbb{R}^{n-1}}e^{i(x'-y')\xi'/h}\cdot(h\partial_tu_j(t,x',\xi')+d_j(t,x',\xi'))d\xi'.\end{aligned}\tag{30}$$

From the order h^j terms, we get equations

$$h\partial_tu_j(t,x',\xi')+a_0(t,x',\xi')u_j(t,x',\xi')+e_j(t,x',\xi')=0\quad\forall j\ge 0,\tag{31}$$

where $e_0=0$ and for $j\ge 1$ we have

$$\begin{aligned}e_j(t,x',\xi')&=d_j(t,x',\xi')-a_0(t,x',\xi')u_j(t,x',\xi')\\ &=\sum_{k=1}^{j}\sum_{l=0}^{k}\frac{1}{l!}(i\langle D_{\xi'},D_{y'}\rangle)^l(a_{k-l}(x',\xi')u_{j-k}(y',\eta'))|_{y'=x',\eta'=\xi'}.\end{aligned}$$

Notice that the term e_j involves $u_k, k<j$. So we will solve these equations iteratively. Equation (31) comes with initial conditions. At $t=0$, we get

$$U_0(0,x',hD_{x'})=\mathrm{Id},\quad U_j(t,x',hD_{x'})=0,\quad j\ge 1.$$

This implies that

$$u_0(0,x',\xi')=1,\quad u_j(0,x',\xi')=0,\quad j\ge 1.\tag{32}$$

To solve (31) with (32), we look for solutions of the form

$$u_j(t,x',\xi')=(2\pi i)^{-1}\int_\gamma e^{\rho zt/h}k_j(t,x',\xi',z)dz,\tag{33}$$

where $\rho = \langle \xi' \rangle^m$, $z \in \mathbb{C}$ is a complex parameter, and γ is a contour so that the integrand is holomorphic in a neighborhood of γ for t, x' in a compact set. Plugging this into (31), we obtain

$$(2\pi i)^{-1} \int_\gamma e^{\rho z t/h} \mathcal{L} k_j dz = 0,$$
$$\mathcal{L} k_j = \rho z k_j + h D_t k_j + a_0 k_j + e_j.$$

Now $hD_t k_j$ has an additional h. So we should look at the asymptotics in (30). After rearrangement, we get from the order h^j term the equations

$$\widetilde{\mathcal{L}} k_0 = \rho z k_0 + a_0 k_0,$$
$$\widetilde{\mathcal{L}} k_j = \rho z k_j + a_0 k_j + e_j + D_t k_{j-1}, \quad j \geq 1.$$

We aim to solve k_j from $\widetilde{\mathcal{L}} k_j = \rho$. Then we justify that the choices solve (31) and (32).

First, we recall that $a_0 \in S^m$ is elliptic. Thus $\rho^{-1} a_0 \in S^0$ is also elliptic and we see that for $\xi' \in \mathbb{R}^{n-1}$, $z - \rho^{-1} a_0$ is nonzero for $\operatorname{Im} z < 0$. Let $E(z) = (z - \rho^{-1} a_0)^{-1}$. For $j = 0$, we solve $\widetilde{\mathcal{L}} k_0 = \rho$ to get $k_0 = E(z)$. For $j \geq 1$, we get

$$k_j = -E(z) \rho^{-1} [D_t k_{j-1} + e_j].$$

It follows from the argument on page 137 of [23] that k_j are smooth in (t, z) and valued in S^{m_j} and $m_j \leq -j \min(1, m)$. This finishes the construction. Finally, we prove the regularizing properties of the parametrix.

LEMMA 9.1. *The operators $U_j(t)$ with u_j defined in* (33) *are smoothing operators and belong to $O(h^\infty)$ for $t > 0$.*

Proof. We look at

$$u_j(t, x', \xi') = (2\pi i)^{-1} \int_\gamma e^{\rho z t/h} k_j(t, x', \xi', z) dz$$

in which $k_j \in C^\infty([0, \infty); S^{m_j})$. We estimate

$$|\partial_{x'}^\alpha \partial_{\xi'}^\beta \partial_t^\gamma k_j(t, x', \xi')| \leq C \langle \xi' \rangle^{m_j - |\beta|}.$$

Also, we estimate

$$|\partial_t^\gamma \partial_{\xi'}^\beta e^{\rho z t/h}| \leq C (t/h)^{-N} (1 + |\xi'|)^{-N}.$$

Putting the estimate together, we obtain that the operators $U_j(t)$ are indeed smoothing operators and of h^∞. ∎

Now we apply the construction to the two heat equations in (24) to conclude that $U(t, h), W(-t, h)$ are smoothing operators of order h^∞ for $t > 0$. We can finish the proof of Proposition 8.1.

Proof of Proposition 8.1. For the DtN map Λ, we know that modulo a smooth $O(h^\infty)$ term, the Schwartz kernel is

$$\Lambda(x', y') = h^{-1} (2\pi i h)^{-(n-1)} \int e^{i(x'-y')\cdot\xi'/h} a(x', \xi', h) d\xi'$$
$$= h^{-1} (2\pi i)^{-n-1} \int e^{i(x'-y')\cdot\xi'} a(x', h\xi', h) d\xi'.$$

Because a is a symbol of order 1, the Schwartz kernel has a singularity like $|x' - y'|$ and $|x' - y'| \Lambda(x', y')$ is bounded in h. ∎

10. Proof of Theorem 1.1. Let's consider two potentials $V, \widetilde{V}$ on M, with $\Lambda, \widetilde{\Lambda}$ the corresponding DtN maps. Suppose $\Lambda = \widetilde{\Lambda}$. We use (7)

$$\partial_\nu \widetilde{\mathcal{R}}(z,z',h) - \partial_\nu \mathcal{R}(z,z',h) = \int_{\partial M} \widetilde{\mathcal{R}}(z,z'',h)\widetilde{\Lambda}(z'',z')dz'' - \int_{\partial M} \mathcal{R}(z,z'',h)\Lambda(z'',z')dz''$$

and (16)

$$\mathcal{R}(z,z',h) - \widetilde{\mathcal{R}}(z,z',h) = \mathcal{F}_{lead} + \mathcal{F}_{para} + \mathcal{F}_{res}$$

and the estimates of the remainder terms to finish the proof.

We start from the left-hand side of (7). Using the results in section 6, we get

$$\begin{aligned}\partial_\nu \widetilde{\mathcal{R}}(z,z',h) - \partial_\nu \mathcal{R}(z,z',h) &= Ch^{-2}\partial_\nu r(z,z'')e^{i\sigma r(z,z'')/h}\mathcal{X}^w W(z,z') \\ &+h^{-1}e^{i\sigma r(z,z'')/h}\partial_\nu \mathcal{X}^w W(z,z') + \partial_\nu \mathcal{F}_{lead,1} + \partial_\nu \mathcal{F}_{res} + \partial_\nu \mathcal{F}_{para}.\end{aligned} \tag{34}$$

For the right-hand side of (7), we write it as

$$\begin{aligned}&\int_{\partial M} \widetilde{\mathcal{R}}(z,z'',h)\widetilde{\Lambda}(z'',z')dz'' - \int_{\partial M} \mathcal{R}(z,z'',h)\Lambda(z'',z')dz'' \\ &\quad= \int_{\partial M}(\widetilde{\mathcal{R}}(z,z'',h) - \mathcal{R}(z,z'',h))\widetilde{\Lambda}(z'',z')dz'' \\ &\quad\quad+ \int_{\partial M} \mathcal{R}(z,z'',h)(\widetilde{\Lambda}(z'',z') - \Lambda(z'',z'))dz''.\end{aligned} \tag{35}$$

Since $\Lambda = \widetilde{\Lambda}$, we only need to consider the first term. But we remark that one can derive a stability estimate from the second term, although we do not pursue it here. We have

$$\begin{aligned}&\int_{\partial M}(\widetilde{\mathcal{R}}(z,z'',\sigma,h) - \mathcal{R}(z,z'',\sigma,h))\widetilde{\Lambda}(z'',z')dz'' \\ &\quad= \int_{\partial M} h^{-1}e^{i\sigma r(z,z'')/h}\mathcal{X}^w W(z,z'')\widetilde{\Lambda}(z'',z')dz'' \\ &\quad\quad+ \int_{\partial M}(\mathcal{F}_{lead,1}(z,z'') + \mathcal{F}_{res}(z,z'') + \mathcal{F}_{para}(z,z''))\widetilde{\Lambda}(z'',z')dz''.\end{aligned} \tag{36}$$

Now we can use the estimate of $\widetilde{\mathcal{R}} - \mathcal{R}$ and the kernel estimate of $\widetilde{\Lambda}$ in Proposition 8.1. Also, we will use the stability estimate of $\mathcal{X}^w$ in Theorem 6.1. From (34) and (36), we get

$$\begin{aligned}\|\mathcal{X}^w W\|_{L^2} &\le Ch^2(h^{-1}\|\partial_\nu \mathcal{X}^w W\|_{L^2} + |\partial_\nu \mathcal{F}_{lead,1} + \partial_\nu \mathcal{F}_{res} + \partial_\nu \mathcal{F}_{para}|_{C^0}) \\ &+Ch|\mathcal{X}^w W|_{C^0} + Ch^2|\mathcal{F}_{lead,1} + \mathcal{F}_{res} + \mathcal{F}_{para}|_{C^0}.\end{aligned} \tag{37}$$

The estimates of these terms are done in Lemmas 7.1 and 7.2 and (22). Also, we used that the boundary ∂M is strictly convex and that we only consider $z, z' \in \partial M$ and away from $z = z'$ in (7), so the absolute value of the $\partial_\nu r$ term in (34) is bounded from below. By the continuity of $\mathcal{X}^w : H^1 \to H^1$, we get from (37) that

$$\|W\|_{L^2(M)} \le Ch|W|_{C^2(M)} + Ch|W|^2_{C^0(M)} \le Ch|W|_{C^2(M)} \tag{38}$$

if $|W|_{C^0(M)} \le C_0$.

To finish the proof of Theorem 1.1, we use $|W|_{C^2(M)} \le C_1 \|W\|_{L^2(M)}$ for some C_1 (depending on $V, \widetilde{V}$). Then we conclude that $\|W\|_{L^2(M)} \le Ch\|W\|_{L^2(M)}$. So for h sufficiently small (depending on $V, \widetilde{V}$), we get $W = 0$ and complete the proof. ∎

Remark 10.1. Let K be a compact set of M. For $C_0, C_1 > 0$, we define

$$\mathscr{W} = \{W \in C_0^\infty(M) : \text{supp } W \subset K, |W|_{C^2(M)} \le C_0, |W|_{C^2(M)} \le C_1\|W\|_{L^2(M)}\}. \tag{39}$$

For potentials $V, \widetilde{V}$ such that $V - \widetilde{V} \in \mathscr{W}$, it follows from the proof of Theorem 1.1 that such potentials are uniquely determined by their DtN map for a sufficiently small h which only depends on C_0, C_1. Actually, one can obtain Hölder type stability estimates by examining the last term in (35). This agrees with the phenomena of increased stability for high frequency Schrödinger operators on $\mathbb{R}^n$. See the recent work [8] and the references therein. We remark that the last inequality in (39) resembles the so-called inverse inequality in numerical methods; see, for example, [9, section 6.2].

Remark 10.2. Our proof leads to an approximate reconstruction method. From (34), we get that

$$\mathcal{X}^w(\widetilde{V} - V) = \frac{Ch^2}{\partial_\nu r(z,z')} \int_{\mathbb{R}} \mathcal{R}(z, z'', h)(\widetilde{\Lambda}(z'', z') - \Lambda(z'', z'))dz'' + O(h),$$

where the function $\mathcal{W}$ in $\mathcal{X}^w$ and the constant can be found explicitly from the proof. Note that they only depend on the background manifold (M, g). Thus, to reconstruct a potential $\widetilde{V}$ from $\widetilde{\Lambda}$, we can choose a reference potential $V = 0$ with corresponding Λ which can be computed for the manifold (M, g). Therefore, for h sufficiently small, we can find $\widetilde{V}$ approximately by inverting the geodesic ray transform.

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