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4 Inverse problems in cosmological X-ray tomography

Abstract: We consider the recovery of space-time structures from the Cosmic Microwave Background (CMB) using tomography methods. On the linearization level, the problem concerns an X-ray transform in Lorentzian geometry, called the light ray transform. We review recent results on the mathematical properties of the transform, and their applications to the CMB inverse problem for various physical models.

Keywords: Cosmic microwave background, light ray transform, microlocal analysis, Lorentzian geometry, integral geometry

MSC 2010: 35Q85, 35A27, 44A12

4.1 Introduction

The purpose of this paper is to review recent progresses on the inverse problem of recovering spacetime structures from the Cosmic Microwave Background (CMB). The study of CMB has a rich history in astrophysics, and there is a large literature on both theoretical and experimental results. Recently, the inverse problem has been explored from the tomography point of view, which is relatively new to the field. The new perspective has lead to many interesting results and challenging mathematical problems.

We briefly recall that CMB is the remnant microwave radiation from the Big Bang. It was discovered by Penzias and Wilson in 1964 and soon became a major source of information regarding the early universe; see Figure 4.1. Two main aspects of the CMB have been explored. First, the CMB temperature is highly smooth and isotropic. The famous EGS theorem [11] says that the isotropy of the CMB implies the isotropy and spatial homogeneity of the universe. Second, the CMB contain faint anisotropies, which can now be mapped by sensitive satellite detectors such as COBE, WMAP and Planck Surveyor. The anisotropies contain rich information regarding the early universe. More precisely, as demonstrated in a seminar paper of Sachs and Wolfe [38] in 1966, primordial perturbations produce anisotropies in the CMB. On the linearization level, the anisotropy is

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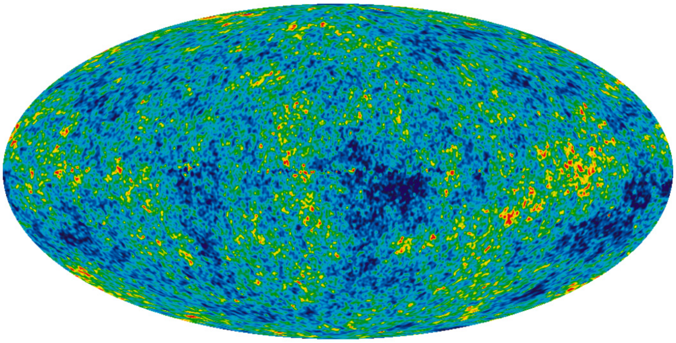

Figure 4.1: All-sky picture of the infant universe created from nine years of Wilkinson Microwave Anisotropy Probe (WMAP) data. Picture courtesy to NASA. The image reveals 13.77 billion-year-old temperature fluctuations (shown as color differences) that correspond to the seeds that grew to become the galaxies. This image shows a temperature range of ± 200 microKelvin.

related to an integral transform of the gravitational perturbations along null geodesics from the "surface of last scattering," known as the (integrated) Sachs–Wolfe effect. The integral transform is the "cosmological X-ray transform" (or the light ray transform) in the title. The inverse problem we investigate is to recover information by "inverting" the transform. By nature, the problem is similar to the famous Radon transform that is widely used in medical imaging. One can say that we are performing an X-ray CT for our universe.

Despite the similarity, the inversion of the cosmological X-ray transform is much more challenging than the Radon transform. In particular, the inverse problem is severely ill-posed. Perhaps Guillemin was the first to note the issue when he encountered the transform in the study of the Lorentzian version of the Zoll problem. In his 1989 monograph [19], Guillemin phrased the issue as "no observer in spacetime can be privy to events occurring beyond his own causal horizon." The instability can be understood well using techniques from microlocal analysis. We review recent results in [30, 31, 51, 52]. The results imply what type of singularities in the gravitational perturbation can be recovered. One immediate application is to find cosmic strings from the CMB, which was one motivation of [30]. The study of cosmic strings has a long history (see, e. g., [50]), although their existence has not been confirmed yet.

Another fascinating object that has been suggested to look for in the CMB is the gravitational waves generated in the early universe, called primordial gravitational waves; see, for example, [8, 10, 25]. Unlike the gravitational waves generated from compact binary collisions, which can be detected by LIGO nowadays, primordial gravitational waves, quoted from [25], "will involve waves today whose wave lengths will extend all the way up to our present cosmological horizon (the distance out to which we can currently observe in principle) and that are likely to be well beyond the reach of any direct detectors for the foreseeable future." Theoretical study has shown that these gravitational waves should leave indirect signatures in the CMB, known as polarizations. Is it possible to identify these waves from the cosmological X-ray transform? The answer is very likely to be yes, at least for scalar-type gravitational perturbations as demonstrated in [49, 53] and [54] for the kinetic model.

There are more to explore, and we mention a few developments that we are not able to discuss in this article. For example, we mainly consider the CMB inverse problem on the linearization level. The nonlinear problem remains open, but there are interesting results on the much related Lorentzian scattering rigidity problem; see [12, 13, 42, 55]. Also, from a practical point of view, it makes sense to assume that CMB is measured near a freely falling observer instead of on a whole Cauchy surface. The partial data inverse problem was studied in [30] for recovering singularities. Yet another practical consideration is to develop numerical algorithms for inverting the light ray transform; see [6]. The ill-posedness makes the problem particularly challenging. Finally, we remark that in addition to the CMB inverse problem, the light ray transform plays an important role in other applications; see, for example, [47] for the hyperbolic Dirichlet-to-Neumann map problem, and [4] for the recovery of bulk geometry in the AdS/CFT correspondence.

This paper is organized as follows. In Section 4.2, we formulate three inverse problems from the physical problem. Then we describe the mathematical results in Sections 4.3–4.5. In Section 4.3, we review the microlocal results for the light ray transform. In Section 4.4, we review results for the light ray transform under the wave equation constraint. In Section 4.5, we consider the inverse source problem for the linear Boltzmann equation. Finally, we propose some open problems in Section 4.6.

4.2 The inverse problems

In this section, we formulate three inverse problems from the physical problem. We refer to [8, 10] for the detailed physical backgrounds on CMB. Our basic setup is the Friedman–Lemaîte–Robertson–Walker (FLRW) model for the universe:

$$\mathcal{M} = (0, \infty) \times \mathbb{R}^3, \quad g_0 = -dt^2 + a^2(t)dx^2 \tag{4.1}$$

where (t, x), $t \in (0, \infty)$, $x \in \mathbb{R}^3$ are coordinates. The factor $a(t)$ is assumed to be positive and smooth in t. It represents the rate of expansion of the universe. As we concern the linearized problem, we assume that the actual universe is a smooth one parameter family of metric perturbations $g_\epsilon = g_0 + \epsilon g_1 + O(\epsilon^2)$ on $\mathcal{M}$. For the ease of elaboration, we will make a few simplifications. First, we take $a(t) = 1$ in g_0 so g_0 becomes the Minkowski metric. In fact, the FLRW metric in (4.1) is conformal to a metric isometric to Minkowski and one can prescribe g_ϵ after the conformal transformation. Most of the results that we discuss in this work hold for general $a(t)$. Second, in the literature, the metric perturbations are often classified to scalar, vector and tensor type. We refer to [10, Section

2.3] for a discussion of the classification. We will focus on the scalar-type perturbations of the form

$$g_\epsilon = -(1+\epsilon\Phi)dt^2 + (1-\epsilon\Psi)dx^2 + O(\epsilon^2) \tag{4.2}$$

where Φ, Ψ are scalar functions on $\mathcal{M}$. In Section 4.6, we will briefly discuss the problems for tensor perturbations, which are mostly open.

4.2.1 The cosmological X-ray transform

Consider the measurement of CMB. Let $\mathcal{M}_0 = \{t_0\} \times \mathbb{R}^3$, $t_0 > 0$ be the surface of last scattering. This is the moment after which photons stopped interaction and started to travel freely in $\mathcal{M}$. Let $\mathcal{M}_1 = \{t_1\} \times \mathbb{R}^3$, $t_1 > t_0$ be the surface where we make observation of the photons. Because we are mostly interested in the region between $\mathcal{M}_0$ and $\mathcal{M}_1$, we will take $\mathcal{M} = (t_0, t_1) \times \mathbb{R}^3$ from now on.

Let $\gamma_\epsilon(\tau)$ be a null geodesic from $\mathcal{M}_0$ to $\mathcal{M}_1$ in metric g_ϵ where $\tau \in [0, \tau_\epsilon]$, $\tau_\epsilon > 0$. It represents the trajectory of photons in $\mathcal{M}$. The photon energies observed at $\mathcal{M}_0$, $\mathcal{M}_1$ are defined by

$$E_0 = (\dot\gamma_\epsilon(0), \partial_t)_{g_\epsilon}, \quad E_1 = (\dot\gamma_\epsilon(\tau_\epsilon), \partial_t)_{g_\epsilon}.$$

Here, the observer is represented by the flow of the vector field ∂_t. The CMB redshift R_ϵ is defined via $1 + R_\epsilon = E_1/E_0$. It is proved in Lemma 3.2 of [30] that for $\epsilon > 0$ sufficiently small,

$$R_\epsilon(x, v) = (\dot\gamma_\epsilon(\tau_\epsilon(z,\theta); z, \theta), \partial_t)_{g_\epsilon} - 1$$

where $z = (t_1, x)$, $\theta = -(1, v)$. Here, we parametrized null geodesics γ_ϵ on $(\mathcal{M}, g_\epsilon)$ using $(x, v) \in \mathbb{R}^3 \times \mathbb{S}^2$.

Now we proceed to find the linearization of R_ϵ; see Section 7 of [30] for details. First,

$$\partial_\epsilon R_\epsilon = \partial_\epsilon\big((\dot\gamma_\epsilon(\tau_\epsilon; z, \theta), \partial_t)_{g_\epsilon} - (\dot\gamma_\epsilon(0; z, \theta), \partial_t)_{g_\epsilon}\big). \tag{4.3}$$

Next, we use the geodesic equation for γ_ϵ in the form

$$\partial_\tau(g_{\epsilon,ij}\dot\gamma_\epsilon^k) = \frac{1}{2}(\partial_j g_{\epsilon,lm})\dot\gamma_\epsilon^l\dot\gamma_\epsilon^m \tag{4.4}$$

Hereafter, we use Einstein summation convention with indices running from $0, 1, 2, 3$. Consider $g_1 = \partial_\epsilon g_\epsilon|_{\epsilon=0}$. We find that

$$\partial_\epsilon((\partial_j g_{\epsilon,lm})\dot\gamma_\epsilon^l\dot\gamma_\epsilon^m)|_{\epsilon=0} = (\partial_j g_{1,lm})\dot\gamma_0^l\dot\gamma_0^m$$

where we used the fact that g_0 is a constant metric. Thus, we get by using (4.4) and (4.3) that

$$\partial_\epsilon R_\epsilon|_{\epsilon=0} = \frac{1}{2}\int_0^{\tau_0} (\partial_t g_{1,lm}(\gamma_0(\tau)))\dot{\gamma}_0^l(\tau)\dot{\gamma}_0^m(\tau)d\tau. \tag{4.5}$$

This is essentially what Sachs and Wolfe derived in [38, Equation (39)], and the term is called the integrated Sachs–Wolfe effect. We remark that in the derivation of (4.3), we actually assumed $g_\epsilon = g_0$ at $\mathcal{M}_1$. Otherwise, there will be another term in (4.5) called the ordinary Sachs–Wolfe effect. The integrated Sachs–Wolfe effect can be extracted from the CMB and other astrophysical data; see, for example, [34].

For scalar perturbations in (4.2), (4.5) becomes

$$\partial_\epsilon R_\epsilon|_{\epsilon=0} = \frac{1}{2}\int_0^{\tau_0} (\partial_t \Phi(\gamma_0(\tau)) + \partial_t \Psi(\gamma_0(\tau)))d\tau. \tag{4.6}$$

The inverse problem is to recover Φ, Ψ from (4.6). We remark that in the derivation of (4.6), we assumed that R_ϵ can be observed at the whole Cauchy surface $\mathcal{M}_1$. In reality, we can only hope to observe CMB along the world-line of a satellite. Thus, the more realistic model should be the inversion of (4.6) for null geodesic γ that intersects a neighborhood of a time-like curve; see the local formulation in [30].

4.2.2 The primordial gravitational waves

For the evolution of the universe, it is reasonable to assume within Einstein's general relativity theory that g_ϵ satisfies the Einstein equations with certain source fields and initial perturbations at $\mathcal{M}_0$. On the linearization level, this means that g_1 satisfies the linearized Einstein equations. The formulation of CMB in this setup has been studied in cosmological literatures; see, for example, [8, Section 5.1] and [10]. Let $R^\mu{}_\nu, \mu, \nu = 0,1,2,3$ be the Ricci curvature tensor and R the scalar curvature on $(\mathcal{M}, g_0)$. Let $T^\mu{}_\nu$ denote the stress-energy tensor of certain source fields. The Einstein equations are

$$G^\mu{}_\nu = 8\pi G T^\mu{}_\nu, \quad G^\mu{}_\nu = R^\mu{}_\nu - \frac{1}{2}\delta^\mu{}_\nu R$$

where G is Newton's gravitational constant. The explicit form of the linearized Einstein equations can be found in [37, Sections 4–6]. We consider two important examples of the sources: the perfect fluid and the scalar field.

First, consider the perfect fluid sources. Let u be the four fluid velocity of a fluid source. The stress-energy tensor for a perfect fluid is

$$T^\mu{}_\nu = (\epsilon + p)u^\mu u_\nu - p\delta^\mu{}_\nu$$

see [37, Equation (5.2)], Here, ϵ is the energy density and p is the pressure of the fluid. We assume that $\epsilon = \epsilon_0 + \delta\epsilon$, $p = p_0 + \delta p$ where 0 denotes the quantity for the background and δ denotes the perturbations. For fluid source, one deduces that the perturbations $\Phi = \Psi$. In the case of adiabatic perturbations, Φ satisfies the following equation, called Bardeen's equation:

$$\Phi'' - c_s^2 \Delta\Phi = 0, \tag{4.7}$$

where $'$ denotes t derivative and $c_s < 1$ is the speed of sound; see [37, Equation (5.22)]. We remark that the equation is simplified because we only consider the Minkowski background. Also in general, the right-hand side of the equation can have a nonzero term related to the entropy perturbations. Prescribing Cauchy data of Φ at $\mathcal{M}_0$, one can solve the Cauchy problem of (4.7) to get Φ in $\mathcal{M}$.

Next, let us consider the universe governed by a scalar field ϕ. The stress energy tensor is

$$T^\mu{}_\nu = \nabla^\mu\phi\nabla_\nu\phi - \left[\frac{1}{2}\nabla^\alpha\phi\nabla_\alpha\phi - V(\phi)\right]\delta^\mu{}_\nu$$

see [37, Equation (6.2)]. Here, V is the potential function for the scalar field ϕ. The field itself satisfies the Klein–Gordon equation $\Box\phi + \partial_\phi V(\phi) = 0$. Now assume that $\phi = \phi_0 + \delta\phi$ where ϕ_0 is the scalar field, which drives the background model and $\delta\phi$ denotes the perturbation. Again, one finds that $\Phi = \Psi$ and it satisfies the equation

$$\Phi'' - 2(\phi_0''/\phi_0')\Phi' - \Delta\Phi = 0; \tag{4.8}$$

see [37, Equation (6.48)]. This is a damped wave equation with wave speed $c = 1$.

For the above two scenarios, the inverse problem is to recover information of Φ from the integrated Sachs–Wolfe effect

$$\partial_\epsilon R_\epsilon|_{\epsilon=0} = \int_0^{\tau_0} \partial_t\Phi(\gamma_0(\tau))d\tau, \tag{4.9}$$

assuming that Φ is a solution of the Cauchy problem for wave equations in (4.7) or (4.8).

4.2.3 The kinetic model

In the derivation of the previous two problems, we assumed that photons travel freely in $\mathcal{M}$. This pure transport regime serves as a good model for the standard universe after the decoupling time. Before the decoupling time, photon interactions cannot be ignored and a kinetic model based on the Boltzmann equation is appropriate. As is well known in cosmology literatures (e. g., [8, 10]), the linearization of the Boltzmann equation on a

FLRW universe with respect to small metric perturbations naturally leads to a source problem for the Boltzmann equation in which the source term is related to the metric perturbation. We briefly discuss the derivation in [54].

Let f_ϵ be the photon distribution function, which is a function of z, p variables where $z \in \mathbb{R}^{3+1}$ and p is on the mass shell $\Sigma_z = \{p \in T_z\mathbb{R}^{3+1} : g_\epsilon(p,p) = 0\}$. We assume that f_ϵ satisfies the linear Boltzmann equation; see [10, Section 4.5]. This means that along γ_ϵ,

$$\frac{d}{ds} f_\epsilon(\gamma_\epsilon(s), p_\epsilon(s)) = C[f_\epsilon], \tag{4.10}$$

where $C[f]$ denotes the interaction term

$$C[f] = -\sigma(z)f(z,p) + \int k(z,\theta,\theta')f(z,v(1,\theta'))d\theta' \tag{4.11}$$

where σ denotes absorption coefficients, k is the scattering kernel and the integration is over $\{\theta : v(1,\theta) \in \Sigma_z \text{ for } v > 0\}$. The terms in (4.11) accounts for photon interactions in Thomson scattering, for example. When $C[f_\epsilon] = 0$, we essentially return to the model in Section 4.2.1.

From (4.10) and (4.11), we get the equation

$$\begin{aligned}&\sum_{i=0}^{3} \frac{\partial f_\epsilon}{\partial z^i}(z,p)\frac{\partial \gamma_\epsilon^i}{\partial s} + \frac{\partial f_\epsilon}{\partial p}(z,p)\frac{\partial p_\epsilon}{\partial s}\\ &\quad = -\sigma(z)f_\epsilon(z,p) + \int k(z,\theta,\theta')f_\epsilon(z,v(1,\theta'))d\theta' \end{aligned}\tag{4.12}$$

Now we consider f_ϵ as a perturbation of some background distribution with an expansion

$$f_\epsilon(z,p) = f_0(v) + \epsilon f_1(z,v,\theta) + O(\epsilon^2) \tag{4.13}$$

Here, f_0 is the background photon distribution. When modeling the CMB, it is reasonable to assume that f_0 satisfies the Planck distribution

$$f_0(v) = (e^{v/T_0} + 1)^{-1};$$

see [10, p. 149]. Here, $T_0 > 0$ is the background temperature of the universe. Also, f_1 in (4.13) is the first-order perturbation term and θ is taken over $\mathbb{S}^2$. In particular, $(1,\theta)$ is a future pointing light-like vector for the background Minkowski metric g_0. Now we fix $v = 1$ and derive

$$\begin{aligned}&\frac{\partial f_1}{\partial t}(z,\theta) + \sum_{j=1}^{3}\theta^j\frac{\partial f_1}{\partial z^j}(z,\theta) + \sigma(z)f_1(z,\theta) - \int_{\mathbb{S}^2} k(z,\theta,\theta')f_1(z,\theta')d\theta'\\ &\quad = C\left(\frac{1}{2}\frac{\partial \Psi}{\partial t} - \frac{1}{2}\sum_{j=1}^{3}\frac{\partial \Phi}{\partial z^j}\theta^j\right)\end{aligned}\tag{4.14}$$

where C is a nonzero constant and σ, k are changed by a scalar factor. The right-hand side comes from the linearization term $\partial_\epsilon \frac{\partial p_\epsilon^0}{\partial t}|_{\epsilon=0}$.

Now the CMB inverse problem is to determine Φ, Ψ from the measurement of f_1 at $t = T$, which is essentially an inverse source problem for the linear Boltzmann equation (4.14). Here, one can also consider the setup in Section 4.2.2 that $\Phi = \Psi$ is a solution of the Cauchy problem of the wave equations.

4.3 Recovery of singularities

We start with the inverse problem in Section 4.2.1. More generally, let $(\mathcal{M}, g)$ be an $n+1, n \geq 2$ dimensional smooth Lorentzian manifold. Let γ be a complete light-like (or null) geodesic, which means that $\gamma(s)$ is defined for $s \in \mathbb{R}$ and $\dot\gamma(s)$ satisfies $g(\dot\gamma(s), \dot\gamma(s)) = 0$. We consider the light ray transform

$$(Lf)(\gamma) = \int_{\mathbb{R}} f(\gamma(s))ds, \quad f \in C_0^\infty(\mathcal{M}) \tag{4.15}$$

when the integral is well-defined. Note that even for C_0^∞ functions, the integral may not converge because γ may be trapped in the support of f. There are very few results on the injectivity of L, and we will discuss them in Section 4.6. In this section, we review results for the recovery of microlocal singularities of f from Lf.

4.3.1 The space-like singularities

To understand the microlocal structure of L, let us start from the light ray transform for Minkowski spacetime where explicit calculations can be done. Let g be the Minkowski metric on $\mathcal{M} = \mathbb{R}^{n+1}$, $n \geq 2$. We parametrize null geodesics as follows: let $y \in \mathcal{M}_0$ and $v \in \mathbb{S}^2$ the unit sphere in $\mathbb{R}^3$. Then a light ray from $(0, y)$ in direction $(1, v)$ is $\gamma(\tau) = (0, y) + \tau(1, v)$, $\tau \in \mathbb{R}$. The set of light rays $\mathcal{C}$ can be identified with $\mathbb{R}^3 \times \mathbb{S}^2$. The light ray transform for scalar functions on $(\mathcal{M}, g)$ is defined by

$$L(f)(y, v) = \int_{\mathbb{R}} f(\tau, y + \tau v)d\tau, \quad f \in C_0^\infty(\mathcal{M}). \tag{4.16}$$

See Figure 4.2. Let L^* be the adjoint of L. Consider the normal operator $N = L^*L$. It is computed in [31, Theorem 2.1] that

$$Nf(t, x) = \int_{\mathbb{R}^{n+1}} K_N(t, x, t', x')f(t', x')dt'dx'$$

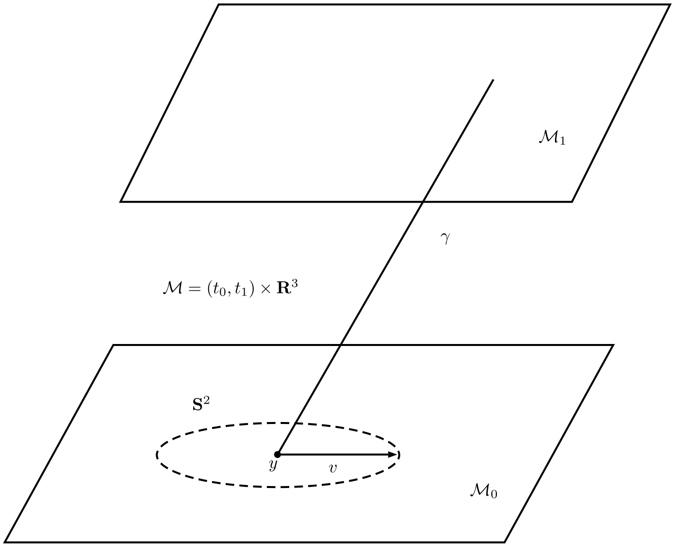

Figure 4.2: The setup of the light ray transform for the Minkowski spacetime.

where the Schwartz kernel

$$K_N(t, x, t', x') = \frac{\delta(t - t' - |x - x'|) + \delta(t - t' + |x - x'|)}{|x - x'|^{n-1}} \tag{4.17}$$

In particular, K_N can be written as an oscillatory integral

$$K_N(t, x, t', x') = \int\limits_{\mathbb{R}^{n+1}} e^{i(t-t')\tau + i(x-x')\cdot\xi} k(\tau, \xi) d\tau d\xi \tag{4.18}$$

where

$$k(\tau, \xi) = C_n \frac{(|\xi|^2 - \tau^2)_+^{\frac{n-3}{2}}}{|\xi|^{n-2}}, \quad C_n = 2\pi|\mathbb{S}^{n-2}|. \tag{4.19}$$

Here, for $s \in \mathbb{R}$, s_+^a, $\operatorname{Re} a > -1$ denotes the distribution defined by $s_+^a = s^a$ if $s > 0$ and $s_+^a = 0$ if $s \leq 0$.

We see that K_N is close to but not exactly a pseudodifferential operator. When properly restricted to $|\xi| > |\tau|$, that is, the cone of space-like covectors, it is an elliptic pseudodifferential operator. Here, we recall that our convention for the signature of the metric g is $(-, +, \cdots, +)$. A covector $\zeta \in T_z^*\mathcal{M}$ is called space-like if $g(\zeta, \zeta) > 0$, time-like if $g(\zeta, \zeta) < 0$ and light-like if $g(\zeta, \zeta) = 0$. The set of space-like, time-like and light-like vectors are denoted by Γ^{sp}, Γ^{tm} and Γ^{lt}, respectively. In general relativity, space-like singularities corresponds to particles moving slower than the speed of light, and light-like singularities corresponds to objects moving at the speed of light such as photons and gravitational waves. From the microlocal structure of K_N, we can conclude that space-like singularities of f can be recovered from Lf for the Minkowski space.

The picture also holds for general Lorentzian manifolds studied in [31]. For simplicity, we recall the result for globally hyperbolic manifold but there is a local statement [31, Theorem 3.1]. Also, the result can be stated for the light ray transform with weights.

Theorem 4.3.1 (Corollary 3.1 of [31]). *Let $(\mathcal{M}, g)$ be a globally hyperbolic Lorentzian manifold on which there are no conjugate points on light-like geodesics. Let $\mathcal{K} \subset \Gamma^{\mathrm{sp}}$ be compact. Then there is a zeroth order pseudodifferential operator χ on $\mathcal{M}$ such that $L^*\chi L$ is a pseudodifferential operator of order -1 with essential support in the space-like cone. Moreover, $L^*\chi L$ is elliptic in $\mathcal{K}$ and the principal symbol is homogeneous and nonnegative.*

From this microlocal result, one can conclude that space-like singularities in f can be recovered from Lf. More precisely, for $f \in \mathcal{E}'(\mathcal{M})$ compactly supported distributions on $\mathcal{M}$, if $\mathrm{WF}(f) \subseteq \Gamma^{\mathrm{sp}}$, then $q \in \mathrm{WF}(Nf)$ if and only if $q \in \mathrm{WF}(f)$.

The proof of Theorem 4.3.1 is based on Guillemin's double fibration approach [20]. First, we recall the microlocal structure of L. Let $\mathcal{F}$ be the set of all geodesics on $(\mathcal{M}, g)$, and $\mathcal{C}$ be the set of light-like geodesics, so $\mathcal{C} \subseteq \mathcal{F}$. Provided there is no conjugate points on $(\mathcal{M}, g)$, $\mathcal{F}$ is a $2n$-dimensional smooth manifold and $\mathcal{C}$ is a codimension one submanifold. We view the light ray transform as an operator $L : C_0^\infty(\mathcal{M}) \to C^\infty(\mathcal{C})$. It is known that the Schwartz kernel K_L is the delta distribution supported on the point-geodesic relation $\mathcal{Z} = \{(z, \gamma) \in \mathcal{M} \times \mathcal{C} : z \in \gamma\}$. Therefore, L is an Fourier integral operator and the kernel has conormal singularities to $\mathcal{Z}$. The canonical relation can be described using Jacobi fields as in the Riemmanian setting; see [52]. Using Hörmander's notion, the Schwartz kernel $K_L \in I^{-n/4}(\mathcal{C} \times \mathcal{M}; C')$. Next, to analyze the microlocal structure of the normal operator L^*L, we look at the double fibration

$$
\begin{array}{ccccc}
 & & C & & \\
 & \swarrow^{\pi_{\mathcal{M}}} & & \searrow^{\pi_{\mathcal{C}}} & \\
T^*\mathcal{M}\backslash 0 & & & & T^*\mathcal{C}\backslash 0
\end{array}
$$

If $\pi_{\mathcal{C}}$ is an injective immersion, then C is said to satisfy the Bolker condition and the composition L^*L can be studied using Duistermaat and Guillemin's clean FIO calculus; see [20]. What was shown in [31] is that when ξ in $\mathcal{C}$ is space-like, the projection $\pi_{\mathcal{C}}$ is indeed injective.

4.3.2 The light-like singularities

Let us consider the microlocal picture up to light-like directions, staring from the Minkowski spacetime. From (4.18) and (4.19), we see that the normal operator is a pseudodifferential operator with symbols singular at the boundary of the light cone

$|\xi|^2 = \tau^2$. The Schwartz kernel is a typical example of the paired Lagrangian distributions developed in [21, 36], see also [7]. This has been noted in several works; see [40, 30], for example.

Let $\mathcal{X}$ be a C^∞ manifold of dimension n and $w_{\mathcal{X}}$ be the symplectic form on $T^*\mathcal{X}$. Let Λ_0, Λ_1 be conic Lagrangian submanifolds of $T^*(\mathcal{X}\times\mathcal{X})\backslash 0$ with symplectic form $\pi_1^* w_{\mathcal{X}} + \pi_2^* w_{\mathcal{X}}$. Here, $\pi_1, \pi_2 : \mathcal{X}\times\mathcal{X} \to \mathcal{X}$ denotes the projections to the first, second copy of $\mathcal{X}$. Suppose that Λ_1 intersects Λ_0 cleanly at a codimension k, $1 \le k \le 2n-1$ submanifold $\Sigma = \Lambda_0 \cap \Lambda_1$, namely $T_p(\Lambda_0 \cap \Lambda_1) = T_p(\Lambda_0) \cap T_p(\Lambda)$, $\forall p \in \Sigma$. From [21, Proposition 2.1], we know that all such intersecting pairs (Λ_0, Λ_1) are locally symplectic diffeomorphic to each other. So, it suffices to define paired Lagrangian distributions for the following model problem. Let $\tilde{\mathcal{X}} = \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$, $1 \le k \le n-1$, and use coordinates $x = (x', x'')$, $x' \in \mathbb{R}^k$, $x'' \in \mathbb{R}^{n-k}$. Let $\tilde{\Lambda}_0 = \{(x,\xi,x,-\xi) \in T^*(\tilde{\mathcal{X}}\times\tilde{\mathcal{X}})\backslash 0 : \xi \neq 0\}$ be the punctured conormal bundle of Diag in $T^*(\tilde{\mathcal{X}}\times\tilde{\mathcal{X}})$, and

$$\tilde{\Lambda}_1 = \{(x,\xi,y,\eta) \in T^*(\tilde{\mathcal{X}}\times\tilde{\mathcal{X}})\backslash 0 : x'' = y'', \xi' = \eta' = 0, \xi'' = \eta'' \neq 0\},$$

which is the punctured conormal bundle to $\{(x,y) \in \tilde{\mathcal{X}}\times\tilde{\mathcal{X}} : x'' = y''\}$. The two Lagrangians intersect cleanly at $\tilde{\Sigma} = \{(x,\xi,y,\eta) \in T^*(\tilde{\mathcal{X}}\times\tilde{\mathcal{X}})\backslash 0 : x'' = y'', \xi'' = \eta'', x' = y', \xi' = \eta' = 0\}$, which is of codimension k. For this model pair, the paired Lagrangian distribution $I^{p,l}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{\Lambda}_0, \tilde{\Lambda}_1)$ consists of oscillatory integrals

$$u(x,y) = \int e^{i[(x'-y'-s)\cdot\eta' + (x''-y'')\cdot\eta'' + s\cdot\sigma]} a(s,x,y,\eta,\sigma)\,d\eta\, d\sigma\, ds \tag{4.20}$$

where a is a product type symbol, which is a C^∞ function and satisfies

$$\left|\partial_\eta^\alpha \partial_\sigma^\beta \partial_s^\theta \partial_x^\gamma \partial_y^\delta a(s,x,y,\eta,\sigma)\right| \le C(1+|\eta|)^{p+k/2-|\alpha|}(1+|\sigma|)^{l-k/2-|\beta|} \tag{4.21}$$

for multiindices α, β, θ, γ, δ over each compact set $\mathcal{K}$ of $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^k$. The constant C depends on the indices and $\mathcal{K}$. The set of product type symbols is denoted by $S^{p,l}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n; \mathbb{R}^k)$.

It is proved in Theorem 3.1 of [52] (see also [51]) that the Schwartz kernel of the normal operator $K_N \in I^{-n/2, n/2-1}(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}; \Lambda_0, \Lambda_1)$, in which Λ_0, Λ_1 are two cleanly intersection Lagrangians defined as follows:

$$\begin{aligned}\Lambda_0 = \{&(t,x,\tau,\xi; t',x',\tau',\xi') \in T^*\mathbb{R}^{n+1}\backslash 0 \times T^*\mathbb{R}^{n+1}\backslash 0 :\\ &t' = t, x' = x, \tau' = -\tau, \xi' = -\xi\},\end{aligned} \tag{4.22}$$

which is the punctured conormal bundle of the diagonal in $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ and

$$\begin{aligned}\Lambda_1 = \{&(t,x,\tau,\xi; t',x',\tau',\xi') \in T^*\mathbb{R}^{n+1}\backslash 0 \times T^*\mathbb{R}^{n+1}\backslash 0 :\\ &x = x' + (t-t')\xi/|\xi|,\ \tau = \pm|\xi|, \tau' = -\tau, \xi' = -\xi\}.\end{aligned} \tag{4.23}$$

The proof of this result is based on the explicit form of the kernel in (4.19) and (4.18). In particular, one can find a symplectic transformation so that (4.18) is transformed to the model problem; see [52, Section 3].

The result has also been generalized to globally hyperbolic Lorentzian manifolds without conjugate points.

Theorem 4.3.2 (Theorem 1.1 of [52]). *Let $(\mathcal{M}, g)$ be a globally hyperbolic Lorentzian manifold of dimension $n+1, n \geq 2$. Suppose $(\mathcal{M}, g)$ is null-geodesic complete without conjugate points. Consider the normal operator $N = L^*L$ of the light ray transform L. Then the Schwartz kernel $K_N \in I^{-n/2,n/2-1}(\mathcal{M} \times \mathcal{M}; \Lambda_0, \Lambda_1)$, in which Λ_0, Λ_1 are two cleanly intersecting Lagrangians. Let $\Sigma = \Lambda_0 \cap \Lambda_1$. The principal symbols of K_N on $\Lambda_0 \backslash \Sigma$, $\Lambda_1 \backslash \Sigma$ are nonvanishing.*

As a consequence, one can derive Sobolev estimates for the light ray transform. More precisely, $L : H^s_{\mathrm{comp}}(\mathcal{M}) \to H^{s+s_0/2}_{\mathrm{loc}}(\mathcal{C})$ is continuous with s_0 such that $\max(-n/2 + 1/2, -1) \leq -s_0$, $n \geq 2$; see [52, Theorem 1.2]. For the Minkowski spacetime, related estimates were obtained by Greenleaf and Seeger [15].

Using the improved microlocal picture, we can say something about recovery of light-like singularities. It is proved in [52, Theorem 1.3] that one may not be able to determine light-like singularities of f using singularities of Nf under the assumptions of Theorem 4.3.2. Related examples are known 2 + 1 dimensional Minkowski spacetime; see [17, Section 2]. Under stronger conditions, for example, if the singularities of f are of conormal type with principal symbols of a fixed sign, it is proved in [52, Theorem 6.2] that the wave front set of f can be determined from Nf. Finally, we remark that conjugate points can cause cancellation of singularities; see the discussion in [31, Section 4] for an example.

The proof of Theorem 4.3.2 does not use the double fibration approach, although there are composition results for the fold-type singularities; see, for example, Greenleaf and Uhlmann [16]. Instead, one can analyze the Schwartz kernel, as in the approach of Stefanov and Uhlmann [44, 46] for the Riemannian geodesic ray transform. For simplicity, we consider $(\mathcal{M}, g)$ a standard static spacetime of the form

$$\mathcal{M} = \mathbb{R} \times \mathcal{N}, \quad g = -dt^2 + h(x, dx) \tag{4.24}$$

and assume that there is no conjugate points on $(\mathcal{M}, g)$. Here, h is a Riemannian metric on $\mathcal{N}$. In this case, light-like geodesics on $(\mathcal{M}, g)$ are lifts of geodesics on $(\mathcal{N}, h)$. More precisely, let $(x, \theta) \in S\mathcal{N}$ so that $h(\theta, \theta) = 1$. Then we have

$$\gamma_{x,\theta}(s) = \exp_{(0,x)} s(1, \theta) = \left(s, \exp^h_x(s\theta)\right) \tag{4.25}$$

where $\exp^h$ denotes the exponential map on $(\mathcal{N}, h)$. Using $(x, \theta) \in S\mathcal{N}$ to parametrize the light rays, the light ray transform becomes

$$L(f)(x,\theta) = \int_{\mathbb{R}} f(s, \exp_x^h(s\theta))ds, \quad f \in C_0^\infty(\mathcal{M}). \tag{4.26}$$

The Schwartz kernel was found in [52, Section 4].

Proposition 4.3.3. *For the light ray transform* (4.26) *on a static spacetime* (4.24) *of dimension* $n+1, n \geq 2$ *and without conjugate points, the Schwartz kernel* K_N *of the normal operator* $N = L^*L$ *is*

$$K_N(t,x,t',x') = \frac{\delta(t-t'-dist^h(x,x')) + \delta(t-t'+dist^h(x,x'))}{(dist^h(x,x'))^{n-1}} J(x,x') \tag{4.27}$$

for $(t,x),(t',x') \in \mathcal{M}$. *Here,* $dist^h : \mathcal{N} \times \mathcal{N} \to [0,\infty)$ *is the distance function on* $(\mathcal{N},h)$ *and* J *is a smooth nonvanishing function on* $\mathcal{N} \times \mathcal{N}$ *with* $J(x,x) = 1$, $x \in \mathcal{N}$.

Here, the measure on $\mathcal{M}$ is $\sqrt{\det h}dtdx$, and $J(x,x')$ is in fact a Jacobian factor similar to the result Proposition 1 in [44]. Proposition 4.3.3 is a generalization of (4.17). One can analyze the microlocal structure near the intersection of $t = t' \pm \mathrm{dist}^h(x,x')$ and $\{t = t', x = x'\}$ via Fourier transform and carry out similar analysis as in the Minkowski case. We remark that this approach can also be used to analyze the structure of the kernel when certain type of conjugate points are present following the idea in [46]; see [52, Section 7]. In particular, for standard static spacetimes with time-like conjugate points of fold type, it was shown in [52] that the Schwartz kernel is the sum of a paired Lagrangian distribution and a Lagrangian distribution associated with the conjugate points.

4.4 Recovery of wave equation solutions

In this section, we review the results in [49, 53] for the inverse problem in Section 4.2.2. Below, we take $\mathcal{M} = (0,T) \times \mathbb{R}^3$ and use $(t,x), t \in (0,T), x \in \mathbb{R}^3$ as the local coordinates. Let g be the Minkowski metric on $\mathcal{M}$. We use $\mathcal{M}_0 = \{0\} \times \mathbb{R}^3$ and $\mathcal{M}_1 = \{T\} \times \mathbb{R}^3$. We consider general wave operators of the form

$$P(x,t,D_x,\partial_t) = \partial_t^2 + c^2 \sum_{i=1}^{3} D_{x_i}^2 + P_1(x,t,iD_x,\partial_t) + P_0(x,t) \tag{4.28}$$

where P_1 is a first-order differential operator with real valued smooth coefficients and P_0 is smooth. Here, we assume c is a constant speed. Then we consider the Cauchy problem

$$\begin{aligned} P(x,t,D_x,\partial_t)f &= 0 \quad \text{on } \mathcal{M}^\circ \\ f = f_1, \quad \partial_t f &= f_2, \quad \text{on } \mathcal{M}_0. \end{aligned} \tag{4.29}$$

The inverse problem we study is to determine the Cauchy data (f_1,f_2) from the light ray data Lf where f is the solution of (4.29) and L is the light ray transform defined in (4.16).

We will see that with the wave equation constraint, one can obtain better result of stable recovery of f.

Theorem 4.4.1 (Theorem 1.1 of [49], [53]). *Suppose* $0 < c \le 1$ *is constant. Assume that* $(f_1, f_2) \in \mathcal{N}^s \stackrel{\text{def}}{=} H^{s+1}_{\text{comp}}(\mathcal{M}_0) \times H^s_{\text{comp}}(\mathcal{M}_0)$, $s \ge 0$, *and* f_1, f_2 *are supported in a compact set* $\mathcal{K}$ *of* $\mathcal{M}_0$. *Then* Lf *uniquely determines* f *and* f_1, f_2, *which satisfy* (4.29). *Moreover, there exists a* $C > 0$ *such that*

$$\|(f_1, f_2)\|_{\mathcal{N}^s} \le C\|Lf\|_{H^{s+2}(\mathcal{C})} \quad \textit{and} \quad \|f\|_{H^{s+1}(M)} \le C\|Lf\|_{H^{s+2}(\mathcal{C})}$$

where $\mathcal{C}$ *is the set of light rays on* $\mathcal{M}$.

Because of the stability estimate, we can generalize the result to include small metric perturbations. We remark that for smooth metric perturbations of the Minkowski metric, the injectivity of the light ray transform is not yet known; see Section 4.6. Let us consider metric perturbations $g_\delta = g + h$ where h is a symmetric two tensor smooth on $\mathcal{M}$, and for $\delta > 0$ small, the seminorm $\|h_{ij}\|_{C^3} < \delta, i, j = 0, 1, 2, 3$. In this case, light rays may not be straight lines but the light ray transform L_δ on $(\mathcal{M}, g_\delta)$ can be parametrized similar to L. Let $\Box_{g_\delta}$ be the d'Alembert operator on $(\mathcal{M}, g_\delta)$. Consider the Cauchy problem

$$\begin{aligned} \Box_{g_\delta} f &= 0 \quad \text{on } \mathcal{M}^\circ \\ f = f_1, \quad \partial_t f &= f_2, \quad \text{on } \mathcal{M}_0. \end{aligned} \tag{4.30}$$

Then we have

Theorem 4.4.2 (Theorem 1.2 of [49]). *Consider* $(\mathcal{M}, g_\delta)$ *described above. Assume that* $(f_1, f_2) \in \mathcal{N}^s$, $s \ge 0$ *and* f_1, f_2 *are supported in a compact set* $\mathcal{K}$ *of* $\mathcal{M}_0$. *For* $\delta \ge 0$ *sufficiently small,* $L_\delta f$ *uniquely determines* f *and* f_1, f_2, *which satisfy* (4.30). *Moreover, there exists* $C > 0$ *such that*

$$\|(f_1, f_2)\|_{\mathcal{N}^s} \le C\|L_\delta f\|_{H^{s+2}(\mathcal{C}_\delta)} \quad \textit{and} \quad \|f\|_{H^{s+1}(M)} \le C\|L_\delta f\|_{H^{s+2}(\mathcal{C}_\delta)}$$

where $\mathcal{C}_\delta$ *is the set of light rays on* $(\mathcal{M}, g_\delta)$.

Roughly speaking, the reason that we are able to get a stable determination is the restriction of singularities of f. We have seen in Section 4.3 that time-like singularities in f are lost after taking the light ray transform. So, we do not expect Theorem 4.4.1 and 4.4.2 to hold for $c > 1$. There is a fundamental difference in the treatment between the $c < 1$ and $c = 1$ cases. The former only needs a good understanding of the normal operator L^*L, while the latter relies on a thorough analysis of the operator LE where E is the fundamental solution or parametrix for the Cauchy problem. Below, we will focus on the more difficult case of $c = 1$.

We will see soon that there are some technicalities related to the behavior of f at $t = 0, T$. For simplicity, we replace Lf by $L(\chi_\epsilon f)$ where χ_ϵ is a smooth cut-off function supported in $[0,T]$. For $\epsilon > 0$ small, let $\chi_\epsilon(t)$ be a smooth cut-off function on $\mathbb{R}$ such that $\chi_\epsilon(t) = 1$ for $2\epsilon < t < t_1 - 2\epsilon$ and $\chi_\epsilon(t) = 0$ for $t < \epsilon$ and $t > t_1 - \epsilon$. In fact, by the continuity of L, the difference of $L\chi_{[0,T]}f$ and $L\chi_\epsilon f$ can be made arbitrarily small in a proper sense. Now we discuss two approaches in [49] and [53].

4.4.1 The first approach

For simplicity, we consider below the Cauchy problem for the standard wave equation.

$$\begin{aligned} \Box f &= 0 \quad \text{on } \mathcal{M} \\ f = f_1, \quad \partial_t f &= f_2, \quad \text{on } \mathcal{M}_0. \end{aligned} \tag{4.31}$$

Using Fourier transform in the x variable, we get

$$\begin{aligned} u(t,x) &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x\cdot\xi + t|\xi|)} \hat{h}_1(\xi) d\xi + (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x\cdot\xi - t|\xi|)} \hat{h}_2(\xi) d\xi \\ &= E_+ h_1 + E_- h_2, \end{aligned} \tag{4.32}$$

where

$$\hat{h}_1 = \frac{1}{2}\left(\hat{f}_1 + \frac{1}{i|\xi|}\hat{f}_2\right), \quad \hat{h}_2 = \frac{1}{2}\left(\hat{f}_1 - \frac{1}{i|\xi|}\hat{f}_2\right).$$

Here, h_1, h_2 are the reparametrized Cauchy data for the Cauchy problem. Thus, $E_\pm$ are represented by oscillatory integrals

$$E_\pm f(t,x) = (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i((x-y)\cdot\xi \pm t|\xi|)} f(y) dy d\xi. \tag{4.33}$$

The phase functions are $\phi_\pm(t,x,y,\xi) = (x-y)\cdot\xi \pm t|\xi|$ and amplitude function $a(t,x,\xi) = 1$. The oscillatory integral representation also works for (4.28) but more generally, the parametrix of Cauchy problem can be constructed as Fourier integral operators; see [9].

We consider the composition $L\chi_\epsilon E_\pm$. Let φ be a smooth function on $\mathbb{S}^2$, and I^φ be the integration operator on $C^\infty(\mathbb{R}^3 \times \mathbb{S}^2)$ defined by

$$I^\varphi f(y) = \int_{\mathbb{S}^2} \varphi(v) f(y,v) dv.$$

Then we consider the composition $K_\pm = I^\varphi \circ L \circ \chi_\epsilon E_\pm$ as an operator from $C^\infty(\mathcal{M}_0)$ to $C^\infty(\mathcal{M}_0)$. A key result is [49, Proposition 7.1], which says that $K_\pm \in \Psi^{-1}(\mathcal{M}_0)$ are pseudodifferential operators of order -1 with complete symbol $k_\pm(\xi)$, $\xi \in \mathbb{R}^3\backslash 0$ and the principal symbols are given by

$$k_{+,-1}(\xi) = 2\pi i c_\epsilon |\xi|^{-1}\varphi(-\xi/|\xi|), \quad k_{-,-1}(\xi) = -2\pi i c_\epsilon |\xi|^{-1}\varphi(\xi/|\xi|),$$

$$\text{where } c_\epsilon = \int_0^{t_1} t^{-1}\chi_\epsilon(t)dt$$

Note that $K_\pm$ are elliptic and we can use them to solve for h_1, h_2 up to smooth terms. More precisely, we write

$$L\chi_\epsilon f = L\chi_\epsilon E_+ h_1 + L\chi_\epsilon E_- h_2.$$

Applying I^φ, we get

$$I^\varphi L\chi_\epsilon f = I^\varphi L\chi_\epsilon E_+ h_1 + I^\varphi L\chi_\epsilon E_- h_2 = K^{\varphi,+} h_1 + K^{\varphi,-} h_2$$

where we added φ to the notation of $K_\pm$ to emphasize the dependency. Then one can show that by varying φ, one can construct parametrices A_1, A_2 such that

$$A_1 L\chi_\epsilon f = h_1 + R_1 h_1 + R_1' h_2, \quad A_2 L\chi_\epsilon f = h_2 + R_2 h_1 + R_2' h_2$$

where $R_i, R_i', i = 1, 2$ are smoothing operators. In particular, we have the estimate

$$\begin{aligned} \|h_1\|_{H^s(\mathbb{R}^3)} + \|h_2\|_{H^s(\mathbb{R}^3)} \leq \|A_1 L\chi_\epsilon f\|_{H^{s+1}(\mathbb{R}^3)} + \|A_2 L\chi_\epsilon f\|_{H^s(\mathbb{R}^3)} \\ + C_\rho\big(\|h_1\|_{H^{s-\rho}(\mathbb{R}^3)} + \|h_2\|_{H^{s-\rho}(\mathbb{R}^3)}\big) \end{aligned}$$

Now we can use a known argument (see, e. g., [43]) to remove the last term with the fact that L is injective on compactly supported functions.

Finally, we discuss what needs to be changed when the smooth cut-off function χ_ϵ is replaced by the characteristic function $\chi_{[0,T]}$ of the interval $[0, T]$ in $\mathbb{R}$. In this case, the operators $K_\pm$ contain additional Fourier integral operators (FIO). For example, we can write $K_+ = K_+^0 + K_+^\epsilon + K_+^{t_1}$ where $K_+^0 \in \Psi^{-1}(\mathbb{R}^3)$, and $K_+^\epsilon \in I^{-2}(\mathbb{R}^3, \mathbb{R}^3; C_\epsilon)$, $K_+^{t_1} \in I^{-2}(\mathbb{R}^3, \mathbb{R}^3; C_{t_1})$ are FIOs of order -2. Here, for $\alpha \in \mathbb{R}$ we have

$$C_\alpha = \{(y, \eta, z, \zeta) \in T^*\mathbb{R}^3\backslash 0 \times T^*\mathbb{R}^3\backslash 0 : y = z + 2\alpha\xi/|\xi|, \xi = \eta\};$$

see [49] for details. Note that C_α is a graph of a canonical transformation. Thus, standard FIO estimates (see [22, Section 25.3]) indicate that the additional FIOs are more regular, and the above argument can work through with some modifications.

4.4.2 The second approach

Consider the operator $L\chi_\epsilon E$. It is natural to apply the "backprojection" and consider the normal operator $E^*L^*L\chi_\epsilon E$. It turns out that the composition is not good as it stands. In fact, the issue is related to the microlocal structure of the normal operator $N = L^*L$. We have seen that the Schwartz kernel of N is a paired Lagrangian distribution. By judicious use of the kernel on one of the Lagrangians, we show that the composition E^*NE can be slightly modified to behave well within the clean FIO calculus of Duistermaat and Guillemin, yielding a pseudodifferential operator on $\mathcal{M}_0$.

We start the general microlocal construction of parametrix E. A linear differential operator $P : C^\infty(\mathbb{R}^{n+1}) \to C^\infty(\mathbb{R}^{n+1})$ of second order is called *normally hyperbolic* if the principal symbol $\mathscr{P}(z,\zeta) \doteq \sigma(P)(z,\zeta) = g^*(\zeta,\zeta)$, $(z,\zeta) \in T^*M$, see [3, p. 33]. Note that P in (4.29) is exactly the normally hyperbolic operator on $(\mathbb{R}^{n+1}, g)$. The operator is strictly hyperbolic of multiplicity one with respect to the Cauchy hypersurfaces $\mathcal{M}_t = \{t\} \times \mathbb{R}^n$, $t \in \mathbb{R}$; see [9, Definition 5.1.1]. This means that all bicharacteristic curves of P are transversal to $\mathcal{M}_t$ and for $(\bar{z},\bar{\zeta}) \in T^*\mathcal{M}_t\backslash 0$, $\mathscr{P}(\bar{z},\zeta) = 0$, $\zeta|_{T_{\bar{z}}\mathcal{M}} = \bar{\zeta}$ has exactly one solution. For the Cauchy problem (4.29), we use Duistermaat–Hörmander's parametrix construction; see, for example, [9]. Let ρ_0 be the restriction operator $\rho_0 : C^\infty(\mathcal{N}) \to C^\infty(\mathcal{M})$, which is in fact an FIO. We consider the canonical relation C_{wv} defined by

$$C_{wv} = \{(w,\iota,\bar{z},\bar{\zeta}) \in T^*\mathcal{N}\backslash 0 \times T^*\mathcal{M}\backslash 0 : (w,\iota) \text{ is on the bicharacteristic strip through some } (\bar{z},\zeta) \text{ such that } \bar{\zeta} = \zeta|_{T_{\bar{z}}\mathcal{M}} \text{ and } \mathscr{P}(\bar{z},\zeta) = 0\} \tag{4.34}$$

It follows from [9, Theorem 5.1.2] that there exists $E_1 \in I^{-1/4}(\mathcal{N},\mathcal{M};C_{wv})$, $E_2 \in I^{-5/4}(\mathcal{N},\mathcal{M};C_{wv})$ such that

$$\begin{aligned} &P(z,D)E_k \in C^\infty(\mathcal{N}), \quad k = 1,2 \\ &\rho_0 E_1 - \mathrm{Id} \in C^\infty(\mathcal{M}), \quad \rho_0 E_2 \in C^\infty(\mathcal{M}) \\ &\rho_0 D_t E_1 \in C^\infty(\mathcal{M}), \quad \rho_0 D_t E_2 - \mathrm{Id} \in C^\infty(\mathcal{M}) \end{aligned} \tag{4.35}$$

Now we can represent the solution of (4.29) as $u = E_1 f_1 + E_2 f_2$ modulo a smooth term.

To analyze $E^*N\chi_\epsilon E$ where $E = E_1, E_2$, first we choose a smooth cut-off function $\chi \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp}\chi \subset (T,T')$, $\chi \geq 0$ and not vanishing identically. Then we consider the composition $E^*\chi L^*L\chi_\epsilon E$. Because $\chi \cdot \chi_\epsilon = 0$, we know that $\chi N\chi_\epsilon \in I^{-n/2}(\mathbb{R}^{n+1},\mathbb{R}^{n+1};\Lambda_1)$. Note that the role of χ is to keep the kernel of N away from the diagonal Λ_0 where the principal symbol is singular.

Next, we can show that Λ_1 intersects $\Lambda = C'_{wv}$ cleanly with excess one so the composition $\chi N\chi_\epsilon E$ is a FIO in $I^*(\mathcal{N},\mathcal{M};C_{wv})$ as a result of Duistermaat–Guillemin's clean FIO calculus with the order $*$ to be determined. Roughy speaking, the reason that the clean calculus works is that both Lagrangians Λ_1 and Λ are the flow out of the same Hamiltonian. Finally, we can compose the operator with E^* by using clean FIO calculus again to conclude that $E^*\chi N\chi_\epsilon E \in \Psi^*(\mathcal{M})$. In fact, we can show that the operator is elliptic. Now one can construct parametrices for the operator and continue with the argument in the first approach.

4.5 The inverse source problem

In this section, we consider the inverse problem in Section 4.2.3. Mathematically, we consider the inverse source problem for the linear Boltzmann equation (or non-stationary transport equation) on $\mathcal{M} = (0, T) \times \mathbb{R}^3$, $T > 0$:

$$\begin{aligned} &\partial_t u(t, x, \theta) + \theta \cdot \nabla_x u(t, x, \theta) + \sigma(t, x, \theta) u(t, x, \theta) \\ &= \int_{\mathbb{S}^2} k(t, x, \theta, \theta') u(t, x, \theta') d\theta' + f(t, x), \end{aligned} \tag{4.36}$$

where $t \in (0, T)$, $x \in \mathbb{R}^3$, $\theta \in \mathbb{S}^2$. Here, σ is the absorption coefficient, k is the scattering kernel and f is the source term. We consider the zero initial condition

$$u(0, x, \theta) = 0. \tag{4.37}$$

The inverse problem we study is to determine the source term f from the measurement of u at $t = T > 0$,

$$u(T, x, \theta) = u_T(x, \theta). \tag{4.38}$$

The inverse problem for (4.36) and its stationary version has a rich history; see [24, Section 7.4]. Both the determination of σ, k and the source term f have been investigated. In particular, there are lots of interest due to its application in optical imaging; see, for example, review papers [1, 40]. Most of the work concern the inverse problem for the so-called albedo operator, which involves many boundary measurements. For the source problem, we have the boundary measurement for a single source and there are fewer results; see [26, 32, 45]. We remark that recently, the inverse problem for the nonlinear Boltzmann-type equations has drawn a lot of attention; see, for instance, [2, 27–29]. The results are interesting because one can use the nonlinear effect to help resolving some difficulties in the linear problem.

In [54], two results on the stable determination of the source term in (4.36) are obtained. Let ϕ be the characteristic function of Γ^{sp}. We define $\phi(D)$ to be a Fourier multiplier $\phi(D)f = \mathscr{F}^{-1}(\phi \mathscr{F} f)$, $f \in L^2(\mathbb{R}^4)$ where $\mathscr{F}$, $\mathscr{F}^{-1}$ denote the Fourier and inverse Fourier transform in t, x variables. We set $\mathcal{V} = (0, T) \times \Omega$ where Ω is a relatively compact set of $\mathbb{R}^3$. The first result is the following.

Theorem 4.5.1 (Theorem 1.1 of [54]). *Let $\sigma \in C^6$ be independent of the x and θ variable. There exists an open dense subset $\mathcal{U}$ of $C_0^6(\mathcal{V} \times \mathbb{S}^2 \times \mathbb{S}^2)$ such that the following is true. Consider the source problem* (4.36) *and* (4.37) *with $k \in \mathcal{U}$ and $f \in H^2_{\mathrm{comp}}(\mathcal{V})$. Then f is uniquely determined by u_T in* (4.38). *Moreover, we have the following stability estimate:*

$$\|\phi(D) f\|_{H^2(\mathcal{M})} \le C \|u_T\|_{H^{5/2}(\mathbb{R}^3 \times \mathbb{S}^2)} \tag{4.39}$$

for some $C > 0$ depending on σ, k.

The second result is the stable determination of f from u_T assuming that f is a solution of the wave equation. The setup is relevant for the inverse problem in Section 4.2.3 and shows that determination of scalar type metric perturbations from the linearized CMB is possible with the presence of kinetic effects.

Theorem 4.5.2 (Theorem 1.2 of [54]). *Let f be the solution of* (4.28) *on $\mathcal{M}$ with Cauchy data $f_1 \in H^2(\mathcal{M}_0)$, $f_2 \in H^1(\mathcal{M}_0)$ supported in a compact set $\mathcal{X}$ of $\mathcal{M}_0$ such that f is supported in $\mathcal{V}$. Suppose that the coefficients $A_j(z)$ in* (4.28) *are real valued smooth functions. Let u be the solution of* (4.36), (4.37) *with source $\chi_0 f$.*

Then there exists an open dense set $\mathcal{U}$ of $C_0^\infty(\mathcal{V} \times \mathbb{S}^2) \times C_0^6(\mathcal{V} \times \mathbb{S}^2 \times \mathbb{S}^2)$ such that for any $(\sigma, k) \in \mathcal{U}$, f_1, f_2 is uniquely determined by u_T and there exists $C > 0$ such that

$$\|f\|_{H^2(\mathcal{M})} \le C\|(f_1, f_2)\|_{H^2(\mathbb{R}^3)\times H^1(\mathbb{R}^3)} \le C\|u_T\|_{H^{5/2}(\mathbb{R}^3\times\mathbb{S}^2)} \tag{4.40}$$

We remark that the stability estimates suggest that the results can be generalized via perturbation arguments to other scenarios such as small metric perturbations of the Minkowski spacetime as in [49], small perturbations of σ for Theorem 4.5.2 and possibly nonlinear perturbations in the Boltzmann equation.

4.5.1 The integral geometry approach

To prove the two theorems, the main idea in [54] is to consider the source problem as the time-dependent version of the inverse source problem studied in Stefanov and Uhlmann [45]. In particular, one treats the map $f \to u_T$ as a perturbation of the light ray transform on the Minkowski spacetime. The difficulty is that, unlike the geodesic ray transform in the Riemmanian setting, the normal operator of the light ray transform is not an elliptic pseudodifferential operator, as we already saw in Section 4.3. Thus, the key is to restore the ellipticity by using either $\phi(D)$ or the parametrix of the Cauchy problem. Below we briefly describe the proof of Theorem 4.5.1.

We start with the expression of u_T. Let

$$T_0 = \partial_t + \theta \cdot \nabla_x, \quad T_1 = T_0 + \sigma, \quad T = T_1 - K \tag{4.41}$$

where σ is regarded as the multiplication operator and K is the integral operator in (4.36). For $k = 0$, the equation $T_1 u = f$ with $u = 0$ at $t = 0$ can be solved explicitly. For $\theta \in \mathbb{S}^{n-1}$, $t > 0$, $x \in \mathbb{R}^n$, consider $u(t, x, \theta) = u(t, x + t\theta)$, which satisfies

$$\frac{d}{dt}u(t, x + t\theta) + \sigma(t, x + t\theta)u(t, x + t\theta) = f(t, x + t\theta) \tag{4.42}$$

An integrating factor is $E(t, x, \theta) = e^{\int_0^t \sigma(s, x+s\theta)ds}$. We solve (4.42) that

$$u(t, x + t\theta) = \int_0^t e^{-\int_s^t \sigma(\tilde{s}, x+\tilde{s}\theta)d\tilde{s}} f(s, x + s\theta)ds$$

Thus, we can write T_1^{-1} as

$$T_1^{-1}f(t, x, \theta) = \int_0^t \kappa(t, x, s, \theta)f(s, x + s\theta)ds,$$

$$\text{with } \kappa(t, x, s, \theta) = e^{-\int_s^t \sigma(\tilde{s}, x+\tilde{s}\theta)d\tilde{s}} \quad (4.43)$$

Next, for $Tu = (T_1 - K)u = f$, we apply T_1^{-1} and get $(\mathrm{Id} - T_1^{-1}K)u = T_1^{-1}f$. It takes some effort to show that $\mathrm{Id} - T_1^{-1}K$ is invertible for suitable k so

$$u = \left(\mathrm{Id} - T_1^{-1}K\right)^{-1} T_1^{-1} f = T_1^{-1}(\mathrm{Id} - KT_1^{-1})^{-1} f \quad (4.44)$$

Now we set $Xf = u|_{t=T}$. We can use (4.44) to obtain a representation for X. In particular, let ρ_T be the restriction operator to $t = T$. Then

$$X = \rho_T T_1^{-1}(\mathrm{Id} - KT_1^{-1})^{-1} \quad (4.45)$$

We observe that $\rho_T T_1^{-1} = L_\kappa$ is a light ray transform with weight:

$$L_\kappa f(x, \theta) = \int_0^T \kappa(T, x, s, \theta)f(s, x + s\theta)ds$$

where κ is defined in (4.43). Of course, when $\sigma = k = 0$, we see that Xf is exactly the light ray transform on the Minkowski spacetime. For analytic weight, a support theorem and injectivity result for the transform was obtained in [40]. For smooth weights, the microlocal structure of the normal operator was studied in [30] and [54]. These results are needed for proving Theorem 4.5.2.

For Theorem 4.5.1, we assume $\sigma(z) = \sigma(t)$ only depends on the t variable. Then we have

$$Xf = \int_0^T \kappa(s)f(s, x + s\theta)ds = L(\kappa f) \quad \text{where } \kappa(s) = e^{-\int_s^T \sigma(\tilde{s})d\tilde{s}} \quad (4.46)$$

In this case, it suffices to look at the light ray transform L. Now we can write $u_T = Xf$ with $X = L\kappa + E$ where E is some operator. To "invert" X, we apply L^* to X to get $L^*X = L^*L\kappa + L^*E$. The idea is to show that $N = L^*L$ is invertible in a proper sense and L^*E is compact. Then one can resort to Fredholm theory.

It is known that L is injective on C_0^∞ functions. However, when acting on say Schwartz functions, L has a nontrivial kernel consisting of functions whose Fourier transform is supported in Γ^{tm}; see, for instance, [23]. It is easy to see that $\phi(D)$: $H^s(\mathbb{R}^{3+1}) \to H^s(\mathbb{R}^{3+1})$, $s \in \mathbb{R}$ is bounded. Also, $\phi^2(D) = \phi(D)$ so $\phi(D)$ is a projection on $H^s(\mathbb{R}^{3+1})$. We denote the range of $\phi(D)$ on $H^s(\mathbb{R}^{3+1})$ by $\mathcal{H}^s$, which is a closed subspace of $H^s(\mathbb{R}^{3+1})$, hence a Hilbert space. For $n = 3$, we see from (4.18) and (4.19) that

$$k(\tau,\xi) = 4\pi^2 \frac{\phi(\tau,\xi)}{|\xi|} \tag{4.47}$$

It follows from (4.18) that $Nf = N\phi(D)f$. Let Q be defined by a Fourier multiplier $\mathscr{F}(Qf)(\tau,\xi) = q(\tau,\xi)\hat{f}(\tau,\xi)$ where

$$q(\tau,\xi) = \left(4\pi^2\right)^{-1}\phi(\tau,\xi)|\xi|^{-1} \tag{4.48}$$

We observe that N is invertible on $\mathcal{H}^s$. Using these constructions, one can derive from $Xf = L\phi(D)\kappa f + E\phi(D)\kappa f$ that

$$Q\phi(D)L^*Xf = \phi(D)\kappa f + Q\phi(D)L^*E\phi(D)\kappa f \tag{4.49}$$

Regarding the right-hand side of (4.49) as acting on functions in $\mathcal{H}^s$, it finally takes some effort to show that $Q\kappa\phi(D)L^*E : \mathcal{H}^2 \to \mathcal{H}^2$ is compact to complete the argument.

The approach gives the stability estimate

$$\|\phi(D)\kappa f\|_{H^2(\mathcal{M})} \le C\|Xf\|_{H^{5/2}(\mathcal{C})}$$

If $Xf = 0$, we get $\phi(D)\kappa f = 0$. By taking Fourier transform, we see that $\mathscr{F}(\kappa f)(\zeta) = 0$ for $\zeta \in \Gamma^{\mathrm{sp}}$. But κf is compactly supported so $\mathscr{F}(\kappa f)(\zeta)$ is analytic in ζ. We conclude that $\kappa f = 0$ so $f = 0$. This proves the uniqueness.

4.5.2 Further discussions on stability

In the literature, there are interesting work on stability of the radiative transport equations based on the method of Carleman estimates; see [26, 32]. Here, we want to review the results from the integral geometry perspective. Usually the problems are formulated using boundary measurements. Consider (4.36) on $\mathcal{M} = (0,T) \times \Omega$ and assume that f is compactly supported in $\mathcal{M}$. Let u be the solution of (4.36). We consider boundary measurements $u|_{[0,T]\times\partial\Omega}$ and study the inverse problem of determining f from $u_{[0,T]\times\partial\Omega}$.

We recall the following simplified result from [32]. Let Ω be a bounded domain of $\mathbb{R}^n$, $n \ge 2$ with the C^1 boundary $\partial\Omega$. Let $V \subset \mathbb{R}^n$ be a bounded subdomain or a measurable subset of $\{v \in \mathbb{R}^n : |v| = 1\}$. Also, we assume that $k(x,v,v') = \sigma_s(x,v)p(x,v,v')$, where $\sigma_s \in L^\infty(\Omega \times V)$ and $p \in L^\infty(\Omega \times V \times V)$ and $p > 0$.

Theorem 4.5.3 (Theorem 1.3 of [32]). *We consider*

$$\partial_t u + v \cdot \nabla u + \sigma_t u - \int_V k(x, v, v')u(x, v', t)dv' = f(x, v)F(x, v, t), \quad x \in \Omega, v \in V, 0 < t < T,$$
$$u(x, v, 0) = 0, \quad x \in \Omega, v \in V. \tag{4.50}$$

We assume

$$k \in L^\infty(\Omega \times V \times V), \quad F, \partial_t F \in L^2(0, T; L^\infty(\Omega \times V)), \quad \sigma_t, \sigma_s \in L^\infty(\Omega \times V),$$

and $u \in \mathcal{U}$. *For an arbitrarily fixed constant* $a_0 > 0$, *we further assume*

$$F(x, v, 0) > a_0, \quad \text{almost all } (x, v) \in \Omega \times V$$

and

$$T > \frac{\max_{x \in \overline{\Omega}}(\gamma \cdot x) - \min_{x \in \overline{\Omega}}(\gamma \cdot x)}{\min_{v \in \overline{V}}(\gamma \cdot v)}$$

There exists a constant $C > 0$, *which depends on* $\|\sigma_t\|_{L^\infty(\Omega \times V)}$, $\|k\|_{L^\infty(\Omega \times V \times V)}$ *and* $\|F\|_{H^1(0,T;L^\infty(\Omega \times V))}$ *such that*

$$\|f\|_{L^2(\Omega \times V)} \le C\left(\int_0^T \int_{\partial\Omega} \int_V |(\nu \cdot v)| |\partial_t u|^2 dv dS dt\right)^{\frac{1}{2}}$$

for all $f \in L^2(\Omega \times V)$.

Actually, this is the key result in [32] from which the determination of σ_t, σ_s can be derived; see [32, Theorem 1.1 and 1.2]. Related results were obtained in [26, 33]. These results are obtained by using the Carleman estimate. Here, we outline another approach, which could help understand the necessity of the condition that f is independent of t. Below, we assume that σ_t, σ_s and f are functions of t, x variables.

First, we solve the forward problem of (4.50) using the operators in (4.41). The solution on $\mathcal{M}$ can still be expressed as in (4.44),

$$u = T_1^{-1}(\mathrm{Id} - KT_1^{-1})^{-1}(fF)$$

Let ρ be the restriction operator to $t = [0, T] \times \partial\Omega$. Then we set

$$X = \rho T_1^{-1}(\mathrm{Id} - KT_1^{-1})^{-1}, \tag{4.51}$$

so $X(fF) = u|_{[0,T] \times \partial\Omega}$. Observe that $\rho T_1^{-1} = L_\kappa$ is still a weighted light ray transform provided that the support of f is sufficiently small and T is large; see Figure 4.3.

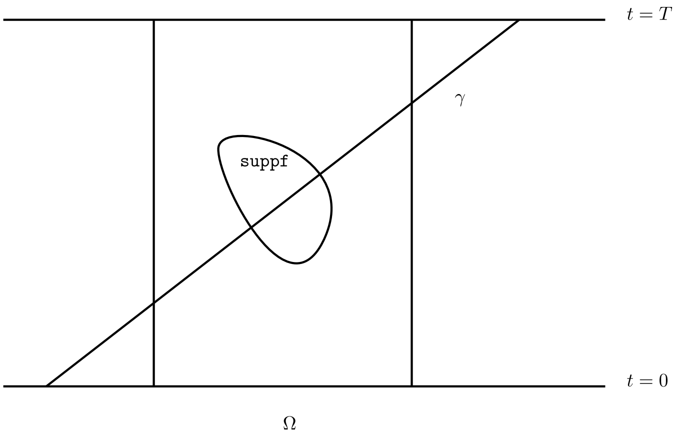

Figure 4.3: The inverse source problem with boundary measurements.

For the moment, let us assume $\sigma_s = 0$ so $K = 0$ in (4.51). Thus, the inverse problem is to recover f from the weighted light ray transform $L_\kappa(f)$ assuming $\kappa > 0$ in $\mathcal{M}$. Here, we can think of F as part of the weight. As we mentioned previously, the microlocal structure of the normal operator of the weighted light ray transform was obtained in [31, 55]. In particular, $L_\kappa^* L_\kappa$ is microlocally elliptic in Γ^{sp}. Thus, if the wave front set of f is contained in Γ^{sp}, then we can stably recover f modulo a smooth term just as explained in Section 4.3. Note that this is the case when f is independent of t variable. Furthermore, for analytic weights, an injectivity result for the weighted light ray transform was obtained in [40] for functions whose support expands slower than the speed of light; see [40, Definition] for the precise statement. These results suggest that Theorem 4.5.3 should hold for generic σ_t. For $\sigma_s \neq 0$, we expect that one can show compactness of the remaining term in (4.51) in view of the argument in Section 4.5.1.

4.6 Open problems

4.6.1 The injectivity problem

It is an important question whether the light ray transform is injective on, for example, C_0^∞ functions. So far, there are only a few known results. For the Minkowski spacetime, the injectivity can be seen from the Fourier slice theorem plus the analyticity of the Fourier transform of f. Under a strictly foliation condition, Stefanov [41] obtained a support theorem for the light ray transform on analytic Lorentzian manifolds; see also [35] for a recent development under the no conjugate point assumption. For certain static and stationary spacetime, Feizmohammadi, Ilmavirta and Oksanen proved in [14] that the transform is injective. For some pseudo-Riemannian manifolds, Ilmavirta [23] obtained injectivity result by using Pestov's energy method. Because of the lack of good stability, it is not known whether the injectivity results mentioned above still hold under small C^∞ metric perturbations.

Also, it is intriguing to consider the injectivity of the weighted light ray transform. The only known injectivity result is for analytic weights obtained in [41]. In many ways, the transform has similar behavior to the limited angle or local Radon transform in dimension two. We know from the work of Boman [5] that there are weights for which the local Radon transform is not injective. It would be interesting to find out whether the phenomena happens for the light ray transform.

In this article, we focused on the scalar type perturbations. In fact, the tensor problem is probably more interesting from the physical point of view. For a light-like geodesic $\gamma(\tau), \tau \in \mathbb{R}$ on a Lorentzian manifold $(\mathcal{M}, g)$, we can define the light-ray transform of a smooth symmetric two tensor field f by

$$L(f)(\gamma) = \int \sum_{i,j=0}^{n} f_{ij}(\gamma(\tau))\dot{\gamma}^i(\tau)\dot{\gamma}^j(\tau)d\tau$$

when the integral makes sense. The transform (4.15) has a nontrivial kernel. The complete description of the kernel is known for the Minkowski space in [30] and some static and stationary spacetimes in [14]. The result is wide open for general Lorentzian manifolds.

4.6.2 The scattering rigidity problem

We consider the possibility of determining spacetime structures by using observation of light signals on a Cauchy surface. Let $\mathcal{M} = [0, T] \times \mathbb{R}^3$, $T > 0$ and g be a globally hyperbolic Lorentzian metric on $\mathcal{M}$ such that each hypersurface $\mathcal{M}_t = \{t\} \times \mathbb{R}^3$ is a Cauchy surface. In this case, every future pointing null geodesic $\gamma(\tau), \tau \in \mathbb{R}$ intersects $\mathcal{M}_0, \mathcal{M}_T$ at one point. We thus have a well-defined scattering relation for null geodesics

$$S(\gamma(0), \dot{\gamma}(0)) = (\gamma(\tau_0), \dot{\gamma}(\tau_0)) \tag{4.52}$$

where $\gamma(0) \in \mathcal{M}_0$, $\gamma(\tau_0) \in \mathcal{M}_T$; see Figure 4.4. It is natural to ask what information of g can be recovered from S. Recently, there are several interesting work by Eskin [12, 13] and Stefanov [42] on related problems; see also [48] for the similar problem for time-like curves.

This problem can be regarded as the nonlinear version of the inverse problems in Section 4.2. Also, the problem is related to Guillemin's work [18, 19] on the Zollfrei deformation of the compactified 2+1 dimensional Minkowski spacetime, which in some sense concerns the scattering relation defined from the past null infinity to the future null infinity. From another perspective, the problem can be regarded as the Lorentzian version

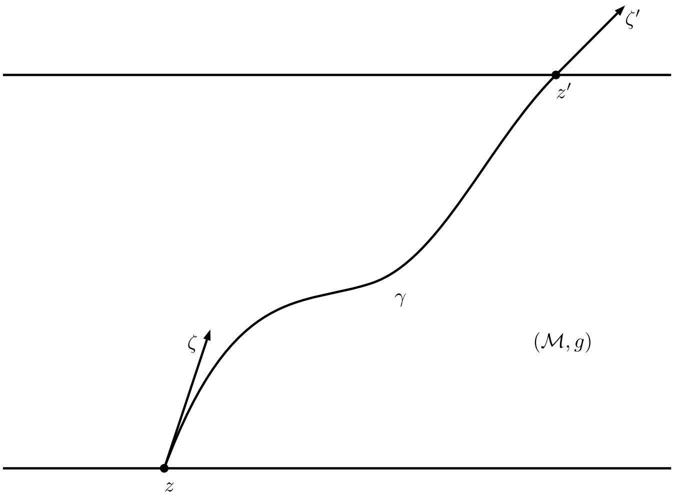

Figure 4.4: The scattering relation for light-like geodesics.

of the scattering rigidity problem for compact Riemannian manifolds with boundary; see [39].

In [55], the author studied the problem for one parameter family of metrics near the Minkowski metric. Roughly speaking, the author followed the approach in [43] for the boundary rigidity problem near the Euclidean metric. The main difficulty is the instability of the weighted light ray transform in the pseudolinearization identity. In view of the result in Section 4.4, the rigidity result is promising for Einstein spacetimes. For example, the linearized problem near Minkowski metric is closely related to the CMB inverse problem for tensor-type metric perturbations. In addition, the metric perturbation satisfies the linearized Einstein equations. We expect the light ray transform to have good stability with a proper gauge choice.

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