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GENERIC PROPERTIES OF CONJUGATE POINTS IN OPTIMAL CONTROL PROBLEMS

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Dedicated to Hélène Frankowska

Abstract. The first part of the paper studies a class of optimal control problems in Bolza form, where the dynamics is linear w.r.t. the control function. A necessary condition is derived, for the optimality of a trajectory which starts at a conjugate point. The second part is concerned with a classical problem in the Calculus of Variations, with free terminal point. For a generic terminal cost $\psi \in \mathcal{C}^4(\mathbb{R}^n)$, applying the previous necessary condition we show that the set of conjugate points is contained in the image of an $(n-2)$-dimensional manifold and has locally bounded $(n-2)$-dimensional Hausdorff measure.

1. Introduction.

Conjugate points play a key role in the study of necessary conditions, for problems in the Calculus of Variations and optimal control [5, 6, 7, 13]. The present paper intends to be a contribution to the analysis of conjugate points, from the point of view of generic theory. Given a family of optimal control problems, with various terminal costs, we seek properties of the set of conjugate points which are true for nearly all terminal costs $\psi \in \mathcal{C}^4(\mathbb{R}^n)$. Here "nearly all" is meant in the topological sense of Baire category: these properties should be true on a $\mathcal{G}_\delta$ set, i.e., on the intersection of countably many open dense subsets. As usual, $\mathcal{C}^k(\mathbb{R}^n)$ denotes the Banach space of all bounded functions with bounded, continuous partial derivatives up to order k, see for example [1, 8].

Our basic setting is as follows. Consider an optimal control problem of the form

$$\text{minimize:} \qquad J^{\tau,y}[u] \doteq \int_\tau^T L\big(x(t),u(t)\big)\,dt + \psi\big(x(T)\big), \tag{1}$$

where $t \mapsto x(t) \in \mathbb{R}^n$ is the solution to the Cauchy problem with dynamics linear w.r.t. the control:

$$\dot{x}(t) = f\big(x(t),u(t)\big) = f_0(x(t)) + \sum_{i=1}^m f_i(x(t))\,u_i(t), \tag{2}$$

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and initial data

$$x(\tau) = y. \tag{3}$$

Here and in the sequel, the upper dot denotes a derivative w.r.t. time. In (1), the minimum cost is sought among all measurable functions $u : [\tau, T] \mapsto \mathbb{R}^m$. For $(\tau, y) \in [0, T] \times \mathbb{R}^n$, the associated value function V is defined as

$$V(\tau, y) \doteq \inf_{u(\cdot)} J^{\tau,y}[u]. \tag{4}$$

To fix ideas, we shall consider a couple (f, L) satisfying the following hypotheses.

(A1) *In (2) the vector fields f_i, $i = 0, \ldots, m$, are three times continuously differentiable and satisfy the sublinear growth condition*

$$\big|f_i(x)\big| \leq c_1 \big(|x| + 1\big) \tag{5}$$

for some constant $c_1 > 0$ and all $x \in \mathbb{R}^n$.

(A2) *The running cost $L : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}$ is three times continuously differentiable and uniformly convex w.r.t. u. Namely, for some $\delta_L > 0$, the $m \times m$ matrix of second derivatives w.r.t. u satisfies*

$$L_{uu}(x, u) - \delta_L \cdot \mathbb{I}_m \geq 0 \qquad \text{for all } x, u. \tag{6}$$

Here $\mathbb{I}_m$ denotes the $m \times m$ identity matrix.

The Pontryagin necessary conditions [2, 6, 10] take the form

$$\begin{cases} \dot{x} = f\big(x, u(x, p)\big), \\ \dot{p} = -p \cdot f_x\big(x, u(x, p)\big) - L_x(x, u(x, p)), \end{cases} \tag{7}$$

where $u(x, p)$ is determined as the pointwise minimizer

$$u(x, p) = \arg\min_{\omega \in \mathbb{R}^m} \Big\{L(x, \omega) + p \cdot f(x, \omega)\Big\}. \tag{8}$$

The assumptions in **(A1)-(A2)** guarantee that the minimizer in (8) is unique and solves

$$p \cdot f_u(x, \omega) + L_u(x, \omega) = 0. \tag{9}$$

Therefore the map $(x, p) \mapsto u(x, p)$ is well defined and continuously differentiable, and the system of ODEs (7) has continuously differentiable right hand side. In particular, for any $z \in \mathbb{R}^n$, the system (7) with terminal conditions

$$x(T) = z, \qquad p(T) = \nabla\psi(z), \tag{10}$$

admits a unique solution $t \mapsto (x, p)(t, z)$ defined on $[0, T]$. In turn, this uniquely determines the control

$$t \mapsto u(t, z) = u\big(x(t, z), p(t, z)\big). \tag{11}$$

In the following we mainly focus on the case $\tau = 0$.

Definition 1.1. Given an initial point $\bar{x} \in \mathbb{R}^n$, we say that a control $u^* : [0, T] \mapsto \mathbb{R}^m$ is a **weak local minimizer** of the cost functional

$$J^{\bar{x}}[u] \doteq \int_0^T L\big(x(t), u(t)\big)\, dt + \psi\big(x(T)\big), \tag{12}$$

subject to

$$\dot{x} = f(x, u), \qquad x(0) = \bar{x}, \tag{13}$$

if there exists $\delta > 0$ such that $J^{\bar{x}}[u^*] \leq J^{\bar{x}}[u]$ for every measurable control $u(\cdot)$ such that $\|u - u^*\|_{\mathbf{L}^\infty} < \delta$.

Consider again the maps

$$z \mapsto x(\cdot, z), \qquad z \mapsto p(\cdot, z), \qquad z \mapsto u(\cdot, z)$$

as in (11), obtained by solving the backward Cauchy problem (7)-(8). Following [3, 4] we shall adopt

Definition 1.2. For the optimization problem (12)-(13) a point $\bar{x} \in \mathbb{R}^n$ is a **conjugate point** if there exists $\overline{z} \in \mathbb{R}^n$ such that $\bar{x} = x(0, \overline{z})$, the control $u(\cdot, \overline{z})$ is a weak local minimizer of (12)-(13), and moreover

$$\det\big(x_z(0, \overline{z})\big) \;=\; 0. \tag{14}$$

Here x_z denotes the $n \times n$ Jacobian matrix of partial derivatives of the map $z \mapsto x(0, z)$.

Our main goal is to understand the structure of the set of conjugate points, for a generic terminal cost $\psi \in \mathcal{C}^4(\mathbb{R}^n)$ in (12). The present paper provides two results in this direction. In Section 2 we prove a necessary condition for the optimality of a trajectory starting at a conjugate point. We recall that, by classical results [6, 7], a trajectory $t \mapsto x(t)$ is not optimal if it contains a conjugate point $x(\tau)$ for some $0 < \tau < T$. However, the case $\tau = 0$ is more delicate. A necessary condition that covers this case is given in Theorem 2.2. Relying on this more precise result, in Section 3 we study a classical problem in the Calculus of Variations:

$$\text{Minimize:} \quad \int_0^T L\big(\dot{x}(t)\big)\, dt + \psi\big(x(T)\big) \qquad \text{subject to } x(0) = \bar{x}.$$

Assuming that the Lagrangian function $L = L(u)$ is smooth and uniformly convex, we study the structure of the set of conjugate points, for a generic terminal cost $\psi \in \mathcal{C}^4(\mathbb{R}^n)$. In particular, we show that its $(n-2)$-dimensional Hausdorff measure is locally finite. In the 1-dimensional case, the set of conjugate points is empty.

2. Necessary conditions for conjugate points.

In this section we derive a necessary condition for conjugate points. For a given $\overline{z} \in \mathbb{R}^n$, we consider the map $z \mapsto g(z, \overline{z})$, defined by

$$g(z, \overline{z}) \;\doteq\; \int_0^T L\big(\tilde{x}(t, z), u(t, z)\big)dt + \psi\big(\tilde{x}(T, z)\big), \tag{15}$$

where $u(t, z) \doteq u\big(x(t, z), p(t, z)\big)$ is the control corresponding to the solution of the backward Cauchy problem (7)–(10), while $\tilde{x}(\cdot, z)$ is the solution of

$$\dot{x}(t) \;=\; f(x(t), u(t, z)), \qquad x(0) \;=\; x(0, \overline{z}). \tag{16}$$

In other words, $g(z, \overline{z})$ is the cost of the trajectory $\tilde{x}(\cdot, z)$ which

(i) starts at the initial point $x(0, \overline{z})$ of the solution to the Pontryagin equations (7) ending at $\overline{z}$,

(ii) but uses the control $u(\cdot, z)$, corresponding to the solution of (7) ending at z.

Lemma 2.1. *Let $\overline{z} \in \mathbb{R}^n$ and $\mathbf{v} \in \mathbb{R}^n$ be a unit vector such that $x_z(0, \overline{z})\mathbf{v} = 0$. Then the map $g_{\mathbf{v}} : \mathbb{R} \to \mathbb{R}$, defined by*

$$g_{\mathbf{v}}(\theta) \;\doteq\; g(\overline{z} + \theta\mathbf{v}, \overline{z}),$$

has first and second derivatives which vanish at $\theta = 0$:

$$g'_{\mathbf{v}}(0) \;=\; 0, \qquad\qquad g''_{\mathbf{v}}(0) \;=\; 0. \tag{17}$$

Proof. **1.** For a given solution to (7)–(10), we denote by

$$x_z, p_z : [0,T] \mapsto \mathbb{R}^{n\times n}, \qquad u_z : [0,T] \mapsto \mathbb{R}^{m\times n},$$

the matrix representations of the differentials w.r.t. the terminal point z. Differentiating (7) and (16) one obtains

$$\begin{cases} \dfrac{d}{dt} x_z(t,z) &= f_x(x,u)x_z + f_u(x,u)u_z\,, \\[2mm] \dfrac{d}{dt} \tilde{x}_z(t,z) &= f_x(\tilde{x},u)\tilde{x}_z + f_u(\tilde{x},u)u_z\,, \\[2mm] \dfrac{d}{dt} p_z(t,z) &= -p_z f_x - p\,(f_{xx}x_z + f_{xu}u_z) - L_{xx}x_z - L_{xu}u_z\,. \end{cases} \tag{18}$$

Moreover, set $\Gamma(t,z) \doteq L(x(t,z),u(t,z))$ for all $(t,z)\in[0,T]\times\mathbb{R}^n$. By (7) and (9) we have

$$\begin{aligned} \frac{d}{dt}\big[p(t,z)\cdot x_z(t,z)\big] \;&=\; \big[-pf_x - L_x\big]x_z + p\big[f_x x_z + f_u u_z\big] \\ &=\; -L_x x_z - L_u u_z \;=\; -\Gamma_z(t,z). \end{aligned} \tag{19}$$

Observing that

$$x(\cdot,\overline{z}) \;=\; \tilde{x}(\cdot,\overline{z}), \qquad x_z(0,\overline{z})\,\mathbf{v} \;=\; 0, \tag{20}$$

we have

$$\tilde{x}_z(t,\overline{z})\mathbf{v} \;=\; x_z(t,\overline{z})\mathbf{v}, \qquad \text{for all } t\in[0,T]. \tag{21}$$

Recalling (15), we now compute

$$\begin{aligned} g'_{\mathbf{v}}(\theta) &= \int_0^T \frac{d}{d\theta} L\big(\tilde{x}(t,\overline{z}+\theta\mathbf{v}),u(t,\overline{z}+\theta\mathbf{v})\big)\,dt + \nabla\psi\big(\tilde{x}(T,\overline{z}+\theta\mathbf{v})\big)\cdot\frac{d}{d\theta}\tilde{x}(T,\overline{z}+\theta\mathbf{v}) \\ &= \int_0^T \Gamma_z(t,\bar{z}+\theta\mathbf{v})\mathbf{v}\,dt + \nabla\psi\big(\tilde{x}(T,\bar{z}+\theta\mathbf{v})\big)\cdot\tilde{x}_z(T,\bar{z}+\theta\mathbf{v})\mathbf{v}. \end{aligned} \tag{22}$$

Therefore, (19)-(21) yield

$$\begin{aligned} g'_{\mathbf{v}}(0) \;&=\; \int_0^T \Gamma_z(t,\overline{z})\mathbf{v}dt + \nabla\psi(\overline{z})\cdot x_z(T,\overline{z})\mathbf{v} \\ &=\; -\int_0^T \frac{d}{dt}\big[p(t,\overline{z})x_z(t,\overline{z})\big]\mathbf{v}\,dt + p(T,\overline{z})\,x_z(T,\overline{z})\mathbf{v} \;=\; p(0,\overline{z})\,x_z(0,\overline{z})\mathbf{v} \;=\; 0. \end{aligned}$$

2. To prove the second identity in (17) one needs to differentiate (22) once more. In the following, second order differentials such as $\psi_{zz} = D_z^2\psi$ and $\tilde{x}_{zz}$ are regarded as symmetric bilinear maps, sending a couple of vectors $\mathbf{v_1}\otimes\mathbf{v_2}\in\mathbb{R}^n\times\mathbb{R}^n$ into $\mathbb{R}$ and into $\mathbb{R}^n$, respectively. We compute

$$\begin{aligned} g''_{\mathbf{v}}(\theta) \;&= \int_0^T \tfrac{d^2}{d\theta^2} L\big(\tilde{x}(t,\overline{z}+\theta\mathbf{v}),u(t,\overline{z}+\theta\mathbf{v})\big)dt \\ &\quad +\psi_{zz}\big(\tilde{x}(T,\overline{z}+\theta\mathbf{v})\big)(\tilde{\mathbf{w}}^\theta(T)\otimes\tilde{\mathbf{w}}^\theta(T)) \\ &\quad +\nabla\psi\big(\tilde{x}(T,\overline{z}+\theta\mathbf{v})\big)\cdot\Big(\tilde{x}_{zz}(T,\overline{z}+\theta\mathbf{v})(\mathbf{v}\otimes\mathbf{v})\Big), \end{aligned} \tag{23}$$

where

$$\tilde{\mathbf{w}}^\theta(t) \;\doteq\; \tilde{x}_z(t,\overline{z}+\theta\mathbf{v})\mathbf{v}, \qquad t\in[0,T].$$

Differentiating the first equation in (18) once again w.r.t. z, one obtains

$$\frac{d}{dt}x_{zz} \;=\; f_{xx}(x_z\otimes x_z)+2f_{xu}(x_z\otimes u_z)+f_{uu}(u_z\otimes u_z)+f_x\,x_{zz}+f_u\,u_{zz}\,. \tag{24}$$

Setting

$$\mathbf{w}^\theta(t) \;\doteq\; x_z(t,\bar z+\theta\mathbf{v})\mathbf{v}, \qquad \mathbf{b}^\theta(t) \;\doteq\; u_z(t,\bar z+\theta\mathbf{v})\mathbf{v}, \qquad t\in[0,T],$$

from (24) it follows

$$\begin{aligned}\frac{d}{dt}x_{zz}(t,\bar z)(\mathbf{v}\otimes\mathbf{v}) \;&=\; f_{xx}\big(x(t,\bar z),u(t,\bar z)\big)\big(\mathbf{w}^0(t)\otimes\mathbf{w}^0(t)\big)\\ &+2f_{xu}\big(x(t,\bar z),u(t,\bar z)\big)\big(\mathbf{w}^0(t)\otimes\mathbf{b}^0(t)\big)+f_{uu}\big(x(t,\bar z),u(t,\bar z)\big)\big(\mathbf{b}^0(t)\otimes\mathbf{b}^0(t)\big)\\ &+f_x\big(x(t,\bar z),u(t,\bar z)\big)x_{zz}(t,\bar z)(\mathbf{v}\otimes\mathbf{v})+f_u\big(x(t,\bar z),u(t,\bar z)\big)u_{zz}(t,\bar z)(\mathbf{v}\otimes\mathbf{v}),\end{aligned} \tag{25}$$

$$\begin{aligned}\frac{d}{dt}\tilde x_{zz}(t,\bar z)(\mathbf{v}\otimes\mathbf{v}) \;&=\; f_{xx}\big(\tilde x(t,\bar z),u(t,\bar z)\big)\big(\tilde{\mathbf{w}}^0(t)\otimes\tilde{\mathbf{w}}^0(t)\big)\\ &+2f_{xu}\big(\tilde x(t,\bar z),u(t,\bar z)\big)\big(\tilde{\mathbf{w}}^0(t)\otimes\mathbf{b}^0(t)\big)+f_{uu}\big(\tilde x(t,\bar z),u(t,\bar z)\big)\big(\mathbf{b}^0(t)\otimes\mathbf{b}^0(t)\big)\\ &+f_x\big(\tilde x(t,\bar z),u(t,\bar z)\big)\tilde x_{zz}(t,\bar z)(\mathbf{v}\otimes\mathbf{v})+f_u\big(\tilde x(t,\bar z),u(t,\bar z)\big)u_{zz}(t,\bar z)(\mathbf{v}\otimes\mathbf{v}).\end{aligned} \tag{26}$$

By (21) one has

$$\mathbf{w}^0(t) \;=\; \tilde{\mathbf{w}}^0(t) \qquad\qquad \text{for all } t\in[0,T].$$

Comparing the two equations (25)-(26), we see that by (20) the only difference between the right hand sides is the term involving x_{zz}. Therefore we can write

$$\tilde x_{zz}(t,\bar z)(\mathbf{v}\otimes\mathbf{v}) \;=\; x_{zz}(t,\bar z)(\mathbf{v}\otimes\mathbf{v})+w(t,\bar z), \tag{27}$$

where $w(\cdot,\bar z):[0,T]\mapsto\mathbb{R}^n$ is the solution to the linear ODE

$$\dot w(t) \;=\; f_x\big(x(t,\bar z),u(t,\bar z)\big)\cdot w(t), \qquad\qquad w(0) \;=\; -x_{zz}(0,\bar z)(\mathbf{v}\otimes\mathbf{v}). \tag{28}$$

Using (27), we now compute

$$\begin{aligned}&\left[\frac{d^2}{d\theta^2}L\big(\tilde x(t,\bar z+\theta\mathbf{v}),u(t,\bar z+\theta\mathbf{v})\big)\right]_{\theta=0}\\ &\quad=\; \Gamma_{zz}(t,\bar z)(\mathbf{v}\otimes\mathbf{v})+L_x\big(x(t,\bar z),u(t,\bar z)\big)\,w(t,\bar z).\end{aligned} \tag{29}$$

By (28) and the second equation in (7) it follows

$$\frac{d}{dt}\big[p(t,\bar z)w(t,\bar z)\big] \;=\; \dot p\,w+pf_xw \;=\; \big(-pf_x-L_x+pf_x\big)w \;=\; -L_xw\,.$$

Hence

$$L_x(x(t,\bar z),u(t,\bar z))\,w(t,\bar z) \;=\; -\frac{d}{dt}\big[p(t,\bar z)w(t,\bar z)\big]. \tag{30}$$

From (23), using (19), (28) and (30), and recalling that $\mathbf{w}^0(T) = \mathbf{v}$ while $\psi_z(\bar z) = p(T, \bar z)$, we obtain

$$\begin{aligned}
g''_{\mathbf{v}}(0) &= \int_0^T \Gamma_{zz}(t,\bar z)(\mathbf{v}\otimes\mathbf{v})\,dt + \int_0^T L_x(x(t,\bar z),u(t,\bar z))w(t,\bar z)dt \\
&\qquad + \psi_{zz}(\bar z)(\mathbf{v}\otimes\mathbf{v}) + \psi_z(\bar z)\cdot\Big[x_{zz}(T,\bar z)(\mathbf{v}\otimes\mathbf{v}) + w(T,\bar z)\Big] \\
&= -\int_0^T \frac{d}{dt}\left(\frac{d}{dz}\big[p(t,\bar z)\,x_z(t,\bar z)\big]\right)(\mathbf{v}\otimes\mathbf{v})\,dt - \int_0^T \frac{d}{dt}\left[p(t,\bar z)w(t,\bar z)\right]dt \\
&\qquad + \frac{d}{dz}[p(T,\bar z)x_z(T,\bar z)](\mathbf{v}\otimes\mathbf{v}) + p(T,\bar z)w(T,\bar z) \\
&= \frac{d}{dz}\big[p(0,\bar z)\cdot x_z(0,\bar z)\big](\mathbf{v}\otimes\mathbf{v}) + p(0,\bar z)w(0,\bar z) \\
&= p(0,\bar z)\big[x_{zz}(0,\bar z)(\mathbf{v}\otimes\mathbf{v}) + w(0,\bar z)\big] + \big(p_z(0,\bar z)\,\mathbf{v}\big)\cdot\big(x_z(0,\bar z)\,\mathbf{v}\big) \;=\; 0.
\end{aligned}$$

The proof is complete. □

In view of (17), if $\bar x = x(0,\bar z)$ is a conjugate point the optimality assumption implies the vanishing of the third derivative:

$$g'''_{\mathbf{v}}(0) \;=\; 0. \tag{31}$$

This yields the following necessary condition:

Theorem 2.2. *Given a conjugate point $\bar x = x(0,\bar z) \in \mathbb{R}^n$, with $\bar z \in \mathbb{R}^n$ associated to a weak local minimizer $u(\cdot,\bar z)$ of the optimization problem (12)-(13), let $\mathbf{v} \in \mathbb{R}^n$ be a unit vector such that $x_z(0,\bar z)\mathbf{v} = 0$. Then one has*

$$\big(p_z(0,\bar z)\mathbf{v}\big)\cdot x_{zz}(0,\bar z)(\mathbf{v}\otimes\mathbf{v}) \;=\; 0. \tag{32}$$

Proof. Differentiating (23), we compute

$$\begin{aligned}
g'''_{\mathbf{v}}(\theta) &= \int_0^T \frac{d^3}{d\theta^3} L\big(\tilde x(t,\bar z+\theta\mathbf{v}), u(t,\bar z+\theta\mathbf{v})\big)\,dt \\
&\qquad + \psi_{zzz}\big(\tilde x(T,\bar z+\theta\mathbf{v})\big)\big(\tilde{\mathbf{w}}^\theta(T)\otimes\tilde{\mathbf{w}}^\theta(T)\otimes\tilde{\mathbf{w}}^\theta(T)\big) \\
&\qquad + 3\psi_{zz}\big(\tilde x(T,\bar z+\theta\mathbf{v})\big)\big(\tilde{\mathbf{w}}^\theta(T)\otimes\tilde x_{zz}(T,\bar z+\theta\mathbf{v})(\mathbf{v}\otimes\mathbf{v})\big) \\
&\qquad + \psi_z\big(\tilde x(T,\bar z+\theta\mathbf{v})\big)\cdot\Big(\tilde x_{zzz}(T,\bar z+\theta\mathbf{v})(\mathbf{v}\otimes\mathbf{v}\otimes\mathbf{v})\Big).
\end{aligned} \tag{33}$$

The third order differentials ψ_{zzz} and $\tilde x_{zzz}$ are here regarded as tri-linear maps, sending a triple of vectors $\mathbf{v}_1\otimes\mathbf{v}_2\otimes\mathbf{v}_3 \in \mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n$ into $\mathbb{R}$ and into $\mathbb{R}^n$, respectively. Differentiating the identities (25)-(26) once more w.r.t. z, we obtain

$$\begin{aligned}
&\frac{d}{dt}x_{zzz}(t,\bar z)(\mathbf{v}\otimes\mathbf{v}\otimes\mathbf{v})\\
&= f_{xxx}\big(x(t,\bar z),u(t,\bar z)\big)\big(\mathbf{W}_0(t)\big)+3f_{xxu}\big(x(t,\bar z),u(t,\bar z)\big)\big(\mathbf{W}_1(t)\big)\\
&\quad+3f_{xuu}\big(x(t,\bar z),u(t,\bar z)\big)\big(\mathbf{W}_2(t)\big)+f_{uuu}\big(x(t,\bar z),u(t,\bar z)\big)\big(\mathbf{W}_3(t)\big)\\
&\quad+3f_{xx}\big(x(t,\bar z),u(t,\bar z)\big)\big(\mathbf{w}^0(t)\otimes\mathbf{x}(t)\big)+3f_{uu}\big(x(t,\bar z),u(t,\bar z)\big)\big(\mathbf{u}(t)\otimes\mathbf{b}^0(t)\big)\\
&\quad+3f_{xu}\big(x(t,\bar z),u(t,\bar z)\big)\big(\mathbf{x}(t)\otimes\mathbf{b}^0(t)+\mathbf{u}(t)\otimes\mathbf{w}^0(t)\big)\\
&\quad+f_u\big(x(t,\bar z),u(t,\bar z)\big)\mathbf{U}(t)+f_x\big(x(t,\bar z),u(t,\bar z)\big)\mathbf{X}(t)
\end{aligned}\tag{34}$$

and

$$\begin{aligned}
&\frac{d}{dt}\tilde x_{zzz}(t,\bar z)(\mathbf{v}\otimes\mathbf{v}\otimes\mathbf{v})\\
&= f_{xxx}\big(x(t,\bar z),u(t,\bar z)\big)\big(\mathbf{W}_0(t)\big)+3f_{xxu}\big(x(t,\bar z),u(t,\bar z)\big)\big(\mathbf{W}_1(t)\big)\\
&\quad+3f_{xuu}\big(x(t,\bar z),u(t,\bar z)\big)\big(\mathbf{W}_2(t)\big)+f_{uuu}\big(x(t,\bar z),u(t,\bar z)\big)\big(\mathbf{W}_3(t)\big)\\
&\quad+3f_{xx}\big(x(t,\bar z),u(t,\bar z)\big)\big(\mathbf{w}^0(t)\otimes\tilde{\mathbf{x}}(t)\big)+3f_{uu}\big(x(t,\bar z),u(t,\bar z)\big)\big(\mathbf{u}(t)\otimes\mathbf{b}^0(t)\big)\\
&\quad+3f_{xu}\big(x(t,\bar z),u(t,\bar z)\big)\big(\tilde{\mathbf{x}}(t)\otimes\mathbf{b}^0(t)+\mathbf{u}(t)\otimes\mathbf{w}^0(t)\big)\\
&\quad+f_u\big(x(t,\bar z),u(t,\bar z)\big)\mathbf{U}(t)+f_x\big(x(t,\bar z),u(t,\bar z)\big)\tilde{\mathbf{X}}(t),
\end{aligned}\tag{35}$$

with

$$\begin{cases}
\mathbf{W}_0(t) \doteq \mathbf{w}^0(t)\otimes\mathbf{w}^0(t)\otimes\mathbf{w}^0(t), & \mathbf{W}_1(t) \doteq \mathbf{w}^0(t)\otimes\mathbf{w}^0(t)\otimes\mathbf{b}^0(t),\\
\mathbf{W}_2(t) \doteq \mathbf{w}^0(t)\otimes\mathbf{b}^0(t)\otimes\mathbf{b}^0(t), & \mathbf{W}_3(t) \doteq \mathbf{b}^0(t)\otimes\mathbf{b}^0(t)\otimes\mathbf{b}^0(t),\\
\mathbf{X}(t) \doteq x_{zzz}(t,\bar z)(\mathbf{v}\otimes\mathbf{v}\otimes\mathbf{v}), & \tilde{\mathbf{X}}(t) \doteq \tilde x_{zzz}(t,\bar z)(\mathbf{v}\otimes\mathbf{v}\otimes\mathbf{v}),\\
\mathbf{x}(t) \doteq x_{zz}(t,\bar z)(\mathbf{v}\otimes\mathbf{v}), & \tilde{\mathbf{x}}(t) \doteq \tilde x_{zz}(t,\bar z)(\mathbf{v}\otimes\mathbf{v}),\\
\mathbf{u}(t) \doteq u_{zz}(t,\bar z)(\mathbf{v}\otimes\mathbf{v}), & \mathbf{U}(t) \doteq u_{zzz}(t,\bar z)(\mathbf{v}\otimes\mathbf{v}\otimes\mathbf{v}).
\end{cases}$$

Comparing the results, we eventually obtain

$$\tilde x_{zzz}(t,\bar z)(\mathbf{v}\otimes\mathbf{v}\otimes\mathbf{v}) = x_{zzz}(t,\bar z)(\mathbf{v}\otimes\mathbf{v}\otimes\mathbf{v})+W(t,\bar z),\tag{36}$$

where $w(\cdot)$ is the function constructed at (28), while $W(\cdot,\bar z):[0,T]\to\mathbb{R}^n$ is the solution to the linear ODE

$$\begin{aligned}
\dot W(t) &= f_x\big(x(t,\bar z),u(t,\bar z)\big)\cdot W(t)+3f_{xx}\big(x(t,\bar z),u(t,\bar z)\big)\big(\mathbf{w}^0(t)\otimes w(t,\bar z)\big)\\
&\qquad+3f_{xu}(x(t,\bar z),u(t,\bar z))\big(w(t,\bar z)\otimes\mathbf{b}^0(t)\big),
\end{aligned}\tag{37}$$

with initial data

$$W(0,\bar z) = -x_{zzz}(0,\bar z)(\mathbf{v}\otimes\mathbf{v}\otimes\mathbf{v}).\tag{38}$$

In this case, we have

$$
\begin{aligned}
&\Big[\frac{d^3}{d\theta^3} L\big(\tilde x(t,\bar z+\theta \mathbf{v}), u(t,\bar z+\theta\mathbf{v})\big)\Big]_{\theta=0}\\
&\quad = \Gamma_{zzz}(t,\bar z)(\mathbf{v}\otimes\mathbf{v}\otimes\mathbf{v}) + L_x\, W(t,\bar z)\\
&\qquad + 3L_{xu}\,\big(w(t,\bar z)\otimes \mathbf{b}^0(t)\big) + 3L_{xx}\,\big(w(t,\bar z)\otimes\mathbf{w}^0(t)\big),\\
&\frac{d}{dt}\Big[p(t,\bar z)\,W(t,\bar z)\Big]\\
&\quad = -L_x\, W(t,\bar x) + 3p(t,\bar z) f_{xx}\,\big(\mathbf{w}^0(t)\otimes w(t,\bar z)\big)\\
&\qquad + 3p(t,\bar z) f_{xu}\,\big(w(t,\bar z)\otimes\mathbf{b}^0(t)\big),\\
&\frac{d}{dt}\Big[\big(p_z(t,\bar z)\mathbf{v}\big)\cdot w(t,\bar z)\Big]\\
&\quad = -p(t,\bar z) f_{xx}\,\big(\mathbf{w}^0(t)\otimes w(t,\bar z)\big) - p(t,\bar z) f_{xu}\,\big(\mathbf{b}^0(t)\otimes w(t,\bar z)\big)\big)\\
&\qquad - L_{xx}\,\big(\mathbf{w}^0(t)\otimes w(t,\bar z)\big) - L_{xu}\,\big(w(t,\bar z)\otimes\mathbf{b}^0(t)\big).
\end{aligned}
$$

In the above formulas, it is understood that the functions f, L and all their partial derivatives are computed at the point $\big(x(t,\bar z), u(t,\bar z)\big)$.

Using the above identities together with (19) and (28), from (33) we obtain

$$
\begin{aligned}
& g_{\mathbf{v}}'''(0)\\
&= \Big[-\int_0^T \tfrac{d}{dt}\Big(\tfrac{d^2}{dz^2}\big[p(t,\bar z)\cdot x_z(t,\bar z)\big](\mathbf{v}\otimes\mathbf{v}\otimes\mathbf{v})\Big)\,dt + \tfrac{d^2}{dz^2}\big[p(T,\bar z)\cdot x_z(T,\bar z)\big](\mathbf{v}\otimes\mathbf{v}\otimes\mathbf{v})\Big]\\
&\quad -\left[\int_0^T \frac{d}{dt}\,[pW]\,(t,\bar z)dt - [pW]\,(T,\bar z)\right]\\
&\quad +3\Big[\big(p_z(T,\bar z)\mathbf{v}\big)\cdot w(T,\bar z) - \int_0^T \tfrac{d}{dt}\big[\big(p_z(t,\bar z)\mathbf{v}\big)\cdot w(t,\bar z)\big]\,dt\Big]\\
&= \frac{d^2}{dz^2}\big[p(0,\bar z)\cdot x_z(0,\bar z)\big](\mathbf{v}\otimes\mathbf{v}\otimes\mathbf{v}) + [pW](0,\bar z) + 3\big(p_z(0,\bar z)\,\mathbf{v}\big)\cdot w(0,\bar z)\\
&= 2\big(p_z(0,\bar z)\,\mathbf{v}\big)\cdot x_{zz}(0,\bar z)(\mathbf{v}\otimes\mathbf{v}) + 3\big(p_z(0,\bar z)\,\mathbf{v}\big)\cdot w(0,\bar z)\\
&= -\big(p_z(0,\bar z)\,\mathbf{v}\big)\cdot x_{zz}(0,\bar z)(\mathbf{v}\otimes\mathbf{v}).
\end{aligned}
$$

Since $g_{\mathbf{v}}$ attains a local minimum at $\theta = 0$ and $g'_{\mathbf{v}}(0) = g''_{\mathbf{v}}(0) = 0$, this yields (32). □

3. **Conjugate points for a generic problem in the Calculus of Variations.**

In this section, the necessary condition stated in Theorem 2.2 will be used to study a generic property of the set of conjugate points for a classical problem in the Calculus of Variations. Namely, we seek to minimize (12) in the special case where

$$
\dot x = u, \qquad L(x,u) = L(u). \tag{39}
$$

In this case (see for example [2]), the value function V is the unique viscosity solution to the Hamilton-Jacobi equation

$$
\begin{cases}
-V_t(t,x) - H\big(\nabla V(t,x)\big) = 0, & (t,x)\in[0,T]\times\mathbb{R}^n,\\
V(T,x) = \psi(x), & x\in\mathbb{R}^n,
\end{cases} \tag{40}
$$

with

$$
H(p) = \min_{\omega\in\mathbb{R}^n}\big\{L(\omega) + p\cdot\omega\big\}. \tag{41}
$$

By (7) and (10) it follows

$$x(0,z) \;=\; z - T\cdot DH(\nabla\psi(z)), \qquad p(0,z) \;=\; \nabla\psi(z), \qquad z\in\mathbb{R}^n. \tag{42}$$

By Theorem 2.2, the conjugate points are thus contained in the set $\big\{x(0,z)\,;\ (z,\bar v)\in\Omega_\psi\big\}$, with

$$\Omega_\psi \;\doteq\; \Big\{(z,\mathbf{v})\in\mathbb{R}^n\times S^{n-1}\,;\ x_z(0,z)\mathbf{v}=0, \quad \big(D^2\psi(z)\mathbf{v}\big)\cdot x_{zz}(0,z)(\mathbf{v}\otimes\mathbf{v})=0\Big\}, \tag{43}$$

where S^{n-1} denotes the set of unit vectors in $\mathbb{R}^n$.

Theorem 3.1. *Let the function $L=L(u)$ be smooth and uniformly convex. Then there exists a $\mathcal{G}_\delta$ subset $\mathcal{M}\subseteq\mathcal{C}^4(\mathbb{R}^n)$ such that for every $\psi\in\mathcal{M}$, the set Ω_ψ at (43) is an embedded manifold of dimension $n-2$.*

Proof. **1.** Given a terminal cost $\psi\in\mathcal{C}^4(\mathbb{R}^n)$, defining (x,p) as in (42), we have

$$x_z(0,z) \;=\; \mathbb{I}_n - T\cdot D^2H(\nabla\psi(z))D^2\psi(z), \qquad p_z(0,z) \;=\; D^2\psi(z). \tag{44}$$

Thus, if $x_z(0,z)\mathbf{v}=0$ then $\mathbf{v}=T\cdot D^2H(\nabla\psi(z))D^2\psi(z)\mathbf{v}$ and

$$p_z(0,z)\mathbf{v} \;=\; D^2\psi(z)\mathbf{v} \;=\; \frac{1}{T}\cdot\big[D^2H(\nabla\psi(z))\big]^{-1}(\mathbf{v}). \tag{45}$$

This implies

$$\Omega_\psi \;=\; \Big\{(z,\mathbf{v})\in\mathbb{R}^n\times S^{n-1}\,;\ x_z(0,z)\mathbf{v}=0,\ \big[D^2H(\nabla\psi(z))\big]^{-1}\mathbf{v}\cdot x_{zz}(0,z)(\mathbf{v}\otimes\mathbf{v})=0\Big\}. \tag{46}$$

Define the $\mathcal{C}^1$ map $\Phi^\psi:\mathbb{R}^n\times S^{n-1}\mapsto\mathbb{R}^n\times\mathbb{R}$ by setting

$$\Phi^\psi(z,\mathbf{v}) \;\doteq\; \Big(x_z(0,z)\mathbf{v}\,,\ \big[D^2H(\nabla\psi(z))\big]^{-1}\mathbf{v}\cdot x_{zz}(0,z)(\mathbf{v}\otimes\mathbf{v})\Big). \tag{47}$$

For $k\geq 1$, let $\overline{B}_k\subset\mathbb{R}^n$ be the closed ball centered at the origin with radius k, and consider the open subset of $\mathcal{C}^4(\mathbb{R}^n)$

$$\mathcal{M}_k \;\doteq\; \big\{\psi\in\mathcal{C}^4(\mathbb{R}^n)\ :\ \Phi^\psi|_{\overline{B}_k\times S^{n-1}} \text{ is transversal to } \{\mathbf{0}\}\big\}. \tag{48}$$

Here $\{\mathbf{0}\}$ denotes the zero-dimensional manifold containing the single point $(0,0)\in\mathbb{R}^n\times\mathbb{R}$.

If $\mathcal{M}_k$ is dense in $\mathcal{C}^4(\mathbb{R}^n)$ for all $k\in\mathbb{Z}^+$, then the set $\mathcal{M}\doteq\bigcap_{k\geq1}\mathcal{M}_k$ is a $\mathcal{G}_\delta$ subset of $\mathcal{C}^4(\mathbb{R}^n)$ such that for every $\psi\in\mathcal{M}$, Φ^ψ is transverse to $\{\mathbf{0}\}$. By the implicit function theorem, the set Ω_ψ is an embedded manifold of dimension $n-2$.

2. Next, we show that $\mathcal{M}_k$ is dense in $\mathcal{C}^4(\mathbb{R}^n)$. For this purpose, fix any $\tilde\psi\in\mathcal{C}^4(\mathbb{R}^n)$. For every $\varepsilon>0$, we first approximate $\tilde\psi$ by a smooth function ψ with $\|\tilde\psi-\psi\|_{\mathcal{C}^4}<\varepsilon$. Then we need to construct a perturbed function ψ^θ arbitrarily close to ψ in the $\mathcal{C}^4$ norm, which lies in $\mathcal{M}_k$. Toward this goal, for any point $(\bar z,\mathbf{v})\in\overline{B}_k\times S^{n-1}$, we consider the family of perturbed functions of the form

$$\psi^\theta(z) \;\doteq\; \psi(z)+\eta(z-\bar z)\cdot\left[\sum_{i,j=1}^n\theta_{ij}(z_i-\bar z_i)(z_j-\bar z_j)+\sum_{k=1}^n\theta_k(z_k-\bar z_k)^3\right]. \tag{49}$$

Here $\eta:\mathbb{R}^n\mapsto[0,1]$ is a smooth cutoff function, such that

$$\eta(y) \;=\; \begin{cases} 1 & \text{if } |y|\leq 1,\\ 0 & \text{if } |y|\geq 2.\end{cases} \tag{50}$$

Moreover, $\theta = (\theta_{ij}, \theta_k) \in \mathbb{R}^{n^2+n}$. We claim that the map

$$(z, \mathbf{v}, \theta) \;\mapsto\; \Phi^{\psi^\theta} \;=\; \Big(x_z^\theta(0,z)\mathbf{v},\; \big[D^2 H(\nabla\psi^\theta(z))\big]^{-1}\mathbf{v}\cdot x_{zz}^\theta(0,z)(\mathbf{v}\otimes\mathbf{v})\Big)$$

is transversal to $\{\mathbf{0}\} \subset \mathbb{R}^n \times \mathbb{R}$ at the point $(\bar z, \mathbf{v}, 0)$. This will certainly be true if the Jacobian matrix $D_\theta \Phi^{\psi^\theta}$ of partial derivatives w.r.t. θ_{ij}, θ_k has maximum rank $n+1$.

Writing $z = (z_1, \ldots, z_n) \in \mathbb{R}^n$ and recalling (49), for $|z - \bar z| < 1$ we compute the partial derivatives

$$\frac{\partial}{\partial z_i}\psi^\theta(z) \;=\; \frac{\partial}{\partial z_i}\psi(z) + \sum_{j=1}^n (\theta_{ij}+\theta_{ji})(z_j - \bar z_j) + 3\theta_i(z_i - \bar z_i)^2, \tag{51}$$

$$\frac{\partial^2}{\partial z_i \partial z_j}\psi^\theta(z) \;=\; \frac{\partial^2}{\partial z_i\partial z_j}\psi(z) + \begin{cases} 2\theta_{ii} + 6\theta_i(z_i - \bar z_i) & \text{if} \quad i = j, \\ \theta_{ij} + \theta_{ji} & \text{if} \quad i \neq j. \end{cases} \tag{52}$$

Calling $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ the standard basis of $\mathbb{R}^n$, we have

$$D^2\psi^\theta(z)\mathbf{v} \;=\; D^2\psi(z)\mathbf{v} + \sum_{i=1}^n\sum_{j=1}^n (\theta_{ij}+\theta_{ji})\mathbf{v}_j\cdot\mathbf{e}_i + 6\sum_{i=1}^n \theta_i(z_i - \bar z_i)\mathbf{v}_i\cdot\mathbf{e}_i\,,$$

$$\frac{\partial}{\partial\theta_{ij}} D^2\psi^\theta(z)\mathbf{v} \;=\; \mathbf{v}_j\cdot\mathbf{e}_i + \mathbf{v}_i\cdot\mathbf{e}_j\,.$$

Thus, for every $i \in \{1, \cdots, n\}$, the matrix $D_\theta[D^2\psi^\theta(\bar z)\mathbf{v}]$ contains the $n\times n$ submatrix

$$S_i \;\doteq\; \left[\frac{\partial}{\partial\theta_{ij}} D^2\psi^\theta(\bar z)\mathbf{v}\right]_{j=1}^n \;=\; \begin{bmatrix}\mathbf{r}_1\\ \vdots \\ \mathbf{r}_i \\ \vdots \\ \mathbf{r}_n\end{bmatrix} \;=\; \begin{bmatrix} \mathbf{v}_i & 0 & \cdots & 0 & \cdots & \cdots & 0 \\ 0 & \mathbf{v}_i & \cdots & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \cdots & \cdots & 0 \\ \mathbf{v}_1 & \cdots & \mathbf{v}_{i-1} & 2\mathbf{v}_i & \mathbf{v}_{i+1} & \cdots & \mathbf{v}_n \\ 0 & \cdots & 0 & 0 & \mathbf{v}_i & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & \mathbf{v}_i \end{bmatrix}.$$

Notice that this implies

$$\det(S_i) \;=\; 2\mathbf{v}_i^n. \tag{53}$$

Recalling (44) and (51), we have

$$D_\theta\big[x_z^\theta(0,\bar z)\mathbf{v}\big] \;=\; -T\cdot D^2H(\nabla\psi^\theta(\bar z))D_\theta[D^2\psi^\theta(\bar z)\mathbf{v}\big], \tag{54}$$

and (47) implies that the matrix $D_\theta\big[\Phi^{\psi^\theta}(\bar z, \mathbf{v}, \theta)\big]$ contains the $(n+1)\times n$ submatrix

$$\tilde S_i \;\doteq\; -T\cdot\begin{bmatrix} D^2H(\nabla\psi^\theta(\bar z))S_i \\ 0_{1\times n}\end{bmatrix}. \tag{55}$$

Furthermore, we compute

$$\begin{aligned} x_{zz}^\theta(0,z)(\mathbf{v}\otimes\mathbf{v}) \;&=\; -\,TD^2H(\nabla\psi^\theta(z))D^3\psi^\theta(z)(\mathbf{v}\otimes\mathbf{v}) \\ &\qquad -\,TD^3H(\nabla\psi^\theta(z))(D^2\psi^\theta(z)\mathbf{v}\otimes D^2\psi^\theta(z)\mathbf{v}), \end{aligned}$$

with

$$\begin{cases} \dfrac{\partial}{\partial\theta_i} D\psi^\theta(z) \ = \ 3(z_i - \bar z_i)^2 \cdot \mathbf{e}_i, \qquad \dfrac{\partial}{\partial\theta_i} D^2\psi^\theta(z)\mathbf{v} \ = \ 6(z_i - \bar z_i)\mathbf{v}_i \cdot \mathbf{e}_i, \\ \dfrac{\partial}{\partial\theta_i} D^3\psi^\theta(z)(\mathbf{v}\otimes\mathbf{v}) \ = \ 6\mathbf{v}_i^2 \cdot \mathbf{e}_i, \qquad \text{for all } i \in \{1, \cdots, n\}. \end{cases}$$

In particular, we have

$$\frac{\partial}{\partial\theta_k} x^\theta_{zz}(\bar z)(\mathbf{v}\otimes\mathbf{v}) \ = \ -6T\mathbf{v}_k^2 D^2H(\nabla\psi^\theta(\bar z))\mathbf{e}_k\,,$$

and this yields

$$\begin{aligned} &\frac{\partial}{\partial\theta_k}\Big(\big[D^2H(\nabla\psi^\theta(\bar z))\big]^{-1}\mathbf{v}\cdot x^\theta_{zz}(0,\bar z)(\mathbf{v}\otimes\mathbf{v})\Big) \\ &= \ \big[D^2H(\nabla\psi^\theta(\bar z))\big]^{-1}\mathbf{v}\cdot \frac{\partial}{\partial\theta_k} x^\theta_{zz}(\bar z)(\mathbf{v}\otimes\mathbf{v}) \\ &= \ -6T\mathbf{v}_k^2 \cdot \big[D^2H(\nabla\psi^\theta(\bar z))\big]^{-1}\mathbf{v}\cdot D^2H(\nabla\psi^\theta(\bar z))\mathbf{e}_k \\ &= \ -6T\mathbf{v}_k^2\cdot\mathbf{v}\cdot\mathbf{e}_k \ = \ -6T\mathbf{v}_k^3. \end{aligned} \tag{56}$$

By the previous analysis we conclude that, for every $i \in \{1, \cdots, n\}$, the Jacobian matrix $D_\theta\Phi^{\psi^\theta}$ of partial derivatives w.r.t. θ_{ij}, θ_i contains $n+1$ columns which form the $(n+1)\times n$ submatrix

$$\Lambda_i \ \doteq \ [\tilde S_i, b_i] \qquad \text{with} \qquad b_i = (*\,,\ \cdots\,,\ *\,,\ -6T\mathbf{v}_i^3)^T. \tag{57}$$

By (53) and (55), it follows

$$\begin{aligned} \det(\Lambda_i) \ &= \ -6T\mathbf{v}_i^3 \cdot \det\big(-T\cdot D^2H(\nabla\psi^\theta(\bar z))S_i\big) \\ &= \ -12T\mathbf{v}_i^{n+3}\cdot\det\big(-T\cdot D^2H(\nabla\psi^\theta(\bar z))\big)\,. \end{aligned}$$

By the strict convexity of H and since $\mathbf{v}\in S^{n-1}$, we have that $\mathrm{rank}(\Lambda_i) = n+1$ for some $i \in \{1, \cdots, n\}$ and this yields

$$\mathrm{rank}\, D_\theta\Phi^{\psi^\theta}(\bar z, \mathbf{v}, 0) \ = \ n+1. \tag{58}$$

3. By continuity, there exists a neighborhood $\mathcal{N}_{\bar z,\mathbf{v}}$ of $(\bar z, \mathbf{v})$ such that

$$\mathrm{rank}\, D_\theta\Phi^{\psi^\theta}(\bar z', \mathbf{v}', 0) \ = \ n+1 \qquad \text{for all } (\bar z', \mathbf{v}') \in \mathcal{N}_{\bar z,\mathbf{v}}.$$

Covering the compact set $\overline{B}_k \times S^{n-1}$ with finitely many open neighborhoods $\mathcal{N}_\ell = \mathcal{N}_{\bar z^\ell,\mathbf{v}^\ell}$, $\ell = 1, \ldots, N$, we consider the family of combined perturbations

$$\psi^\theta(z) \ = \ \psi(z) + \sum_{\ell=1}^N \left\{\eta(z-\bar z^\ell)\cdot\left[\sum_{i,j=1}^n \theta^\ell_{ij}(z_i - \bar z^\ell_i)(z_j - \bar z^\ell_j) + \sum_{k=1}^n \theta^\ell_k (z_k - \bar z^\ell_k)^3\right]\right\}. \tag{59}$$

By construction, the matrix of partial derivatives w.r.t. all combined variables $\theta = (\theta^\ell_{ij}, \theta^\ell_k)$ satisfies (58) at every point $(\bar z, \mathbf{v}) \in \overline{B}_k \times S^{n-1}$.

Again by continuity, we still have

$$\mathrm{rank}\, D_\theta\Phi^{\psi^\theta}(\bar z, \mathbf{v}, \theta) \ = \ n+1$$

for all $(\bar z, \mathbf{v}, \theta) \in \overline{B}_k \times S^{n-1} \times \mathbb{R}^{N(n^2+n)}$ with $|\theta| < \varepsilon$ sufficiently small. By the transversality theorem [11, Lemma II.4.6], this implies that, for a dense set of values θ, the smooth map Φ^{ψ^θ} at (47) is transversal to $\{\mathbf{0}\}$, restricted to the domain $(\bar z, \mathbf{v}) \in \overline{B}_k \times S^{n-1}$. We conclude that the set $\mathcal{M}_k$ is dense and the proof is complete. □

Corollary 3.2. *In the same setting of Theorem 3.1, there exists a $\mathcal{G}_\delta$ subset $\mathcal{M} \subseteq \mathcal{C}^4(\mathbb{R}^n)$ with the following property. For every $\psi \in \mathcal{M}$, the set $\Gamma_\psi \subset \mathbb{R}^n$ of all conjugate points has locally bounded $(n-2)$-dimensional Hausdorff measure.*

Proof. Call $\pi : \mathbb{R}^n \times S^{n-1} \to \mathbb{R}^n$ the projection on the first component, so that $\pi(z, \mathbf{v}) = z$. Then the set of all conjugate points satisfies the inclusion

$$\Gamma_\psi \subseteq \Big\{ x\big(0, \pi(z, \mathbf{v})\big) \doteq z - T \cdot DH\big(\nabla \psi(z)\big)\,;\ (z, \mathbf{v}) \in \Omega_\psi \Big\}.$$

By Theorem 3.1, there exists a $\mathcal{G}_\delta$ set $\mathcal{M} \subset \mathcal{C}^4(\mathbb{R}^n)$ such that, for $\psi \in \mathcal{M}$, the set Ω_ψ is an embedded manifold of dimension $n-2$.

We now observe that the map $(z, \mathbf{v}) \mapsto x(0, \pi(z, \mathbf{v}))$ is Lipschitz continuous. Moreover, for every $z \in \mathbb{R}^n$, one has

$$|z| \leq \big|x(0,z)\big| + LT\,, \qquad \text{with} \qquad L \doteq \max_{|p| \leq \|\nabla \psi\|_\infty} |DH(p)|,$$

and this implies

$$\Gamma_\psi \cap \overline{B}_r \subseteq \Big\{ x(0,z)\,;\ (z, \mathbf{v}) \in \Omega_\psi\,,\ z \in \overline{B}_{r+LT} \Big\} \qquad \text{for all } r > 0.$$

Since Ω_ψ is an embedded manifold, and the map $(z, \mathbf{v}) \mapsto x(0, z)$ is Lipschitz continuous, by the properties of Hausdorff measures [9] we conclude that the set Γ_ψ has locally bounded $(n-2)$-dimensional Hausdorff measure. □

Remark 3.3. Using the original version of Sard's theorem [12], the smoothness assumption on L can be somewhat relaxed. Indeed, one can check that both Theorem 3.1 and Corollary 3.2 still hold for a uniformly convex Lagrangian function $L \in \mathcal{C}^{n+2}$.

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