Bivariate splines on a triangulation with a single totally interior edge*

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Abstract. We derive an explicit formula, valid for all integers $r, d \geq 0$, for the dimension of the vector space $C^r_d(\Delta)$ of piecewise polynomial functions continuously differentiable to order r and whose constituents have degree at most d, where Δ is a planar triangulation that has a single totally interior edge. This extends previous results of Tohăneanu, Mináč, and Sorokina. Our result is a natural successor of Schumaker's 1979 dimension formula for splines on a planar vertex star. Indeed, there has not been a dimension formula in this level of generality (valid for all integers $d, r \geq 0$ and any vertex coordinates) since Schumaker's result. We derive our results using commutative algebra.

Key words. Dimension of spline spaces, Gröbner bases, linear programming, lattice point enumeration

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1. Introduction.

Suppose Δ is a planar triangulation. A *spline* on Δ is a function on Δ which restricts to a polynomial on each triangle; we call such a spline *bivariate* to emphasize that the domain of the spline is two-dimensional. The *degree* of a bivariate spline is the maximum (total) degree of the polynomial constituents. Given integers $r, d \geq 0$, we denote by $C^r_d(\Delta)$ the vector space of splines on Δ which are continuously differentiable of order r and whose degree is bounded by d.

Bivariate splines have grown in importance since around 1980, when the study of univariate splines could be regarded as more or less complete, and problems arising in engineering and computer-aided design made clear the need to understand multivariate splines. The modern significance of multivariate splines is marked by the authoritative monograph of Lai and Schumaker [16]. Today, bivariate splines are a fundamental tool in areas such as computer-aided geometric design, data fitting, and numerical solutions to partial differential equations. In such applications, it is important to find sufficiently rich spline spaces capable of modeling complex data, but not so large that they are computationally intractable. In terms of the parameters r and d for the spline space $C^r_d(\Delta)$, one seeks for the parameter d to be large enough (compared to r) to yield a sufficiently flexible spline space but not so large that computations are impossible. It is thus a fundamental problem in numerical analysis to determine the dimension of (and a basis for) $C^r_d(\Delta)$, especially when d is relatively small compared to r [16].

It turns out that even finding the dimension of $C^r_d(\Delta)$ is extraordinarily difficult. We briefly summarize the salient results on this problem. It was known to Strang (see [3]) that $\dim C^r_d(\Delta)$ depends on *local geometry* (the number of slopes that meet at each interior vertex). This dependence is standard by now, and is a fundamental part of a well-known lower bound for $\dim C^r_d(\Delta)$ derived by Schumaker [29]. Hong proves in [15] that Schumaker's lower bound

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coincides with $\dim C^r_d(\Delta)$ for $d \geq 3r+2$. Under a genericity condition, Alfeld and Schumaker show that $\dim C^r_d(\Delta)$ also coincides with Schumaker's lower bound for $d = 3r+1$ [2].

The '$2r+1$' conjecture of Schenck, appearing in his doctoral thesis [27] (see also [28, 23]), is that $\dim C^r_d(\Delta)$ coincides with Schumaker's lower bound for $d \geq 2r+1$. When $r \geq 2$, this has recently been disproved by the second author together with Schenck and Stillman [37, 26]. If $r = 1$, Schenck's conjecture reduces to a formula for the dimension of $C^1_3(\Delta)$ that has been conjectured since at least 1991 by Alfeld and Manni [23, 1]. This remains one of the most difficult open problems in bivariate splines. To lend further credence to this problem, piecewise cubic C^1 splines are possibly the most commonly used splines in practice.

For $r+1 \leq d < 3r+1$, there are relatively few general statements known about $\dim C^r_d(\Delta)$. In this range it is possible that $\dim C^r_d(\Delta)$ depends upon *global geometry* of Δ – illustrated in the *Morgan-Scott split* [1]. The issue of geometric dependence can be sidestepped by assuming that Δ is suitably *generic*. If $r = 1$, Billera [3] shows that the generic dimension of $\dim C^1_d(\Delta)$ coincides with Schumaker's lower bound for all $d \geq 2$, proving a conjecture of Strang [33]. Whiteley computes certain generic dimension formulas for $r > 1$ in [36], but generic dimension formulas for $\dim C^r_d(\Delta)$ for all $d \geq r+1$ and $r \geq 2$ are (as of yet) out of reach. Given the difficulty of computing the dimension of $C^r_d(\Delta)$ for $d < 3r+1$, it is natural and useful to have a complete characterization of $\dim C^r_d(\Delta)$ for interesting examples, which is the motivation for our work. While we consider a limited class of triangulations, we hope that our new approach will be of use to address the difficult problem of determining the dimension of spline spaces in the range $r+1 \leq d \leq 3r$, which is especially important for applications.

In this paper we derive in Theorem 6.1 an explicit formula for $\dim C^r_d(\Delta)$ for all $r, d \geq 0$ whenever Δ is a triangulation that has a single totally interior edge – that is, an edge connecting two interior vertices (see Figure 1). Another formulation of our main result, described in terms of lattice points in a polytope, appears in Theorem 4.1. Our formula applies to any choice of vertex coordinates for Δ, and only depends on the number of distinct slopes of edges meeting at each interior vertex. This is the first non-trivial dimension formula for bivariate splines that applies in this level of generality (all $r, d \geq 0$ and any choice of vertex coordinates) since Schumaker computed the formula for splines on a planar vertex star in 1979 [29]. In particular, we affirm the conjecture of Alfeld and Manni on $\dim C^1_3(\Delta)$ for triangulations with two interior vertices and one totally interior edge in Corollary 4.4.

Our work directly extends results in previous papers of Tohăneanu [34], Mináč and Tohăneanu [21], and Sorokina [32] which study the dimension of splines on a particular triangulation with a single totally interior edge. As a consequence of our work, we see that Schenck's '$2r+1$' conjecture is satisfied for triangulations with a single totally interior edge for all $r \geq 0$ (Corollary 4.3). Moreover, it is clear from our result that the dimension of splines on a triangulation with a single totally interior edge only depends on local geometry and not global geometry. Thus the dependence of $\dim C^r_d(\Delta)$ upon global geometry indicated by the Morgan-Scott split does not manifest unless there is more than one totally interior edge.

We briefly outline the paper. In Section 2 we recall background on splines and dimension formulas from previous papers. Section 3 is a largely technical section in which we prove a few results in commutative algebra, possibly of independent interest, for use in future sections. We then prove the first formulation of our main result – Theorem 4.1 – in Section 4, stated in terms of lattice points. In Section 5 we characterize in what degrees the spline space does not change upon removal of the totally interior edge (this is related to the phenomenon of supersmoothness explored in [32]). We give the fully explicit dimension formula in Theorem 6.1 of Section 6 and illustrate the result with several examples. We conclude with additional remarks and open problems in Section 7.

2. Splines on planar triangulations. We call a domain $\Omega \subset \mathbb{R}^2$ *polygonal* if it consists of a simple closed polygon and its interior. The simple closed polygon is the boundary of Ω, which we denote by $\partial\Omega$. Throughout we assume Δ is a triangulation of a polygonal domain; we denote the domain which Δ triangulates by $|\Delta|$. A triangulation is a collection of triangles in which each pair σ, σ' of triangles satisfies $\sigma \cap \sigma' = \emptyset$ or $\sigma \cap \sigma'$ is either an edge or vertex of both σ and σ'.

We write Δ_0 for the set of vertices of Δ, Δ_1 for the set of edges of Δ, and Δ_2 for the set of triangles of Δ. An *interior edge* of Δ is an edge that is a common edge of two triangles of Δ. A *boundary edge* of Δ is an edge that is only contained in a single triangle of Δ. An *interior vertex* of Δ is a vertex that is not contained in any boundary edges. Put Δ_0° and Δ_1° for the set of interior vertices and interior edges of Δ, respectively. A *totally interior edge* of Δ is an edge that connects two interior vertices of Δ.

Let $r \geq 0$ be an integer. We define $C^r(\Delta)$ to be the set of C^r-differentiable piecewise polynomial functions on Δ. These functions are called *splines*. More explicitly:

Definition 2.1. *$C^r(\Delta)$ is the set of functions $F : |\Delta| \to \mathbb{R}$ such that:*

1. *For all triangles $\sigma \in \Delta$, $F_\sigma := F|_\sigma$ is a polynomial in $\mathbb{R}[x, y]$.*
2. *F is differentiable of order r.*

For each integer $d \geq 0$, we define

$$C^r_d(\Delta) := \{F \in C^r(\Delta) : \deg(F_\sigma) \leq d, \text{ for all } \sigma \in \Delta_2\}.$$

For an edge $\tau \in \Delta_1$, we write $\tilde{\ell}_\tau$ for a choice of affine linear form that vanishes on the affine span of τ.

Proposition 2.1 (Algebraic spline criterion). *[4, Corollary 1.3] Suppose Δ is a triangulation of a polygonal domain and $F : |\Delta| \to \mathbb{R}$ is a piecewise polynomial function. Then $F \in C^r(\Delta)$ if and only if*

$$\tilde{\ell}_\tau^{r+1} \mid F_{\sigma_1} - F_{\sigma_2}$$

for every pair $\sigma_1, \sigma_2 \in \Delta_2$ so that $\sigma_1 \cap \sigma_2 = \tau \in \Delta_1$.

The space $C^r_d(\Delta)$ is a finite dimensional $\mathbb{R}$-vector space. One of the key problems in spline theory is to determine $\dim C^r_d(\Delta)$ for all (Δ, r, d). To study this problem, we use a standard coning construction due to Billera and Rose [4]. Namely, for any set $U \subset \mathbb{R}^2$, define $\hat{U} \subset \mathbb{R}^3$ by $\hat{U} := \{(sa, sb, s) : (a, b) \in U \text{ and } 0 \leq s \leq 1\}$. Define $\hat{\Delta}$ to be the tetrahedral complex whose tetrahedra are $\{\widehat{\sigma} : \sigma \in \Delta_2\}$. All above definitions for triangulations in $\mathbb{R}^2$ carry over in the expected way to tetrahedral complexes in $\mathbb{R}^3$. For a triangle $\hat{\tau} \in \Delta_2$, we write ℓ_τ for the linear form defining the linear span of $\hat{\tau}$ (ℓ_τ is the homogenization $\tilde{\ell}_\tau$). We put

$$[C^r(\hat{\Delta})]_d := \{F \in C^r(\hat{\Delta}) : F \in \mathbb{R}[x, y, z]_d\},$$

where $\mathbb{R}[x, y, z]_d$ is the vector space of *homogeneous* polynomials of degree d. We relate this space to $C^r_d(\Delta)$ using:

Proposition 2.2. *[4, Theorem 2.6]* $[C^r(\hat{\Delta})]_d \cong C^r_d(\Delta)$ *as real vector spaces.*

The Billera-Schenck-Stillman (BSS) chain complex $\mathcal{R}_\bullet/\mathcal{J}_\bullet$, which we define below, is introduced by Billera in [3] and modified by Schenck and Stillman in [25].

Definition 2.2. *For an edge* $\tau \in \Delta_1^\circ$ *and vertex* $\gamma \in \Delta_0^\circ$, *define*

- $J(\tau) := \langle \ell_\tau^{r+1} \rangle$ *(the principal ideal of* $\mathbb{R}[x,y,z]$ *generated by* ℓ_τ^{r+1}*) and*
- $J(\gamma) := \sum_{\gamma\in\tau} J(\tau)$ *(the ideal of* $\mathbb{R}[x,y,z]$ *generated by* $\{\ell_\tau^{r+1} : \tau \in \Delta_1^\circ \text{ and } \gamma \in \tau\}$.

Let $R = \mathbb{R}[x,y,z]$. We define the chain complex $\mathcal{R}_\bullet$ by

$$\mathcal{R}_\bullet := \bigoplus_{\sigma\in\Delta_2} R \xrightarrow{\partial_2} \bigoplus_{\tau\in\Delta_1^\circ} R \xrightarrow{\partial_1} \bigoplus_{\tau\in\Delta_0^\circ} R,$$

where the differentials ∂_2 and ∂_1 are the differentials in the simplicial chain complex of Δ relative to $\partial\Delta$ with coefficients in R. By definition, the ith homology $H_i(\mathcal{R}_\bullet)$ is isomorphic to $H_i(\Delta, \partial\Delta; R)$, where the latter is the ith simplicial homology group of Δ relative to $\partial\Delta$ with coefficients in R. We also define the subcomplex $\mathcal{J}_\bullet \subset \mathcal{R}_\bullet$ by

$$0 \xrightarrow{\partial_2} \bigoplus_{\tau\in\Delta_1^\circ} J(\tau) \xrightarrow{\partial_1} \bigoplus_{\gamma\in\Delta_0^\circ} J(\gamma)$$

and the quotient complex (which we call the Billera-Schenck-Stillman chain complex)

$$\mathcal{R}_\bullet/\mathcal{J}_\bullet = \bigoplus_{\sigma\in\Delta_2} R \xrightarrow{\overline{\partial_2}} \bigoplus_{\tau\in\Delta_1^\circ} R/J(\tau) \xrightarrow{\overline{\partial_1}} \bigoplus_{\tau\in\Delta_0^\circ} R/J(\gamma).$$

We use the following result of Schenck and Stillman.

Theorem 2.3 (Schenck and Stillman [24, 25]). *The dimension of* $C^r_d(\Delta)$ *is*

$$\dim C^r_d(\Delta) = L(\Delta, d, r) + \dim H_1(\mathcal{R}_\bullet/\mathcal{J}_\bullet)_d, \tag{2.1}$$

where $L(\Delta, d, r)$ *is Schumaker's lower bound [29]. In fact,* $L(\Delta, d, r)$ *coincides with the Euler characteristic, in degree* d*, of the Billera-Schenck-Stillman chain complex* $\mathcal{R}_\bullet/\mathcal{J}_\bullet$.

We set some additional notation to give an expression for Schumaker's lower bound $L(\Delta, d, r)$. If $A, B \in \mathbb{Z}$ are non-negative integers, we use the following convention for the binomial coefficient $\binom{A}{B}$:

$$\binom{A}{B} = \begin{cases} \dfrac{A!}{B!(A-B)!} & B \le A \\ 0 & \text{otherwise} \end{cases}.$$

Proposition 2.4 (Schumaker's lower bound [29]). *For each vertex* $\gamma \in \Delta_0^\circ$ *we let* s_γ *be the number of slopes of edges containing* γ*. Let* α_γ *and* ν_γ *be the quotient and remainder when* $s_\gamma(r+1)$ *is divided by* $s_\gamma - 1$*; that is,* $s_\gamma(r+1) = \alpha_\gamma(s_\gamma - 1) + \nu_\gamma$*, with* $\alpha_\gamma, \nu_\gamma \in \mathbb{Z}$ *and* $0 \le \nu_\gamma < s_\gamma - 1$*. Put* $\mu_\gamma = s_\gamma - 1 - \nu_\gamma$.

Using the above notation, let

$$L(\Delta,d,r) = \binom{d+2}{2} + \left(|\Delta_1^\circ| - \sum_{\gamma\in\Delta_0^\circ} s_\gamma\right)\binom{d+1-r}{2} + \sum_{\gamma\in\Delta_0^\circ}\left(\mu_\gamma\binom{d+2-\alpha_\gamma}{2} + \nu_\gamma\binom{d+1-\alpha_\gamma}{2}\right).$$

Then $L(\Delta,d,r) \le C_d^r(\Delta)$.

Proof. Schumaker [29] gave a lower bound on the dimension of the spline space which Schenck and Stillman show in [24] is equivalent to the above expression $L(\Delta,d,r)$. ∎

There is also an upper bound for $\dim C_d^r(\Delta)$ due to Schumaker [30], which requires a certain ordering of Δ_0°. The bound is extended to any ordering on Δ_0° by Mourrain and Villamizar in [22], and we record their formulation of Schumaker's upper bound below.

Proposition 2.5. *[30, 22] Let Δ be a triangulation with a fixed ordering $\gamma_1,\ldots,\gamma_{|\Delta_0^\circ|}$ on its interior vertices. For each interior vertex $\gamma=\gamma_i$ we let $\tilde{s}_\gamma$ be the number of different slopes of edges containing γ_i which connect γ_i either to a vertex γ_j with $j<i$ or to a vertex on the boundary of Δ. We then define $\tilde{\alpha}_\gamma,\tilde{\nu}_\gamma,\tilde{\mu}_\gamma$ in exactly the way α_γ,ν_γ, and μ_γ are defined in Proposition 2.4, except we replace s_γ with $\tilde{s}_\gamma$. Let $U(\Delta,d,r)$ be the expression resulting from substituting $s_\gamma\to\tilde{s}_\gamma,\alpha_\gamma\to\tilde{\alpha}_\gamma,\nu_\gamma\to\tilde{\nu}_\gamma$, and $\mu_\gamma\to\tilde{\mu}_\gamma$ in Schumaker's lower bound $L(\Delta,d,r)$. Then $\dim C_d^r(\Delta)\le U(\Delta,d,r)$.*

We will also consider splines on a partition Δ which is not a triangulation but a *rectilinear partition* – in this case the polygonal domain $|\Delta|$ is subdivided into polygons which meet along edges. All definitions and results stated thus far carry over to rectilinear partitions. The class of rectilinear partitions we will have occasion to use are *quasi-cross-cut* partitions.

Definition 2.3. *A rectilinear partition Δ is a quasi-cross-cut partition if every edge of Δ is connected to the boundary of Δ by a sequence of adjacent edges that all have the same slope.*

We give a formulation of a result of Chui and Wang [5] which appears in [24].

Proposition 2.6. *If Δ is a quasi-cross-cut partition then $\dim C_d^r(\Delta) = L(\Delta,d,r)$ for all $d,r\ge 0$.*

2.1. Case of a single totally interior edge.

In this section we specialize to the case of interest in this paper. That is, Δ is a triangulation with only two interior vertices v_1 and v_2 connected by a single totally interior edge τ. We further assume that the triangulation consists only of triangles which contain either v_1 or v_2 or both. There are two cases in which the dimension formula on such a triangulation is trivial.

Proposition 2.7. *Let Δ be a triangulation with a single totally interior edge τ connecting interior vertices v_1 and v_2. Suppose that either*

- *the interior edge τ has the same slope as another edge meeting τ at either v_1 or v_2 or*
- *the number of slopes of edges meeting at either v_1 or v_2 is at least $r+3$.*

Then $\dim C_d^r(\Delta) = L(\Delta,d,r)$ for all integers $d,r\ge 0$.

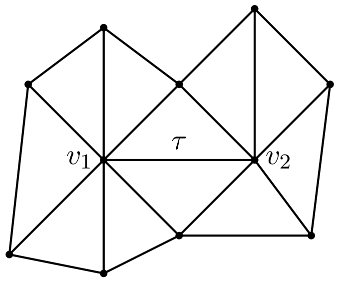

Figure 1. *A triangulation with a single totally interior edge,* $p = 6$*,* $s = 3$*,* $q = 5$*, and* $t = 4$*. A choice of coordinates that realizes this data is* $v_1 = (-1,0), v_2 = (1,0)$ *and, for the remaining vertices (read counterclockwise around the boundary, starting with the vertex northeast of* v_2*),* $(2,1),(1,2),(0,1),(-1,7/4),(-2,1),(-9/4,-5/4),(-1,-3/2),(0,-1),$ *and* $(1,7/4)$*.*

Proof. The result follows from [24, Theorem 5.2]. In either case, $H_1(\mathcal{R}_\bullet/\mathcal{J}_\bullet) = 0$ and $C^r(\hat{\Delta})$ is a free module over the polynomial ring. This also follows from the upper bound $U(\Delta, d, r)$ of Schumaker [30] – see Proposition 2.5 – since either hypothesis forces $L(\Delta, d, r) = U(\Delta, d, r)$ for $d \geq 0$. ■

Assumptions 2.1. *In the remainder of the paper, we use the following notation and assumptions whenever we have a triangulation* Δ *with a single totally interior edge* τ *connecting interior vertices* v_1 *and* v_2*.*

- *We assume no edge adjacent to* v_1 *or* v_2 *has the same slope as* τ*.*
- *We write* p *(respectively* q*) for the number of edges different from* τ *which are adjacent to* v_1 *(respectively* v_2*).*
- *We write* s *(respectively* t*) for the number of different slopes achieved by the edges different from* τ *which contain* v_1 *(respectively* v_2*).*
- *We assume (without loss) that* $2 \leq s \leq t \leq r+1$*.*

Remark 1. *We explain the last bullet point in Assumptions 2.1. Since we assume no other edge besides* τ *has a slope equal to the slope of* τ*,* v_1 *is surrounded by* $p+1$ *edges taking on* $s+1$ *different slopes and* v_2 *is surrounded by* $q+1$ *edges taking on* $t+1$ *different slopes. See Figure 1. We obtain* $s \leq t$ *by relabeling* v_1 *and* v_2 *if necessary. If* $s = 1$ *then either* $p = 1$ *or* $p = 2$*. In either case it is not possible for* Δ *to be a triangulation. (If* $p = 2$ *it would be possible to have a so-called* T*-juncture or 'hanging vertex' at* v_1*, but we do not allow these under our definition of a triangulation.) Hence* $2 \leq s, t$*. We can also assume that* $s+1$ *and* $t+1$ *are both at most* $r+2$ *by Proposition 2.7. Putting these all together, we arrive at* $2 \leq s \leq t \leq r+1$*.*

We now consider the homology module $H_1(\mathcal{R}_\bullet/\mathcal{J}_\bullet)$.

Lemma 2.8. *If* Δ *has only one totally interior edge* τ *, then*

$$H_1(\mathcal{R}_\bullet/\mathcal{J}_\bullet) \simeq R/(J_1 : J(\tau) + J_2 : J(\tau))(-r-1), \tag{2.2}$$

where

$$J_i = \sum_{\substack{\varepsilon \in \Delta_1^\circ \\ v_i \in \varepsilon, \varepsilon \neq \tau}} J(\varepsilon) \text{ for } i = 1, 2. \tag{2.3}$$

This lemma is a consequence of presentation for $H_1(\mathcal{R}_\bullet/\mathcal{J}_\bullet)$ due to Schenck and Stillman. We recall this presentation before proceeding to the proof.

Lemma 2.9. *[25, Lemma 3.8] Let $\bigoplus_{\varepsilon\in\Delta_1^\circ} R[e_\varepsilon]$ be the free R module with summands indexed by the formal basis symbols $\{[e_\varepsilon] \mid \varepsilon \in \Delta_1^\circ\}$ which each have degree $r+1$. Define $K^r \subset \bigoplus_{\varepsilon\in\Delta_1^\circ} R[e_\varepsilon]$ to be the submodule of $\bigoplus_{\varepsilon\in\Delta_1^\circ} R[e_\varepsilon]$ generated by*

$$\{[e_\varepsilon] \mid \varepsilon \in \Delta_1^\circ \textit{ is not totally interior}\}$$

and, for each $\gamma \in \Delta_0^\circ$,

$$\left\{ \sum_{\gamma\in\varepsilon} a_\varepsilon [e_\varepsilon] \;\middle|\; \sum_{\gamma\in\varepsilon} a_\varepsilon \ell_\varepsilon^{r+1} = 0 \right\}.$$

The R-module $H_0(\mathcal{J}_\bullet)$ is given by generators and relations by

$$0 \to K^r \to \bigoplus_{\varepsilon\in\Delta_1^\circ} R[e_\varepsilon] \to H_0(\mathcal{J}_\bullet) \to 0.$$

Proof of Lemma 2.8. First, since Δ has no holes, $H_1(\mathcal{R}_\bullet/\mathcal{J}_\bullet) \cong H_0(\mathcal{J}_\bullet)$. This follows from the long exact sequence in homology associated to the short exact sequence of chain complexes $0 \to \mathcal{J}_\bullet \to \mathcal{R}_\bullet \to \mathcal{R}_\bullet/\mathcal{J}_\bullet \to 0$ and the fact that $H_1(\mathcal{R}_\bullet) = H_0(\mathcal{R}_\bullet) = 0$ (see [25]).

Thus we may use Lemma 2.9. Since Δ has only one totally interior edge τ, K^r is generated by the free module $F = \{[e_\varepsilon] \mid \varepsilon \in \Delta_1^\circ, \varepsilon \neq \tau\}$ and the syzygy modules $K_i = \{\sum_{v_i\in\varepsilon} a_\varepsilon[e_\varepsilon] \mid \sum_{v_i\in\varepsilon} a_\varepsilon \ell_\varepsilon^{r+1} = 0\}$ for $i = 1, 2$. Since all factors of R indexed by an interior edge different from τ are quotiented out, after trimming the presentation in Lemma 2.9 we are left with

$$0 \to \overline{K^r} \to R[e_\tau] \to H_0(\mathcal{J}_\bullet) \to 0,$$

where $\overline{K^r} = K^r/F$. Observe that $\overline{K^r}$ is the internal sum of the submodules

$$K_i = \{a_\tau[e_\tau] \mid a_\tau \ell_\tau^{r+1} \in J_i\}$$

for $i = 1, 2$, where $J_i = \sum_{\substack{\varepsilon\in\Delta_1^\circ \\ v_i\in\varepsilon, \varepsilon\neq\tau}} J(\varepsilon)$ for $i = 1, 2$. Thus $\overline{K^r} = J_1 : \ell_\tau^{r+1} + J_2 : \ell_\tau^{r+1} = J_1 : J(\tau) + J_2 : J(\tau)$. Recalling that $[e_\tau]$ has degree $r+1$, this proves that

$$H_1(\mathcal{R}_\bullet/\mathcal{J}_\bullet) \cong R/(J_1 : J(\tau) + J_2 : J(\tau))(-r-1).$$ ■

After coning, we apply a change of coordinates $T : \mathbb{R}^3 \to \mathbb{R}^3$ so that $T(\hat{v}_1)$ points in the direction of $(0, 1, 0)$ and $T(\hat{v}_2)$ points in the direction of $(1, 0, 0)$. With respect to this new choice of coordinates we may choose linear forms vanishing on the interior triangle so that:

$$\begin{aligned} J(\tau) &= \langle z^{r+1}\rangle, \\ J_1 &= \langle (x+b_1z)^{r+1}, (x+b_2z)^{r+1}, \ldots, (x+b_sz)^{r+1}\rangle, \text{ and} \\ J_2 &= \langle (y+c_1z)^{r+1}, (y+c_2z)^{r+1}, \ldots, (y+c_tz)^{r+1}\rangle. \end{aligned}$$

In Section 3 we study ideals of this type, returning to the study of the homology module in Section 4.

3. The initial ideal of a power ideal in two variables. This section is largely a technical section in which we derive some results from commutative algebra – possibly of independent interest – to use in our analysis for $\dim C^r_d(\Delta)$ in future sections. The reader will not lose much by skipping this section for now and returning later as needed or desired. Our references are [6] for Gröbner bases, [13] for apolarity, and [20, Chapter 2] for lex-segment ideals.

Suppose we are given a set of points $\mathcal{X} = \{p_1, \cdots, p_s\} \subset \mathbb{P}^1$, where $p_i = [b_i : c_i]$ for $1 \le i \le s$, and a sequence $\mathbf{a} = (a_1, \ldots, a_s)$ of *multiplicities* for these points. We will assume that the points are ordered so that $a_1 \le a_2 \le \cdots \le a_s$. We associate two ideals to this set of points. First, the *power ideal*

$$J(\mathcal{X}, \mathbf{a}) := \langle (b_1x + c_1y)^{a_1+1}, \ldots, (b_sx + c_sy)^{a_s+1} \rangle$$

in the polynomial ring $R = \mathbb{R}[x, y]$. Secondly, the *fat point* ideal

$$I(\mathcal{X}, \mathbf{a}) := \bigcap_{i=1}^{s} \langle b_iY - c_iX \rangle^{a_i} = \langle \prod_{i=1}^{s} (b_iY - c_iX)^{a_i} \rangle$$

in the polynomial ring $S = \mathbb{R}[X, Y]$ (S is the coordinate ring of $\mathbb{P}^1$). The ideal $I_{\mathbf{a}}(\mathcal{X})$ consists of all polynomials which vanish to order a_i at p_i, for $i = 1, \ldots, s$.

Our objective is to show that, under the assumption that $b_i \neq 0$ for $i = 1, \ldots, s$, the initial ideal $\mathrm{In}(J(\mathcal{X}, \mathbf{a}))$, with respect to either graded lexicographic or graded reverse lexicographic order, is a lex-segment ideal. Since the graded lexicographic and graded reverse lexicographic order coincide in two variables, we focus on the lexicographic order since it is consistent with the lex-segment definition.

Definition 3.1. *A monomial ideal $I \subseteq R$ is called a lex-segment ideal if, whenever a monomial $m \in R$ of degree d satisfies $m >_{lex} n$ for some monomial $n \in I$ of degree d, then $m \in I$.*

Lex-segment ideals play an important role in Macaulay's classification of Hilbert functions [18]. Before proceeding to the proof, we introduce the notion of *apolarity*; see [13] for an excellent survey. Define an action of S on R by

$$(X^aY^b) \circ f = \frac{\partial f}{\partial x^a \partial y^b},$$

and extend linearly. That is, S acts on R as *partial differential operators.* It is straightforward to see that this action induces a perfect pairing $R_d \times S_d \to \mathbb{R}$ via $(f, F) \to F \circ f$. For an $\mathbb{R}$-vector subspace $U \subset R_d$ we thus define

$$U^\perp := \{F \in S : F \circ f = 0 \text{ for all } f \in U\}.$$

Write $J_d(\mathcal{X}, \mathbf{a})$ for the $\mathbb{R}$-vector space spanned by homogeneous polynomials in $J(\mathcal{X}, \mathbf{a})$ of degree d (this definition clearly extends to any homogeneous ideal). A result of Emsalem and Iarrobino describes $J_d(\mathcal{X}, \mathbf{a})^\perp$ in terms of fat point ideals. In the statement of the result below, we put $[m]_+ = \max\{m, 0\}$ and $[d - \mathbf{a}]_+ = ([d - a_1]_+, [d - a_2]_+, \ldots, [d - a_s]_+)$.

Theorem 3.1 (Emsalem and Iarrobino [11]). $J_d(\mathcal{X}, \mathbf{a})^\perp = I_d(\mathcal{X}, [d - \mathbf{a}]_+)$

As a corollary, the Hilbert function $\mathrm{HF}(d, J(\mathcal{X}, \mathbf{a})) = \dim J_d(\mathcal{X}, \mathbf{a})$ can be derived.

Corollary 3.2 (Geramita and Schenck [14]). $\dim J_d(\mathcal{X}, \mathbf{a}) = \min\{d+1, \sum_{i=1}^{s}[d-a_i]_+\}$

This shows that the Hilbert function of $J(\mathcal{X}, \mathbf{a})$ has the maximal growth possible for its number of generators. We take this analysis one step further.

Corollary 3.3. *Suppose that no point of $\mathcal{X}$ has a vanishing x-coordinate. Then the initial ideal $In(J(\mathcal{X}, \mathbf{a}))$ is a lex-segment ideal.*

Proof. Fix a degree d. Put $F = \prod_{i=1}^{s}(b_iY - c_iX)^{[d-a_i]_+}$ and $\alpha = \deg(F) = \sum_{i=1}^{s}[d-a_i]_+$. By assumption, $b_i \neq 0$ for any $i = 1, \dots, s$, so the monomial Y^α appears with non-zero coefficient in F.

Since $I(\mathcal{X}, [d-\mathbf{a}]_+)$ is principle, a basis for $I_d(\mathcal{X}, [d-\mathbf{a}]_+)$ is given by

$$\{X^{d-\alpha-b}Y^bF : 0 \leq b \leq d-\alpha\}$$

(Coupled with Theorem 3.1, this proves that $\dim J_d(\mathcal{X}, \mathbf{a}) = \alpha$, which is Corollary 3.2.) Observe that the given basis for $I_d(\mathcal{X}, \mathbf{a})$ has a polynomial whose lex-last term involves the monomial $X^{d-\alpha-b}Y^{b+\alpha}$ for $0 \leq b \leq d-\alpha$.

If $d < \min\{a_i \mid 1 \leq i \leq s\}$ then $J_d(\mathcal{X}, \mathbf{a}) = 0$. So suppose $d \geq \min\{a_i \mid 1 \leq i \leq s\}$ and that the leading term of some polynomial $f \in J_d(\mathcal{X}, \mathbf{a})$ with respect to lex order is $Cx^{d-\alpha-b}y^{b+\alpha}$ for some $b \geq 0$ and $C \neq 0$. Then every other term of f involves a power of y which is larger than $b+\alpha$. From our above observation, the lex-last (or lex-least) monomial in the basis polynomial $X^{d-\alpha-b}Y^bF$ is $X^{d-\alpha+b}Y^{b+\alpha}$. Thus $X^{d-\alpha-b}Y^bF \circ f \neq 0$. In fact, $X^{d-\alpha-b}Y^bF \circ f = \prod_{i=1}^{s} b_iX^{d-\alpha-b}Y^{b+\alpha} \circ (Cx^{d-\alpha+b}y^{b+\alpha})$, so we can compute it exactly as:

$$X^{d-\alpha-b}Y^bF \circ f = C(\prod_{i=1}^{s} b_i)(d-\alpha+b)!(b+\alpha)!,$$

which is non-zero because the b_i's are all non-vanishing and $C \neq 0$. This contradicts Theorem 3.1, since $X^{d-\alpha-b}Y^bF \in I_d(\mathcal{X}, [d-\mathbf{a}]_+)$ but $X^{d-\alpha-b}Y^bF \circ f \neq 0$.

It follows that the initial terms of $J_d(\mathcal{X}, \mathbf{a})$ can only involve the monomials x^Ay^B, where $0 \leq B < \alpha$. Since $\dim J_d(\mathcal{X}, \mathbf{a}) = \alpha$ by Corollary 3.2, it follows that $\mathrm{In}(J(\mathcal{X}, \mathbf{a}))_d$ consists of the α lex-largest monomials of degree d. Thus $\mathrm{In}(J(\mathcal{X}, \mathbf{a}))$ is a lex-segment ideal. ∎

In the following corollary we use the ordering $a_1 \leq a_2 \leq \cdots \leq a_s$.

Corollary 3.4. *With the same setup as Corollary 3.3, The initial ideal $In(J(\mathcal{X}, \mathbf{a}))$ consists of the monomials x^Ay^B, where $A \geq 0$, $B \geq 0$, and one of the strict inequalities $\sum_{i=1}^{j} a_i < jA + (j-1)B$, $1 \leq j \leq s$, is satisfied.*

Proof. It suffices to show that $x^Ay^B \notin \mathrm{In}(J(\mathcal{X}, \mathbf{a}))$ if and only if $A \geq 0, B \geq 0$ and $\sum_{i=1}^{j} a_i \geq jA + (j-1)B$ is satisfied for every $j = 1, \dots, s$.

Since $\mathrm{In}(J(\mathcal{X}, \mathbf{a}))$ is a lex-segment ideal with Hilbert function $\dim \mathrm{In}(J(\mathcal{X}, \mathbf{a}))_d = \min\{d+1, \sum_{i=1}^{s}[d-a_i]_+$, $x^Ay^B \notin \mathrm{In}(J(\mathcal{X}, \mathbf{a}))$ if and only if

$$B \geq \dim \mathrm{In}(J(\mathcal{X}, \mathbf{a}))_{A+B} = \min\left\{A+B+1, \sum_{i=1}^{s}[A+B-a_i]_+\right\}.$$

Since $A, B \geq 0$ it is not possible that $B \geq A+B+1$. So we are left with the condition

$$B \geq \sum_{i=1}^{s}[A+B-a_i]_+.$$

Now, since $a_1 \leq a_2 \leq \cdots \leq a_s$, $A+B-a_1 \geq A+B-a_2 \geq \cdots A+B-a_s$. The 'plus' subscript means only positive contributions to the sum on the right hand side are taken. So we can interpret the above inequality as

$$B \geq \max\left\{\sum_{i=1}^{j}(A+B-a_i) : j=1,\ldots,s\right\}.$$

Equivalently, $B \geq \sum_{i=1}^{j}(A+B-a_i)$ is satisfied for $j=1,\ldots,s$. Re-arranging, we get $x^Ay^B \notin \mathrm{In}(J(\mathcal{X},\mathbf{a}))$ if and only if $jA+(j-1)B \leq \sum_{i=1}^{j} a_i$ for $i=1,\ldots,s$. ■

Remark 2. *Given non-negative integers $a_1 \leq a_2 \leq \cdots \leq a_s$, the inequalities $A \geq 0$, $B \geq 0$, and $jA+(j-1)B \leq \sum_{i=1}^{j} a_i$ for $1 \leq j \leq s$ define a convex polygon in $\mathbb{R}^2$. Corollary 3.4 says that the initial ideal of $J(\mathcal{X},\mathbf{a})$ consists of monomials which are in bijection with the lattice points in the first quadrant of $\mathbb{R}^2$ and are additionally **not** contained in this polygon. Equivalently, the monomials which are **not** in the initial ideal of $J(\mathcal{X},\mathbf{a})$ are in bijection with the lattice points of this polygon.*

In the next result, and following, if r is a non-negative integer we write $J(\mathcal{X},r)$ and $I(\mathcal{X},r)$ for the case where $\mathbf{a}=(r,r,\ldots,r)$ consists of s copies of r.

Corollary 3.5. *With the same setup as Corollary 3.3, the initial ideal $In(J(\mathcal{X},r))$ consists of those monomials x^Ay^B satisfying $A \geq 0, B \geq 0$, and $sr < sA+(s-1)B$.*

Proof. Due to Corollary 3.4, it suffices to show that the inequality $sr < sA+(s-1)B$ is implied by the inequality $jr < jA+(j-1)B$ for any $j \leq s$. This is clear by multiplying both sides of $jr < jA+(j-1)B$ by s/j. ■

3.1. Behavior under colon. In this section we discuss the behavior of $J(\mathcal{X},\mathbf{a})$ under coloning with a power of y. We continue to assume that no point of $\mathcal{X}$ has a vanishing x-coordinate. We use the following fact about graded reverse lexicographic order.

Proposition 3.6. *If $I \subset R = \mathbb{R}[x_1,\ldots,x_n]$ under graded reverse lexicographic order, then $In(I : x_n) = In(I) : x_n$. In particular, for any integer $e \geq 0$, $In(J(\mathcal{X},\mathbf{a}) : y^e) = In(J(\mathcal{X},\mathbf{a})) : y^e$.*

Proof. This is a special case of [10, Proposition 15.12]. ■

Corollary 3.7. *For any integer $e \geq 0$, $In(J(\mathcal{X},\mathbf{a}) : y^e)$ is a lex-segment ideal with Hilbert function*

$$\dim In(J(\mathcal{X},\mathbf{a}) : y^e)_d = [\dim In(J(\mathcal{X},\mathbf{a}))_{d+e} - e]_+ = [\min\{d+1, \sum_{i=1}^{s}[d+e-a_i]_+\} - e]_+$$

The monomial x^Ay^B is in $In(J(\mathcal{X},\mathbf{a}) : y^e)$ if and only if $A \geq 0, B \geq 0$, and the inequality $\sum_{i=1}^{j} a_i - (j-1)e < jA+(j-1)B$ is satisfied for some $j=1,\ldots,s$.

Proof. Due to Proposition 3.6 and the fact that graded lexicographic and graded reverse lexicographic orders coincide in two variables, we have $\mathrm{In}(J(\mathcal{X},\mathbf{a}) : y^e) = \mathrm{In}(J(\mathcal{X},\mathbf{a})) : y^e$. Now, a well-known identity is that $(\mathrm{In}(J(\mathcal{X},\mathbf{a})) : y^e)y^e = \mathrm{In}(J(\mathcal{X},\mathbf{a})) \cap \langle y^e \rangle$. Said otherwise, the monomials in $\mathrm{In}(J(\mathcal{X},\mathbf{a})) : y^e$ of degree d are in bijection with the monomials of degree $d+e$ in $\mathrm{In}(J(\mathcal{X},\mathbf{a}))$ which are divisible by y^e. Since $(\mathrm{In}(J(\mathcal{X},\mathbf{a})))_{d+e}$ is spanned by lex-largest monomials, $\mathrm{In}(J(\mathcal{X},\mathbf{a})) : y^e$ is either empty or consists of the $\dim(\mathrm{In}(J(\mathcal{X},\mathbf{a})))_{d+e} - e$ lex-largest monomials of degree d. This establishes both that $\mathrm{In}(J(\mathcal{X},\mathbf{a})) : y^e$ is lex-segment and the claimed form of the Hilbert function.

For the description of the monomials $x^A y^B$ which are in $\mathrm{In}(J(\mathcal{X},\mathbf{a}) : y^e) = \mathrm{In}(J(\mathcal{X},\mathbf{a})) : y^e$, it suffices to observe that $x^A y^B \in \mathrm{In}(J(\mathcal{X},\mathbf{a})) : y^e$ if and only if $x^A y^{B+e} \in \mathrm{In}(J(\mathcal{X},\mathbf{a}))$. Then apply Corollary 3.4. ■

3.2. A summation property of Gröbner bases. We would like to prove a general fact, which will be useful in later sections. We refer the reader to [6, Chapter 2] for basics on Gröbner bases and the Buchberger algorithm, and we follow the same notation.

Lemma 3.8. *Let R be the polynomial ring $\mathbb{R}[x,y,z]$. Assume I is a homogeneous ideal generated by polynomials in the variables x and z and J is a homogeneous ideal generated by polynomials in the variables y and z, then a Gröbner basis for $I+J$ with respect to graded lexicographic (or graded reverse lexicographic) order can be obtained by taking the union of the Gröbner bases of I and J with respect to the graded lexicographic (or graded reverse lexicographic) order. In particular,* $\mathrm{In}(I+J) = \mathrm{In}(I) + \mathrm{In}(J)$.

Proof. Let $\mathcal{G}_1$ be a Gröbner basis for I and $\mathcal{G}_2$ be a Gröbner basis for J, both taken with respect to either graded lexicographic order or graded reverse lexicographic order. It suffices to show that $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$ satisfies Buchberger's criterion - that is, the S-pair $S(f,g)$ of any two $f,g \in \mathcal{G}$ reduces to zero under the division algorithm. This is clearly true if both f and g are in $\mathcal{G}_1$ or both f and g are in $\mathcal{G}_2$. So we assume $f \in \mathcal{G}_1, g \in \mathcal{G}_2$. We further assume the leading coefficients of f and g are normalized to 1. Let $\mathrm{LT}(f) = x^A z^C$ and $\mathrm{LT}(g) = y^B z^D$. Then

$$f = x^A z^C + \text{terms in } x,z \text{ divisible by } z^C \text{ and}$$
$$g = y^B z^D + \text{terms in } y,z \text{ divisible by } z^D.$$

Put $f' = f - x^A z^C$ and $g' = g - y^B z^D$. Assume $C \geq D$ (the case $D \geq C$ is entirely analogous). Then we can write the S-pair of f and g as

$$S(f,g) = \frac{g-g'}{z^D} f - \frac{f-f'}{z^D} g = \frac{f'}{z^D} g - \frac{g'}{z^D} f,$$

where $\frac{f'}{z^D}$ and $\frac{g'}{z^D}$ are both polynomials because every term of f' is divisible by z^C (and hence z^D since $C \geq D$) and every term of g' is divisible by z^D. There is no cancellation between the lead terms of $f'g/z^D$ and $g'f/z^D$ since the lead term of $f'g/z^D$ has a higher power of y in it than fg'/z^D. Thus $\mathrm{LT}(S(f,g)) = \max\left\{\mathrm{LT}\left(\frac{f'}{z^D} g\right), \mathrm{LT}\left(\frac{g'}{z^D} f\right)\right\}$. Since $\mathrm{LT}\left(\frac{f'}{z^D} g\right) \leq \mathrm{LT}(S(f,g))$ and $\mathrm{LT}\left(\frac{g'}{z^D} f\right) \leq \mathrm{LT}(S(f,g))$,

$$S(f,g) = \frac{f'}{z^D} g - \frac{g'}{z^D} f$$

is what is called a *standard representation* of $S(f,g)$ in [6, Section 9]. It is shown in [6, Section 9] that if every S-pair of $\mathcal{G}$ has a standard representation, then $\mathcal{G}$ is a Gröbner basis, and so the result follows. ■

4. The dimension formula expressed via lattice points. In this section we prove our first version of the dimension formula for $C^r_d(\Delta)$ when Δ is a triangulation with a single totally interior edge. We also characterize when $\dim C^r_d(\Delta)$ begins to agree with Schumaker's lower bound. We record these as two separate results, and their proofs are based on results before and in the section.

Theorem 4.1. *Let Δ be a triangulation with a single totally interior edge τ satisfying Assumptions 2.1. Then for all integers $d \geq 0$,*

$$\dim C^r_d(\Delta) = L(\Delta, d, r) + \#(\mathcal{P} \cap \mathbb{Z}^3 \cap H_d)$$

where $H_d = \{(A,B,C) \in \mathbb{R}^3 : A+B+C = d-r-1\}$ and $\mathcal{P}$ is the polytope in $\mathbb{R}^3$ defined by $A,B,C \geq 0$, $sA+(s-1)C \leq r+1-s$, and $tB+(t-1)C \leq r+1-t$. Equivalently, for all integers $d \geq 0$,

$$\dim C^r_d(\Delta) = L(\Delta, d, r) + \#(\mathcal{P}_d \cap \mathbb{Z}^2),$$

where $\mathcal{P}_d$ is the polygon in $\mathbb{R}^2$ defined by the inequalities $A \geq 0, B \geq 0$, $A - B(s-1) \leq sr - d(s-1)$, $B - A(t-1) \leq tr - d(t-1)$, and $A + B \leq d - r - 1$.

Proof. The first equation in Theorem 4.1 follows from Lemma 4.6, Theorem 2.3, and Lemma 2.9. The second equation follows from the first and Proposition 4.7. ■

Theorem 4.2. *Let Δ be a triangulation with a single totally interior edge τ satisfying Assumptions 2.1. If $r+1 \equiv s-1 \pmod{s}$ and $r+1 \equiv t-1 \pmod{t}$ then*

$$\dim C^r_d(\Delta) > L(\Delta,d,r) \text{ for } d = \left\lfloor \frac{r+1}{s} \right\rfloor + \left\lfloor \frac{r+1}{t} \right\rfloor + r, \text{ and}$$
$$\dim C^r_d(\Delta) = L(\Delta,d,r) \text{ for } d \geq \left\lfloor \frac{r+1}{s} \right\rfloor + \left\lfloor \frac{r+1}{t} \right\rfloor + r + 1.$$

Otherwise,

$$\dim C^r_d(\Delta) > L(\Delta,d,r) \text{ for } d = \left\lfloor \frac{r+1}{s} \right\rfloor + \left\lfloor \frac{r+1}{t} \right\rfloor + r - 1, \text{ and}$$
$$\dim C^r_d(\Delta) = L(\Delta,d,r) \text{ for } d \geq \left\lfloor \frac{r+1}{s} \right\rfloor + \left\lfloor \frac{r+1}{t} \right\rfloor + r.$$

Proof. Theorem 4.2 follows from Theorem 2.3 and Proposition 4.8. ■

Remark 3. *In case there are three slopes that meet at each endpoint of the interior edge τ (so $s = t = 2$), Theorem 4.2 yields that $\dim C^r_d(\Delta) = L(\Delta,d,r)$ for $d \geq 2r+1$, which recovers the main result of Tohăneanu and Mináč in [21]. We say more on this in Example 6.1.*

Corollary 4.3. *If Δ has a single totally interior edge, then $\dim C^r_d(\Delta) = L(\Delta,d,r)$ for $d \geq 2r+1$, so Δ satisfies the '$2r+1$' conjecture of Schenck [27] (see also [28, Conjecture 2.1]).*

Proof. This is immediate from Proposition 2.7 and Theorem 4.2, coupled with the fact that $t \geq s \geq 2$. ■

Since the case $r = 1, d = 3$ receives lots of attention, we treat it separately and give an explicit formula for $\dim C^1_3(\Delta)$.

Corollary 4.4. *Assume Δ has two interior vertices and one totally interior edge. Then $\dim C^1_d(\Delta) = L(\Delta, d, 1)$ for $d \geq 3$, so Δ satisfies the conjecture of Alfeld and Manni for $\dim C^1_3(\Delta)$ (see [1, Conjecture 3] or [23]). Explicitly,*

$$\dim C^1_3(\Delta) = 10 \cdot |\Delta_2| - 7 \cdot |\Delta^\circ_1| + 6.$$

Proof. Corollary 4.3 yields $\dim C^1_3(\Delta) = L(\Delta, 3, 1)$. We know that $L(\Delta, 3, 1)$ is the Euler characteristic of $\mathcal{R}_\bullet/\mathcal{J}_\bullet$ (see Theorem 2.3) in degree 3. A straightforward calculation then yields the explicit formula. Any equivalent expression for Schumaker's lower bound (e.g. Proposition 2.4) may also be used to obtain the formula for $L(\Delta, 3, 1)$, although it may differ from the exact expression above by identities involving the number of triangles, interior vertices, and interior edges. ■

We shall use Theorem 2.3 to prove Theorem 4.1 and Theorem 4.2, hence we spend the remainder of this section analyzing the homology module $H_1(\mathcal{R}_\bullet/\mathcal{J}_\bullet)$, where $\mathcal{R}_\bullet/\mathcal{J}_\bullet$ is the Billera-Schenck-Stillman chain complex from Section 2. We use Assumptions 2.1 throughout this section. As we observed in Subsection 2.1, we may change coordinates so that

$$\begin{aligned} J(\tau) &= \langle z^{r+1} \rangle, \\ J_1 &= \langle (x + b_1 z)^{r+1}, (x + b_2 z)^{r+1}, \ldots, (x + b_s z)^{r+1} \rangle, \text{ and} \\ J_2 &= \langle (y + c_1 z)^{r+1}, (y + c_2 z)^{r+1}, \ldots, (y + c_t z)^{r+1} \rangle. \end{aligned}$$

Using Lemma 2.8, Proposition 3.6, and Lemma 3.8, we obtain the following corollary.

Corollary 4.5. *With the above definition of J_1, J_2 and $J(\tau)$,*

$$\mathrm{In}(J_i : J(\tau)) = \mathrm{In}(J_i) : J(\tau), \text{ for } i = 1, 2 \text{ and} \tag{4.1}$$

$$\mathrm{In}(J_1 : J(\tau) + J_2 : J(\tau)) = \mathrm{In}(J_1 : J(\tau)) + \mathrm{In}(J_2 : J(\tau)), \tag{4.2}$$

where the initial ideal is taken with respect to graded lexicographic order or graded reverse lexicographic order.

Proof. The equation (4.1) follows from Proposition 3.6. Because $J_1 : J(\tau)$ is only generated in polynomials in x and z, and $J_2 : J(\tau)$ is only generated in polynomials in y and z, we may apply Lemma 3.8 here and obtain (4.2). ■

Lemma 4.6. *A basis for $R/(J_1 : J(\tau) + J_2 : J(\tau))$ as an $\mathbb{R}$-vector space is given by the monomials $x^A y^B z^C$ which satisfy the inequalities $A \geq 0, B \geq 0, C \geq 0, r+1-s \geq sA+(s-1)C$, and $r + 1 - t \geq tB + (t - 1)C$.*

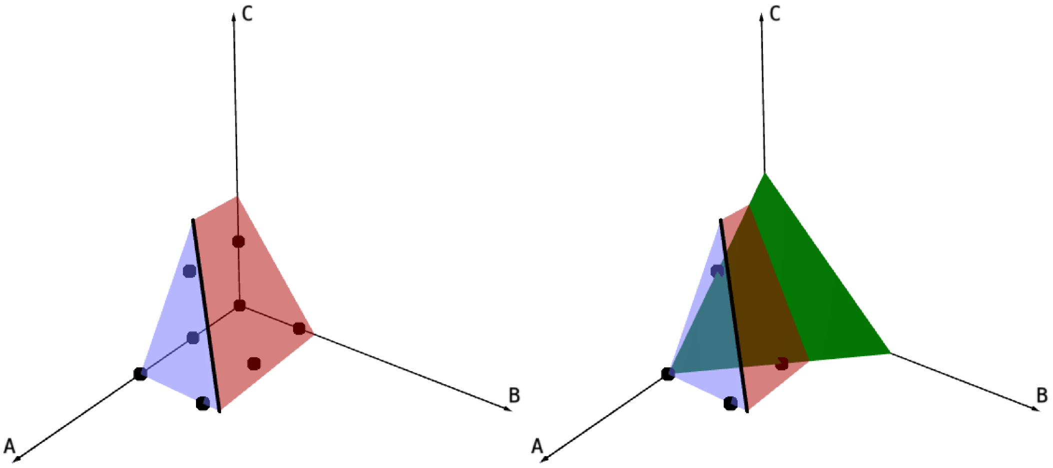

Figure 2. *Exponent vectors of monomials outside $R/(J_1 : J(\tau) + J_2 : J(\tau))$ when $p = 6, s = 3, q = 5, t = 4$, and $r = 8$. The blue shaded plane (parallel to the B-axis) has equation $3A+2C = 6$, while the red shaded plane (parallel to the A-axis) has equation $4B + 3C = 5$. The picture at right shows the slice of this polytope when $A + B + C = 2$. There are three lattice points of the polytope in this slice.*

Proof. A common use of initial ideals is that the monomials outside of $\text{In}(I)$ form a basis for R/I [6, Section 5.3]. Thus it suffices to show that $x^Ay^Bz^C \notin \text{In}(J_1 : J(\tau) + J_2 : J(\tau))$ if and only if A, B, C satisfy the claimed inequalities. Since $\text{In}(J_1 : J(\tau) + J_2 : J(\tau)) = \text{In}(J_1 : J(\tau)) + \text{In}(J_2 : J(\tau))$ by (4.2), it suffices to show that $x^Ay^Bz^C \notin \text{In}(J_1 : J(\tau))$ and $x^Ay^Bz^C \notin \text{In}(J_2 : J(\tau))$ if and only if the claimed inequalities hold. Since the initial ideals are monomial, $x^Ay^Bz^C \notin \text{In}(J_1 : J(\tau)) \iff x^Az^C \notin \text{In}(J_1 : J(\tau))$ and $x^Ay^Bz^C \notin \text{In}(J_2 : J(\tau)) \iff y^Bz^C \notin \text{In}(J_2 : J(\tau))$. Thus we reduce in both cases to two variables, and the result now follows from Corollary 3.7. ∎

Example 4.1. *Let Δ be the triangulation in Figure 1, with $s = 3$ and $t = 4$. When $r = 8$, a basis for $R/(J_1 : J(\tau) + J_2 : J(\tau))$ as an $\mathbb{R}$-vector space is given by the monomials $x^Ay^Bz^C$ with $A \geq 0, B \geq 0, C \geq 0, 3A + 2C \leq 6$, and $4B + 3C \leq 5$. The lattice points $(A, B, C) \in \mathbb{Z}^3$ satisfying these inequalities are shown in Figure 2. When $A+B+C = 2$ the only lattice points satisfying these inequalities are $(2, 0, 0)$, $(1, 1, 0)$, and $(1, 0, 1)$ (see the plot at right in Figure 2). Thus $\dim(R/(J_1 : J(\tau) + J_2 : J(\tau)))_2 = 3$ and $\dim H_1(R_\bullet/J_\bullet)_{r+1+2} = \dim H_1(R_\bullet/J_\bullet)_{11} = 3$.*

Proposition 4.7. *The dimension of $H_1(\mathcal{R}_\bullet/J_\bullet)$ in degree d is given by the number of lattice points $(A, B) \in \mathbb{Z}^2$ satisfying the inequalities $A \geq 0, B \geq 0$, $A - B(s - 1) \leq sr - d(s - 1)$, $B - A(t - 1) \leq tr - d(t - 1)$, and $A + B \leq d - r - 1$.*

Proof. The graded isomorphism (2.2) shows that the dimension of $H_1(\mathcal{R}_\bullet/J_\bullet)$ in degree d is equal to the dimension of $R/(J_1 : J(\tau) + J_2 : J(\tau))$ in degree $d - r - 1$. From Lemma 4.6, this is the number of lattice points $(A, B, C) \in \mathbb{Z}^3_{\geq 0}$ satisfying $A + B + C = d - r - 1$, $A \geq 0$, $B \geq 0$, $C \geq 0$, $sA + (s - 1)C \leq r - s + 1$, and $tB + (t - 1)C \leq r - t + 1$. We get the result by substituting $C = d - r - 1 - A - B$ and simplifying the ensuing inequalities. ∎

If M is a graded module of finite length, recall that the *(Castelnuovo-Mumford) regularity* of M, written $\text{reg}\, M$, is defined by $\text{reg}\, M := \max\{d \mid M_d \neq 0\}$.

Proposition 4.8. *The regularity of* $H_1(\mathcal{R}_\bullet/\mathcal{J}_\bullet)$ *is bounded by*

$$\left\lfloor\frac{r+1}{s}\right\rfloor+\left\lfloor\frac{r+1}{t}\right\rfloor+r-1\leq \text{reg } H_1(\mathcal{R}_\bullet/\mathcal{J}_\bullet)\leq\left\lfloor\frac{r+1}{s}+\frac{r+1}{t}\right\rfloor+r-1. \tag{4.3}$$

More precisely,

$$\text{reg } H_1(\mathcal{R}_\bullet/\mathcal{J}_\bullet)=\begin{cases}\left\lfloor\frac{r+1}{s}\right\rfloor+\left\lfloor\frac{r+1}{t}\right\rfloor+r & \text{if } r+1\equiv s-1 \pmod s \\ & \text{and } r+1\equiv t-1 \pmod t\\ \left\lfloor\frac{r+1}{s}\right\rfloor+\left\lfloor\frac{r+1}{t}\right\rfloor+r-1 & \text{otherwise.}\end{cases} \tag{4.4}$$

Thus $\dim C^r_d(\Delta)=L(\Delta,d,r)$ *for* $d>\frac{r+1}{s}+\frac{r+1}{t}+r-1$*, where* $L(\Delta,d,r)$ *is Schumaker's lower bound [29].*

To prove Proposition 4.8, we use the following lemma:

Lemma 4.9. *Assume* $2\leq s\leq t\leq r+1$*. Let* $\mathcal{P}$ *be the polytope in* $\mathbb{R}^3$ *defined by the inequalities* $A\geq 0, B\geq 0, C\geq 0, r+1-s\geq sA+(s-1)C$*, and* $r+1-t\geq tB+(t-1)C$*. Let* H *be the plane defined by* $A+B+C=\lfloor\frac{r+1}{s}\rfloor+\lfloor\frac{r+1}{t}\rfloor-1$*. Then* $\mathcal{P}\cap H\cap\mathbb{Z}^3\neq\emptyset$ *if and only if* $r+1\equiv s-1 \pmod s$ *and* $r+1\equiv t-1 \pmod t$*. Moreover, if* $r+1\equiv s-1 \pmod s$ *and* $r+1\equiv t-1 \pmod t$ *then*

- *If* $t\geq 3$ *then* $\mathcal{P}\cap H\cap\mathbb{Z}^3=\{(\lfloor\frac{r+1}{s}\rfloor-1,\lfloor\frac{r+1}{t}\rfloor-1,1)\}$
- *If* $s=t=2$ *then* $\mathcal{P}\cap H\cap\mathbb{Z}^3=\{(t,t,r-1-2t)\mid t=0,1,\ldots,r/2-1\}$

Proof. We first treat the case $t\geq 3$ and $s\geq 2$. Assume $P=(A_0,B_0,C_0)\in\mathcal{P}\cap H\cap\mathbb{Z}^3$. If $C_0=0$, then $A_0\leq\lfloor\frac{r+1}{s}\rfloor-1$ and $B_0\leq\lfloor\frac{r+1}{t}\rfloor-1$. Hence, $(A_0,B_0,0)\notin H$, contradiction. Therefore, $C_0\geq 1$.

Next, we show that $C_0\leq 1$. Let $d_0=\lfloor\frac{r+1}{s}\rfloor+\lfloor\frac{r+1}{t}\rfloor-1$. Substituting $A_0=d_0-B_0-C_0$ to $sA_0+(s-1)C_0\leq r+1-s$, we know that $(B_0,C_0)\in\mathbb{Z}^2_{\geq 0}$ must satisfy $sB_0+C_0\geq s+sd_0-(r+1)$ and $tB_0+(t-1)C_0\leq r+1-t$. Eliminating B_0 and simplifying, we obtain

$$\left(1-\frac{1}{s}-\frac{1}{t}\right)C_0\leq\left\{\frac{r+1}{s}\right\}+\left\{\frac{r+1}{t}\right\}-1.$$

where $\{\frac{r+1}{s}\}=\frac{r+1}{s}-\lfloor\frac{r+1}{s}\rfloor$ and $\{\frac{r+1}{t}\}=\frac{r+1}{t}-\lfloor\frac{r+1}{t}\rfloor$. Because $\{\frac{r+1}{s}\}\leq 1-\frac{1}{s}$ and $\{\frac{r+1}{t}\}\leq 1-\frac{1}{t}$, so $\left(1-\frac{1}{s}-\frac{1}{t}\right)C_0\leq 1-\frac{1}{s}-\frac{1}{t}$. Since $s\geq 2$ and $t\geq 3$, this implies $C_0\leq 1$.

Therefore, $C_0=1$, and hence $A_0\leq\lfloor\frac{r+2}{s}\rfloor-2\leq\lfloor\frac{r+1}{s}\rfloor-1$ and $B_0\leq\lfloor\frac{r+2}{t}\rfloor-2\leq\lfloor\frac{r+1}{t}\rfloor-1$. Observe that if $\lfloor\frac{r+2}{s}\rfloor-2<\lfloor\frac{r+1}{s}\rfloor-1$ or $\lfloor\frac{r+2}{t}\rfloor-2<\lfloor\frac{r+1}{t}\rfloor-1$ then $H\cap\mathcal{P}\cap\mathbb{Z}^3=\emptyset$. Therefore if $H\cap\mathcal{P}\cap\mathbb{Z}^3\neq\emptyset$ then $\lfloor\frac{r+2}{s}\rfloor=\lfloor\frac{r+1}{s}\rfloor+1$ and $\lfloor\frac{r+2}{t}\rfloor=\lfloor\frac{r+1}{t}\rfloor+1$ which in turn happens if and only if $r+1\equiv s-1 \pmod s$ and $r+1\equiv t-1 \pmod t$. In case both congruences are satisfied, it is clear from the above reasoning that $(A_0,B_0,C_0)=(\lfloor\frac{r+1}{s}\rfloor-1,\lfloor\frac{r+1}{t}\rfloor-1,1)$ is the only point in $\mathcal{P}\cap H\cap\mathbb{Z}^3$.

Now we treat the case $s=t=2$. First suppose r is odd, so $r=2k-1$ for some integer $k\geq 1$. Then $d_0=2k-1$ and the polytope $\mathcal{P}$ is defined by $A\geq 0$, $B\geq 0$, $C\geq 0$, $2A+C\leq 2k-2$, and $2B+C\leq 2k-2$. From the final two inequalities we deduce that $A+B+C\leq 2k-2$ and thus $H \cap \mathcal{P}$ is empty. Now suppose $r = 2k$ for some integer $k \geq 1$. Then $d_0 = 2k-1$ again, and $\mathcal{P}$ is defined by the inequalities $A \geq 0$, $B \geq 0$, $C \geq 0$, $2A+C \leq 2k-1$, and $2B+C \leq 2k-1$. From $A+B+C = 2k-1$, $2A+C \leq 2k-1$, and $2B+C \leq 2k-1$, we deduce that $A = B$. Since $A \geq 0$ and $B \geq 0$, we deduce that $H \cap \mathcal{P} \cap \mathbb{Z}^3 = \{(t,t,r-1-2t) \mid t = 0,\dots,r/2-1\}$. ■

Now we are ready to prove Proposition 4.8.

Proof of Proposition 4.8. We first prove the bounds (4.3). By Lemma 2.8, the regularity of $H_1(\mathcal{R}_\bullet/\mathcal{J}_\bullet)$ is the largest degree of a monomial in $R/(J_1 : J(\tau) + J_2 : J(\tau))(-r-1)$. By Lemma 4.6, $x^A y^B \in R/(J_1 : J(\tau) + J_2 : J(\tau))$ if $r+1 \geq s(A+1)$ and $r+1 \geq t(B+1)$. In particular, $A = \lfloor \frac{r+1}{s} \rfloor - 1$ and $B = \lfloor \frac{r+1}{t} \rfloor - 1$ satisfy this condition. By Lemma 2.8, we obtain

$$\left\lfloor \frac{r+1}{s} \right\rfloor + \left\lfloor \frac{r+1}{t} \right\rfloor + r - 1 \leq \operatorname{reg} H_1(\mathcal{R}_\bullet/\mathcal{J}_\bullet).$$

Note that the region of $\mathbb{R}^3$ bounded by inequalities in Lemma 4.6 is a polytope, which we denote by $\mathcal{P}$ as in Lemma 4.9. Thus the largest degree of a monomial in $R/(J_1 : J(\tau) + J_2 : J(\tau))$ is obtained by maximizing the linear functional $A+B+C$ over $\mathcal{P} \cap \mathbb{Z}^3$. It is well-known in linear programming that the maximum of this linear functional on $\mathcal{P} \cap \mathbb{R}^3$ occurs at one of the vertices of $\mathcal{P}$. Therefore, to prove the upper bound, it suffices to verify that evaluating $A+B+C$ at the vertices achieves a value of at most $\frac{r+1}{s} + \frac{r+1}{t} - 2$. The vertices of $\mathcal{P}$ are

$$\left(\frac{t-s}{s(t-1)}r, 0, \frac{r}{t-1} - 1\right), \quad \left(\frac{r+1}{s} - 1, \frac{r+1}{t} - 1, 0\right),$$

$$\left(0, 0, \frac{r}{t-1} - 1\right), \quad \left(\frac{r+1}{s} - 1, 0, 0\right), \quad \left(0, \frac{r+1}{t} - 1, 0\right), \quad \text{and} \quad (0,0,0),$$

respectively. Computing $A+B+C$ for each of them, we have

$$\frac{t}{s(t-1)}r - 1, \quad \frac{r+1}{s} + \frac{r+1}{t} - 2, \quad \frac{r}{t-1} - 1, \quad \frac{r+1}{s} - 1, \quad \frac{r+1}{t} - 1, \text{ and } \ 0,$$

respectively. We want to show that $\frac{r+1}{s} + \frac{r+1}{t} - 2$ is the largest among all of them. It is clear that

$$0 \leq \frac{r+1}{t} - 1 \leq \frac{r+1}{s} - 1 \leq \frac{r+1}{s} + \frac{r+1}{t} - 2,$$

and that $\dfrac{r}{t-1} - 1 \leq \frac{t}{s(t-1)}r - 1$. We only need to show that

$$\frac{t}{s(t-1)}r - 1 \leq \frac{r+1}{s} + \frac{r+1}{t} - 2. \tag{4.5}$$

We have

$$\frac{t}{s(t-1)}r - 1 - \left[\frac{r+1}{s} + \frac{r+1}{t} - 2\right] = \left[\frac{1-(t-1)(s-1)}{t(t-1)s}\right] r + 1 - \frac{1}{s} - \frac{1}{t}.$$

Because $t \geq s \geq 2$, so $1-(t-1)(s-1) \leq 0$, where equality holds if and only if $t = s = 2$. If $t = s = 2$, then equality in (4.5) holds. Otherwise, $1-(t-1)(s-1) < 0$ and

$$\left[\frac{1-(t-1)(s-1)}{t(t-1)s}\right] r + 1 - \frac{1}{s} - \frac{1}{t} \leq \left[\frac{1-(t-1)(s-1)}{t(t-1)s}\right](t-1) + 1 - \frac{1}{s} - \frac{1}{t} = 0.$$

Thus, we have proved (4.5). This means $\operatorname{reg} H_1(\mathcal{R}_\bullet/\mathcal{J}_\bullet) \leq \left\lfloor \frac{r+1}{s} + \frac{r+1}{t} \right\rfloor + r - 1$. Therefore, the inequality (4.3) holds.

Now we prove Equation (4.4). If $\operatorname{reg} H_1(\mathcal{R}_\bullet/\mathcal{J}_\bullet) \neq \lfloor \frac{r+1}{s} \rfloor + \lfloor \frac{r+1}{t} \rfloor + r - 1$, then by (4.3), $\operatorname{reg} H_1(\mathcal{R}_\bullet/\mathcal{J}_\bullet) = \lfloor \frac{r+1}{s} \rfloor + \lfloor \frac{r+1}{t} \rfloor + r$. By Lemma 2.8 and Lemma 4.6, this is equivalent to saying that $\mathcal{P} \cap H \cap \mathbb{Z}^3 \neq \emptyset$, where $\mathcal{P}$ and H are as in the setup of Lemma 4.9. Applying Lemma 4.9 completes the proof. ■

Example 4.2. *Assume that* $(s,t) = (3,4)$ *and* $r = 6$. *Then*

$$\begin{aligned}\operatorname{In}(J_1) =& \langle x^7, x^6z, x^5z^2, x^4z^4, x^3z^5, x^2z^7, xz^8, z^{10}\rangle \text{ and}\\ \operatorname{In}(J_2) =& \langle y^7, y^6z, y^5z^2, y^4z^3, y^3z^5, y^2z^6, yz^7, z^9\rangle.\end{aligned}$$

Therefore, $\operatorname{In}(J_1 \colon J(\tau) + J_2 \colon J(\tau)) = \langle x^2, xz, y, z^2\rangle$. *Every monomial of degree two or more is in* $\langle x^2, xz, y, z^2\rangle$, *but* x *is not in this ideal. Therefore* $\operatorname{reg} R/\langle x^2, xz, y, z^2\rangle = 1$ *and thus* $\operatorname{reg} H_1(\mathcal{R}_\bullet/\mathcal{J}_\bullet) = 8$ *by Lemma 2.8. The bounds given by* (4.3) *are* $8 \leq \operatorname{reg} H_1(\mathcal{R}_\bullet/\mathcal{J}_\bullet) \leq 9$. *In this case,* $r+1 \not\equiv s-1 \pmod{s}$, *so by Proposition 4.8,* $\operatorname{reg} H_1(\mathcal{R}_\bullet/\mathcal{J}_\bullet) = 8$, *which aligns with what we have found already. On the other hand, if* $(s,t) = (3,4)$ *and* $r = 10$, *then*

$$\begin{aligned}\operatorname{In}(J_1) =& \langle x^{11}, x^{10}z, x^9z^2, x^8z^4, x^7z^5, \ldots, x^4z^{10}, x^3z^{11}, x^2z^{13}, xz^{14}, z^{16}\rangle \text{ and}\\ \operatorname{In}(J_2) =& \langle y^{11}, y^{10}z, y^9z^2, y^8z^3, y^7z^5, \ldots, y^4z^9, y^3z^{10}, y^2z^{11}, yz^{13}, z^{14}\rangle.\end{aligned}$$

Therefore, $\operatorname{In}(J_1 \colon J(\tau) + J_2 \colon J(\tau)) = \langle x^3, x^2z^2, y^2, yz^2, z^3\rangle$. *We can see by inspection that any monomial of degree five or more is in* $\langle x^3, x^2z^2, y^2, yz^2, z^3\rangle$, *while* x^2yz *is a monomial of degree four not in this ideal. Thus* $\operatorname{reg} R/\operatorname{In}(J_1 \colon J(\tau) + J_2 \colon J(\tau)) = 4$ *and so, by Lemma 2.8,* $\operatorname{reg} H_1(\mathcal{R}_\bullet/\mathcal{J}_\bullet) = 15$. *In this case,* (4.3) *specializes to* $14 \leq \operatorname{reg} H_1(\mathcal{R}_\bullet/\mathcal{J}_\bullet) \leq 15$. *Since* $r+1 \equiv s-1 \pmod{s}$ *and* $r+1 \equiv t-1 \pmod{t}$, *Proposition 4.8 yields* $\operatorname{reg} H_1(\mathcal{R}_\bullet/\mathcal{J}_\bullet) = 15$, *which aligns with what we found by inspection.*

5. Comparison to quasi-cross-cut. In this section we address the phenomenon that, for certain pairs (r,d), $\dim C^r_d(\Delta) = \dim C^r_d(\Delta')$ where Δ' is obtained by removing the unique totally interior edge from Δ to get a quasi-cross-cut partition (see Definition 2.3), as shown in Figure 3. In [32], Sorokina discusses this phenomenon using the Bernstein-Bézier form in the case $s = t = 2$ (which she calls the Tohăneanu partition due to its appearance in [34]). A main result of [32] is that $\dim C^r_d(\Delta) = \dim C^r_d(\Delta')$ for $d \leq 2r$ when $s = t = 2$. In this section, we extend Sorokina's result to arbitrary s and t. This equality of dimensions upon removal of an edge is related to the phenomenon of *supersmoothness* [32, 12], although we will not go into details about this.

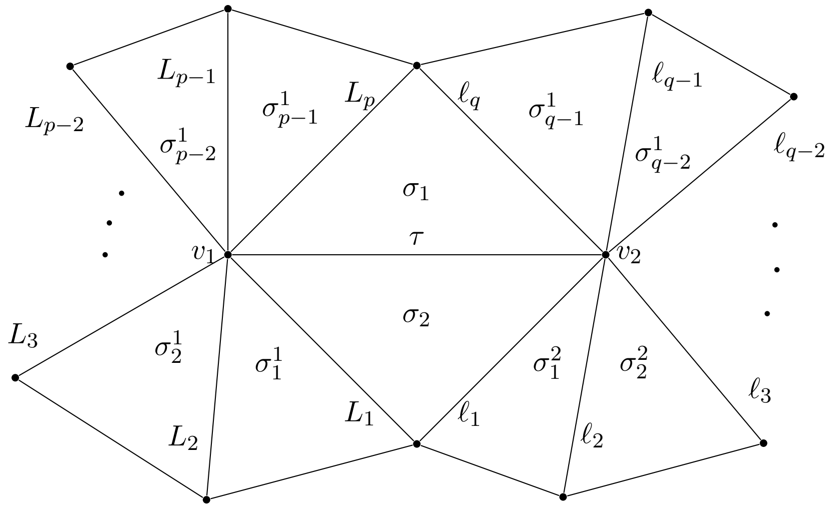

Figure 3. *The triangulation Δ with edges and faces labeled for use in the proof of Lemma 5.1. The quasi-cross-cut partition Δ' is obtained by removing the totally interior edge τ.*

Let Δ be a triangulation with a single totally interior edge τ and let Δ' be the partition formed by removing the edge τ from Δ as in Figure 3. As in Section 4, put $J_i = \sum_{v_i \in \varepsilon, \varepsilon \neq \tau} J(\varepsilon)$ for $i = 1, 2$.

Furthermore, for any homogeneous ideal $I \subseteq R$, define $\operatorname{initdeg} I := \min\{d : I_d \neq 0\}$.

Lemma 5.1. *There is a short exact sequence*

$$0 \to C^r(\Delta') \xrightarrow{\iota} C^r(\Delta) \xrightarrow{\delta} J(\tau) \cap J_1 \cap J_2 \to 0 \tag{5.1}$$

where ι is the natural inclusion and $\delta(F) = F_{\sigma_2} - F_{\sigma_1}$ is the difference of F restricted to the triangles σ_2 and σ_1 shown in Figure 3. In particular, if $d < \operatorname{initdeg} J_1 \cap J_2 \cap J(\tau)$ then $\dim C^r_d(\Delta) = \dim C^r_d(\Delta')$.

Proof. We prove that (5.1) is a short exact sequence; the final statement follows immediately. Let τ be the totally interior edge of Δ, with corresponding linear form L_τ. Let $L_1, \ldots, L_p$ be the linear forms defining the edges which surround the interior vertex v_1, in *clockwise* order. Likewise suppose that the linear forms defining the edges which surround the interior vertex v_2 in *counterclockwise* order are $\ell_1, \ldots, \ell_q$. See Figure 3, where the edges are labeled by the corresponding linear forms. With this convention, $J(\tau) = \langle L_\tau^{r+1} \rangle$, $J_1 = \langle L_1^{r+1}, \ldots, L_p^{r+1} \rangle$, and $J_2 = \langle \ell_1^{r+1}, \ldots, \ell_q^{r+1} \rangle$.

It is clear that $C^r(\Delta')$ is the kernel of the map δ. It follows from the algebraic spline criterion that if $F \in C^r(\Delta)$ then $\delta(F) \in J(\tau) \cap J_1 \cap J_2$. We show that δ is surjective. Suppose that $f \in J(\tau) \cap J_1 \cap J_2$. We define a spline $F \in C^r(\Delta)$ so that $\delta(F) = f$ as follows. Let $F_{\sigma_1} = 0$ and $F_{\sigma_2} = f$. Write $\sigma^1_1, \ldots, \sigma^1_{p-1}$ for the remaining triangles surrounding the vertex v_1 (in clockwise order) and $\sigma^2_1, \ldots, \sigma^2_{q-1}$ for the remaining triangles surrounding the vertex v_2 (in counterclockwise order). See Figure 3.

Then the linear forms defining the interior edges adjacent to σ^1_i are L_i and L_{i+1} for $i = 1, \ldots, p-1$ and the linear forms defining the interior edges adjacent to σ^2_j are ℓ_j and ℓ_{j+1}

for $j = 1, \dots, q-1$.

Now we continue to define F. Since $f \in J_1, f = \sum_{i=1}^{p} g_i L_i^{r+1}$ for some polynomials $g_1, \dots, g_p$. Define $F_{\sigma_i^1}$ by $f - \sum_{j=1}^{i} g_j L_j^{r+1}$ for $i = 1, \dots, p-1$. Likewise, since $f \in J_2$, $f = \sum_{i=1}^{q} h_i \ell_i^{r+1}$ for some polynomials $h_1, \dots, h_q$. Define $F_{\sigma_i^2}$ by $f - \sum_{j=1}^{i} h_j \ell_j^{r+1}$ for $i = 1, \dots, q-1$. One readily checks, using Proposition 2.1, that $F \in C^r(\Delta)$. Clearly $\delta(F) = f$, so we are done. ■

Since $J(\tau)$ is principal, $J_1 \cap J_2 \cap J(\tau) = [(J_1 \cap J_2) : J(\tau)]J(\tau)$ and $(J_1 \cap J_2) : J(\tau) = (J_1 : J(\tau)) \cap (J_2 : J(\tau))$. Hence, the $\operatorname{initdeg}(J_1 \cap J_2 \cap J(\tau)) = (r+1) + \operatorname{initdeg}(J_1 : J(\tau) \cap J_2 : J(\tau))$.

Lemma 5.2. *For $i = 1, 2$, let $J_i' = J_i : J(\tau)$. We have $\operatorname{In}(J_1' \cap J_2') = \operatorname{In}(J_1') \cap \operatorname{In}(J_2')$. Moreover, the monomial $x^A y^B z^C \in \operatorname{In}(J_1') \cap \operatorname{In}(J_2')$ if and only if $A, B, C \geq 0$, $r+1-s < sA + (s-1)C$, and $r+1-t < tB + (t-1)C$.*

Proof. The second part follows immediately from Corollary 3.7. So we only verify the first part. It is clear that $\operatorname{In}(J_1' \cap J_2') \subseteq \operatorname{In}J_1' \cap \operatorname{In}J_2'$. We only need to show that $\dim \operatorname{In}(J_1' \cap J_2')_d = \dim(\operatorname{In}J_1' \cap \operatorname{In}J_2')_d$ for all degrees $d \geq 0$. By Corollary 4.5, we know that $\dim \operatorname{In}(J_1' + J_2')_d = \dim(\operatorname{In}J_1' + \operatorname{In}J_2')_d$. We also know that $\dim(\operatorname{In}I)_d = \dim I_d$ for any ideal I. Because

$$\dim \operatorname{In}(J_1' \cap J_2')_d = \dim(J_1')_d + \dim(J_2')_d - \dim(J_1' + J_2')_d \text{ and}$$
$$\dim(\operatorname{In}J_1' \cap \operatorname{In}J_2')_d = \dim(J_1')_d + \dim(J_2')_d - \dim(\operatorname{In}J_1' + \operatorname{In}J_2')_d,$$

we must have $\dim \operatorname{In}(J_1' \cap J_2')_d = \dim(\operatorname{In}J_1' \cap \operatorname{In}J_2')_d$ for all degree $d \geq 0$. ■

Corollary 5.3. *Let Δ be a triangulation with a single totally interior edge satisfying Assumptions 2.1. For $d \leq \frac{t}{s(t-1)}r + r$, $\dim C_d^r(\Delta) = \dim C_d^r(\Delta')$.*

Proof. As in Lemma 5.2, we let $J_i' = J_i : J(\tau)$ for $i = 1, 2$. By Lemma 5.1, it suffices to prove that $\frac{t}{s(t-1)}r + r < \operatorname{initdeg}(J_1 \cap J_2 \cap J(\tau))$. From the discussion just prior to Lemma 5.2 coupled with the lemma itself, it suffices to prove that $\frac{t}{s(t-1)}r - 1 < \operatorname{initdeg}(\operatorname{In}(J_1') \cap \operatorname{In}(J_2'))$.

Let $\mathcal{Q}$ be the collection of points $(A, B, C) \in \mathbb{R}^3$ defined by the inequalities $A, B, C \geq 0$, $r-s+1 < sA+(s-1)C$ and $r-t+1 < tB+(t-1)C$. Then its closure $\overline{\mathcal{Q}}$ (in the usual topology on $\mathbb{R}^3$) is the polyhedron in $\mathbb{R}^3$ defined by the inequalities $A, B, C \geq 0$, $r-s+1 \leq sA+(s-1)C$ and $r-t+1 \leq tB+(t-1)C$. Using Lemma 5.2 again,

$$\operatorname{initdeg}(\operatorname{In}(J_1') \cap \operatorname{In}(J_2')) = \min\{A + B + C : (A, B, C) \in \mathcal{Q} \cap \mathbb{Z}^3\}.$$

We first show that $\frac{t}{s(t-1)}r - 1$ is the smallest value achieved by $A+B+C$ on the polyhedron $\overline{\mathcal{Q}}$. Since we assume that $s \leq t \leq r+1$, it is not possible for any $(A, B, C) \in \mathcal{Q}$ to satisfy $A = C = 0$ or $B = C = 0$. The vertices of the polyhedron $\overline{\mathcal{Q}}$ are

$$Q_1 = \left(\frac{t-s}{s(t-1)}r,\ 0,\ \frac{r}{t-1} - 1\right), \quad Q_2 = \left(\frac{r+1}{s} - 1,\ \frac{r+1}{t} - 1,\ 0\right), \quad \text{and}$$
$$Q_3 = \left(0,\ 0,\ \frac{r}{s-1} - 1\right).$$

We have proved (4.5), which implies that $A+B+C$ evaluated at Q_2 is at least as large as $A+B+C$ evaluated at Q_1. Since $\frac{t}{s(t-1)}r - 1 \leq \frac{r}{s-1} - 1$, $A+B+C$ evaluated at Q_1 is at most $A+B+C$ evaluated at Q_3. Therefore, over the real numbers, $A+B+C$ is minimized over $\overline{\mathcal{Q}}$ at the vertex Q_1, with a value of $\frac{t}{s(t-1)}r-1$. Let H' be the affine hyperplane defined by $A+B+C=\frac{t}{s(t-1)}r-1$. A straightforward calculation with the inequalities also shows that $H'\cap\overline{\mathcal{Q}}=Q_1$, hence $H'\cap\mathcal{Q}=\emptyset$ and also $H'\cap\mathcal{Q}\cap\mathbb{Z}^3=\emptyset$. It follows that

$$\operatorname{initdeg}(\operatorname{In}(J_1')\cap\operatorname{In}(J_2'))=\min\{A+B+C:(A,B,C)\in\mathcal{Q}\cap\mathbb{Z}^3\}>\frac{t}{s(t-1)}r-1. \qquad \blacksquare$$

6. The explicit dimension formula.

In this section we use the preceding sections to give an explicit formula for $\dim C^r_d(\Delta)$, where Δ is a planar triangulation with a single totally interior edge, for any $d\geq 0$ and $r\geq 0$. We then illustrate the formula in a few examples.

Theorem 6.1. *Let Δ be a triangulation with a single totally interior edge τ satisfying Assumptions 2.1 and Δ' the partition formed by removing τ. Then*

$$\dim C^r_d(\Delta)=\begin{cases} L(\Delta',d,r) & d\leq \frac{t}{s(t-1)}r+r\\ L(\Delta,d,r)+f(\Delta,d,r) & \frac{t}{s(t-1)}r+r<d\leq\frac{r+1}{s}+\frac{r+1}{t}+r-1\\ L(\Delta,d,r) & d>\frac{r+1}{s}+\frac{r+1}{t}+r-1,\end{cases}$$

where

$$f(\Delta,d,r):=\sum_{i=\left\lceil\frac{2st(d-r)-(s+t)d}{(s-1)(t-1)-1}\right\rceil}^{d-r-1}\left(\left\lfloor\frac{(i-d)(s-1)}{s}+r\right\rfloor-\left\lceil\frac{i+d(t-1)}{t}-r\right\rceil+1\right).$$

Moreover, put $\mathfrak{r}=\lfloor(r+1)/s\rfloor+\lfloor(r+1)/t\rfloor+r$. If $t\geq 3$, $r+1\equiv s-1\,(\mathrm{mod}\,s)$, and $r+1\equiv t-1\,(\mathrm{mod}\,t)$, then $f(\Delta,\mathfrak{r},r)=1$. Otherwise $f(\Delta,\mathfrak{r},r)=0$.

Proof. First, it follows from $t\leq r+1$ that $\frac{t}{s(t-1)}\leq\frac{r+1}{s}+\frac{r+1}{t}+r-1$. Now, if $d\leq\frac{t}{s(t-1)}r+r$ then $\dim C^r_d(\Delta)=\dim C^r_d(\Delta')$ by Corollary 5.3. Since Δ' is a quasi-cross-cut partition, it follows from Proposition 2.6 that $\dim C^r_d(\Delta')=L(\Delta',d,r)$ for all $d\geq 0$.

Likewise, if $d>\frac{r+1}{s}+\frac{r+1}{t}+r-1$ then $\dim C^r_d(\Delta)=L(\Delta,d,r)$ by Theorem 4.2. Observe that these first two cases allow us to dispense of the case $s=t=2$ (which we consider in more detail in Example 6.1). So henceforth we assume $t\geq 3$.

According to Theorem 4.1, it remains to show that, when $\frac{t}{s(t-1)}r+r<d\leq\frac{r+1}{s}+\frac{r+1}{t}+r-1$, $f(\Delta,d,r)=\#(\mathcal{P}_d\cap\mathbb{Z}^2)$, where $\mathcal{P}_d$ is the polytope defined by the inequalities $A\geq 0$, $B\geq 0$, $A\leq B(s-1)-d(s-1)+sr$, $B\leq A(t-1)-d(t-1)+tr$, and $A+B\leq d-r-1$.

We first show that, in the given range for d, $\mathcal{P}_d$ is a triangle bounded by $A\leq B(s-1)-d(s-1)+sr$, $B\leq A(t-1)-d(t-1)+tr$, and $A+B\leq d-r-1$. For this observe that

$$B\leq A(t-1)-d(t-1)+tr\leq B(s-1)(t-1)+sr(t-1)-d(s-1)(t-1)+tr-d(t-1)$$

from which we deduce that $ds(t-1)-sr(t-1)-tr\leq B[(s-1)(t-1)-1]$. Since $t\geq 3$, we need only show that $0\leq ds(t-1)-sr(t-1)-tr$. Re-arranging, we see this is equivalent to

$$\frac{tr}{s(t-1)}+r\leq d,$$

which is precisely our assumption. So $B \geq 0$ is a consequence of $A \leq B(s-1) - d(s-1) + sr$ and $B \leq A(t-1) - d(t-1) + tr$. Since $B \geq 0$, we obtain

$$0 \leq A(t-1) - d(t-1) + tr$$

or $d(t-1) - tr \leq A(t-1)$. Using the given bound on d, we obtain $r(t/s) - r \leq d(t-1) - tr$. Since $s \leq t$, we thus have $0 \leq A(t-1)$ and so $0 \leq A$.

It follows that $\mathcal{P}_d$ is the triangle in the first quadrant bounded by $A \leq B(s-1) - d(s-1) + sr$, $B \leq A(t-1) - d(t-1) + tr$, and $A + B \leq d - r - 1$. Now we count the lattice points $(A, B) \in \mathbb{Z}^2 \cap \mathcal{P}_d$. We do this by counting the lattice points on the line segments defined by the intersection of $A + B = i$ with $\mathcal{P}_d$, for $0 \leq i \leq d - r - 1$. The two lines defined by the equations $A = B(s-1) - d(s-1) + sr$ and $B = A(t-1) - d(t-1) + tr$ intersect at the point

$$\left(\frac{t(s-1)(d-r) - sr}{(s-1)(t-1) - 1}, \frac{s(t-1)(d-r) - tr}{(s-1)(t-1) - 1} \right),$$

where $A + B$ (restricted to $\mathcal{P}_d$) achieves its minimum value of

$$\frac{2st(d-r) - (s+t)d}{(s-1)(t-1) - 1}.$$

Thus we start our count at $i = \lceil (2st(d-r) - (s+t)d)/((s-1)(t-1) - 1) \rceil$, which is the lower index of summation for the definition of $f(\Delta, d, r)$ in the theorem statement. Clearly the maximum is $i = d - r - 1$. Now put $A + B = i$, so $B = i - A$. We have

$$A \leq B(s-1) + sr - d(s-1) = (i - A)(s-1) + sr - d(s-1)$$

yielding $sA \leq (i-d)(s-1) + sr$ or $A \leq (i-d)(s-1)/s + r$. Likewise we have

$$i - A = B \leq A(t-1) + tr - d(t-1)$$

which yields $i - tr + d(t-1) \leq tA$ or $(i + d(t-1))/t - r \leq A$. Putting these together, the number of lattice points $(A, B) \in \mathcal{P}_d \cap \mathbb{Z}^2$ with $A + B = i$ is the same as the number of integers $A \in \mathbb{Z}$ in the interval

$$(i + d(t-1))/t - r \leq A \leq (i-d)(s-1)/s + r,$$

which is counted by

$$\left\lfloor \frac{(i-d)(s-1)}{s} + r \right\rfloor - \left\lceil \frac{i + d(t-1)}{t} - r \right\rceil + 1.$$

Summing this over the appropriate range for i yields the expression for $f(\Delta, d, r)$.

Now put $\mathfrak{r} = \lfloor (r+1)/s \rfloor + \lfloor (r+1)/t \rfloor + r$. If $t \geq 3$, $r + 1 \equiv s - 1 \pmod{s}$, and $r + 1 \equiv t - 1 \pmod{t}$ then $f(\Delta, \mathfrak{r}, r) = 1$ by Lemma 4.9. Otherwise $f(\Delta, \mathfrak{r}, r) = 0$, also by Lemma 4.9. ■

In the following examples we compute explicit formulas for certain triangulations with a single totally interior edge. We assume that the triangulation Δ satisfies Assumptions 2.1 and we introduce some additional notation to explicitly write out Schumaker's lower bound. Let α_1 and ν_1 (respectively α_2 and ν_2) be the quotient and remainder when $(s+1)(r+1)$ is divided by s (respectively $(t+1)(r+1)$ is divided by t). That is, $(s+1)(r+1)=\alpha_1 s+\nu_1$ and $(t+1)(r+1)=\alpha_2 t+\nu_2$, where $0\le \nu_1 < s$ and $0\le\nu_2<t$. Furthermore, put $\mu_1=s-\nu_1$ and $\mu_2=t-\nu_2$. From Proposition 2.4 we have

$$\begin{aligned} L(\Delta,d,r) :=& \binom{d+2}{2}+(p-s+q-t-1)\binom{d+1-r}{2} \\ &+\sum_{i=1,2}\mu_i\binom{d+2-\alpha_i}{2}+\nu_i\binom{d+1-\alpha_i}{2}. \end{aligned} \tag{6.1}$$

Now let α'_1 and ν'_1 (respectively α'_2 and ν'_2) be the quotient and remainder when $s(r+1)$ is divided by $s-1$ (respectively $t(r+1)$ is divided by $t-1$). That is, $s(r+1)=\alpha'_1(s-1)+\nu'_1$ and $t(r+1)=\alpha'_2(t-1)+\nu'_2$, where $0\le\nu'_1<s-1$ and $0\le\nu'_2<t-1$. Furthermore, put $\mu'_1=s-1-\nu'_1$ and $\mu'_2=t-1-\nu'_2$. Again from Proposition 2.4 we have

$$\begin{aligned} L(\Delta',d,r) :=& \binom{d+2}{2}+(p-s+q-t)\binom{d+1-r}{2} \\ &+\sum_{i=1,2}\mu'_i\binom{d+2-\alpha'_i}{2}+\nu'_i\binom{d+1-\alpha'_i}{2}. \end{aligned} \tag{6.2}$$

Schumaker's upper bound $U(\Delta,d,r)$ – see Proposition 2.5 – depends on an ordering of the two interior vertices of Δ. The optimal upper bound is obtained by ordering the vertex with the larger number of slopes first, and this amounts to replacing t by $t-1$ in (6.1) (this also affects μ_i,α_i, and ν_i since they are defined using t).

Example 6.1. *Consider the triangulation with $p=q=4$ and $s=t=2$. This triangulation is studied in [32], [34], and [21]. According to Theorem 6.1, we have*

$$\dim C^r_d(\Delta)=\begin{cases} L(\Delta',d,r) & d\le 2r\\ L(\Delta,d,r) & d\ge 2r+1.\end{cases}$$

The first case (for $d\le 2r$) recovers [32, Theorem 3.1]. The second case (for $d\ge 2r+1$) recovers the main result of [21]. From Equation (6.2) we have

$$L(\Delta',d,r)=\binom{d+2}{2}+4\binom{d+1-r}{2}+2\binom{d-2r}{2}.$$

Since the final term of $L(\Delta',d,r)$ vanishes for $d\le 2r$, we have $\dim C^r_d(\Delta)=\binom{d+2}{2}+4\binom{d+1-r}{2}$ for $d\le 2r$, recovering [32, Theorem 3.2]. Furthermore, $\dim C^r_{2r}(\Delta)>L(\Delta,2r,r)$ by Theorem 4.2, recovering the main result of [34].

We observe that Schumaker's upper bound $U(\Delta,d,r)$ satisfies $U(\Delta,d,r)-L(\Delta',d,r)=\binom{d+2-\lceil(3(r+1))/2\rceil}{2}+\binom{d+2-\lfloor(3(r+1))/2\rfloor}{2}-\binom{d-2r}{2}$. Thus if $d<\lfloor 3(r+1)/2\rfloor$ we see that $\dim C^r_d(\Delta)=U(\Delta,d,r)$ while if $d\ge\lfloor 3(r+1)/2\rfloor$ we have $\dim C^r_d(\Delta)<U(\Delta,d,r)$.

Example 6.2. *Consider the triangulation* Δ *shown in Figure 1, with* $p=6, s=3, q=5$, *and* $t=4$. *If* $r\le 5$ *then* $C^r(\Delta)$ *is free and* $\dim C^r_d(\Delta)=L(\Delta,d,r)$ *for all integers* $d\ge 0$ *by Proposition 2.7. For* $r\ge 6$, *according to Theorem 6.1,* $\dim C^r_d(\Delta)=L(\Delta',d,r)$ *for* $d\le 13r/9$ *and* $\dim C^r_d(\Delta)=L(\Delta,d,r)$ *for* $d>(19r-5)/12$. *For* $13r/9<d\le(19r-5)/12$,

$$\begin{aligned}\dim C^r_d(\Delta) =& L(\Delta,d,r)+f(\Delta,d,r)\\ =& L(\Delta,d,r)+\sum_{i=\lceil(17d-24r)/5\rceil}^{d-r-1}\left(\lfloor 2/3(i-d)+r\rfloor-\lceil (i+3d)/4-r\rceil+1\right).\end{aligned}$$

We now use Equations (6.1) and (6.2) to compute dimension formulas for $r=6,7,8$. *When* $r=6$, *we have*

$$\dim C^6_d(\Delta)=\begin{cases} L(\Delta',d,6)=\binom{d+2}{2}+4\binom{d-5}{2}+2\binom{d-7}{2}+2\binom{d-8}{2}+\binom{d-9}{2} & d\le 8\\ L(\Delta,d,6)=\binom{d+2}{2}+3\binom{d-5}{2}+\binom{d-6}{2}+5\binom{d-7}{2}+\binom{d-8}{2} & d\ge 10\end{cases}.$$

When $d=9$, $f(\Delta,d,r)=\sum_{i=2}^{2}\left(\lfloor 2/3(i-9)+6\rfloor-\lceil(i+3\cdot 9)/4-6\rceil+1\right)=0$, *which simply means that the triangle defined by the inequalities in Proposition 4.7 does not contain any lattice points. Thus* $\dim C^6_9(\Delta)=L(\Delta,9,6)$. *This also is expected by Theorem 6.1 since* $\mathfrak{r}=\lfloor 7/3\rfloor+\lfloor 7/4\rfloor+6=9$ *and* $r+1=7\not\equiv 2\pmod 3$.

Observing that the last three terms of $L(\Delta',d,r)$ *vanish when* $d\le 8$, *we conclude that*

$$\dim C^6_d(\Delta)=\begin{cases}\binom{d+2}{2}+4\binom{d-5}{2} & d\le 8\\ \binom{d+2}{2}+3\binom{d-5}{2}+\binom{d-6}{2}+5\binom{d-7}{2}+\binom{d-8}{2} & d\ge 9\end{cases}.$$

We also have $U(\Delta,d,6)=\binom{d+2}{2}+4\binom{d-5}{2}+4\binom{d-7}{2}+2\binom{d-8}{2}$, *so Schumaker's upper bound coincides with* $L(\Delta',d,6)$ *for* $d\le 8$ *and gives the dimension of the spline space. However* $U(\Delta,9,6)=28=\dim C^6_9(\Delta)+2$ *and* $U(\Delta,d,6)>\dim C^6_d(\Delta)$ *for* $d\ge 9$.

When $r=7$, *there is no integer* d *so that* $91/9=13r/9<d\le(19r-5)/12=128/12$, *so we simply have*

$$\dim C^7_d(\Delta)=\begin{cases} L(\Delta',d,7)=\binom{d+2}{2}+4\binom{d-6}{2}+\binom{d-8}{2}+2\binom{d-9}{2}+2\binom{d-10}{2} & d\le 10\\ L(\Delta,d,7)=\binom{d+2}{2}+3\binom{d-6}{2}+5\binom{d-8}{2}+2\binom{d-9}{2} & d\ge 11\end{cases}$$

We also have $U(\Delta,d,7)=\binom{d+2}{2}+4\binom{d-6}{2}+2\binom{d-8}{2}+4\binom{d-9}{2}$, *which gives the dimension of the spline space for* $d\le 9$ *but exceeds it for* $d\ge 10$.

When $r=8$, *we hit our first non-zero contribution from* $f(\Delta,d,r)$. *Namely, when* $d=12$, $f(\Delta,12,8)=1$ *(this comes from the single lattice point pictured on the right in Figure 2). Notice that* $\mathfrak{r}=\lfloor 9/3\rfloor+\lfloor 9/4\rfloor+8=13$, *so we must compute* $f(\Delta,12,8)$ *directly. Thus*

$$\dim C^8_d(\Delta)=\begin{cases} L(\Delta',d,r)=\binom{d+2}{2}+4\binom{d-7}{2}+3\binom{d-10}{2}+\binom{d-11}{2}+\binom{d-12}{2} & d\le 11\\ L(\Delta,12,r)+f(\Delta,12,r)=134+1=135 & d=12\\ L(\Delta,d,r)=\binom{d+2}{2}+3\binom{d-7}{2}+3\binom{d-9}{2}+4\binom{d-10}{2} & d\ge 13\end{cases}$$

We also have $U(\Delta,d,8)=\binom{d+2}{2}+4\binom{d-7}{2}+6\binom{d-10}{2}$, *which gives the dimension of the spline space for* $d\le 11$ *but exceeds it for* $d\ge 12$.

Remark 4. *The smallest values of* s, t, r, *and* d *where we see a non-zero contribution from* $f(\Delta, d, r)$ *in Theorem 6.1 are* $s = 2, t = 3, r = 5$, *and* $d = 9$, *where* $f(\Delta, 9, 5) = 1$.

7. Concluding remarks and open problems. We close with a number of remarks on connections to the literature and open problems.

Remark 5. *It should be possible to use our techniques to analyze both additional 'super-smoothness' across the totally interior edge (as Sorokina does in [32]) and varying smoothness conditions across each edge (see also Remark 8).*

Remark 6. *We can apply the methods of this paper to determine* $\dim C^r_d(\Delta)$ *whenever the only non-trivial generators of* $H_1(\mathcal{R}_\bullet/\mathcal{J}_\bullet)$ *correspond to totally interior edges which do not meet each other. In this case the dimension of* $H_1(\mathcal{R}_\bullet/\mathcal{J}_\bullet)_d$ *would be obtained by simply adding together the contributions from the different totally interior edges.*

The recent counterexample to Schenck's '$2r + 1$' conjecture in [37, 26] is a triangulation with two totally interior edges which meet at a vertex. Thus, contrary to our result for triangulations with a single totally interior edge in Corollary 4.3, we might not have $\dim C^r_d(\Delta) = L(\Delta, d, r)$ for $d \geq 2r + 1$ when Δ is a triangulation with two totally interior edges meeting at a vertex.

Remark 7. *The well-known Morgan-Scott split* Δ_{MS}, *for which* $\dim C^r_d(\Delta_{MS})$ *depends on the global geometry of* Δ_{MS}, *has three totally interior edges which form a triangle. In [7], Diener proves that all the spline spaces* $C^r_{2r}(\Delta_{MS})$ *for* $r \geq 1$ *have the same instability coming from global geometry (see also [8] where Diener considers the instability of a wider class of rectilinear partitions). In a remarkable preprint [35], Whiteley shows that the process of vertex splitting applied to the Morgan-Scott split leads to infinitely many triangulations for which the dimension of* C^1 *quadratic splines depends on global geometry. Vertex splitting results in a triangulation with additional triangles all of whose edges are totally interior edges. For each* $r \geq 2$, *there is a variation* Δ^r_{MS} *of the Morgan-Scott split so that* $C^r_{r+1}(\Delta^r_{MS})$ *exhibits dependence on global geometry [17]. Each of these has three totally interior edges forming a triangle as well. Given that Theorem 6.1 implies that a triangulation with a single totally interior edge depends only on local geometry, we pose Problem 1.*

Problem 1. *If no triangle of* Δ *is surrounded by totally interior edges – equivalently, the dual graph has no interior vertex – does the dimension of* $C^r_d(\Delta)$ *depend only on local geometry (that is, the number of slopes meeting at each interior vertex)?*

Remark 8. *If* Δ *is a rectilinear partition, a mixed spline space on* Δ, *written* $C^\alpha(\Delta)$, *is one where different orders of smoothness are imposed across different edges according to a function* $\alpha : \Delta^\circ_1 \to \mathbb{Z}_{\geq 0}$. *Generally speaking, decreasing the order of smoothness across certain edges of a partition enriches the resulting spline space, while increasing the smoothness coarsens the spline space.*

In [9] it is shown that the (Castelnuovo-Mumford) regularity of the mixed spline space $C^\alpha(\Delta)$ *on a rectilinear partition* Δ *can be bounded by the maximum regularity of the space of mixed splines on the union of two adjacent polygonal cells of* Δ *–that is, the star of an edge –where vanishing is imposed (to the order prescribed by* α*) across all edges which the polygonal cells do not have in common. It may be possible that the methods of this paper can be used* *to improve the regularity bounds derived in [9] for mixed splines on the star of an edge with vanishing imposed across the boundary. Improving the regularity bound for splines on the star of an edge with vanishing across the boundary will give a better bound on the degree d needed for the formula $\dim C^{\alpha}_d(\Delta)$ to stabilize.*

Remark 9. *A generalized quasi-cross-cut partition (see [19]) is defined as follows. We call a sequence of adjacent edges of Δ a cross-cut if they all have the same slope and both endpoints of the sequence touch the boundary of Δ. We call a sequence of adjacent edges of Δ a quasi-cross-cut if all edges have the same slope, one endpoint of the sequence touches the boundary, and the other endpoint cannot be extended to include another adjacent edge of the same slope. It is possible that a cross-cut or quasi-cross-cut consists of only a single edge – for instance, any edge which is not totally interior is either a quasi-cross-cut or it can be extended to a quasi-cross-cut. For a vertex $\gamma \in \Delta^{\circ}_0$, we define C_γ to be the number of cross-cuts passing through γ and F_γ to be the number of quasi-cross-cuts passing through γ. The rectilinear partition Δ is a generalized quasi-cross-cut partition if $C_\gamma + F_\gamma \geq 2$ for every $\gamma \in \Delta^{\circ}_0$. Generalized quasi-cross-cut partitions are studied by Manni in [19] and Shi, Wang, and Yin in [31].*

If Δ has a single totally interior edge, it is clearly a generalized quasi-cross-cut partition. If Δ has a single totally interior edge connecting vertices v_1 and v_2 with $s+1$ different slopes meeting at v_1 and $t+1$ different slopes meeting at v_2, it follows from [19, Theorem 2.2] that $\dim C^r_d(\Delta) = L(\Delta, d, r)$ for $d \geq r + 1 + 2\lceil (r+1)/(s-1)\rceil$ and from [31, Theorem 5] that $\dim C^r_d(\Delta) = L(\Delta, d, r)$ for $d \geq r + \lfloor r/(s-1)\rfloor + \lfloor r/(t-1)\rfloor$. Theorem 4.2 shows an improvement on both of these bounds. This leads us to pose Problem 2, inspired by the result of Shi, Wang, and Yin and our Theorem 4.2.

Problem 2. *If Δ is a generalized quasi-cross-cut partition, define for each edge $\tau = \{u, v\} \in \Delta_1$ the quantity $\xi_\tau = (r+1)/(C_u + N_u) + (r+1)/(C_v + N_v)$. Let $\xi_\Delta = \max\{\xi_\tau : \tau \in \Delta^{\circ}_1\}$. Is it true that $\dim C^r_d(\Delta) = L(\Delta, d, r)$ for $d > \xi_\Delta + r - 1$?*

If Problem 2 has a positive answer, it would imply that all generalized quasi-cross-cut partitions satisfy Schenck's '$2r+1$' conjecture (and the conjecture of Alfeld and Manni for $\dim C^1_3(\Delta)$). Notice that the only known counterexample to Schenck's conjecture in [37, 26] is not a generalized quasi-cross-cut partition since the central vertex has only a single cross-cut passing through it, and no quasi-cross-cuts.

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