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Optimal superresolution of two point sources via Schmidt basis

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Abstract: We investigate superresolution of two general point sources using continuous rotation of the observation basis. Optimal superresolution with maximum estimation accuracy is achieved when measurements are performed in the Schmidt basis. © 2024 The Author(s)

Recently, it has been demonstrated that the conventional Abbe-Rayleigh resolution limit can be overcome for passive point sources when the spatial domain of the signal is analyzed in Hermite Gaussian (HG) mode basis instead of direct intensity measurement [1, 2]. This is due to the fact that measurements in the decomposed HG basis allows opportunities for more accurate estimation. It is now natural to ask: what about perform measurements in another basis? Given that there are infinite number of bases, does an optimal basis exist? To answer these questions, here we perform a systematic investigation of analyzing the estimation accuracy quantity (in terms of quantum Fisher information) in a continuously varying basis. Interestingly, it is found that there do exist such an optimal basis with maximum quantum Fisher information. More surprisingly, this optimal basis is exactly the Schmidt basis that spans simultaneously the spatial degree of freedom (DoF) and the non-spatial DoFs. Our results provide important guidance for the optimum realization of two-point source superresolution.

To illustrate our approach, we consider the superresolution of two arbitrarily unbalanced and partially coherent point sources. The full optical field of a point source can always be described by its spatial dependence (in terms of spatial domain vectors $|h_{\pm}\rangle$) and the remaining DoFs (in terms of generic vector basis $|\phi_{1,2}\rangle$ that can represent the temporal modes, polarization, etc.). Then the total state of the two sources can be in general described as

$$|\Psi\rangle = a\,|h_+\rangle\,|\phi_1\rangle + b\,|h_-\rangle\,|\phi_2\rangle\,, \tag{1}$$

where a and b are arbitrary amplitudes (permitting any unbalanceness) of the two sources respectively. In the most general case, $|\phi_{1,2}\rangle$ are non orthogonal, i.e., $\alpha = \langle\phi_1|\phi_2\rangle$ representing arbitrary partial coherence. Thus one can always express $|\phi_2\rangle = \alpha\,|\phi_1\rangle + \beta\,|\phi_1^{\perp}\rangle$ with $\langle\phi_1|\phi_1^{\perp}\rangle = 0$ and $|\alpha|^2 + |\beta|^2 = 1$. The spatial states $|h_{\pm}\rangle$, taken as amplitudes of Gaussian point spread functions, are also non-orthogonal at finite separation with $d = \langle h_-|h_+\rangle$. The spatial DoF and the non-spatial DoF are apparently entangled in (1), which can be quantified with concurrence [3] to be $C = 2|ab|\sqrt{1-|d\alpha|}$.

Then the two source state can be rewritten as

$$|\Psi\rangle = (a\,|h_+\rangle + \alpha|h_-\rangle)\,|\phi_1\rangle + b\beta\,|h_-\rangle\,|\phi_1^{\perp}\rangle\,. \tag{2}$$

To explore the effect of continuous basis rotation, we now apply an arbitrary rotation of the non-spatial states, i.e.,

$$|\phi_1^{\theta}\rangle = \cos\theta|\phi_1\rangle + \sin\theta|\phi_1^{\perp}\rangle, \tag{3}$$

$$|\phi_2^{\theta}\rangle = -\sin\theta|\phi_1\rangle + \cos\theta|\phi_1^{\perp}\rangle. \tag{4}$$

The rotation angle θ is the continuously varying parameter for arbitrary basis. With this rotated basis, one can further express the state $|\Psi\rangle$ as

$$|\Psi\rangle = |h_1^{\theta}\rangle\,|\phi_1^{\theta}\rangle + h_2^{\theta}\rangle\,|\phi_2^{\theta}\rangle\,, \tag{5}$$

where we have defined two new spatial states, $|h_1^{\theta}\rangle = \cos\theta(a\,|h_+\rangle + \alpha|h_-\rangle) + \sin\theta b\beta|h_-\rangle$, and $|h_2^{\theta}\rangle = -\sin\theta(a\,|h_+\rangle + \alpha|h_-\rangle) + \cos\theta b\beta|h_-\rangle$.

Now one is ready to analyze the Fisher information (FI) of estimation the unknown separation s. For normalization consideration, we adopt the weighted Fisher information proposed in Ref. [4]. It is calculated based on the

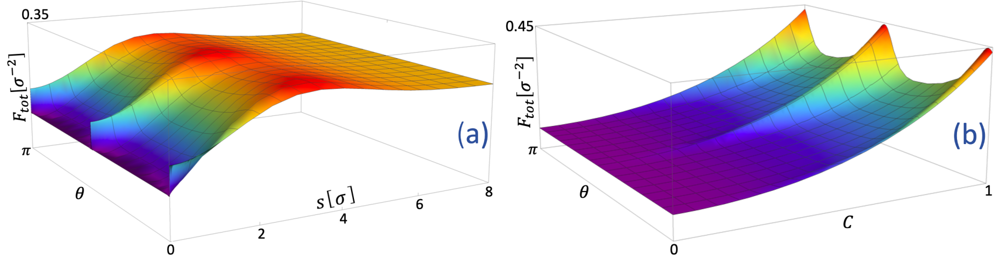

Fig. 1. For a given unbalancencess $a = \sqrt{3}/2$ and $b = 1/2$ (a) Illustration of F_{tot} with respect to separation s and rotating basis parameter θ. (b) Illustration of F_{tot} with respect to θ and entanglement C for a fixed separation $s = 0.5\sigma$.

conditional measurement of finding the entangled partner on $|\phi_1^\theta\rangle$ or in $|\phi_2^\theta\rangle$. Information will be stored in two sub-systems considering the weights for each of $|h_1^\theta\rangle$ and $|h_2^\theta\rangle$, i.e, the total Fisher information can be achieved as $F_{tot} = \langle h_1^\theta | h_1^\theta \rangle F_{\rho_1} + \langle h_2^\theta | h_2^\theta \rangle F_{\rho_2}$, where F_{ρ_1} and F_{ρ_2} are FI for $\rho_1 = |h_1^\theta\rangle\langle h_1^\theta|$ and $\rho_2 = |h_2^\theta\rangle\langle h_2^\theta|$ respectively [5,6].

Through some tedious algebraic calculations, the total FI can be obtained as

$$F_{tot} = \frac{1}{4\sigma^2}(\Gamma_1^2+\Gamma_2^2+\Gamma_1'^2+\Gamma_2'^2) - (\Gamma_1\Gamma_2+\Gamma_1'\Gamma_2')\frac{d(-s^2+4\sigma^2)}{8\sigma^4} + \frac{\frac{1}{4\sigma^2}d^2s^2(\Gamma_1\Gamma_2)^2}{(\Gamma_1^2+\Gamma_2^2)+2\Gamma_1\Gamma_2 d} + \frac{\frac{1}{4\sigma^2}d^2s^2(\Gamma_1'\Gamma_2')^2}{(\Gamma_1'^2+\Gamma_2'^2)+2\Gamma_1'\Gamma_2' d},$$

where σ is the width of the point spread function, $\Gamma_1 = (\sqrt{\lambda_1}\cos\theta c_1 - \sqrt{\lambda_2}\sin\theta c_2) - \frac{d}{\sqrt{1-d^2}}(\sqrt{\lambda_1}\cos\theta c_2 + \sqrt{\lambda_2}\sin\theta c_1)$, and $\Gamma_2 = \frac{1}{\sqrt{1-d^2}}(\sqrt{\lambda_1}\cos\theta c_2 + \sqrt{\lambda_2}\sin\theta c_1)$, and $\Gamma_1' = (\sqrt{\lambda_1}\sin\theta c_1 + \sqrt{\lambda_2}\cos\theta c_2) - \frac{d}{\sqrt{1-d^2}}(\sqrt{\lambda_1}\sin\theta c_2 - \sqrt{\lambda_2}\cos\theta c_1)$ and $\Gamma_2' = \frac{1}{\sqrt{1-d^2}}(\sqrt{\lambda_1}\sin\theta c_2 - \sqrt{\lambda_2}\cos\theta c_1)$. Here the coefficients c_1 and c_2 are expressed as $c_1 = \sqrt{\frac{1+\cos\Omega}{2}}$, $c_2 = \sqrt{\frac{1-\cos\Omega}{2}}$ where $\cos\Omega = \frac{\cos 2\eta}{\sqrt{1-(bd)^2\sin^2 2\eta}}$ and $\sin\eta = \sqrt{b^2(1-d^2)}\alpha$. The parameter $\lambda_{1,2}$ takes the form $[1+\sqrt{1\pm(bd)^2\sin^2 2\eta}]/2$ respectively.

Obviously, the total Fisher information F_{tot} is a function of the two-source separation s, entanglement C, and the rotational parameter θ for any unbalancencess a, b. For a given unbalancencess $a = \sqrt{3}/2$ and $b = 1/2$, Fig. 1 (a) illustrates the behavior of F_{tot} with respect to the two-source separation s and the rotating basis parameter θ, and Fig. 1 (b) shows the behavior of F_{tot} with respect to θ and the entanglement C. Interestingly, there exists an optimal angle $\cos\theta = \frac{1}{\lambda_1}c_1\cos\eta$ that enables the total Fisher information to reach its maximum. At its maximum in Fig. 1 (a), the Fisher information is finite even when the separation s decreases to zero. From Fig. 1 (b), one also notes that entanglement C is beneficial to increase the Fisher information [6].

Surprisingly, this particular optimal basis is exactly the Schmidt basis of the two vector spaces $\{|h_1^\theta\rangle, |h_2^\theta\rangle\}$ and $\{|\phi_1^\theta\rangle, |\phi_2^\theta\rangle\}$, i.e., $\langle h_1^\theta | h_2^\theta \rangle = \langle \phi_1^\theta | \phi_2^\theta \rangle = 0$. In other words, the optimal estimation occurs when the measurement is exactly in the Schmidt basis of the entangled state $|\Psi\rangle$.

In conclusion, we have carried out a systematic analysis of the effect of continuous basis change on the superresolution accuracy of two partially coherent and unbalanced point sources. The well-known Schmidt basis is found to be the optimal basis for maximum Fisher information even at the zero separation case. Our result provides important guidance to the realization of optimal two-source superresolution. Future works will be dedicated to reveal the fundamental significance of Schmidt decomposition in estimation theory. Our work is supported by NSF Grant No. PHY-2316878 and the U.S. Army under Contact No. W15QKN-18-D-0040.

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