

# MEMOIRS

of the  
American Mathematical Society

ISSN 0065-9266 (print)  
ISSN 1947-6221 (online)

Number 1536

## Foundations of Matroids, Part 1: Matroids without Large Uniform Minors

Matthew Baker  
Oliver Lorscheid

January 2025 • Volume 305 • Number 1536 (second of 7 numbers)

# MEMOIRS

of the

American Mathematical Society

ISSN 0065-9266 (print)

ISSN 1947-6221 (online)

---

Number 1536

---

## Foundations of Matroids, Part 1: Matroids without Large Uniform Minors

Matthew Baker  
Oliver Lorscheid

---

January 2025 • Volume 305 • Number 1536 (second of 7 numbers)

---

---

## Library of Congress Cataloging-in-Publication Data

Cataloging-in-Publication Data has been applied for by the AMS.

See <http://www.loc.gov/publish/cip/>.

DOI: <https://doi.org/10.1090/memo/1536>

---

## Memoirs of the American Mathematical Society

This journal is devoted entirely to research in pure and applied mathematics.

**Subscription information.** Beginning in 2024, *Memoirs* will be published monthly through 2026. *Memoirs* is also accessible from [www.ams.org/journals](http://www.ams.org/journals). The 2025 subscription begins with volume 305 and consists of twelve mailings, each containing one or more numbers. Individual subscription prices for 2025 are as follows. For electronic only: US\$1044. For paper delivery: US\$1197. Add US\$65 for delivery within the United States; US\$200 for surface delivery outside the United States. Upon request, subscribers to paper delivery of this journal are also entitled to receive electronic delivery. For information on institutional pricing, please visit <https://www.ams.org/publications/journals/subscriberinfo>. Subscription renewals are subject to late fees. See [www.ams.org/journal-faq](http://www.ams.org/journal-faq) for more journal subscription information. Each number may be ordered separately; please specify number when ordering an individual number.

**Back number information.** For back issues see [www.ams.org/backvols](http://www.ams.org/backvols).

Subscriptions and orders should be addressed to the American Mathematical Society, P. O. Box 845904, Boston, MA 02284-5904 USA. *All orders must be accompanied by payment.* Other correspondence should be addressed to 201 Charles Street, Providence, RI 02904-2213 USA.

**Copying and reprinting.** Individual readers of this publication, and nonprofit libraries acting for them, are permitted to make fair use of the material, such as to copy select pages for use in teaching or research. Permission is granted to quote brief passages from this publication in reviews, provided the customary acknowledgment of the source is given.

Republication, systematic copying, or multiple reproduction of any material in this publication is permitted only under license from the American Mathematical Society. Requests for permission to reuse portions of AMS publication content are handled by the Copyright Clearance Center. For more information, please visit [www.ams.org/publications/pubpermissions](http://www.ams.org/publications/pubpermissions).

Send requests for translation rights and licensed reprints to [reprint-permission@ams.org](mailto:reprint-permission@ams.org).

Excluded from these provisions is material for which the author holds copyright. In such cases, requests for permission to reuse or reprint material should be addressed directly to the author(s). Copyright ownership is indicated on the copyright page, or on the lower right-hand corner of the first page of each article within proceedings volumes.

---

*Memoirs of the American Mathematical Society* (ISSN 0065-9266 (print); 1947-6221 (online)) is published bimonthly (each volume consisting usually of more than one number) by the American Mathematical Society at 201 Charles Street, Providence, RI 02904-2213 USA. Periodicals postage paid at Providence, RI. Postmaster: Send address changes to *Memoirs*, American Mathematical Society, 201 Charles Street, Providence, RI 02904-2213 USA.

© 2025 by the American Mathematical Society. All rights reserved.

This publication is indexed in *Mathematical Reviews*®, *Zentralblatt MATH*, *Science Citation Index*®, *Science Citation Index*™-Expanded, *ISI Alerting Services*™, *SciSearch*®, *Research Alert*®, *CompuMath Citation Index*®, *Current Contents*®/Physical, Chemical & Earth Sciences.

This publication is archived in *Portico* and *CLOCKSS*.

Printed in the United States of America.

∞ The paper used in this book is acid-free and falls within the guidelines established to ensure permanence and durability.

Visit the AMS home page at <https://www.ams.org/>

10 9 8 7 6 5 4 3 2 1      30 29 28 27 26 25

# Contents

Introduction	1
A quick introduction to pastures	2
Representations of matroids over pastures and the foundation of a matroid	4
Products and coproducts	6
Universal cross ratios	7
The structure theorem for foundations of matroids without large uniform minors	8
Consequences of the structure theorem	9
Content overview	10
Acknowledgments	11
Chapter 1. Background	13
1.1. Notation	13
1.2. The Tutte group	13
1.3. Embedded minors	14
Chapter 2. Pastures	17
2.1. Definition and first properties	17
2.2. Matroid representations	22
2.3. Matroid representations via hyperplane functions	23
2.4. The universal pasture	24
2.5. The Tutte group and the universal pasture	26
Chapter 3. Cross ratios	27
3.1. Cross ratios of $P$ -matroids	27
3.2. Cross ratios for hyperplanes	29
3.3. Universal cross ratios	30
3.4. Fundamental elements	31
3.5. Compatibility with the Tutte group formulation of Dress and Wenzel	32
Chapter 4. Foundations	35
4.1. Definition and basic facts	35
4.2. Universal cross ratios as generators of the foundation	36
4.3. The foundation of the dual matroid	36
4.4. Foundations of embedded minors	37
4.5. The foundation of $U_4^2$	40
4.6. The tip and cotip relations	42
4.7. Relations for parallel elements	43
4.8. The foundation of the Fano matroid and its dual	44
4.9. A presentation of the foundation by hyperplanes	45

4.10.	A presentation of the foundation by bases	47
4.11.	A presentation of the foundation by embedded minors	50
Chapter 5.	The structure theorem	53
5.1.	Foundations of matroids on 5 elements	53
5.2.	Symmetry quotients	56
5.3.	The structure theorem for matroids without large uniform minors	59
Chapter 6.	Applications	63
6.1.	Forbidden minors for regular, binary and ternary matroids	64
6.2.	Uniqueness of the rescaling class over $\mathbb{F}_3$	65
6.3.	Criteria for representability over certain fields	65
6.4.	Oriented matroids without large minors are uniquely dyadic	66
6.5.	Positively oriented matroids without large uniform minors are near-regular	68
6.6.	Representation classes of matroids without large uniform minors	71
6.7.	Characterization of classes of matroids	76
Bibliography		83

# Abstract

The *foundation* of a matroid is a canonical algebraic invariant which classifies, in a certain precise sense, all representations of the matroid up to rescaling equivalence. Foundations of matroids are *pastures*, a simultaneous generalization of partial fields and hyperfields which are special cases of both tracts (as defined by the first author and Bowler) and ordered blue fields (as defined by the second author).

Using deep results due to Tutte, Dress–Wenzel, and Gelfand–Rybnikov–Stone, we give a presentation for the foundation of a matroid in terms of generators and relations. The generators are certain “cross-ratios” generalizing the cross-ratio of four points on a projective line, and the relations encode dependencies between cross-ratios in certain low-rank configurations arising in projective geometry.

Although the presentation of the foundation is valid for all matroids, it is simplest to apply in the case of matroids *without large uniform minors*. i.e., matroids having no minor corresponding to five points on a line or its dual configuration. For such matroids, we obtain a complete classification of all possible foundations.

We then give a number of applications of this classification theorem, for example:

- (1) We prove the following strengthening of a 1997 theorem of Lee and Scobee: every orientation of a matroid without large uniform minors comes from a dyadic representation, which is unique up to rescaling.
- (2) For a matroid  $M$  without large uniform minors, we establish the following strengthening of a 2017 theorem of Ardila–Rincón–Williams: if  $M$  is positively oriented then  $M$  is representable over every field with at least 3 elements.
- (3) Two matroids are said to belong to the same *representation class* if they are representable over precisely the same pastures. We prove that there are precisely 12 possibilities for the representation class of a matroid without large uniform minors, exactly three of which are not representable over any field.

---

Received by the editor September 28, 2020, and, in revised form, August 22, 2022.

Article electronically published on January 14, 2025.

DOI: <https://doi.org/10.1090/memo/1536>

2020 *Mathematics Subject Classification*. Primary 05B35; Secondary 20Axx.

The first author was supported in part by the NSF Research Grants DMS-1529573 and DMS-2155224. Parts of this project have been carried out while the second author was hosted by the Max Planck Institute in Bonn.

©2025 American Mathematical Society



# Introduction

Matroids are a combinatorial abstraction of the notion of linear independence in vector spaces. If  $K$  is a field and  $n$  is a positive integer, any linear subspace of  $K^n$  gives rise to a matroid; such matroids are called *representable* over  $K$ . The task of deciding whether or not certain families of matroids are representable over certain kinds of fields has occupied a plethora of papers in the matroid theory literature.

Dress and Wenzel [14, 15] introduced the *Tutte group* and the *inner Tutte group* of a matroid. These are abelian groups which, in a certain precise sense, can be used to understand representations of  $M$  over all so-called *fuzzy rings* (which in particular include fields). Dress and Wenzel gave several different presentations for these groups in terms of generators and relations, and Gelfand–Rybnikov–Stone [17] subsequently gave additional presentations for the inner Tutte group of  $M$ . The Dress–Wenzel theory of Tutte groups, inner Tutte groups, and fuzzy rings is powerful but lacks simple definitions and characterizations in terms of universal properties.

In their 1996 paper [29], Semple and Whittle generalized the notion of matroid representations to *partial fields* (which are special cases of fuzzy rings); this allows one to consider certain families of matroids (e.g. regular or dyadic) as analogous to matroids over a field, and to prove new theorems in the spirit of Tutte’s theorem that a matroid is both binary and ternary if and only if it is regular. Pendavingh and van Zwam [24, 25] subsequently introduced the *universal partial field* of a matroid  $M$ , which governs the representations of  $M$  over all partial fields. Unfortunately, most matroids (asymptotically 100%, in fact, by a theorem of Nelson [21]) are not representable over *any* partial field, and in this case the universal partial field gives no information. One can view non-representable matroids as the “dark matter” of matroid theory: they are ubiquitous but somehow mysterious.

Using the theory of matroids over partial hyperstructures presented in [3] (which has been continued in [1], [9] and [23]), we introduced in [5] a generalization of the universal partial field which we call the *foundation* of a matroid. The foundation is a kind of algebraic object which we call a *pasture*; pastures include both hyperfields and partial fields and form a natural class of “field-like” objects within the second author’s theory of *ordered blueprints* in [19]. The category of pastures has various desirable categorical properties (e.g., the existence of products and co-products) which makes it a natural context in which to study algebraic invariants of matroids. Pastures are closely related to fuzzy rings, but they are axiomatically much simpler.

One advantage of the foundation over the universal partial field is that the foundation exists for *every* matroid  $M$ , not just matroids that are representable over some field. Moreover, unlike the inner Tutte group, the foundation of a matroid



is characterized by a universal property which immediately clarifies its importance and establishes its naturality.

More precisely, the foundation of a matroid  $M$  represents the functor taking a pasture  $F$  to the set of *rescaling equivalence classes* of  $F$ -representations of  $M$ ; in particular,  $M$  is representable over a pasture  $F$  if and only if there is a morphism from the foundation of  $M$  to  $F$ .

Our first main result (Theorem 4.21) gives a precise and useful description of the foundation of a matroid in terms of generators and relations. Although this theorem applies to *all* matroids, it is easiest to apply in the case of matroids *without large uniform minors*, by which we mean matroids which do not have minors isomorphic to either  $U_5^2$  or  $U_5^3$ .<sup>1</sup> For such matroids, we obtain a complete classification (Theorem 5.9) of all possible foundations, from which one can read off just about any desired representability property. We also show, for example, that there are precisely three different representation classes of matroids without large uniform minors which are not representable over any field. The applications of Theorem 5.9 which we present in Chapter 6 are merely a representative sample of the kinds of things one can deduce from this structural result.

We now give a somewhat more precise introduction to the main concepts, definitions, and results in the present paper.

### A quick introduction to pastures

A field  $K$  can be thought of as an abelian group  $G = (K^\times, \cdot, 1)$ , a multiplicatively absorbing element  $0$ , and a binary operation  $+$  on  $K = G \cup \{0\}$  which satisfies certain additional natural axioms (e.g. commutativity, associativity, distributivity, and the existence of additive inverses). Pastures are a generalization of the notion of field in which we still have a multiplicative abelian group  $G$ , an absorbing element  $0$ , and an “additive structure”, but we relax the requirement that the additive structure come from a binary operation. The following two examples are illustrative of the type of relaxations we have in mind.

EXAMPLE (Krasner hyperfield). As a pasture, the Krasner hyperfield  $\mathbb{K}$  consists of the multiplicative monoid  $\{0, 1\}$  with  $0 \cdot x = 0$  and  $1 \cdot 1 = 1$  and the additive relations  $0 + x = x$ ,  $1 + 1 = 1$ , and  $1 + 1 = 0$ . Note, in particular, that *both*  $1 + 1 = 1$  and  $1 + 1 = 0$  are true, and in particular the additive structure is not derived from a binary operation. The fact that  $1 + 1$  is equal to two different things may seem counterintuitive at first, but if we think of  $1$  as a symbol meaning “non-zero”, it is simply a reflection of the fact that the sum of two non-zero elements (in a field, say) can be either non-zero or zero.

EXAMPLE (Regular partial field). As a pasture, the regular partial field  $\mathbb{F}_1^\pm$  consists of the multiplicative monoid  $\{0, 1, -1\}$  with  $0 \cdot x = 0$ ,  $1 \cdot 1 = 1$ ,  $1 \cdot (-1) = -1$ , and  $(-1) \cdot (-1) = 1$ , together with the additive relations  $0 + x = x$  and  $1 + (-1) = 0$ . Note, in particular, that there is no additive relation of the form  $1 + 1 = x$  or  $(-1) + (-1) = x$ , so that once again the additive structure is not derived from a binary operation (but for a different reason: here,  $1 + 1$  is undefined rather than being multi-valued). We think of  $\mathbb{F}_1^\pm$  as encoding the restriction of addition and multiplication in the ring  $\mathbb{Z}$  to the multiplicative subset  $\{0, \pm 1\}$ .

---

<sup>1</sup>Note that if  $M$  has no minor of type  $U_5^2$  or  $U_5^3$ , then  $M$  also has no uniform minor  $U_n^r$  with  $n \geq 5$  and  $2 \leq r \leq n - 2$ , hence the term “large”.

In general, we will require that a pasture  $P$  has an involution  $x \mapsto -x$  (which is trivial in the case of  $\mathbb{K}$ ), and we can use this involution to rewrite additive relations of the form  $x + y = z$  as  $x + y - z = 0$ . It turns out to be more convenient to define pastures using this formalism, and from now on we view the expression  $x + y = z$  as shorthand for  $x + y + (-z) = 0$ . For additional notational convenience, we identify relations of the form  $x + y + z = 0$  with triples  $(x, y, z)$ ; the set of all such triples will be denoted  $N_P$  and called the *null set* of the pasture.

More formally, a *pasture* is a multiplicative monoid-with-zero  $P$  such that  $P^\times = P \setminus \{0\}$  is an abelian group, an involution  $x \mapsto -x$  on  $P$  fixing 0, and a subset  $N_P$  of  $P^3$  such that:

- (1) (Symmetry)  $N_P$  is invariant under the natural action of  $S_3$  on  $P^3$ .
- (2) (Weak Distributivity)  $N_P$  is invariant under the diagonal action of  $P^\times$  on  $P^3$ .
- (3) (Unique Weak Inverses)  $(0, x, y) \in N_P$  if and only if  $y = -x$ .

If we set  $x \boxplus y = \{z \in P : x + y = z\}$ , then the pasture  $P$  corresponds to a field if and only if  $\boxplus$  is an associative binary operation. If  $x \boxplus y$  contains *at least one* element for all  $x, y \in P$  and  $\boxplus$  is associative (in the sense of set-wise addition), we call  $P$  a *hyperfield*. If  $x \boxplus y$  contains *at most one* element for all  $x, y \in P$  and satisfies a suitable associative law, we call  $P$  a *partial field*. Pastures generalize (and simplify) both hyperfields and partial fields by imposing no conditions on the size of the sets  $x \boxplus y$  and no associativity conditions.

EXAMPLE (Hyperfields). Let  $K$  be a field and let  $G \leq K^\times$  be a multiplicative subgroup. Then the quotient monoid  $K/G = (K^\times/G) \cup \{0\}$  is naturally a hyperfield: the additive relations are all expressions of the form  $[x] + [y] = [z]$  for which there exist  $a, b, c \in G$  such that  $ax + by = cz$ . For example,  $\mathbb{R}/\mathbb{R}^\times$  is isomorphic to the Krasner hyperfield  $\mathbb{K}$ ,  $\mathbb{R}/\mathbb{R}_{>0}$  is isomorphic to the sign hyperfield  $\mathbb{S}$  (cf. [3, Example 2.13]), and if  $p \geq 7$  is a prime number with  $p \equiv 3 \pmod{4}$  then  $\mathbb{F}_p/(\mathbb{F}_p^\times)^2$  is isomorphic to the weak sign hyperfield  $\mathbb{W}$  (cf. [3, Example 2.13]). However, not every hyperfield arises in this way (cf. [4, 20]).

EXAMPLE (Partial fields). Let  $R$  be a commutative ring and let  $G \leq R^\times$  be a subgroup of the unit group of  $R$  containing  $-1$ . Then  $P = G \cup \{0\}$  is naturally a partial field: the additive relations are all expressions of the form  $x + y = z$  with  $x, y, z \in G \cup \{0\}$  such that  $x + y = z$  in  $R$ . Unlike the situation with hyperfields, every partial field arises from this construction (cf. [25, Theorem 2.16]).

EXAMPLE (Partial fields, continued). If  $R$  is a commutative ring, let  $P(R)$  be the partial field corresponding to  $R^\times \subset R$ . In this paper, we will make extensive use of the following partial fields:

- (1)  $\mathbb{F}_1^\pm = P(\mathbb{Z})$ . We call this the *regular partial field*.
- (2)  $\mathbb{D} = P(\mathbb{Z}[\frac{1}{2}])$ . We call this the *dyadic partial field*.
- (3)  $\mathbb{H} = P(\mathbb{Z}[\zeta_6])$ , where  $\zeta_6 \in \mathbb{C}$  is a primitive sixth root of unity. We call this the *hexagonal partial field*.<sup>2</sup>
- (4)  $\mathbb{U} = P(\mathbb{Z}[x, \frac{1}{x}, \frac{1}{1-x}])$ , where  $x$  is an indeterminate. We call this the *near-regular partial field*.

<sup>2</sup>In [25] the partial field  $\mathbb{H}$  is denoted  $\mathbb{S}$ , but in our context that would conflict with the established terminology for the sign hyperfield, so we re-christen it as  $\mathbb{H}$ . The partial field  $\mathbb{U}$  is denoted  $\mathbb{U}_1$  in [25].

EXAMPLE (Fields). It is perhaps worth pointing out explicitly that fields are special cases of both hyperfields and partial fields; in fact, they are precisely the pastures which are both hyperfields and partial fields. Since we will be making extensive use of the finite fields  $\mathbb{F}_2$  and  $\mathbb{F}_3$  in this paper, here is how to explicitly realize these fields as pastures:

- (1)  $\mathbb{F}_2$  has as its underlying monoid  $\{0, 1\}$  with the usual multiplication. The involution  $x \mapsto -x$  is trivial, and the 3-term additive relations are  $0 + 0 + 0 = 0$  and  $0 + 1 + 1 = 0$  (and all permutations thereof).
- (2)  $\mathbb{F}_3$  has as its underlying monoid  $\{0, 1, -1\}$  with the usual multiplication. The involution  $x \mapsto -x$  sends 0 to 0 and 1 to  $-1$ . The 3-term additive relations are  $0 + 0 + 0 = 0$ ,  $1 + (-1) + 0 = 0$  (and all permutations thereof), and  $1 + 1 + 1 = 0$ .

A *morphism* of pastures is a multiplicative map  $f : P \rightarrow P'$  of monoids such that  $f(0) = 0$ ,  $f(1) = 1$  and  $f(x) + f(y) + f(z) = 0$  in  $P'$  whenever  $x + y + z = 0$  in  $P$ . Pastures form a category whose initial object is  $\mathbb{F}_1^\pm$  and whose final object is  $\mathbb{K}$ .

### Representations of matroids over pastures and the foundation of a matroid

Let  $M$  be a matroid of rank  $r$  on the finite set  $E$ , and let  $P$  be a pasture.

A  $P$ -*representation* of  $M$  is a function  $\Delta : E^r \rightarrow P$  such that:

- (1)  $\Delta(e_1, \dots, e_r) \neq 0$  if and only if  $\{e_1, \dots, e_r\}$  is a basis of  $M$ .
- (2)  $\Delta(\sigma(e_1), \dots, \sigma(e_r)) = \text{sign}(\sigma) \cdot \Delta(e_1, \dots, e_r)$  for all permutations  $\sigma \in S_r$ .
- (3)  $\Delta$  satisfies the *3-term Plücker relations*: for all  $\mathbf{J} \in E^{r-2}$  and all  $(e_1, e_2, e_3, e_4) \in E^4$ , the null set  $N_P$  of  $P$  contains the additive relation

$$\Delta(\mathbf{J}e_1e_2) \cdot \Delta(\mathbf{J}e_3e_4) - \Delta(\mathbf{J}e_1e_3) \cdot \Delta(\mathbf{J}e_2e_4) + \Delta(\mathbf{J}e_1e_4) \cdot \Delta(\mathbf{J}e_2e_3) = 0,$$

where  $\mathbf{J}e_ie_j := (j_1, \dots, j_{r-2}, e_i, e_j)$ .

DEFINITION.

- (1)  $M$  is *representable* over  $P$  if there is at least one  $P$ -representation of  $M$ .
- (2) Two  $P$ -representations  $\Delta$  and  $\Delta'$  are *isomorphic* if there exists  $c \in P^\times$  such that  $\Delta'(e_1, \dots, e_r) = c\Delta(e_1, \dots, e_r)$  for all  $(e_1, \dots, e_r) \in E^r$ .<sup>3</sup>
- (3)  $\Delta$  and  $\Delta'$  are *rescaling equivalent* if there exist  $c \in P^\times$  and a map  $d : E \rightarrow P^\times$  such that  $\Delta'(e_1, \dots, e_r) = c \cdot d(e_1) \cdots d(e_r) \cdot \Delta(e_1, \dots, e_r)$  for all  $(e_1, \dots, e_r) \in E^r$ .
- (4) We denote by  $\mathcal{X}_M^I(P)$  (resp.  $\mathcal{X}_M^R(P)$ ) the set of isomorphism classes (resp. rescaling classes) of  $P$ -representations of  $M$ .<sup>4</sup>

EXAMPLE. By the results in [3] and [5], we have:

- (1) If  $K$  is a field, the isomorphism classes of  $K$ -representations of  $M$  are naturally in bijection with  $r$ -dimensional subspaces of  $K^E$  (the  $K$ -vector space of functions from  $E$  to  $K$ ) whose underlying matroid is  $M$ .
- (2) Every matroid has a unique representation over the Krasner hyperfield  $\mathbb{K}$ .

---

<sup>3</sup>An isomorphism class of  $P$ -representations of  $M$  is the same thing as a *weak  $P$ -matroid* whose support is  $M$ , in the terminology of [3].

<sup>4</sup>In [5], these sets are denoted  $\mathcal{X}_M^w(P)$  and  $\mathcal{X}_M^f(P)$ , respectively.

- (3) If  $P$  is a partial field,  $M$  is representable over  $P$  if and only if it is representable by a  $P$ -matrix in the sense of [25]. In particular, a matroid is regular (i.e., representable over  $\mathbb{Z}$  by a totally unimodular matrix) if and only if it is representable over the partial field  $\mathbb{F}_1^\pm$ . A regular matroid will in general have many different (non-isomorphic) representations over  $\mathbb{F}_1^\pm$ , but there is a unique rescaling class of such representations.
- (4) A matroid is orientable if and only if it is representable over the sign hyperfield  $\mathbb{S}$ . An *orientation* of  $M$  is the same thing as an  $\mathbb{S}$ -representation, and in this case rescaling equivalence is usually called *reorientation equivalence*.

For fixed  $M$  the map taking a pasture  $P$  to the set  $\mathcal{X}_M^I(P)$  (resp.  $\mathcal{X}_M^R(P)$ ) is a *functor*. In particular, if  $f : P_1 \rightarrow P_2$  is a morphism of pastures, there are natural maps  $\mathcal{X}_M^I(P_1) \rightarrow \mathcal{X}_M^I(P_2)$  and  $\mathcal{X}_M^R(P_1) \rightarrow \mathcal{X}_M^R(P_2)$ .

We now come to the key result from [5] motivating the present paper:

**THEOREM.** *Given a matroid  $M$ , the functor taking a pasture  $P$  to the set  $\mathcal{X}_M^I(P)$  is representable by a pasture  $P_M$  which we call the universal pasture of  $M$ . In other words, we have a natural isomorphism*

$$(1) \quad \text{Hom}(P_M, -) \simeq \mathcal{X}_M^I.$$

*The functor taking a pasture  $P$  to the set  $\mathcal{X}_M^R(P)$  is representable by a subpasture  $F_M$  of  $P_M$  which we call the foundation of  $M$ , i.e. there is a natural isomorphism*

$$(2) \quad \text{Hom}(F_M, -) \simeq \mathcal{X}_M^R.$$

For various reasons, including the fact that the foundation can be presented by generators and relations “induced from small minors”, we will mainly focus in this paper on studying the foundation of  $M$  rather than the universal pasture. Note that both  $P_M$  and  $F_M$  have the property that  $M$  is representable over a pasture  $P$  if and only if there is a morphism from  $P_M$  (resp.  $F_M$ ) to  $P$ .

**REMARK.**

- (1) The universal partial field and foundation behave nicely with respect to various matroid operations. For example, the universal partial fields (resp. foundations) of  $M$  and its dual matroid  $M^*$  are canonically isomorphic. And there is a natural morphism from the universal partial field (resp. foundation) of a minor  $N = M \setminus I / J$  of  $M$  to the universal partial field (resp. foundation) of  $M$ .
- (2) The multiplicative group  $P_M^\times$  (resp.  $F_M^\times$ ) of the universal partial field (resp. foundation) of  $M$  is isomorphic to the *Tutte group* (resp. *inner Tutte group*) of Dress and Wenzel [14, Definition 1.6].

If we take  $P = P_M$  in (1), the identity map is a distinguished element of  $\text{Hom}(P_M, P_M)$ . It therefore corresponds to a distinguished element  $\hat{\Delta}_M \in \mathcal{X}_M^I(P_M)$ , which (by abuse of terminology) we call the *universal representation* of  $M$ . (Technically speaking, the universal representation is actually an isomorphism class of representations.)

**REMARK.** When  $F_M$  is a partial field, the foundation coincides with the universal partial field of [24]. However, when  $M$  is not representable over any field, the universal partial field does not exist. On the other hand, the foundation of  $M$

is always well-defined; this is one sense in which the theory of pastures helps us explore the “dark matter” of the matroid universe.

### Products and coproducts

The category of pastures admits finite products and co-products (a.k.a. tensor products). This is a key advantage of pastures over the categories of fields, partial fields, and hyperfields, none of which admit both products and co-products. The relevance of such considerations to matroid theory is illustrated by the following observations:

- (1)  $M$  is representable over both  $P_1$  and  $P_2$  if and only if  $M$  is representable over the product pasture  $P_1 \times P_2$ . (This is immediate from the universal property of the foundation and of categorical products.)
- (2) If  $M_1$  and  $M_2$  are matroids, the foundation of the direct sum  $M_1 \oplus M_2$  is canonically isomorphic to the tensor product  $F_{M_1} \otimes F_{M_2}$ , and similarly for the 2-sum of  $M_1$  and  $M_2$ . (These facts, along with some applications, will be discussed in detail a follow-up paper.)
- (3) Tensor products of pastures are needed in order to state and apply the main theorem of this paper, the classification theorem for foundations of matroids without large uniform minors (Theorem 5.9 below).

In order to illustrate the utility of categorical considerations for studying matroid representations, we briefly discuss a couple of key examples.

EXAMPLE. The product of the fields  $\mathbb{F}_2$  and  $\mathbb{F}_3$ , considered as pastures, is isomorphic to the regular partial field  $\mathbb{F}_1^\pm$ . As an immediate consequence, we obtain Tutte’s celebrated result that a matroid  $M$  is representable over every field if and only if  $M$  is regular. (Proof: If  $M$  is regular then since  $\mathbb{F}_1^\pm$  is an initial object in the category of pastures,  $M$  is representable over every pasture, and in particular over every field. Conversely, if  $M$  is representable over every field, then it is in particular representable over both  $\mathbb{F}_2$  and  $\mathbb{F}_3$ , hence over their product  $\mathbb{F}_1^\pm$ , and thus  $M$  is regular.)

One can, in the same way, establish Whittle’s theorem that a matroid is representable over both  $\mathbb{F}_3$  and  $\mathbb{F}_4$  if and only if it is hexagonal, i.e., representable over the partial field  $\mathbb{H}$ .

These kind of arguments are well-known in the theory of partial fields; however, the theory of pastures is more flexible. For example, the product of the field  $\mathbb{F}_2$  and the hyperfield  $\mathbb{S}$  is also isomorphic to the partial field  $\mathbb{F}_1^\pm$ . In this way, we obtain a unified proof of the result of Tutte just mentioned and the theorem of Bland and Las Vergnas that a matroid is regular if and only if it is both binary and orientable [8].

EXAMPLE. If we try to extend this type of argument to more general pastures, we run into some intriguing complications. As an illuminating example, consider the theorem of Lee and Scobee [18] that a matroid is both ternary and orientable if and only if it is dyadic, i.e., representable over the partial field  $\mathbb{D}$ . In this case, the product of  $\mathbb{F}_3$  and  $\mathbb{S}$  is *not* isomorphic to  $\mathbb{D}$ ; there is merely a morphism from  $\mathbb{D}$  to  $\mathbb{F}_3 \times \mathbb{S}$ . The theorem of Lee and Scobee therefore lies deeper than the theorems mentioned in the previous example; proving it requires establishing, in particular, that  $\mathbb{F}_3 \times \mathbb{S}$  is *not the foundation of any matroid*.

To do this, one needs a structural understanding of foundations, which we obtain by utilizing highly non-trivial results of Tutte, Dress–Wenzel, and Gelfand–Rybnikov–Stone. The result of our analysis, in the context of matroids which are both ternary and orientable, is that *every morphism from the foundation of some matroid to  $\mathbb{F}_3 \times \mathbb{S}$  lifts uniquely to  $\mathbb{D}$* . More precisely, we prove that if  $M$  is a matroid without large uniform minors (e.g. if  $M$  is ternary), then the morphism  $\mathbb{D} \rightarrow \mathbb{S}$  induces a canonical bijection  $\text{Hom}(F_M, \mathbb{D}) \rightarrow \text{Hom}(F_M, \mathbb{S})$ . This gives a new and non-trivial strengthening of the Lee–Scobee theorem. The proof goes roughly as follows: by Theorem B we have  $F_M \cong F_1 \otimes \cdots \otimes F_s$ , where each  $F_i$  belongs to an explicit finite set  $\mathcal{P}$  of pastures. By categorical considerations, the statement that a morphism  $f : F_M \rightarrow \mathbb{S}$  lifts uniquely to  $\mathbb{D}$  is equivalent to the statement that  $\text{Hom}(P, \mathbb{S}) = \text{Hom}(P, \mathbb{D})$  for all  $P \in \mathcal{P}$ , and this can be checked by concrete elementary computations.

### Universal cross ratios

In order to explain why the “large” uniform minors  $U_5^2$  and  $U_5^3$  play a special role in the theory of foundations, we need to first explain the concept of a universal cross ratio, which is intimately related to  $U_4^2$ -minors.

Let  $M$  be a matroid of rank  $r$ , let  $P$  be a pasture, and let  $\Delta$  be a  $P$ -representation of  $M$ . Let  $\mathbf{J} \in E^{r-2}$  have distinct coordinates and let  $J$  be the corresponding unordered subset of  $E$  of size  $r-2$ . If  $\Delta(\mathbf{J}e_1e_4)$  and  $\Delta(\mathbf{J}e_2e_3)$  are both non-zero (i.e., if  $J \cup \{e_1, e_4\}$  and  $J \cup \{e_2, e_3\}$  are both bases of  $M$ ), then we can rewrite the 3-term Plücker relation

$$\Delta(\mathbf{J}e_1e_2)\Delta(\mathbf{J}e_3e_4) - \Delta(\mathbf{J}e_1e_3)\Delta(\mathbf{J}e_2e_4) + \Delta(\mathbf{J}e_1e_4)\Delta(\mathbf{J}e_2e_3) = 0$$

as

$$\frac{\Delta(\mathbf{J}e_1e_3)\Delta(\mathbf{J}e_2e_4)}{\Delta(\mathbf{J}e_1e_4)\Delta(\mathbf{J}e_2e_3)} + \frac{\Delta(\mathbf{J}e_1e_2)\Delta(\mathbf{J}e_3e_4)}{\Delta(\mathbf{J}e_1e_4)\Delta(\mathbf{J}e_3e_2)} = 1.$$

Moreover, as one easily checks, the quantities

$$\frac{\Delta(\mathbf{J}e_1e_3)\Delta(\mathbf{J}e_2e_4)}{\Delta(\mathbf{J}e_1e_4)\Delta(\mathbf{J}e_2e_3)} \quad \text{and} \quad \frac{\Delta(\mathbf{J}e_1e_2)\Delta(\mathbf{J}e_3e_4)}{\Delta(\mathbf{J}e_1e_4)\Delta(\mathbf{J}e_3e_2)}$$

are invariant under rescaling equivalence and do not depend on the choice of ordering of elements of  $J$ . In particular,

$$\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{\Delta, \mathbf{J}} := \frac{\Delta(\mathbf{J}e_1e_3)\Delta(\mathbf{J}e_2e_4)}{\Delta(\mathbf{J}e_1e_4)\Delta(\mathbf{J}e_2e_3)}$$

depends only on  $J$  and on the rescaling class  $[\Delta]$  of  $\Delta$  in  $\mathcal{X}_M^R(P)$ .

The cross ratio associated to the universal representation  $\hat{\Delta}_M : E^r \rightarrow P_M$  plays an especially important role in our theory. For notational convenience, we set

$$\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{M, J} := \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{\hat{\Delta}_M, \mathbf{J}}.$$

We will write  $\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_J$  instead of  $\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{M, J}$  when  $M$  is understood.

Using the fact that  $\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{\hat{\Delta}_M, J}$  depends only on the rescaling class of  $\hat{\Delta}_M$ , one sees easily that  $\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_J$ , which *a priori* is an element of the universal pasture  $P_M$ , in fact belongs to the foundation  $F_M$ .

We call elements of  $F_M$  of the form  $\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_J$  *universal cross ratios* of  $M$ . When  $J = \emptyset$  we omit the subscript entirely. By [5, Lemma 7.7], we have:

LEMMA. *The foundation of  $M$  is generated by its universal cross ratios.*

REMARK.

- (1) When  $J = \emptyset$  and  $M = U_4^2$  is the uniform matroid of rank 2 on the 4-element set  $\{1, 2, 3, 4\}$ , the quantity  $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$  can be viewed as a “universal” version of the usual cross-ratio of four points on a projective line. The fact that the cross-ratio is the *only* projective invariant of four points on a line corresponds to the fact that the foundation of  $U_4^2$  is isomorphic to the partial field  $\mathbb{U} = P(\mathbb{Z}[x, \frac{1}{x}, \frac{1}{1-x}])$  described above. The six different values of  $\begin{bmatrix} \sigma(1) & \sigma(2) \\ \sigma(3) & \sigma(4) \end{bmatrix}$  for  $\sigma \in S_4$  correspond to the elements  $x, 1-x, \frac{1}{x}, 1-\frac{1}{x}, \frac{1}{1-x}$ , and  $1-\frac{1}{1-x}$  of  $\mathbb{U}$ .
- (2) More generally, we can associate a universal cross ratio to each  $U_4^2$ -minor  $N = M \setminus I/J$  of  $M$  (together with an ordering of the ground set of  $N$ ) via the natural map from  $F_N$  to  $F_M$ , and every universal cross ratio arises from this construction.

### The structure theorem for foundations of matroids without large uniform minors

In order to calculate and classify foundations of matroids, in addition to knowing that the universal cross ratios generate  $F_M$ , we need to understand the relations between these generators.

EXAMPLE. The universal cross ratios of the uniform matroid  $U_5^2$  on  $\{1, 2, 3, 4, 5\}$  satisfy certain *tip relations* of the form

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 5 & 3 \end{bmatrix} = 1.$$

By duality, the universal cross ratios of  $U_5^3$  satisfy similar identities which we call the *cotip relations*.

The theoretical tool which allows one to understand *all* relations between universal cross ratios is Tutte’s Homotopy Theorem [32, 33] (or, more specifically, [17, Theorem 4], whose proof is based on Tutte’s Homotopy Theorem). We give an informal description here; a more precise version is given in Theorem 4.21 below. To state the result, we say that a relation between universal cross-ratios of  $M$  is *inherited* from a minor  $N = M \setminus I/J$  if it is the image (with respect to the natural inclusion  $F_N \subseteq F_M$ ) of a relation between universal cross ratios in  $F_N$ .

THEOREM A. *Every relation between universal cross ratios of a matroid  $M$  is inherited from a minor on a 6-element set. The foundation of  $M$  is generated as an  $\mathbb{F}_1^\pm$ -algebra by such generators and relations, together with the relation  $-1 = 1$  if  $M$  has a minor isomorphic to either the Fano matroid  $F_7$  or its dual.*

The most complicated relations between universal cross ratios come from the tip and cotip relations in  $U_5^2$  and  $U_5^3$ , respectively (six-element minors and non-uniform five-element minors only contribute additional relations identifying certain cross ratios with one another). In the absence of such minors, we can completely classify all possible foundations. Roughly speaking, the conclusion is that the foundation of a matroid is the tensor product of copies of  $\mathbb{F}_2$  and quotients of  $\mathbb{U}$  (the foundation of  $U_4^2$ ) by groups of automorphisms. By calculating all possible quotients of  $\mathbb{U}$  by automorphisms, we obtain the following result (Theorem 5.9):



THEOREM B. *Let  $M$  be a matroid without large uniform minors and  $F_M$  its foundation. Then*

$$F_M \simeq F_1 \otimes \cdots \otimes F_r$$

*for some  $r \geq 0$  and pastures  $F_1, \dots, F_r \in \{\mathbb{U}, \mathbb{D}, \mathbb{H}, \mathbb{F}_3, \mathbb{F}_2\}$ .*

REMARK. In a sequel paper, we will show that every pasture of the form  $F_1 \otimes \cdots \otimes F_r$  with  $F_1, \dots, F_r \in \{\mathbb{U}, \mathbb{D}, \mathbb{H}, \mathbb{F}_3, \mathbb{F}_2\}$  is the foundation of some matroid.

### Consequences of the structure theorem

A matroid  $M$  is representable over a pasture  $P$  if and only if there is a morphism from the foundation  $F_M$  of  $M$  to  $P$ . If  $M$  is without large uniform minors (which is automatic if  $M$  is binary or ternary), then by Theorem 5.9 its foundation is isomorphic to a tensor product of copies  $F_i$  of  $\mathbb{U}$ ,  $\mathbb{D}$ ,  $\mathbb{H}$ ,  $\mathbb{F}_3$  and  $\mathbb{F}_2$ . There is a morphism from  $F_M$  to  $P$  if and only if there is a morphism from each  $F_i$  to  $P$ , so one readily obtains various theorems about representability of such matroids.

We mention just a selection of sample applications from the more complete list of results in Chapter 6. For instance, our method yields short proofs of the excluded minor characterizations of regular, binary and ternary matroids (Theorems 6.3 and 6.4). We find a similarly short proof for Brylawski-Lucas's result that every matroid has at most one rescaling class over  $\mathbb{F}_3$  (Theorem 6.5 and Remark 6.6).

As already mentioned, we derive a strengthening of a theorem by Lee and Scobee ([18]) on lifts of oriented matroids. The lifting result assumes a particularly strong form in the case of *positively oriented matroids*, improving on a result by Ardila, Rincón and Williams ([2]). The following summarizes Theorems 6.9 and 6.15:

THEOREM C. *Let  $M$  be an oriented matroid whose underlying matroid is without large uniform minors. Then  $M$  is uniquely dyadic up to rescaling. If  $M$  is positively oriented, then  $M$  is near-regular.*

In Theorem 6.7, we derive similar statements for the weak hyperfield of signs  $\mathbb{W}$  and the phase hyperfield  $\mathbb{P}$ ; cf. section 2.1.2 for definitions. Namely, a matroid  $M$  without large uniform minors is ternary if it is representable over  $\mathbb{W}$ , and is representable over  $\mathbb{D} \otimes \mathbb{H}$  if it is representable over  $\mathbb{P}$ .

We define the *representation class* of a matroid  $M$  as the class  $\mathcal{P}_M$  of all pastures  $P$  over which  $M$  is representable. Two matroids  $M$  and  $M'$  are *representation equivalent* if  $\mathcal{P}_M = \mathcal{P}_{M'}$ . The following is Theorem 6.20.

THEOREM D. *Let  $M$  be a matroid without large uniform minors. Then there are precisely 12 possibilities for the representation class of  $M$ . Nine of these classes are representable over some field, and the other three are not.*

The structure theorem also provides short proofs of various characterizations (some new, some previously known by other methods) of certain classes of matroids. The following summarizes Theorems 6.26–6.34:

THEOREM E. *Let  $M$  be a matroid without large uniform minors and  $F_M$  its foundation. Then all conditions in a given row in Table 0.1 are equivalent, where the conditions should be read as follows:*

- (1) *The first column describes the class by name (cf. Definition 2.14 for any unfamiliar terms).*



TABLE 0.1. Characterizations of classes of matroids without large uniform minors

class	possible factors of $F_M$	representable over
regular	–	$\mathbb{U} / \mathbb{F}_2 \times P$ with $-1 \neq 1$ in $P$
binary	$\mathbb{F}_2$	$\mathbb{F}_2$
ternary	$\mathbb{U}, \mathbb{D}, \mathbb{H}, \mathbb{F}_3$	any field extension $k$ of $\mathbb{F}_3$ / $\mathbb{W}$
quaternary	$\mathbb{U}, \mathbb{H}, \mathbb{F}_2$	any field extension $k$ of $\mathbb{F}_4$
near-regular	$\mathbb{U}$	$\mathbb{U} / \mathbb{F}_3 \times \mathbb{F}_8 / \mathbb{F}_4 \times \mathbb{F}_5 / \mathbb{F}_4 \times \mathbb{S} / \mathbb{F}_8 \times \mathbb{W} / \mathbb{D} \times \mathbb{H}$
dyadic	$\mathbb{U}, \mathbb{D}$	$\mathbb{D} / \mathbb{F}_3 \times \mathbb{Q} / \mathbb{F}_3 \times \mathbb{S} / \mathbb{F}_3 \times \mathbb{F}_q$ with $2 \nmid q$ and $3 \nmid q - 1$
hexagonal	$\mathbb{U}, \mathbb{H}$	$\mathbb{H} / \mathbb{F}_3 \times \mathbb{F}_4 / \mathbb{F}_4 \times \mathbb{W}$
$\mathbb{D} \otimes \mathbb{H}$ -representable	$\mathbb{U}, \mathbb{D}, \mathbb{H}$	$\mathbb{F}_3 \times \mathbb{C} / \mathbb{F}_3 \times \mathbb{P} / \mathbb{F}_3 \times \mathbb{F}_q$ with $2 \nmid q$ and $3 \mid q - 1$
representable	$\mathbb{U}, \mathbb{D}, \mathbb{H}, \mathbb{F}_3$ or $\mathbb{U}, \mathbb{H}, \mathbb{F}_2$	either $\mathbb{F}_3$ or $\mathbb{F}_4$

- (2) The second column characterizes the class in terms of the factors  $F_i$  that may appear in a decomposition  $F_M \simeq \bigotimes F_i$ , as in Theorem B.
- (3) The third column lists various classifying pastures  $P$ , separated by slashes, which means that  $M$  is contained in the class in question if and only if it is representable over  $P$ .

Another consequence of the structure theorem for foundations of matroids without large uniform minors is the following result, which will be the theme of a forthcoming paper.

**THEOREM F.** *Let  $M$  be a ternary matroid. Then up to rescaling equivalence,*

- (1) *every quaternary representation of  $M$  lifts uniquely to  $\mathbb{H}$ ;*
- (2) *every quinary representation of  $M$  lifts uniquely to  $\mathbb{D}$ ;*
- (3) *every septenary representation of  $M$  lifts uniquely to  $\mathbb{D} \otimes \mathbb{H}$ ;*
- (4) *every octonary representation of  $M$  lifts uniquely to  $\mathbb{U}$ .*

### Content overview

In Chapter 1, we introduce embedded minors and review basic facts concerning the Tutte group of a matroid. In Chapter 2, we discuss matroid representations over pastures and explain the concept of the universal pasture of a matroid. In Chapter 3, we extend the concept of cross ratios to matroid representations over pastures and define universal cross ratios. In Chapter 4, we introduce the foundation of a matroid and exhibit a complete set of relations between cross ratios, which culminates in Theorem A. In Chapter 5, we focus on matroids without large uniform minors and prove Theorem B. In Chapter 6, we explain several consequences of Theorem B, such as Theorems C, D and E.

### Acknowledgments

The authors thank Nathan Bowler and Rudi Pendavingh for helpful discussions; in particular, we thank Rudi Pendavingh for suggesting that a result along the lines of Theorem 4.21 should follow from [17]. We also thank the anonymous referee for a careful reading and numerous helpful comments. Finally, the authors thank their respective muses Camille and Cecilia for inspiring the name of the matroid  $C_5$ .



## CHAPTER 1

# Background

### 1.1. Notation

In this paper, we assume that the reader is familiar with basic concepts from matroid theory.

Typically,  $M$  denotes a matroid of rank  $r$  on the ground set  $E = \{1, \dots, n\}$ . We denote its set of bases by  $\mathcal{B} = \mathcal{B}_M$  and its set of hyperplanes by  $\mathcal{H} = \mathcal{H}_M$ . We denote the closure of a subset  $J$  of  $E$  by  $\langle J \rangle$ . We denote the dual matroid of  $M$  by  $M^*$ .

Given two subsets  $I$  and  $J$  of  $E$ , we denote by  $I - J = \{i \in I \mid i \notin J\}$  the complement of  $J$  in  $I$ . For an ordered tuple  $\mathbf{J} = (j_1, \dots, j_s)$  in  $E^s$ , we denote by  $|\mathbf{J}|$  the subset  $\{j_1, \dots, j_s\}$  of  $E$ . Given  $k$  elements  $e_1, \dots, e_k \in E$ , we denote by  $\mathbf{J}e_1 \cdots e_k$  the  $s + k$ -tuple  $(j_1, \dots, j_s, e_1, \dots, e_k) \in E^{s+k}$ . If  $J$  is a subset of  $E$ , then we denote by  $Je_1 \cdots e_k$  the subset  $J \cup \{e_1, \dots, e_k\}$  of  $E$ . In particular, we have  $|\mathbf{J}e_1 \cdots e_k| = |\mathbf{J}|e_1 \cdots e_k$  for  $\mathbf{J} \in E^s$ .

### 1.2. The Tutte group

The Tutte group is an invariant of a matroid that was introduced and studied by Dress and Wenzel in [14]. We will review the parts of this theory that are relevant for the present text in the following.

**DEFINITION 1.1.** Let  $M$  be a matroid of rank  $r$  on  $E$  with Grassmann-Plücker function  $\Delta : E^r \rightarrow \mathbb{K}$ . The multiplicatively written abelian group  $\mathbb{T}_M^{\mathcal{B}}$  is generated by symbols  $-1$  and  $X_{\mathbf{I}}$  for every  $\mathbf{I} \in \text{supp}(\Delta)$  modulo the relations

$$(T1) \quad (-1)^2 = 1;$$

$$(T2) \quad X_{(e_{\sigma(1)}, \dots, e_{\sigma(r)})} = \text{sign}(\sigma) X_{(e_1, \dots, e_r)}$$

for every permutation  $\sigma \in S_r$ , where we consider  $\text{sign}(\sigma)$  as an element of  $\{\pm 1\} \subset \mathbb{T}_M^{\mathcal{B}}$ ;

$$(T3) \quad X_{\mathbf{J}e_1e_3} X_{\mathbf{J}e_2e_4} = X_{\mathbf{J}e_1e_4} X_{\mathbf{J}e_2e_3}$$

for  $\mathbf{J} = (j_1, \dots, j_{r-2}) \in E^{r-2}$  and  $e_1, \dots, e_4 \in E$  such that  $\mathbf{J}e_i e_j \in \text{supp}(\Delta)$  for all  $i = 1, 2$  and  $j = 3, 4$  but  $\Delta(\mathbf{J}e_1 e_2) \Delta(\mathbf{J}e_3 e_4) = 0$ .

The group  $\mathbb{T}_M^{\mathcal{B}}$  comes with a morphism  $\deg : \mathbb{T}_M^{\mathcal{B}} \rightarrow \mathbb{Z}$  that sends  $X_{\mathbf{I}}$  to 1 for every  $\mathbf{I} \in \text{supp}(\Delta)$ . The *Tutte group* of  $M$  is the kernel  $\mathbb{T}_M = \ker(\deg)$  of this map.

By definition, the Tutte group  $\mathbb{T}_M$  is generated by ratios  $X_{\mathbf{I}}/X_{\mathbf{J}}$  of generators of  $X_{\mathbf{I}}, X_{\mathbf{J}}$  of  $\mathbb{T}_M^{\mathcal{B}}$ . Since the basis exchange graph of a matroid is connected, it follows that  $\mathbb{T}_M$  is generated by elements of the form  $X_{\mathbf{J}e}/X_{\mathbf{J}e'}$ , where  $\mathbf{J} \in E^{r-1}$  and  $e, e' \in E$  are such that both  $\mathbf{J}e$  and  $\mathbf{J}e'$  are in the support of  $\Delta$ .

The Tutte group can equivalently be defined in terms of hyperplanes, as explained in the following.

DEFINITION 1.2. Let  $M$  be a matroid and  $\mathcal{H}$  its set of hyperplanes. We define  $\mathbb{T}_M^{\mathcal{H}}$  as the abelian group generated by symbols  $-1$  and  $X_{H,e}$  for all  $H \in \mathcal{H}$  and  $e \in E - H$  modulo the relations

$$(TH1) \quad (-1)^2 = 1;$$

$$(TH2) \quad \frac{X_{H_1,e_2} X_{H_2,e_3} X_{H_3,e_1}}{X_{H_1,e_3} X_{H_2,e_1} X_{H_3,e_2}} = -1,$$

where  $H_1, H_2, H_3 \in \mathcal{H}$  are pairwise distinct such that  $F = H_1 \cap H_2 \cap H_3$  is a flat of rank  $r - 2$  and  $e_i \in H_i - F$  for  $i = 1, 2, 3$ .

This group comes with a map  $\deg_{\mathcal{H}} : \mathbb{T}_M^{\mathcal{H}} \rightarrow \mathbb{Z}^{\mathcal{H}}$  that sends an element  $X_{H,e}$  to the characteristic function  $\chi_H : \mathcal{H} \rightarrow \mathbb{Z}$  of  $\{H\} \subset \mathcal{H}$ , i.e.  $\chi_H(H') = \delta_{H,H'}$  for  $H' \in \mathcal{H}$ .

The relation between  $\mathbb{T}_M$  and  $\mathbb{T}_M^{\mathcal{H}}$  is explained in [14, Thms. 1.1 and 1.2], which is as follows.

THEOREM 1.3 (Dress-Wenzel '89). *Let  $M$  be a matroid and  $\mathcal{B}$  its set of bases. Then the association  $-1 \mapsto -1$  and  $X_{\mathbf{J}e}/X_{\mathbf{J}e'} \mapsto X_{H,e}/X_{H,e'}$ , where  $\mathbf{J} \in E^{r-1}$ ,  $e, e' \in E$  with  $|\mathbf{J}e|, |\mathbf{J}e'| \in \mathcal{B}$  and  $H = \langle |\mathbf{J}| \rangle$ , defines an injective group homomorphism  $\mathbb{T}_M \rightarrow \mathbb{T}_M^{\mathcal{H}}$  whose image is  $\ker(\deg_{\mathcal{H}})$ .*

### 1.3. Embedded minors

In this section, we review some basic facts about minors of a matroid and introduce the concept of an embedded minor.

Let  $M$  and  $N$  be matroids with respective ground sets  $E_M$  and  $E_N$ . An *isomorphism*  $\varphi : N \rightarrow M$  of matroids is a bijection  $\varphi : E_N \rightarrow E_M$  such that  $B \subset E_N$  is a basis of  $N$  if and only if  $\varphi(B)$  is a basis of  $M$ .

DEFINITION 1.4. Let  $M$  be a matroid on  $E$ . A *minor* of  $M$  is a matroid isomorphic to  $M \setminus I / J$ , where  $I$  and  $J$  are disjoint subsets of  $E$ ,  $M \setminus I$  denotes the deletion of  $I$  in  $M$  and  $M \setminus I / J$  denotes the contraction of  $J$  in  $M \setminus I$ .

Note that there are in general different pairs of subsets  $(I, J)$  and  $(I', J')$  as above that give rise to isomorphic minors  $M \setminus I / J \simeq M \setminus I' / J'$ . In particular, [22, Prop. 3.3.6] shows that there is a co-independent subset  $J$  and an independent subset  $I$  of  $E$  for every minor  $N$  of  $M$  such that  $I \cap J = \emptyset$  and  $N \simeq M \setminus I / J$ . Still, such  $I$  and  $J$  are in general not uniquely determined by  $N$ , cf. Example 1.8.

If we fix  $I$  and  $J$  as above, then we can identify the ground set  $E_N$  of  $N$  with  $E - (I \cup J)$ , which yields an inclusion  $\iota : E_N \rightarrow E$ . Since  $I$  is co-independent and  $J$  is independent, the set of bases of  $N$  is

$$\mathcal{B}_N = \{ B - J \mid B \in \mathcal{B}_M \text{ such that } J \subset B \subset E - I \},$$

where  $\mathcal{B}_M$  is the set of bases of  $M$ . Consequently, the difference between the rank  $r$  of  $M$  and the rank  $r_N$  of  $N$  is  $r - r_N = \#J$ . Moreover, the inclusion  $E_N \rightarrow E$  induces an inclusion

$$\begin{aligned} \iota : \mathcal{B}_N &\longrightarrow \mathcal{B}_M \\ B &\longmapsto B \cup J. \end{aligned}$$

DEFINITION 1.5. An *embedded minor* of  $M$  is a minor  $N = M \setminus I / J$  together with the pair  $(I, J)$ , where  $I$  is a co-independent subset and  $J$  is an independent subset  $J$  of  $E$  such that  $I \cap J = \emptyset$ . By abuse of notation, we say that  $\iota : N \hookrightarrow M$  is an *embedded minor*, where  $N = M \setminus I / J$  for fixed subsets  $I$  and  $J$  as above and where  $\iota : \mathcal{B}_N \rightarrow \mathcal{B}_M$  is the induced inclusion of the respective set of bases.

Let  $N'$  be a matroid. Then we say that an embedded minor  $\iota : N \hookrightarrow M$  is of *type*  $N'$ , or is an *embedded  $N'$ -minor*, if  $N$  is isomorphic to  $N'$ .

Let  $N$  and  $M$  be matroids. A *minor embedding* of  $N$  into  $M$  is an isomorphism  $N \simeq M \setminus I / J$  of  $N$  together with an embedded minor  $M \setminus I / J \hookrightarrow M$  of  $M$ .

Given two minor embeddings  $\iota : N = M \setminus J / I \hookrightarrow M$  and  $\iota' : N' = N \setminus I' / J' \hookrightarrow N$ , we define the *composition*  $\iota \circ \iota'$  of  $\iota'$  with  $\iota$  as the minor embedding  $N' = M \setminus (I \cup I') / (J \cup J') \hookrightarrow M$ .

EXAMPLE 1.6 (Embedded minors of type  $U_4^2$ ). Let  $M$  be a matroid and  $\iota : N \rightarrow M$  an embedded minor of type  $U_4^2$ . Let  $I$  and  $J$  be as above. Then  $\#J = r - 2$  since the rank of  $N$  is 2, and  $E_N = E - (I \cup J)$  has 4 elements  $e_1, \dots, e_4$ . The set of bases  $\mathcal{B}_N$  of  $N$  consists of all 2-subsets of  $E_N$ , and thus

$$\iota(\mathcal{B}_N) = \left\{ J e_i e_j \mid \{i, j\} \subset \{1, \dots, 4\} \text{ and } i \neq j \right\}.$$

REMARK 1.7. Note that a composition  $N' = N \setminus I' / J' \hookrightarrow N = M \setminus J / I \hookrightarrow M$  of minor embeddings induces a composition  $\mathcal{B}_{N'} \rightarrow \mathcal{B}_N \rightarrow \mathcal{B}_M$  of inclusions of sets of bases. On the other hand, a minor embedding  $\iota : N = M \setminus J / I \rightarrow M$  decomposes into  $\iota = \iota_1 \circ \iota_2$  with  $\iota_1 : N' = M \setminus I_1 / J_1 \rightarrow M$  and  $\iota_2 : N = N' \setminus I_2 / J_2 \rightarrow N'$  for every pair of partitions  $I = I_1 \cup I_2$  and  $J = J_1 \cup J_2$ .

Note further that it is slightly inaccurate to suppress the subsets  $I$  and  $J$  from the notation of an embedded minor  $\iota : N \rightarrow M$  since they are in general not uniquely determined by the isomorphism type of  $N$  and the injection  $\iota : \mathcal{B}_N \rightarrow \mathcal{B}_M$ , cf. Example 1.9. However, there is always a maximal choice for  $I$  and  $J$  for a given injection  $\iota : \mathcal{B}_N \rightarrow \mathcal{B}_M$ .

More precisely, for two disjoint subsets  $I$  and  $J$  of  $E$  and  $\mathcal{B} = \mathcal{B}_M$ , let  $\mathcal{B} \setminus I / J = \{B \in \mathcal{B} \mid J \subset B \subset E - I\}$ . If  $\mathcal{B} \setminus I / J$  is not empty, then  $I$  is co-independent and  $J$  is independent and  $\mathcal{B} \setminus I / J$  is the image  $\iota(\mathcal{B}_{M \setminus I / J}) \subset \mathcal{B}$  for the embedded minor  $M \setminus I / J$  of  $M$ . Tautologically,

$$I_{\max} = E - \bigcup_{B \in \mathcal{B} \setminus I / J} B \quad \text{and} \quad J_{\max} = \bigcap_{B \in \mathcal{B} \setminus I / J} B$$

are the maximal co-independent and independent subsets of  $E$  such that  $\mathcal{B} \setminus I / J = \mathcal{B} \setminus I_{\max} / J_{\max} = \iota(\mathcal{B}_{M \setminus I_{\max} / J_{\max}})$ .

EXAMPLE 1.8. In the following, we illustrate how different choices of disjoint subsets  $I$  and  $J$  of  $E$  lead to different injections  $\iota : \mathcal{B}_{M \setminus I / J} \rightarrow \mathcal{B}_M$ .

Let  $M$  be the matroid on  $E = \{1, 2, 3\}$  whose set of bases is  $\mathcal{B}_M = \{\{1, 2\}, \{1, 3\}\}$ . Let  $N = M \setminus \{23\}$  be the restriction of  $M$  to  $\{1\}$ , whose set of bases is  $\mathcal{B}_N = \{\{1\}\}$ . Since there is no canonical map  $\mathcal{B}_N \rightarrow \mathcal{B}_M$ , it is clear that not every pair of disjoint subsets  $I$  and  $J$  leads to an embedding  $\mathcal{B}_{M \setminus I / J} \rightarrow \mathcal{B}_M$ .

The minor  $N$  is isomorphic to both  $N_2 = M \setminus \{2\} / \{3\}$  and  $N_3 = M \setminus \{3\} / \{2\}$ , which are embedded minors with respect to the inclusions  $\iota_2 : \mathcal{B}_{N_2} \rightarrow \mathcal{B}_M$  with  $\iota_2(\{1\}) = \{1, 2\}$  and  $\iota_3 : \mathcal{B}_{N_3} \rightarrow \mathcal{B}_M$  with  $\iota_3(\{1\}) = \{1, 3\}$ , respectively.

EXAMPLE 1.9. The contrary effect to that illustrated in Example 1.8 can also happen: different embedded minors can give rise to the same inclusions of sets of bases.

For instance, consider the matroid  $M$  on  $E = \{1, 2\}$  with  $\mathcal{B}_M = \{\{1, 2\}\}$  and the embedded minor  $N = M \setminus \{2\}$ . Then  $\mathcal{B}_N = \{\{1\}\}$  and the induced embedding  $\iota : \mathcal{B}_N \rightarrow \mathcal{B}_M$  is a bijection. This is obviously also the case for the trivial minor  $N' = M = M \setminus \emptyset / \emptyset$ . This shows that  $N$  is not determined by  $\iota : \mathcal{B}_N \rightarrow \mathcal{B}_M$ .

## CHAPTER 2

# Pastures

### 2.1. Definition and first properties

By a *monoid with zero* we mean a multiplicatively written commutative monoid  $P$  with an element  $0$  that satisfies  $0 \cdot a = 0$  for all  $a \in P$ . We denote the unit of  $P$  by  $1$  and write  $P^\times$  for the group of invertible elements in  $P$ . We denote by  $\text{Sym}_3(P)$  all elements of the form  $a + b + c$  in the monoid semiring  $\mathbb{N}[P]$ , where  $a, b, c \in P$ .

DEFINITION 2.1. A *pasture* is a monoid  $P$  with zero such that  $P^\times = P - \{0\}$ , together with a subset  $N_P$  of  $\text{Sym}_3(P)$  such that for all  $a, b, c, d \in P$

- (P1)  $a + 0 + 0 \in N_P$  if and only if  $a = 0$ ,
- (P2) if  $a + b + c \in N_P$ , then  $ad + bd + cd$  is in  $N_P$ ,
- (P3) there is a unique element  $\epsilon \in P^\times$  such that  $1 + \epsilon + 0 \in N_P$ .

We call  $N_P$  the *nullset* of  $P$ , and say that  $a + b + c$  is *null*, and write symbolically  $a + b + c = 0$ , if  $a + b + c \in N_P$ . For  $a \in P$ , we call  $\epsilon a$  the *weak inverse* of  $a$ .

The element  $\epsilon$  plays the role of an additive inverse of  $1$ , and the relations  $a + b + c = 0$  express that certain sums of elements are zero, even though the multiplicative monoid  $P$  does not carry an addition. For this reason, we will write frequently  $-a$  for  $\epsilon a$  and  $a - b$  for  $a + \epsilon b$ . In particular, we have  $\epsilon = -1$ . Moreover, we shall write  $a + b = c$  or  $c = a + b$  for  $a + b + \epsilon c = 0$ .

REMARK 2.2. As a word of warning, note that  $-1$  is not an additive inverse of  $1$  if considered as elements in the semiring  $\mathbb{N}[P]$ , i.e.  $1 - 1 = 1 + \epsilon \neq 0$  as elements of  $\mathbb{N}[P]$ . Psychologically, it is better to think of “ $-$ ” as an involution on  $P$ .

DEFINITION 2.3. A *morphism of pastures* is a multiplicative map  $f : P_1 \rightarrow P_2$  with  $f(0) = 0$  and  $f(1) = 1$  such that  $f(a) + f(b) + f(c) = 0$  in  $N_{P_2}$  whenever  $a + b + c = 0$  in  $N_{P_1}$ . This defines the category Pastures.

DEFINITION 2.4. A *subpasture* of a pasture  $P$  is a submonoid  $P'$  of  $P$  together with a subset  $N'_P \subset \text{Sym}_3(P')$  such that  $a^{-1} \in P'$  for every nonzero  $a \in P'$  and  $a + b + c \in N_{P'}$  for all  $a + b + c \in N_P$  with  $a, b, c \in P'$ .

Given a subset  $S$  of  $P^\times$ , the *subpasture generated by  $S$*  is the submonoid  $P' = \{0\} \cup \langle S \rangle$ , where  $\langle S \rangle$  denotes the subgroup of  $P^\times$  generated by  $S$ , together with the nullset  $N_{P'} = N_P \cap \text{Sym}_3(P')$ .

LEMMA 2.5. *Let  $P$  be a pasture. Then  $a + b = 0$  if and only if  $b = \epsilon a$ . In particular, we have  $\epsilon^2 = 1$ . Let  $f : P_1 \rightarrow P_2$  be a morphism of pastures. Then  $f(\epsilon) = \epsilon$ .*

PROOF. Note that  $\epsilon$  is uniquely determined by the relation  $1 + \epsilon + 0 = 0$ . By (P2), this implies that  $\epsilon^{-1} + 1 + 0 = 0$  and thus by (P3), we conclude that  $\epsilon^{-1} = \epsilon$ , or equivalently,  $\epsilon^2 = 1$ .



Given a morphism  $f : P_1 \rightarrow P_2$  be a morphism of pastures, the null relation  $1 + \epsilon + 0 = 0$  in  $P_1$  yields the relation  $f(1) + f(\epsilon) + 0 = 0$  in  $P_2$ . Thus  $f(\epsilon)$  is the weak inverse of  $f(1) = 1$ , which is  $\epsilon$ .  $\square$

**2.1.1. Free algebras and quotients.** Let  $P$  be a pasture with null set  $N_P$ . We define the *free  $P$ -algebra in  $x_1, \dots, x_s$*  as the pasture  $P\langle x_1, \dots, x_s \rangle$  whose unit group is  $P\langle x_1, \dots, x_s \rangle^\times = P^\times \times \langle x_1, \dots, x_s \rangle$ , where  $\langle x_1, \dots, x_s \rangle$  is the free abelian group generated by the symbols  $x_1, \dots, x_s$ , and whose null set is

$$N_{P\langle x_1, \dots, x_s \rangle} = \{da + db + dc \mid d \in \langle x_1, \dots, x_s \rangle, a + b + c \in N_P\},$$

where  $da$  stands for  $(a, d) \in P\langle x_1, \dots, x_s \rangle^\times$  if  $a \neq 0$  and for 0 if  $a = 0$ . This pasture comes with a canonical morphism  $P \rightarrow P\langle x_1, \dots, x_s \rangle$  of pastures that sends  $a$  to  $1a$ .

Let  $S \subset \text{Sym}_3(P)$  be a set of relations of the form  $a + b + c$  with  $ab \neq 0$ . We define the *quotient  $P \parallel S$  of  $P$  by  $S$*  as the following pasture. Let  $\tilde{N}_{P/S}$  be the smallest subset of  $\text{Sym}_3(P)$  that is closed under property (P2) and that contains  $N_P$  and  $S$ . Since all elements  $a + b + c$  in  $S$  have at least two nonzero terms by assumption,  $\tilde{N}_{P/S}$  also satisfies (P1). But it might fail to satisfy (P3), necessitating the following quotient construction for  $P^\times$ .

We define the unit group  $(P \parallel S)^\times$  of  $P \parallel S$  as the quotient of the group  $P^\times$  by the subgroup generated by all elements  $a$  for which  $a - 1 + 0 \in \tilde{N}_{P/S}$ . The underlying monoid of  $P \parallel S$  is, by definition,  $\{0\} \cup (P \parallel S)^\times$ , and it comes with a surjection  $\pi : P \rightarrow P \parallel S$  of monoids. We denote the image of  $a \in P$  by  $\bar{a} = \pi(a)$ , and define the null set of  $P \parallel S$  as the subset

$$N_{P \parallel S} = \{\bar{a} + \bar{b} + \bar{c} \mid a + b + c \in \tilde{N}_{P/S}\}$$

of  $\text{Sym}_3(P \parallel S)$ . The quotient  $P \parallel S$  of  $P$  by  $S$  comes with a canonical map  $P \rightarrow P \parallel S$  that sends  $a$  to  $\bar{a}$  and is a morphism of pastures.

If  $S \subset \text{Sym}_3(P\langle x_1, \dots, x_s \rangle)$  is a subset of relations of the form  $a + b + c$  with  $ab \neq 0$ , then the composition of the canonical morphisms for the free algebra and for the quotient yields a canonical morphism

$$\pi : P \longrightarrow P\langle x_1, \dots, x_s \rangle \longrightarrow P\langle x_1, \dots, x_s \rangle \parallel S.$$

We denote by  $\pi_0 : \{x_1, \dots, x_s\} \rightarrow P\langle x_1, \dots, x_s \rangle \parallel S$  the map that sends  $x_i$  to  $\bar{x}_i$ .

The following result describes the universal property of  $P\langle x_1, \dots, x_s \rangle \parallel S$ , which is analogous to the universal property of the quotient  $k[T_1^{\pm 1}, \dots, T_r^{\pm 1}]/(S)$  of the algebra of Laurent polynomials over a field  $k$  by the ideal  $(S)$  generated by a set  $S$  of Laurent polynomials (each with only two or three terms). Note that the special case  $S = \emptyset$  yields the universal property of the free algebra  $P\langle x_1, \dots, x_s \rangle$  and the special case  $s = 0$  yields the universal property of the quotient  $P \parallel S$ .

**PROPOSITION 2.6.** *Let  $P$  be a pasture,  $s \geq 0$  and  $S \subset \text{Sym}_3(P\langle x_1, \dots, x_s \rangle)$  a subset of relations of the form  $a + b + c$  with  $ab \neq 0$ . Let  $f : P \rightarrow Q$  be a morphism of pastures and  $f_0 : \{x_1, \dots, x_s\} \rightarrow Q^\times$  a map with the property that  $a \prod x_i^{\alpha_i} + b \prod x_i^{\beta_i} + c \prod x_i^{\gamma_i} \in S$  with  $a, b, c \in P$  and  $\alpha_i, \beta_i, \gamma_i \in \mathbb{Z}$  for  $i = 1, \dots, s$  implies that*

$$f(a) \prod f_0(x_i)^{\alpha_i} + f(b) \prod f_0(x_i)^{\beta_i} + f(c) \prod f_0(x_i)^{\gamma_i} \in N_Q.$$

Then there is a unique morphism  $\hat{f} : P\langle x_1, \dots, x_s \rangle // S \rightarrow Q$  such that the diagrams

$$\begin{array}{ccc} P & \xrightarrow{f} & Q \\ \pi \downarrow & \searrow \hat{f} & \\ P\langle x_1, \dots, x_s \rangle // S & & \end{array} \quad \text{and} \quad \begin{array}{ccc} \{x_1, \dots, x_s\} & \xrightarrow{f_0} & Q \\ \pi_0 \downarrow & \searrow \hat{f} & \\ P\langle x_1, \dots, x_s \rangle // S & & \end{array}$$

commute.

PROOF. We claim that the association

$$\hat{f} : P\langle x_1, \dots, x_s \rangle // S \longrightarrow Q \\ a \prod x_i^{\alpha_i} \longmapsto f(a) \prod f_0(x_i)^{\alpha_i}$$

is a morphism of pastures. Once we have proven this, it is clear that  $f = \hat{f} \circ \pi$  and  $f_0 = \hat{f} \circ \pi_0$ . Since the unit group of  $\hat{P} = P\langle x_1, \dots, x_s \rangle // S$  is generated by  $\{ax_i \mid a \in P^\times, i = 1, \dots, s\}$ , it follows that  $\hat{f}$  is uniquely determined by the conditions  $f = \hat{f} \circ \pi$  and  $f_0 = \hat{f} \circ \pi_0$ .

We are left with the verification that  $\hat{f}$  is a morphism. As a first step, we show that the restriction  $\hat{f}^\times : \hat{P}^\times \rightarrow Q^\times$  defines a group homomorphism. Note that  $N_{\hat{P}} = \{yz + yz' + yz'' \mid y \in \hat{P}^\times, z + z' + z'' \in S\}$ . Thus we have an equality  $a \prod x_i^{\alpha_i} = b \prod x_i^{\beta_i}$  in  $\hat{P}^\times$  if and only if  $da \prod x_i^{\alpha_i + \delta_i} - db \prod x_i^{\beta_i + \delta_i} \in S$  for some  $d \prod x_i^{\delta_i} \in \hat{P}^\times$ . By our assumptions, we have  $f(da) \prod f_0(x_i)^{\alpha_i + \delta_i} - f(db) \prod f_0(x_i)^{\beta_i + \delta_i} \in N_Q$ , and thus multiplying with  $f(d^{-1}) \prod f_0(x_i)^{-\delta_i}$  yields  $\hat{f}(a \prod x_i^{\alpha_i}) = \hat{f}(b \prod x_i^{\beta_i})$ . This verifies that  $\hat{f}^\times : \hat{P}^\times \rightarrow Q^\times$  is well-defined as a map. It is clear from the definition that it is a group homomorphism.

For showing that  $\hat{f} : \hat{P} \rightarrow Q$  is a morphism of pastures, we need to verify that for every element  $z + z' + z''$  in  $N_{\hat{P}}$ , the element  $\hat{f}(z) + \hat{f}(z') + \hat{f}(z'')$  is in  $N_Q$ . This can be done by a similar argument as before. We omit the details.  $\square$

**2.1.2. Examples.** The *regular partial field* is the pasture  $\mathbb{F}_1^\pm = \{0, 1, -1\} // \{1 - 1\}$  whose multiplication is determined by  $(-1)^2 = 1$ .

Let  $K$  be a field and  $K^\bullet$  its multiplicative monoid. Then we can associate with  $K$  the pasture  $K^\bullet // \{a + b + c \mid a + b + c = 0 \text{ in } K\}$ . We can recover the addition of  $K$  by the rule  $-c = a + b$  if  $a + b + c = 0$ . In particular, we can identify the finite field with 2 elements with the pasture  $\mathbb{F}_2 = \mathbb{F}_1^\pm // \{1 + 1\}$ , which implies that  $-1 = 1$ , and the finite field with 3 elements with the pasture  $\mathbb{F}_3 = \mathbb{F}_1^\pm // \{1 + 1 + 1\}$ .

Let  $F$  be a hyperfield and  $F^\bullet$  its multiplicative monoid. Then we can associate with  $F$  the pasture  $F^\bullet // \{a + b + c \mid 0 \in a \boxplus b \boxplus c \text{ in } F\}$ . In particular, we can realize the *Krasner hyperfield* as  $\mathbb{K} = \mathbb{F}_1^\pm // \{1 + 1, 1 + 1 + 1\}$ , and the *sign hyperfield* as  $\mathbb{S} = \mathbb{F}_1^\pm // \{1 + 1 - 1\}$ .

The *near-regular partial field* is

$$\mathbb{U} = \mathbb{F}_1^\pm \langle x, y \rangle // \{x + y - 1\}.$$

The *dyadic partial field* is

$$\mathbb{D} = \mathbb{F}_1^\pm \langle z \rangle // \{z + z - 1\}.$$

The *hexagonal partial field* is

$$\mathbb{H} = \mathbb{F}_1^\pm \langle z \rangle // \{z^3 + 1, z - z^2 - 1\}.$$

It is a straightforward exercise to verify that these descriptions of  $\mathbb{U}, \mathbb{D}, \mathbb{H}$  agree with the definitions given in the introduction.

As final examples, the *weak sign hyperfield* is the pasture

$$\mathbb{W} = \mathbb{F}_1^\pm // \langle 1 + 1 + 1, 1 + 1 - 1 \rangle$$

and the *phase hyperfield* is the pasture  $\mathbb{P}$  whose unit group  $\mathbb{P}^\times$  is the subgroup of norm 1-elements in  $\mathbb{C}^\times$  and whose null set is

$$N_{\mathbb{P}} = \left\{ a + b + c \in \text{Sym}_3(P) \mid \langle a, b, c \rangle_{>0} \text{ is an } \mathbb{R}\text{-linear subspace of } \mathbb{C} \right\}$$

where  $\langle a, b, c \rangle_{>0}$  is the smallest cone in  $\mathbb{C}$  that contains  $a, b$  and  $c$ . In fact,  $\mathbb{P}$  is isomorphic to the quotient of the pasture associated with  $\mathbb{C}$  by the action of  $\mathbb{R}_{>0}$  by multiplication.

**2.1.3. Initial and final objects.** The category Pastures admits both initial and final objects. The initial object of Pastures is the regular partial field  $\mathbb{F}_1^\pm$ . Given a pasture  $P$ , we denote by  $i_P$  the unique *initial morphism*  $i_P : \mathbb{F}_1^\pm \rightarrow P$ .

The final object of Pastures is the Krasner hyperfield  $\mathbb{K}$ . Given a pasture  $P$ , we denote by  $t_P$  the unique *terminal morphism*  $t_P : P \rightarrow \mathbb{K}$  sending 0 to 0 and every nonzero element of  $P$  to 1.

**2.1.4. Products and coproducts.** The category Pastures admits both a product and coproduct.

Let  $P_1, P_2$  be pastures. The (categorical) product  $P_1 \times P_2$  can be constructed explicitly as follows. As sets, we have  $P_1 \times P_2 = (P_1^\times \oplus P_2^\times) \cup \{0\}$ , endowed with the coordinatewise multiplication on  $P_1^\times \oplus P_2^\times$ , extended by the rule  $(a_1, a_2) \cdot 0 = 0 \cdot (a_1, a_2) = 0$ , and the nullset is the subset

$$N_{P_1 \times P_2} = \left\{ (a_1, a_2) + (b_1, b_2) + (c_1, c_2) \mid a_i + b_i + c_i \in N_{P_i} \text{ for } i = 1, 2 \right\}$$

of  $\text{Sym}^3(P_1 \times P_2)$ , where  $(a_1, a_2), (b_1, b_2)$  and  $(c_1, c_2)$  are in  $P_1 \times P_2$  (i.e.,  $a_1 a_2 \neq 0$  or  $a_1 = a_2 = 0$ , etc.) and we identify  $(0, 0)$  with 0 in  $P_1 \times P_2$ .

The categorical coproduct is given by the *tensor product*  $P_1 \otimes P_2$  defined as follows. As sets, we have  $P_1 \otimes P_2 = (P_1 \times P_2) / \sim$ , where  $P_1 \times P_2$  denotes the Cartesian product (not the underlying set of the product in the category of pastures) and  $(x_1, x_2) \sim (y_1, y_2)$  if and only if either:

- At least one of  $x_1, x_2$  is zero and at least one of  $y_1, y_2$  is zero; or
- $x_1 = y_1$  and  $x_2 = y_2$ ; or
- $x_1 = -y_1$  and  $x_2 = -y_2$ .

Denoting the equivalence class of  $(x_1, x_2)$  by  $x_1 \otimes x_2$ , the additive relations are given by:

- $a \otimes y + b \otimes y + c \otimes y \in N_{P_1 \otimes P_2}$  for  $y \in P_2$  and  $a, b, c \in P_1$  with  $a + b + c \in N_{P_1}$ .
- $x \otimes a + x \otimes b + x \otimes c \in N_{P_1 \otimes P_2}$  for  $x \in P_1$  and  $a, b, c \in P_2$  with  $a + b + c \in N_{P_2}$ .

**LEMMA 2.7.** *The tensor product of pastures satisfies the universal property of a coproduct with respect to the morphisms  $f_1 : P_1 \rightarrow P_1 \otimes P_2$  and  $f_2 : P_2 \rightarrow P_1 \otimes P_2$  given by  $x \mapsto x \otimes 1$  and  $y \mapsto 1 \otimes y$ , respectively.*

**PROOF.** Given a pasture  $P$  and morphisms  $g_i : P_i \rightarrow P$  for  $i = 1, 2$ , we must show that there is a unique morphism  $g : P_1 \otimes P_2 \rightarrow P$  such that  $g_i = g \circ f_i$  for  $i = 1, 2$ .

Define  $g$  by the formula  $g(x_1 \otimes x_2) = g_1(x_1) \cdot g_2(x_2)$ . To see that this is well-defined, suppose  $(x_1, x_2) \sim (y_1, y_2)$ . If  $x_1 x_2 = 0$  and  $y_1 y_2 = 0$ , then  $g(x_1 \otimes x_2) = g(y_1 \otimes y_2) = 0$ . Otherwise  $x_i = (-1)^k y_i$  for  $i = 1, 2$  with  $k \in \{0, 1\}$ , and we have

$$g(x_1 \otimes x_2) = (-1)^k g_1(x_1) (-1)^k g_2(x_2) = g_1(y_1) g_2(y_2) = g(y_1 \otimes y_2).$$

Hence  $g$  is well-defined.

It is straightforward to verify that  $g \circ f_i = g_i$  for  $i = 1, 2$  and that  $g$  is a morphism.

To see that  $g$  is unique, suppose  $g'$  is another such morphism. Then  $g'(x_1 \otimes 1) = g_1(x_1)$  and  $g'(1 \otimes x_2) = g_2(x_2)$ , and since  $g'$  is a morphism we have

$$g'(x_1 \otimes x_2) = g'((x_1 \otimes 1)(1 \otimes x_2)) = g'(x_1 \otimes 1)g'(1 \otimes x_2) = g_1(x_1)g_2(x_2)$$

for all  $x_1 \in P_1$  and  $x_2 \in P_2$ . Thus  $g' = g$ .  $\square$

By comparison, the category of fields (which is a full subcategory of Pastures) does not have an initial object, a final object, products, or coproducts.

EXAMPLE 2.8. We have  $\mathbb{F}_2 \times \mathbb{F}_3 \cong \mathbb{F}_1^\pm$  and  $\mathbb{F}_2 \otimes \mathbb{F}_3 \cong \mathbb{K}$ . The first isomorphism follows easily from our formula for the product of two pastures, and the second is an immediate consequence of the following lemma, which in turn follows easily from the universal property of the coproduct.

LEMMA 2.9. *If  $P_2 \cong \mathbb{F}_1^\pm // S$ , where  $S \subseteq \text{Sym}_3(\mathbb{F}_1^\pm)$ , then  $P_1 \otimes P_2 \cong P_1 // S$ .*

EXAMPLE 2.10. We have  $\mathbb{F}_3 \times \mathbb{S} \simeq \mathbb{D} // \{z^2 - 1\}$  and  $\mathbb{F}_3 \otimes \mathbb{S} \simeq \mathbb{F}_1^\pm // \{1 + 1 + 1, 1 + 1 - 1\}$ . For the first isomorphism, note that the underlying set of  $\mathbb{F}_3 \times \mathbb{S}$  is  $(\{\pm 1\} \times \{\pm 1\}) \cup \{0\}$  while the underlying set of  $\mathbb{D} // \{z^2 - 1\}$  is  $(\{\pm 1\} \times \{\pm z\}) \cup \{0\}$ . One checks easily that the map sending  $(1, 1)$  to 1 and  $(-1, 1)$  to  $z$  is an isomorphism of pastures. The second isomorphism is a consequence of Lemma 2.9.

EXAMPLE 2.11. Here (without proof) are a few more examples of products and coproducts:

- $\mathbb{F}_1^\pm = \mathbb{F}_2 \times \mathbb{S} = \mathbb{F}_2 \times \mathbb{W}$ .
- $\mathbb{K} = \mathbb{F}_2 \otimes \mathbb{S} = \mathbb{F}_2 \otimes \mathbb{W}$ .
- $\mathbb{H} = \mathbb{F}_3 \times \mathbb{F}_4$ .

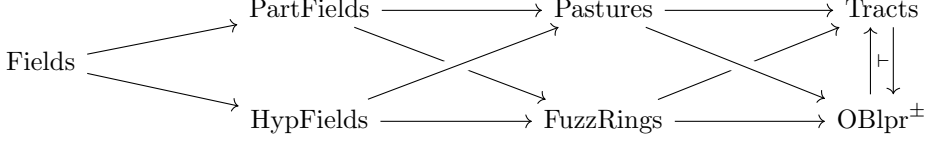
REMARK 2.12. More generally, one can show that the category Pastures is complete and co-complete, i.e., it admits all small limits and colimits. In particular, one can form arbitrary fiber products and fiber coproducts in Pastures. This is proven in [12].

**2.1.5. Comparison with partial fields, hyperfields, fuzzy rings, tracts and ordered blueprints.** The definitions of partial fields, hyperfields, fuzzy rings, tracts and ordered blueprints, and a comparison thereof, can be found in [5]. We are not aiming at repeating all definitions, but we will explain how the category of pastures fits within the landscape of these types of algebraic objects.

We have already explained how partial fields and hyperfields give rise to pastures. The tract associated with a pasture  $P$  is defined as  $F = (P^\times, N_F)$ , where  $N_F$  is the ideal generated by  $N_P$  in  $\mathbb{N}[P^\times]$ . The ordered blueprint associated to a pasture  $P$  is defined as  $B = P // \{0 \leq u + v + w \mid u + v + w \in N_P\}$ .

These associations yield fully faithful embeddings of the category PartFields of partial fields and the category HypFields of hyperfields into Pastures, and of

Pastures into the category **Tracts** of tracts and into the category  $\mathbf{OBlpr}^\pm$  of ordered blueprints with unique weak inverses. This completes the diagram of [5, Theorem 2.21] to



where **FuzzRings** is the category of fuzzy rings. This diagram commutes and all functors are fully faithful, with exception of the adjunction between **Tracts** and  $\mathbf{OBlpr}^\pm$ . We omit the details of these claims.

Note that fuzzy rings, seen as objects in either **Tracts** or  $\mathbf{OBlpr}^\pm$ , are not pastures in general since the ideal  $I$  of the fuzzy ring might not be generated by 3-term elements of  $\mathbb{N}[P^\times]$ . Conversely, not every pasture, seen as a tract or as an ordered blueprint, gives rise to a fuzzy ring since the axiom (FR2) (in the notation of [5, Section 2.4]) might not be satisfied. An example of a pasture for which (FR2) fails to hold is the pasture  $\mathbb{F}_1^\pm\langle z \rangle // \{z^2 + 1, 1 + 1 + z\}$ ; cf. [5, Ex. 2.11] for more details on this example.

## 2.2. Matroid representations

We recall the notion of weak matroids over pastures from [3]. Let  $P$  be a pasture. A *weak Grassmann–Plücker function* of rank  $r$  on  $E$  with values in  $P$  is a function  $\Delta : E^r \rightarrow P$  such that:

- (1) The set of  $r$ -element subsets  $\{e_1, \dots, e_r\} \subseteq E$  such that  $\Delta(e_1, \dots, e_r) \neq 0$  is the set of bases of a matroid  $\underline{M}$ .
- (2)  $\Delta(\sigma(e_1), \dots, \sigma(e_r)) = \text{sign}(\sigma) \cdot \Delta(e_1, \dots, e_r)$  for all permutations  $\sigma \in S_r$ .
- (3)  $\Delta$  satisfies the *3-term Plücker relations*: for all  $\mathbf{J} \in E^{r-2}$  and all  $(e_1, e_2, e_3, e_4) \in E^4$ ,

$$\Delta(\mathbf{J}e_1e_2) \cdot \Delta(\mathbf{J}e_3e_4) - \Delta(\mathbf{J}e_1e_3) \cdot \Delta(\mathbf{J}e_2e_4) + \Delta(\mathbf{J}e_1e_4) \cdot \Delta(\mathbf{J}e_2e_3) = 0.$$

Two weak Grassmann–Plücker functions  $\Delta, \Delta'$  are *isomorphic* if there is a  $c \in P^\times$  such that  $\Delta'(e_1, \dots, e_r) = c\Delta(e_1, \dots, e_r)$  for all  $(e_1, \dots, e_r) \in E^r$ .

A *weak  $P$ -matroid*  $M$  of rank  $r$  on  $E$  is an isomorphism class of weak Grassmann–Plücker functions  $\Delta : E^r \rightarrow P$ .

We call  $\underline{M}$  the *underlying matroid* of  $M$ , and we refer to  $\Delta$  as a  *$P$ -representation* of  $\underline{M}$ .

We say that a matroid  $\underline{M}$  is *representable* over a pasture  $P$  if there is at least one  $P$ -representation of  $\underline{M}$ .

REMARK 2.13. In [3] one also finds a definition of strong  $P$ -matroids, but this will not play a role in the present paper. We therefore omit the adjective “weak” when talking about  $P$ -representations.

With this terminology, we introduce the following subclasses of matroids:

DEFINITION 2.14. A matroid  $M$  is

- *regular* if it is representable over  $\mathbb{F}_1^\pm$ ;
- *binary* if it is representable over  $\mathbb{F}_2$ ;
- *ternary* if it is representable over  $\mathbb{F}_3$ ;

- *quaternary* if it is representable over  $\mathbb{F}_4$ ;
- *near-regular* if it is representable over  $\mathbb{U}$ ;
- *dyadic* if it is representable over  $\mathbb{D}$ ;
- *hexagonal* if it is representable over  $\mathbb{H}$ ;
- $\mathbb{D} \otimes \mathbb{H}$ -*representable*<sup>1</sup> if it is representable over  $\mathbb{D} \otimes \mathbb{H}$ ;
- *representable* if it is representable over some field;
- *orientable* if it is representable over  $\mathbb{S}$ ;
- *weakly orientable* if it is representable over  $\mathbb{W}$ .

Note that hexagonal matroids are also called  $\sqrt[6]{1}$ -matroids or sixth-root-of-unity-matroids in the literature, cf. [25] and [28].

### 2.3. Matroid representations via hyperplane functions

There are various “cryptomorphic” descriptions of weak  $P$ -matroids, for example in terms of “weak  $P$ -circuits”, cf. [3]. For the purposes of the present paper, it will be more convenient to reformulate things in terms of *hyperplanes* rather than circuits.

DEFINITION 2.15. Let  $P$  be a pasture and let  $\underline{M}$  be a matroid on the finite set  $E$ . Let  $\mathcal{H}$  be the set of hyperplanes of  $\underline{M}$ .

- (1) Given  $H \in \mathcal{H}$ , we say that  $f_H : E \rightarrow P$  is a  $P$ -hyperplane function for  $H$  if  $f_H(e) = 0$  if and only if  $e \in H$ .
- (2) Two  $P$ -hyperplane functions  $f_H, f'_H$  for  $H$  are *projectively equivalent* if there exists  $c \in P^\times$  such that  $f'_H(e) = cf_H(e)$  for all  $e \in E$ .
- (3) A triple of hyperplanes  $(H_1, H_2, H_3) \in \mathcal{H}^3$  is *modular* if  $F = H_1 \cap H_2 \cap H_3$  is a flat of corank 2 such that  $F = H_i \cap H_j$  for all distinct  $i, j \in \{1, 2, 3\}$ .
- (4) A *modular system* of  $P$ -hyperplane functions for  $\underline{M}$  is a collection of  $P$ -hyperplane functions  $f_H : E \rightarrow P$ , one for each  $H \in \mathcal{H}$ , such that whenever  $H_1, H_2, H_3$  is a modular triple of hyperplanes in  $\mathcal{H}$ , the corresponding functions  $f_{H_i}$  are linearly dependent, i.e., there exist constants  $c_1, c_2, c_3$  in  $P$ , not all zero, such that

$$c_1 f_{H_1}(e) + c_2 f_{H_2}(e) + c_3 f_{H_3}(e) = 0$$

for all  $e \in E$ .

- (5) Two modular systems of  $P$ -hyperplane functions  $\{f_H\}$  and  $\{f'_H\}$  are *equivalent* if  $f_H$  and  $f'_H$  are projectively equivalent for all  $H \in \mathcal{H}$ .

The following result can be viewed as a generalization of “Tutte’s representation theorem” [33, Theorem 5.1] (compare with [16, Theorem 3.5]). One can also view it as adding to the collection of cryptomorphisms for weak matroids established in [3].

THEOREM 2.16. Let  $P$  be a pasture and let  $\underline{M}$  be a matroid of rank  $r$  on  $E$ . Let  $\mathcal{H}$  be the set of hyperplanes of  $\underline{M}$ . There is a canonical bijection

$$\Xi : \left\{ P\text{-representations of } \underline{M} \right\} \longrightarrow \left\{ \begin{array}{c} \text{modular systems of} \\ P\text{-hyperplane functions for } \underline{M} \end{array} \right\}.$$

<sup>1</sup>In [25, p. 55], the partial field  $\mathbb{D} \otimes \mathbb{H}$  is denoted  $\mathbb{Y}$ .

If  $\Delta : E^r \rightarrow P$  is a  $P$ -representation of  $\underline{M}$  and  $\mathcal{H} = \Xi(\Delta)$ , then

$$\frac{f_H(e)}{f_H(e')} = \frac{\Delta(\mathbf{I}e)}{\Delta(\mathbf{I}e')}$$

for every  $f_H \in \mathcal{H}$ , elements  $e, e' \in E - H$  and  $\mathbf{I} \in E^{r-1}$  such that  $|\mathbf{I}|$  is an independent set which spans  $H$ .

PROOF. Let  $M$  be a weak  $P$ -matroid with underlying matroid  $\underline{M}$ . Let  $H$  be a hyperplane of  $\underline{M}$ . The complement of  $H$  in  $E$  is a cocircuit  $\underline{D}$  of  $\underline{M}$ ; choose a  $P$ -cocircuit  $D$  of  $M$  whose support is  $\underline{D}$ . Now define  $f_H : E \rightarrow P$  by  $f_H(e) = D(e)$ . Then  $f_H(e) = 0$  iff  $D(e) = 0$  iff  $e \notin \underline{D}$  iff  $e \in H$ , so  $f_H$  is a  $P$ -hyperplane function for  $H$ .

Suppose  $H_1, H_2, H_3$  is a modular triple of hyperplanes of  $\underline{M}$  with intersection  $F$ , a flat of corank 2. Let  $e$  be an element of  $H_3 - F$ . Then  $e \in H_3 - (H_1 \cup H_2)$  by the covering axiom for flats [22, Exercise 1.4.11, Axiom (F3)]. Let  $D_1$  and  $D_2$  be the  $P$ -cocircuits of  $M$  corresponding to  $H_1$  and  $H_2$ , respectively, and let  $\alpha_1 = D_2(e), \alpha_2 = -D_1(e) \in P$ . Then  $\alpha_1 D_1(e) = -\alpha_2 D_2(e)$ , so by [3, Axiom (C3)'], there is a  $P$ -cocircuit  $D_3$  of  $M$  such that  $D_3(e) = 0$  and  $\alpha_1 D_1(f) + \alpha_2 D_2(f) - D_3(f) = 0$  for all  $f \in E$ . By [3, Lemma 3.7], the support of  $D_3$  is  $E - H_3$ . By [3, Axiom (C2)],  $D_3$  is a scalar multiple of  $f_{H_3}$ , say  $D_3 = -\alpha_3 f_{H_3}$ . Then  $\alpha_1 f_{H_1} + \alpha_2 f_{H_2} + \alpha_3 f_{H_3} = 0$ , so  $\{f_H\}$  is a modular system of  $P$ -hyperplane functions for  $\underline{M}$ .

Conversely, a similar argument shows that given a modular system of  $P$ -hyperplane functions  $\{f_H\}$  for  $\underline{M}$ , there is a corresponding family of  $P$ -cocircuits  $\mathcal{D}$  defining a weak  $P$ -matroid  $M$ . These operations are inverse to one another by construction, and this establishes the desired bijection.

We turn to the second claim, which is obvious for  $e = e'$ , so we may assume that  $e \neq e'$ . Let  $n = \#E$  and choose  $\mathbf{I}' \in E^{n-r-1}$  such that  $E = |\mathbf{I}'| \cup \{e, e'\}$ . Note that since  $|\mathbf{I}e'|$  is a basis of  $\underline{M}$ , the complement  $|\mathbf{I}'e|$  is a basis for  $\underline{M}^*$ . If  $\mathbf{I} = (i_1, \dots, i_{r-1})$  and  $\mathbf{I}' = (i'_1, \dots, i'_{n-r-1})$ , we define a total order on  $E$  by

$$i'_1 < \dots < i'_{n-r-1} < e < i_1 < \dots < i_{r-1} < e'.$$

By [3, Lemma 4.1], there is a dual Grassmann-Plücker function  $\Delta^* : E^{n-r} \rightarrow P$  to  $\Delta$  that satisfies

$$\Delta^*(\mathbf{I}'e) = \text{sign}(\text{id}_E) \cdot \Delta(\mathbf{I}e') = \Delta(\mathbf{I}e')$$

and

$$\Delta^*(\mathbf{I}'e') = \text{sign}(\tau_{e,e'}) \cdot \Delta(\mathbf{I}e) = -\Delta(\mathbf{I}e),$$

where  $\text{id}_E : E \rightarrow E$  is the identity and  $\tau_{e,e'} : E \rightarrow E$  is the transposition that exchanges  $e$  with  $e'$ . This implies that

$$\frac{f_H(e)}{f_H(e')} = -\frac{\Delta^*(\mathbf{I}'e')}{\Delta^*(\mathbf{I}'e)} = \frac{\Delta(\mathbf{I}e)}{\Delta(\mathbf{I}e')}$$

as desired, where we use [3, Def. 4.6 and Lemma 4.7] for the first equality.  $\square$

## 2.4. The universal pasture

The universal pasture of a matroid was introduced in [5] as a tool to control the representations of a matroid  $M$  over other pastures. We review this in the following.

The symmetric group  $S_r$  on  $r$  elements acts by permutation of coefficients on  $E^r$ . In the following, we understand the sign  $\text{sign}(\sigma)$  of a permutation  $\sigma \in S_r$  as an element of  $(\mathbb{F}_1^\pm)^\times = \{\pm 1\}$ .

DEFINITION 2.17. Let  $M$  be a matroid with Grassmann-Plücker function  $\Delta : E^r \rightarrow \mathbb{K}$ . The *extended universal pasture* of  $M$  is the pasture  $P_M^+ = \mathbb{F}_1^\pm \langle T_{\mathbf{I}} \mid \Delta(\mathbf{I}) \neq 0 \rangle // \{S\}$ , where  $S$  is the set of the relations  $T_{\sigma(\mathbf{I})} = \text{sign}(\sigma)T_{\mathbf{I}}$  for all  $\mathbf{I} \in E^r$  and  $\sigma \in S_r$ , together with the 3-term Plücker relations

$$T_{\mathbf{J}_{e_1 e_2}} T_{\mathbf{J}_{e_3 e_4}} - T_{\mathbf{J}_{e_1 e_3}} T_{\mathbf{J}_{e_2 e_4}} + T_{\mathbf{J}_{e_1 e_4}} T_{\mathbf{J}_{e_2 e_3}} = 0$$

for all  $\mathbf{J} \in E^{r-2}$  and  $e_1, \dots, e_4 \in E$ .

The pasture  $P_M^+$  is naturally graded by the rule that  $T_{\mathbf{I}}$  has degree 1 for every  $\mathbf{I} \in \text{supp}(\Delta)$ . The *universal pasture* of  $M$  is the subpasture  $P_M$  of degree 0-elements of  $P_M^+$ .

The relevance of the universal pasture is that it represents the set of isomorphism classes of  $P$ -representations of  $M$ . This is derived in [5] by means of the algebraic geometry of the moduli space of matroids. We include an independent, and more elementary, proof in the following.

THEOREM 2.18 ([5, Prop. 6.22]). *Let  $M$  be a matroid of rank  $r$  on  $E$  and  $P$  a pasture. Then there is a functorial bijection between the set of isomorphism classes of  $P$ -representations of  $M$  and  $\text{Hom}(P_M, P)$ . In particular,  $M$  is representable over  $P$  if and only if there is a morphism  $\chi_M : P_M \rightarrow P$ .*

PROOF. Let  $\Delta : E^r \rightarrow P$  be a  $P$ -representation of  $M$  and  $P_M^+$  the extended universal pasture of  $M$ . Define the map  $\chi_{\Delta,0}^+ : T_{\mathbf{I}} \mapsto \Delta(\mathbf{I})$  from the set  $\{T_{\mathbf{I}} \mid \mathbf{I} \in \text{supp}(\Delta)\}$  of generators of  $P_M^+$  to  $P$ . Let  $S$  be the set of 3-term Plücker relations

$$T_{\mathbf{J}_{e_1 e_2}} T_{\mathbf{J}_{e_3 e_4}} - T_{\mathbf{J}_{e_1 e_3}} T_{\mathbf{J}_{e_2 e_4}} + T_{\mathbf{J}_{e_1 e_4}} T_{\mathbf{J}_{e_2 e_3}},$$

where  $\mathbf{J} \in E^{r-2}$  and  $e_1, \dots, e_4 \in E$  such that  $|\mathbf{J}_{e_1 \dots e_4}|$  has  $r+2$  elements. Applying  $\chi_{\Delta,0}^+$  to this relation, with the convention that  $\chi_{\Delta,0}^+(T_{\mathbf{I}}) = 0$  if  $\Delta(\mathbf{I}) = 0$ , yields

$$\begin{aligned} & \chi_{\Delta,0}^+(T_{\mathbf{J}_{e_1 e_2}}) \chi_{\Delta,0}^+(T_{\mathbf{J}_{e_3 e_4}}) - \chi_{\Delta,0}^+(T_{\mathbf{J}_{e_1 e_3}}) \chi_{\Delta,0}^+(T_{\mathbf{J}_{e_2 e_4}}) + \chi_{\Delta,0}^+(T_{\mathbf{J}_{e_1 e_4}}) \chi_{\Delta,0}^+(T_{\mathbf{J}_{e_2 e_3}}) \\ &= \Delta(\mathbf{J}_{e_1 e_2}) \Delta(\mathbf{J}_{e_3 e_4}) - \Delta(\mathbf{J}_{e_1 e_3}) \Delta(\mathbf{J}_{e_2 e_4}) + \Delta(\mathbf{J}_{e_1 e_4}) \Delta(\mathbf{J}_{e_2 e_3}), \end{aligned}$$

which is an element of  $N_P$  since  $\Delta$  is a Grassmann-Plücker function. Thus, by Proposition 2.6, the map  $\chi_{\Delta,0}^+$  together with the unique morphism  $\mathbb{F}_1^\pm \rightarrow P$  define a morphism

$$\chi_\Delta^+ : P_M^+ = \mathbb{F}_1^\pm \langle T_{\mathbf{I}} \mid \mathbf{I} \in \text{supp}(\Delta) \rangle // S \longrightarrow P$$

with  $\chi_\Delta^+(T_{\mathbf{I}}) = \Delta(\mathbf{I})$  for  $\mathbf{I} \in \text{supp}(\Delta)$ . We define  $\chi_\Delta : P_M \rightarrow P$  as the composition of the inclusion  $P_M \rightarrow P_M^+$  with  $\chi_\Delta^+$ . Since every element of  $P_M$  has degree 0, we have  $\chi_\Delta = \chi_{a\Delta}$  for every  $a \in P^\times$ , which shows that  $\chi_\Delta$  depends only on the isomorphism class of  $\Delta$ .

This yields a canonical map

$$\begin{array}{ccc} \left\{ \text{isomorphism classes of } P\text{-representations of } M \right\} & \longrightarrow & \text{Hom}(P_M, P), \\ [\Delta] & \longmapsto & \chi_\Delta \end{array}$$



which turns out to be a bijection whose inverse can be described as follows. Let  $\chi : P_M \rightarrow P$  be a morphism. Choose an  $\mathbf{I}_0 \in E^r$  such that  $|\mathbf{I}_0|$  is a basis of  $M$  and define the map

$$\begin{aligned} \Delta_\chi : E^r &\longrightarrow P, \\ \mathbf{I} &\longmapsto \begin{cases} \chi(T_{\mathbf{I}}/T_{\mathbf{I}_0}) & \text{if } |\mathbf{I}| \text{ is a basis of } M; \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

This is a Grassmann-Plücker function, since

$$\begin{aligned} &\Delta_\chi(\mathbf{J}e_1e_2)\Delta_\chi(\mathbf{J}e_3e_4) - \Delta_\chi(\mathbf{J}e_1e_3)\Delta_\chi(\mathbf{J}e_2e_4) + \Delta_\chi(\mathbf{J}e_1e_4)\Delta_\chi(\mathbf{J}e_2e_3) \\ &= \chi\left(\frac{T_{\mathbf{J}e_1e_2}}{T_{\mathbf{I}_0}}\right)\chi\left(\frac{T_{\mathbf{J}e_3e_4}}{T_{\mathbf{I}_0}}\right) - \chi\left(\frac{T_{\mathbf{J}e_1e_3}}{T_{\mathbf{I}_0}}\right)\chi\left(\frac{T_{\mathbf{J}e_2e_4}}{T_{\mathbf{I}_0}}\right) + \chi\left(\frac{T_{\mathbf{J}e_1e_4}}{T_{\mathbf{I}_0}}\right)\chi\left(\frac{T_{\mathbf{J}e_2e_3}}{T_{\mathbf{I}_0}}\right) \end{aligned}$$

is in the nullset of  $P_M$ . Note that the isomorphism class of  $\Delta_\chi$  is independent of the choice of  $\mathbf{I}_0$ , since any two such choices yield Grassmann-Plücker functions that are constant multiples of each other.

It is straightforward to verify that the associations  $\chi \mapsto [\Delta_\chi]$  and  $[\Delta] \mapsto \chi_\Delta$  are mutually inverse, and that both maps are functorial in  $P$ ; we omit the details.  $\square$

**DEFINITION 2.19.** The morphism  $\chi_\Delta : P_M \rightarrow P$  in Theorem 2.18 that is associated with the (isomorphism class of the)  $P$ -representation  $\Delta$  is called the *characteristic morphism* of  $\Delta$ .

**REMARK 2.20.** The proof of Theorem 2.18 also shows that the set of  $P$ -representations of  $M$  are in functorial bijection with  $\text{Hom}(P_M^+, P)$ . Under this identification, the identity morphism  $P_M^+ \rightarrow P_M^+$  defines a  $P_M^+$ -representation  $\widehat{\Delta} : E^r \rightarrow P_M^+$  of  $M$ , which we call the *universal Grassmann-Plücker function* of  $M$ . It satisfies  $\widehat{\Delta}(\mathbf{I}) = T_{\mathbf{I}}$  if  $|\mathbf{I}|$  is a basis of  $M$  and  $\widehat{\Delta}(\mathbf{I}) = 0$  otherwise, and  $t_{P_M^+} \circ \widehat{\Delta} : E^r \rightarrow \mathbb{K}$  is a Grassmann-Plücker function for  $M$  where  $t_{P_M^+} : P_M^+ \rightarrow \mathbb{K}$  is the terminal morphism, cf. Section 2.1.3.

## 2.5. The Tutte group and the universal pasture

The connection between the Tutte group and the universal pasture is explained in Theorem 6.26 of [5], which is as follows:

**THEOREM 2.21.** *Let  $M$  be a matroid with Grassmann-Plücker function  $\Delta : E^r \rightarrow \mathbb{K}$ . The association  $-1 \mapsto -1$  and  $T_{\mathbf{I}} \mapsto X_{\mathbf{I}}$  for  $\mathbf{I} \in \text{supp}(\Delta)$  defines an isomorphism of groups  $(P_M^+)^{\times} \rightarrow \mathbb{T}_M^{\mathcal{B}}$  that restricts to an isomorphism  $P_M^{\times} \rightarrow \mathbb{T}_M$ .*

**REMARK 2.22.** Dress and Wenzel show in [16, Thm. 3.7] that a matroid  $M$  is representable over a fuzzy ring  $R$  if and only if there is a group homomorphism  $\mathbb{T}_M \rightarrow R^{\times}$  that preserves the Plücker relations. This can be seen as an analogue of Theorem 2.18 in the formalism of Dress and Wenzel, but it also lets us explain the advantage of our formulation.

Namely, the foundation of a matroid is an object in the same category Pastures as the coefficient domains for matroid representations. We can thus use standard arguments from category theory to deduce results about the representability of a matroid. For example, if the foundation of a matroid  $M$  is the tensor product  $F_1 \otimes F_2$  of two pastures  $F_1$  and  $F_2$ , then  $M$  is representable over a third pasture  $P$  if and only if there exist morphisms  $F_1 \rightarrow P$  and  $F_2 \rightarrow P$ . We will make a frequent use of this observation in Chapter 6.

## CHAPTER 3

# Cross ratios

In this chapter, we review the theory of cross ratios for matroids from different angles, and explain the connection between these viewpoints, which are derived from cryptomorphic descriptions of a matroid in terms of bases and hyperplanes. There are two principally different types of cross ratios: cross ratios for  $P$ -matroids, which are elements of  $P$ , and universal cross ratios of a matroid  $M$ , which are elements of the universal pasture  $P_M$  of  $M$ . It turns out that there is a close relation between these two types of cross ratios and their different incarnations in terms of bases and hyperplanes. In particular, we identify in a concluding subsection the set of universal cross ratios with the set of fundamental elements in  $P_M$ .

### 3.1. Cross ratios of $P$ -matroids

Let  $E = \{1, \dots, n\}$  and  $0 \leq r \leq n$ . Let  $P$  be a pasture and  $M$  a  $P$ -matroid with Grassmann-Plücker function  $\Delta : E^r \rightarrow P$ .

Define  $\Omega_M$  to be the set of tuples  $(J; e_1, \dots, e_4)$  for which there exists a  $\mathbf{J} \in E^{r-2}$  with underlying set  $|\mathbf{J}| = J$  such that

$$\Delta(\mathbf{J}e_1e_4) \Delta(\mathbf{J}e_2e_3) \Delta(\mathbf{J}e_1e_3) \Delta(\mathbf{J}e_2e_4) \neq 0,$$

where  $\mathbf{J}e_ke_l = (j_1, \dots, j_{r-2}, e_k, e_l)$ .

**DEFINITION 3.1.** Let  $M$  be a  $P$ -matroid with Grassmann-Plücker function  $\Delta : E^r \rightarrow P$  and  $(J; e_1, \dots, e_4) \in \Omega_M$ . The *cross ratio* of  $(J; e_1, \dots, e_4)$  in  $M$  is the element

$$\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{M,J} = \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{\Delta, \mathbf{J}} = \frac{\Delta(\mathbf{J}e_1e_3)\Delta(\mathbf{J}e_2e_4)}{\Delta(\mathbf{J}e_1e_4)\Delta(\mathbf{J}e_2e_3)}$$

of  $P$  for any  $\mathbf{J} \in E^{r-2}$  with  $|\mathbf{J}| = J$ .

Note that the value of the cross ratio  $\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{M,J}$  does not depend on the ordering of  $\mathbf{J}$ , nor on the choice of Grassmann-Plücker function  $\Delta$  for  $M$ , which justifies our notation.

We find the following relations between cross ratios with permuted arguments. Let  $(J; e_1, \dots, e_4) \in \Omega_M$  and  $\mathbf{J} \in E^{r-2}$  be such that  $J = |\mathbf{J}|$ . We say that  $(J; e_1, \dots, e_4)$  is *non-degenerate* if

$$\Delta(\mathbf{J}e_1e_2)\Delta(\mathbf{J}e_3e_4) \neq 0,$$

or equivalently, if  $\begin{bmatrix} e_{\sigma(1)} & e_{\sigma(2)} \\ e_{\sigma(3)} & e_{\sigma(4)} \end{bmatrix}_{M,J}$  is defined and nonzero for every permutation  $\sigma$  of  $\{1, \dots, 4\}$ . We define  $\Omega_M^\diamond$  to be the subset of  $\Omega_M$  consisting of all non-degenerate  $(J; e_1, \dots, e_4)$ . We call a cross ratio  $\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{M,J}$  *non-degenerate* if  $(J; e_1, \dots, e_4)$  is non-degenerate. We call  $(J; e_1, \dots, e_4) \in \Omega_M$  *degenerate* if it is not in  $\Omega_M^\diamond$ .

One finds some relations that follow immediately from the definition, such as the fact that permuting rows and columns has no effect on the value of the cross ratio, i.e.

$$\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{M,J} = \begin{bmatrix} e_2 & e_1 \\ e_4 & e_3 \end{bmatrix}_{M,J} = \begin{bmatrix} e_3 & e_4 \\ e_1 & e_2 \end{bmatrix}_{M,J} = \begin{bmatrix} e_4 & e_3 \\ e_2 & e_1 \end{bmatrix}_{M,J};$$

that permuting the last two entries inverts the cross ratio, i.e.

$$\begin{bmatrix} e_1 & e_2 \\ e_4 & e_3 \end{bmatrix}_{M,J} = \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{M,J}^{-1};$$

and that a cyclic rotation of the last three entries yields the relation

$$\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{M,J} \cdot \begin{bmatrix} e_1 & e_3 \\ e_4 & e_2 \end{bmatrix}_{M,J} \cdot \begin{bmatrix} e_1 & e_4 \\ e_2 & e_3 \end{bmatrix}_{M,J} = -1$$

if  $(J; e_1, \dots, e_4) \in \Omega_M^\diamond$  is non-degenerate. We will discuss these relations and others in detail in Theorem 4.21.

The cross ratios keep track of the Plücker relations

$$(3) \quad \Delta(\mathbf{J}e_1e_2)\Delta(\mathbf{J}e_3e_4) - \Delta(\mathbf{J}e_1e_3)\Delta(\mathbf{J}e_2e_4) + \Delta(\mathbf{J}e_1e_4)\Delta(\mathbf{J}e_2e_3) = 0$$

satisfied by the Grassmann-Plücker function  $\Delta: E^r \rightarrow P$ . Namely, if  $(J; e_1, \dots, e_4) \in \Omega_M^\diamond$  and  $\mathbf{J} \in E^{r-2}$  are such that  $J = |\mathbf{J}|$ , then dividing both sides of the Plücker relation (3) by  $-\Delta(\mathbf{J}e_1e_4)\Delta(\mathbf{J}e_2e_3)$  yields the *Plücker relation for cross ratios*

$$\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{M,J} + \begin{bmatrix} e_1 & e_3 \\ e_2 & e_4 \end{bmatrix}_{M,J} = 1,$$

where the notation  $a + b = c$  in a pasture  $P$  is short-hand for  $a + b - c \in N_P$ .

If  $(J; e_1, \dots, e_4) \in \Omega_M$  is degenerate, then  $\Delta(\mathbf{J}e_1e_2)\Delta(\mathbf{J}e_3e_4) = 0$  and dividing the Plücker relation by  $-\Delta(\mathbf{J}e_1e_4)\Delta(\mathbf{J}e_2e_3)$  yields  $\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{M,J} - 1 = 0$ , and thus

$$\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{M,J} = 1$$

by the uniqueness of additive inverses in  $P$ .

LEMMA 3.2. *Let  $P$  be a pasture and  $M$  a  $P$ -matroid of rank  $r$  on  $E$  with dual  $M^*$ . Let  $(J; e_1, \dots, e_4) \in \Omega_{\underline{M}}$  and  $I = E - Je_1 \dots e_4$ . Then*

$$\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{M^*, I} = \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{M, J}$$

as elements of  $P$ .

PROOF. Let  $n = \#E$ . Choose  $\mathbf{J} = (j_1, \dots, j_{r-2})$  with  $|\mathbf{J}| = J$  and  $\mathbf{I} = (i_1, \dots, i_{n-r-2})$  with  $|\mathbf{I}| = I$ . Choose a total order on  $E$ . Let  $\Delta: E^r \rightarrow P$  be a Grassmann-Plücker function for  $M$ . Then by [3, Lemma 4.2], there is a Grassmann-Plücker function  $\Delta^*: E^{n-r} \rightarrow P$  for  $M^*$  such that for all identifications  $\{i, j, k, l\} = \{1, 2, 3, 4\}$ , we have

$$\Delta^*(\mathbf{I}e_ie_k) = \text{sign}(\pi_{i,j,k,l}) \cdot \Delta(\mathbf{J}e_je_l),$$

where  $\pi = \pi_{i,j,k,l}$  is the permutation of  $E$  such that

$$\pi(i_1) < \dots < \pi(i_{n-r-2}) < \pi(e_i) < \pi(e_k) < \pi(j_1) < \dots < \pi(j_{r-2}) < \pi(e_j) < \pi(e_l)$$

in the chosen total order of  $E$ . Since  $\pi_{i,j,l,k} = \pi_{i,j,k,l} \circ \tau_{k,l}$  for the transposition  $\tau_{k,l}$  that exchanges  $e_k$  and  $e_l$ , we have  $\text{sign}(\pi_{i,j,k,l})/\text{sign}(\pi_{i,j,l,k}) = -1$ . Thus we obtain

$$\begin{aligned} \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{M^*, I} &= \frac{\Delta^*(\mathbf{I}e_1e_3)\Delta^*(\mathbf{I}e_2e_4)}{\Delta^*(\mathbf{I}e_1e_4)\Delta^*(\mathbf{I}e_2e_3)} \\ &= \frac{\text{sign}(\pi_{1,2,3,4})}{\text{sign}(\pi_{1,2,4,3})} \cdot \frac{\text{sign}(\pi_{2,1,4,3})}{\text{sign}(\pi_{2,1,3,4})} \cdot \frac{\Delta(\mathbf{J}e_2e_4)\Delta(\mathbf{J}e_1e_3)}{\Delta(\mathbf{J}e_2e_3)\Delta(\mathbf{J}e_1e_4)} = \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{M, J} \end{aligned}$$

as claimed.  $\square$

### 3.2. Cross ratios for hyperplanes

There is a different, but closely related, notion of cross ratios associated to certain quadruples of hyperplanes.

**DEFINITION 3.3.** Let  $M$  be a matroid of rank  $r$  on  $E$  and  $\mathcal{H}$  be its set of hyperplanes. A quadruple of hyperplanes  $(H_1, \dots, H_4) \in \mathcal{H}^4$  is *modular* if  $F = H_1 \cap H_2 \cap H_3 \cap H_4$  is a flat of corank 2 such that  $F = H_i \cap H_j$  for all  $i \in \{1, 2\}$  and  $j \in \{3, 4\}$ . A modular quadruple  $(H_1, \dots, H_4)$  is *non-degenerate* if  $F = H_i \cap H_j$  for all distinct  $i, j \in \{1, \dots, 4\}$ . Otherwise it is called *degenerate*.<sup>1</sup> We denote the set of all modular quadruples of hyperplanes by  $\Theta_M$  and the subset of all non-degenerate modular quadruples by  $\Theta_M^\diamond$ .

**DEFINITION 3.4.** Let  $P$  be a pasture and  $M$  a  $P$ -matroid with underlying matroid  $\underline{M}$ . Let  $(H_1, \dots, H_4) \in \Theta_{\underline{M}}$ . The *cross ratio* of  $(H_1, \dots, H_4)$  in  $M$  is the element

$$\begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}_M = \frac{f_1(e_3)f_2(e_4)}{f_1(e_4)f_2(e_3)}$$

of  $P$ , where  $f_i : E \rightarrow P$  is a  $P$ -hyperplane function for  $H_i$  for  $i = 1, 2$  (cf. Definition 2.15), and where  $e_k \in H_k - F$  for  $k = 3, 4$  with  $F = H_1 \cap \dots \cap H_4$ .

Since  $f_1$  and  $f_2$  are determined by  $H_1$  and  $H_2$  up to a factor in  $P^\times$ , the definition of  $\begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}_M$  is independent of the choices of  $f_1$  and  $f_2$ . It follows from [3, Theorem 3.21, Lemma 4.5, and Definition 4.6] that it is also independent of the choices of  $e_3$  and  $e_4$ .

We continue with a comparison of the two notions of cross ratios.

**LEMMA 3.5.** *Let  $M$  be a matroid of rank  $r$  on  $E$ . The association  $(J; e_1, \dots, e_4) \mapsto (H_1, \dots, H_4)$  with  $H_i = \langle Je_i \rangle$  for  $i = 1, \dots, 4$  defines a surjective map  $\Psi : \Omega_M \rightarrow \Theta_M$ , which restricts to a surjective map  $\Psi^\diamond : \Omega_M^\diamond \rightarrow \Theta_M^\diamond$ .*

**PROOF.** The flat  $F = H_1 \cap \dots \cap H_4 = \langle J \rangle$  is of rank  $r - 2$  since  $J$  is an independent set of rank  $r - 2$ . We have  $H_i \cap H_j = F$  for all  $i = 1, 2$  and  $j = 3, 4$  since  $\Delta(Je_i e_j) \neq 0$  and thus  $\langle H_i \cup H_j \rangle = E$ . This shows that  $(H_1, \dots, H_4)$  is indeed a modular quadruple. By the same reasoning applied to arbitrary distinct  $i, j \in \{1, \dots, 4\}$ , we conclude that  $\Psi$  restricts to a map  $\Psi^\diamond : \Omega_M^\diamond \rightarrow \Theta_M^\diamond$ .

Given  $(H_1, \dots, H_4) \in \Theta_M$  and  $F = H_1 \cap \dots \cap H_4$ , choose an independent subset  $J \subset F$  with  $r - 2$  elements and  $e_i \in H_i - F$  for  $i = 1, \dots, 4$ . Since  $H_i \cap H_k = F$  for  $i \in \{1, 2\}$  and  $k \in \{3, 4\}$ , the closure of  $Je_i e_k$  is  $E$ , i.e.  $Je_i e_k$  is a basis of  $M$ . Thus  $(J; e_1, \dots, e_4) \in \Omega_M$  and  $\Psi(J; e_1, \dots, e_4) = (H_1, \dots, H_4)$ , which establishes

<sup>1</sup>Note that in some papers the term “modular quadruple” is used for what we call a non-degenerate quadruple; e.g. see [3], [7, Def. 5.1] and [26, Def. 3.18].

the surjectivity of  $\Psi$ . If  $(H_1, \dots, H_4) \in \Theta_M^\diamond$ , then  $H_i \cap H_k = F$  and thus  $Je_i e_k$  is a basis of  $M$  for all distinct  $i, k \in \{1, \dots, 4\}$ . Thus  $(J; e_1, \dots, e_4) \in \Omega_M^\diamond$  and  $\Psi^\diamond(J; e_1, \dots, e_4) = (H_1, \dots, H_4)$ , which establishes the surjectivity of  $\Psi^\diamond$ .  $\square$

PROPOSITION 3.6. *Let  $P$  be a pasture and  $M$  a  $P$ -matroid with underlying matroid  $\underline{M}$ . Let  $(J; e_1, \dots, e_4) \in \Omega_{\underline{M}}$  and  $(H_1, \dots, H_4) = \Psi(J; e_1, \dots, e_4)$ . Then we have*

$$\begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}_M = \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{M,J}$$

as elements of  $P$ .

PROOF. Since  $|\mathbf{J}e_i|$  is an  $(r-1)$ -set that generates  $H_i$  and  $e_j \notin H_i$  for  $i \in \{1, 2\}$  and  $j \in \{3, 4\}$ , we can apply Theorem 2.16 to conclude that

$$\begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}_M = \frac{f_1(e_3)f_2(e_4)}{f_1(e_4)f_2(e_3)} = \frac{\Delta(\mathbf{J}e_1 e_3)\Delta(\mathbf{J}e_2 e_4)}{\Delta(\mathbf{J}e_1 e_4)\Delta(\mathbf{J}e_2 e_3)} = \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{M,J}$$

as claimed.  $\square$

Our comparison of different notions of cross ratios has the following immediate consequence.

COROLLARY 3.7. *Let  $M$  be a matroid and  $(J; e_1, \dots, e_4), (J'; f_1, \dots, f_4) \in \Omega_M$ . If  $\langle Je_i \rangle = \langle J'f_i \rangle$  for  $i = 1, \dots, 4$ , then  $\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_J = \begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \end{bmatrix}_{J'}$ .*

PROOF. By Proposition 3.6, we have  $\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_J = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} = \begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \end{bmatrix}_{J'}$  if  $H_i = \langle Je_i \rangle = \langle J'f_i \rangle$  for  $i = 1, \dots, 4$ .  $\square$

### 3.3. Universal cross ratios

Let  $M$  be a matroid of rank  $r$  on  $E = \{1, \dots, n\}$  with Grassmann-Plücker function  $\Delta : E^r \rightarrow \mathbb{K}$ .

Recall from Section 2.4 the definition of the extended universal pasture

$$P_M^+ = \mathbb{F}_1^\pm \langle T_{\mathbf{I}} \mid \Delta(\mathbf{I}) \neq 0 \rangle // \{S\}$$

of  $M$ , where  $S$  contains the relations  $T_{\sigma(\mathbf{I})} = \text{sign}(\sigma)T_{\mathbf{I}}$  and the 3-term Plücker relations

$$T_{\mathbf{J}e_1 e_2} T_{\mathbf{J}e_3 e_4} - T_{\mathbf{J}e_1 e_3} T_{\mathbf{J}e_2 e_4} + T_{\mathbf{J}e_1 e_4} T_{\mathbf{J}e_2 e_3} = 0$$

for all  $\mathbf{J} \in E^{r-2}$  and  $e_1, \dots, e_4 \in E$ , where we use the convention  $T_{\mathbf{I}} = 0$  if  $\Delta(\mathbf{I}) = 0$ . The universal Grassmann-Plücker function  $\widehat{\Delta} : E^r \rightarrow P_M^+$  for  $M$  sends  $\mathbf{I} \in E^r$  to  $T_{\mathbf{I}}$  if  $|\mathbf{I}|$  is a basis of  $M$ , and to 0 otherwise. The universal  $P_M$ -matroid  $\widehat{M}$  for  $M$  is defined by the Grassmann-Plücker function  $T_{\mathbf{I}}^{-1} \widehat{\Delta} : E^r \rightarrow P_M$ , where  $\mathbf{I} \in E^r$  is any  $r$ -tuple with  $\Delta(\mathbf{I}) \neq 0$ .

DEFINITION 3.8. Let  $M$  be a matroid with universal  $P_M$ -matroid  $\widehat{M}$ . Let  $(J; e_1, \dots, e_4) \in \Omega_M$  and  $(H_1, \dots, H_4) \in \Theta_M$ . The *universal cross ratio* of  $(J; e_1, \dots, e_4)$  is the element

$$\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_J := \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{\widehat{M}, J}$$

of  $P_M$ , and the *universal cross ratio* of  $(H_1, \dots, H_4)$  is the element

$$\begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} := \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}_{\widehat{M}}$$

of  $P_M$ .

The relation between cross ratios of a  $P$ -matroid and the universal cross ratio of the underlying matroid  $\underline{M}$  is explained in the following statement.

**PROPOSITION 3.9.** *Let  $P$  be a pasture and  $M$  a  $P$ -matroid with Grassmann Plücker function  $\Delta : E^r \rightarrow P$ . Let  $\underline{M}$  be the underlying matroid and  $P_{\underline{M}}$  its universal pasture. Let  $\chi_M : P_{\underline{M}} \rightarrow P$  be the universal morphism associated with  $M$ , which maps  $T_{\mathbf{I}}/T_{\mathbf{I}'}$  to  $\Delta(\mathbf{I})/\Delta(\mathbf{I}')$ . Then*

$$\chi_M \left( \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_J \right) = \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{M,J}$$

as elements of  $P$  for every  $(J; e_1, \dots, e_4) \in \Omega_{\underline{M}}$ .

**PROOF.** This follows directly from the definitions of  $\chi_M$ ,  $\widehat{\Delta}$  and the (universal) cross ratios.  $\square$

### 3.4. Fundamental elements

Universal cross ratios can be characterized intrinsically as the fundamental elements of the universal pasture of a matroid. To the best of our knowledge, the importance of fundamental elements in the study of matroid representations goes back to Semple's paper [27], where this concept was introduced in the context of partial fields. We extend the notion of fundamental elements to pastures and explain its relation to universal cross ratios in the following.

The property of cross ratios that lead to the definition of fundamental elements are the 3-term Plücker relations

$$\Delta(\mathbf{J}e_1e_2)\Delta(\mathbf{J}e_3e_4) - \Delta(\mathbf{J}e_1e_3)\Delta(\mathbf{J}e_2e_4) + \Delta(\mathbf{J}e_1e_4)\Delta(\mathbf{J}e_2e_3) = 0$$

for a Grassmann-Plücker function  $\Delta : E^r \rightarrow P$ , where  $\mathbf{J} \in E^{r-2}$  and  $e_1, \dots, e_4 \in E$ . If  $\Delta(\mathbf{J}e_ie_j) \neq 0$  for all distinct  $i, j \in \{1, \dots, 4\}$ , then division by  $-\Delta(\mathbf{J}e_1e_4)\Delta(\mathbf{J}e_2e_3)$  yields

$$\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{\Delta, \mathbf{J}} + \begin{bmatrix} e_1 & e_3 \\ e_2 & e_4 \end{bmatrix}_{\Delta, \mathbf{J}} = \frac{\Delta(\mathbf{J}e_1e_3)\Delta(\mathbf{J}e_2e_4)}{\Delta(\mathbf{J}e_1e_4)\Delta(\mathbf{J}e_2e_3)} + \frac{\Delta(\mathbf{J}e_1e_2)\Delta(\mathbf{J}e_3e_4)}{\Delta(\mathbf{J}e_1e_4)\Delta(\mathbf{J}e_2e_3)} = 1$$

for the non-degenerate cross ratios  $\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{\Delta, \mathbf{J}}$  and  $\begin{bmatrix} e_1 & e_3 \\ e_2 & e_4 \end{bmatrix}_{\Delta, \mathbf{J}}$  in  $P^\times$ .

**DEFINITION 3.10.** Let  $P$  be a pasture. A *fundamental element* of  $P$  is an element  $z \in P^\times$  such that  $z + z' = 1$  for some  $z' \in P^\times$ .

**PROPOSITION 3.11.** *Let  $M$  be a matroid. For an element  $z \in P_M$ , the following are equivalent:*

- (1)  $z$  is a fundamental element of  $P_M$ ;
- (2)  $z = \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_J$  for some  $(J; e_1, \dots, e_4) \in \Omega_M^\diamond$ ;
- (3)  $z = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}$  for some  $(H_1, \dots, H_4) \in \Theta_M^\diamond$ .

**PROOF.** Our preceding discussion shows that  $\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_J + \begin{bmatrix} e_1 & e_3 \\ e_2 & e_4 \end{bmatrix}_J = 1$  for  $(J; e_1, \dots, e_4) \in \Omega_M^\diamond$ . Thus (2) $\Rightarrow$ (1). The equivalence of (2) and (3) follows from Proposition 3.6.

We are left with (1) $\Rightarrow$ (2). Assume that  $z \in P_M^\times$  is a fundamental element, i.e.  $z + z' - 1 = 0$  for some  $z' \in P_M^\times$ . Since the nullset of the extended universal pasture  $P_M^+$  is generated by the 3-term Plücker relations, there must be an element  $a \in (P_M^+)^{\times}$  such that  $az + az' - a = 0$  is of the form

$$T_{\mathbf{J}e_1e_2}T_{\mathbf{J}e_3e_4} - T_{\mathbf{J}e_1e_3}T_{\mathbf{J}e_2e_4} + T_{\mathbf{J}e_1e_4}T_{\mathbf{J}e_2e_3} = 0$$

for some  $\mathbf{J} \in E^{r-2}$  and  $e_1, \dots, e_4 \in E$  such that  $|\mathbf{J}e_i e_j|$  is a basis of  $M$  for all distinct  $i, j \in \{1, \dots, 4\}$ , i.e.  $(J; e_1, \dots, e_4) \in \Omega_M^\diamond$  where  $J = |\mathbf{J}|$ . After a suitable permutation of  $e_1, \dots, e_4$ , we can assume that  $-a = T_{\mathbf{J}e_1 e_4} T_{\mathbf{J}e_2 e_3}$  and  $az = -T_{\mathbf{J}e_1 e_3} T_{\mathbf{J}e_2 e_4}$ . Thus

$$z = \frac{-az}{-a} = \frac{T_{\mathbf{J}e_1 e_3} T_{\mathbf{J}e_2 e_4}}{T_{\mathbf{J}e_1 e_4} T_{\mathbf{J}e_2 e_3}} = \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_J$$

is a cross ratio, as claimed.  $\square$

### 3.5. Compatibility with the Tutte group formulation of Dress and Wenzel

We provide a comparison of the different types of universal cross ratios, as introduced above, with the cross ratios introduced by Dress and Wenzel in [15, Def. 2.3].

The image of a universal cross ratio  $\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_J$  under the isomorphism  $P_M^\times \rightarrow \mathbb{T}_M$  from Theorem 2.21 appears implicitly already in [14, Prop. 2.2], and is as follows.

LEMMA 3.12. *Let  $M$  be a matroid with Grassmann-Plücker function  $\Delta : E^r \rightarrow \mathbb{K}$ , Tutte group  $\mathbb{T}_M$  and universal pasture  $P_M$ . Let  $\varphi : P_M^\times \rightarrow \mathbb{T}_M$  be the isomorphism of groups that sends  $T_{\mathbf{I}}/T_{\mathbf{I}'}$  to  $X_{\mathbf{I}}/X_{\mathbf{I}'}$  for  $\mathbf{I}, \mathbf{I}' \in \text{supp}(\Delta)$ . Then*

$$\varphi\left(\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_J\right) = \frac{X_{\mathbf{J}e_1 e_3} X_{\mathbf{J}e_2 e_4}}{X_{\mathbf{J}e_1 e_4} X_{\mathbf{J}e_2 e_3}}$$

for all  $(J; e_1, \dots, e_4) \in \Omega_M$  and  $\mathbf{J} \in E^{r-2}$  with  $|\mathbf{J}| = J$ .

PROOF. Note that the ratio  $(X_{\mathbf{J}e_1 e_3} X_{\mathbf{J}e_2 e_4}) (X_{\mathbf{J}e_1 e_4} X_{\mathbf{J}e_2 e_3})^{-1}$  does not depend on the ordering of  $\mathbf{J}$ . The rest follows immediately from the definitions.  $\square$

Let  $(H_1, \dots, H_4)$  be a modular quadruple of hyperplanes of  $M$  and  $F$  the corank 2 flat contained in all  $H_i$ . Let  $e_3 \in H_3 - F$  and  $e_4 \in H_4 - F$ . The Dress–Wenzel universal cross ratio of  $(H_1, \dots, H_4)$  is the element

$$\begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}_{\mathbb{T}} := \frac{X_{H_1, e_3} X_{H_2, e_4}}{X_{H_2, e_3} X_{H_1, e_4}}$$

of the group  $\mathbb{T}_M^{\mathcal{H}}$ .

As shown in [15, Lemma 2.1], this definition is independent of the choices of  $e_3$  and  $e_4$ . Since  $\deg_{\mathcal{H}}\left(\begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}_{\mathcal{H}}\right) = 0$ , it follows from Theorem 1.3 that  $\begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}_{\mathcal{H}}$  is contained in the image of the injection  $\iota : \mathbb{T}_M \rightarrow \mathbb{T}_M^{\mathcal{H}}$ .

LEMMA 3.13. *Let  $\psi : P_M^\times \rightarrow \mathbb{T}_M^{\mathcal{H}}$  be the group homomorphism that maps  $T_{\mathbf{I}e} T_{\mathbf{I}e'}^{-1}$  to  $X_{H, e} X_{H, e'}^{-1}$  where  $\mathbf{I} \in E^{r-1}$ ,  $e, e' \in E$ ,  $I = |\mathbf{I}|$ ,  $H = \langle I \rangle$ , and  $Ie, Ie'$  are bases of  $M$ . Let  $(H_1, \dots, H_4) \in \Theta_M$  be a modular quadruple of hyperplanes of  $M$ . Then*

$$\psi\left(\begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}\right) = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}_{\mathbb{T}}.$$

PROOF. It is clear from the definitions that  $\psi = \iota \circ \varphi$ . By Lemma 3.5, there is an element  $(J; e_1, \dots, e_4) \in \Omega_M$  with  $\Psi(J; e_1, \dots, e_4) = (H_1, \dots, H_4)$ , i.e.  $H_i = \langle Je_i \rangle$

for  $i = 1, \dots, 4$ . Using Proposition 3.6, we obtain

$$\begin{aligned}
 \psi\left(\begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}\right) &= \iota \circ \varphi\left(\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_J\right) \\
 &= \iota\left(\frac{X_{\mathbf{J}e_1e_3}X_{\mathbf{J}e_2e_4}}{X_{\mathbf{J}e_1e_4}X_{\mathbf{J}e_2e_3}}\right) \\
 &= \frac{X_{H_1,e_3}X_{H_2,e_4}}{X_{H_1,e_4}X_{H_2,e_3}} \\
 &= \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}_{\mathbb{T}}
 \end{aligned}$$

as claimed. □





## CHAPTER 4

# Foundations

The foundation  $F_M$  of a matroid  $M$  is the subpasture of multidegree 0-elements of the universal pasture  $P_M$ , and it represents the functor taking a pasture  $P$  to the set of  $P$ -rescaling classes of  $M$ . In particular, just as with  $P_M$ , the foundation can detect whether or not a matroid is representable over a given pasture  $P$  in terms of the existence of a morphism from  $F_M$  to  $P$ .

One advantage of the foundation over the universal pasture is that, because of some deep theorems due to Tutte, Dress–Wenzel, and Gelfand–Rybnikov–Stone, there is an explicit presentation of  $F_M$  in terms of generators and relations in which the relations are all inherited from “small” embedded minors. More precisely, the foundation of  $M$  is generated by the universal cross ratios of  $M$ , and all relations between these cross ratios are generated by a small list of relations stemming from embedded minors of  $M$  having at most 7 elements.

We begin our discussion of foundations by reviewing some facts which were proved in the authors’ previous paper [5]. Next we explain the role of embedded minors in the study of foundations. We then exhibit, through very explicit computations, the relations between universal cross ratios inherited from small minors which enter into the presentation by generators and relations alluded to above. Finally, we use the aforementioned result of Gelfand, Rybnikov and Stone to prove that these relations generate *all* relations in  $F_M$  between universal cross ratios.

REMARK 4.1. The term “foundation” is derived from the fact that  $F_M$  is the subpasture of the universal pasture  $P_M$  that is generated by the fundamental elements of  $P_M$ .

### 4.1. Definition and basic facts

Let  $M$  be a matroid of rank  $r$  on  $E$  with extended universal pasture  $P_M^+$ . For a subset  $I$  of  $E$ , let  $\delta_I : E \rightarrow \mathbb{Z}$  be the characteristic function of  $I$ , which is an element of  $\mathbb{Z}^E$ . The *multidegree* is the group homomorphism

$$\begin{aligned} \deg_E : (P_M^+)^{\times} &\longrightarrow \mathbb{Z}^E \\ T_{\mathbf{I}} &\longmapsto \delta_I, \end{aligned}$$

where  $I = |\mathbf{I}|$ . It is easily verified that this map is well-defined, cf. [5, section 7.3]. The *degree in  $i$*  is the function  $\deg_i : (P_M^+)^{\times} \rightarrow \mathbb{Z}$  that is the composition of  $\deg_E : (P_M^+)^{\times} \rightarrow \mathbb{Z}^E$  with the canonical projection to the  $i$ -th component, i.e.  $\deg_i(T_{\mathbf{I}}) = 1$  if  $i \in I$  and  $\deg_i(T_{\mathbf{I}}) = 0$  if  $i \notin I$ . The *total degree* is the function  $\deg : (P_M^+)^{\times} \rightarrow \mathbb{Z}$  that is the sum over  $\deg_i$  for all  $i \in E$ , i.e.  $\deg(T_{\mathbf{I}}) = \sum_{i \in E} \deg_i(T_{\mathbf{I}}) = \#I = r$ .

DEFINITION 4.2. Let  $M$  be a matroid with extended universal pasture  $P_M^+$ . The *foundation of  $M$*  is the subpasture  $F_M$  of  $P_M^+$  that consists of 0 and all elements of multidegree 0.

Note that the universal pasture  $P_M$  of  $M$  is the subpasture of  $P_M^+$  that is generated by all units of total degree 0. Since  $\deg(x) = 0$  if  $\deg_E(x) = 0$ , the foundation  $F_M$  of  $M$  is a subpasture of  $P_M$ .

The relevance of the foundation of  $M$  is the fact that it represents the rescaling class space

$$\mathcal{X}_M^R(P) := \left\{ \text{rescaling classes of } M \text{ over } P \right\}$$

considered as a functor in  $P$ .

**THEOREM 4.3** ([5, Cor. 7.26]). *Let  $M$  be a matroid and  $P$  a pasture. Then there is a functorial bijection  $\mathcal{X}_M^R(P) = \text{Hom}(F_M, P)$ . In particular,  $M$  is representable over  $P$  if and only if there is a morphism  $F_M \rightarrow P$ .*

Recall from [14] that the inner Tutte group  $\mathbb{T}_M^{(0)}$  of a matroid  $M$  is defined as the subgroup of the Tutte group  $\mathbb{T}_M$  of  $M$  that consists of all elements of multidegree 0, where the multidegree  $\deg : \mathbb{T}_M \rightarrow \mathbb{Z}^E$  is defined in the same way as the multidegree  $\deg : P_M \rightarrow \mathbb{Z}^E$ . This yields at once the following consequence of Theorem 2.21 (cf. [5, Cor. 7.11]).

**COROLLARY 4.4.** *The canonical isomorphism  $\mathbb{P}_M^\times \rightarrow \mathbb{T}_M$  restricts to an isomorphism  $F_M^\times \rightarrow \mathbb{T}_M^{(0)}$ .*

**REMARK 4.5.** Wenzel observes in [34, Thm. 6.3] that a matroid representation over a fuzzy ring  $K$  induces a group homomorphism  $\mathbb{T}_M^{(0)} \rightarrow K^\times$ , and that this homomorphism detects the rescaling class of a representation. This can be seen as a partial analogue of Theorem 4.3 for fuzzy rings (cf. Remark 2.22).

## 4.2. Universal cross ratios as generators of the foundation

Let  $M$  be a matroid of rank  $r$  on  $E$  and  $P_M^+$  its extended universal pasture. The simplest type of elements of  $P_M^+$  with multidegree 0 are universal cross ratios

$$\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_J = \frac{T_{\mathbf{J}e_1e_3}T_{\mathbf{J}e_2e_4}}{T_{\mathbf{J}e_1e_4}T_{\mathbf{J}e_2e_3}}$$

where  $(J; e_1, \dots, e_4) \in \Omega_M$  and  $\mathbf{J} \in E^{r-2}$  such that  $|\mathbf{J}| = J$ . This formula shows that the universal cross ratios are elements of the foundation  $F_M$  of  $M$ . It is proven in [5, Cor. 7.11] that the foundation is generated by the universal cross ratios. To summarize, we have:

**THEOREM 4.6.** *Let  $M$  be a matroid. Then  $\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_J \in F_M^\times$  for every  $(J; e_1, \dots, e_4) \in \Omega_M$ , and  $F_M^\times$  is generated by the collection of all such universal cross ratios.*

Using Proposition 3.6, we obtain:

**COROLLARY 4.7.** *Let  $M$  be a matroid. Then  $\begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} \in F_M^\times$  for every  $(H_1, \dots, H_4) \in \Theta_M$ , and  $F_M^\times$  is generated by the collection of all such hyperplane universal cross ratios.*

## 4.3. The foundation of the dual matroid

Let  $M$  be a matroid of rank  $r$  on  $E$  and  $P_M$  its universal pasture. By definition the identity morphism  $\text{id} : P_M \rightarrow P_M$  is the characteristic morphism of the universal  $P_M$ -matroid  $\widehat{M}$ ; cf. Theorem 2.18. The underlying matroid of  $\widehat{M}$  is  $\underline{\widehat{M}} = M$ . The

underlying matroid of the dual  $P_M$ -matroid  $\widehat{M}^*$  of  $\widehat{M}$  is the dual  $\widehat{\widehat{M}^*} = M^*$  of  $M$ , cf. [3, Thm. 3.24]. Let  $\omega_M : P_{M^*} \rightarrow P_M$  be the characteristic morphism of  $\widehat{M}^*$ .

**PROPOSITION 4.8.** *Let  $M$  be a matroid of rank  $r$  on  $E$ . Then  $\omega_M : P_{M^*} \rightarrow P_M$  is an isomorphism of pastures that restricts to an isomorphism  $F_{M^*} \rightarrow F_M$  between the respective foundations of  $M^*$  and  $M$ . Let  $n = \#E$ . For every  $\mathbf{I} \in E^{n-r-1}$ ,  $\mathbf{J} \in E^{r-1}$  and  $e, f \in E$  such that  $E = |\mathbf{I}| \cup |\mathbf{J}| \cup \{e, f\}$ , we have*

$$\omega_M \left( \frac{T_{\mathbf{I}e}}{T_{\mathbf{I}f}} \right) = - \frac{T_{\mathbf{J}f}}{T_{\mathbf{J}e}},$$

and for every  $(J; e_1, \dots, e_4) \in \Omega_M$  and  $I = E - Je_1 \dots e_4$ , we have  $(I; e_1, \dots, e_4) \in \Omega_{M^*}$  and

$$\omega_M \left( \left[ \begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix} \right]_{\widehat{M}^*, I} \right) = \left[ \begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix} \right]_{\widehat{M}, J},$$

where  $\widehat{M}$  is the universal  $P_M$ -matroid of  $M$  and  $\widehat{M}^*$  is the universal  $P_{M^*}$ -matroid of  $M^*$ .

**PROOF.** The construction of  $\omega_M$ , applied to  $M^*$  in place of  $M$ , yields a morphism  $\omega_{M^*} : P_{M^{**}} \rightarrow P_{M^*}$ . Since  $M^{**} = M$ , we have  $P_{M^{**}} = P_M$ . The composition  $\omega_M \circ \omega_{M^*} : P_M = P_{M^{**}} \rightarrow P_{M^*} \rightarrow P_M$  is the characteristic morphism of the double dual  $\widehat{\widehat{M}^{**}}$  of  $\widehat{M}$ , which is equal to  $\widehat{M}$  by [3, Thm. 3.24], and thus  $\omega_M \circ \omega_{M^*}$  is the identity of  $P_M$ . Similarly, the composition  $\omega_{M^*} \circ \omega_M$  is the identity of  $P_{M^*}$ . This shows that  $\omega_M$  and  $\omega_{M^*}$  are mutually inverse isomorphisms.

Let  $\Delta : E^r \rightarrow P_M$  be a Grassmann-Plücker function for  $\widehat{M}$ . Endow  $E$  with a total order and define  $\text{sign}(i_1, \dots, i_n) = \text{sign}(\pi)$  as the sign of the permutation  $\pi$  of  $E$  such that  $\pi(i_1) < \dots < \pi(i_n)$  if  $i_1, \dots, i_n \in E$  are pairwise distinct. Then by [3, Lemma 4.1], there is a Grassmann-Plücker function  $\Delta^* : E^{n-r} \rightarrow P_{M^*}$  for  $\widehat{M}^*$  that satisfies

$$\Delta^*(i_1, \dots, i_{n-r}) = \text{sign}(i_1, \dots, i_n) \Delta(i_{n-r+1}, \dots, i_n)$$

for all pairwise distinct  $i_1, \dots, i_n \in E$ . Thus if  $\mathbf{I} = (i_1, \dots, i_{n-r-1})$ ,  $\mathbf{J} = (j_1, \dots, j_{r-1})$  and  $e, f \in E$  are as in the hypothesis of the theorem, then

$$\omega_M \left( \frac{T_{\mathbf{I}e}}{T_{\mathbf{I}f}} \right) = \frac{\Delta^*(\mathbf{I}e)}{\Delta^*(\mathbf{I}f)} = \frac{\text{sign}(i_1, \dots, i_{n-r-1}, e, j_1, \dots, j_{r-1}, f) \Delta(\mathbf{J}f)}{\text{sign}(i_1, \dots, i_{n-r-1}, f, j_1, \dots, j_{r-1}, e) \Delta(\mathbf{J}e)} = - \frac{T_{\mathbf{J}f}}{T_{\mathbf{J}e}},$$

as claimed. If  $(J; e_1, \dots, e_4) \in \Omega_M$  and  $I = E - Je_1 \dots e_4$ , then  $Je_i e_k$  is a basis for  $M$ , and thus  $Ie_j e_l$  is a basis for  $M^*$  for all  $i, j \in \{1, 2\}$  and  $k, l \in \{3, 4\}$ . Thus  $(I; e_1, \dots, e_4) \in \Omega_{M^*}$ . The image of the corresponding cross ratio under  $\omega_M$  is

$$\omega_M \left( \left[ \begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix} \right]_I \right) = \frac{\Delta^*(\mathbf{I}e_1 e_3) \Delta^*(\mathbf{I}e_2 e_4)}{\Delta^*(\mathbf{I}e_1 e_4) \Delta^*(\mathbf{I}e_2 e_3)} = \left[ \begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix} \right]_{\widehat{M}^*, I} = \left[ \begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix} \right]_{\widehat{M}, J}$$

where  $\mathbf{I} \in E^{n-r-2}$  such that  $|\mathbf{I}| = I$  and where we use Lemma 3.2 for the last equality. Since the foundations of  $M$  and  $M^*$  are generated by cross ratios, it follows at once that  $\omega_M$  restricts to an isomorphism  $F_{M^*} \rightarrow F_M$ .  $\square$

#### 4.4. Foundations of embedded minors

Let  $M$  be a matroid of rank  $r$  on  $E$ , and let  $\widehat{M}$  be the universal  $P_M$ -matroid associated with  $M$ , whose characteristic function is the identity map on  $P_M$ ; cf. Theorem 2.18. Let  $\Delta : E^r \rightarrow P_M$  be a Grassmann-Plücker function for  $\widehat{M}$ ; e.g. we

can choose some  $\mathbf{I}_0 \in E^r$  such that  $|\mathbf{I}_0|$  is a basis of  $M$  and define  $\Delta(\mathbf{I}) = T_{\mathbf{I}}/T_{\mathbf{I}_0}$  if  $|\mathbf{I}|$  is a basis of  $M$  and  $\Delta(\mathbf{I}) = 0$  if not.

Let  $N = M \setminus I/J$  be an embedded minor of  $M$ . Let  $s$  be its rank and  $E_N = E - (I \cup J)$  its ground set. Choose an ordering  $J = \{j_{s+1}, \dots, j_r\}$  of the elements of  $J$ . By [3, Lemma 4.4], the function

$$\begin{array}{ccc} \Delta \setminus I/J : & E_N^s & \longrightarrow & P_M \\ & \mathbf{I} & \longmapsto & \Delta(\mathbf{I}j_{s+1} \dots j_r) \end{array}$$

is a Grassmann-Plücker function that represents  $N = M \setminus I/J$  and its isomorphism class  $\widehat{N} = \widehat{M} \setminus I/J$  is independent of the choice of ordering of  $J$ . The characteristic function of the  $P_M$ -matroid  $\widehat{N}$  is a morphism  $\psi_{M \setminus I/J} : P_N \rightarrow P_M$ ; once again cf. Theorem 2.18.

**PROPOSITION 4.9.** *Let  $M$  be a matroid of rank  $r$  on  $E$  and  $N = M \setminus I/J$  an embedded minor of rank  $s$  on  $E_N = E - (I \cup J)$ . Let  $J = \{j_{s+1}, \dots, j_r\}$ . Then the morphism  $\psi_{M \setminus I/J} : P_N \rightarrow P_M$  satisfies the following properties.*

- (1) *For all  $\mathbf{I}, \mathbf{J} \in E_N^s$  such that  $|\mathbf{I}|$  and  $|\mathbf{J}|$  are bases of  $N$ , we have*

$$\psi_{M \setminus I/J} \left( \frac{T_{\mathbf{I}}}{T_{\mathbf{J}}} \right) = \frac{T_{\mathbf{I}j_{s+1} \dots j_r}}{T_{\mathbf{J}j_{s+1} \dots j_r}}.$$

- (2) *The identification  $N^* = M^* \setminus J/I$  yields a commutative diagram*

$$\begin{array}{ccc} P_{N^*} & \xrightarrow{\psi_{M^* \setminus J/I}} & P_{M^*} \\ \omega_N \downarrow & & \downarrow \omega_M \\ P_N & \xrightarrow{\psi_{M \setminus I/J}} & P_M \end{array}$$

*of pastures, where  $\omega_N$  and  $\omega_M$  are the isomorphisms from Proposition 4.8.*

- (3) *The morphism  $\psi_{M \setminus I/J} : P_N \rightarrow P_M$  restricts to a morphism  $\varphi_{M \setminus I/J} : F_N \rightarrow F_M$  between the foundations of  $N$  and  $M$ . For  $(J'; e_1, \dots, e_4) \in \Omega_N$ , we have  $(J' \cup J; e_1, \dots, e_4) \in \Omega_M$  and*

$$\varphi_{M \setminus I/J} \left( \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{J'} \right) = \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{J \cup J'}.$$

- (4) *If every element in  $I$  is a loop or parallel to an element in  $E_N$  and if every element in  $J$  is a coloop or coparallel to an element in  $E_N$ , then  $\varphi_{M \setminus I/J}$  is an isomorphism.*

**PROOF.** Property (1) follows from the direct computation

$$\psi_{M \setminus I/J} \left( \frac{T_{\mathbf{I}}}{T_{\mathbf{J}}} \right) = \frac{\Delta \setminus I/J(\mathbf{I})}{\Delta \setminus I/J(\mathbf{J})} = \frac{T_{\mathbf{I}j_{s+1} \dots j_r}}{T_{\mathbf{J}j_{s+1} \dots j_r}}.$$

We continue with (2). Let  $r^*$  be the corank of  $M$  and  $s^*$  the corank of  $N$ . Choose an ordering  $I = \{i_{s^*+1}, \dots, i_{r^*}\}$ . Let  $\mathbf{I} \in E_N^{s^*-1}$ ,  $\mathbf{J} \in E_N^{s^*-1}$  and  $e, f \in E_N$  be such that  $E_N = |\mathbf{I}| \cup |\mathbf{J}| \cup \{e, f\}$ , which are the assumptions needed to apply Proposition 4.8 to  $\omega_N$ . Since  $P_{N^*}$  is generated by elements of the form  $T_{\mathbf{I}e}/T_{\mathbf{I}f}$ , the

commutativity of the diagram in question follows from

$$\begin{aligned}
 \psi_{M \setminus I/J} \circ \omega_N \left( \frac{T_{\mathbf{I}e}}{T_{\mathbf{I}f}} \right) &= \psi_{M \setminus I/J} \left( -\frac{T_{\mathbf{J}f}}{T_{\mathbf{J}e}} \right) \\
 &= -\frac{T_{\mathbf{J}f j_{s+1} \dots j_r}}{T_{\mathbf{J}e j_{s+1} \dots j_r}} \\
 &= \omega_M \left( \frac{T_{\mathbf{I}e i_{s^*+1} \dots i_{r^*}}}{T_{\mathbf{I}f i_{s^*+1} \dots i_{r^*}}} \right) \\
 &= \omega_M \circ \psi_{M^* \setminus J/I} \left( \frac{T_{\mathbf{I}e}}{T_{\mathbf{I}f}} \right).
 \end{aligned}$$

Note that we can apply Proposition 4.8 to  $\omega_M$  since  $E = |\mathbf{I}| \cup |\mathbf{J}| \cup \{e, f\} \cup I \cup J$ .

We continue with (3). If  $(J'; e_1, \dots, e_4) \in \Omega_N$ , then for all  $i \in \{1, 2\}$  and  $k \in \{3, 4\}$ , the set  $J'e_i e_k$  is a basis of  $N$  and thus  $J' \cup J \cup \{e_i, e_k\}$  is a basis of  $M$ . Thus  $(J' \cup J; e_1, \dots, e_4) \in \Omega_M$ . Let  $\mathbf{J}' \in E_N^S$  such that  $|\mathbf{J}'| = J'$ . Then

$$\begin{aligned}
 \psi_{M \setminus I/J} \left( \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{J'} \right) &= \frac{\Delta \setminus I/J(\mathbf{J}'e_1e_3) \cdot \Delta \setminus I/J(\mathbf{J}'e_2e_4)}{\Delta \setminus I/J(\mathbf{J}'e_1e_4) \cdot \Delta \setminus I/J(\mathbf{J}'e_2e_3)} \\
 &= \frac{T_{\mathbf{J}'e_1e_3 j_{s+1} \dots j_r} T_{\mathbf{J}'e_2e_4 j_{s+1} \dots j_r}}{T_{\mathbf{J}'e_1e_4 j_{s+1} \dots j_r} T_{\mathbf{J}'e_2e_3 j_{s+1} \dots j_r}} \\
 &= \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{J \cup J'}.
 \end{aligned}$$

By Theorem 4.6, the foundation of a matroid is generated by its cross ratios. Thus the previous calculation shows that  $\psi_{M \setminus I/J}$  restricts to a morphism  $\varphi_{M \setminus I/J} : F_N \rightarrow F_M$  which maps  $\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{J'}$  to  $\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{J' \cup J}$ .

We continue with (4). By successively deleting or contracting one element at a time, it suffices to prove the claim for  $\#(I \cup J) = 1$ . Using (2), we can assume that  $I = \{e\}$  and  $J = \emptyset$ . If  $e$  is a loop, then  $I' \mapsto I'$  defines a bijection between the set of bases  $I' \subset E_N = E - \{e\}$  of  $N$  and the set of bases of  $M$ . Moreover, for every  $(J'; e_1, \dots, e_4) \in \Omega_M$ , we have  $e \notin J'e_1 \dots e_4$ , which provides an identification  $\Omega_N = \Omega_M$ . Thus  $P_N$  and  $P_M$  have the same generators and the same 3-term Plücker relations, so  $\psi_{M \setminus I/J} : P_N \rightarrow P_M$  is an isomorphism. This argument also shows that  $\varphi_{M \setminus I/J} : F_N \rightarrow F_M$  is an isomorphism.

If  $e$  is parallel to an element  $f \in E_N$ , then  $\langle J'e \rangle = \langle J'f \rangle$  for every subset  $J'$  of  $E_N$ . Thus for  $e_1, \dots, e_4 \in E$  and  $f_1, \dots, f_4 \in E_N$  with either  $e_i = f_i$  or  $(e_i, f_i) = (e, f)$  for  $i = 1, \dots, 4$ , we have  $(J'; e_1, \dots, e_4) \in \Omega_M$  if and only if  $(J'; f_1, \dots, f_4) \in \Omega_N$ , and  $\varphi_{M \setminus I/J} \left( \begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \end{bmatrix}_{J'} \right) = \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{J'}$ . This shows that  $\varphi_{M \setminus I/J} : F_N \rightarrow F_M$  is an isomorphism, which completes the proof.  $\square$

An immediate consequence of Proposition 4.9 is the following.

**COROLLARY 4.10.** *The foundation of a matroid is isomorphic to the foundation of its simplification and isomorphic to the foundation of its cosimplification.*

**PROOF.** This follows at once from Proposition 4.9, since the simplification of a matroid  $M$  is an embedded minor of  $M$  of the form  $M \setminus I$ , where  $I$  consists of all loops of  $M$  and a choice of all but one element in each class of parallel elements. Similarly, the cosimplification of  $M$  is an embedded minor of  $M$  of the form  $M/J$ , where  $J$  consists of all coloops of  $M$  and a choice of all but one element in each class of coparallel elements.  $\square$

Another consequence of Proposition 4.9, which we will utilize constantly in the upcoming sections, is the following observation. Since a universal cross ratio  $\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_J$  involves only bases  $Je_ie_k$  that contain  $J$  and have a trivial intersection with  $I = E - Je_1e_2e_3e_4$ , we have

$$\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_J = \frac{T_{Je_1e_3}T_{Je_2e_4}}{T_{Je_1e_4}T_{Je_2e_3}} = \frac{\varphi(T_{(e_1,e_3)})\varphi(T_{(e_2,e_4)})}{\varphi(T_{(e_1,e_4)})\varphi(T_{(e_2,e_3)})} = \varphi\left(\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_\emptyset\right)$$

for the morphism  $\varphi = \varphi_{M \setminus I/J} : F_{M \setminus I/J} \rightarrow F_M$  from Proposition 4.9. Thus every universal cross ratio in  $F_M$  is the image of a universal cross ratio of an embedded minor  $N = M \setminus I/J$  of rank 2 on a 4-element set  $\{e_1, e_2, e_3, e_4\} = E - (I \cup J)$ .

#### 4.5. The foundation of $U_4^2$

Let  $M = U_4^2$  be the uniform minor of rank 2 on the set  $E = \{1, \dots, 4\}$ , which is represented by the Grassmann-Plücker function  $\Delta : E^2 \rightarrow \mathbb{K}$  with  $\Delta(i, j) = 1 - \delta_{i,j}$ . The cross ratios of  $M$  are of the form

$$\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix} := \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_\emptyset$$

for some permutation  $e : i \mapsto e_i$  of  $E$ . Since permuting columns and rows in  $\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}$  does not change the cross ratio, as pointed out in section 3.1, we have

$$(R\sigma^*) \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}.$$

Thus we can assume that  $e_1 = 1$ , and with this convention, we find that each of the 24 possible cross ratios is equal to one of the following six:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}.$$

They satisfy the following two types of multiplicative relations

$$(R1^*) \quad \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1}, \quad \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}^{-1}, \quad \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}^{-1};$$

$$(R2^*) \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = -1, \quad \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} = -1;$$

and the Plücker relations

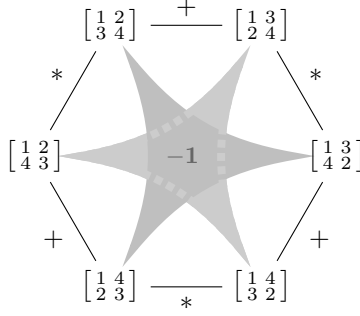
$$(R+^*) \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = 1, \quad \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} = 1, \quad \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} = 1.$$

These relations can be illustrated in the form of a hexagon, see Figure 4.1. The three edges with label  $*$  refer to relations of type  $(R1^*)$ , the three edges with label  $+$  refer to the Plücker relations  $(R+^*)$ , and the two inner triangles refer to the relations of type  $(R2^*)$ .

Note that we can rewrite the relations of type  $(R1^*)$  as  $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} = 1$ , and so forth, which highlights an analogy with the Plücker relations  $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = 1$ . This makes the meaning of the edge labels  $*$  and  $+$  easy to remember.

**PROPOSITION 4.11.** *Let  $x = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$  and  $y = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ . Then the foundation of  $M = U_4^2$  is*

$$F_M = \mathbb{U} = \mathbb{F}_1^\pm \langle x, y \rangle // \{x + y - 1\}.$$

FIGURE 4.1. The hexagon of cross ratios of  $U_4^2$ 

In particular, we have

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} = x^{-1}, \quad \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} = y^{-1}, \quad \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} = -xy^{-1}, \quad \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = -x^{-1}y.$$

PROOF. By relation  $(R\sigma^*)$ ,  $F_M$  is generated by the 6 cross ratios

$$x = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad y = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}.$$

By relation  $(R1^*)$ , we have

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = x^{-1} \quad \text{and} \quad \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}^{-1} = y^{-1}.$$

Relation  $(R2^*)$ , paired with  $(R1^*)$ , yields

$$\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}^{-1} = - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} = -xy^{-1}.$$

Applying  $(R1^*)$  once again yields

$$\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}^{-1} = -x^{-1}y.$$

By  $(R+^*)$ , we have  $x + y - 1 = 0$ . This shows that the foundation  $F_M$  of  $M = U_4^2$  is a quotient of  $\mathbb{U} = \mathbb{F}_1^\pm \langle x, y \rangle // \{x + y - 1\}$ .

There are several different ways to show that there are no further relations in  $F_M$  aside from those already present in  $\mathbb{U}$ , for example:

- (1) One can work this out “by hand”.
- (2) One can utilize the fact that  $U_4^2$  is near-regular, which implies that there is a morphism  $F_M \rightarrow \mathbb{U}$ .
- (3) One can apply Theorem 4.21, whose proof does not rely on Proposition 4.11.

We explain a fourth route, which uses a theorem of Dress and Wenzel determining the inner Tutte group of a uniform matroid. In the case of  $M = U_4^2$ , [14, Thm. 8.1], paired with Corollary 4.4, shows that  $F_M^\times \simeq \mathbb{T}^{(0)} \simeq (\mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}^2 \simeq \mathbb{U}^\times$ . We conclude that the quotient map  $\mathbb{U} \rightarrow F_M$  is an isomorphism between the underlying monoids. We are left with showing that every relation in the nullset of  $F_M$  comes



from  $\mathbb{U}$ , which is the intersection of the nullset  $N_{P_M^+}$  of  $P_M^+$  with  $\text{Sym}^3(F_M)$ . Since  $N_{P_M^+}$  is generated by the single term

$$T_{1,2}T_{3,4} - T_{1,3}T_{2,4} + T_{1,4}T_{2,3} = -T_{1,4}T_{2,3} \cdot (x + y - 1),$$

where we use the short-hand notation  $T_{i,j} = T_{(i,j)}$ , every term in  $N_{F_M}$  is a multiple of  $x + y - 1$ . This shows that  $\mathbb{U} \rightarrow F_M$  is an isomorphism.  $\square$

Morphisms from  $\mathbb{U}$  into another pasture can be studied in terms of pairs of fundamental elements:

**DEFINITION 4.12.** A pair of fundamental elements in  $P$  is an ordered pair  $(z, z')$  of elements  $z, z' \in P^\times$  such that  $z + z' = 1$ .

**LEMMA 4.13.** Let  $P$  be a pasture. Then there is a bijection between  $\text{Hom}(\mathbb{U}, P)$  and the set of pairs of fundamental elements.

**PROOF.** Every morphism  $f : \mathbb{U} = \mathbb{F}_1^\pm \langle x, y \rangle // \{x + y = 1\} \rightarrow P$  maps  $x$  and  $y$  to invertible elements in  $P$ . Since  $x + y = 1$ , we have  $f(x) + f(y) = 1$  in  $P$ , which shows that  $(f(x), f(y))$  is a pair of fundamental elements. This defines a map  $\Phi : \text{Hom}(\mathbb{U}, P) \rightarrow \mathcal{F}_P$ , where  $\mathcal{F}_P$  is the set of pairs of fundamental elements in  $P$ .

Since  $f$  is determined by the images of  $x$  and  $y$ , we see that  $\Phi$  is injective. On the other hand, for every pair  $(u, v)$  of fundamental elements in  $P$ , the map  $x \mapsto u$  and  $y \mapsto v$  extends to a morphism  $f : \mathbb{U} \rightarrow P$ . Thus  $\Phi$  is surjective as well.  $\square$

Recall that a reorientation class is a rescaling class over the sign hyperfield  $\mathbb{S}$ . The following corollary is well known:

**COROLLARY 4.14.** The rescaling classes of  $U_4^2$  over a field  $k$  are in bijection with  $k - \{0, 1\}$ , and  $U_4^2$  has 3 reorientation classes.

**PROOF.** If  $P = k$  is a field, then  $y = 1 - x$  is uniquely determined by  $x$ , and  $x, y$  both belong to  $k^\times$  precisely when  $x \in k - \{0, 1\}$ , which establishes the first claim. The second claim follows from the observation that  $a + b = 1$  in  $\mathbb{S}$  if and only if  $(a, b)$  is one of the 3 pairs  $(1, 1)$ ,  $(1, -1)$  and  $(-1, 1)$ .  $\square$

#### 4.6. The tip and cotip relations

In this section, we exhibit two types of relations that occur for matroids of ranks 2 and 3, respectively, on the five element set  $E = \{1, \dots, 5\}$ .

As in the case of the uniform matroid  $U_4^2$ , we write  $\begin{bmatrix} i & j \\ k & l \end{bmatrix}$  for  $\begin{bmatrix} i & j \\ k & l \end{bmatrix}_\emptyset$  in the case of a rank 2-matroid  $M$ . We also use the shorthand notation  $T_{i,j} = T_{(i,j)}$  and  $T_{i,j,k} = T_{(i,j,k)}$ .

**LEMMA 4.15.** Let  $M$  be a matroid of rank 2 on  $E = \{1, \dots, 5\}$ . Assume that  $\{i, j\}$  is a basis of  $M$  for all  $i \in \{1, 2\}$  and all  $j \in \{3, 4, 5\}$ . Then

$$(R3^*) \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 5 & 3 \end{bmatrix} = 1.$$

**PROOF.** Equation (R3\*) follows from the direct computation

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 5 & 3 \end{bmatrix} = \frac{T_{1,3}T_{2,4}}{T_{1,4}T_{2,3}} \cdot \frac{T_{1,4}T_{2,5}}{T_{1,5}T_{2,4}} \cdot \frac{T_{1,5}T_{2,3}}{T_{1,3}T_{2,5}} = 1. \quad \square$$

We call equation (R3\*) the *tip relation with tip*  $\{1, 2\}$  and *cyclic orientation*  $(3, 4, 5)$ . The reason for this terminology is that in the case of the uniform matroid  $M = U_5^2$ , the three cross ratios in equation (R3\*) stem from three octahedrons in the basis exchange graph of  $M$ , which share exactly one common vertex, or *tip*, which is  $\{1, 2\}$ .

Note that if  $M$  is not uniform, i.e. some 2-subsets  $\{i, j\}$  of  $E$  are not bases, then some of the cross ratios in equation (R3\*) are trivial. We will examine this situation in more detail in Section 5.1.

In the case of a matroid of rank 3, we write  $\begin{bmatrix} i & j \\ k & l \end{bmatrix}_m$  for  $\begin{bmatrix} i & j \\ k & l \end{bmatrix}_{\{m\}}$ .

LEMMA 4.16. *Let  $M$  be a matroid of rank 3 on  $E = \{1, \dots, 5\}$ . Assume that  $\{i, j, k\}$  is a basis of  $M$  for all  $i \in \{1, 2\}$  and all  $j, k \in \{3, 4, 5\}$  with  $j \neq k$ . Then*

$$(R4^*) \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_5 \cdot \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}_3 \cdot \begin{bmatrix} 1 & 2 \\ 5 & 3 \end{bmatrix}_4 = 1.$$

PROOF. Equation (R4\*) follows from the direct computation

$$\begin{aligned} & \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_5 \cdot \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}_3 \cdot \begin{bmatrix} 1 & 2 \\ 5 & 3 \end{bmatrix}_4 \\ &= \frac{T_{5,1,3} \cdot T_{5,2,4}}{T_{5,1,4} \cdot T_{5,2,3}} \cdot \frac{T_{3,1,4} \cdot T_{3,2,5}}{T_{3,1,5} \cdot T_{3,2,4}} \cdot \frac{T_{4,1,5} \cdot T_{4,2,3}}{T_{4,1,3} \cdot T_{4,2,5}} \\ &= \frac{T_{4,1,5}}{-T_{4,1,5}} \cdot \frac{T_{3,2,5}}{-T_{3,2,5}} \cdot \frac{T_{5,1,3}}{-T_{5,1,3}} \cdot \frac{T_{4,2,3}}{-T_{4,2,3}} \cdot \frac{T_{3,1,4}}{-T_{3,1,4}} \cdot \frac{T_{5,2,4}}{-T_{5,2,4}} \\ &= (-1)^6 = 1. \quad \square \end{aligned}$$

We call equation (R4\*) the *cotip relation with cotip*  $\{1, 2\}$  and *cyclic orientation*  $(3, 4, 5)$ . Similar to the rank 2-case, we use this terminology since in the case of the uniform matroid  $M = U_5^3$ , the three cross ratios in equation (R4\*) stem from three octahedrons in the basis exchange graph of  $M$ , which share exactly one common vertex, which is  $\{3, 4, 5\}$ . Therefore we call the complement  $\{1, 2\}$  of this common vertex the *cotip*.

Note that the tip and cotip relations are both invariant under even permutations of  $\{1, \dots, 5\}$  that leave  $\{1, 2\}$  and  $\{3, 4, 5\}$  invariant. An odd permutation that leaves  $\{1, 2\}$  and  $\{3, 4, 5\}$  invariant leads to an inversion of all cross ratios in the tip or cotip relation. Any other permutation of  $E$  leads to a significantly different tip or cotip relation, provided that all involved values of  $\Delta$  are nonzero.

#### 4.7. Relations for parallel elements

In this section, we exhibit a type of relation between universal cross ratios that stems from parallel elements. As in the previous section, we write  $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_5$  for  $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{\{5\}}$ .

LEMMA 4.17. *Let  $M$  be a matroid of rank 3 on  $E = \{1, \dots, 6\}$  and assume that 5 and 6 are parallel elements, i.e.  $\{5, 6\}$  is a circuit of  $M$ . If  $(\{k\}; 1, \dots, 4) \in \Omega_M$  for  $k = 5, 6$ , then*

$$(R5^*) \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_5 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_6.$$

PROOF. By our assumptions, every subset of the form  $\{i, j, k\}$  with  $i \in \{1, 2\}$ ,  $j \in \{3, 4\}$  and  $k \in \{5, 6\}$  is a basis of  $M$ , but no basis contains both 5 and 6. Thus

$(\{1\}; 3, 4, 6, 5)$  and  $(\{2\}; 3, 4, 5, 6)$  are degenerate tuples in  $\Omega_M$ , and thus  $\begin{bmatrix} 3 & 4 \\ 6 & 5 \end{bmatrix}_1 = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}_2 = 1$ . With this, equation (R5\*) follows from the computation

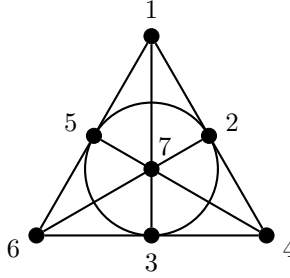
$$\begin{aligned}
 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_5 &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_5 \cdot \begin{bmatrix} 3 & 4 \\ 6 & 5 \end{bmatrix}_1 \cdot \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}_2 \\
 &= \frac{T_{5,1,3} \cdot T_{5,2,4}}{T_{5,1,4} \cdot T_{5,2,3}} \cdot \frac{T_{1,3,6} \cdot T_{1,4,5}}{T_{1,3,5} \cdot T_{1,4,6}} \cdot \frac{T_{2,3,5} \cdot T_{2,4,6}}{T_{2,3,6} \cdot T_{2,4,5}} \\
 &= \frac{T_{1,4,5}}{T_{1,4,6}} \cdot \frac{T_{2,3,5}}{T_{2,3,6}} \cdot \frac{T_{5,1,3}}{T_{5,1,4}} \cdot \frac{T_{6,1,3} \cdot T_{6,2,4}}{T_{6,1,4} \cdot T_{6,2,3}} \cdot \frac{T_{5,2,4}}{T_{5,2,3}} \\
 &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_6.
 \end{aligned}$$

□

#### 4.8. The foundation of the Fano matroid and its dual

In this section, we show that the Fano matroid  $F_7$  and its dual  $F_7^*$  impose the relation  $-1 = 1$  on their foundation, which is  $\mathbb{F}_2$ . This already follows from [5, Thms. 7.30 and 7.33], using the fact that  $F_7$  and  $F_7^*$  are not regular. Here we offer a proof in terms of a direct calculation that does not rely on knowledge of the representability of  $F_7$ .

The Fano matroid  $F_7$  is the rank 3 matroid on  $E = \{1, \dots, 7\}$  represented by the Grassmann-Plücker function  $\Delta : E^3 \rightarrow \mathbb{K}$  with  $\Delta(i, i+1, i+3) = 0$  for  $i \in E$ , where we read  $i+1$  and  $i+3$  modulo 7, and  $\Delta(i, j, k) = 1$  otherwise. Thus the family of circuits is  $\left\{ \{i, i+1, i+3\} \mid i \in E \right\}$ , together with all 4-element subsets that do not contain one of these, which can be illustrated as follows:



LEMMA 4.18. *The foundation of both the Fano matroid  $F_7$  and its dual  $F_7^*$  is  $\mathbb{F}_2$ .*

PROOF. Since the foundation of  $F_7^*$  is isomorphic to the foundation of  $F_7$ , it is enough to prove the lemma for the Fano matroid. Throughout the proof, we read expressions like  $i+k$  and  $i-k$  modulo 7 for all  $i, k \in E$ .

Since the rank of  $F_7$  is 3, the set  $J$  of a tuple  $(J; e_1, \dots, e_4) \in \Omega_M$  is a singleton, i.e.  $J = \{j\}$  for some  $j \in E$ . The element  $j$  is contained in the three circuits  $C_1 = \{j, j+1, j+3\}$ ,  $C_2 = \{j-1, j, j+2\}$  and  $C_3 = \{j-3, j-2, j\}$  whose union is equal to  $E$ . By the pigeonhole principle, we must have  $e_k, e_l \in C_i$  for some  $i$  and  $k \neq l$ . Since  $j, e_k, e_l$  are pairwise distinct,  $C_i = \{j, e_k, e_l\}$  is not a basis. This shows that every  $(J; e_1, \dots, e_4) \in \Omega_M$  is degenerate, and thus  $\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_J = 1$ . We conclude that  $F_M$  is a quotient of  $\mathbb{F}_1^\pm$ .

We use the shorthand notations  $\begin{bmatrix} i & j \\ k & l \end{bmatrix}_m = \begin{bmatrix} i & j \\ k & l \end{bmatrix}_{\{m\}}$  and  $T_{j,k,l}^i = T_{(i+j, i+k, i+l)}$  in the following. Note that  $T_{j+m, k+m, l+m}^{i-m} = T_{j,k,l}^i$  and  $T_{\sigma(j), \sigma(k), \sigma(l)}^i = \text{sign}(\sigma) T_{j,k,l}^i$

for every permutation  $\sigma$  of  $\{j, k, l\}$ . We calculate that

$$\begin{aligned}
1 &= \prod_{i=1}^7 \begin{bmatrix} i+1 & i+3 \\ i+2 & i+4 \end{bmatrix}_i \cdot \begin{bmatrix} i+2 & i+6 \\ i+5 & i+4 \end{bmatrix}_i \\
&= \prod_{i=1}^7 \frac{T_{0,1,2}^i \cdot T_{0,3,4}^i}{T_{0,1,4}^i \cdot T_{0,3,2}^i} \cdot \frac{T_{0,2,5}^i \cdot T_{0,6,4}^i}{T_{0,2,4}^i \cdot T_{0,6,5}^i} \\
&= \prod_{i=1}^7 \frac{T_{0,1,2}^i \cdot T_{0,3,4}^i \cdot T_{0,2,5}^i \cdot T_{0,6,4}^i}{T_{3,4,0}^{i-3} \cdot T_{4,0,6}^{i-4} \cdot T_{5,0,2}^{i-5} \cdot T_{2,1,0}^{i-2}} \\
&= \prod_{i=1}^7 \frac{T_{0,3,4}^i}{T_{0,3,4}^{i-3}} \cdot \frac{T_{0,6,4}^i}{T_{0,6,4}^{i-4}} \cdot \frac{T_{0,2,5}^i}{T_{0,2,5}^{i-5}} \cdot \frac{T_{0,1,2}^i}{-T_{0,1,2}^{i-2}} \\
&= (-1)^7 = -1.
\end{aligned}$$

This shows that the foundation  $F_M$  of  $F_7$  is a quotient of  $\mathbb{F}_2 = \mathbb{F}_1^\pm // \{-1 = 1\}$ . Since  $F_7$  does not contain any  $U_4^2$ -minors, all cross ratios are degenerate and thus the nullset of  $F_M$  does not contain any 3-term relations. We conclude that  $F_M = \mathbb{F}_2$ .  $\square$

#### 4.9. A presentation of the foundation by hyperplanes

Gelfand, Rybnikov and Stone exhibit in [17, Thm. 4] a complete set of multiplicative relations in the inner Tutte group of  $M$  between the cross ratios  $\begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}$  of modular quadruples  $(C_1, \dots, C_4)$  of circuits, which results in essence from Tutte's homotopy theorem. Since hyperplanes are just complements of circuits of the dual matroid, this set of relations yields at once a complete set of relations for cross ratios  $\begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}$  of modular quadruples  $(H_1, \dots, H_4)$  of hyperplanes.

**THEOREM 4.19.** *Let  $M$  be a matroid with foundation  $F_M$ . Then*

$$F_M = \mathbb{F}_1^\pm \langle \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} \mid (H_1, \dots, H_4) \in \Theta_M \rangle // S,$$

where  $S$  is defined by the multiplicative relations

$$(H-) \quad -1 = 1$$

if the Fano matroid  $F_7$  or its dual  $F_7^*$  is a minor of  $M$ ;

$$(H\sigma) \quad \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} = \begin{bmatrix} H_2 & H_1 \\ H_4 & H_3 \end{bmatrix} = \begin{bmatrix} H_3 & H_4 \\ H_1 & H_2 \end{bmatrix} = \begin{bmatrix} H_4 & H_3 \\ H_2 & H_1 \end{bmatrix}$$

for all  $(H_1, \dots, H_4) \in \Theta_M^\diamond$ ;

$$(H0) \quad \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} = 1$$

for all degenerate  $(H_1, \dots, H_4) \in \Theta_M$ ;

$$(H1) \quad \begin{bmatrix} H_1 & H_2 \\ H_4 & H_3 \end{bmatrix} = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}^{-1}$$

for all  $(H_1, \dots, H_4) \in \Theta_M^\diamond$ ;

$$(H2) \quad \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} \cdot \begin{bmatrix} H_1 & H_3 \\ H_4 & H_2 \end{bmatrix} \cdot \begin{bmatrix} H_1 & H_4 \\ H_2 & H_3 \end{bmatrix} = -1$$

for all  $(H_1, \dots, H_4) \in \Theta_M^\diamond$ ;

$$(H3) \quad \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} \cdot \begin{bmatrix} H_1 & H_2 \\ H_4 & H_5 \end{bmatrix} \cdot \begin{bmatrix} H_1 & H_2 \\ H_5 & H_3 \end{bmatrix} = 1$$

for all  $(H_1, H_2, H_3, H_4), (H_1, H_2, H_4, H_5), (H_1, H_2, H_5, H_3) \in \Theta_M^\diamond$ ; and

$$(H4) \quad \begin{bmatrix} H_{15} & H_{25} \\ H_{35} & H_{45} \end{bmatrix} \cdot \begin{bmatrix} H_{13} & H_{23} \\ H_{43} & H_{53} \end{bmatrix} \cdot \begin{bmatrix} H_{14} & H_{24} \\ H_{54} & H_{34} \end{bmatrix} = 1,$$

where  $H_{ij} = \langle F_i \cup F_j \rangle$  for five pairwise distinct corank 2-flats  $F_1, \dots, F_5$  that contain a common flat of corank 3 such that  $(H_{15}, H_{25}, H_{35}, H_{45}), (H_{14}, H_{24}, H_{54}, H_{34}) \in \Theta_M^\diamond$  and  $(H_{13}, H_{23}, H_{43}, H_{53}) \in \Theta_M$ , as well as the additive Plücker relations

$$(H+) \quad \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} + \begin{bmatrix} H_1 & H_3 \\ H_2 & H_4 \end{bmatrix} = 1$$

for all  $(J; e_1, \dots, e_4) \in \Theta_M^\diamond$ .

PROOF. The theorem follows from a translation of [17, Thm. 4] to our context, by replacing a cocycle  $C$  by the hyperplane  $H = E - C$ . To pass from the inner Tutte group to the foundation, we employ Lemma 3.13, which identifies  $\begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}_{\mathbb{T}}$  with  $\begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}$  under the canonical isomorphism  $\mathbb{P}_M^\times \rightarrow \mathbb{T}_M^{(0)}$ .

Using this translation, relation (H−) is equivalent to (TG0) and (CR5) in [17]. Relation (Hσ) is equivalent to (CR3). Relation (H0) is equivalent to (CR1). Relation (CR4) is equivalent to (H1) (in the case that one cross ratio is degenerate) and (H3) (in the case that all cross ratios are non-degenerate). Relation (H2) is equivalent to (CR4). Relation (H4) is equivalent to (CR6), where we observe that the degenerate case  $L = L'$  in [17] reduces (CR6) to (CR1). Finally note that the 3-term Plücker relations of  $F_M$  are captured in (H+).  $\square$

REMARK 4.20. We include a discussion of relation (H4), which has the most complicated formulation among the relations of Theorem 4.19. Since all flats contain a common flat of corank 3, this constellation comes from a minor of rank 3, which has 5 corank 2-flats corresponding to  $F_1, \dots, F_5$ . In the non-degenerate situation where all hyperplanes  $H_{ij}$  are pairwise distinct, this minor is of type  $U_5^3$ , and the containment relation of the  $F_i$  and  $H_{ij}$  can be illustrated as on the right-hand side of Figure 4.2.

The original formulation of Gelfand, Rybnikov and Stone concerns *points*, which are circuits, and *lines*, which are unions of circuits having projective dimension 1. To pass from our formulation to that of Gelfand-Rybnikov-Stone, we take the complement of a hyperplane  $H_{ij}$ , which is a circuit  $C_{ij}$  of the dual matroid. Thus, in the non-degenerate case, axiom (CR6) expresses the point-line configuration of  $U_5^2$ , which we illustrate on the left-hand side of Figure 4.2. The lines  $L_i$  are the complements of the flats  $F_i$ , and therefore the union of the circuits  $C_{ij}$  (with varying  $j$ ).

Note that there are two degenerate situations that (CR6) allows for: (a) three lines, say  $L_1, L_2$  and  $L_3$ , intersect in one point  $C_{12} = C_{13} = C_{23}$ ; this case corresponds to the point-line arrangement of a parallel extension of  $U_4^2$ , which we denote by  $C_5^*$  in Section 5.1.3 and in this case,  $(H_{13}, H_{23}, H_{43}, H_{53}) \in \Theta_M$  is degenerate; and (b) two lines agree; this case corresponds to the point-line arrangement of  $U_4^2$ . Note that this latter relation corresponds to (H3), but with one of the quadruples

of hyperplanes being degenerate. Since in this case, (H3) is equivalent to (H1), we can assume in (H3) that all quadruples of hyperplanes are non-degenerate.

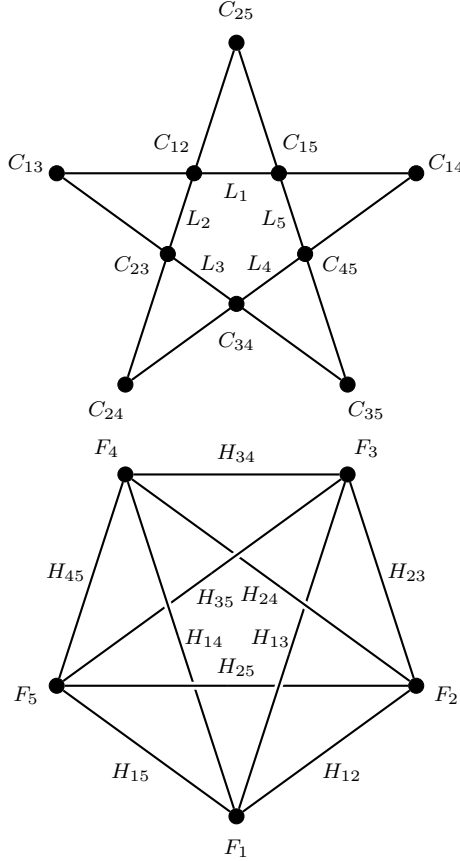


FIGURE 4.2. Point-line configuration for  $U_5^2$  and flat configuration for  $U_5^3$

#### 4.10. A presentation of the foundation by bases

Using the relation between cross ratios  $\begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}$  for modular quadruples  $(H_1, \dots, H_4)$  of hyperplanes and universal cross ratios  $\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_J$  for  $(J; e_1, \dots, e_4) \in \Omega_M$ , as exhibited in Proposition 3.6, we derive from Theorem 4.19 the following description of a complete set of relations between universal cross ratios. The possibility of such a deduction was observed and communicated to us by Rudi Pendavingh, who proves a similar result in the joint work [10] with Brettell.

**THEOREM 4.21.** *Let  $M$  be a matroid with foundation  $F_M$ . Then*

$$F_M = \mathbb{F}_1^\pm \left\langle \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_J \mid (J; e_1, \dots, e_4) \in \Omega_M \right\rangle // S,$$

where  $S$  is defined by the multiplicative relations

$$(R-) \quad -1 = 1$$

if the Fano matroid  $F_7$  or its dual  $F_7^*$  is a minor of  $M$ ;

$$(R\sigma) \quad \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_J = \begin{bmatrix} e_2 & e_1 \\ e_4 & e_3 \end{bmatrix}_J = \begin{bmatrix} e_3 & e_4 \\ e_1 & e_2 \end{bmatrix}_J = \begin{bmatrix} e_4 & e_3 \\ e_2 & e_1 \end{bmatrix}_J$$

for all  $(J; e_1, \dots, e_4) \in \Omega_M^\diamond$ ;

$$(R0) \quad \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_J = 1$$

for all degenerate  $(J; e_1, \dots, e_4) \in \Omega_M$ ;

$$(R1) \quad \begin{bmatrix} e_1 & e_2 \\ e_4 & e_3 \end{bmatrix}_J = \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_J^{-1}$$

for all  $(J; e_1, \dots, e_4) \in \Omega_M^\diamond$ ;

$$(R2) \quad \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_J \cdot \begin{bmatrix} e_1 & e_3 \\ e_4 & e_2 \end{bmatrix}_J \cdot \begin{bmatrix} e_1 & e_4 \\ e_2 & e_3 \end{bmatrix}_J = -1$$

for all  $(J; e_1, \dots, e_4) \in \Omega_M^\diamond$ ;

$$(R3) \quad \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_J \cdot \begin{bmatrix} e_1 & e_2 \\ e_4 & e_5 \end{bmatrix}_J \cdot \begin{bmatrix} e_1 & e_2 \\ e_5 & e_3 \end{bmatrix}_J = 1$$

for all  $e_1, \dots, e_5 \in E$  and  $J \subset E$  such that each of  $(J; e_1, e_2, e_3, e_4)$ ,  $(J; e_1, e_2, e_4, e_5)$  and  $(J; e_1, e_2, e_5, e_3)$  is in  $\Omega_M$ ;

$$(R4) \quad \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{Je_5} \cdot \begin{bmatrix} e_1 & e_2 \\ e_4 & e_5 \end{bmatrix}_{Je_3} \cdot \begin{bmatrix} e_1 & e_2 \\ e_5 & e_3 \end{bmatrix}_{Je_4} = 1$$

for all  $e_1, \dots, e_5 \in E$  and  $J \subset E$  such that  $(Je_5; e_1, e_2, e_3, e_4)$ ,  $(Je_3; e_1, e_2, e_4, e_5)$  and  $(Je_4; e_1, e_2, e_5, e_3)$  are in  $\Omega_M$ ;

$$(R5) \quad \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{Je_5} = \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{Je_6}$$

for all  $e_1, \dots, e_6 \in E$  and  $J \subset E$  such that  $\langle Je_5 \rangle = \langle Je_6 \rangle$  and such that  $(Je_5; e_1, e_2, e_3, e_4)$  and  $(Je_6; e_1, e_2, e_3, e_4)$  are in  $\Omega_M^\diamond$ ; as well as the additive Plücker relations

$$(R+) \quad \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_J + \begin{bmatrix} e_1 & e_3 \\ e_2 & e_4 \end{bmatrix}_J = 1$$

for all  $(J; e_1, \dots, e_4) \in \Omega_M^\diamond$ .

PROOF. By Proposition 3.6, we have  $\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_J = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}$  for every  $(J; e_1, \dots, e_4) \in \Omega_M$  and  $H_i = \langle Je_i \rangle$  for  $i = 1, \dots, 4$ . Thus (R−)–(R3) follow from (H−)–(H3), and (R+) follows from (H+). To see that (R4) implies (H4), define for given  $j_1, \dots, j_{r-3}, e_1, \dots, e_5 \in E$  and  $J = \{j_1, \dots, j_{r-3}\}$  as in (R4) the corank 2 flats  $F_i = \langle Je_i \rangle$  for  $i = 1, \dots, 5$ , which are pairwise distinct and contain the common flat  $\langle J \rangle$  of corank 3, as required. For  $i \neq j$ , we define hyperplanes  $H_{ij} = \langle F_i \cup F_j \rangle = \langle Je_i e_j \rangle$ . Then we have for all identifications  $\{i, j, k\} = \{3, 4, 5\}$  that

$$\begin{bmatrix} e_1 & e_2 \\ e_i & e_j \end{bmatrix}_{Je_k} = \begin{bmatrix} H_{1k} & H_{2k} \\ H_{ik} & H_{jk} \end{bmatrix},$$

which shows that (H4) implies (R4). The relation (R5) follows from

$$\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{Je_5} = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} = \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{Je_6},$$

where  $H_i = \langle Je_5 e_i \rangle = \langle Je_6 e_i \rangle$  is the  $i$ -th coefficient of the common image  $(H_1, \dots, H_4)$  of  $(Je_5; e_1, \dots, e_4)$  and  $(Je_6; e_1, \dots, e_4)$  under  $\Psi : \Omega_M \rightarrow \Theta_M$ .

We are left to show that (R–)–(R5) imply that  $\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_J = \begin{bmatrix} e'_1 & e'_2 \\ e'_3 & e'_4 \end{bmatrix}_{J'}$  if  $\Psi(J; e_1, \dots, e_4) = \Psi(J'; e'_1, \dots, e'_4)$ , i.e. if  $\langle J e_i \rangle = \langle J' e'_i \rangle$  for  $i = 1, \dots, 4$ . We will prove this by replacing one element of  $J e_1 \dots e_4$  by an element of  $J' e'_1 \dots e'_4$  at a time. Note that both  $J$  and  $J'$  are bases of the restriction  $M|_F = M \setminus (E - F)$ , where  $F = \langle J \rangle = \langle J' \rangle$  is the flat of rank  $r - 2$  generated by  $J$  and  $J'$ . Since the basis exchange graph of  $M|_F$  is connected, we find a sequence  $J = J_0, J_1, \dots, J_{s-1}, J_s = J'$  of bases for  $M|_F$  such that  $J_k = I_k j_k$  and  $J_{k+1} = I_k j'_k$  for  $I_k = J_k \cap J_{k+1}$  and some  $j_k \in J_k$  and  $j'_k \in J_{k+1}$ . Considered as subsets of  $M$ , we have  $\langle J_k \rangle = F$  and thus  $(J_k; e'_1, \dots, e'_4) \in \Omega_M^\diamond$  for all  $k = 0, \dots, s$ . Thus we can apply (R5), which yields

$$\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{J_k} = \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{I_k j_k} = \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{I_k j'_k} = \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{J_{k+1}}.$$

We conclude that  $\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_J = \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{J'}$ .

Next we replace the  $e_i$  by the  $e'_i$ , one at a time. After permuting rows and columns appropriately, which does not change the value of the cross ratio by (R $\sigma$ ), we are reduced to studying cross ratios of the forms  $\begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \end{bmatrix}_{J'}$  and  $\begin{bmatrix} f_1 & f_2 \\ f_3 & f'_4 \end{bmatrix}_{J'}$  such that  $\langle J' f_4 \rangle = \langle J' f'_4 \rangle$  is a hyperplane. By (R3), we have

$$\begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \end{bmatrix}_{J'} \cdot \begin{bmatrix} f_1 & f_2 \\ f_4 & f'_4 \end{bmatrix}_{J'} \cdot \begin{bmatrix} f_1 & f_2 \\ f'_4 & f_3 \end{bmatrix}_{J'} = 1.$$

Since  $\langle J' f_4 \rangle = \langle J' f'_4 \rangle$  is a hyperplane, the subset  $J' f_4 f'_4$  of  $M$  has rank  $r - 1$  and is not a basis of  $M$ . Thus  $\begin{bmatrix} f_1 & f_2 \\ f_4 & f'_4 \end{bmatrix}_{J'} = 1$  by (R0), which shows that

$$\begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \end{bmatrix}_{J'} = \begin{bmatrix} f_1 & f_2 \\ f'_4 & f_3 \end{bmatrix}_{J'}^{-1} = \begin{bmatrix} f_1 & f_2 \\ f_3 & f'_4 \end{bmatrix}_{J'},$$

where we use (R1) for the last equality. We conclude that

$$\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_J = \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{J'} = \begin{bmatrix} e'_1 & e'_2 \\ e'_3 & e'_4 \end{bmatrix}_{J'},$$

as desired. This completes the proof of the theorem.  $\square$

**COROLLARY 4.22.** *The foundation  $F_M$  of a matroid  $M$  is naturally isomorphic to a quotient*

$$F_M \simeq \left( \bigotimes_{\substack{N \rightarrow M \\ \text{of type } U_4^2}} F_N \right) // S$$

of a tensor product of foundations  $F_N \simeq \mathbb{U}$ , where the set  $S$  is generated by the relations of type (R–) in the presence of an  $F_7$  or  $F_7^*$ -minor and of types (R3)–(R5) that are induced by embedded minors  $M \setminus I/J \rightarrow M$  on at most 6 elements  $\{e_1, \dots, e_6\} = E - (I \cup J)$ .

**PROOF.** By Theorem 4.21, the foundation is generated by the universal cross ratios  $\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_J$  of  $M$ , which are the images  $\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_J = \varphi_{M \setminus I/J} \left( \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix} \right)$  of the universal cross ratios  $\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}$  of minors  $N = M \setminus I/J$  on 4 elements  $e_1, \dots, e_4$ ; cf. Proposition 4.9. The morphisms  $\varphi_{M \setminus I/J} : F_N \rightarrow F_M$  testify that all relations of  $F_N$  also hold in  $F_M$ , and therefore we conclude that  $F_M$  is of the form

$$F_M \simeq \left( \bigotimes_{\substack{N \rightarrow M \\ \text{with } \#E_N = 4}} F_N \right) // S$$



for some set of 3-term relations  $S$ , where  $E_N$  denotes the ground set of  $N$ . *A priori*, this holds if we include all relations (R $-$ )–(R $+$ ) of Theorem 4.21 in  $S$ . To reduce this to the assertion of the corollary, we observe the following.

If  $N = M \setminus I/J$  is a minor on 4 elements that is not of type  $U_4^2$ , then  $N$  is regular and  $F_N = \mathbb{F}_1^\pm$ . Thus we can omit these factors from the tensor product. Note that (R0) assures that the cross ratios coming from such a minor are trivial in  $F_M$ . Therefore we can omit (R0) from  $S$ .

Each of (R $\sigma$ ), (R1), (R2) and (R $+$ ) involve only cross ratios that come from the same  $U_4^2$ -minor  $N = M \setminus I/J$ . Therefore the analogous relations hold already in  $F_N$ , and we can omit them from the set  $S$ .

By Theorem 4.21, the relation (R $-$ ) holds if  $M$  has a minor of type  $F_7$  or  $F_7^*$ . Each of the relations (R3)–(R5) involve cross ratios that come from the same minor on 5 or 6 elements. This shows all assertions of the corollary.  $\square$

#### 4.11. A presentation of the foundation by embedded minors

Let  $N = M \setminus I/J$  and  $N' = M \setminus I'/J'$  be two embedded minors of a matroid  $M$ . If  $I' \subset I$  and  $J' \subset J$ , then  $N = N' \setminus (I - I')/(J - J')$  is an embedded minor of  $N'$ . We write  $\iota : N \rightarrow N'$  for the inclusion as embedded minors and  $\iota_* : F_N \rightarrow F_{N'}$  for the induced morphism between the respective foundations.

**THEOREM 4.23.** *Let  $M$  be a matroid with foundation  $F_M$ . Let  $\mathcal{E}$  be the collection of all embedded minors  $N = M \setminus I/J$  of  $M$  on at most 7 elements with the following properties:*

- *if  $N$  has at most 6 elements, then it contains a minor of type  $U_4^2$ ;*
- *if  $N$  has exactly 6 elements, then it contains two parallel elements;*
- *if  $N$  has 7 elements, then it is isomorphic to  $F_7$  or  $F_7^*$ .*

*Then*

$$F_M \simeq \left( \bigotimes_{N \in \mathcal{E}} F_N \right) // S,$$

*where the set  $S$  is generated by the relations  $a = \iota_*(a)$  for every inclusion  $\iota : N \rightarrow N'$  of embedded minors  $N$  and  $N'$  in  $\mathcal{E}$ .*

**PROOF.** It is clear that the morphisms  $\varphi_{M \setminus I/J} : F_{M \setminus I/J} \rightarrow F_M$  from Proposition 4.9 induce a canonical morphism  $\left( \bigotimes_{N \in \mathcal{E}} F_N \right) // S \rightarrow F_M$ , and since  $\mathcal{E}$  contains all embedded  $U_4^2$ -minors of  $M$ , this morphism is surjective. Thus we are left with showing that  $S$  contains all defining relations of  $M$ .

Let us define  $\mathcal{E}_i = \{N \in \mathcal{E} \mid \#E_N = i\}$  for  $i = 4, \dots, 7$  where  $E_N$  denotes the ground set of the embedded minor  $N$ . Then  $\mathcal{E} = \mathcal{E}_4 \cup \dots \cup \mathcal{E}_7$ . The set  $\mathcal{E}_4$  consists of the embedded  $U_4^2$ -minors of  $M$ , and thus

$$F_M \simeq \left( \bigotimes_{N \in \mathcal{E}_4} F_N \right) // S'$$

by Corollary 4.22, where  $S'$  contains all relations of types (R $-$ ) (in the presence of an  $F_7$  or  $F_7^*$ -minor) and (R3)–(R5).

The relations (R3) and (R4) stem from embedded minors  $N = M \setminus I/J$  on 5 elements, and these relations involve a nondegenerate cross ratio only if  $N$  contains a  $U_4^2$ -minor, i.e.  $N \in \mathcal{E}_5$ . Thus (R3) and (R4) can be replaced by tensoring with

$F_N$  and including the relations  $a = \iota_*(a)$  for every minor embedding  $\iota : N' = N \setminus I'/J' \rightarrow N$  with  $N' \in \mathcal{E}_4$ .

Similarly, (R5) stems from embedded minors  $N = M \setminus I/J$  on 6 elements with two parallel elements, and involves a nondegenerate cross ratio only if  $N$  contains a  $U_4^2$ -minor, i.e.  $N \in \mathcal{E}_6$ . Thus (R5) can be replaced by tensoring with  $F_N$  and including the relations  $a = \iota_*(a)$  for every minor embedding  $\iota : N' = N \setminus I'/J' \rightarrow N$  with  $N' \in \mathcal{E}_4$ .

The set  $\mathcal{E}_7$  consists of all embedded minors of types  $F_7$  and  $F_7^*$ . Since  $F_{F_7} = F_{F_7^*} = \mathbb{F}_2$  and  $P // \langle 1 = -1 \rangle \simeq P \otimes \mathbb{F}_2$  for every pasture  $P$ , we can replace the relation (R-) by  $- \otimes F_N$  if  $N \in \mathcal{E}_7$ . This recovers all relations in  $S'$  and completes the proof.  $\square$



## CHAPTER 5

### The structure theorem

In this section, we prove the central result of this paper, Theorem 5.9, which asserts that the foundation of a matroid  $M$  without large uniform minors is isomorphic to a tensor product of finitely many copies of the pastures  $\mathbb{U}$ ,  $\mathbb{D}$ ,  $\mathbb{H}$ ,  $\mathbb{F}_3$  and  $\mathbb{F}_2$ .

This is done by first showing that in the absence of large uniform minors, the tip and cotip relations are of a particularly simple form, which eventually leads to the conclusion that the foundation of  $M$  is the tensor product of quotients of  $\mathbb{U}$  by automorphism groups, and possibly  $\mathbb{F}_2$ . The quotients of  $\mathbb{U}$  by automorphisms are precisely  $\mathbb{U}$ ,  $\mathbb{D}$ ,  $\mathbb{H}$  and  $\mathbb{F}_3$ .

#### 5.1. Foundations of matroids on 5 elements

By Theorem 4.23, the foundation of a matroid is determined completely by its minors on at most 5 elements and the embedded minors on 6 elements with two parallel elements.

In this section, we will determine the foundations of all matroids on at most 5 elements. Most of these matroids are regular and have foundation  $\mathbb{F}_1^\perp$  by [5, Thm. 7.33]. There is only a small number of non-regular matroids on at most 5 elements, which we will inspect in detail.

Let  $0 \leq r \leq n \leq 5$  and  $M$  be a matroid of rank  $r$  on  $E = \{1, \dots, n\}$ .

**5.1.1. Regular matroids.** A matroid  $M$  is regular if and only if there is no nontrivial cross ratio, which is the case if and only if the matroid  $M$  does not contain any minor of type  $U_4^2$ .

This is the case in exactly one of the following situations: (a)  $r \in \{0, 1, n-1, n\}$ ; (b)  $n = 4$ ,  $r = 2$  and  $M$  is not uniform; (c)  $n = 5$ ,  $r = 2$  and  $M \setminus i$  is not uniform for every  $i \in E$ ; (d)  $n = 5$ ,  $r = 3$  and  $M/i$  is not uniform for every  $i \in E$ .

**5.1.2. Matroids with exactly one embedded  $U_4^2$ -minor.** There are several isomorphism classes of matroids with exactly one  $U_4^2$ -minor, which we list in the following.

Since the tip and cotip relations involve cross ratios from different embedded  $U_4^2$ -minors, they do not appear for matroids with only one embedded  $U_4^2$ -minor.

If  $n = 4$ , then there is exactly one such matroid, namely  $M = U_4^2$  itself, which has foundation  $\mathbb{U}$  by Proposition 4.11.

**PROPOSITION 5.1.** *Let  $M$  be a matroid on 5 elements with exactly one embedded  $U_4^2$ -minor. Then  $M$  is isomorphic to  $U_4^2 \oplus N$  where  $N$  is a matroid on 1 element. The foundation of  $M$  is isomorphic to  $\mathbb{U}$ .*

**PROOF.** In order to have an  $U_4^2$ -minor,  $M$  must have rank 2 or 3. Since the embedded minors  $N \rightarrow M$  of  $M$  correspond bijectively to the embedded minors

$N^* \rightarrow M^*$  and since  $U_4^2$  is self-dual, the matroids  $M$  and  $M^*$  have the same number of  $U_4^2$ -minors. Once we have shown that every rank 2-matroid with exactly one embedded  $U_4^2$ -minor is isomorphic to  $U_4^2 \oplus N$  for a matroid  $N$  on one element, which has to be of rank 0, then we can conclude that  $M^*$  is isomorphic to  $U_4^2 \oplus N^*$ . To complete this reduction to the rank 2-case, we note that the foundation of  $M^*$  is canonically isomorphic to the foundation of  $M$ , cf. Proposition 4.8.

Assume that the rank 2-matroid  $M$  on  $E = \{1, \dots, 5\}$  has an embedded  $U_4^2$ -minor. After a permutation of  $E$ , we can assume that this embedded  $U_4^2$ -minor is  $M \setminus 5 = M \setminus \{5\}$ , i.e. that all of the following 2-subsets

$$\{1, 2\}, \quad \{1, 3\}, \quad \{1, 4\}, \quad \{2, 3\}, \quad \{2, 4\} \quad \text{and} \quad \{3, 4\}$$

of  $E$  are bases. If these are all bases of  $M$ , then 5 is a loop and  $M$  is isomorphic to  $U_4^2 \oplus N$ , as claimed.

We indicate why  $M$  cannot have more bases of the form  $\{i, 5\}$ . If  $M$  has exactly one additional basis element, say  $\{1, 5\}$ , then the basis exchange property is violated by exchanging 1 by an element of the basis  $\{3, 4\}$ . The same reason excludes the possibility that  $M$  has exactly two additional basis elements, say  $\{1, 5\}$  and  $\{2, 5\}$ . If  $M$  has 9 or more basis elements, say all 2-subsets of  $E$  but possibly  $\{4, 5\}$ , then both minors  $M \setminus 4$  and  $M \setminus 5$  are isomorphic to  $U_4^2$ . Thus in this case,  $M$  has at least two embedded  $U_4^2$ -minors.

This shows that  $M$  has to be isomorphic to  $U_4^2 \oplus N$ . Since 5 is a loop, the conditions for the tip relations are not satisfied, which means that all relations stem from the unique embedded  $U_4^2$ -minor  $M \setminus 5$ . This shows that the foundation of  $M$  is isomorphic to  $F_{M \setminus 5} \simeq \mathbb{U}$ , as claimed.  $\square$

**5.1.3. Matroids with exactly two embedded  $U_4^2$ -minors.** If  $M$  has two embedded  $U_4^2$ -minors, then the ground set must be  $E = \{1, \dots, 5\}$ . As explained in Section 5.1.2,  $M$  must have rank 2 or 3 if  $M$  has an  $U_4^2$ -minor. We will show that if  $M$  has exactly two embedded  $U_4^2$ -minors, then it must be isomorphic to the following matroid, or its dual.

**DEFINITION 5.2.** We denote by  $C_5$  the rank 3-matroid on  $E = \{1, \dots, 5\}$  whose set of bases is  $\binom{E}{3} - \{3, 4, 5\}$ .

**PROPOSITION 5.3.** A matroid  $M$  on 5 elements has exactly two embedded  $U_4^2$ -minors if and only if  $M$  is isomorphic to either  $C_5$  or its dual. The cross ratios of  $C_5$  satisfy

$$\begin{bmatrix} i & j \\ k & 4 \end{bmatrix}_5 = \begin{bmatrix} i & j \\ k & 5 \end{bmatrix}_4,$$

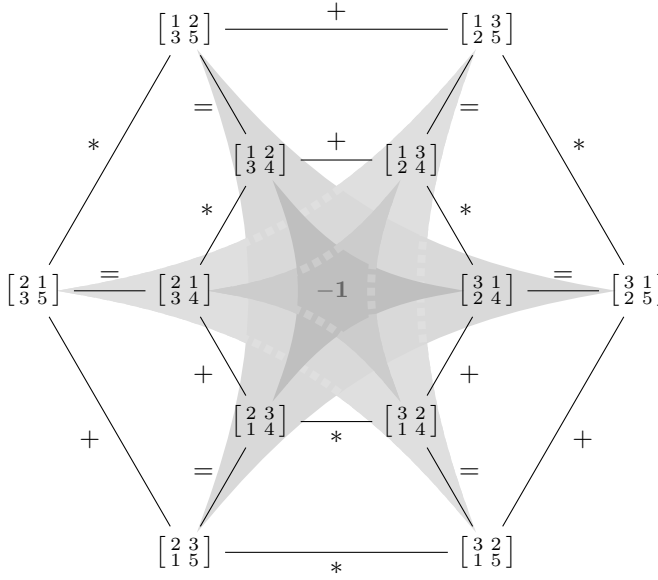
and the cross ratios of  $C_5^*$  satisfy

$$\begin{bmatrix} i & j \\ k & 4 \end{bmatrix} = \begin{bmatrix} i & j \\ k & 5 \end{bmatrix}$$

for all identifications  $\{i, j, k\} = \{1, 2, 3\}$ . The foundations of both  $C_5$  and  $C_5^*$  are isomorphic to  $\mathbb{U}$ .

We illustrate all non-degenerate cross ratios of  $C_5^*$  and their relations in Figure 5.1.

**PROOF.** The claims for  $C_5$  and  $C_5^*$  follow from each other by the duality result Proposition 4.8. Thus we can restrict ourselves to the proof of all claims for  $C_5^*$ .

FIGURE 5.1. The cross ratios of  $C_5^*$  and their relations

In the proof of Proposition 5.1, we saw that  $C_5^*$  has at least two embedded  $U_4^2$ -minors, which correspond to the  $U_4^2$ -minors  $C_5^* \setminus 4$  and  $C_5^* \setminus 5$ . All other minors of rank 2 on 4 elements of  $C_5^*$  are of the form  $C_5^* \setminus i$  for  $i \in \{1, 2, 3\}$ . But since  $\{4, 5\}$  is not a basis of  $C_5^*$ , none of these minors is isomorphic to  $U_4^2$ . This shows that  $C_5^*$  has exactly two embedded  $U_4^2$ -minors, as has every matroid  $M$  that is isomorphic to  $C_5^*$ .

Conversely, assume that  $M$  is a rank 2 matroid on 5 elements with exactly two embedded  $U_4^2$ -minors. Since duality preserves  $U_4^2$ -minors, can assume that  $M$  is of rank 2. After a permutation of  $E$ , we can assume that these two embedded  $U_4^2$ -minors are  $M \setminus 4$  and  $M \setminus 5$ . Thus all of the 2-subsets

$$\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\} \text{ and } \{3, 5\}$$

are bases. If  $\{4, 5\}$  was also a basis of  $M$ , then  $M$  would be the uniform matroid  $U_5^2$ , which has five  $U_4^2$  minors  $U_5^2 \setminus i$  for  $i = 1, \dots, 5$ . Thus  $M$  is isomorphic to  $C_5^*$ . This proves our first claim.

Let us choose an identification  $\{i, j, k\} = \{1, 2, 3\}$ . The tip relation (R3) in Theorem 4.21 with tip  $\{i, j\}$  and cyclic orientation  $(k, 4, 5)$  for  $C_5^*$  is

$$\begin{bmatrix} i & j \\ k & 4 \end{bmatrix} \cdot \begin{bmatrix} i & j \\ 4 & 5 \end{bmatrix} \cdot \begin{bmatrix} i & j \\ 5 & k \end{bmatrix} = 1.$$

Since  $\begin{bmatrix} i & j \\ 4 & 5 \end{bmatrix} = 1$  is degenerate, we obtain the claimed relation

$$\begin{bmatrix} i & j \\ k & 4 \end{bmatrix} = \begin{bmatrix} i & j \\ 5 & k \end{bmatrix}^{-1} = \begin{bmatrix} i & j \\ k & 5 \end{bmatrix},$$

where the second equality is relation (R1). Since  $C_5^*$  is a parallel extension of  $U_4^2$ , the foundation of  $C_5^*$  is  $\mathbb{U}$  by Corollary 4.10, which concludes the proof.  $\square$

**5.1.4. Matroids with five embedded  $U_4^2$ -minors.** The only matroids on at most five elements that do not appear among the previous cases with at most two embedded  $U_4^2$ -minors are the uniform matroids  $U_5^2$  and  $U_5^3$ , which have five embedded  $U_4^2$ -minors.

For completeness, we describe their foundations. However, we postpone the proof to a sequel to this paper where we develop more sophisticated methods to calculate the foundations of matroids. Note that the results of this first part are independent from the following result since we consider matroids without large uniform minors.

PROPOSITION 5.4. *The foundations of  $U_5^2$  and  $U_5^3$  are isomorphic to*

$$\mathbb{F}_1^\pm \langle x_1, \dots, x_5 \rangle // \{x_i + x_{i-1}x_{i+1} - 1 \mid i = 1, \dots, 5\}$$

where  $x_0 = x_5$  and  $x_6 = x_1$ .

## 5.2. Symmetry quotients

The classification of foundations of matroids on up to five elements in Section 5.1 shows that in a matroid without large uniform minors, all relations between cross ratios of different embedded  $U_4^2$ -minors arise from minors of type  $C_5$  or  $C_5^*$ . Proposition 5.3 shows that these types of minors identify the two hexagons of cross ratios, which implies an identification of two copies of the near-regular partial field  $\mathbb{U}$ ; cf. Figure 5.1. The same happens for relations of type R5: they identify two copies of  $\mathbb{U}$ .

It can, and it will, happen that a matroid contains a chain of such minors, which creates a self-identification of the cross ratios belonging to an embedded  $U_4^2$ -minor of  $M$ . By Proposition 5.3, this self-identification must respect the relations between the cross ratios in each hexagon, and induces an automorphism of  $\mathbb{U}$ . Therefore we are led to study the quotients of  $\mathbb{U}$  by such automorphisms.

**5.2.1. Automorphisms of the near-regular partial field.** In the following, we determine all automorphisms of the near-regular partial field  $\mathbb{U} = \mathbb{F}_1^\pm \langle x, y \rangle // \{x + y = 1\}$ . By Lemma 4.13, it suffices to determine the images of  $x$  and  $y$  to describe an automorphism of  $\mathbb{U}$ . A result equivalent to the following is also proved in [25, Lemma 4.4].

LEMMA 5.5. *The elements of the form  $z + z' - 1$  in the nullset  $N_{\mathbb{U}}$  of  $\mathbb{U}$  with  $z, z' \in \mathbb{U}^\times$  are*

$$x + y - 1, \quad x^{-1} - x^{-1}y - 1 \quad \text{and} \quad y^{-1} - xy^{-1} - 1.$$

*Thus the fundamental elements of  $\mathbb{U}$  are  $x, y, x^{-1}, -x^{-1}y, y^{-1}, -xy^{-1}$ .*

PROOF. Note that the only element  $z$  with  $z + 1 - 1 = 0$  is  $z = 0$ . Thus to find all fundamental elements, it suffices to search for relations of the form  $z + z' - 1 \in N_{\mathbb{U}}$  with  $z, z' \in \mathbb{U}^\times$ . Since  $N_{\mathbb{U}}$  is generated by  $1 - 1 + 0$  and  $x + y - 1$ , and since all terms have to be nonzero and at least one term has to be equal to  $-1$  to find a relation for fundamental elements, we find exactly three relations of the form  $z + z' - 1 = 0$ , which are

$$x + y - 1, \quad x^{-1} - x^{-1}y - 1 \quad \text{and} \quad y^{-1} - xy^{-1} - 1.$$

Thus the claim of the lemma. □

PROPOSITION 5.6. *The associations*

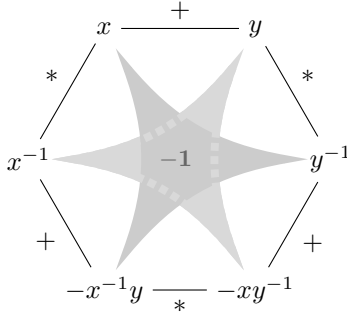
$$\begin{array}{ccc} \rho: \mathbb{U} & \longrightarrow & \mathbb{U} \\ x & \longmapsto & y^{-1} \\ y & \longmapsto & -xy^{-1} \end{array} \quad \text{and} \quad \begin{array}{ccc} \sigma: \mathbb{U} & \longrightarrow & \mathbb{U}, \\ x & \longmapsto & y \\ y & \longmapsto & x \end{array}$$

define automorphisms of  $\mathbb{U}$  that generate the automorphism group of  $\mathbb{U}$  and satisfy the relations  $\rho^3 = \sigma^2 = (\rho\sigma)^2 = \text{id}$ . In particular,  $\text{Aut}(\mathbb{U}) \simeq S_3$ .

PROOF. By Lemma 5.5, both  $(y^{-1}, -xy^{-1})$  and  $(y, x)$  are pairs of fundamental elements in  $\mathbb{U}$ . Thus, by Lemma 4.13,  $\rho$  and  $\sigma$  define morphisms from  $\mathbb{U}$  to  $\mathbb{U}$ . Since  $\rho^3(x) = x$  and  $\rho^3(y) = y$ , we conclude that  $\rho$  defines a group automorphism of  $\mathbb{U}^\times$  of order 3. Similarly,  $\sigma$  defines a group automorphism of  $\mathbb{U}^\times$  of order 2. The relation  $(\rho\sigma)^2 = \text{id}$  can be easily verified by evaluation on  $x$  and  $y$ .

We conclude that the automorphism group of  $\mathbb{U}$  contains  $\langle \rho, \sigma \rangle \simeq S_3$  as a subgroup. By Lemma 5.5,  $\mathbb{U}$  contains precisely 6 fundamental elements, which implies easily that  $\text{Aut}(\mathbb{U})$  is generated by  $\rho$  and  $\sigma$ .  $\square$

REMARK 5.7. It follows from Lemma 5.5 that the isomorphism  $F_{U_4^2} \rightarrow \mathbb{U}$  from Proposition 4.11 maps the cross ratios of  $U_4^2$  bijectively to the fundamental elements of  $\mathbb{U}$ . We can arrange these fundamental elements in a hexagon



in the same way as we arrange the cross ratios in Figure 4.1. It follows from Proposition 5.6 that the automorphisms of  $\mathbb{U}$  correspond bijectively to the symmetries of this hexagon that preserve the edge labels and the inner triangles.

**5.2.2. Classification of the symmetry quotients of  $\mathbb{U}$ .** A *symmetry quotient* of  $\mathbb{U}$  is the quotient of  $\mathbb{U}$  by a group of automorphisms. More precisely, if  $H$  is a subgroup of  $\text{Aut}(\mathbb{U})$ , then the *quotient of  $\mathbb{U}$  by  $H$*  is

$$\mathbb{U}/H = \mathbb{U} // \{x = \tau(x), y = \tau(y) \mid \tau \in H\}.$$

In fact, we have  $\mathbb{U}/H = \mathbb{U} // \{x = \tau(x), y = \tau(y) \mid \tau \in S\}$  if  $S$  is a set of generators of  $H$ .

Recall from section 2.1.2 that  $\mathbb{F}_3 = \mathbb{F}_1^\pm // \{1 + 1 + 1\}$ ,

$$\mathbb{D} = \mathbb{F}_1^\pm \langle z \rangle // \{z + z - 1\} \quad \text{and} \quad \mathbb{H} = \mathbb{F}_1^\pm \langle z \rangle // \{z^3 + 1, z - z^2 - 1\}.$$

Note that this implies that  $z^3 = -1$  and  $z^6 = 1$  in  $\mathbb{H}$ .

PROPOSITION 5.8. *The symmetry quotients of  $\mathbb{U}$  are, up to isomorphism,*

$$\mathbb{U}/\langle \text{id} \rangle \simeq \mathbb{U}, \quad \mathbb{U}/\langle \sigma \rangle \simeq \mathbb{D}, \quad \mathbb{U}/\langle \rho \rangle \simeq \mathbb{H}, \quad \mathbb{U}/\langle \rho, \sigma \rangle \simeq \mathbb{F}_3.$$



PROOF. In the following, we show that the quotients of  $\mathbb{U}$  by different subgroups  $H$  of  $\text{Aut}(\mathbb{U}) \simeq S_3$  are exactly the pastures  $\mathbb{U}$ ,  $\mathbb{D}$ ,  $\mathbb{H}$  and  $\mathbb{F}_3$ , up to isomorphism. Clearly  $\mathbb{U} = \mathbb{U}/\langle \text{id} \rangle$  is the quotient of  $\mathbb{U}$  by the trivial subgroup.

Note that if  $H'$  is a subgroup conjugate to  $H$ , i.e.  $H' = \tau H \tau^{-1}$  for some  $\tau \in \text{Aut}(\mathbb{U})$ , then the quotient of  $\mathbb{U}$  by  $H'$  equals the quotient of  $\tau(\mathbb{U}) = \mathbb{U}$  by  $H$ . This means that it suffices to determine the isomorphism classes of the quotients of  $\mathbb{U}$  by the groups  $\langle \sigma \rangle$ ,  $\langle \rho \rangle$  and  $\text{Aut}(\mathbb{U}) = \langle \rho, \sigma \rangle$ , which represent all conjugacy classes of nontrivial subgroups of  $\text{Aut}(\mathbb{U})$ .

Let  $H = \langle \sigma \rangle$ . We denote the residue classes of  $x$  and  $y$  in  $\mathbb{U}/\langle \sigma \rangle$  by  $\bar{x}$  and  $\bar{y}$ , respectively. We claim that the association

$$\begin{array}{ccc} f : \mathbb{U}/\langle \sigma \rangle & \longrightarrow & \mathbb{D} \\ \bar{x} & \longmapsto & z \\ \bar{y} & \longmapsto & z \end{array}$$

defines an isomorphism of pastures. We begin with the verification that  $f$  defines a morphism. The map  $\hat{f} : \mathbb{U} \rightarrow \mathbb{D}$  with  $\hat{f}(x) = \hat{f}(y) = z$  is a morphism, since the generator  $x + y - 1$  of the nullset of  $\mathbb{U}$  is mapped to  $z + z - 1$ , which is in the nullset of  $\mathbb{D}$ . Since  $\hat{f}(\sigma(x)) = z = \hat{f}(x)$  and  $\hat{f}(\sigma(y)) = z = \hat{f}(y)$ , the morphism  $\hat{f}$  induces a morphism  $f : \mathbb{U}/\langle \sigma \rangle \rightarrow \mathbb{D}$  by the universal property of the quotient  $\mathbb{U}/\langle \sigma \rangle = \mathbb{U} // \{\sigma(x) = y, \sigma(y) = x\}$ , cf. Proposition 2.6.

We define the inverse to  $f$  as the association  $g : z \mapsto \bar{x}$ . This defines a multiplicative map since  $\mathbb{D}^\times$  is freely generated by  $z$ . Since

$$g(z) + g(z) - 1 = \bar{x} + \bar{x} - 1 = \bar{x} + \bar{y} - 1$$

is null in  $\mathbb{U}/\langle \sigma \rangle$ , this defines a morphism  $g : \mathbb{D} \rightarrow \mathbb{U}/\langle \sigma \rangle$ . It is obvious that  $g$  is an inverse to  $f$ , which shows that  $f$  is an isomorphism.

We continue with the automorphism group  $H = \langle \rho \rangle$ . We claim that the association

$$\begin{array}{ccc} f : \mathbb{U}/\langle \rho \rangle & \longrightarrow & \mathbb{H} \\ \bar{x} & \longmapsto & z \\ \bar{y} & \longmapsto & -z^2 \end{array}$$

defines an isomorphism of pastures. We begin with the verification that  $f$  defines a morphism. The map  $\hat{f} : \mathbb{U} \rightarrow \mathbb{H}$  with  $\hat{f}(x) = z$  and  $\hat{f}(y) = -z^2$  is a morphism, since the generator  $x + y - 1$  of the nullset of  $\mathbb{U}$  is mapped to  $z - z^2 - 1$ , which is in the nullset of  $\mathbb{H}$ . Since  $\hat{f}(\rho(x)) = \hat{f}(y^{-1}) = z = \hat{f}(x)$  and  $\hat{f}(\rho(y)) = \hat{f}(-xy^{-1}) = -z^2 = \hat{f}(y)$ , the morphism  $\hat{f}$  induces a morphism  $f : \mathbb{U}/\langle \rho \rangle \rightarrow \mathbb{H}$  by the universal property of the quotient  $\mathbb{U}/\langle \rho \rangle = \mathbb{U} // \{\rho(x) = y, \rho(y) = x\}$ .

We define the inverse of  $f$  as follows. Let  $\hat{g} : \mathbb{F}_1^\pm \langle z \rangle \rightarrow \mathbb{U}/\langle \rho \rangle$  be the morphism that maps  $z$  to  $\bar{x}$ . The defining relations of  $\mathbb{U}/\langle \rho \rangle$  are  $\bar{x} = \bar{y}^{-1}$  and  $\bar{y} = -\bar{x}\bar{y}^{-1}$ . Thus

$$\hat{g}(z^3) + \hat{g}(1) = \bar{x}^3 + 1 = \bar{y}^{-2}\bar{x} + 1 = -\bar{x}^{-1}\bar{y}\bar{y}^{-1}\bar{x} + 1 = -1 + 1,$$

which is in the nullset of  $\mathbb{U}/\rho$ . Since  $z^3 = -1$  in  $\mathbb{H}$ , we have  $-z^2 = z^{-1}$  and thus

$$\hat{g}(z) + \hat{g}(-z^2) - 1 = \bar{x} + \bar{x}^{-1} - 1 = \bar{x} + \bar{y} - 1,$$

which is also in the nullset of  $\mathbb{U}/\langle \rho \rangle$ . This shows that the morphism  $\hat{g}$  defines a morphism  $g : \mathbb{H} \rightarrow \mathbb{U}/\langle \rho \rangle$ , which is obviously inverse to  $f$ .

Finally we show that  $\mathbb{U}/\langle\rho, \sigma\rangle$  is isomorphic to  $\mathbb{F}_3$ . Since  $\mathbb{U}/\langle\rho, \sigma\rangle \simeq (\mathbb{U}/\langle\rho\rangle)/\langle\sigma\rangle$ , it suffices to show that the association

$$\begin{array}{ccc} f: \mathbb{H}/\langle\sigma\rangle & \longrightarrow & \mathbb{F}_3 \\ \bar{z} & \longmapsto & -1 \end{array}$$

is an isomorphism. Since  $\sigma(z) = \sigma(\bar{x}) = \bar{y} = z^{-1}$  and  $f(\bar{z}) = f(\bar{z}^{-1})$ , and since  $f(z^6) = (-1)^6 = 1 = f(1)$ , the assignment  $f(\bar{z}) = -1$  extends to a multiplicative map. Since  $f(z^3) + f(1) = (-1)^3 + 1 = -1 + 1$  and  $f(z) + f(-z^2) - 1 = -1 - 1 - 1$  are null in  $\mathbb{F}_3$ , the map  $f$  is a morphism. Note that in  $\mathbb{H}/\langle\sigma\rangle$ , we have  $\bar{z}^3 = -1$  and  $\bar{z} = \bar{z}^{-1}$ , and thus  $\bar{z} = -1$ . We conclude that the assignment  $g: 1 \mapsto 1 = -\bar{z}$  defines a morphism  $g: \mathbb{F}_3 \rightarrow \mathbb{H}/\langle\sigma\rangle$ , since

$$g(1) + g(1) + g(1) = 1 + 1 + 1 = -(\bar{z} - \bar{z}^2 - 1)$$

is null in  $\mathbb{H}/\langle\sigma\rangle$ . It is clear that  $g$  is an inverse of  $f$ , which shows that  $f$  is an isomorphism. This concludes the proof of the proposition.  $\square$

### 5.3. The structure theorem for matroids without large uniform minors

We are prepared to prove the central result of this paper. In the following, the empty tensor product stands for the initial object in Pastures, which is  $\mathbb{F}_1^\pm$ .

**THEOREM 5.9.** *Let  $M$  be a matroid without large uniform minors and  $F_M$  its foundation. Then*

$$F_M \simeq F_1 \otimes \cdots \otimes F_r$$

for some  $r \geq 0$  and pastures  $F_1, \dots, F_r \in \{\mathbb{U}, \mathbb{D}, \mathbb{H}, \mathbb{F}_3, \mathbb{F}_2\}$ .

**PROOF.** Let  $\mathcal{E}$  be the collection of embedded minors  $N$  of  $M$  from Theorem 4.23. Then

$$F_M \simeq \left( \bigotimes_{N \in \mathcal{E}} F_N \right) // S,$$

where the set  $S$  is generated by the relations  $a = \iota_*(a)$  for every inclusion  $\iota: N \rightarrow N'$  of embedded minors  $N$  and  $N'$  in  $\mathcal{E}$ .

From the analysis in section 5.1, it follows that the foundation  $F_N$  of every embedded minor  $N$  of  $M$  with at most 5 elements is either  $\mathbb{F}_1^\pm$  or  $\mathbb{U}$ , where we use the assumption that  $M$  is without minors of types  $U_5^2$  and  $U_5^3$ . A matroid with foundation  $\mathbb{F}_1^\pm$  is regular and has thus no minor of type  $U_4^2$ . We conclude that every embedded minor in  $\mathcal{E}$  on at most 5 elements has foundation  $\mathbb{U}$ .

If an embedded minor  $N$  in  $\mathcal{E}$  has 6 elements, and thus two of them are parallel, then deleting one of these parallel elements yields an embedded minor  $N' = N \setminus e$  of  $N$ , and the induced morphism  $F_{N'} \rightarrow F_N$  is an isomorphism. Thus also every embedded minor in  $\mathcal{E}$  with 6 elements has foundation  $\mathbb{U}$ .

Since neither  $F_7$  nor  $F_7^*$  contains a minor of type  $\mathbb{U}$ , an embedded minor  $N$  in  $\mathcal{E}$  with 7 elements cannot contain another embedded minor  $N'$  in  $\mathcal{E}$ . Consequently the isomorphism of Theorem 4.23 implies that

$$F_M \simeq \bigotimes_{N \in \mathcal{E}_7} F_N \otimes \left( \bigotimes_{N \in \mathcal{E}'} F_N \right) // S',$$

where  $\mathcal{E}_7$  is the subset of  $\mathcal{E}$  that contains all embedded minors with 7 elements,  $\mathcal{E}'$  is the subset of  $\mathcal{E}$  that contains all embedded minors with at most 6 elements and

$S$  is the set generated by the relations  $a = \iota_*(a)$  for every inclusion  $\iota : N \rightarrow N'$  of embedded minors  $N$  and  $N'$  in  $\mathcal{E}'$ .

By what we have seen, an inclusion  $N \rightarrow N'$  of embedded minors in  $\mathcal{E}'$  is an isomorphism, and either foundation is isomorphic to  $\mathbb{U}$ . Thus all identifications in  $S'$  stem from isomorphisms between some factors  $F_N$  of the tensor product. What can, and does, happen is that a chain of such isomorphisms imposes a self-identification of a factor  $F_N \simeq \mathbb{U}$  with itself by a non-trivial automorphism. This leads to a symmetry quotient of  $\mathbb{U}$ , which is one of  $\mathbb{U}$ ,  $\mathbb{D}$ ,  $\mathbb{H}$  and  $\mathbb{F}_3$ . Thus

$$\left( \bigotimes_{N \in \mathcal{E}'} F_N \right) // S'$$

is a tensor product of copies of  $\mathbb{U}$ ,  $\mathbb{D}$ ,  $\mathbb{H}$  and  $\mathbb{F}_3$ .

This leaves us with the factors  $F_N$  for  $N \in \mathcal{E}_7$ . By Theorem 4.21, we have  $-1 = 1$ , and all cross ratios are trivial since there are no  $U_4^2$ -minors. Thus  $F_N \simeq \mathbb{F}_1^\pm // \{1 = -1\} = \mathbb{F}_2$ . This concludes the proof of the theorem.  $\square$

Theorem 5.9 can be reformulated as follows, which expresses the dependencies of the factors  $F_i$  on  $M$ .

**COROLLARY 5.10.** *Let  $M$  be a matroid without large uniform minors,  $F_M$  its foundation. Then*

$$F_M \simeq F_0 \otimes F_1 \otimes \cdots \otimes F_r$$

for a uniquely determined  $r \geq 0$  and uniquely determined pastures  $F_0 \in \{\mathbb{F}_1^\pm, \mathbb{F}_2, \mathbb{F}_3, \mathbb{K}\}$  and  $F_1, \dots, F_r \in \{\mathbb{U}, \mathbb{D}, \mathbb{H}\}$ , up to a permutation of the indices  $1, \dots, r$ . We have  $F_0 = \mathbb{F}_2$  or  $F_0 = \mathbb{K}$  if and only if  $M$  contains a minor of type  $F_7$  or  $F_7^*$ .

**PROOF.** By Theorem 5.9, the foundation  $F_M$  of a matroid  $M$  without large uniform minors is isomorphic to a tensor product of copies of  $\mathbb{U}$ ,  $\mathbb{D}$ ,  $\mathbb{H}$ ,  $\mathbb{F}_3$  and  $\mathbb{F}_2$ .

Since morphisms from  $\mathbb{F}_2$  and  $\mathbb{F}_3$  into other pastures are uniquely determined, if they exist, we conclude that  $\mathbb{F}_2 \otimes \cdots \otimes \mathbb{F}_2 = \mathbb{F}_2$  and  $\mathbb{F}_3 \otimes \cdots \otimes \mathbb{F}_3 = \mathbb{F}_3$ . Thus the pasture

$$\underbrace{\mathbb{F}_2 \otimes \cdots \otimes \mathbb{F}_2}_{r \text{ times}} \otimes \underbrace{\mathbb{F}_3 \otimes \cdots \otimes \mathbb{F}_3}_{s \text{ times}}$$

is isomorphic to

$$\mathbb{F}_1^\pm \text{ if } r = s = 0; \quad \mathbb{F}_2 \text{ if } r > s = 0; \quad \mathbb{F}_3 \text{ if } s > r = 0; \quad \mathbb{F}_2 \otimes \mathbb{F}_3 = \mathbb{K} \text{ if } r, s > 0;$$

cf. Example 2.8 for the equality  $\mathbb{F}_2 \otimes \mathbb{F}_3 = \mathbb{K}$ . This explains the list of possible isomorphism types for  $F_0$ . Since  $\mathbb{F}_2$  appears as a factor of  $F_M$  if and only if  $M$  has a minor of type  $F_7$  or  $F_7^*$ , this verifies the last claim of the corollary.

It follows that  $F_M$  is isomorphic to a tensor product of  $F_0$  with pastures  $F_1, \dots, F_r \in \{\mathbb{U}, \mathbb{D}, \mathbb{H}\}$ .

We are left with establishing the uniqueness claims. To begin with,  $F_0$  is uniquely determined by the presence or absence of the relations  $1 + 1 = 0$  and  $1 + 1 + 1 = 0$ , which correspond to the relations  $r > 0$  and  $s > 0$ , respectively, in our previous case consideration. Thus  $F_0$  is uniquely determined.

The factors  $F_1, \dots, F_r$  are determined by the fundamental elements of  $F_M$ , as we explain in the following. Let  $\iota_i : F_i \rightarrow \bigotimes F_j \simeq F_M$  be the canonical inclusion. By the construction of the tensor product, the nullset of  $F_M$  consists of all terms of the form  $d\iota_i(a) + d\iota_i(b) + d\iota_i(c)$  for some  $i \in \{0, \dots, r\}$ ,  $d \in \bigotimes F_j$  and  $a, b, c \in F_i$  such that  $a + b + c$  is in the nullset of  $F_i$ . The fundamental elements of  $F_M$  stem

from such equations for which  $d\iota_i(a)$  and  $d\iota_i(b)$  are nonzero and  $d\iota_i(c) = -1$ . Thus  $d = -\iota_i(c)^{-1} = \iota_i(-c^{-1})$  is in the image of  $\iota_i$ , and therefore  $d\iota_i(a) = \iota_i(-c^{-1}a)$  and  $d\iota_i(b) = \iota_i(-c^{-1}b)$ . Since  $-c^{-1}a - c^{-1}b - 1$  is in the nullset of  $F_i$ , we conclude that all fundamental elements in  $F_M$  are of the form  $\iota_i(z)$  for some  $i$  and some fundamental element  $z$  of  $F_i$ .

To make a distinction between the different isomorphism types of the factors, we note that every fundamental element  $x$  with relation  $x + y - 1 = 0$  gives rise to a set  $\{x, x^{-1}, y, y^{-1}, -x^{-1}y, -xy^{-1}\}$  of fundamental elements. If these six fundamental elements come from a factor  $F_i \simeq \mathbb{U}$ , then they are pairwise different. If they come from a factor  $F_i \simeq \mathbb{D}$ , then

$$\{x, x^{-1}, y, y^{-1}, -x^{-1}y, -xy^{-1}\} = \{x, y^{-1}, -x^{-1}y\}$$

is a set with three distinct elements. If they come from a factor  $F_i \simeq \mathbb{D}$ , then

$$\{x, x^{-1}, y, y^{-1}, -x^{-1}y, -xy^{-1}\} = \{x, y\}$$

is a set with two distinct elements. Note that if  $F_0 = \mathbb{F}_3$  or  $F_0 = \mathbb{K}$ , then  $x = -1$  is also a fundamental element, and in this case  $x^{-1} = y = y^{-1} = -x^{-1}y = -xy^{-1} = -1$  are all equal. This shows that the number of factors of types  $\mathbb{U}$ ,  $\mathbb{D}$  and  $\mathbb{H}$  are determined by the fundamental elements of  $F_M$ , which completes the proof of our uniqueness claims.  $\square$

REMARK 5.11. In a sequel to this paper, we will show that for all  $r \geq 0$  and  $F_1, \dots, F_r \in \{\mathbb{U}, \mathbb{D}, \mathbb{H}, \mathbb{F}_3, \mathbb{F}_2\}$ , there is a matroid  $M$  without large uniform minors whose foundation is isomorphic to the tensor product  $F_1 \otimes \dots \otimes F_r$ .



## CHAPTER 6

# Applications

In this concluding part of the paper, we explain various applications of our central result Theorem 5.9. Along with some new results and strengthenings of known facts, we also present short conceptual proofs for a number of established theorems which illustrate the versatility of our structure theory for foundations.

The main technique in most of the upcoming proofs is the following. A matroid  $M$  is representable over a pasture  $P$  if and only there is a morphism from the foundation  $F_M$  of  $M$  to  $P$ . If  $M$  is without large uniform minors, then we know by Theorem 5.9 that  $F_M$  is isomorphic to the tensor product of copies  $F_i$  of  $\mathbb{U}$ ,  $\mathbb{D}$ ,  $\mathbb{H}$ ,  $\mathbb{F}_3$  and  $\mathbb{F}_2$ . Thus a morphism from  $F_M$  to  $P$  exists if and only there is a morphism from each  $F_i$  to  $P$ , which in practice is quite easy to determine.

For reference in the later sections, we will provide some general criteria for such morphisms in the following result, and list the outcome for a series of prominent pastures in Table 6.1.

LEMMA 6.1. *Let  $P$  be a pasture.*

- (1) *There is a morphism  $\mathbb{U} \rightarrow P$  if and only if  $P$  contains a fundamental element. For a field  $k$ , this is the case if and only if  $\#k \geq 3$ .*
- (2) *There is a morphism  $\mathbb{D} \rightarrow P$  if and only if there is an element  $u \in P^\times$  such that  $u + u = 1$ . For a field  $k$ , this is the case if and only if  $\text{char } k \neq 2$ .*
- (3) *There is a morphism  $\mathbb{H} \rightarrow P$  if and only if there is an element  $u \in P^\times$  such that  $u^3 = -1$  and  $u - u^2 = 1$ . For a field  $k$ , this is the case if and only if  $\text{char } k = 3$  or if  $k$  contains a primitive third root of unity.*
- (4) *There is a morphism  $\mathbb{F}_3 \rightarrow P$  if and only if  $1 + 1 + 1 = 0$  in  $P$ . For a field  $k$ , this is the case if and only if  $\text{char } k = 3$ .*
- (5) *There is a morphism  $\mathbb{F}_2 \rightarrow P$  if and only if  $-1 = 1$  in  $P$ . For a field  $k$ , this is the case if and only if  $\text{char } k = 2$ .*

*There exist morphisms from  $\mathbb{U}$ ,  $\mathbb{D}$ ,  $\mathbb{H}$ ,  $\mathbb{F}_3$  and  $\mathbb{F}_2$  into the pastures  $\mathbb{U}$ ,  $\mathbb{D}$ ,  $\mathbb{H}$ ,  $\mathbb{F}_q$  for  $q = 2, \dots, 8$ ,  $\mathbb{Q}$ ,  $\mathbb{C}$ ,  $\mathbb{S}$ ,  $\mathbb{P}$  and  $\mathbb{W}$  where Table 6.1 contains a check mark—a dash indicates that there is no morphism.*

PROOF. We briefly indicate the reasons for claims (1)–(5). We begin with claim (1). The universal property from Proposition 2.6 implies that there is a morphism from  $\mathbb{U} = \mathbb{F}_1^\pm \langle x, y \rangle // \{x + y - 1\}$  to  $P$  if and only if there are  $u, v \in P$  such that  $u + v = 1$ . By definition, such elements are fundamental elements of  $P$ . If  $P = k$  is a field, then a pair  $(u, v)$  of fundamental elements is a point of the line  $L = \{(w, 1 - w) | w \in k\}$  in  $k^2$ . Since  $L$  contains precisely two points  $(0, 1)$  and  $(1, 0)$  with vanishing coordinates, the elements of  $L \cap (k^\times)^2$  are in bijection with  $k - \{0, 1\}$ . Thus  $k$  has a fundamental element if and only if  $\#k \geq 3$ .

TABLE 6.1. Existence of morphisms from  $\mathbb{U}$ ,  $\mathbb{D}$ ,  $\mathbb{H}$ ,  $\mathbb{F}_3$  and  $\mathbb{F}_2$  into other pastures

	$\mathbb{U}$	$\mathbb{D}$	$\mathbb{H}$	$\mathbb{F}_2$	$\mathbb{F}_3$	$\mathbb{F}_4$	$\mathbb{F}_5$	$\mathbb{F}_7$	$\mathbb{F}_8$	$\mathbb{Q}$	$\mathbb{C}$	$\mathbb{S}$	$\mathbb{P}$	$\mathbb{W}$
$\mathbb{U}$	✓	✓	✓	—	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
$\mathbb{D}$	—	✓	—	—	✓	—	✓	✓	—	✓	✓	✓	✓	✓
$\mathbb{H}$	—	—	✓	—	✓	✓	—	✓	—	—	✓	—	✓	✓
$\mathbb{F}_3$	—	—	—	—	✓	—	—	—	—	—	—	—	—	✓
$\mathbb{F}_2$	—	—	—	✓	—	✓	—	—	✓	—	—	—	—	—

We continue with claim (2). The first assertion follows at once from the universal property for  $\mathbb{D} = \mathbb{F}_1^\pm \langle z \rangle // \{z + z - 1\}$ . A field  $P = k$  contains an element  $u$  with  $u + u = 1$  if and only if  $1 + 1$  is invertible in  $k$ , which is the case if and only if  $k$  is of characteristic different from 2.

We continue with claim (3). The first assertion follows at once from the universal property for  $\mathbb{H} = \mathbb{F}_1^\pm \langle z \rangle // \{z^3 - 1, z - z^2 - 1\}$ . In a field  $P = k$  of characteristic 3, the element  $u = -1$  satisfies  $u^3 = -1$  and  $u - u^2 = 1$ . If  $k$  has characteristic different from 3, then  $v = -u$  satisfies the equation  $v^2 + v + 1 = 0$ , which characterizes a primitive third root of unity. Note that we have automatically  $u^3 = -v^3 = -1$  in a field if  $v$  is a third root of unity.

Claims (4) and (5) are obvious. The existence or non-existence of morphisms as displayed in Table 6.1 can be easily verified using (1)–(5).  $\square$

### 6.1. Forbidden minors for regular, binary and ternary matroids

The techniques of this paper allow for short arguments to re-establish the known characterizations of regular, binary and ternary matroids in terms of forbidden minors, as they have been proven by Tutte in [32] for regular and binary matroids, and independently by Bixby in [6] and by Seymour in [30] for ternary matroids.

We spell out the following basic fact for its importance for many of the upcoming theorems.

LEMMA 6.2. *Binary matroids and ternary matroids are without large uniform minors.*

PROOF. All minors of a binary or ternary matroid are binary or ternary, respectively. Since  $U_5^2$  and  $U_5^3$  are neither binary nor ternary, the result follows.  $\square$

Next we turn to the proofs of the excluded minor characterizations of regular, binary and ternary matroids.

THEOREM 6.3 (Tutte '58). *A matroid is regular if and only if it contains no minor of types  $U_4^2$ ,  $F_7$  or  $F_7^*$ . A matroid is binary if and only if it contains no minor of type  $U_4^2$ .*

PROOF. By Corollary 4.14,  $U_4^2$  is not binary and therefore also not regular. It follows from Theorem 4.21 that the foundations of  $F_7$  and  $F_7^*$  contain the relation  $-1 = 1$ , which means that they do not admit a morphism to  $\mathbb{F}_1^\pm$ . Thus  $F_7$  and  $F_7^*$  are not regular.

We are left with showing that the respective lists of forbidden minors are complete. If a matroid  $M$  does not contain a minor of type  $U_4^2$ , then Corollary 4.22

implies that the foundation  $F_M$  of  $M$  is equal to  $\mathbb{F}_1^\pm$  or  $\mathbb{F}_1^\pm // \{-1 = 1\} = \mathbb{F}_2$ . In either case, there is a morphism from  $F_M$  to  $\mathbb{F}_2$ , which shows that  $M$  is binary if it has no minor of type  $U_4^2$ .

If, in addition,  $M$  has no minor of types  $F_7$  or  $F_7^*$ , then Corollary 4.22 implies that  $F_M = \mathbb{F}_1^\pm$ , and thus  $M$  is regular.  $\square$

**THEOREM 6.4** (Bixby '79, Seymour '79). *A matroid is ternary if and only if it does not contain a minor of type  $U_5^2$ ,  $U_5^3$ ,  $F_7$  or  $F_7^*$ .*

**PROOF.** If  $M$  is ternary, then it does not have a minor of type  $U_5^2$  or  $U_5^3$  by Lemma 6.2. Thus Theorem 4.21 applies, and since  $-1 \neq 1$  in  $\mathbb{F}_3$ ,  $M$  does not have a minor of type  $F_7$  or  $F_7^*$ . This establishes all forbidden minors as listed in the theorem.

To show that the list of forbidden minors is complete, we assume that  $M$  contains no minors of these types. Then Corollary 5.10 implies that the foundation of  $M$  is isomorphic to  $F_1 \otimes \cdots \otimes F_r$  with  $F_i \in \{\mathbb{U}, \mathbb{D}, \mathbb{H}, \mathbb{F}_3\}$ . Since each of  $\mathbb{U}$ ,  $\mathbb{D}$ ,  $\mathbb{H}$ ,  $\mathbb{F}_3$  admits a morphism to  $\mathbb{F}_3$ , there is a morphism  $F_M \rightarrow \mathbb{F}_3$ , which shows that  $M$  is ternary.  $\square$

## 6.2. Uniqueness of the rescaling class over $\mathbb{F}_3$

Brylawski and Lucas show in [11] that a representation of a matroid over  $\mathbb{F}_3$  is uniquely determined up to rescaling. Our method yields a short proof of the following generalization.

**THEOREM 6.5.** *Let  $P$  be a pasture with at most one fundamental element. Then every matroid has at most one rescaling class over  $P$ .*

**PROOF.** Let  $M$  be a matroid with foundation  $F_M$ . Since the rescaling classes of  $M$  over  $P$  are in bijective correspondence with the morphisms  $F_M \rightarrow P$ , it suffices to show that there is at most one such morphism.

By Proposition 3.11, every cross ratio of  $F_M$  is a fundamental element of  $F_M$ , and thus must be mapped to a fundamental element  $z$  of  $P$ . By the uniqueness of  $z$  (if it exists), the image of every cross ratio is uniquely determined. Since  $F_M$  is generated over  $\mathbb{F}_1^\pm$  by cross ratios, the result follows.  $\square$

**REMARK 6.6.** Examples of pastures with at most one fundamental element are  $\mathbb{F}_1^\pm$ ,  $\mathbb{F}_2$ ,  $\mathbb{F}_3$  and  $\mathbb{K}$ . In fact it is not hard to prove that every pasture with at most one fundamental element contains one of these pastures as a subpasture, and that the fundamental element is  $-1$  (if it exists). Note that Brylawski and Lucas's theorem concerns the case  $P = \mathbb{F}_3$ .

## 6.3. Criteria for representability over certain fields

Our theory allows us to deduce at once that matroids without large minors that are representable over certain pastures are automatically representable over certain (partial) fields. For instance, we find such criteria in the cases of the sign hyperfield  $\mathbb{S}$ , the phase hyperfield  $\mathbb{P}$  and the weak sign hyperfield  $\mathbb{W}$ .

Note that the proof of Criterion (1) in the following theorem strengthens Lee and Scobee's result that every ternary and orientable matroid is dyadic; see [18, Cor. 1]. In fact, we further improve on this result in Theorem 6.9 where we show that every orientation is uniquely liftable to  $\mathbb{D}$  up to rescaling.



In the statement of the following theorem, recall that a matroid is said to be *weakly orientable* if it is representable over  $\mathbb{W}$ .

**THEOREM 6.7.** *Let  $M$  be a matroid without large uniform minors.*

- (1) *If  $M$  is orientable, then it is representable over every field of characteristic different from 2.*
- (2) *If  $M$  is representable over  $\mathbb{P}$ , then it is representable over fields of every characteristic except possibly 2.*
- (3) *If  $M$  is weakly orientable, then it is ternary.*

**PROOF.** Let  $F_M$  be the foundation of  $M$  and  $F_M \simeq F_1 \otimes \cdots \otimes F_r$  the decomposition from Theorem 5.9 into factors  $F_i \in \{\mathbb{U}, \mathbb{D}, \mathbb{H}, \mathbb{F}_3, \mathbb{F}_2\}$ . If  $M$  is representable over a pasture  $P$ , then there is a morphism  $F_M \rightarrow P$ , and thus there is a morphism  $F_i \rightarrow P$  for every  $i = 1, \dots, r$ . Conversely, if one of the building blocks  $\mathbb{U}$ ,  $\mathbb{D}$ ,  $\mathbb{H}$ ,  $\mathbb{F}_3$  and  $\mathbb{F}_2$  does not map to  $P$ , we conclude that this building block does not occur among the  $F_i$ .

Claim (1) follows since there are no morphisms from  $\mathbb{H}$ ,  $\mathbb{F}_3$  or  $\mathbb{F}_2$  to  $\mathbb{S}$ , and both  $\mathbb{U}$  and  $\mathbb{D}$  map to every field of characteristic different from 2. Claim (2) follows since there are no morphisms from  $\mathbb{F}_3$  or  $\mathbb{F}_2$  to  $\mathbb{P}$ , and since each of  $\mathbb{U}$ ,  $\mathbb{D}$  and  $\mathbb{H}$  maps to a field  $k$  if its characteristic is 3 or if it is different from 2 and if  $k$  contains a primitive third root of unity. Claim (3) follows since there is no morphism from  $\mathbb{F}_2$  to  $\mathbb{W}$ , and each of  $\mathbb{U}$ ,  $\mathbb{D}$ ,  $\mathbb{H}$  and  $\mathbb{F}_3$  maps to  $\mathbb{F}_3$ .  $\square$

**REMARK 6.8.** The proof of Theorem 6.7 shows that similar conclusions can be formulated for other pastures  $P$  that do not receive morphisms from some of the building blocks of the foundation  $F_M$  of a matroid  $M$  without large uniform minors. If  $M$  is representable over  $P$ , then we can conclude the following, for instance:

- if there is no morphism from  $\mathbb{D}$  to  $P$ , then  $M$  is quaternary;
- if there is no morphism from either  $\mathbb{F}_2$  or  $\mathbb{D}$  to  $P$ , then  $M$  is hexagonal.

#### 6.4. Oriented matroids without large minors are uniquely dyadic

Our techniques allow us to strengthen the result of Lee and Scobee ([18, Thm. 1]) that an oriented matroid is dyadic if its underlying matroid is ternary. At the end of this section, we deduce Lee and Scobee's result from ours.

An *oriented matroid* is an  $\mathbb{S}$ -matroid, i.e. the class  $M = [\Delta]$  of a Grassmann-Plücker function  $\Delta : E^r \rightarrow \mathbb{S}$ , where  $r$  is the rank of  $M$  and  $E$  its ground set. The *underlying matroid* of  $M$  is the matroid  $\underline{M} = t_{\mathbb{S},*}(M)$ , where  $t_{\mathbb{S}} : \mathbb{S} \rightarrow \mathbb{K}$  is the terminal morphism, cf. section 2.1.3. Recall that a reorientation class is a rescaling class over  $\mathbb{S}$ .

Let  $\text{sign} : \mathbb{D} \rightarrow \mathbb{S}$  be the morphism from the dyadic partial field  $\mathbb{D} = \mathbb{F}_1^\pm \langle z \rangle // \{z + z - 1\}$  to  $\mathbb{S}$  that maps  $z$  to 1. An oriented matroid  $M = [\Delta]$  is *dyadic* if there is a  $\mathbb{D}$ -matroid  $\widehat{M}$  such that  $M = \text{sign}_*(\widehat{M})$ . We call  $\widehat{M}$  a *lift of  $M$  along  $\text{sign} : \mathbb{D} \rightarrow \mathbb{S}$* .

**THEOREM 6.9.** *Let  $M$  be an oriented matroid whose underlying matroid  $\underline{M}$  is without large uniform minors. Then there is a unique rescaling class  $[\widehat{M}]$  of dyadic matroids such that  $\text{sign}_*(\widehat{M}) = M$ .*

**PROOF.** Let  $F_{\underline{M}}$  be the foundation of  $\underline{M}$ . The oriented matroid  $M$  determines a reorientation class  $[M]$  and thus a morphism  $f : F_{\underline{M}} \rightarrow \mathbb{S}$ . Since rescaling classes of  $\underline{M}$  over  $\mathbb{D}$  correspond bijectively to morphisms  $F_{\underline{M}} \rightarrow \mathbb{D}$ , we need to show that

the morphism  $f : F_{\underline{M}} \rightarrow \mathbb{S}$  lifts uniquely to  $\mathbb{D}$ , i.e. that there is a unique morphism  $\hat{f} : F_{\underline{M}} \rightarrow \mathbb{D}$  such that the diagram

$$\begin{array}{ccc} F_{\underline{M}} & \xrightarrow{\hat{f}} & \mathbb{D} \\ & \searrow f & \downarrow \text{sign} \\ & & \mathbb{S} \end{array}$$

commutes.

Note that this implies only that there is a unique rescaling class  $[\widehat{M}]$  such that the reorientation classes  $[\text{sign}_*(\widehat{M})]$  and  $[M]$  are equal. In order to conclude that we can choose  $\widehat{M}$  such that  $\text{sign}_*(\widehat{M}) = M$ , we note that the morphism  $\text{sign} : \mathbb{D} \rightarrow \mathbb{S}$  is surjective, and thus any reorientation  $M' = \text{sign}_*(\widehat{M})$  of  $M$  can be inverted by a rescaling of  $\widehat{M}$  over  $\mathbb{D}$ . This shows that we have proven everything, once we show that  $f$  lifts uniquely to  $\mathbb{D}$ .

Since  $\underline{M}$  is without large uniform minors, Theorem 5.9 implies that  $F_{\underline{M}}$  is isomorphic to  $F_1 \otimes \cdots \otimes F_r$  for some  $F_1, \dots, F_r \in \{\mathbb{U}, \mathbb{D}, \mathbb{H}, \mathbb{F}_3, \mathbb{F}_2\}$ . Composing  $f : F_{\underline{M}} \rightarrow \mathbb{S}$  with the canonical inclusions  $\iota_i : F_i \rightarrow F_{\underline{M}}$  yields morphisms  $f_i = f \circ \iota_i : F_i \rightarrow \mathbb{S}$  for  $i = 1, \dots, r$ . As visible in Table 6.1, there are no morphisms from  $\mathbb{H}$ ,  $\mathbb{F}_3$  or  $\mathbb{F}_2$  to  $\mathbb{S}$ . This means that  $F_1, \dots, F_r \in \{\mathbb{U}, \mathbb{D}\}$ .

By the universal property of the tensor product, the morphisms  $F_{\underline{M}} \rightarrow \mathbb{D}$  correspond bijectively to the tuples of morphisms  $f_i : F_i \rightarrow \mathbb{D}$ . Thus there is a unique lift of  $f$  to  $\mathbb{D}$  if and only if for every  $i$ , there is a unique lift of  $f_i$  to  $\mathbb{D}$ . This reduces our task to an inspection of the two cases  $F_i = \mathbb{D}$  and  $F_i = \mathbb{U}$ .

Consider the case  $f_i : F_i = \mathbb{D} \rightarrow \mathbb{S}$ . Since  $z + z = 1$  in  $\mathbb{D}$ , we must have  $f(z) + f(z) = 1$  in  $\mathbb{S}$ , which is only possible if  $f(z) = 1$ . Thus  $f_i = \text{sign}$ , which means that the identity morphism  $\hat{f}_i = \text{id} : \mathbb{D} \rightarrow \mathbb{D}$  lifts  $f_i$ , i.e.

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{\hat{f}_i = \text{id}} & \mathbb{D} \\ & \searrow f_i & \downarrow \text{sign} \\ & & \mathbb{S} \end{array}$$

commutes. This lift is unique since  $u + u = 1$  is only satisfied by  $u = z \in \mathbb{D}$ , and thus  $\hat{f}_i(z) = z$  is determined.

We are left with the case  $f_i : F_i = \mathbb{U} \rightarrow \mathbb{S}$ , for which we inspect the possible images of the fundamental elements  $x$  and  $y$  of  $\mathbb{U}$  in  $\mathbb{S}$  and  $\mathbb{D}$ . The relations of the form  $u + v - 1 = 0$  in  $\mathbb{S}$  are  $1 + 1 - 1 = 0$  and  $1 - 1 - 1 = 0$ . Thus  $f_i$  maps  $(x, y)$  to one of  $(1, 1)$ ,  $(1, -1)$  and  $(-1, 1)$ . This means that there are precisely 3 morphisms  $\mathbb{U} \rightarrow \mathbb{S}$ , and  $f_i$  has to be one of them.

The relations of the form  $u + v - 1 = 0$  in  $\mathbb{D}$  are  $z + z - 1 = 0$  and  $z^{-1} - 1 - 1 = 0$ . Thus the morphisms  $\mathbb{U} \rightarrow \mathbb{U}$  correspond to a choice of mapping  $(x, y)$  to one of  $(z, z)$ ,  $(z^{-1}, -1)$  and  $(-1, z^{-1})$ . Considering the respective images  $\text{sign}(z) = \text{sign}(z^{-1}) = 1$  and  $\text{sign}(-1) = -1$  in  $\mathbb{S}$ , we conclude that every morphism  $f_i : \mathbb{U} \rightarrow \mathbb{S}$  lifts uniquely to a morphism  $\hat{f}_i : \mathbb{U} \rightarrow \mathbb{D}$ , i.e.

$$\begin{array}{ccc} \mathbb{U} & \xrightarrow{\hat{f}_i} & \mathbb{D} \\ & \searrow f_i & \downarrow \text{sign} \\ & & \mathbb{S} \end{array}$$

commutes. This completes the proof of the theorem.  $\square$

As an application, we show how Theorem 6.9 implies the result [18, Thm. 1] of Lee and Scobee.

**THEOREM 6.10** (Lee–Scobee ’99). *An oriented matroid is dyadic if and only if its underlying matroid is ternary.*

**PROOF.** Let  $M$  be an oriented matroid and let  $\underline{M}$  be its underlying matroid. If  $\underline{M}$  is ternary, then it is without large uniform minors. Thus  $M$  is dyadic by Theorem 6.9.

Conversely, assume that  $M$  is dyadic, i.e. it has a lift  $\widehat{M}$  along  $\text{sign} : \mathbb{D} \rightarrow \mathbb{S}$ . Since there is a morphism  $f : \mathbb{D} \rightarrow \mathbb{F}_3$ , and since  $t_{\mathbb{F}_3} \circ f = t_{\mathbb{S}} \circ \text{sign}$ , the  $\mathbb{F}_3$ -matroid  $f_*(\widehat{M})$  is a representation of  $\underline{M} = t_{\mathbb{S},*}(M)$  over  $\mathbb{F}_3$ . Thus  $\underline{M}$  is ternary.  $\square$

### 6.5. Positively oriented matroids without large uniform minors are near-regular

In their 2017 paper [2], Ardila, Rincón and Williams prove that every positively oriented matroid can be represented over  $\mathbb{R}$  (and *a posteriori*, by a theorem of Postnikov, over  $\mathbb{Q}$ ), which solves a conjecture from da Silva’s thesis [13] from 1987. A second proof has recently been obtained by Speyer and Williams in [31]. Neither of these proofs yields information about the structure of the lifts of positive orientations to  $\mathbb{Q}$  or  $\mathbb{R}$ .

With our techniques, we can recover and strengthen the result for positively oriented matroids whose underlying matroid is without large uniform minors. To begin with, let us recall the definition of positively oriented matroids.

**DEFINITION 6.11.** Let  $M$  be a matroid of rank  $r$  on the ground set  $E = \{1, \dots, n\}$ . A *positive orientation of  $M$  (with respect to  $E$ )* is a Grassmann–Plücker function  $\Delta : E^r \rightarrow \mathbb{S}$  such that  $t_{*,\mathbb{S}}([\Delta]) = M$  and such that  $\Delta(j_1, \dots, j_r) \in \{0, 1\}$  for every  $(j_1, \dots, j_r) \in E^r$  with  $j_1 < \dots < j_r$ .

An oriented matroid  $M$  of rank  $r$  on  $E$  is *positively oriented* if its underlying matroid has a positive orientation  $\Delta : E^r \rightarrow \mathbb{S}$  with respect to some identification  $E \simeq \{1, \dots, n\}$  such that  $M = [\Delta]$ .

A key tool for proof of Theorem 6.15 is the following notion.

**DEFINITION 6.12.** Let  $M$  be a matroid of rank  $r$  on the ground set  $E = \{1, \dots, n\}$ . Let  $V$  be the Klein 4-group, considered as a subgroup of  $S_4$ . The  *$\Omega$ -signature of  $M$  (with respect to  $E$ )* is the map

$$\Sigma : \Omega_M^\diamond \longrightarrow S_4/V$$

that sends  $(J; e_1, \dots, e_4) \in \Omega_M^\diamond$  to the class  $[\epsilon] \in S_4/V$  of the uniquely determined permutation  $\epsilon \in S_4$  that

$$\begin{array}{ccc} \{e_1, \dots, e_4\} & \longrightarrow & \{1, \dots, 4\} \\ e_i & \longmapsto & \epsilon(i) \end{array}$$

is an order-preserving bijection.

**EXAMPLE 6.13.** The key example to understand the relevance of the  $\Omega$ -signature is the uniform matroid  $M = U_4^2$ , whose foundation is  $F_M = \mathbb{U}$ . In this case,  $\Omega_M^\diamond$  consists of the tuples  $(\emptyset; e_1, \dots, e_4)$  for which  $(e_1, \dots, e_4)$  is a permutation of  $(1, \dots, 4)$ . Since the cross ratio  $\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix} \in F_M$  determines  $(e_1, e_2, e_3, e_4)$  up to a permutation

in  $V$ , which corresponds to a permutation of the rows and the columns of the cross ratio, the  $\Omega$ -signature induces a well-defined bijection

$$\begin{aligned} \left\{ \text{cross ratios in } F_M \right\} &\longrightarrow S_4/V \\ \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix} &\longmapsto \Sigma(\emptyset; e_1, \dots, e_4). \end{aligned}$$

LEMMA 6.14. *Let  $M$  be a matroid of rank  $r$  on the ground set  $E = \{1, \dots, n\}$  and let  $\Delta : E^r \rightarrow \mathbb{S}$  be a positive orientation of  $M$ . Let  $(J; e_1, \dots, e_4) \in \Omega_M^\diamond$  and  $\epsilon \in S_4$  be such that  $[\epsilon] = \Sigma(J; e_1, \dots, e_4)$ . Then*

$$\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{\Delta, J} = (-1)^{\epsilon(1)+\epsilon(2)+1}.$$

PROOF. Choose  $\mathbf{J} = (j_1, \dots, j_{r-2}) \in E^{r-2}$  so that  $|\mathbf{J}| = J$ . Since  $\Delta$  is a positive orientation, we have for all  $i \in \{1, 2\}$  and  $j \in \{3, 4\}$  that  $\Delta(\mathbf{J}e_ie_j) = \text{sign } \pi_{i,j}$ , where  $\pi_{i,j} : \mathbf{J}e_ie_j \rightarrow \mathbf{J}e_ie_j$  is the unique permutation such that

$$\pi_{i,j}(j_1) < \dots < \pi_{i,j}(j_{r-2}) < \pi_{i,j}(e_i) < \pi_{i,j}(e_j).$$

Since the cross ratio  $\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{\Delta, J}$  is invariant under permutations of  $J$ , we can assume that  $j_1 < \dots < j_{r-2}$ . Thus we can write  $\pi_{i,j} = \sigma_{i,j} \circ \epsilon_{i,j}$  as the composition of  $\sigma_{i,j} = \pi_{i,j} \circ \epsilon_{i,j}^{-1}$  with the permutation  $\epsilon_{i,j}$  of  $\mathbf{J}e_ie_j$  that fixes  $j_1, \dots, j_{r-2}$  and satisfies  $\epsilon_{i,j}(e_i) < \epsilon_{i,j}(e_j)$ . A minimal decomposition of  $\sigma_{i,j}$  into transpositions is

$$\sigma_{i,j} = (j_{k_j} e_j) \cdots (j_{r-2} e_j) (j_{k_i} e_i) \cdots (j_{r-2} e_i),$$

where  $k_i$  is such that  $j_{k_i-1} < e_i < j_{k_i}$ . Thus

$$\text{sign}(\sigma_{i,j}) = (-1)^{(r-1-k_i)+(r-1-k_j)} = (-1)^{k_i+k_j},$$

and

$$\begin{aligned} \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{\Delta, J} &= \frac{\Delta(\mathbf{J}e_1e_3)\Delta(\mathbf{J}e_2e_4)}{\Delta(\mathbf{J}e_1e_4)\Delta(\mathbf{J}e_2e_3)} \\ &= \frac{\text{sign}(\pi_{1,3})\text{sign}(\pi_{2,4})}{\text{sign}(\pi_{1,4})\text{sign}(\pi_{2,3})} \\ &= \frac{(-1)^{k_1+k_3}(-1)^{k_2+k_4}}{(-1)^{k_1+k_4}(-1)^{k_2+k_3}} \cdot \frac{\text{sign}(\epsilon_{1,3})\text{sign}(\epsilon_{2,4})}{\text{sign}(\epsilon_{1,4})\text{sign}(\epsilon_{2,3})} \\ &= \text{sign}(\epsilon_{1,3})\text{sign}(\epsilon_{2,4})\text{sign}(\epsilon_{1,4})\text{sign}(\epsilon_{2,3}). \end{aligned}$$

Since the parity of  $\epsilon'(1) + \epsilon'(2) + 1$  is even for every  $\epsilon' \in V$ , we can assume that  $\epsilon$  is the representative that occurs in the definition of  $\Sigma$ , i.e. we can assume that  $e_i \mapsto \epsilon(i)$  defines an order preserving bijection  $\{e_1, \dots, e_4\} \rightarrow \{1, \dots, 4\}$ . Then  $\epsilon_{i,j}$  is the identity if  $\epsilon(i) < \epsilon(j)$  and  $\epsilon_{i,j} = (e_i e_j)$  if  $\epsilon(i) > \epsilon(j)$ . Thus  $\text{sign}(\epsilon_{i,j}) = 1$  if  $\epsilon(i) < \epsilon(j)$  and  $\text{sign}(\epsilon_{i,j}) = -1$  if  $\epsilon(i) > \epsilon(j)$ .

Since  $\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{\Delta, J}$  is invariant under exchanging rows and columns, we can assume that  $e_1$  is the minimal element in  $\{e_1, \dots, e_4\}$ , i.e.  $\epsilon(1) = 1$  and  $\text{sign}(\epsilon_{1,j}) = 1$  for  $j \in \{3, 4\}$ . We verify the claim of the lemma by a case consideration for the value of  $\epsilon(2)$ .

If  $\epsilon(2) = 2$ , then  $e_2$  is minimal in  $\{e_2, e_3, e_4\}$  and  $\text{sign}(\epsilon_{2,j}) = 1$  for all  $j \in \{3, 4\}$ . Thus

$$\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{\Delta, J} = 1 = (-1)^{1+2+1} = (-1)^{\epsilon(1)+\epsilon(2)+1}.$$

If  $\epsilon(2) = 3$ , then  $e_3 < e_2 < e_4$  or  $e_4 < e_2 < e_3$ . Thus  $\text{sign}(\epsilon_{2,3}) \text{sign}(\epsilon_{2,4}) = -1$  and

$$\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{\Delta, J} = -1 = (-1)^{1+3+1} = (-1)^{\epsilon(1)+\epsilon(2)+1}.$$

If  $\epsilon(2) = 4$ , then  $e_2$  is maximal in  $\{e_2, e_3, e_4\}$  and  $\text{sign}(\epsilon_{2,j}) = -1$  for all  $j \in \{3, 4\}$ . Thus

$$\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{\Delta, J} = (-1)^2 = (-1)^{1+4+1} = (-1)^{\epsilon(1)+\epsilon(2)+1},$$

which completes the proof.  $\square$

Let  $f : P \rightarrow \mathbb{S}$  be a morphism of pastures. A *lift of  $M$  to  $P$  (along  $f$ )* is a  $P$ -matroid  $\widehat{M}$  such that  $f_*(\widehat{M}) = M$ . In the following result, we will implicitly understand that a subfield  $k$  of  $\mathbb{R}$  comes with the sign map  $\text{sign} : k \rightarrow \mathbb{S}$ .

As explained in Corollary 4.14, the near-regular partial field  $\mathbb{U} = \mathbb{F}_1^\pm \langle x, y \rangle // \{x + y - 1\}$  admits three morphisms to  $\mathbb{S}$ . Since the automorphism group  $\text{Aut}(\mathbb{U})$  acts transitively on these three morphisms, we can fix one of them without restricting the generality of our results. Thus we will implicitly understand that  $\mathbb{U}$  comes with the morphism  $\text{sign} : \mathbb{U} \rightarrow \mathbb{S}$  given by  $\text{sign}(x) = \text{sign}(y) = 1$ .

**THEOREM 6.15.** *Let  $M$  be a positively oriented matroid whose underlying matroid  $\underline{M}$  is without large uniform minors. Then  $\underline{M}$  is near-regular and  $F_{\underline{M}} \simeq \mathbb{U}^{\otimes r}$  for some  $r \geq 0$ . Up to rescaling equivalence, there are precisely  $2^r$  lifts of  $M$  to  $\mathbb{U}$ , and for every subfield  $k$  of  $\mathbb{R}$ , the lifts of  $M$  to  $k$  modulo rescaling equivalence correspond bijectively to  $\left((0, 1) \cap k\right)^r$ .*

**PROOF.** By Theorem 5.9, the foundation  $F_{\underline{M}}$  is isomorphic to a tensor product  $F_1 \otimes \cdots \otimes F_r$  of copies  $F_i$  of  $\mathbb{F}_2$  and symmetry quotients of  $\mathbb{U}$ . The rescaling class of  $M$  induces a morphism  $F_{\underline{M}} \rightarrow \mathbb{S}$ . Since there is no morphism from  $\mathbb{F}_2$  to  $\mathbb{S}$ , each of the factors  $F_i$  has to be a symmetry quotient of  $\mathbb{U}$ .

From the proof of Theorem 5.9, it follows that each symmetry quotient  $F_i = \mathbb{U}/H_i$  of  $\mathbb{U}$  is the image of the induced morphism  $\mathbb{U} \simeq F_N \rightarrow F_M$  of foundations for an embedded  $U_4^2$ -minor  $N = M \setminus I/J$  of  $M$ . This means that for every  $\sigma \in H_i$  and every  $(J; e_1, \dots, e_4) \in \Omega_M$ , we have an identity of universal cross ratios

$$\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_J = \begin{bmatrix} \sigma(e_1) & \sigma(e_2) \\ \sigma(e_3) & \sigma(e_4) \end{bmatrix}_J.$$

We claim that if  $\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_J = \begin{bmatrix} e'_1 & e'_2 \\ e'_3 & e'_4 \end{bmatrix}_J$  then  $\Sigma(e_1, \dots, e_4) = \Sigma(e'_1, \dots, e'_4)$ , where  $\Sigma : \Omega_M^\diamond \rightarrow S_4/V$  is the  $\Omega$ -signature. We verify this in the following for all the defining relations of  $F_{\underline{M}}$  that involve non-degenerate cross ratios, as they appear in Theorem 4.21.

The relations (R-) and (R0) do not involve non-degenerate cross ratios (and (R-) does not occur in our case since neither the Fano matroid nor its dual are orientable). The relations (R $\sigma$ ), (R1), (R2) and (R+) are already incorporated in  $\mathbb{U}$  and can thus be ignored. For relation (R5), it is obvious that both involved cross ratios have the same  $\Omega$ -signature.

Thus we are left with the relations (R3) and (R4). Since  $\underline{M}$  is without large uniform minors, each of these relations reduces to an identity of two universal cross ratios. We begin with the tip relation (R3), which is of the form

$$\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_J = \begin{bmatrix} e_1 & e_2 \\ e_3 & e_5 \end{bmatrix}_J$$

in our case, where we use (R1) to express  $\begin{bmatrix} e_1 & e_2 \\ e_5 & e_3 \end{bmatrix}_J^{-1}$  as  $\begin{bmatrix} e_1 & e_2 \\ e_3 & e_5 \end{bmatrix}_J$ . After a permutation of  $\{e_1, \dots, e_4\}$ , we can assume that  $e_1 < e_4 < e_2 < e_3$ , and thus

$$\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{\Delta, J} = (-1)^{1+3+1} = -1$$

by Lemma 6.14. Therefore also  $\begin{bmatrix} e_1 & e_2 \\ e_3 & e_5 \end{bmatrix}_{\Delta, J} = -1$ , which means that the unique order preserving bijection  $\pi : \{e_1, e_2, e_3, e_5\} \rightarrow \{1, \dots, 4\}$  must satisfy  $\pi(e_1) = \pi(e_2)$  according to Lemma 6.14. Since  $e_1 < e_2 < e_3$  by our assumptions, this implies that  $e_1 < e_5 < e_2$ . Thus  $\Sigma(e_1, e_2, e_3, e_4) = \Sigma(e_1, e_2, e_3, e_5)$ .

The cotip relations (R4) are in our case of the form

$$\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{Je_5} = \begin{bmatrix} e_1 & e_2 \\ e_3 & e_5 \end{bmatrix}_{Je_4}.$$

As before, we can assume that  $e_1 < e_4 < e_2 < e_3$  and thus  $\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{\Delta, Je_5} = -1$ . By the same reasoning, this implies that  $e_1 < e_5 < e_2 < e_3$  and thus  $\Sigma(e_1, e_2, e_3, e_4) = \Sigma(e_1, e_2, e_3, e_5)$ . This establishes our claim that  $\Sigma(e_1, \dots, e_4) = \Sigma(e'_1, \dots, e'_4)$  whenever  $\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_J = \begin{bmatrix} e'_1 & e'_2 \\ e'_3 & e'_4 \end{bmatrix}_J$ .

In particular, if  $\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_J = \begin{bmatrix} \sigma(e_1) & \sigma(e_2) \\ \sigma(e_3) & \sigma(e_4) \end{bmatrix}_J$  then  $\Sigma(e_1, \dots, e_4) = \Sigma(\sigma(e_1), \dots, \sigma(e_4))$ , which means that  $\sigma$  is in  $V$ . These are precisely the relations in (R $\sigma$ ), which are already satisfied in  $\mathbb{U}$ . We conclude that  $\sigma$  is the identity on  $\mathbb{U}$ .

This shows that every factor  $F_i$  of  $F_{\underline{M}}$  is a trivial quotient of  $\mathbb{U}$  and thus  $F_{\underline{M}} \simeq \mathbb{U}^{\otimes r}$ , as claimed in the theorem. It also implies at once that  $\underline{M}$  is near-regular.

Let  $\chi_M : F_{\underline{M}} \rightarrow \mathbb{S}$  be the morphism of pastures induced by the rescaling class of  $M$ . The lifts of  $M$  to  $\mathbb{U}$  and  $k$ , up to rescaling, correspond to the lifts of  $\chi_M$  to  $\mathbb{U}$  and  $k$ , respectively. We can study this question for each factor  $F_i = \mathbb{U}$  of  $F_{\underline{M}}$  individually.

A lift of  $f : \mathbb{U} \rightarrow \mathbb{S}$  to  $\mathbb{U}$  is a morphism  $\hat{f} : \mathbb{U} \rightarrow \mathbb{U}$  such that  $\text{sign}(\hat{f}(x)) = \text{sign}(\hat{f}(y)) = 1$ . This determines  $\hat{f}$  up to a permutation of  $x$  and  $y$ , which shows that there are precisely two lifts of  $f : \mathbb{U} \rightarrow \mathbb{S}$  to  $\mathbb{U}$ . Thus there are precisely  $2^r$  lifts of  $M$  to  $\mathbb{U}$  up to rescaling equivalence.

A lift of  $f : \mathbb{U} \rightarrow \mathbb{S}$  to  $k$  is a morphism  $\hat{f} : \mathbb{U} \rightarrow k$  such that  $\text{sign}(\hat{f}(x)) = \text{sign}(\hat{f}(y)) = 1$ . Since  $\hat{f}(y) = 1 - \hat{f}(x)$ , this means that  $\hat{f}(x) \in ((0, 1) \cap k)$  and, conversely, every choice of image  $\hat{f}(x) \in ((0, 1) \cap k)$  determines a lift  $\hat{f}$  of  $f$  to  $k$ . Thus the lifts of  $M$  to  $k$  up to rescaling equivalence correspond bijectively to  $((0, 1) \cap k)^r$ . This completes the proof of the theorem.  $\square$

## 6.6. Representation classes of matroids without large uniform minors

Given a matroid  $M$ , we can ask over which pastures  $M$  is representable. This defines a class of pastures that we call the representation class of  $M$ .

For cardinality reasons, it is clear that not every class of pastures can be the representation class of a matroid. The theorems in Section 6.7 make clear that this fails in an even more drastic way—for example, a matroid that is representable over  $\mathbb{F}_2$  and  $\mathbb{F}_3$  is representable over all pastures; cf. Theorem 6.26.

In this section, we determine the representation classes that are defined by matroids without large uniform minors. It turns out that there are only twelve of them; see Table 6.2 for a characterization.

DEFINITION 6.16. Let  $M$  be a matroid. The *representation class* of  $M$  is the class  $\mathcal{P}_M$  of all pastures  $P$  over which  $M$  is representable. Two matroids  $M$  and  $M'$  are *representation equivalent* if  $\mathcal{P}_M = \mathcal{P}_{M'}$ .

Note that the representation class  $\mathcal{P}_M$  of a matroid  $M$  consists of precisely those pastures for which there is a morphism from the foundation  $F_M$  of  $M$  to  $P$ . This means that the representation class of a matroid is determined by its foundation. Evidently,  $\mathcal{P}_M = \mathcal{P}_{M'}$  if  $M$  and  $M'$  are representation equivalent, which justifies the notation  $\mathcal{P}_C = \mathcal{P}_M$  where  $C$  is the representation class of  $M$ .

Often there are simpler pastures than the foundation that characterize representation classes in the same way, which leads to the following notion.

DEFINITION 6.17. Let  $M$  be a matroid with representation class  $\mathcal{P}_M$ . A *characteristic pasture* for  $M$  is a pasture  $\Pi$  for which a pasture  $P$  is in  $\mathcal{P}_M$  if and only if there is a morphism  $\Pi \rightarrow P$ . A matroid  $M$  is *strictly representable over a pasture*  $P$  if  $P$  is a characteristic pasture for  $M$ .

By the existence of the identity morphism  $\text{id} : \Pi \rightarrow \Pi$ , strictly representable implies representable. And the foundation of a matroid  $M$  is clearly a characteristic pasture for  $M$ . The following result characterizes all characteristic pastures:

LEMMA 6.18. *Let  $M$  be a matroid with foundation  $F_M$ . A pasture  $\Pi$  is a characteristic pasture of  $M$  if and only if there exist morphisms  $F_M \rightarrow \Pi$  and  $\Pi \rightarrow F_M$ .*

PROOF. Assume that  $\Pi$  is a characteristic pasture for  $M$ . Since also  $F_M$  is a characteristic pasture, we have  $F_M, \Pi \in \mathcal{P}_M$ , and by the defining property of characteristic pastures, there are morphisms  $F_M \rightarrow \Pi$  and  $\Pi \rightarrow F_M$ .

Conversely, assume that there are morphisms  $F_M \rightarrow \Pi$  and  $\Pi \rightarrow F_M$ . If  $P \in \mathcal{P}_M$ , then there is a morphism  $F_M \rightarrow P$ , which yields a morphism  $\Pi \rightarrow F_M \rightarrow P$ . If there is a morphism  $\Pi \rightarrow P$ , then there is a morphism  $F_M \rightarrow \Pi \rightarrow P$ , and thus  $P \in \mathcal{P}_M$ . This shows that  $\Pi$  is a characteristic pasture for  $M$ .  $\square$

The next result describes an explicit condition for representation equivalent matroids.

LEMMA 6.19. *Let  $M$  and  $M'$  be two matroids with respective representation classes  $\mathcal{P}_M$  and  $\mathcal{P}_{M'}$  and respective characteristic pastures  $\Pi$  and  $\Pi'$ . Then  $\mathcal{P}_{M'}$  is contained in  $\mathcal{P}_M$  if and only if there is a morphism  $\Pi \rightarrow \Pi'$ . In particular,  $M$  and  $N$  are representation equivalent if and only if there exist morphisms  $\Pi \rightarrow \Pi'$  and  $\Pi' \rightarrow \Pi$ .*

PROOF. If there is a morphism  $f : \Pi \rightarrow \Pi'$ , then we can compose every morphism  $\Pi' \rightarrow P$  with  $f$ , which implies that  $\mathcal{P}_{M'} \subset \mathcal{P}_M$ . Assume conversely that  $\mathcal{P}_{M'} \subset \mathcal{P}_M$ . Then  $\Pi' \in \mathcal{P}_M$ , which means that there is a morphism  $\Pi \rightarrow \Pi'$ . The additional claim of the lemma is obvious.  $\square$

In the following, we say that a matroid  $M$  is

- *strictly binary* if  $\mathbb{F}_2$  is a characteristic pasture for  $M$ ;
- *strictly ternary* if  $\mathbb{F}_3$  is a characteristic pasture for  $M$ ;

TABLE 6.2. The equivalence classes of matroids without large uniform minors

$C$	Name	minimal $\Pi_C$	add. $F_i$	$P \in \mathcal{P}_C$ iff. $\exists u, v \in P^\times$ s.t.	field?
$C_1$	regular	$\mathbb{F}_1^\pm$			yes
$C_2$	str. near-regular	$\mathbb{U}$	$\mathbb{U}$	$u + v = 1$	yes
$C_3$	strictly dyadic	$\mathbb{D}$	$\mathbb{U}, \mathbb{D}$	$u + u = 1$	yes
$C_4$	str. hexagonal	$\mathbb{H}$	$\mathbb{U}, \mathbb{H}$	$v - v^2 = -v^3 = 1$	yes
$C_5$	str. $\mathbb{D} \otimes \mathbb{H}$ -repr.	$\mathbb{D} \otimes \mathbb{H}$	$\mathbb{U}, \mathbb{D}, \mathbb{H}$	$u + u = v - v^2 = -v^3 = 1$	yes
$C_6$	strictly ternary	$\mathbb{F}_3$	$\mathbb{U}, \mathbb{D}, \mathbb{H}$	$1 + 1 = -1$	yes
$C_7$	strictly binary	$\mathbb{F}_2$		$-1 = 1$	yes
$C_8$		$\mathbb{F}_2 \otimes \mathbb{U}$	$\mathbb{U}$	$-1 = u + v = 1$	yes
$C_9$		$\mathbb{F}_2 \otimes \mathbb{D}$	$\mathbb{U}, \mathbb{D}$	$-1 = u + u = 1$	no
$C_{10}$		$\mathbb{F}_2 \otimes \mathbb{H}$	$\mathbb{U}, \mathbb{H}$	$-1 = v - v^2 = v^3 = 1$	yes
$C_{11}$		$\mathbb{F}_2 \otimes \mathbb{D} \otimes \mathbb{H}$	$\mathbb{U}, \mathbb{D}, \mathbb{H}$	$-1 = u + u = v - v^2 = v^3 = 1$	no
$C_{12}$	idempotent	$\mathbb{F}_2 \otimes \mathbb{F}_3$	$\mathbb{U}, \mathbb{D}, \mathbb{H}$	$-1 = 1 + 1 = 1$	no

- *strictly near-regular* if  $\mathbb{U}$  is a characteristic pasture for  $M$ ;
- *strictly dyadic* if  $\mathbb{D}$  is a characteristic pasture for  $M$ ;
- *strictly hexagonal* if  $\mathbb{H}$  is a characteristic pasture for  $M$ ;
- *strictly  $\mathbb{D} \otimes \mathbb{H}$ -representable* if  $\mathbb{D} \otimes \mathbb{H}$  is a characteristic pasture for  $M$ ;
- *idempotent* if  $\mathbb{K}$  is a characteristic pasture for  $M$ .

Note that an idempotent matroid  $M$  is representable over a pasture  $P$  if and only if  $P$  is *idempotent*, by which we mean that both  $-1 = 1$  and  $1 + 1 = 1$  hold in  $P$ .

THEOREM 6.20. *Let  $M$  be a matroid without large uniform minors. Then  $M$  belongs to precisely one of the 12 classes that are described in Table 6.2. The six columns of Table 6.2 describe the following information:*

- (1) a label for each class  $C$ ;
- (2) a name (as far as we have introduced one);
- (3) a characteristic pasture  $\Pi_C$  that is minimal in the sense that the foundation of every matroid  $M$  in the class  $C$  is of isomorphism type  $F_M \simeq \Pi_C \otimes F_1 \otimes \cdots \otimes F_r$  for some  $r \geq 0$  and  $F_1, \dots, F_r \in \{\mathbb{U}, \mathbb{D}, \mathbb{H}\}$ ;
- (4) the type of factors  $F_i$  that can occur in the expression  $F_M \simeq \Pi_C \otimes F_1 \otimes \cdots \otimes F_r$  for  $M$  in  $C$ ;
- (5) a characterization of the pastures  $P$  in the representation class  $\mathcal{P}_C$ ;
- (6) whether the matroids in this class are representable over some field.

The left diagram in Figure 6.1 illustrates the existence of morphisms between the different characteristic pastures  $\Pi_C$  in Table 6.2. The right diagram illustrates the inclusion relation between the representation classes  $\mathcal{P}_i = \mathcal{P}_{C_i}$  (for  $i = 1, \dots, 12$ )—an edge indicates that the class on the bottom end of the edge is contained in the class at the top end of the edge.



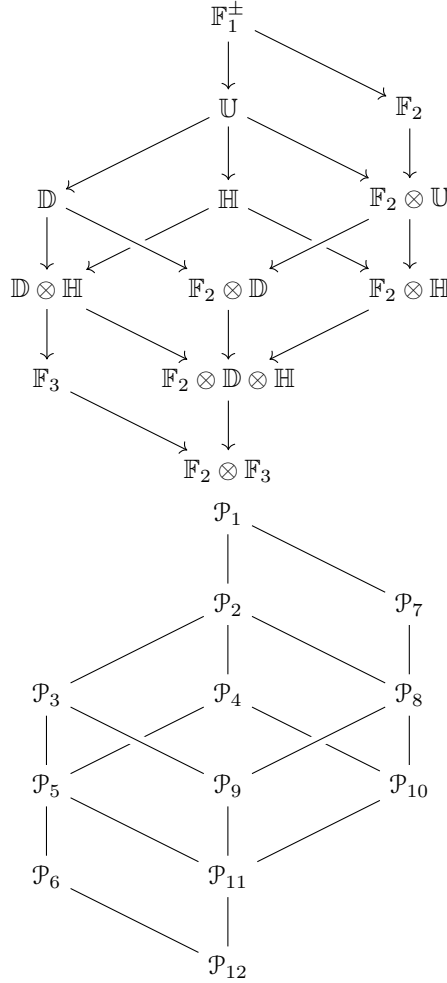


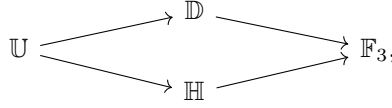
FIGURE 6.1. Morphisms between characteristic pastures and containment of the representation classes for matroids without large uniform minors

PROOF. For the sake of this proof, we say that two pastures  $P$  and  $P'$  are *equivalent*, and write  $P \sim P'$ , if there are morphisms  $P \rightarrow P'$  and  $P' \rightarrow P$ .

If there is a morphism  $P' \rightarrow P$ , then there are morphisms  $P \rightarrow P \otimes P'$  and  $P \otimes P' \rightarrow P$ , which means that  $P \otimes P' \sim P$ . This applies in particular to  $P' = P$ . This shows that  $P_1 \otimes \cdots \otimes P_r \sim P_1 \otimes \cdots \otimes P_s$  for  $s \leq r$  and pastures  $P_1, \dots, P_r$  if, for every  $i \in \{s+1, \dots, r\}$ , there is a  $j \in \{1, \dots, s\}$  and a morphism  $P_i \rightarrow P_j$ .

Let  $M$  be a matroid without large uniform minors and  $F_M$  its foundation. By Theorem 5.9,  $F_M \simeq F_1 \otimes \cdots \otimes F_r$  for some  $F_1, \dots, F_r \in \{\mathbb{U}, \mathbb{D}, \mathbb{H}, \mathbb{F}_3, \mathbb{F}_2\}$ , where we can assume that  $\mathbb{F}_2$  appears at most once as a factor. By the previous considerations,  $F_M \sim F_1 \otimes \cdots \otimes F_s$  for pairwise distinct  $F_1, \dots, F_s \in \{\mathbb{U}, \mathbb{D}, \mathbb{H}, \mathbb{F}_3, \mathbb{F}_2\}$ . Since there

are morphisms



we have  $\mathbb{D} \otimes \mathbb{U} \sim \mathbb{D}$ ,  $\mathbb{H} \otimes \mathbb{U} \sim \mathbb{H}$  and  $\mathbb{F}_3 \otimes F \sim \mathbb{F}_3$  for  $F \in \{\mathbb{U}, \mathbb{D}, \mathbb{H}\}$ . Thus we can assume that in the expression  $F_1 \otimes \cdots \otimes F_s$  at most one of  $\mathbb{U}$ ,  $\mathbb{D}$ ,  $\mathbb{H}$  and  $\mathbb{F}_3$  appears, with the exception of  $\mathbb{D} \otimes \mathbb{H}$ .

Thus we are limited to the twelve different expressions for  $F_1 \otimes \cdots \otimes F_s$  that appear in Figure 6.1. We conclude that  $F_M$  is equivalent to one of those and that  $\Pi = F_1 \otimes \cdots \otimes F_s$  is a characteristic pasture for  $M$ .

An easy case-by-case verification based on Table 6.1, which we shall not carry out, shows that there is a morphism between two pastures if and only if there is a directed path between these pastures in the diagram on the left hand side of Figure 6.1. By Lemma 6.19, this diagram determines at once the inclusion behaviour of the associated representation classes  $\mathcal{P}_1$ – $\mathcal{P}_{12}$  as illustrated on the right hand side of Figure 6.1.

Note that the way we found the twelve characteristic pastures  $\Pi$  shows that they are minimal in the sense of part (3) of the theorem, and it shows that the types of additional factors displayed in the forth column of Table 6.2 are correct. The conditions in the fifth column of Table 6.2 follows at once from Lemma 6.1.

For the verification of the last column, note that there is a morphism  $\Pi_C \rightarrow \mathbb{F}_3$  for the classes  $C \in \{C_1, \dots, C_6\}$  and that there is a morphism  $\Pi \rightarrow \mathbb{F}_4$  for  $C \in \{C_7, C_8, C_{10}\}$ . Thus the matroids in the classes  $C_1$ – $C_8$  and  $C_{10}$  are representable over a field. There is no morphism from  $\mathbb{F}_2 \otimes \mathbb{D}$  to any field since in a field only one of  $1 + 1 = 0$  and  $1 + 1 = u^{-1}$  for some  $u \neq 0$  can hold. Thus matroids in the classes  $C_9$ ,  $C_{11}$  and  $C_{12}$  are not representable over any field, which concludes the proof of the theorem.  $\square$

As a sample application, we formulate the following strengthening of the result [37, Thm. 3.3] by Whittle. Recall that a matroid is called *representable* if it is representable over some field.

**THEOREM 6.21.** *Let  $\mathcal{P}_{\leq 8} = \{\mathbb{F}_q \mid q \leq 8 \text{ a prime power}\}$ . Then two representable matroids  $M$  and  $M'$  without large uniform minors are representation equivalent if and only if  $\mathcal{P}_M \cap \mathcal{P}_{\leq 8} = \mathcal{P}_{M'} \cap \mathcal{P}_{\leq 8}$ . More precisely, for  $i \in \{1, \dots, 8, 10\}$  and  $p_i$  and  $q_i$  as in Table 6.3, the class  $\mathcal{P}_{C_i}$  is the intersection of the representation classes  $\mathcal{P}_M$  of all matroids  $M$  without large uniform minors that are representable over  $\mathbb{F}_{p_i}$  and  $\mathbb{F}_{q_i}$ .*

TABLE 6.3. Prime powers such that  $\mathcal{P}_{C_i} = \bigcap \{\mathcal{P}_M \mid M \text{ is representable over } \mathbb{F}_{p_i} \text{ and } \mathbb{F}_{q_i}\}$

$i$	1	2	3	4	5	6	7	8	10
$p_i$	2	3	3	3	3	3	2	8	4
$q_i$	3	8	5	4	7	3	2	8	4

PROOF. For  $i \in \{1, \dots, 8, 10\}$  and  $M$  in  $C_i$ , let  $\mathcal{U}_i$  be the subset of  $\{\mathbb{U}, \mathbb{D}, \mathbb{H}, \mathbb{F}_3, \mathbb{F}_2\}$  such that  $\Pi_i = \bigotimes_{P \in \mathcal{U}_i} P$  is a characteristic pasture for  $M$ , cf. Table 6.2. Then we can read off from Table 6.1 that there are morphisms  $P \rightarrow \mathbb{F}_{p_i}$  and  $P \rightarrow \mathbb{F}_{q_i}$  for all  $P \in \mathcal{U}_i$ , and that for all  $P \in \{\mathbb{U}, \mathbb{D}, \mathbb{H}, \mathbb{F}_3, \mathbb{F}_2\}$  that are not in  $\mathcal{U}_i$ , there is either no morphism from  $P$  to  $\mathbb{F}_{p_i}$  or no morphism from  $P$  to  $\mathbb{F}_{q_i}$ . This shows that the existence of morphisms into  $\mathbb{F}_{p_i}$  and  $\mathbb{F}_{q_i}$  characterize the factors of the characteristic pasture  $\Pi_i$  and establishes the claims of the theorem.  $\square$

REMARK 6.22. Note that the representation class  $\mathcal{P}_1$  of regular matroids contains all pastures and is therefore the largest possible representation class. The representation class  $\mathcal{P}_{12}$  of idempotent pastures is the smallest representation class, since every matroid is by definition representable over  $\mathbb{K}$  and thus over every idempotent pasture. (Recall that a pasture  $P$  is called idempotent if there is a morphism from  $\mathbb{K}$  to  $P$ .) Every other representation class thus lies between  $\mathcal{P}_{12}$  and  $\mathcal{P}_1$ .

REMARK 6.23. We will show in a sequel to this paper that every tensor product of copies of the pastures  $\mathbb{U}, \mathbb{D}, \mathbb{H}, \mathbb{F}_3$  and  $\mathbb{F}_2$  occurs as the foundation of a matroid. Consequently each of the classes  $C_1$ – $C_{12}$  is nonempty.

Alternatively, we can use known results to deduce this. Since there are matroids that are regular, strictly near-regular (e.g.  $U_4^2$ ), strictly dyadic (e.g. the non-Fano matroid  $F_7^-$ ), strictly hexagonal (e.g. the ternary affine plane  $AG(2, 3)$ ), strictly ternary (e.g. the matroid  $T_8$  from Oxley's book [22]) and strictly binary (e.g. the Fano matroid  $F_7$ ), the classes  $C_1, C_2, C_3, C_4, C_6$  and  $C_7$  are nonempty.

Since the characteristic pastures of the remaining classes in Table 6.2 are tensor products of characteristic pastures of one of the aforementioned matroids, we can deduce that the other classes are also nonempty by observing that

$$\{P \mid F_M \otimes F_{M'} \xrightarrow{\exists} P\} = \{P \mid F_M \xrightarrow{\exists} P\} \cap \{P \mid F_{M'} \xrightarrow{\exists} P\} = \mathcal{P}_M \cap \mathcal{P}_{M'} = \mathcal{P}_{M \oplus M'}$$

for two matroids  $M$  and  $M'$ .

REMARK 6.24. Since all binary and ternary matroids are without large uniform minors, all matroids in the classes  $C_1$ – $C_7$  are without large uniform minors. This is not true for all classes though. For instance the direct sum of an idempotent matroid with  $U_5^2$  is also idempotent and thus in  $C_{12}$ , but has a minor of type  $U_5^2$ ; cf. Remark 6.23 for the existence of idempotent matroids.

In fact, a similar construction yield matroids with  $U_5^2$ -minors in the classes  $C_{10}$  and  $C_{11}$ . By contrast, all matroids in  $C_8$  and  $C_9$  are without large uniform minors. This latter fact can be proven as follows: a class  $C_i$  contains a matroid  $M$  with a  $U_5^2$ - or a  $U_5^3$ -minor if and only if there is morphism from the foundation of  $U_5^2$  (cf. Proposition 5.4) to the minimal characteristic pasture for  $M$ . There is no morphism from the foundation of  $U_5^2$  to  $\mathbb{F}_2 \otimes \mathbb{U}$  or to  $\mathbb{F}_2 \otimes \mathbb{D}$ , but there are morphisms to  $\mathbb{F}_2 \otimes \mathbb{H}$  and  $\mathbb{F}_2 \otimes \mathbb{D} \otimes \mathbb{H}$ .

## 6.7. Characterization of classes of matroids

In this section, we use our results to provide different characterizations of some prominent classes of matroids, such as regular, near-regular, binary, ternary, quaternary, dyadic, and hexagonal matroids. In particular, we find new proofs for results by Tutte, Bland and Las Vergnas, and Whittle, which we refer to in detail at the beginnings of the appropriate sections. Moreover, we obtain new characterizations, which often involve the pastures  $\mathbb{S}, \mathbb{P}$  and  $\mathbb{W}$ .

All these characterizations are immediate applications of Theorem 5.9 in combination with Table 6.1. It is possible to work out additional descriptions for the classes of matroids under consideration, or to study other classes with the same techniques. For example, our technique allows for an easy proof of the following results found in Theorems 5.1 and 5.2 of Semple and Whittle's paper [28].

**THEOREM 6.25** (Semple–Whittle '96). *Let  $\mathcal{C}_P$  denote the class of matroids without large uniform minors that are representable over a pasture  $P$ . Then the following hold true.*

- (1)  $\mathcal{C}_{\mathbb{F}_{2^r}} \cap \mathcal{C}_{\mathbb{F}_3} = \mathcal{C}_{\mathbb{U}}$  for odd  $r \geq 2$ .
- (2)  $\mathcal{C}_{\mathbb{F}_{2^r}} \cap \mathcal{C}_{\mathbb{F}_3} = \mathcal{C}_{\mathbb{H}}$  for even  $r \geq 2$ .
- (3)  $\mathcal{C}_k \subset \mathcal{C}_{\mathbb{F}_3}$  for every field  $k$  of characteristic different from 2, and  $\mathcal{C}_k = \mathcal{C}_{\mathbb{D}}$  if, in addition,  $k$  does not contain a primitive sixth root of unity.

**6.7.1. Regular matroids.** The following theorem extends a number of classical results that characterize regular matroids, namely as binary matroids that are representable over a field  $k$  with  $\text{char } k \neq 2$  by Tutte in [32] (use  $P = k$  in (5)) and as binary and orientable matroids by Bland and Las Vergnas in [8] (use  $P = \mathbb{S}$  in (5)). Up to the characterization (3), the authors of this paper have proven Theorem 6.26 in its full generality in [5, Thm. 7.33] with a slightly different proof.

**THEOREM 6.26.** *Let  $M$  be a matroid with foundation  $F_M$ . Then the following assertions are equivalent:*

- (1)  $M$  is regular.
- (2)  $F_M = \mathbb{F}_1^\pm$ .
- (3)  $M$  belongs to  $C_1$ .
- (4)  $M$  is representable over all pastures.
- (5)  $M$  is representable over  $\mathbb{F}_2$  and a pasture with  $-1 \neq 1$ .

**PROOF.** The logical structure of this proof is  $(1) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (2) \Rightarrow (1)$ . The implications  $(2) \Rightarrow (1) \Rightarrow (3) \Rightarrow (4)$  follow from Theorem 6.20 and  $(4) \Rightarrow (5)$  is trivial.

We close the circle by showing  $(5) \Rightarrow (2)$ . If  $M$  is binary, then it is without large uniform minors by Lemma 6.2. Thus, by Theorem 5.9,  $F_M$  is a tensor product of copies of  $\mathbb{U}$ ,  $\mathbb{D}$ ,  $\mathbb{H}$ ,  $\mathbb{F}_3$  and  $\mathbb{F}_2$ . But none of  $\mathbb{U}$ ,  $\mathbb{D}$ ,  $\mathbb{H}$  or  $\mathbb{F}_3$  admits a morphism to  $\mathbb{F}_2$ , and  $\mathbb{F}_2$  admits no morphism into a pasture  $P$  with  $-1 \neq 1$ . Thus  $F_M = \mathbb{F}_1^\pm$ , as claimed.  $\square$

**6.7.2. Binary matroids.** We find the following equivalent characterizations of binary matroids.

**THEOREM 6.27.** *Let  $M$  be a matroid with foundation  $F_M$ . Then the following assertions are equivalent:*

- (1)  $M$  is binary.
- (2)  $F_M \simeq \mathbb{F}_1^\pm$  or  $F_M \simeq \mathbb{F}_2$ .
- (3)  $M$  belongs to  $C_1$  or  $C_7$ .
- (4)  $M$  is representable over every pasture for which  $-1 = 1$ .
- (5) All fundamental elements of  $F_M$  are trivial.

**PROOF.** We prove  $(1) \Rightarrow (3) \Rightarrow (2) \Rightarrow (5) \Rightarrow (4) \Rightarrow (1)$ . Steps  $(1) \Rightarrow (3) \Rightarrow (2)$  follow from Theorem 6.20, step  $(5) \Rightarrow (2)$  follows from part (1) of Lemma 6.1 and Corollary 5.10, and steps  $(2) \Rightarrow (5)$  and  $(2) \Rightarrow (4) \Rightarrow (1)$  are trivial.  $\square$

**6.7.3. Ternary matroids.** We find the following equivalent characterizations of ternary matroids.

**THEOREM 6.28.** *Let  $M$  be a matroid with foundation  $F_M$ . Then the following assertions are equivalent:*

- (1)  $M$  is ternary.
- (2)  $F_M \simeq F_1 \otimes \cdots \otimes F_r$  for  $r \geq 0$  and  $F_1, \dots, F_r \in \{\mathbb{U}, \mathbb{D}, \mathbb{H}, \mathbb{F}_3\}$ .
- (3)  $M$  belongs to one of  $C_1$ – $C_6$ .
- (4)  $M$  is representable over every pasture for which  $1 + 1 + 1 = 0$ .
- (5)  $M$  is without large uniform minors and representable over a field of characteristic 3.
- (6)  $M$  is without large uniform minors and weakly orientable.
- (7)  $M$  is without large uniform minors and there is no morphism from  $\mathbb{F}_2$  to  $F_M$ .

**PROOF.** We show  $(2) \Leftrightarrow (3)$ ,  $(1) \Leftrightarrow (4)$  and  $(2) \Rightarrow (1) \Rightarrow (5) / (6) / (7) \Rightarrow (2)$ . The implications  $(2) \Rightarrow (1) \Leftrightarrow (4)$  are trivial. The equivalence  $(2) \Leftrightarrow (3)$  follows from Theorem 6.20.

Assuming (1), then  $M$  is without large uniform minors by Lemma 6.2. Since there are morphisms  $\mathbb{F}_3 \rightarrow k$  for every field  $k$  of characteristic 3 and  $\mathbb{F}_3 \rightarrow \mathbb{W}$ , this implies (5) and (6).

If  $M$  is without large uniform minors, then Theorem 5.9 implies that  $F_M$  is the tensor product of copies of  $\mathbb{U}, \mathbb{D}, \mathbb{H}, \mathbb{F}_3$  and  $\mathbb{F}_2$ . Thus (1) and the fact that  $\mathbb{F}_2$  does not map to  $\mathbb{F}_3$  implies (7). Conversely, each condition of (5), (6) and (7) implies that  $\mathbb{F}_2$  cannot occur as a building block of  $F_M$ , and thus (2).  $\square$

**6.7.4. Quaternary matroids without large uniform minors.** We find the following equivalent characterizations of quaternary matroids without large uniform minors.

**THEOREM 6.29.** *Let  $M$  be a matroid without large uniform minors and  $F_M$  its foundation. Then the following assertions are equivalent:*

- (1)  $M$  is quaternary.
- (2)  $F_M \simeq F_1 \otimes \cdots \otimes F_r$  for  $r \geq 0$  and  $F_1, \dots, F_r \in \{\mathbb{U}, \mathbb{H}, \mathbb{F}_2\}$ .
- (3)  $M$  belongs to  $C_1, C_2, C_4, C_7, C_8$  or  $C_{10}$ .
- (4)  $M$  is representable over every pasture for which  $1 + 1 = 0$  and that contains an element  $u$  for which  $u^2 + u + 1 = 0$ .
- (5)  $M$  is representable over all field extensions of  $\mathbb{F}_4$ .
- (6) There is no morphism from  $\mathbb{D}$  to  $F_M$ .

**PROOF.** We show  $(2) \Leftrightarrow (3)$  and  $(2) \Rightarrow (4) \Rightarrow (1) \Rightarrow (5) \Rightarrow (6) \Rightarrow (2)$ . The equivalence  $(2) \Leftrightarrow (3)$  follows from Theorem 6.20. The implications  $(2) \Rightarrow (4) \Rightarrow (1) \Rightarrow (5)$  are trivial. The implication  $(5) \Rightarrow (6)$  follows since there is no morphism from  $\mathbb{D}$  to  $\mathbb{F}_4$  by Lemma 6.1. The implication  $(6) \Rightarrow (2)$  follows by Theorem 5.9, together with the fact that there is a morphism  $\mathbb{D} \rightarrow \mathbb{F}_3$  but there are no morphisms from  $\mathbb{D}$  to  $\mathbb{U}, \mathbb{H}$  and  $\mathbb{F}_2$ , and thus only the latter three pastures can occur as factors of  $F_M$ .  $\square$

**6.7.5. Near-regular matroids.** In this section, we provide several characterizations of near-regular matroids. The descriptions (5) and (6) appear in Whittle's paper [36, Thm. 1.4].

**THEOREM 6.30.** *Let  $M$  be a matroid with foundation  $F_M$ . Then the following assertions are equivalent:*

- (1)  $M$  is near-regular.
- (2)  $F_M \simeq F_1 \otimes \cdots \otimes F_r$  for  $r \geq 0$  and  $F_1 = \cdots = F_r = \mathbb{U}$ .
- (3)  $M$  belongs to  $C_1$  or  $C_2$ .
- (4)  $M$  is representable over all pastures with a fundamental element.
- (5)  $M$  is representable over fields with at least 3 elements.
- (6)  $M$  is representable over  $\mathbb{F}_3$  and  $\mathbb{F}_8$ .
- (7)  $M$  is without large uniform minors and representable over  $\mathbb{F}_4$  and  $\mathbb{F}_5$ .
- (8)  $M$  is without large uniform minors and representable over  $\mathbb{F}_4$  and  $\mathbb{S}$ .
- (9)  $M$  is without large uniform minors and representable over  $\mathbb{F}_8$  and  $\mathbb{W}$ .
- (10)  $M$  is dyadic and hexagonal.
- (11)  $M$  is without large uniform minors and there are no morphisms  $\mathbb{F}_2 \rightarrow F_M$ ,  $\mathbb{D} \rightarrow F_M$ , or  $\mathbb{H} \rightarrow F_M$ .

**PROOF.** We show  $(2) \Leftrightarrow (3)$ ,  $(2) \Rightarrow (1) \Rightarrow (4) \Rightarrow (5) \Rightarrow (2)$  and the equivalence of (2) with each of (6)–(11). The equivalence  $(2) \Leftrightarrow (3)$  follows from Theorem 6.20,  $(2) \Rightarrow (1)$  and  $(4) \Rightarrow (5)$  are trivial and  $(1) \Rightarrow (4)$  follows from Lemma 6.1. That (2) implies (6)–(11) can be read off from Table 6.1. Conversely, each of (5)–(11) implies that  $M$  is without large uniform minors and thus Theorem 5.9 applies. In turn, each of (5)–(11) excludes that any of  $\mathbb{D}$ ,  $\mathbb{H}$ ,  $\mathbb{F}_3$  and  $\mathbb{F}_2$  occur as a factor  $F_M$ , and thus (2).  $\square$

**6.7.6. Dyadic matroids.** In this section, we provide several characterizations of dyadic matroids. Description (6) has been given by Whittle in [35, Thm. 7.1]. Descriptions (4) and (5) have been given by Whittle in [36, Thm. 1.1].

**THEOREM 6.31.** *Let  $M$  be a matroid with foundation  $F_M$ . Then the following assertions are equivalent:*

- (1)  $M$  is dyadic.
- (2)  $F_M \simeq F_1 \otimes \cdots \otimes F_r$  for  $r \geq 0$  and  $F_1, \dots, F_r \in \{\mathbb{U}, \mathbb{D}\}$ .
- (3)  $M$  belongs to  $C_1$ ,  $C_2$  or  $C_3$ .
- (4)  $M$  is representable over every pasture  $P$  such that  $1 + 1 = u$  for some  $u \in P^\times$ .
- (5)  $M$  is representable over every field of characteristic different from 2.
- (6)  $M$  is representable over  $\mathbb{F}_3$  and  $\mathbb{F}_q$ , where  $q$  is an odd prime power such that  $q - 1$  is not divisible by 3.
- (7)  $M$  is representable over  $\mathbb{F}_3$  and  $\mathbb{Q}$ .
- (8)  $M$  is representable over  $\mathbb{F}_3$  and  $\mathbb{S}$ .
- (9)  $M$  is without large uniform minors and there are no morphisms  $\mathbb{F}_2 \rightarrow F_M$  or  $\mathbb{H} \rightarrow F_M$ .

**PROOF.** We show  $(2) \Leftrightarrow (3)$ ,  $(2) \Rightarrow (1) \Rightarrow (4) \Rightarrow (5) \Rightarrow (2)$  and the equivalence of (2) with each of (6)–(9). The equivalence  $(2) \Leftrightarrow (3)$  follows from Theorem 6.20,  $(2) \Rightarrow (1)$  and  $(4) \Rightarrow (5)$  are trivial and  $(1) \Rightarrow (4)$  follows from Lemma 6.1. That (2) implies (6)–(9) follows from Lemma 6.1 and Table 6.1. Conversely, each of (5)–(9) implies that  $M$  is without large uniform minors and thus Theorem 5.9 applies. In turn, each of (5)–(9) excludes that any of  $\mathbb{H}$ ,  $\mathbb{F}_3$  and  $\mathbb{F}_2$  occur as a factor  $F_M$ , and thus (2).  $\square$

**6.7.7. Hexagonal matroids.** In this section, we provide several characterizations of hexagonal matroids. Description (5) has been given by Whittle in [36, Thm. 1.2].

**THEOREM 6.32.** *Let  $M$  be a matroid with foundation  $F_M$ . Then the following assertions are equivalent:*

- (1)  $M$  is hexagonal.
- (2)  $F_M \simeq F_1 \otimes \cdots \otimes F_r$  for  $r \geq 0$  and  $F_1, \dots, F_r \in \{\mathbb{U}, \mathbb{H}\}$ .
- (3)  $M$  belongs to  $C_1$ ,  $C_2$  or  $C_4$ .
- (4)  $M$  is representable over every pasture that contains an element  $u$  with  $u^3 = -1$  and  $u^2 - u + 1 = 0$ .
- (5)  $M$  is representable over every field that is of characteristic 3 or contains a primitive sixth root of unity.
- (6)  $M$  is representable over  $\mathbb{F}_3$  and  $\mathbb{F}_4$ .
- (7)  $M$  is without large uniform minors, weakly orientable, and representable over  $\mathbb{F}_4$ .
- (8)  $M$  is without large uniform minors and there are no morphisms  $\mathbb{F}_2 \rightarrow F_M$  or  $\mathbb{D} \rightarrow F_M$ .

**PROOF.** We show  $(2) \Leftrightarrow (3)$ ,  $(2) \Rightarrow (1) \Rightarrow (4) \Rightarrow (5) \Rightarrow (2)$  and the equivalence of (2) with each of (6)–(8). The equivalence  $(2) \Leftrightarrow (3)$  follows from Theorem 6.20,  $(2) \Rightarrow (1)$  and  $(4) \Rightarrow (5)$  are trivial and  $(1) \Rightarrow (4)$  follows from Lemma 6.1. That (2) implies (6)–(8) follows from Lemma 6.1 and Table 6.1. Conversely, each of (5)–(8) implies that  $M$  is without large uniform minors and thus Theorem 5.9 applies. In turn, each of (5)–(8) excludes that any of  $\mathbb{D}$ ,  $\mathbb{F}_3$  and  $\mathbb{F}_2$  occur as a factor  $F_M$ , and thus (2).  $\square$

**6.7.8.  $\mathbb{D} \otimes \mathbb{H}$ -representable matroids.** Whittle describes in [36, Thm. 1.3] equivalent conditions that are satisfied by  $\mathbb{D} \otimes \mathbb{H}$ -representable matroids, which are conditions (4) and (5) below. We augment Whittle’s result with the following theorem.

**THEOREM 6.33.** *Let  $M$  be a matroid with foundation  $F_M$ . Then the following assertions are equivalent:*

- (1)  $M$  is  $\mathbb{D} \otimes \mathbb{H}$ -representable.
- (2)  $F_M \simeq F_1 \otimes \cdots \otimes F_r$  for  $r \geq 0$  and  $F_1, \dots, F_r \in \{\mathbb{U}, \mathbb{D}, \mathbb{H}\}$ .
- (3)  $M$  belongs to one of  $C_1$ – $C_5$ .
- (4)  $M$  is representable over  $\mathbb{F}_3$  and  $\mathbb{C}$ .
- (5)  $M$  is representable over  $\mathbb{F}_3$  and  $\mathbb{F}_q$ , where  $q$  is an odd prime power congruent to 1 modulo 3.
- (6)  $M$  is representable over  $\mathbb{F}_3$  and  $\mathbb{P}$ .

**PROOF.** We show  $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$  and the equivalence of (2) with each of (4)–(6). The implications  $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$  follow from Theorem 6.20. That (2) implies (4)–(6) follows from Lemma 6.1 and Table 6.1. Conversely, each of (4)–(6) implies that  $M$  is without large uniform minors by Lemma 6.2, and thus Theorem 5.9 applies. In turn, each of (4)–(6) excludes the possibility that either  $\mathbb{F}_3$  or  $\mathbb{F}_2$  occurs as a factor  $F_M$ , and thus (2).  $\square$

**6.7.9. Representable matroids without large uniform minors.** As a final application, we find the following equivalent characterization of matroids without large uniform minors which are representable over some field.

**THEOREM 6.34.** *Let  $M$  be a matroid without large uniform minors and  $F_M$  its foundation. Then the following assertions are equivalent:*

- (1)  $M$  is representable over some field.

- (2)  $F_M \simeq F_1 \otimes \cdots \otimes F_r$  for  $r \geq 0$  and either  $F_1, \dots, F_r \in \{\mathbb{U}, \mathbb{D}, \mathbb{H}, \mathbb{F}_3\}$  or  $F_1, \dots, F_r \in \{\mathbb{U}, \mathbb{H}, \mathbb{F}_2\}$ .
- (3)  $M$  belongs to one of  $C_1$ – $C_8$  or  $C_{10}$ .
- (4)  $M$  is ternary or quaternary.
- (5) There is no morphism from  $\mathbb{F}_2 \otimes \mathbb{D}$  to  $F_M$ .

PROOF. The equivalences  $(1) \Leftrightarrow (2) \Leftrightarrow (3)$  follow from Theorem 6.20. The implications  $(2) \Rightarrow (4) \Rightarrow (5) \Rightarrow (2)$  can be derived by combining the implications  $(2) \Rightarrow (1) \Rightarrow (7) \Rightarrow (2)$  from Theorem 6.28 and  $(2) \Rightarrow (1) \Rightarrow (6) \Rightarrow (2)$  from Theorem 6.29.  $\square$





# Bibliography

- [1] Laura Anderson, *Vectors of matroids over tracts*, J. Combin. Theory Ser. A **161** (2019), 236–270, DOI 10.1016/j.jcta.2018.08.002. MR3861778
- [2] Federico Ardila, Felipe Rincón, and Lauren Williams, *Positively oriented matroids are realizable*, J. Eur. Math. Soc. (JEMS) **19** (2017), no. 3, 815–833, DOI 10.4171/JEMS/680. MR3612868
- [3] Matthew Baker and Nathan Bowler, *Matroids over partial hyperstructures*, Adv. Math. **343** (2019), 821–863, DOI 10.1016/j.aim.2018.12.004. MR3891757
- [4] Matthew Baker and Tong Jin, *On the structure of hyperfields obtained as quotients of fields*, Proc. Amer. Math. Soc. **149** (2021), no. 1, 63–70, DOI 10.1090/proc/15207. MR4172586
- [5] Matthew Baker and Oliver Lorscheid, *The moduli space of matroids*, Adv. Math. **390** (2021), Paper No. 107883, 118, DOI 10.1016/j.aim.2021.107883. MR4292957
- [6] Robert E. Bixby, *On Reid’s characterization of the ternary matroids*, J. Combin. Theory Ser. B **26** (1979), no. 2, 174–204, DOI 10.1016/0095-8956(79)90056-X. MR532587
- [7] Robert G. Bland and David L. Jensen, *Weakly oriented matroids*, Cornell University School of OR/IE Technical Report No. 732, 1987.
- [8] Robert G. Bland and Michel Las Vergnas, *Orientability of matroids*, J. Combinatorial Theory Ser. B **24** (1978), no. 1, 94–123, DOI 10.1016/0095-8956(78)90080-1. MR485461
- [9] Nathan Bowler and Rudi Pendavingh, *Perfect matroids over hyperfields*, Preprint, [arXiv:1908.03420](https://arxiv.org/abs/1908.03420), 2019.
- [10] Nick Brettell and Rudi Pendavingh, *Computing excluded minors for classes of matroids representable over partial fields*, Electron. J. Combin. **31** (2024), no. 3, Paper No. 3.20, 23, DOI 10.37236/12078. MR4790340
- [11] Tom Brylawski and Dean Lucas, *Uniquely representable combinatorial geometries*, Proceedings of the International Colloquium on Combinatorial Theory. Rome, 1973.
- [12] Steven Creech, *Limits and colimits in the category of pastures*, Preprint, [arXiv:2103.08655](https://arxiv.org/abs/2103.08655), 2021.
- [13] Ilda P.F. da Silva, *Quelques propriétés des matroides orientés*, Ph. D. Thesis, Université Paris VI, 1987.
- [14] Andreas W. M. Dress and Walter Wenzel, *Geometric algebra for combinatorial geometries*, Adv. Math. **77** (1989), no. 1, 1–36, DOI 10.1016/0001-8708(89)90013-3. MR1014071
- [15] Andreas W. M. Dress and Walter Wenzel, *On combinatorial and projective geometry*, Geom. Dedicata **34** (1990), no. 2, 161–197, DOI 10.1007/BF00147324. MR1061287
- [16] Andreas W. M. Dress and Walter Wenzel, *Perfect matroids*, Adv. Math. **91** (1992), no. 2, 158–208, DOI 10.1016/0001-8708(92)90015-D. MR1149622
- [17] Israel M. Gelfand, Grigori L. Rybnikov, and David A. Stone, *Projective orientations of matroids*, Adv. Math. **113** (1995), no. 1, 118–150, DOI 10.1006/aima.1995.1036. MR1332809
- [18] Jon Lee and Matt Scobee, *A characterization of the orientations of ternary matroids*, J. Combin. Theory Ser. B **77** (1999), no. 2, 263–291, DOI 10.1006/jctb.1999.1914. MR1719344
- [19] Oliver Lorscheid, *A unifying approach to tropicalization*, Trans. Amer. Math. Soc. **376** (2023), no. 5, 3111–3189, DOI 10.1090/tran/8870. MR4577330
- [20] Ch. G. Massouros, *Methods of constructing hyperfields*, Internat. J. Math. Math. Sci. **8** (1985), no. 4, 725–728, DOI 10.1155/S0161171285000813. MR821630
- [21] Peter Nelson, *Almost all matroids are nonrepresentable*, Bull. Lond. Math. Soc. **50** (2018), no. 2, 245–248, DOI 10.1112/blms.12141. MR3830117
- [22] James G. Oxley, *Matroid theory*, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1992. MR1207587

- [23] Rudi Pendavingh, *Field extensions, derivations, and matroids over skew hyperfields*, Preprint, [arXiv:1802.02447](#), 2018.
- [24] R. A. Pendavingh and S. H. M. van Zwam, *Confinement of matroid representations to subsets of partial fields*, J. Combin. Theory Ser. B **100** (2010), no. 6, 510–545, DOI 10.1016/j.jctb.2010.04.002. MR2718674
- [25] R. A. Pendavingh and S. H. M. van Zwam, *Lifts of matroid representations over partial fields*, J. Combin. Theory Ser. B **100** (2010), no. 1, 36–67, DOI 10.1016/j.jctb.2009.03.006. MR2563513
- [26] Rudi A. Pendavingh and Stefan H. M. van Zwam, *Representing some non-representable matroids*, Preprint, [arXiv:1106.3088](#), 2011.
- [27] Charles Semple, *k-regular matroids*, Combinatorics, complexity, & logic (Auckland, 1996), Springer Ser. Discrete Math. Theor. Comput. Sci., Springer, Singapore, 1997, pp. 376–386. MR1647324
- [28] Charles Semple and Geoff Whittle, *On representable matroids having neither  $U_{2,5}$ - nor  $U_{3,5}$ -minors*, Matroid theory (Seattle, WA, 1995), Contemp. Math., vol. 197, Amer. Math. Soc., Providence, RI, 1996, pp. 377–386, DOI 10.1090/conm/197/02535. MR1411699
- [29] Charles Semple and Geoff Whittle, *Partial fields and matroid representation*, Adv. in Appl. Math. **17** (1996), no. 2, 184–208, DOI 10.1006/aama.1996.0010. MR1390574
- [30] P. D. Seymour, *Matroid representation over  $\text{GF}(3)$* , J. Combin. Theory Ser. B **26** (1979), no. 2, 159–173, DOI 10.1016/0095-8956(79)90055-8. MR532586
- [31] David Speyer and Lauren K. Williams, *The positive Dressian equals the positive tropical Grassmannian*, Trans. Amer. Math. Soc. Ser. B **8** (2021), 330–353, DOI 10.1090/btran/67. MR4241765
- [32] W. T. Tutte, *A homotopy theorem for matroids. I, II*, Trans. Amer. Math. Soc. **88** (1958), 144–174, DOI 10.2307/1993243. MR101526
- [33] W. T. Tutte, *Lectures on matroids*, J. Res. Nat. Bur. Standards Sect. B **69B** (1965), 1–47. MR179781
- [34] Walter Wenzel, *Projective equivalence of matroids with coefficients*, J. Combin. Theory Ser. A **57** (1991), no. 1, 15–45, DOI 10.1016/0097-3165(91)90003-Y. MR1101745
- [35] Geoff Whittle, *A characterisation of the matroids representable over  $\text{GF}(3)$  and the rationals*, J. Combin. Theory Ser. B **65** (1995), no. 2, 222–261, DOI 10.1006/jctb.1995.1052. MR1358987
- [36] Geoff Whittle, *On matroids representable over  $\text{GF}(3)$  and other fields*, Trans. Amer. Math. Soc. **349** (1997), no. 2, 579–603, DOI 10.1090/S0002-9947-97-01893-X. MR1407504
- [37] Geoff Whittle, *Recent work in matroid representation theory*, Discrete Math. **302** (2005), no. 1-3, 285–296, DOI 10.1016/j.disc.2004.07.039. MR2179649

## Editorial Information

To be published in the *Memoirs*, a paper must be correct, new, nontrivial, and significant. Further, it must be well written and of interest to a substantial number of mathematicians. Piecemeal results, such as an inconclusive step toward an unproved major theorem or a minor variation on a known result, are in general not acceptable for publication.

Papers appearing in *Memoirs* are generally at least 80 and not more than 200 published pages in length. Papers less than 80 or more than 200 published pages require the approval of the Managing Editor of the Transactions/Memoirs Editorial Board. Published pages are the same size as those generated in the style files provided for  $\mathcal{A}\mathcal{M}\mathcal{S}$ - $\mathcal{L}\mathcal{T}\mathcal{E}\mathcal{X}$ .

Information on the backlog for this journal can be found on the AMS website starting from <http://www.ams.org/memo>.

A Consent to Publish is required before we can begin processing your paper. After a paper is accepted for publication, the Providence office will send a Consent to Publish and Copyright Agreement to all authors of the paper. By submitting a paper to the *Memoirs*, authors certify that the results have not been submitted to nor are they under consideration for publication by another journal, conference proceedings, or similar publication.

## Information for Authors

*Memoirs* is an author-prepared publication. Once formatted for print and on-line publication, articles will be published as is with the addition of AMS-prepared frontmatter and backmatter. Articles are not copyedited; however, confirmation copy will be sent to the authors.

**Initial submission.** The AMS uses Centralized Manuscript Processing for initial submissions. Authors should submit a PDF file using the Initial Manuscript Submission form found at [www.ams.org/submission/memo](http://www.ams.org/submission/memo), or send one copy of the manuscript to the following address: Centralized Manuscript Processing, MEMOIRS OF THE AMS, 201 Charles Street, Providence, RI 02904-2294 USA. If a paper copy is being forwarded to the AMS, indicate that it is for *Memoirs* and include the name of the corresponding author, contact information such as email address or mailing address, and the name of an appropriate Editor to review the paper (see the list of Editors below).

The paper must contain a *descriptive title* and an *abstract* that summarizes the article in language suitable for workers in the general field (algebra, analysis, etc.). The *descriptive title* should be short, but informative; useless or vague phrases such as “some remarks about” or “concerning” should be avoided. The *abstract* should be at least one complete sentence, and at most 300 words. Included with the footnotes to the paper should be the 2020 *Mathematics Subject Classification* representing the primary and secondary subjects of the article. The classifications are accessible from [www.ams.org/msc/](http://www.ams.org/msc/). The Mathematics Subject Classification footnote may be followed by a list of *key words and phrases* describing the subject matter of the article and taken from it. Journal abbreviations used in bibliographies are listed in the latest *Mathematical Reviews* annual index. The series abbreviations are also accessible from [www.ams.org/msnhtml/serials.pdf](http://www.ams.org/msnhtml/serials.pdf). To help in preparing and verifying references, the AMS offers MR Lookup, a Reference Tool for Linking, at [www.ams.org/mrlookup/](http://www.ams.org/mrlookup/).

**Electronically prepared manuscripts.** The AMS encourages electronically prepared manuscripts, with a strong preference for  $\mathcal{A}\mathcal{M}\mathcal{S}$ - $\mathcal{L}\mathcal{T}\mathcal{E}\mathcal{X}$ . To this end, the Society has prepared  $\mathcal{A}\mathcal{M}\mathcal{S}$ - $\mathcal{L}\mathcal{T}\mathcal{E}\mathcal{X}$  author packages for each AMS publication. Author packages include instructions for preparing electronic manuscripts, samples, and a style file that generates the particular design specifications of that publication series.

Authors may retrieve an author package for *Memoirs of the AMS* from [www.ams.org/journals/memo/memoauthorpac.html](http://www.ams.org/journals/memo/memoauthorpac.html). The *AMS Author Handbook* is available in PDF format from the author package link. The author package can also be obtained free of charge by sending email to [tech-support@ams.org](mailto:tech-support@ams.org) or from the Publication Division,

American Mathematical Society, 201 Charles St., Providence, RI 02904-2294, USA. When requesting an author package, please specify the publication in which your paper will appear. Please be sure to include your complete mailing address.

**After acceptance.** The source files for the final version of the electronic manuscript should be sent to the Providence office immediately after the paper has been accepted for publication. The author should also submit a PDF of the final version of the paper to the editor, who will forward a copy to the Providence office.

Accepted electronically prepared files can be submitted via the web at [www.ams.org/submit-book-journal/](http://www.ams.org/submit-book-journal/), sent via FTP, or sent on CD to the Electronic Prepress Department, American Mathematical Society, 201 Charles Street, Providence, RI 02904-2294 USA. T<sub>E</sub>X source files and graphic files can be transferred over the Internet by FTP to the Internet node [ftp.ams.org](ftp://ftp.ams.org) (130.44.1.100). When sending a manuscript electronically via CD, please be sure to include a message indicating that the paper is for the *Memoirs*.

**Electronic graphics.** Comprehensive instructions on preparing graphics are available at [www.ams.org/authors/journals.html](http://www.ams.org/authors/journals.html). A few of the major requirements are given here.

Submit files for graphics as EPS (Encapsulated PostScript) files. This includes graphics originated via a graphics application as well as scanned photographs or other computer-generated images. If this is not possible, TIFF files are acceptable as long as they can be opened in Adobe Photoshop or Illustrator.

Authors using graphics packages for the creation of electronic art should also avoid the use of any lines thinner than 0.5 points in width. Many graphics packages allow the user to specify a “hairline” for a very thin line. Hairlines often look acceptable when proofed on a typical laser printer. However, when produced on a high-resolution laser imagesetter, hairlines become nearly invisible and will be lost entirely in the final printing process.

Screens should be set to values between 15% and 85%. Screens which fall outside of this range are too light or too dark to print correctly. Variations of screens within a graphic should be no less than 10%.

Any graphics created in color will be rendered in grayscale for the printed version unless color printing is authorized by the Managing Editor and the Publisher. In general, color graphics will appear in color in the online version.

**Inquiries.** Any inquiries concerning a paper that has been accepted for publication should be sent to [memo-query@ams.org](mailto:memo-query@ams.org) or directly to the Electronic Prepress Department, American Mathematical Society, 201 Charles St., Providence, RI 02904-2294 USA.

## Editors

This journal is designed particularly for long research papers, normally at least 80 pages in length, and groups of cognate papers in pure and applied mathematics. Papers intended for publication in the *Memoirs* should be addressed to one of the following editors. The AMS uses Centralized Manuscript Processing for initial submissions to AMS journals. Authors should follow instructions listed on the Initial Submission page found at [www.ams.org/memo/memosubmit.html](http://www.ams.org/memo/memosubmit.html).

Managing Editor: Dan Abramovich, Department of Mathematics, Brown University, Providence, RI 02912 USA; e-mail: [Dan\\_Abramovich@TAMS@brown.edu](mailto:Dan_Abramovich@TAMS@brown.edu)

### 1. GEOMETRY, TOPOLOGY & LOGIC

Coordinating Editor: Mariya Ivanova Soslava, Department of Mathematics, University of Wisconsin–Madison, Madison, WI 53706 USA; e-mail: [msoslava@math.wisc.edu](mailto:msoslava@math.wisc.edu)

**Algebraic topology**, Craig Westerland, School of Mathematics, University of Minnesota, 206 Church St. SE, Minneapolis, MN 55455 USA; e-mail: [cwester1@umn.edu](mailto:cwester1@umn.edu)

**Geometric group theory and low-dimensional geometry and topology**, Christopher Leininger, Department of Mathematics, Rice University, Houston, TX 77005 USA; e-mail: [cjl12@rice.edu](mailto:cjl12@rice.edu)

**Geometric topology**, Jennifer Hom, School of Mathematics, Georgia Institute of Technology, 686 Cherry Street, Atlanta, GA 30332 USA; e-mail: [jhom@gatech.edu](mailto:jhom@gatech.edu)

**Homotopy theory, higher category theory, and geometric applications**, Clark Barwick, School of Mathematics, University of Edinburgh, James Clerk Maxwell Building, Peter Guthrie Tait Road, Edinburgh, EH9 3FD, United Kingdom; e-mail: [clarkbar@maths.gla.ac.uk](mailto:clarkbar@maths.gla.ac.uk)

**Logic**, Mariya Ivanova Soslava

**Symplectic geometry**, Yael Karshon, Department of Mathematics, University of Toronto, Toronto, Ontario M5S 2E4, Canada; e-mail: [karshon@math.toronto.edu](mailto:karshon@math.toronto.edu)

### 2. ALGEBRA AND NUMBER THEORY

Coordinating Editor: Radha Kessar, Department of Mathematics, The Alan Turing Building, The University of Manchester, Oxford Road, Manchester M13 9PL, United Kingdom; e-mail: [radha.kesar@manchester.ac.uk](mailto:radha.kesar@manchester.ac.uk)

**Algebra, Algebraic and Arithmetic Geometry**, Daniel Krashen, 3E6 David Rittenhouse Lab., University of Pennsylvania, Department of Mathematics, 209 South 33rd Street, Philadelphia, PA 19104 USA; e-mail: [daniel.krashen@gmail.com](mailto:daniel.krashen@gmail.com)

**Algebraic geometry**, Dan Abramovich, Department of Mathematics, Brown University, Providence, RI 02912 USA; e-mail: [Dan\\_Abramovich@TAMS@brown.edu](mailto:Dan_Abramovich@TAMS@brown.edu)

**Arithmetic algebraic geometry**, Ehud de Shalit, Institute of Mathematics, Hebrew University, Giv'at-Ram, Jerusalem 91904 Israel; e-mail: [deshalit@math.huji.ac.il](mailto:deshalit@math.huji.ac.il)

**Arithmetic geometry and number theory**, Elena Mantovan, Department of Mathematics, PMA, California Institute of Technology, M/C 253-37, 1200 E California Blvd, Pasadena, CA 91125 USA; e-mail: [mantovan@caltech.edu](mailto:mantovan@caltech.edu)

**Associative rings and algebras, category theory, homological algebra, group theory and generalizations**, Daniel Nakano, Department of Mathematics, University of Georgia, Athens, Georgia 30602 USA; e-mail: [nakano@uga.edu](mailto:nakano@uga.edu)

**Commutative algebra**, Irena Peeva, Department of Mathematics, Cornell University, Ithaca, NY 14853 USA; e-mail: [irena@math.cornell.edu](mailto:irena@math.cornell.edu)

**Representation theory and group theory**, Radha Kessar

### 3. GEOMETRIC ANALYSIS & PDE

Coordinating Editor: Joachim Krieger, Bâtiment de Mathématiques, École Polytechnique Fédérale de Lausanne, Station 8, 1015 Lausanne Vaud, Switzerland; e-mail: [joachim.krieger@epfl.ch](mailto:joachim.krieger@epfl.ch)

**Calculus of variations, geometric measure theory, elliptic PDEs**, Francesco Maggi, Department of Mathematics, The University of Texas at Austin, 2515 Speedway, Stop C1200, Austin TX 78712-1202 USA; e-mail: [maggi@math.utexas.edu](mailto:maggi@math.utexas.edu)

**Elliptic and parabolic PDEs, geometric analysis**, Ben Weinkove, Mathematics Department, Northwestern University, 2033 Sheridan Rd, Evanston, IL 60201 USA; e-mail: [weinkove@math.northwestern.edu](mailto:weinkove@math.northwestern.edu)

**Elliptic PDEs, geometric analysis**, Eugenia Malinikova, Department of Mathematics, Stanford University, Stanford, CA 94305 USA; e-mail: [eugeniam@stanford.edu](mailto:eugeniam@stanford.edu)

**Harmonic analysis and partial differential equations**, Monica Visan, Department of Mathematics, University of California Los Angeles, 520 Portola Plaza, Los Angeles, CA 90095 USA; e-mail: [visan@math.ucla.edu](mailto:visan@math.ucla.edu)

**Nonlinear Fourier and harmonic analysis and partial differential equations**, Andrea R. Nahmod, Department of Mathematics and Statistics, University of Massachusetts Amherst, 710 N. Pleasant St. Lederle GRT, Amherst, MA 01003 USA; e-mail: [nahmod@umass.edu](mailto:nahmod@umass.edu)

**Real analysis and partial differential equations**, Joachim Krieger

**Riemannian geometry, metric geometry, mathematical general relativity, and geometric measure theory**, Christina Sormani, Lehman College and CUNY Graduate Center, 250 Bedford Park Boulevard West, Bronx, NY 10468 USA; e-mail: [sormanic@gmail.com](mailto:sormanic@gmail.com)

### 4. ANALYSIS & DYNAMICS

Coordinating Editor: Krzysztof Frączek, Faculty of Math and Computer Science, Nicolaus Copernicus University, Ul. Chopina 12/18 87-100 Toruń, Poland; e-mail: [fraczek@mat.umk.pl](mailto:fraczek@mat.umk.pl)

**Analysis, probability and ergodic theory**, Tim Austin, Department of Mathematics, University of California, Los Angeles, Los Angeles, CA 90095 USA; e-mail: [tim@math.ucla.edu](mailto:tim@math.ucla.edu)

**Ergodic theory and dynamical systems**, Krzysztof Frączek

**Ergodic theory, applications to combinatorics and number theory**, Nikos Frantzikinakis, University of Crete, Rethymno, Crete, Greece; e-mail: [frantzikinakis@gmail.com](mailto:frantzikinakis@gmail.com)

**Functional analysis, groups and operator algebras, analysis in quantum information theory**, Magdalena Musat, Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, 2100 Copenhagen, Denmark; e-mail: [musat@math.ku.dk](mailto:musat@math.ku.dk)

**Operator algebras and ergodic theory**, David Kerr, Mathematisches Institut, University of Münster, Einsteinstrasse 62, 48149 Münster, Germany; e-mail: [kerrd@uni-muenster.de](mailto:kerrd@uni-muenster.de)

### 5. PROBABILITY & COMBINATORICS

Coordinating Editor: Davar Khoshnevisan, Department of Mathematics, The University of Utah, Salt Lake City, UT 84112 USA; e-mail: [davar@math.utah.edu](mailto:davar@math.utah.edu)

**Combinatorics**, Jacob Fox, Department of Mathematics, 450 Jane Stanford Way, Building 380, Stanford, CA 94305 USA; e-mail: [jacobfox@stanford.edu](mailto:jacobfox@stanford.edu)

**Enumeration, algebraic combinatorics, integrable probability, symmetric functions**, Sylvie Corteel, Department of Mathematics, University of California, Berkeley, Berkeley, CA 94720 USA; CNRS, IRIF et Université Paris Cité, Paris, France; e-mail: [corteel@berkeley.edu](mailto:corteel@berkeley.edu)

**Probability theory and statistical physics**, Jian Ding, School of Mathematical Sciences, Peking University, Beijing 100871, China; e-mail: [dingjian@math.pku.edu.cn](mailto:dingjian@math.pku.edu.cn)

**Probability theory and stochastic analysis**, Davar Khoshnevisan

**Probability theory, probabilistic combinatorics, statistical physics**, Eyal Lubetzky, Department of Mathematics, Courant Institute of Mathematical Sciences, New York University, New York, NY 10012 USA; e-mail: [eyal@courant.nyu.edu](mailto:eyal@courant.nyu.edu)

SELECTED PUBLISHED TITLES IN THIS SERIES

- 1534 **Padraig Condon, Alberto Espuny Díaz, António Girão, Daniela Kühn, and Deryk Osthus**, Hamiltonicity of Random Subgraphs of the Hypercube, 2024
- 1533 **Peter Rudzis**, Rough Collisions, 2024
- 1532 **Xiang Yu**, On the Problem of Infinite Spin in Total Collisions of the Planar  $N$ -Body Problem, 2024
- 1531 **Dirk Basson, Florian Breuer, and Richard Pink**, Drinfeld Modular Forms of Arbitrary Rank, 2024
- 1530 **Jianbei An, Gerhard Hiss, and Frank Lübeck**, The Inductive Blockwise Alperin Weight Condition for the Chevalley Groups  $F_4(q)$ , 2024
- 1529 **Tadahiro Oh, Mamoru Okamoto, and Leonardo Tolomeo**, Focusing  $\Phi_3^4$ -Model with a Hartree-Type Nonlinearity, 2024
- 1528 **Dmitry Kubrak and Artem Prikhodko**,  $p$ -Adic Hodge Theory for Artin Stacks, 2024
- 1520 **Guixiang Hong, Simeng Wang, and Xumin Wang**, Pointwise Convergence of Noncommutative Fourier Series, 2024
- 1519 **Katarzyna Mazowiecka, Michał Miśkiewicz, and Armin Schikorra**, On the Size of the Singular Set of Minimizing Harmonic Maps, 2024
- 1518 **L. Alexander Betts**, The Motivic Anabelian Geometry of Local Heights on Abelian Varieties, 2024
- 1517 **Changningphaabi Namoiyam and Matthew A. Papanikolas**, Hyperderivatives of Periods and Quasi-Periods for Anderson  $t$ -Modules, 2024
- 1516 **Daniel Han-Kwan and David Michel**, On Hydrodynamic Limits of the Vlasov–Navier–Stokes System, 2024
- 1515 **Chenchen Mou and Jianfeng Zhang**, Wellposedness of Second Order Master Equations for Mean Field Games with Nonsmooth Data, 2024
- 1514 **Yohann Le Floch and Joseph Palmer**, Semitoric Families, 2024
- 1513 **Alcides Buss, Siegfried Echterhoff, and Rufus Willett**, Amenability and Weak Containment for Actions of Locally Compact Groups on  $C^*$ -Algebras, 2024
- 1512 **Alessandro Audrito and Susanna Terracini**, On the Nodal Set of Solutions to a Class of Nonlocal Parabolic Equations, 2024
- 1511 **Ermerson Araujo, Yuri Lima, and Mauricio Poletti**, Symbolic Dynamics for Nonuniformly Hyperbolic Maps with Singularities in High Dimension, 2024
- 1510 **C. Bleak, P. Cameron, Y. Maissel, A. Navas, and F. Olukoya**, The Further Chameleon Groups of Richard Thompson and Graham Higman: Automorphisms via Dynamics for the Higman-Thompson Groups  $G_{n,r}$ , 2024
- 1509 **Anthony Genevois**, Automorphisms of Graph Products of Groups and Acylindrical Hyperbolicity, 2024
- 1508 **Andrew J. Blumberg and Michael A. Mandell**, The Strong Künneth Theorem for Topological Periodic Cyclic Homology, 2024
- 1507 **Myoungjean Bae, Gui-Qiang G. Chen, and Mikhail Feldman**, Prandtl-Meyer Reflection Configurations, Transonic Shocks, and Free Boundary Problems, 2024
- 1506 **Radu Saghin, Pancho Valenzuela-Henríquez, and Carlos H. Vásquez**, Regularity with Respect to the Parameter of Lyapunov Exponents for Diffeomorphisms with Dominated Splitting, 2024
- 1505 **Deniz Bilman and Peter D. Miller**, Extreme Superposition: High-Order Fundamental Rogue Waves in the Far-Field Regime, 2024
- 1504 **Daniele Alessandrini, Olivier Guichard, Eugen Rogozinnikov, and Anna Wienhard**, Noncommutative Coordinates for Symplectic Representations, 2024

For a complete list of titles in this series, visit the  
AMS Bookstore at [www.ams.org/bookstore/memoseries/](http://www.ams.org/bookstore/memoseries/).

ISBN 978-1-4704-7268-9



**MEMO/305/1536**

