WALSH SPIDER DIFFUSIONS AS TIME CHANGED MULTI-PARAMETER PROCESSES

ERHAN BAYRAKTAR, JINGJIE ZHANG, AND XIN ZHANG

ABSTRACT. Inspired by allocation strategies in multi-armed bandit model, we propose a pathwise construction of Walsh diffusions. For any infinitesimal generator on a star shaped graph, there exists a unique time change associated with a multi-parameter process such that the time change of this multi-parameter process is the desired diffusion. The time change has an interpretation of time allocation of the process on each edge, and it can be derived explicitly from a family of equations.

1. INTRODUCTION

A Walsh spider diffusion is a singular process on a star shaped graph which behaves like a one-dimensional diffusion on the interior of each edge. Once it hits the vertex, it is kicked away from the vertex in a reflecting manner with a random direction according to some fixed distribution. Indicated by its name, such diffusions were first constructed explicitly by Walsh in [32] as an example of one dimensional diffusions with discontinuous local times. Since then, there has been a continuing interest in both theory and applications of such processes. Several constructions of Walsh diffusions have been proposed in the literature, including those based on Feller's semigroup theory [3], resolvents [29], martingale problem [26], and excursion theory [30]. An analogue of Itô's formula has been established in [19], and the ergodic behavior of Walsh diffusions has been studied in [14]. Stochastic control and optimization problems on this star-shaped model have been analyzed by [4, 13, 19]. Moreover, the connection between queueing theory and Walsh Brownian motion has been justified in [1].

In this paper, we focus on Walsh spider diffusion processes, which is a particular type of diffusion on graphs and "spider" in the name is a metaphor of the star-shaped graph. In the end we will mention that our construction can be easily generalized to diffusion on a metric graph with no loops.

In this paper, we propose a new construction of Walsh diffusions as a time change of multi-parameter processes; see Theorem 2.1. An N-parameter process can be viewed a collection of random variables indexed by $\mathbb{R}_+^N$. The theory of multi-parameter processes offers an elegant extension to the existing one-parameter theory, and finds diverse applications in

various fields, including functional analysis, analytic number theory, sequential stochastic optimization, and data science, and we refer readers to [8, 12, 21] for a nice introduction.

For any infinitesimal generator on a star shaped graph with N edges, we assign each edge a proper reflecting diffusion process Y_i and thus obtain a multi-parameter process $\mathbb{R}_+^N \ni (t_1, \ldots, t_N) \mapsto (Y_1(t_1), \ldots, Y_N(t_N))$. Inspired by the allocation strategies in multi-armed bandit problem [20, 25], we design a random time change $T(t) = (T_1(t), \ldots, T_N(t))$ with respect to the multi-parameter filtration, which can be interpreted as time allocation of the process on each edge and is determined uniquely by a family of explicit equations involving local times. The construction of the random time change $T(t)$ ensures that the occupation time of vertex in each direction is distributed according to a given proportion and only one process Y_i is running at any given time t. After establishing the time-homogeneity of the random time change and the strong Markov property of the multi-pamamerater process, we demonstrate that the time changed multi-parameter process $(Y_1(T_1(t)), \ldots, Y_N(T_N(t)))$ is a diffusion characterized by the desired infinitesimal generator. Additionally, our construction enables us to readily obtain the long-term average occupation time of each edge, as shown in Corollary 3.2.

The well-known skew Brownian motion is a simple example of the Walsh spider diffusion, where the graph has 2 edges and the diffusion behaves like a reflecting Brownian motion in the interior of each edge. Different constructions and generalized versions of skew Brownian motion have been studied since the concept was first mentioned by Ito and McKean [15, Section 4.2, Problem 1]; see [24] and references therein. Among them, "follow the leader" construction of variably skew Brownian motion in [2] is closely related to our work. It provided a weak solution to a stochastic differential equation involving local time as a time change of a two-parameter process. While the time change therein is determined by comparing running maximum of two Brownian motions, we derive the time change explicitly through a family of equations of local times. This family of equations represents two conditions imposed on the time change straightforwardly: *(i)* the sum of time allocations on each edge equals the total running time; *(ii)* the local time of each edge at vertex is proportional to the probability of choosing each edge at any time. Our approach not only provides clear intuition, but also enables us to generalize the results from $N = 2$ and reflecting Brownian motion to $N \geq 3$ and general reflecting diffusion. Furthermore, we adopt a different argument in the proof of our main theorem, which is mainly based on time homogeneity of the time change; see Proposition 3.1 *(iii)* and the strong Markov property of multi-parameter process.

In the next section, we introduce some notions of multi-parameter process and our main result. In Section 3, we present an explicit construction of Walsh diffusion and prove our main result. Generalizing our approach to diffusions on a metric graph is also discussed.

2. PRELIMINARY AND MAIN RESULT

Let Γ be a star shaped metric graph with one internal vertex v and edges $e_1, \ldots, e_N$. A Walsh spider diffusion on Γ is characterized by an infinitesimal operator A with domain $\mathcal{D}(A)$. Let us introduce the following notations to define $(A, \mathcal{D}(A))$.

For each $i = 1, \ldots, N$, a coordinate function C_i is chosen on e_i such that e_i is homeomorphic to an interval I_i of real line and $C_i(v) = 0$. I_i can be either $[0, \infty)$ or $[0, l_i]$ where $l_i > 0$ representing the length of edge e_i. Let us denote by $\partial\Gamma$ the boundary of Γ, i.e., $\partial\Gamma = \{x : x \in e_i, C_i(x) = l_i < \infty, i = 1, \ldots, N\}$. We equip the star shaped graph Γ with the tree metric d

$$d(x, x') = \begin{cases} |C_i(x) - C_i(v)| + |C_j(x') - C_j(v)|, & i \neq j,\ x \in e_i,\ x' \in e_j, \\ |C_i(x) - C_i(x')|, & x, x' \in e_i. \end{cases}$$

A function $f : \Gamma \to \mathbb{R}$ is said to be continuous if f is continuous with respect to the tree metric d. Denote by $\mathcal{C}_0(\Gamma)$ the space of continuous functions on Γ that vanish at infinity. We say a function $f \in \mathcal{C}_0(\Gamma)$ belongs to $\mathcal{C}_0^\infty(\Gamma)$ if $f|_{e_i^o}$ is smooth, and all derivatives can be naturally extended to e_i for all $i = 1, \ldots, N$, where e_i^o denotes the interior of e_i.

For a function $f : \Gamma \to \mathbb{R}$, we denote its derivative at vertex v in the direction of edge i by

$$D_i f(v) := \lim_{x \in e_i^o, x \to v} \frac{f(x) - f(v)}{|C_i(x) - C_i(v)|},$$

if the limit exists, and we denote its derivative at $y \in e_i$, $y \neq v$ by

$$Df(y) := \lim_{x \in e_i^o, x \to y} \frac{f(x) - f(y)}{|C_i(x) - C_i(y)|},$$

if the limit exists.

Let $\{b_i, \sigma_i\}_{i=1,\ldots,N}$ be bounded and Lipschitz functions on I_i, and $\{\sigma_i^2\}_{i=1,\ldots,N}$ are uniformly strictly positive. Define the operators $\mathcal{L}_i$ via

$$\mathcal{L}_i f(x) = \frac{1}{2} \sigma_i^2(y_i) \frac{d^2 f}{dy_i^2}(y_i) + b_i(y_i) \frac{df}{dy_i}(y_i), \quad x \in e_i^o,\ C_i(x) = y_i,\ i = 1, \ldots, N.$$

Let $\{\alpha_i : i = 1, \cdots, N\}$ be some fixed positive constants such that $\sum_{i=1}^N \alpha_i = 1$. Define an operator A with domain $\mathcal{D}(A) \subset \mathcal{C}_0^\infty(\Gamma)$ as the following.

$$Af(x) := \mathcal{L}_i f(x),\ x \in e_i,\ i = 1, \ldots, N,$$

$$\mathcal{D}(A) := \left\{ f \in \mathcal{C}_0^\infty(\Gamma) : Af \in \mathcal{C}_0(\Gamma),\ \sum_{j=1}^N \alpha_j D_j f(v) = 0,\ Df(y) = 0,\ \forall y \in \partial\Gamma \right\}. \quad (2.1)$$

Note that the condition $Af \in \mathcal{C}_0(\Gamma)$ implies the infinitesimal changes of f coincide at the vertex v in every direction, i.e., $\mathcal{L}_i f(v) = \mathcal{L}_j f(v)$, $\forall i \neq j$.

It was proved in [3, 11] using Hille-Yosida theorem that there exists a continuous Feller process $X(\cdot)$ on Γ with infinitesimal generator $(A, \mathcal{D}(A))$, i.e.,

$$\lim_{t \to 0} \frac{P_t f(x) - f(x)}{t} = Af(x), \quad \text{for } x \in \Gamma \text{ and } f \in \mathcal{D}(A),$$

where $P_t f(x) := \mathbb{E}[f(X(t)) \,|\, X_0 = x]$ is the semigroup generated by $X(\cdot)$. In words, on the interior of each edge, e_i^o, the process $X(\cdot)$ behaves like a diffusion with drift b_i and

volatility σ_i. And once hitting the vertex v, the process will be assigned independently a direction e_j with probability $\alpha_j > 0$; see e.g. [3, 19].

Remark 2.1. *The domain $\mathcal{D}(A)$ determines the behavior of process at the vertex. In the simplest case $N = 1$, $\alpha_1 = 1$, $b_1 \equiv 0$, $\sigma_1 \equiv 1$, with $\mathcal{D}(A)$ defined in (2.1) we get a reflecting Brownian motion on the interval I_1. Another choice of domain $\tilde{\mathcal{D}}(A) := \{f \in \mathcal{C}_0^\infty(\Gamma) : Af \in \mathcal{C}_0(\Gamma), f(v) = 0\}$ provides a Brownian motion with absorption at v; see e.g. [16].*

Suppose there exists a Walsh diffusion (X_t) with the prescribed behavior at the vertex v. It is not hard to verify that to make the limit $\lim_{t \to 0} \frac{P_t f(v) - f(v)}{t}$ exist, it is necessary to require $f \in \mathcal{D}(A)$; see the proof of Proposition 3.3. In words, the weighted average derivative of f in each direction should be 0 so that the infinitesimal generator is well-defined for such reflecting diffusion.

The purpose of this paper is to provide a pathwise construction of Walsh diffusion, which is achieved by showing that a Walsh diffusion can be represented as a time change of a multi-parameter process. Let us first introduce some notations and terminology about multi-parameter process. All filtrations of stochastic processes are assumed to satisfy the usual conditions in this paper.

Throughout, the i-th coordinate of any point $\mathbf{s} \in \mathbb{R}_+^{\mathbf{N}}$ is written as s_i. The order structure of $\mathbb{R}_+^N$ is explained as the following. For any $\mathbf{t}, \mathbf{s} \in \mathbb{R}_+^N$, we write $\mathbf{s} \leq \mathbf{t}\,(\mathbf{s} < \mathbf{t})$ or equivalently $\mathbf{t} \geq \mathbf{s}\,(\mathbf{t} > \mathbf{s})$ if for all $i = 1, \ldots, N$, $s_i \leq t_i\,(s_i < t_i)$.

On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a collection of sub σ-algebras, $\mathbb{F} = (\mathcal{F}_{\mathbf{t}}, \mathbf{t} \in \mathbb{R}_+^N)$, is an N-parameter filtration if $\mathbf{s} \leq \mathbf{t}$ implies that $\mathcal{F}_{\mathbf{s}} \subset \mathcal{F}_{\mathbf{t}}$. An N-parameter stochastic process $Z = (Z_{\mathbf{t}}, \mathbf{t} \in \mathbb{R}_+^N)$ is adapted to the filtration $\mathbb{F}$ if for all $\mathbf{t} \in \mathbb{R}_+^N$, $Z_{\mathbf{t}}$ is $\mathcal{F}_{\mathbf{t}}$-measurable.

Definition 2.1. *A random variable $\boldsymbol{\tau} = (\tau_1, \cdots, \tau_N) \in \mathbb{R}_+^N$ is a stopping point w.r.t. the filtration $\mathbb{F}$, if the following holds for any $\mathbf{s} \in \mathbb{R}_+^N$*

$$\bigcap_{i=1}^N \{\tau_i \leq s_i\} \in \mathcal{F}_{\mathbf{s}}.$$

For any stopping point $\boldsymbol{\tau}$, we associate it with a σ-algebra

$$\mathcal{F}_{\boldsymbol{\tau}} := \left\{A \in \mathcal{F} : A \cap \{\boldsymbol{\tau} \leq \mathbf{s}\} \in \mathcal{F}_{\mathbf{s}} \text{ for any } \mathbf{s} \in \mathbb{R}_+^N\right\}.$$

The concept of stopping point is a generalization of stoping time in the context of multi-parameter processes. Similarly, we define multi-parameter random time change as a family of stopping points that satisfies certain conditions.

Definition 2.2. *A $\mathbb{R}_+^N$-valued process $\mathbf{T}(\cdot) = (T_1(\cdot), \cdots, T_N(\cdot))$ is called a multi-parameter random time change w.r.t. the filtration $\mathbb{F}$ if the following two properties hold:*

  (i) $\mathbf{T}$ has continuous non-decreasing sample paths on $\mathbb{R}_+^N$, i.e., $t \mapsto T_i(t)$ is non-decreasing and continuous for each $i = 1, \ldots, N$ almost surely;
  (ii) $\mathbf{T}(t)$ is a stopping point w.r.t. the filtration $\mathbb{F}$ for each $t \geq 0$.

We are ready to state our main theorem, which says that a Walsh diffusion can be represented as a time change of a multi-parameter process. Furthermore, the time change is the unique solution to a family of equations.

To this end, we identify each point $x \in \Gamma$ as a vector $\boldsymbol{y} = (y_1, \ldots, y_N)$ in $\mathbb{R}^N$, where for each $i = 1, \ldots, N$, $y_i = C_i(x)$ if $x \in e_i$ and $y_i = 0$ if $x \notin e_i$. It is clear that the map

$$\boldsymbol{y}(x) := (y_1, \ldots, y_N) \tag{2.2}$$

is a continuous injection from Γ to $\mathbb{R}^N$.

In the rest of the paper, we fix a starting point $x^0 \in \Gamma$ with $\boldsymbol{y}(x^0) = (y_1^0, \ldots, y_N^0)$ and an infinitesimal generator $(A, \mathcal{D}(A))$ as in (2.1). For each $i = 1, \ldots, N$, there exists a reflecting diffusion process $Y_i(\cdot)$ on a filtered probability space $(\Omega^i, \mathcal{F}^i, (\mathcal{F}_t^i), \mathbb{P}^i)$ such that $Y_i(\cdot)$ has infinitesimal generator $\mathcal{L}_i$, reflects at end point(s) on interval I_i, and $Y_i(0) = y_i^0$; see e.g. [27] for constructions of reflecting diffusions. Throughout the paper, it is convenient to take $\Omega^i = \mathcal{C}([0, \infty); \mathbb{R})$ and $Y_i(\cdot)$ to be the canonical process on Ω^i. Denote by $L^i(\cdot)$ the local time of semi-martingale Y_i at the vertex v, where we adopt the definition of local time from [28, Chapter VI]. To guarantee the strong existence of Y_i and the recurrence of Y_i at 0, we make the following assumption. Thanks to [18, Proposition 5.5.22], each Y_i is recurrent at 0 under Assumption 2.1 *(ii)*.

Assumption 2.1.

(i) *The coefficients $\{b_i, \sigma_i\}_{i=1,\ldots,N}$ are bounded and Lipschitz continuous, and $\{\sigma_i^2\}_{i=1,\ldots,N}$ are uniformly strictly positive.*

(ii) *For each $i = 1, \ldots, N$, we have either $|I_i| = l_i < \infty$, or*

$$\int_0^\infty \exp\left(-2\int_0^\xi \frac{b_i(h)}{\sigma_i^2(h)}\, dh\right) d\xi = \infty.$$

We take the N-parameter process $(\boldsymbol{Y}(\boldsymbol{s}) = (Y_1(s_1), \ldots, Y_N(s_N)), \boldsymbol{s} \in \mathbb{R}_+^N)$ on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_{\boldsymbol{s}}), \mathbb{P})$ where

$$\Omega := \Omega^1 \times \ldots \times \Omega^N, \qquad \mathcal{F} := \sigma^{\mathbb{P}}\left(\mathcal{F}^1 \otimes \ldots \otimes \mathcal{F}^N\right), \tag{2.3}$$

$$\mathbb{P} := \mathbb{P}^1 \otimes \ldots \otimes \mathbb{P}^N, \qquad \mathcal{F}_{\boldsymbol{s}} := \sigma^{\mathbb{P}}\left(\mathcal{F}_{s_1}^1 \otimes \ldots \otimes \mathcal{F}_{s_N}^N\right), \ \forall \boldsymbol{s} \in \mathbb{R}_+^N.$$

We adopt the convention that $\sigma^{\mathbb{P}}(\mathcal{G})$ represents the complete σ-algebra generated by $\mathcal{G}$ under $\mathbb{P}$ for any $\mathcal{G} \subset \mathcal{F}^1 \otimes \ldots \otimes \mathcal{F}^N$. In Lemma 3.2, we will verify the right-continuity of filtration $(\mathcal{F}_{\boldsymbol{s}})$, i.e., $\mathcal{F}_{\boldsymbol{t}} = \bigcap_{\boldsymbol{s} > \boldsymbol{t}} \mathcal{F}_{\boldsymbol{s}}$ for any $\boldsymbol{t} \in \mathbb{R}_+^N$, given that all $(\mathcal{F}_{s_i}^i), i = 1, 2, \cdots, N$ satisfy the usual conditions.

Theorem 2.1. *Under Assumption 2.1, given the N-parameter process $\boldsymbol{Y}(\cdot)$ there exists a unique random time change $\boldsymbol{T}(\cdot)$ w.r.t. the filtration $(\mathcal{F}_{\boldsymbol{s}})$ such that*

$$X(\cdot) := (Y_1(T_1(\cdot)), \ldots, Y_N(T_N(\cdot)))$$

is a diffusion on Γ with filtration $(\mathcal{F}_{\boldsymbol{T}(t)})$ that starts from position x^0 with infinitesimal generator $(A, \mathcal{D}(A))$.

Furthermore, almost surely for each $t \geq 0$, $(T_i(t) = s_i)_{i=1}^N$ is the unique solution of equations

$$\sum_{i=1}^N s_i = t, \quad and \quad \frac{L^i(s_i)}{\alpha_i} = \frac{L^j(s_j)}{\alpha_j} \quad for \ all \ 1 \leq i, j \leq N.$$

Remark 2.2. *It was proved in* [31] *that Walsh Brownian motion does not generate a Brownian filtration for $N \geq 3$. To this end, it showed that the property of being "cozy" for filtered probability spaces is invariant under morphism. Then the claim follows from the fact that any filtered probability space generated by Brownian motion is "cozy", while Walsh Brownian motion is not. A relatively straightforward proof can been found in* [33, Chapter 1.3]

The multi-parameter filtration (2.3) considered in this paper is not in the same category of one-parameter filtrations, as the time index values in $\mathbb{R}_+^N$ instead of $\mathbb{R}_+$. Hence our result doesn't contradict with [31]. *Actually, our theorem offers an alternative perspective that exemplifies this finding. Specifically, let $(\mathcal{F}_t^i)$, $i = 1, \ldots, N$, be chosen as Brownian filtrations. According to Theorem 2.1, the Walsh Brownian motion X is not adapted to the Brownian filtration $(\otimes_{i=1}^N \mathcal{F}_t^i)$. Instead, X is adapted to $(\mathcal{F}_{\boldsymbol{T}(t)})$, which is a time change of multi-parameter filtration $(\mathcal{F}_{\boldsymbol{s}})$ in (2.3).*

3. Proof of Main Result

We start with the construction of a random time change $\boldsymbol{T}$ with respect to the multi-parameter filtration $(\mathcal{F}_{\boldsymbol{s}})$. Define the left-continuous inverse of $L^i(\cdot)/\alpha_i$ as

$$h_i(l) := \inf \left\{ t \geq 0 : \frac{1}{\alpha_i} L^i(t) \geq l \right\}, \quad l \in \mathbb{R}_+, \tag{3.1}$$

for $i = 1, \ldots, N$, and denote their sum by

$$h(l) := \sum_{i=1}^N h_i(l). \tag{3.2}$$

It is clear that h and h_i, $i = 1, \ldots, N$, are strictly increasing functions and transform local time to real time. Let us also take the left continuous inverse of h, which transform real time to local time.

$$L(t) := \inf\{l \geq 0 : h(l) \geq t\}. \tag{3.3}$$

Since $h(l)$ is a strictly increasing function, its left-continuous inverse $L(t)$ is actually continuous. We will see that $L(t)$ is the local time of process X at the vertex v in (3.15).

We now introduce the random time change $\boldsymbol{T}(t) = (T_1(t), T_2(t), \cdots, T_N(t))$, where each $T_i(t)$ has the interpretation of the total time that the Walsh diffusion process spends on the edge i up to time t. The main idea of our construction is the following: *(i)* impose the condition

$$\frac{L^1(T_1(t))}{\alpha_1} = \ldots = \frac{L^N(T_N(t))}{\alpha_N},$$

so that at each time t the occupation time of vertex v for each Y_i is proportional to α_i; *(ii)* at each time t with the corresponding local time level $l = L(t)$, only the process Y_i with a jump at $h_i(l)$, i.e., $\Delta h_i(l) := h_i(l+) - h_i(l) > 0$, will be running and all other Y_j with $\Delta h_j(l) = 0$ will be frozen. Precisely, we have the following definition

Definition 3.1. *Given the reflecting diffusion processes $(Y_i(\cdot))_{i=1}^N$ and the filtered proba- bility space $(\Omega, \mathcal{F}, (\mathcal{F}_s), \mathbb{P})$ in (2.3), we define $\boldsymbol{T} = (T_1(\cdot), \cdots, T_N(\cdot))$ via*

$$T_i(t) := h_i(L(t)) + \mathbb{1}_{\{\Delta h_i(L(t)) > 0\}}(t - h(L(t))), \quad \forall i \in \{1, 2, \cdots, N\}, \tag{3.4}$$

where $h_i(\cdot), h(\cdot)$ and $L(\cdot)$ are given by (3.1), (3.2) and (3.3), respectively.

Remark 3.1. *Note that here the time change $\boldsymbol{T}$ is expressed in terms of the left-continuous inverse h_i of local times, where $h_i(l)$ stands for the starting time of excursion at the local time level l. Then by (3.4), it is clear that $t \mapsto T_i(t)$ is increasing.*

One can also write $\boldsymbol{T}$ using the right-continuous inverse of local times. We would like to emphasize that the construction of random time change $\boldsymbol{T}$ doesn't depend on the choice of left-continuous inverse or right-continuous inverse of local times.

In the following lemma, we show that almost surely there is at most one process with a jump at $h_i(l)$ for all $l \geq 0$.

Lemma 3.1. *$\{h_i\}_{i=1}^N$ have no simultaneous jumps almost surely. That is for*

$$\Omega_0 := \left\{ \omega \in \Omega : \sum_{l \geq 0} \Delta h_i(l)(\omega) \Delta h_j(l)(\omega) = 0, \forall i \neq j \right\}$$

we have $\mathbb{P}[\Omega_0] = 1$.

Proof. Notice that $\Omega_0 = \Omega_1 \cap \Omega_2$, where

$$\Omega_1 = \left\{ \omega \in \Omega : \sum_{l > 0} \Delta h_i(l)(\omega) \Delta h_j(l)(\omega) = 0, \forall i \neq j \right\};$$

$$\Omega_2 = \left\{ \omega \in \Omega : \Delta h_i(0)(\omega) \Delta h_j(0)(\omega) = 0, \forall i \neq j \right\}.$$

Let us first prove that $\mathbb{P}(\Omega_1) = 1$ using the excursion theory. For each process Y_i, point 0 is regular, i.e., $\mathbb{P}^0(\tau_0 = 0) = 1$ and recurrent, i.e., $\mathbb{P}^{y_i}(\tau_0 < \infty) = 1, \forall y_i \in I_i$, where $\tau_0 := \inf\{t \geq 0 : Y_i(t) = 0\}$ denotes the hitting time of 0. We denote the excursion process of Y_i from point 0 by $(e_i(s))_{s>0}$. The excursion process takes values in the excursion space

$$U_\delta^{(i)} = U^{(i)} \cup \delta,$$

where δ is the function that is identically 0, $U^{(i)} = \cup_{n=1}^\infty U_n^{(i)}$, and

$$U_n^{(i)} = \left\{ \text{continuous path } c : [0, \tau] \to I_i \text{ where } \tau = \inf\{t > 0 : c(t) = 0\} \text{ satisfies } \frac{1}{n} < \tau < \infty \right\}.$$

The excursion process is given by

$$
e_i(s) = \begin{cases} Y_i(h_i(\frac{s}{\alpha_i}) + \cdot) : [0, \Delta h_i(\frac{s}{\alpha_i})] \to I_i & \text{if } \Delta h_i(\frac{s}{\alpha_i}) > 0; \\ \delta & \text{if } \Delta h_i(\frac{s}{\alpha_i}) = 0. \end{cases}
$$

It is known that $e_i(\cdot)$ is a Poisson point process; see e.g. [5] .

For each $n \geq 1$,

$$
N_i^{(n)}(l) := \sum_{s \leq l} \mathbb{1}_{\{e_i(s) \in U_n^{(i)}\}}, \quad l > 0
$$

denotes the number of excursions before local time l whose time duration $\Delta h_i(\frac{s}{\alpha_i})$ is greater than $\frac{1}{n}$. Therefore we have

$$
N_i^{(n)}(l) \leq n h_i\left(\frac{l}{\alpha_i}+\right) < \infty,
$$

and $N_i^{(n)}(\cdot)$ is a Poisson process with finite jump rate. For any $i \in \{1, 2, \cdots, N\}, n \geq 1$, define another Poisson process $\tilde{N}_i^{(n)}(l) := N_i^{(n)}(\alpha_i l)$. By the independence of Y_i, $\tilde{N}_i^{(n)}(\cdot), i \in \{1, 2, \cdots, N\}$ are also independent. According to [28, Chapter XII, Proposition 1.5], independent Poisson processes have no simultaneous jumps, i.e.,

$$
\mathbb{P}\left(\sum_{l>0} \Delta \tilde{N}_i^{(n)}(l) \tilde{N}_j^{(n)}(l) = 0, \forall i \neq j\right) = 1, \quad \forall n \geq 1.
$$

Notice that $\Delta h_i(l)(\omega) > 0$ if and only if $L^i(\cdot)$ has a constant stretch at level $\alpha_i l$, i.e., $e_i(\alpha_i l) \neq \delta$, which is equivalent to that there exists $n \geq 1$ such that $\Delta \tilde{N}_i^{(n)}(l) > 0$. Thus

$$
\begin{aligned}
\mathbb{P}(\Omega_1) &= \mathbb{P}\left(\left\{\sum_{l>0} \Delta h_i(l) \Delta h_j(l) = 0, \forall i \neq j\right\}\right) \\
&= \mathbb{P}\left(\bigcap_{n \geq 1} \left\{\sum_{l>0} \Delta \tilde{N}_i^{(n)}(l) \tilde{N}_j^{(n)}(l) = 0, \forall i \neq j\right\}\right) \\
&= \lim_{n \to \infty} \mathbb{P}\left(\sum_{l>0} \Delta \tilde{N}_i^{(n)}(l) \tilde{N}_j^{(n)}(l) = 0, \forall i \neq j\right) \\
&= 1.
\end{aligned}
$$

Next consider $\Omega_2 = \{\omega \in \Omega : \Delta h_i(0)(\omega) \Delta h_j(0)(\omega), \forall i \neq j\}$. For each Y_i, when $Y_i(0) = 0$, we have $\Delta h_i(0) = 0$. If this does not hold, there exists $\epsilon > 0$ such that for all $s > 0$, $h_i(s) > \epsilon$, which implies $L_i(\epsilon) < \alpha_i s$ for any $s > 0$. However it contradicts with the fact that $\mathbb{P}[L^i(\epsilon) > 0 \,|\, Y_i(0) = 0] = 1$. Therefore in both cases of the initial position $x = v$ and $x \neq v$, there is at most one $i \in \{1, 2, \cdots, N\}$ such that $\Delta h_i(0)(\omega) > 0$. This leads to $\mathbb{P}(\Omega_2) = 1$ and hence

$$
\mathbb{P}(\Omega_0) = 1 - \mathbb{P}(\Omega_1^c \cup \Omega_2^c) = 1.
$$

□

Remark 3.2. *Note that h_i has a jump at l, i.e., $h_i(l) < h_i(l+)$, is equivalent to that the local time $L^i(\cdot)$ is flat at level $\alpha_i l$, i.e., $\mathrm{Leb}(\{t : L^i(t) = \alpha_i l\}) > 0$. Hence the absence of simultaneous jumps in processes $\{h_i\}_{i=1}^N$ is equivalent to the absence of same level of plateaus in processes $\{L^i(\cdot)/\alpha_i\}_{i=1}^N$. It is known that the local time $L^i(\cdot)$ for process Y_i only exhibits a plateau when the process is on an excursion away from v. Therefore, heuristically, this lemma implies that for any $l \geq 0$, at most one process among $\{Y_i\}_{i=1}^N$ is on an excursion at local time $\{\alpha_i l\}_{i=1}^N$.*

Remark 3.3. *Heuristically, $T_i(t)$ is the the total time that the Walsh diffusion process spends on edge i up to time t and we will show in Proposition 3.1 that $\sum_i T_i(t) = t$ almost surely. Moreover (3.4) and Lemma 3.1 indicate that $(T_i(t), i = 1, 2, \cdots, N)$ increase in either one of the following manners.*

(1) *There is exactly one of $(T_i(t), i = 1, 2, \cdots, N)$ that is strictly increasing and it increases at rate 1. This happens when $\Delta h(L(t)) > 0$ and for the one and only one i such that $\Delta h_i(L(t)) > 0$, we have $dT_i(t) = dt$.*

(2) *All $(T_i(t), i = 1, 2, \cdots, N)$ strictly increase simultaneously. This happens when $\Delta h(L(t)) = 0$, equivalently when $L(\cdot)$ strictly increases at t. As the local time of the Walsh diffusion X at vertex v, $L(\cdot)$ only increases at $\{s \geq 0 : X(s) = v\}$, which has Lebesgue measure 0. So we say $(T_i(t), i = 1, 2, \cdots, N)$ increase simultaneously at essentially no time.*

The following lemma shows that if the filtration $(\mathcal{F}^i_{s_i})$ is right-continuous and complete for each $i = 1, \ldots, N$, then the multi-parameter filtration $(\mathcal{F}_{\boldsymbol{s}})$ defined in (2.3) is right-continuous as well.

Lemma 3.2. *The filtration $(\mathcal{F}_{\boldsymbol{s}})$ is right continuous, i.e., $\mathcal{F}_{\boldsymbol{t}} = \bigcap_{\boldsymbol{s} > \boldsymbol{t}} \mathcal{F}_{\boldsymbol{s}}$ for any $\boldsymbol{t} \in \mathbb{R}_+^N$.*

Proof. Let us take a decreasing sequence $(\boldsymbol{s}^k)_{k\in\mathbb{N}}$ such that $\boldsymbol{s}^k > \boldsymbol{t}$ and $\lim_{k\to\infty} \boldsymbol{s}^k = \boldsymbol{t}$. Notice that the claim is equivalent to $\mathbb{P}[A \,|\, \mathcal{F}_{\boldsymbol{t}}] = \mathbb{P}\left[A \,|\, \bigcap_{k\in\mathbb{N}} \mathcal{F}_{\boldsymbol{s}^k}\right]$ for any $A \in \mathcal{F}$. By a standard monotone class argument, it suffices to prove $\mathbb{P}[A \,|\, \mathcal{F}_{\boldsymbol{t}}] = \mathbb{P}\left[A \,|\, \bigcap_{k\in\mathbb{N}} \mathcal{F}_{\boldsymbol{s}^k}\right]$ for all $A = A_1 \times \ldots \times A_N$ with $A_i \in \mathcal{F}^i$, $i = 1, \ldots, N$.

Since each filtration $(F^i_{s_i})_{s_i \in \mathbb{R}_+}$ is complete and right continuous, the probability measure $\mathbb{P}$ is a product of $\mathbb{P}^1, \ldots, \mathbb{P}^N$, we have

$$\mathbb{P}[A \,|\, \mathcal{F}_{\boldsymbol{t}}] = \prod_{i=1}^N \mathbb{P}^i[A_i \,|\, \mathcal{F}^i_{t_i}] = \prod_{i=1}^N \lim_{k\to\infty} \mathbb{P}^i[A_i \,|\, \mathcal{F}^i_{s_i^k}] = \lim_{k\to\infty} \prod_{i=1}^N \mathbb{P}^i[A_i \,|\, \mathcal{F}^i_{s_i^k}],$$

where in the second equality we make use of the backward martingale convergence theorem; see e.g. [23, Appendix A.2]. By the same token, the last term is equal to $\lim_{k\to\infty} \mathbb{P}[A \,|\, \mathcal{F}_{\boldsymbol{s}^k}] = \mathbb{P}\left[A \,|\, \bigcap_{k\in\mathbb{N}} \mathcal{F}_{\boldsymbol{s}^k}\right]$, which completes our proof. $\qquad\square$

We are now ready to present the key properties of the random time change $\boldsymbol{T}(t)$ defined in (3.4). Before that we recall the definition of shift operator $\boldsymbol{\theta}$ on canonical spaces.

Denote the path space by $\Omega = \mathcal{C}([0,\infty); \mathbb{R})^N$, and each element therein by $\omega = (\omega_1, \ldots, \omega_N)$ For any non-negative random variable $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_N) =: \Omega \to \mathbb{R}_+^N$, we define the map

$\boldsymbol{\theta_\tau} : \Omega \to \Omega$ via

$$\boldsymbol{\theta_\tau}(\omega)(t_1, \dots, t_N) = (\omega_1(t_1 + \tau_1(\omega)), \dots, \omega_N(t_N + \tau_N(\omega))),$$

and its corresponding shift operator $\boldsymbol{\theta_\tau}$ via

$$\boldsymbol{\theta_\tau}\xi(\omega) := \xi(\boldsymbol{\theta_\tau}(\omega)) \text{ for any random variable } \xi : \Omega \to \mathbb{R}.$$

Proposition 3.1. *The $\boldsymbol{T}$ defined in* (3.4) *has the following properties.*

(i) *For every $\omega \in \Omega_0$ with Ω_0 defined in Lemma 3.1 and any $t \in \mathbb{R}_+$, $(s_i := T_i(t))_{i=1}^N$ is the unique solution to*

$$\sum_{i=1}^N s_i = t, \text{ and } \frac{L^i(s_i)}{\alpha_i} = \frac{L^j(s_j)}{\alpha_j} \text{ for all } 1 \le i, j \le N. \tag{3.5}$$

(ii) *$\boldsymbol{T}$ is a multi-parameter random time change.*

(iii) *$\boldsymbol{T}$ is time homogeneous. That is for any $t, s \ge 0$, we have that*

$$\boldsymbol{T}(s + t) = \boldsymbol{T}(s) + \boldsymbol{\theta_{T(s)}}\boldsymbol{T}(t) \quad a.s.$$

Proof. (i): First we prove that $\boldsymbol{T}$ defined by (3.4) satisfies (3.5). Recall the definition of T_i

$$T_i(t) := h_i(L(t)) + \mathbb{1}_{\{\Delta h_i(L(t)) > 0\}}(t - h(L(t))).$$

Thanks to Lemma 3.1, for every $\omega \in \Omega_0$, $\Delta h(l) > 0$ if and only of there is exactly one $i \in \{1, 2, \cdots, N\}$ such that $\Delta h_i(l) > 0$. Thus

$$\sum_i \mathbb{1}_{\{\Delta h_i(L(t)) > 0\}} = \mathbb{1}_{\{\Delta h(L(t)) > 0\}}.$$

It follows from the construction of $\{T_i\}_{i=1}^N$ that

$$\sum_i T_i(t) = h(L(t)) + \mathbb{1}_{\{\Delta h(L(t)) > 0\}}(t - h(L(t))) = t.$$

Since h_i is the left-continuous inverse of L^i/α^i, it is straightforward that

$$\frac{L^i(T_i(t))}{\alpha_i} = L(t),$$

which is independent of i.

Then we prove that for any $s \ge 0$, solution to (3.5) is unique. Suppose there is another family of solutions $\{s_i'\}_{i=1}^N$ to (3.5). Let us take

$$c = \frac{L^1(s_1)}{\alpha_1} = \dots = \frac{L^N(s_N)}{\alpha_N},$$

and

$$c' = \frac{L^1(s_1')}{\alpha_1} = \dots = \frac{L^N(s_N')}{\alpha_N}.$$

Since $\sum_{i=1}^{N} s_i = \sum_{i=1}^{N} s_i'$, without loss of generality we assume that $s_1 > s_1'$, $s_2 < s_2'$. Therefore by the monotonicity of local times, we have

$$c = \frac{L^1(s_1)}{\alpha_1} \geq \frac{L^1(s_1')}{\alpha_1} = c' = \frac{L^2(s_2')}{\alpha_2} \geq \frac{L^2(s_2)}{\alpha_2} = c,$$

and hence $c = c'$. Then h_1 and h_2 have jumps simultaneously at $l = c$, which contradicts Lemma 3.1.

(ii): According to the definition (3.4), $T_i(t)$ is increasing in t for every $i \in \{1, 2, \cdots, N\}$. It remains to show that $T_i(t)$ is continuous in t and $\boldsymbol{T}(t)$ is a stopping point for every $t \in \mathbb{R}_+$.

By *(i)* we have that for any $0 \leq s < t$, $\sum_i T_i(t) - \sum_i T_i(s) = t - s$. Then

$$|T_i(t) - T_i(s)| \leq |t - s|, \ \forall t, s \geq 0,$$

which indicates that $t \mapsto T_i(t)$ is continuous. For continuous and increasing T_i, to show that T is a random time change, it suffices to show that

$$\{T_1(t) \leq s_1, \cdots, T_N(t) \leq s_N\} \in \mathcal{F}_{s_1}^1 \otimes \cdots \otimes \mathcal{F}_{s_N}^N, \quad \forall \boldsymbol{s} = (s_1, \ldots, s_N) \in \mathbb{R}_+^N.$$

To this end, we adopt the idea from [25, Theorem 3], showing that T is the limit of stopping points that have exactly only one component increasing at each time t according to a *priority scheme*.

For a fixed $\epsilon > 0$ and each $i = 1, \ldots, N$, define

$$c_k^{(i)} = \inf\left\{t \geq 0 : L^i(t) \geq \epsilon k \alpha_i\right\} - \inf\left\{t \geq 0 : L^i(t) \geq \epsilon(k-1)\alpha_i\right\}, \ \forall k \geq 1,$$

and a sequence of intervals $\left\{U_k^{(i)} : k \in \mathbb{N}, \ i = 1, \ldots, N\right\}$ via

$$U_k^{(i)} := \left[\sum_{n=1}^{k-1}\sum_{j=1}^{N} c_n^{(j)} + \sum_{j=1}^{i-1} c_k^{(j)}, \ \sum_{n=1}^{k-1}\sum_{j=1}^{N} c_n^{(j)} + \sum_{j=1}^{i} c_k^{(j)}\right).$$

Consider a family of stopping points $\boldsymbol{T}^\epsilon = (T_1^\epsilon, \cdots, T_N^\epsilon)$ defined by

$$T_i^\epsilon(t) = \sum_{k=1}^{\infty} \int_0^t \mathbb{1}_{\{u \in U_k^{(i)}\}} \, du.$$

Intuitively, we let T_1^ϵ run with rate 1 until $\frac{L^1(t)}{\alpha_1}$ reaches level ϵ. Proceed sequentially with each T_i^ϵ until $\frac{L^i(t)}{\alpha_i}$ reaches level ϵ for $i = 2, \cdots, N$. When $\frac{L^N(t)}{\alpha_N}$ is at level ϵ, let T_1^ϵ run again until $\frac{L^1(t)}{\alpha_1}$ reaches level 2ϵ. Proceed sequentially with each $i = 2, \cdots, N$ until $\frac{L^N(t)}{\alpha_N}$ is at level 2ϵ, and so on. In the above notation, $\sum_{j=1}^{k} c_j^{(i)} = h_i(k\epsilon)$ and $\sum_{i=1}^{N}\sum_{j=1}^{k} c_j^{(i)} = \sum_{i=1}^{N} h_i(k\epsilon) = h(k\epsilon)$. Thus for all $i \in \{1, \cdots, N\}$, $\frac{L^i(T_i^\epsilon(t))}{\alpha_i} = k\epsilon$ whenever $t = \sum_{j=1}^{N}\sum_{n=1}^{k} c_n^{(j)}$. In other words, $T_i^\epsilon(t) = T_i(t) = h_i(k\epsilon)$ when $t = h(k\epsilon)$.

With $\epsilon = 2^{-n}$, by the continuity of T_i, $\boldsymbol{T}^\epsilon(t)$ converges to $\boldsymbol{T}(t)$ as $n \to \infty$. Since $\boldsymbol{T}^\epsilon(t)$ is a stopping point and the filtration $\mathcal{F}(s)$ is right-continuous by Lemma 3.2, its limit $\boldsymbol{T}$ must be a stopping point as well.

(iii): Take Ω_0 as defined in Lemma 3.1. Let us first prove the result assuming

$$\boldsymbol{\theta}_{\boldsymbol{T}(s)}(\omega) \in \Omega_0 \quad \text{for almost every } \omega \in \Omega_0. \tag{3.6}$$

Thanks to *(i)*, it suffices to show that for almost every $\omega \in \Omega_0$,

$$\sum_i T_i(s)(\omega) + \boldsymbol{\theta}_{\boldsymbol{T}(s)} T_i(t)(\omega) = s + t, \tag{3.7}$$

as well as

$$\frac{L^i(T_i(s) + \boldsymbol{\theta}_{\boldsymbol{T}(s)} T_i(t))(\omega)}{\alpha_i} \quad \text{is independent of } i. \tag{3.8}$$

According to the definition of $\boldsymbol{\theta}$,

$$\boldsymbol{\theta}_{\boldsymbol{T}(s)} \boldsymbol{T}(t)(\omega) = \boldsymbol{T}(t)(\boldsymbol{\theta}_{\boldsymbol{T}(s)}(\omega)),$$

and therefore

$$\sum_i \boldsymbol{\theta}_{\boldsymbol{T}(s)} T_i(t)(\omega) = \sum_i T_i(t)(\boldsymbol{\theta}_{\boldsymbol{T}(s)}(\omega)) = t,$$

which justifies (3.7).

Then we prove (3.8). Recall a property of local time for continuous semi-martingales [28, Chapter VI, Corollary 1.9]

$$L^i(t)(\omega_i) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^t \mathbb{1}_{\{|Y_i(u)(\omega_i)| < \epsilon\}} d[Y_i]_u(\omega_i),$$

where $[Y_i]_u$ stands for the quadratic variation of Y_i at time u. Hence we obtain that

$$L^i \left(T_i(s) + \boldsymbol{\theta}_{\boldsymbol{T}(s)} T_i(t) \right)(\omega)$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^{T_i(s)(\omega) + \boldsymbol{\theta}_{\boldsymbol{T}(s)} T_i(t)(\omega)} \mathbb{1}_{\{|Y_i(u)(\omega)| < \epsilon\}} d[Y_i]_u(\omega)$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^{T_i(s)(\omega)} \mathbb{1}_{\{|Y_i(u)(\omega)| < \epsilon\}} d[Y_i]_u(\omega)$$

$$+ \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^{T_i(t)(\boldsymbol{\theta}_{\boldsymbol{T}(s)}(\omega))} \mathbb{1}_{\{|Y_i(u)(\boldsymbol{\theta}_{\boldsymbol{T}(s)}(\omega))| < \epsilon\}} d[Y_i]_u(\boldsymbol{\theta}_{\boldsymbol{T}(s)}(\omega))$$

$$= L^i \left(T_i(s)(\omega) \right)(\omega) + L^i \left(T_i(t)(\boldsymbol{\theta}_{\boldsymbol{T}(s)}(\omega)) \right) (\boldsymbol{\theta}_{\boldsymbol{T}(s)}(\omega)), \tag{3.9}$$

where we use the strong Markov property of Y_i. In other words, L^i is a a strong additive functional of Y_i; see e.g. [6, Chapter IV, Section 1]. Since both $\omega, \boldsymbol{\theta}_{\boldsymbol{T}(s)}(\omega)$ belong to Ω_0, part *(i)* implies that the following quantity is independent of i

$$\frac{L^i(T_i(s) + \boldsymbol{\theta}_{\boldsymbol{T}(s)} T_i(t))(\omega)}{\alpha_i} = \frac{L^i \left(T_i(s) \right)(\omega)}{\alpha_i} + \frac{L^i \left(T_i(t)(\boldsymbol{\theta}_{\boldsymbol{T}(s)}(\omega)) \right) (\boldsymbol{\theta}_{\boldsymbol{T}(s)}(\omega))}{\alpha_i}.$$

It completes the proof of (3.8).

As a final step, we prove the claim (3.6). We show that for any $\boldsymbol{\theta}_{\boldsymbol{T}(s)}(\omega)$ with $\omega \in \Omega_0$,

$$\{h_i(l)(\boldsymbol{\theta}_{\boldsymbol{T}(s)}(\omega))\}_{i=1,\dots,N}$$

have no simultaneous jumps at any local time level $l \geq 0$. Indeed, by a similar argument as (3.9) we have

$$\frac{L^i(t)(\boldsymbol{\theta}_{\boldsymbol{T}(s)}(\omega))}{\alpha_i} = \frac{L^i(t + T_i(s)(\omega))(\omega)}{\alpha_i} - \frac{L^i(T_i(s)(\omega))(\omega)}{\alpha_i}. \tag{3.10}$$

As $\omega \in \Omega_0$, there are no simultaneous jumps of $\{h_i(l)(\omega)\}_{i=1,\dots,N}$, which can be written as

$$\text{Leb}(\{t : L^i(t)(\omega) = \alpha_i l\}) \cdot \text{Leb}(\{t : L^j(t)(\omega) = \alpha_j l\}) = 0, \quad \forall l \geq 0, \ \forall i \neq j.$$

Together with (3.10) and the fact that $L^i(T_i(s)(\omega))(\omega)/\alpha_i$ is independent of i, one can deduce that

$$\text{Leb}(\{t : L^i(t)(\boldsymbol{\theta}_{\boldsymbol{T}(s)}(\omega)) = \alpha_i l\}) \cdot \text{Leb}(\{t : L^j(t)(\boldsymbol{\theta}_{\boldsymbol{T}(s)}(\omega)\}) = \alpha_j l) = 0, \quad \forall l \geq 0, \ \forall i \neq j,$$

which justifies there are no simultaneous jumps of $\{h_i(l)(\boldsymbol{\theta}_{\boldsymbol{T}(s)}(\omega))\}_{i=1,\dots,N}$ for every $\omega \in \Omega_0$.

$\square$

Remark 3.4. *Notice that we have a priority scheme when defining stopping points $\boldsymbol{T}^\epsilon(t)$ in the proof of Proposition 3.1 (ii). However, according to the construction of $\boldsymbol{T}^\epsilon$, $T_i^\epsilon(h(k\epsilon)) = h_i(k\epsilon)$ is independent of the priority scheme for every $k \in \mathbb{N}$. Therefore, as the limit of $\boldsymbol{T}^\epsilon(t)$, the stopping point $\boldsymbol{T}(t)$ is also independent of the priority scheme.*

Using the random time change $\boldsymbol{T} = (T_1, \cdots, T_N)$ constructed above, we define our diffusion process on Γ as the time change of the multi-parameter process, i.e.,

$$X(t) = (Y_1(T_1(t)), \cdots, Y_N(T_N(t))). \tag{3.11}$$

According to Lemma 3.1, at any time $t \geq 0$, at most one $Y_i(T_i(t))$ is on excursion from v, which means at any time $t \geq 0$, there is at most one $Y_i(T_i(t)) > 0$, and $Y_j(T_j(t)) = 0$ for $j \neq i$.

Expanding on the concept of strong Markov property of a one-parameter process, we define the strong Markov property of multi-parameter process as the following.

Definition 3.2. *$\boldsymbol{Y}(\boldsymbol{t}) = (Y_1(t_1), \cdots, Y_N(t_N))$ is a multi-parameter process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_{\boldsymbol{s}}), \mathbb{P})$ with values in $(\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N))$. Let $\boldsymbol{\tau} = (\tau_1, \cdots, \tau_N)$ be an almost surely finite stopping point with respect to the filtration $(\mathcal{F}_{\boldsymbol{s}})$. We say that the strong Markov property holds at $\boldsymbol{\tau}$ for process $\boldsymbol{Y}$ if for arbitrary k vectors $\boldsymbol{t}_1, \cdots, \boldsymbol{t}_k \in \mathbb{R}_+^N$ and arbitrary k bounded continuous functions $f_1, \dots, f_k \in \mathcal{C}_b(\mathbb{R}^N)$,*

$$\mathbb{E}\left[\prod_{i=1}^k f_i(\boldsymbol{Y}(\boldsymbol{\tau} + \boldsymbol{t}_i)) \,\Big|\, \mathcal{F}_{\boldsymbol{\tau}}\right] = \mathbb{E}_{\boldsymbol{Y}(\boldsymbol{\tau})}\left[\prod_{i=1}^k f_i(\boldsymbol{Y}(\boldsymbol{t}_i))\right]. \tag{3.12}$$

Remark 3.5. *By a standard monotone class argument, the above definition is equivalent to: For any $\boldsymbol{y} \in \mathbb{R}_+^N$ and positive $\mathcal{F}$-measurable random variable $V : \Omega \to \mathbb{R}_+$,*

$$\mathbb{E}_{\boldsymbol{y}}[\boldsymbol{\theta}_{\boldsymbol{\tau}} V | \mathcal{F}_{\boldsymbol{\tau}}] = \mathbb{E}_{\boldsymbol{Y}(\boldsymbol{\tau})}[V]$$

See [9, Proposition 5.29] *for a reference.*

The following lemma provides a sufficient condition for the strong Markov property of multi-parameter processes.

Lemma 3.3. *If Y_i, $i = 1, \cdots, N$ are independent strong Markov processes, then strong Markov property holds at any stopping point $\boldsymbol{\tau} < \infty$ a.s. for the multi-parameter process $\boldsymbol{Y} = (Y_1, \cdots, Y_N)$.*

Proof. By a monotone class argument, it suffices to prove (3.12) for f_i of the form

$$f_i(\boldsymbol{y}) = \prod_{j=1}^{N} f_{ij}(y_j), \quad f_{ij} \in \mathcal{C}_b(\mathbb{R}), \; j = 1, 2, \cdots, N.$$

First we assume that there exists a countable sequence $\{\boldsymbol{s}_n\}_{n \in \mathbb{N}}$ in $\mathbb{R}_+^N$ such that

$$\mathbb{P}(\boldsymbol{\tau} \in \{\boldsymbol{s}_n\}_{n \in \mathbb{N}}) = 1.$$

Thanks to the property of conditional probability,

$$\mathbb{E}\left[\prod_{i=1}^{k} \prod_{j=1}^{N} f_{ij}(Y_j(\tau_j + t_{ij})) \,\middle|\, \mathcal{F}_{\boldsymbol{\tau}}\right] = \sum_{n=1}^{\infty} \mathbb{1}_{\{\boldsymbol{\tau} = \boldsymbol{s}_n\}} \mathbb{E}\left[\prod_{i=1}^{k} \prod_{j=1}^{N} f_{ij}(Y_j(s_{nj} + t_{ij})) \,\middle|\, \mathcal{F}_{\boldsymbol{s}_n}\right]$$

By the independence of $\{Y_i\}_{i \in \{1,2,\cdots,N\}}$ and Markov property of each Y_i, for each $n \in \mathbb{N}$

$$\mathbb{E}\left[\prod_{i=1}^{k} \prod_{j=1}^{N} f_{ij}(Y_j(s_{nj} + t_{ij})) \,\middle|\, \mathcal{F}_{\boldsymbol{s}_n}\right] = \prod_{j=1}^{N} \mathbb{E}_{Y_j(s_{nj})}\left[\prod_{i=1}^{k} f_{ij}(Y_j(t_{ij}))\right]$$

$$= \mathbb{E}_{\boldsymbol{Y}(\boldsymbol{s}_n)}\left[\prod_{i=1}^{k} \prod_{j=1}^{N} f_{ij}(Y_j(t_{ij}))\right].$$

Thus (3.12) holds if $\boldsymbol{\tau}$ takes countable many values.

Passing to the general case we approximate an arbitrary stopping point $\boldsymbol{\tau}$ by elementary stopping points,

$$\boldsymbol{\tau}^{(n)} = \sum_{\boldsymbol{s} \in \mathbb{N}^N} 2^{-n} \boldsymbol{s} \mathbb{1}_{(2^{-n}\boldsymbol{s}, 2^{-n}(\boldsymbol{s}+\boldsymbol{1})]}(\boldsymbol{\tau}),$$

where $\boldsymbol{s} + \boldsymbol{1} := (s_1 + 1, \ldots, s_N + 1)$. For each $n \in \mathbb{N}$, (3.12) holds at $\boldsymbol{\tau}^{(n)}$, $\boldsymbol{\tau}^{(n)}$ is a decreasing sequence of stopping points, and $\lim_{n \to \infty} \boldsymbol{\tau}^{(n)} = \boldsymbol{\tau}$. Noticing that for each $j \in \{1, 2, \cdots, N\}$, Y_j is a Feller process with continuous paths, it is clear that $\lim_{n \to \infty} Y_j(\tau_j^{(n)}) =$

$Y_j(\tau_j)$. By a standard monotone class argument and the definition of Feller process, we get the continuity of $y \mapsto \mathbb{E}_y\left[\prod_{i=1}^k f_i(Y_j(t_{ij}))\right]$, which implies

$$\lim_{n\to\infty} \mathbb{E}_{Y_j(\tau_j^{(n)})}\left[\prod_{i=1}^k f_i(Y_j(t_{ij}))\right] = \mathbb{E}_{Y_j(\tau_j)}\left[\prod_{i=1}^k f_i(Y_j(t_{ij}))\right];$$

see [9, Remark 5.35] for a reference. It gives rise to the equality

$$\lim_{n\to\infty} \mathbb{E}_{\boldsymbol{Y}(\boldsymbol{\tau}^{(n)})}\left[\prod_{i=1}^k f_i(\boldsymbol{Y}(\boldsymbol{t}_i))\right] = \mathbb{E}_{\boldsymbol{Y}(\boldsymbol{\tau})}\left[\prod_{i=1}^k f_i(\boldsymbol{Y}(\boldsymbol{t}_i))\right].$$

To conclude the result, it remains to show that

$$\lim_{n\to\infty} \mathbb{E}\left[\prod_{i=1}^k f_i(\boldsymbol{Y}(\boldsymbol{\tau}^{(n)} + \boldsymbol{t}_i)) \,\middle|\, \mathcal{F}_{\boldsymbol{\tau}^{(n)}}\right] = \mathbb{E}\left[\prod_{i=1}^k f_i(\boldsymbol{Y}(\boldsymbol{\tau} + \boldsymbol{t}_i)) \,\middle|\, \mathcal{F}_{\boldsymbol{\tau}}\right]. \qquad (3.13)$$

To this end, we estimate the difference

$$\left| \mathbb{E}\left[\prod_{i=1}^k f_i(\boldsymbol{Y}(\boldsymbol{\tau}^{(n)} + \boldsymbol{t}_i)) \,\middle|\, \mathcal{F}_{\boldsymbol{\tau}^{(n)}}\right] - \mathbb{E}\left[\prod_{i=1}^k f_i(\boldsymbol{Y}(\boldsymbol{\tau} + \boldsymbol{t}_i)) \,\middle|\, \mathcal{F}_{\boldsymbol{\tau}}\right] \right| \le I + II,$$

where

$$I = \left| \mathbb{E}\left[\prod_{i=1}^k (f_i(\boldsymbol{Y}(\boldsymbol{\tau}^{(n)} + \boldsymbol{t}_i)) - f_i(\boldsymbol{Y}(\boldsymbol{\tau} + \boldsymbol{t}_i))) \,\middle|\, \mathcal{F}_{\boldsymbol{\tau}^{(n)}}\right] \right|,$$

$$II = \left| \mathbb{E}\left[\prod_{i=1}^k f_i(\boldsymbol{Y}(\boldsymbol{\tau} + \boldsymbol{t}_i)) \,\middle|\, \mathcal{F}_{\boldsymbol{\tau}^{(n)}}\right] - \mathbb{E}\left[\prod_{i=1}^k f_i(\boldsymbol{Y}(\boldsymbol{\tau} + \boldsymbol{t}_i)) \,\middle|\, \mathcal{F}_{\boldsymbol{\tau}}\right] \right|.$$

Let us denote

$$Z_n := \mathbb{E}\left[\prod_{i=1}^k (f_i(\boldsymbol{Y}(\boldsymbol{\tau}^{(n)} + \boldsymbol{t}_i)) - f_i(\boldsymbol{Y}(\boldsymbol{\tau} + \boldsymbol{t}_i))) \,\middle|\, \mathcal{F}_{\boldsymbol{\tau}^{(n)}}\right].$$

Then by Jensen's inequality and the tower property of conditional expectation

$$\mathbb{E}[|Z_n|] \le \mathbb{E}\left[\left|\prod_{i=1}^k (f_i(\boldsymbol{Y}(\boldsymbol{\tau}^{(n)} + \boldsymbol{t}_i)) - f_i(\boldsymbol{Y}(\boldsymbol{\tau} + \boldsymbol{t}_i)))\right|\right],$$

which converges to 0 by the dominated convergence theorem. The L^1 convergence of $\lim_{n\to\infty} \mathbb{E}[|Z_n|] = 0$ implies that, there exists a subsequence $\{n_j\}_{j\in\mathbb{N}}$ such that $\lim_{n_j\to\infty} Z_{n_j} = 0$ almost surely. Therefore I converges to 0 as $n_j \to \infty$. For the second term II, we apply the backward martingale convergence theorem to obtain that $II \to 0$ as $n \to \infty$. Hence, we prove (3.13), and (3.12) holds for all stopping point $\boldsymbol{\tau}$. $\qquad\square$

Let us prove that the process $X(\cdot)$ constructed above is a Markov process on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_{\boldsymbol{T}(t)}), \mathbb{P})$.

Proposition 3.2. *The process X defined in* (3.11) *is Markovian with the right-continuous filtration* $(\mathcal{F}_{\boldsymbol{T}(t)})$.

Proof. Since $\{\boldsymbol{Y}(\boldsymbol{t})\}_{\boldsymbol{t}\in\mathbb{R}_+^N}$ is a continuous multi-parameter process and for any $s \in \mathbb{R}_+^N$, $\boldsymbol{T}(s)$ is a stopping time w.r.t. filtration $(\mathcal{F}_{\boldsymbol{t}})$, it can be easily seen that $X(s) = \boldsymbol{Y}(\boldsymbol{T}(s))$ is $\mathcal{F}_{\boldsymbol{T}(s)}$-measurable. Then we verify that the time changed filtration $(\mathcal{F}_{\boldsymbol{T}(s)})$ is right-continuous. To this end, fix $s_0 \geq 0$ and let $A \in \bigcap_{s>s_0} \mathcal{F}_{\boldsymbol{T}(s)}$. By the definition of stopping point, for each $s > s_0$, and $\boldsymbol{t} \in \mathbb{R}_+^N$

$$A \cap \{\boldsymbol{T}(s) \leq \boldsymbol{t}\} \in \mathcal{F}_{\boldsymbol{t}} = \mathcal{F}_{\boldsymbol{t}+},$$

where the last equality is due to the right continuity of filtration $(\mathcal{F}_{\boldsymbol{t}})$ from Lemma 3.2. By standard arguments this condition is equivalent to

$$A \cap \{\boldsymbol{T}(s) < \boldsymbol{t}\} \in \mathcal{F}_{\boldsymbol{t}}, \quad \forall \boldsymbol{t} \in \mathbb{R}_+^N.$$

Thanks to the continuity of stopping points $s \mapsto \boldsymbol{T}(s)$,

$$A \cap \{\boldsymbol{T}(s_0) < \boldsymbol{t}\} = \bigcap_{s>s_0} A \cap \{\boldsymbol{T}(s) < \boldsymbol{t}\} \in \mathcal{F}_{\boldsymbol{t}}, \quad \forall \boldsymbol{t} \in \mathbb{R}_+^N,$$

which concludes that $A \in \mathcal{F}_{\boldsymbol{T}(s_0)}$.

Next, to verify the Markov property of X, it is sufficient to show that

$$\mathbb{E}_x[f(X(t+h)) \,|\, \mathcal{F}_{\boldsymbol{T}(t)}] = \mathbb{E}_{X(t)}[f(X(h))]$$

for any $f \in C_0(\Gamma)$. By Tietze extension theorem, f can be viewed as a bounded continuous function on $\mathbb{R}_+^N$, and therefore $f(\boldsymbol{Y}(\boldsymbol{T}(t)))$ is well-defined. According to the definition of X, it is equivalent to show that

$$\mathbb{E}_{\boldsymbol{y}}[f(\boldsymbol{Y}(\boldsymbol{T}(t+h))) \,|\, \mathcal{F}_{\boldsymbol{T}(t)}] = \mathbb{E}_{\boldsymbol{Y}(\boldsymbol{T}(t))}[f(\boldsymbol{Y}(\boldsymbol{T}(h)))], \qquad (3.14)$$

where $\boldsymbol{y} \in \mathbb{R}_+^N$ is the representation of $x \in \Gamma$; see (2.2).

According to Proposition 3.1 *(iii)*

$$f(\boldsymbol{Y}(\boldsymbol{T}(t+h))) = f(\boldsymbol{Y}(\boldsymbol{T}(t) + \boldsymbol{\theta}_{\boldsymbol{T}(t)}\boldsymbol{T}(h))).$$

Remember that $\boldsymbol{Y}$ is defined as the canonical process on $\Omega = \mathcal{C}([0,\infty);\mathbb{R})^N$, and therefore

$$f(\boldsymbol{Y}(\boldsymbol{T}(t) + \boldsymbol{\theta}_{\boldsymbol{T}(t)}\boldsymbol{T}(h))(\omega)) = f(\boldsymbol{Y}(\boldsymbol{T}(h))(\boldsymbol{\theta}_{\boldsymbol{T}(t)}(\omega))) = \boldsymbol{\theta}_{\boldsymbol{T}(t)}f(\boldsymbol{Y}(\boldsymbol{T}(h)))(\omega),$$

where the last equality is provided by the definition of the shift operator $\boldsymbol{\theta}$. Taking conditional expectation w.r.t. $\mathcal{F}_{\boldsymbol{T}(t)}$, the two equations above yield

$$\mathbb{E}_{\boldsymbol{y}}[f(\boldsymbol{Y}(\boldsymbol{T}(t+h))) \,|\, \mathcal{F}_{\boldsymbol{T}(t)}] = \mathbb{E}_{\boldsymbol{y}}[\theta_{\boldsymbol{T}(t)}f(\boldsymbol{Y}(\boldsymbol{T}(h))) \,|\, \mathcal{F}_{\boldsymbol{T}(t)}].$$

We completes the proof of (3.14) by invoking the strong Markov property of $\boldsymbol{Y}$ from Lemma 3.3.

$\square$

Define the semigroup (P_t) of the Markov process X as the following.

$$P_t f(x) := \mathbb{E}_x[f(X_t)], \quad \forall f \in C_0(\Gamma), \ x \in \Gamma.$$

Next we show that X constructed above is a diffusion process on graph Γ which is characterized by the infinitesimal generator A and the corresponding domain $\mathcal{D}(A)$ as in (2.1).

Proposition 3.3. *For any $x \in e_i$, $i = 1, \dots, N$, it holds that for all $f \in \mathcal{D}(A)$*

$$\lim_{t \to 0} \frac{P_t f(x) - f(x)}{t} = \mathcal{L}_i f(x).$$

Proof. It suffices to prove the result for $x = v$. Let us define $|X(t)| := d(X(t), v)$, where d is the tree metric on graph Γ. Define a sequence of downcrossing times iteratively for any fixed $\delta > 0$

$$\tau_0^\delta = 0,$$
$$\theta_1^\delta = \inf\{t > \tau_0^\delta : |X(t)| = \delta\},$$
$$\tau_1^\delta = \inf\{t > \theta_1^\delta : |X(t)| = 0\},$$
$$\vdots$$
$$\theta_n^\delta = \inf\{t > \tau_{n-1}^\delta : |X(t)| = \delta\},$$
$$\tau_n^\delta = \inf\{t > \theta_n^\delta : |X(t)| = 0\}.$$

Then we split $P_t f(v) - f(v)$ as a telescopic sum

$$\mathbb{E}_v[f(X_t) - f(v)] = \sum_{n \geq 0} \mathbb{E}_v[f(X_{t \wedge \theta_{n+1}^\delta}) - f(X_{t \wedge \tau_n^\delta})]$$
$$+ \sum_{n \geq 1} \mathbb{E}_v[f(X_{t \wedge \tau_n^\delta}) - f(X_{t \wedge \theta_n^\delta})] = (I)^\delta + (II)^\delta.$$

Heuristically, $(I)^\delta$ is an approximation of dynamics of $f(X(t))$ around v, which can be estimated using local times. The term $(II)^\delta$ is the dynamic of $f(X(t))$ when $X(t)$ is away from v, and we can compute it using Itô's formula.

Note that one can characterize local times by downcrossings, and hence

$$\lim_{\delta \to 0} (I)^\delta = \lim_{\delta \to 0} \sum_{i=1}^N D_i f(v) \left(\delta \sum_{n \geq 1} \mathbb{1}_{\{\tau_n^\delta < t, X(\theta_n^\delta) \in e_i\}} \right) = \sum_{i=1}^N D_i f(v) \, L^i(T_i(t)).$$

According to our construction of $\{T_i\}_{i=1}^N$, and the definition of $\mathcal{D}(A)$, we obtain that

$$\sum_{i=1}^N D_i f(v) L^i(T_i(t)) = \sum_{i=1}^N \alpha_i D_i f(v) \frac{L^i(T_i(t))}{\alpha_i} = 0,$$

and hence that $\lim_{\delta \to 0} (I)^\delta = 0$.

Let us now estimate $(II)^\delta$. Denote by $i(t)$ the index of edge which $X(t)$ belongs to, i.e., $i(t) = j$ iff $X(t) \in e_j$. As $\lim_{\delta \to 0} \sum_{n \geq 0} (\theta^\delta_{n+1} - \tau^\delta_n) = 0$, it can be easily verified that

$$\lim_{\delta \to 0} (II)^\delta = \mathbb{E}_v \left[\int_0^t \mathcal{L}_{i(s)} f(X(s)) \, ds \right].$$

Now it is clear that

$$\lim_{t \to 0} \frac{P_t f(v) - f(v)}{t} = \lim_{t \to 0} \frac{\mathbb{E}_v \left[\int_0^t \mathcal{L}_{i(s)} f(X(s)) \, ds \right]}{t} = \mathcal{L}_i f(v), \quad \text{for all } i = 1, \ldots, N,$$

where the last the equality is due to our definition of $\mathcal{D}(A)$ in (2.1).   □

It was shown in [11] that $(A, \mathcal{D}(A))$ is indeed an infinitesimal generator of a Feller semigroup. By the uniqueness part of Hille-Yosida theorem; see e.g. Lemma [17, Lemma 17.5], this Feller semigroup must coincide with (P_t) generated by $X(\cdot)$. Hence, we have the following corollary.

Corollary 3.1. *The semigroup (P_t) generated by $X(\cdot)$ is a Feller semigroup. Furthermore since $t \mapsto \mathcal{F}_{\boldsymbol{T}(t)}$ is right-continuous, it implies that $X(\cdot)$ is a strong Markov process with filtration $(\mathcal{F}_{\boldsymbol{T}(t)})$.*

Let us give some interpretations of positive constants α_i, $L(t)$, and the time change $T(t)$. It is intuitive to view α_i as the probability of choosing edge e_i when the process is at vertex v. We state this observation in Remark 3.6, the proof of which can be found in [10, Corollary 2.4].

Remark 3.6. *For each $i = 1, \ldots, N$, we have that $\lim_{\delta \to 0} \mathbb{P}_v[X(\theta^\delta) \in e_i] = \alpha_i$, where $\theta^\delta := \inf\{t > 0 : |X(t)| = \delta\}$.*

For process X, one can still define its local time as limits of downcrossing times as in the case of semimartingales. Adopting the notation from Proposition 3.3, its local time at the vertex v is given by

$$\lim_{\delta \to 0} \delta \sum_{n \geq 0} \mathbb{1}_{\{\tau^\delta_n < t\}} = \sum_{k=1}^N L^i(T_i(t)) = \sum_{k=1}^N L^i(h_i(L(t)) = \sum_{k=1}^N \alpha_i L(t) = L(t), \tag{3.15}$$

which justifies our notation in (3.3).

Let us turn to the time change $T(t)$. It is straightforward that for each i, $T_i(t)$ is the occupation time of X on edge e_i, i.e.,

$$T_i(t) = \int_0^t \mathbf{1}_{\{X(s) \in e_i\}}(s) \, ds.$$

In Corollary 3.2, we investigate the long-term average occupation time for a special case when all edges of Γ have finite lengths. This result aligns with [14, Remark 3.1], which we are able to obtain using a simpler approach thanks to our construction of time change $\boldsymbol{T}$.

Corollary 3.2. *Suppose each edge e_i is homeomorphic to $[0, l_i]$ for some positive real number l_i, then for each $i \in \{1, 2, \cdots, N\}$, we have that ,*

$$\lim_{t \to \infty} \frac{T_i(t)}{t} = \frac{\alpha_i m_i([0, l_i])}{\sum_{j=1}^{N} \alpha_j m_j([0, l_i])} \quad a.s.,$$

where m_i is the speed measure of process Y_i on the interval $[0, l_i]$, i.e.,

$$m_i([0, l_i]) = \int_0^{l_i} \frac{2}{\sigma_i^2(x)} \exp \left(\int_0^x \frac{2b_i(y)}{\sigma_i^2(y)} dy \right) dx.$$

Proof. By the construction of T_i, we have

$$h_i(l) \le T_i(t) \le h_i(l) + h(l+) - h(l), \forall l \ge 0, t \in [h(l), h(l+)].$$

Thus

$$\frac{h_i(l)}{h(l)} \frac{h(l)}{h(l+)} \le \frac{T_i(t)}{t} \le \frac{h_i(l) + h(l+) - h(l)}{h(l)}, \forall l \ge 0, t \in [h(l), h(l+)].$$

Notice that h and all h_j are strictly increasing functions of l, thus

$$\lim_{l \to \infty} \frac{h_i(l)}{h(l)} \frac{h(l)}{h(l+)} \le \lim_{t \to \infty} \frac{T_i(t)}{t} \le \lim_{l \to \infty} \frac{h_i(l) + h(l+) - h(l)}{h(l)}.$$

The left-continuous inverse of $L^i(\cdot)/\alpha_i$, $h_i(l)$, can be rewritten as $h_i(l) = \rho_i(\alpha_i l)$ where ρ_i is the left-continuous inverse of $L^i(\cdot)$. Thus $h_i(l)$ has independent stationary increment and $\mathbb{E}[h_i(l)] = \mathbb{E}[\rho_i(\alpha_i l)] = \alpha_i l m_i([0, l_i])$, see [7, Chapter II, Section 2.14]. By the strong law of large numbers we have that

$$\lim_{l \to \infty} \frac{h_i(l)}{l} = \lim_{l \to \infty} \frac{h_i(l+)}{l} = \mathbb{E}[h_i(1)] = \alpha_i m_i([0, l_i]) \quad a.s.$$

Using $h(\cdot) = \sum_{j=1}^{N} h_j(\cdot)$, we obtain that

$$\lim_{t \to \infty} \frac{T_i(t)}{t} = \frac{\alpha_i m_i([0, l_i])}{\sum_{j=1}^{N} \alpha_j m_j([0, l_i])} \quad a.s.$$

$\square$

Let us also mention that our construction can be generalized to diffusions on metric graphs, which are singular processes behaving as Walsh diffusions in neighborhood of each vertex; see [11] for the precise definition. Existence of such diffusions has been studies using Feller's semigroup theory. An analog of Itô's formula, and a large derivation principle have also been established in [10]. In [22], Brownian motions on a metric graph are defined rigorously and constructed pathwise. By merging two graphs as described in [22, Section 4], we can glue Walsh diffusions constructed in our paper and obtain any desired diffusion on a general metric graph.

On the other hand, our construction can be generalized directly to diffusions on a metric graph under the assumption that there is no loop in the graph. The key idea is to induct on the number of internal vertices of graph. To elaborate the idea, let us consider a graph with two internal vertices v and u as in Figure 1. We can view Γ_v, the star shaped subgraph

with vertex v, as an edge of the vertex u, and carry out the same construction for Walsh diffusions.
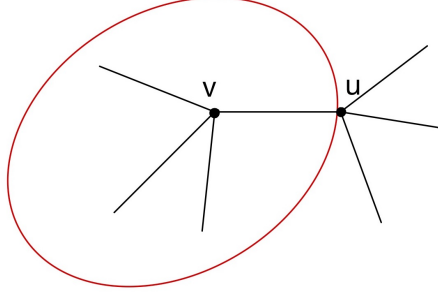

FIGURE 1. A graph with two internal vertices

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Department of Mathematics, University of Michigan
Email address: erhan@umich.edu

Department of Business and Finance, University of International Business and Economics
Email address: jingjie.zhang@uibe.edu.cn

Department of Mathematics, University of Vienna
Email address: xin.zhang@univie.ac.at