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Derived equivalences over base schemes and support of complexes

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Let X and Y be smooth projective varieties over a field k admitting morphisms $f : X \to T$ and $g : Y \to T$ to a third variety T. We formulate conditions on a derived equivalence $\Phi : D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(Y)$ ensuring that Φ is induced by a complex P in $D^b_{\mathrm{coh}}(X \times_T Y)$, defining derived equivalences between the fibers of f and g. We apply our results to the canonical fibration and Albanese fibration.

1. Introduction

For a noetherian scheme X let $D^b_{\mathrm{coh}}(X)$ denote the derived category of complexes of $\mathcal{O}_X$-modules with bounded and coherent cohomology sheaves.

Let X and Y be varieties over a field k, admitting an equivalence

$$\Phi_P : D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(Y)$$

given by a complex $P \in D^b_{\mathrm{coh}}(X \times Y)$. The equivalence Φ_P preserves many natural structures: the canonical ring, the automorphism scheme of the identity functor, etc. This suggests that there are often natural morphisms

$f : X \to T$ and $g : Y \to T$ with a common target that should be somehow respected by Φ_P. The simplest way to "respect T" would be for P to be in the image of the natural pushforward map $D^b_{\mathrm{coh}}(X \times_T Y) \to D^b_{\mathrm{coh}}(X \times Y)$, and it is natural to ask for conditions ensuring this holds. It is that question that we study here.

The typical way in which one would use such a thing is as the natural generalization of the seminal result of Bondal and Orlov [4]: two derived equivalent varieties X and Y are relatively equivalent over isomorphisms of their canonical and anti-canonical models (see §3). This particular idea also appeared in a slighlty different form in [16].

In general, it appears to be a difficult question to decide when a complex that is set-theoretically supported on a closed subscheme $X_0 \subset X$ is the pushforward of a complex from X_0. Analogous questions about affine morphisms $X' \to X$ have an interesting history. As shown in [13, Theorem 8.1], even for base change by field extensions, it is unusual for a complex with an X' structure to be pushed forward from X'. For kernels of derived equivalences, however, we have additional tools at our disposal. First, on the level of ∞-categories a very satisfactory solution was given by Ben-Zvi, Francis, and Nadler [1, 4.7]. Second, to get results on the level of derived categories, rather than ∞-categories, we use variations of the gluing results of Beilinson, Bernstein, and Deligne developed in [10].

Our main technical results are not restricted to derived equivalences and we state them in more general form. Let S be a scheme (in practice this will often be the spectrum of a field) and let

$$f : X \to T, \quad g : Y \to T$$

be morphisms of S-schemes with T/S separated and flat. Let

$$d_0, d_1 : X \times_S Y \to X \times_S T \times_S Y$$

be the morphisms given on scheme-valued points

$$d_0(x, y) = (x, f(x), y), \quad d_1(x, y) = (x, g(y), y).$$

We also consider the maps

$$d^{(1)}_i : X \times T \times Y \to X \times T^2 \times Y, \ i = 0, 1, 2,$$

given on scheme-valued points by

$$d^{(1)}_0 : (x, t, y) \mapsto (x, f(x), t, y),$$

$$d_1^{(1)} : (x,t,y) \mapsto (x,t,t,y),$$
$$d_2^{(1)} : (x,t,y) \mapsto (x,t,g(y),y).$$

Observe that $d_0^{(1)} \circ d_0 = d_1^{(1)} \circ d_0$, $d_1^{(1)} \circ d_1 = d_2^{(1)} \circ d_1$, and $d_0^{(1)} \circ d_1 = d_2^{(1)} \circ d_0$.

We consider a pair (P,φ) where $P \in D_{\mathrm{qcoh}}(X \times_S Y)$ is an object of the derived category of complexes with quasi-coherent cohomology sheaves on $X \times_S Y$ and

$$\varphi : d_{0*}P \to d_{1*}P$$

is an isomorphism in $D_{\mathrm{qcoh}}(X \times_S T \times_S Y)$ such that the diagram

$$\begin{array}{ccccc} d_{0*}^{(1)} d_{0*}P & \xrightarrow{d_{0*}^{(1)}\varphi} & d_{0*}^{(1)} d_{1*}P \simeq d_{2*}^{(1)} d_{0*}P & \xrightarrow{d_{2*}^{(1)}\varphi} & d_{2*}^{(1)} d_{1*}P \\ \downarrow \simeq & & & & \downarrow \simeq \\ d_{1*}^{(1)} d_{0*}P & & \xrightarrow{\qquad\qquad d_{1*}^{(1)}\varphi \qquad\qquad} & & d_{1*}^{(1)} d_{1*}P \end{array} \tag{1.0.1}$$

commutes (we refer to this commutativity as the *cocycle condition*).

Note that if

$$\epsilon : X \times_T Y \to X \times_S Y$$

is the natural inclusion then $d_0 \circ \epsilon = d_1 \circ \epsilon$ and therefore for a complex $P_0 \in D_{\mathrm{qcoh}}(X \times_T Y)$ the pushforward $\epsilon_* P_0$ admits a natural such isomorphism φ over $X \times_S T \times_S Y$.

Theorem 1.1. *Assume that either f or g is flat and that (P,φ) is a pair as above. Assume further that P is a perfect complex and that the derived pushforward*

$$\mathbf{R}\mathrm{pr}_{1*}\, \mathbf{R}\mathscr{H}om(P,P)$$

lies in $D^{\geq 0}(X)$. Then there exists a pair (P_0,λ), unique up to unique isomorphism, consisting of a complex $P_0 \in D^b_{\mathrm{qcoh}}(X \times_T Y)$ and $\lambda : \epsilon_* P_0 \simeq P$ identifying φ with the canonical isomorphism between the pushforwards of P_0.

Remark 1.2. As indicated above the proof of this theorem given in section 2 has two steps. The first step is to identify the derived category $D^-_{\mathrm{qcoh}}(X \times_T Y)$ with the quasi-coherent objects of the total topos of a certain cosimplicial scheme $E(f,g)^\bullet$ (see theorem 2.6). The second step is to apply the results of [10] to lift (uniquely) the object (P,φ) to this total topos.

In this paper, we will also discuss two main applications of Theorem 1.1.

1.3 (Canonical fibration). Let X and Y be smooth projective varieties over a field k related by a derived equivalence $\Phi : D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(Y)$ given by a complex $P \in D^b_{\mathrm{coh}}(X \times Y)$. Let $R_X := \oplus_{n\geq 0}\Gamma(X, K_X^{\otimes n})$ (resp. $R_Y := \oplus_{n\geq 0}\Gamma(Y, K_Y^{\otimes n})$) denote the canonical ring of X (resp. Y) so we have rational maps

$$c_X : X \dashrightarrow \mathrm{Proj}(R_X) , \quad c_Y : Y \dashrightarrow \mathrm{Proj}(R_Y) .$$

It is well-known (this is stated explicitly in [16, 4.4] and attributed to [11] in [6, 6.1]) that Φ induces a canonical isomorphism

$$\tilde{\tau} : R_X \to R_Y, \tag{1.3.1}$$

and therefore also an isomorphism of schemes

$$\tau : \mathrm{Proj}(R_X) \simeq \mathrm{Proj}(R_Y).$$

Let $U_X \subset X$ (resp. $U_Y \subset Y$) be the complement of the base locus of $\{K_X^{\otimes n}\}_{n\geq 0}$ (resp. $\{K_Y^{\otimes n}\}_{n\geq 0}$), so c_X (resp. c_Y) is a morphism on U_X (resp. U_Y). Also, for an open subset $A \subset \mathrm{Proj}(R_Y)$ let $U_{X,A}$ (resp. $U_{Y,A}$) denote the preimage of A under $\tau \circ c_X$ (resp. c_Y).

Theorem 1.4. *There exists a dense open subset $A \subset \mathrm{Proj}(R_Y)$ such that the restriction of P to $U_{X,A} \times Y$ is isomorphic to the the image of an object*

$$P_0 \in D^b_{\mathrm{coh}}(U_{X,A} \times_{\tau\circ c_X, A, c_Y} U_{Y,A})$$

whose support is proper over both $U_{X,A}$ and $U_{Y,A}$.

Remark 1.5. (i) The above generalizes earlier work of Toda [16, 1.1].

(ii) Over fields of characteristic 0 the canonical ring is known to be finitely generated. Over fields of positive characteristic, however, this is not known so a little more care is needed in the arguments presented below.

(iii) We also prove a version of the above theorem replacing U_X with the maximal open subset W_X over which the morphism c_X extends.

1.6 (Albanese fibration). Let X be a smooth projective variety over a field k, let Pic^0_X (resp. Aut^0_X) denote the connected component of the identity of the Picard scheme of X (resp. the automorphism group scheme of X),

and set

$$\mathbf{R}^0_X := \mathrm{Pic}^0_X \times \mathrm{Aut}^0_X.$$

As we recall in section 4, if

$$\Phi : D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(Y)$$

is a derived equivalence given by a complex $P \in D^b_{\mathrm{coh}}(X \times Y)$, then Φ induces an isomorphism

$$\Phi_{\mathbf{R}^0} : \mathbf{R}^0_X \simeq \mathbf{R}^0_Y.$$

Let

$$c_X : X \to \mathbf{T}^0_X, \quad c_Y : Y \to \mathbf{T}^0_Y$$

be the Albanese torsors of X and Y (see section 4 for more discussion). For an open subset $U \subset \mathbf{T}^0_X$ let X_U denote $c_X^{-1}(U)$, and similarly for Y.

Theorem 1.7 (Theorem 4.9). *Assume that* Pic^0_X *and* Pic^0_Y *are reduced and that* $\Phi_{\mathbf{R}^0}$ *sends* Pic^0_X *to* Pic^0_Y *and therefore defines an isomorphism*

$$\Phi_{\mathrm{Pic}^0} : \mathrm{Pic}^0_X \to \mathrm{Pic}^0_Y.$$

Then Φ *induces an isomorphism of schemes*

$$\Phi_{\mathbf{T}^0} : \mathbf{T}^0_X \to \mathbf{T}^0_Y$$

compatible with the actions of the Picard schemes, and there exists a dense open subset $A \subset \mathbf{T}^0_Y$ *such that*

$$P|_{X_{\Phi^{-1}_{\mathbf{T}^0}(A)} \times Y_A}$$

is isomorphic to the image of an object in $D^b_{\mathrm{coh}}(X_{\Phi^{-1}_{\mathbf{T}^0}(A)} \times_A Y_A)$.

Remark 1.8. (a) The assumption that the groups schemes Pic^0_X and Pic^0_Y are reduced holds for example if k has characteristic 0.

(b) The assumption that $\Phi_{\mathbf{R}^0}$ preserves the Picard schemes frequently holds. For example, it holds if the automorphism group schemes are affine. Varieties for which the automorphism group scheme has a nontrivial abelian variety as quotient can be classified; see [12, 2.4].

1.9. The body of the article is divided into three sections. Section 2 is devoted to the proof of 1.1. As noted above the key ingredient is 2.6, which

reduces the proof to a problem of gluing in the derived category of a cosimplicial scheme. Theorem 1.1 is obtained from this and a variant of the BBD gluing lemma for cosimplicial schemes. Section 3 is concerned with the canonical fibration. In this section we prove, in particular, theorem 1.4. In addition to the results of section 2 we use in a key way the Beilinson resolution of the diagonal of projective space. Finally in section 4 we apply a similar analysis to the Albanese fibration proving 1.7.

2. Support of complexes and relativization of equivalences

2.1. Complexes on fiber products

For the convenience of the reader we review the result [1, 4.7] using the more classical language of derived categories.

2.2. We fix a base scheme S, a flat S-scheme T/S, and consider two separated morphisms of S-schemes

$$f : X \to T, \quad g : Y \to T.$$

Let $\gamma_f : X \to X \times_S T$ (resp. $\gamma_g^t : Y \to T \times_S Y$) be the maps given on scheme-valued points by

$$\gamma_f(x) = (x, f(x)), \quad \gamma_g^t(y) = (g(y), y).$$

We then get a cosimplicial scheme

$$E(f,g)^\bullet$$

by the following variant of the bar construction. Set

$$E(f,g)^n = X \times_S T^{\times n} \times_S Y,$$

where $T^{\times n}$ denotes the n-fold fiber product of T with itself over S and we make the convention that $T^0 = S$ so $E(f,g)^0 = X \times_S Y$. Define maps

$$d_i^{(n-1)} : E(f,g)^{n-1} \to E(f,g)^n, \quad i = 0, \ldots, n$$

by

$$d_0^{(n-1)} = \gamma_f \times \mathrm{id}_{T^{\times(n-1)} \times Y}, \quad d_n^{(n-1)} = \mathrm{id}_{X \times T^{\times(n-1)}} \times \gamma_g^t,$$

and, for $i = 1, \ldots, n-1$,

$$d_i^{(n-1)} = \mathrm{id}_{X \times T^{\times(i-1)}} \times \Delta_T \times \mathrm{id}_{T^{\times(n-i-1)} \times Y}.$$

Note that with our earlier notation we have $d_i = d_i^{(0)}$ and the above notation is consistent with that of the introduction. Define maps

$$s_i^{(n)} : E(f,g)^{n+1} \to E(f,g)^n, \quad i = 0, \ldots, n,$$

by the formula

$$s_i^{(n)}(x, t_1, \ldots, t_{n+1}, y) = (x, t_1, \ldots, \hat{t}_{i+1}, \ldots, t_{n+1}, y)$$

on scheme-valued points.

Lemma 2.3. *The maps $d_i^{(n-1)}$ and $s_i^{(n)}$ satisfy the cosimplicial identities [15, Tag 016K] and therefore define a cosimplicial scheme $E(f,g)^\bullet$.*

Proof. This is immediate from the definitions. □

2.4. We have an augmentation

$$\epsilon : X \times_T Y \to E(f,g)^\bullet,$$

that is, a map from $X \times_T Y$ to $E(f,g)^0$ compatible with each map in the cosimplicial structure. Writing

$$\epsilon_n : X \times_T Y \to E(f,g)^n$$

for the induced map in degree n, the map ϵ_n is given on scheme valued points by

$$(x, y) \mapsto (x, f(x), \ldots, f(x), y).$$

Note that since $f(x) = g(y)$ we could also write this formula using $g(y)$. One can also think of the augmentation as a morphism from the constant cosimplicial scheme on $X \times_T Y$.

2.5. Let $\mathrm{Mod}(E(f,g)^\bullet)$ denote the category of systems $(\{\mathscr{F}_n\}, \varphi_\delta)$, where $\mathscr{F}_n$ is a sheaf of $\mathcal{O}_{E(f,g)^n}$-modules on the étale site of $E(f,g)^n$ and for every

morphism $\delta : [n] \to [m]$ with associated morphism $d : E(f,g)^n \to E(f,g)^m$ we have a morphism

$$\varphi_\delta : \mathscr{F}_m \to d_*\mathscr{F}_n,$$

and these morphisms are compatible with compositions. The category $\mathrm{Mod}(E(f,h)^\bullet)$ is abelian, with kernels and cokernels defined level-wise. In particular, restriction to any particular $E(f,g)^n$ is an exact functor. We let

$$D(E(f,g)^\bullet)$$

denote the associated derived category and

$$D^{(-)}(E(f,g)^\bullet) \subset D(E(f,g)^\bullet)$$

the subcategory of complexes whose restriction to each $E(f,g)^n$ is bounded above.

For an object $\mathscr{P} \in D(E(f,g)^\bullet)$ we let $\mathscr{P}_n$ denote its restriction to $E(f,g)^n$. For a morphism $d : E(f,g)^n \to E(f,g)^m$ associated to a morphism δ in Δ we have a map

$$\varphi_\delta : \mathscr{P}_m \to Rd_*\mathscr{P}_n$$

in the derived category of $E(f,g)^m$. We write

$$D_{\mathrm{qcoh}}(E(f,g)^\bullet) \subset D(E(f,g)^\bullet)$$

for the subcategory of complexes for which the sheaves $\mathscr{H}^i(\mathscr{P}_m)$ are quasi-coherent and the maps φ_δ are all isomorphisms, and

$$D^-_{\mathrm{qcoh}}(E(f,g)^\bullet) := D^{(-)}(E(f,g)^\bullet) \cap D_{\mathrm{qcoh}}(E(f,g)^\bullet).$$

Pushforward along the augmentation defines a functor

$$\epsilon_* : D^-_{\mathrm{qcoh}}(X \times_T Y) \to D^-_{\mathrm{qcoh}}(E(f,g)^\bullet). \tag{2.5.1}$$

Theorem 2.6. *If either f or g is flat then the functor* (2.5.1) *is an equivalence.*

Remark 2.7. It seems likely that one can formulate and prove a version of this result without the flatness assumption replacing $X \times_T Y$ by a suitable derived fiber product.

The proof will be in several steps.

2.8. The functor

$$\epsilon_* : \mathrm{Mod}(X \times_T Y) \to \mathrm{Mod}(E(f,g)^\bullet)$$

has a left adjoint ϵ^*: For $(\{\mathscr{F}_n\}, \{\varphi_\delta\}) \in \mathrm{Mod}(E(f,g)^\bullet)$ the pullbacks $\epsilon_n^*\mathscr{F}_n$ form a simplicial object in $\mathrm{Mod}(X \times_T Y)$ and $\epsilon^*(\{\mathscr{F}_n\}, \{\varphi_\delta\})$ is given by the coequalizer of the two maps $\epsilon_1^*\mathscr{F}_1 \rightrightarrows \epsilon_0^*\mathscr{F}_0$. Deriving this left adjoint we get a functor

$$\mathbf{L}\epsilon^* : D^-_{\mathrm{qcoh}}(E(f,g)^\bullet) \to D^-_{\mathrm{qcoh}}(X \times_T Y).$$

We show that the functors

$$\mathbf{L}\epsilon^*\epsilon_* : D^-_{\mathrm{qcoh}}(X \times_T Y) \to D^-_{\mathrm{qcoh}}(X \times_T Y)$$

and

$$\epsilon_*\mathbf{L}\epsilon^* : D^-_{\mathrm{qcoh}}(E(f,g)^\bullet) \to D^-_{\mathrm{qcoh}}(E(f,g)^\bullet)$$

are isomorphic to the respective identity functors by the adjunction maps. Note that the functors and natural transformations are all compatible with restriction to open subschemes. In particular, we can check whether the adjunction maps are isomorphisms locally on $X \times_S Y$.

2.9. We reduce to the case when X, Y, S, and T are all affine as follows.

Since T/S is separated the maps $d_0, d_1 : X \times_S Y \to X \times_S T \times_S Y$ are closed immersions. Indeed, the morphisms γ_f and γ_g^t are closed immersions being sections of separated morphisms, and therefore d_0 and d_1 are also closed immersions being obtained by base change from these maps. For $\mathscr{P} \in D^-_{\mathrm{qcoh}}(E(f,g)^\bullet)$ we have

$$d_0(\mathrm{Supp}(\mathscr{P}_0)) = \mathrm{Supp}(\mathscr{P}_1) = d_1(\mathrm{Supp}(\mathscr{P}_0)). \tag{2.9.1}$$

Now observe that if $U \subset T$ is an open subset then

$$d_1^{-1}(d_0(f^{-1}(U) \times_S Y)) = f^{-1}(U) \times_T g^{-1}(U).$$

Combining this with (2.9.1) we find that if $T = \cup_i T_i$ is an open covering of T and if

$$f_i : X_i \to T_i, \quad g_i : Y_i \to T_i$$

are the restrictions of f and g then $\mathscr{P}$ is supported on

$$\cup_i E(f_i, g_i)^\bullet \subset E(f,g)^\bullet,$$

and it suffices to verify that our adjunction maps are isomorphisms for $D(f_i, g_i)^\bullet$. We may therefore assume that T is affine, say $T = \operatorname{Spec} R$, and that the map $T \to S$ factors through an affine open subset $\operatorname{Spec}(k) \subset S$ for a ring k. Replacing S by $\operatorname{Spec}(k)$ we are reduced to the case when S and T are affine.

Having made this reduction, we can then cover X and Y by affines and verify that the adjunction maps are isomorphisms over corresponding open subsets.

We may therefore assume that $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$ for R-algebras A and B.

2.10. In this case $E(f,g)^\bullet$ is given by the simplicial ring $\mathscr{A}_\bullet$ (with tensor products taken over k)

$$\cdots A \otimes R \otimes R \otimes B \;\substack{\rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow}\; A \otimes R \otimes B \;\substack{\rightarrow\\ \leftarrow\\ \rightarrow}\; A \otimes B.$$

Lemma 2.11. *The augmentation $\mathscr{A}_\bullet \to A \otimes_R B$ induces a quasi-isomorphism on the associated normalized complexes.*

Proof. Let $\mathscr{A}'_\bullet \to R$ be the simplicial ring with augmentation obtained from the above construction taking $A = B = R$.

Since the maps

$$d_i^* : A \otimes R^{\otimes n} \otimes B \to A \otimes R^{\otimes(n-1)} \otimes B$$

are $A \otimes B$-linear, the normalized complex of $\mathscr{A}_\bullet$ is isomorphic to the complex obtained from the normalized complex of $\mathscr{A}'_\bullet$ tensored over $R \otimes R$ with $A \otimes B$. Note also that $\mathscr{A}'_\bullet$ is term-wise flat over $R \otimes R$. It follows that if we show that $\mathscr{A}'_\bullet \to R$ induces a quasi-isomorphism on associated normalized complexes, then the map

$$\mathscr{A}_\bullet \simeq \mathscr{A}'_\bullet \otimes_{R\otimes R} (A \otimes B) \to R \otimes^{\mathbf{L}}_{R\otimes R} (A \otimes B) \simeq A \otimes_R B$$

also induces a quasi-isomorphism (using the flatness of one of f or g). We are therefore reduced to the case when $A = B = R$.

In this case it follows from direct calculation that the maps

$$h_n : R^{\otimes(n+2)} \to R^{\otimes(n+3)}, \quad a_0 \otimes \cdots \otimes a_{n+1} \mapsto a_0 \otimes \cdots \otimes a_{n+1} \otimes 1$$

define a homotopy between the identity and 0. □

2.12. From this and [7, I, 3.3.4.6] we see that if $P \in D^-(A \otimes_R B)$ then the adjunction map

$$\mathbf{L}\epsilon^*\epsilon_* P = (A \otimes_R B) \otimes^{\mathbf{L}}_{\mathscr{A}_\bullet} P \to P$$

is an isomorphism in the derived category.

To verify that the adjunction

$$P \to \epsilon_* \mathbf{L}\epsilon^* P \tag{2.12.1}$$

is an equivalence for $P \in D^-_{\mathrm{qcoh}}(E(f,g)^\bullet)$, note that since all the transition maps in $E(f,g)^\bullet$ are affine, and therefore have exact pushforwards, and ϵ^* is right exact we get by descending induction that it suffices to consider the case when P is a module concentrated in degree 0. In this case P is given by an $A \otimes B$-module for which the two induced actions of $A \otimes R \otimes B$ are equal. That is, P is given by an $A \otimes B$-module on which the R-actions coming from the two factors are the same implying that the $A \otimes B$-module structure descends to an $A \otimes_R B$-module structure. So in this case P is, in fact, of the form $\epsilon_* P_0$ for a $A \otimes_R B$-module P_0 and we have

$$\epsilon_* \mathbf{L}\epsilon^* \epsilon_* P_0 \simeq \epsilon_* P_0$$

by the case already considered. This isomorphism identifies (2.12.1) with the identity map (in the derived category), and therefore (2.12.1) is an isomorphism. This completes the proof of 2.6. □

Example 2.13. Let $X = \mathrm{Spec}(R)$ be an affine scheme over a field k and let $f \in R$ be an element defining an effective Cartier divisor $Z \hookrightarrow X$. Let

$$F : X \to \mathbf{A}^1_k$$

be the morphism defined by f so that $F^{-1}(0) = Z$. We can then apply our setup with $T = \mathbf{A}^1_k$, $Y = \mathrm{Spec}(k)$, and g the zero section $\mathrm{Spec}(k) \hookrightarrow \mathbf{A}^1_k$. The two maps

$$d_0, d_1 : X = X \times Y \to X \times T \times Y = X \times \mathbf{A}^1_k$$

are then given by

$$d_0 = (\mathrm{id}_X, F), \quad d_1 = (\mathrm{id}_X, 0);$$

that is, $d_0 = \gamma_F$ is the graph of F and $d_1 = \gamma_0$ is the graph of the zero map.

2.14. Proof of 1.1

Theorem 2.6 reframes the problem of descending a complex $P \in D^b_{\mathrm{qcoh}}(X \times_S Y)$ to $D^b_{\mathrm{qcoh}}(X \times_T Y)$ to one of extending P to an object of $D(E(f,g)^\bullet)$, the derived category of the cosimplicial scheme $E(f,g)^\bullet$. This is, fundamentally, a problem of gluing objects of the derived category (though not in the classical setting of a covering in a site but instead in the setting of gluing objects in a cosimplicial topos) and we apply the results of [10].

For $m \geq 0$ let $\gamma_i : [0] \to [m]$ $(0 \leq i \leq m)$ be the morphism in Δ sending 0 to i. For a morphism $\delta : [n] \to [m]$ in Δ let $E_\delta : E(f,g)^n \to E(f,g)^m$ be the corresponding morphism, and let

$$\alpha_\delta : [1] \to [m]$$

be the map sending 0 to 0 and 1 to $\delta(0)$ so we have

$$E_{\gamma_0} = E_{\alpha_\delta} \circ d_0, \quad E_{\gamma_{\delta(0)}} = E_{\alpha_\delta} \circ d_1.$$

2.15. Consider again the setup of 2.2. Let $P \in D^b_{\mathrm{qcoh}}(X \times_S Y)$ be a complex equipped with an isomorphism

$$\varphi : d_{0*}P \xrightarrow{\sim} d_{1*}P$$

on $X \times_S T \times_S Y$ such that the diagram (1.0.1) commutes. We show that under the assumptions of 1.1 there exists a unique pair (P_0, λ), where $P_0 \in D^b_{\mathrm{qcoh}}(X \times_T Y)$ is a complex and $\lambda : \epsilon_{0*}P_0 \simeq P$ is an isomorphism identifying φ with the canonical isomorphism between the pushforwards of P_0.

2.16. Define $P_n \in D^b_{\mathrm{qcoh}}(E(f,g)^n)$ to be the pushforward of P along the morphism $X \times_S Y \to E(f,g)^n$ given by the map $[0] \to [n]$ sending 0 to 0.

For every morphism $\delta : [n] \to [m]$ we then get an isomorphism

$$\varphi_\delta : P_m = \mathbf{R}E_{\alpha_\delta *}d_{0*}P \xrightarrow{\varphi} \mathbf{R}E_{\alpha_\delta *}d_{1*}P \simeq \mathbf{R}E_{\delta *}P_n.$$

The maps φ_δ are compatible with composition. For maps

$$[n] \xrightarrow{\delta} [m] \xrightarrow{\rho} [k]$$

we get by applying $\mathbf{R}E_{\rho *}$ to the diagram (1.0.1) an equality

$$\mathbf{R}E_{\rho *}(\varphi_\delta) \circ \varphi_\rho = \varphi_{\rho\delta}.$$

Lemma 2.17. *Let $\delta : [n] \to [m]$ and $\rho : [t] \to [m]$ be morphisms in Δ. Then*

$$\mathrm{Ext}^i_{E(f,g)^m}(\mathbf{R}E_{\rho *}P_t, \mathbf{R}E_{\delta *}P_n) = 0, \quad \textit{for } i < 0.$$

Proof. We have $\mathbf{R}E_{\delta *}P_n \simeq P_m \simeq \mathbf{R}E_{\rho *}P_t$, so it suffices to show that

$$\mathrm{Ext}^s_{E(f,g)^m}(\gamma_{0*}P, \gamma_{0*}P) = 0$$

for $s < 0$.

Let

$$t : X \to X \times_S T^m$$

be the map given on scheme-valued points by

$$x \mapsto (x, f(x), \ldots, f(x)).$$

Then γ_0 is obtained by taking the product of t with Y. By adjunction, and using the fact that P is perfect, we have

$$\begin{aligned}&\mathrm{Ext}^s_{E(f,g)^m}(\gamma_{0*}P, \gamma_{0*}P)\\ &\qquad \simeq H^s(X \times Y, \mathbf{R}\mathscr{H}om(\mathbf{L}\gamma_0^*\gamma_{0*}\mathcal{O}_{X\times Y}, \mathcal{O}_{X\times Y}) \otimes \mathbf{R}\mathscr{H}om(P, P)).\end{aligned}$$

Let $\mathscr{R}$ denote the complex

$$\mathbf{R}\mathscr{H}om(\mathbf{L}t^*t_*\mathcal{O}_X, \mathcal{O}_X).$$

Then we have

$$\mathbf{R}\mathrm{pr}_1^*\mathscr{R} \simeq \mathbf{R}\mathscr{H}om(\mathbf{L}\gamma_0^*\gamma_{0*}\mathcal{O}_{X\times Y}, \mathcal{O}_{X\times Y}).$$

We conclude that

$$\begin{aligned}(2.17.1)\quad &\mathscr{R} \otimes^{\mathbf{L}} \mathbf{R}\mathrm{pr}_{1*}\,\mathbf{R}\mathscr{H}om(P, P)\\ &\qquad \simeq \mathbf{R}\mathrm{pr}_{1*}(\mathbf{R}\mathscr{H}om(\mathbf{L}\gamma_0^*\gamma_{0*}\mathcal{O}_{X\times Y}, \mathcal{O}_{X\times Y}) \otimes \mathbf{R}\mathscr{H}om(P, P)).\end{aligned}$$

Since $\mathscr{R}$ is locally represented by a complex of projective $\mathcal{O}_X$-modules concentrated in degrees ≥ 0 and $\mathbf{R}\mathrm{pr}_{1*}\,\mathbf{R}\mathscr{H}om(P, P)$ is concentrated in degrees ≥ 0 the complex (2.17.1) is in $D^{\geq 0}(X)$. In particular, its cohomology is zero in negative degrees. □

Theorem 1.1 now follows from the lemma and the BBD gluing lemma for a D-topos [10, 1.5 and 1.7]. In our situation these results give the following. With notation of loc. cit. we take $D = \Delta^{\mathrm{op}}$ so that $E(f,g)^\bullet$ defines a functor

$$D^{\mathrm{op}} = \Delta \to (\text{schemes}).$$

Informally (or formally if one works with appropriate ∞-categories) this defines a functor

$$D^{\mathrm{op}} = \Delta \to (\text{triangulated categories}), \quad [n] \mapsto D(E(f,g)^n, \mathcal{O}_{E(f,g)^n}).$$

Let Γ (notation consistent with [10, 1.4]) denote the category of systems

$$(\{M_n\}_{[n]\in\Delta}, \{\varphi_\delta\})$$

as follows:

(i) For each $[n] \in \Delta$ we are given an object $M_n \in D^b(E(f,g)^n, \mathcal{O}_{E(f,g)^n})$.

(ii) For each morphism $\delta : [n] \to [m]$ in Δ we are given a morphism $\varphi_\delta : M_m \to RE_{\delta *}M_n$.

(iii) The maps φ_δ are compatible with composition in the sense that for a composition in Δ

$$[n] \xrightarrow{\delta} [m] \xrightarrow{\delta'} [k]$$

the diagram

$$\begin{array}{ccccc} M_k & \xrightarrow{\varphi_\delta} & RE_{\delta *}M_m & \xrightarrow{RE_{\delta *}(\varphi_{\delta'})} & RE_{\delta' *}RE_{\delta *}M_n \\ & \searrow^{\varphi_{\delta'\delta}} & & & \downarrow \simeq \\ & & & & RE_{\delta'\delta *}M_n \end{array}$$

commutes.

There is a functor

$$\pi : D^b(E(f,g)^\bullet) \to \Gamma$$

sending a complex M to the system given by the restrictions M_n of M to each $E(f,g)^n$ with the natural transition maps. Let $\Gamma' \subset \Gamma$ be the full subcategory of systems for which the following additional conditions hold:

(iv) There exists integers $a < b$ such that $M_n \in D^{[a,b]}(E(f,g)^n, \mathcal{O}_{E(f,g)^n})$ (the category of complexes with cohomology supported in the interval $[a,b]$).

(v) For any two morphisms $\delta : [n] \to [m]$ and $\rho : [t] \to [m]$ in Δ with common target we have

$$\mathrm{Ext}^i_{E(f,g)^m}(\mathbf{R}E_{\rho *}M_t, \mathbf{R}E_{\delta *}M_n) = 0, \quad \text{for } i < 0.$$

Let $\widetilde{\Gamma}' \subset D^b(E(f,g)^\bullet)$ be those complexes K with $\pi(K) \in \Gamma'$. Then [10, 1.5 and 1.7 (i)] imply that the functor $\widetilde{\Gamma}' \to \Gamma'$ is an equivalence of categories.

Lemma 2.17 is therefore the necessary input needed to apply the results of [10] yielding Theorem 1.1. □

3. Canonical fibration

3.1. For a smooth projective variety X over a field k let

$$R_X := \oplus_{n \geq 0} \Gamma(X, K_X^{\otimes n})$$

be its canonical ring. Over fields of characteristic 0 this ring is known to be finitely generated [3].

For an integer $n \geq 1$ for which $H_{X,n} := \Gamma(X, K_X^{\otimes n})$ is nonzero we get a rational map

$$\pi_n : \ X \dashrightarrow \mathbf{P}H_{X,n} \ .$$

We then get open subsets

$$U_{X,n} \subset W_{X,n} \subset X,$$

where $U_{X,n}$ is the maximal open subset over which $H_{X,n}$ generates $K_X^{\otimes n}$ and $W_{X,n}$ is the maximal open subset over which π_n is a morphism. Since X is normal the complement of $W_{X,n}$ in X has codimension ≥ 2 and since X is smooth the invertible sheaf $\pi_n^* \mathcal{O}_{\mathbf{P}H_{X,n}}(1)$ extends uniquely to an invertible subsheaf

$$\pi_n^* \mathcal{O}_{\mathbf{P}H_{X,n}}(1) \hookrightarrow K_X^{\otimes n}$$

over all of X for which there is a map

$$H_{X,n} \to \Gamma(X, \pi_n^* \mathcal{O}_{\mathbf{P}H_{X,n}}(1))$$

whose image generates $\pi_n^* \mathcal{O}_{\mathbf{P}H_{X,n}}(1)$ over $W_{X,n}$.

For an open subset $A \subset \mathbf{P}H_{X,n}$ we write $U_{X,n,A}$ (resp. $W_{X,n,A}$) for the preimage of A in $U_{X,n}$ (resp. $W_{X,n}$).

3.2. If X and Y are two smooth projective varieties over k related by a derived equivalence

$$\Phi : D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(Y)$$

then Φ induces an isomorphism

$$H_{X,n} \simeq H_{Y,n} \tag{3.2.1}$$

for all n; in particular, Φ induces an isomorphism of canonical rings $R_X \simeq R_Y$. This is due to Bondal and Orlov [4].

Let us recall the argument. For an integer $n \in \mathbf{Z}$ define a functor

$$S_n : D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(X), \quad \mathscr{F} \mapsto \mathscr{F} \otimes K_X^{\otimes n}.$$

Define $\mathcal{S}_{K_X}$ to be the category whose objects are the functors S_n and for which the morphisms $S_m \to S_n$ are given by elements of $H^0(X, K_X^{\otimes(n-m)})$. So $\mathcal{S}_{K_X}$ is a subcategory of the category $\mathrm{End}(D^b_{\mathrm{coh}}(X))$ of endofunctors of $D^b_{\mathrm{coh}}(X)$.

Lemma 3.3. *Let X and Y be smooth projective varieties over a field k and let $\Phi : D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(Y)$ be an equivalence of triangulated categories. Then the induced functor*

$$\mathrm{End}(D^b_{\mathrm{coh}}(X)) \to \mathrm{End}(D^b_{\mathrm{coh}}(Y)), \quad F \mapsto \Phi \circ F \circ \Phi^{-1} \tag{3.3.1}$$

sends $\mathcal{S}_{K_X}$ to $\mathcal{S}_{K_Y}$.

Proof. The fact that conjugation by Φ matches up the objects of the categories $\mathcal{S}_{K_X}$ and $\mathcal{S}_{K_Y}$ is due to Bondal and Orlov [4]. Let $P \in D^b_{\mathrm{coh}}(X \times Y)$ be a complex defining Φ. For an integer n and $S_{X,n} \in \mathrm{End}(D^b_{\mathrm{coh}}(X))$ (resp. $S_{Y,n} \in \mathrm{End}(D^b_{\mathrm{coh}}(Y))$) given by tensoring with $K_X^{\otimes n}$ (resp. $K_Y^{\otimes n}$) we have $\Phi \circ S_{X,n}$ given by $P \otimes p_X^* K_X^{\otimes n}$ and $S_{Y,n} \circ \Phi$ given by $P \otimes p_Y^* K_Y^{\otimes n}$, where p_X and p_Y are the projections. The result therefore follows from the standard fact [6, 5.22] that

$$P \otimes p_X^* K_X^{\otimes n} \simeq P \otimes p_Y^* K_Y^{\otimes n}.$$

Since these subcategories are not full, however, a bit more is required to get the compatibility on morphisms. Following [17], let $L_{perf}(X)$ (resp.

L_{perf}) denote the dg-category of perfect complexes of quasi-coherent sheaves on X. The kernel P then defines an equivalence

$$\widetilde{\Phi} : L_{perf}(X) \to L_{perf}(Y).$$

Let $S_X : D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(X)$ be the Serre functor of X. By the uniqueness part of Orlov's theorem, as well as Toën's representability result in [17, 8.15] the functor has a lift

$$\widetilde{S}_X : L_{perf}(X) \to L_{perf}(X)$$

which is unique up to equivalence of dg functors (in the sense of [17]).

In fact, $\widetilde{S}_X$ is given by $\Delta_{X*}\omega_X[\dim(X)] \in L_{perf}(X \times X)$. For integers n and m it therefore makes sense to consider the subspace

$$\mathrm{Hom}'_{\mathrm{End}(D^b_{\mathrm{coh}}(X))}(S^n_X, S^m_X) \subset \mathrm{Hom}_{\mathrm{End}(D^b_{\mathrm{coh}}(X))}(S^n_X, S^m_X)$$

of morphisms of functors $S^n_X \to S^m_X$ which admit liftings to morphisms of dg functors $\widetilde{S}^n_X \to \widetilde{S}^m_X$. By [17, 8.9] the set $\mathrm{Hom}'_{\mathrm{End}(D^b_{\mathrm{coh}}(X))}(S^n_X, S^m_X)$ consists precisely of those morphisms induced by sections of $K_X^{\otimes(m-n)}$. Now for a lift $\widetilde{S}_X$ the functor

$$\widetilde{\Phi} \circ \widetilde{S}_X \circ \widetilde{\Phi}^{-1} : L_{perf}(Y) \to L_{perf}(Y)$$

is a dg lift of the Serre functor S_Y of Y. From this it follows that (3.3.1) sends the subset $\mathrm{Hom}'_{\mathrm{End}(D^b_{\mathrm{coh}}(X))}(S^n_X, S^m_X)$ to $\mathrm{Hom}'_{\mathrm{End}(D^b_{\mathrm{coh}}(Y))}(S^n_Y, S^m_Y)$ which implies the lemma. □

Theorem 3.4. *Let X and Y be smooth projective varieties over k and let*

$$\Phi : D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(Y)$$

be a derived equivalence given by a complex $P \in D^b_{\mathrm{coh}}(X \times Y)$. Let $n \geq 1$ be an integer such that $H_{X,n}$ (and therefore also $H_{Y,n}$) is nonzero.

(a) The support of $P|_{U_{X,n} \times Y}$ (resp. $P|_{W_{X,n} \times Y}$) is contained in $U_{X,n} \times U_{Y,n}$ (resp. $W_{X,n} \times W_{Y,n}$).

(b) There exists a dense open subset $A \subset \mathbf{P}H_{X,n}$ such that $P|_{W_{X,n,A} \times Y}$ is isomorphic to an object in the image of

$$D^b_{\mathrm{coh}}(W_{X,n,A} \times_A W_{Y,n,A}) \to D^b_{\mathrm{coh}}(W_{X,n,A} \times Y).$$

(c) There exists a dense open subset $A \subset \mathbf{P}H_{X,n}$ such that $P|_{U_{X,n,A}\times Y}$ is isomorphic to an object in the image of

$$D^b_{\mathrm{coh}}(U_{X,n,A} \times_A U_{Y,n,A}) \to D^b_{\mathrm{coh}}(U_{X,n,A} \times Y).$$

Here we somewhat abusively write $U_{Y,n,A}$ and $W_{Y,n,A}$ for the preimages in $U_{Y,n}$ and $W_{Y,n}$ of the open subset $A \hookrightarrow \mathbf{P}H_{X,n} \simeq \mathbf{P}H_{Y,n}$ of $\mathbf{P}H_{Y,n}$ defined by A.

The proof occupies the remainder of this section.

Remark 3.5. Note that the open subset $U_X \subset X$ considered in 1.4 is the union over all n of the $U_{X,n}$. Since U_X is quasi-compact, we in fact have $U_X = U_{X,n}$ for $n >> 0$ and therefore 1.4 follows from 3.4.

Remark 3.6. Note that statement (c) follows from (a) and (b). Indeed let P_W in $D^b_{\mathrm{coh}}(W_{X,n,A} \times_A W_{Y,n,A})$ be an object with image in $D^b_{\mathrm{coh}}(W_{X,n,A} \times Y)$ isomorphic to $P|_{W_{X,n,A}\times Y}$. Then the image of P_W in $D^b_{\mathrm{coh}}(U_{X,n,A} \times_A W_{Y,n,A})$ is by (a) supported on $U_{X,n,A} \times_A U_{Y,n,A}$, and therefore is in the image of $D^b_{\mathrm{coh}}(U_{X,n,A} \times_A U_{Y,n,A})$, and has image $P_{U_{X,n,A}\times Y}$ in $D^b_{\mathrm{coh}}(U_{X,n,A} \times Y)$.

3.7. The complex $\mathscr{C}_{X,n}$.

3.8. We can define a complex $\mathscr{C}_{X,n}$ on X with a map of complexes

$$\epsilon_{X,n} : \mathscr{C}_{X,n} \to \pi_n^* \mathcal{O}_{\mathbf{P}H_{X,n}}(1)$$

which restricts to a quasi-isomorphism over $W_{X,n}$. Recall that we write

$$\pi_n^* \mathcal{O}_{\mathbf{P}H_{X,n}}(1)$$

for the line bundle on X obtained by pullback under the rational map π_n. This complex $\mathscr{C}_{X,n}$ will be used to understand the set $W_{X,n}$.

The complex $\mathscr{C}_{X,n}$ is the Koszul complex associated to the map $\rho : H_{X,n} \otimes_k \mathcal{O}_X \to K_X^{\otimes n}$ (note that this map factors through $\pi_n^* \mathcal{O}_{\mathbf{P}H_{X,n}}(1)$). Precisely, we have

$$\mathscr{C}^i_{X,n} := (\wedge^{-i+1} H_{X,n}) \otimes_k K_X^{\otimes(in)}$$

for $i \leq 0$ and $\mathscr{C}^i_{X,n} = 0$ for $i > 0$. The differential

$$d_i : \mathscr{C}^i_{X,n} \to \mathscr{C}^{i+1}_{X,n}$$

is given by the usual formula in local coordinates

$$d_i((h_1 \wedge \cdots \wedge h_{1-i}) \otimes \ell) := \sum_{j=1}^{i} (-1)^{j+1} (h_1 \wedge \cdots \hat{h}_j \cdots \wedge h_{1-i}) \otimes (\rho(h_j) \otimes \ell). \tag{3.8.1}$$

The map $\epsilon_{X,n}$ is defined to be the map induced by the natural map $H_{X,n} \otimes_k \mathcal{O}_X \to \pi_n^* \mathcal{O}_{\mathbf{P}H_{X,n}}(1)$. By standard properties of the Koszul complex the restriction of $\epsilon_{X,n}$ to $W_{X,n}$ is a quasi-isomorphism.

If

$$\Sigma_{X,n} \subset \pi_n^* \mathcal{O}_{\mathbf{P}H_{X,n}}(1) \tag{3.8.2}$$

is the image of $H_{X,n} \otimes_k \mathcal{O}_X$ then a point $z \in X$ lies in $W_{X,n}$ if and only if $\Sigma_{X,n,z}$ is generated by a single element. Indeed if this is the case then Σ_n is a line bundle in a neighborhood of z and the inclusion (3.8.2) restrict to this open subset to an isomorphism, since it is an inclusion of line bundles which is an isomorphism away from a codimension 2 subset.

Proposition 3.9. *Let X and Y be smooth projective varieties over k and let*

$$\Phi : D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(Y)$$

be a derived equivalence given by a kernel $P \in D^b_{\mathrm{coh}}(X \times Y)$. Let $T_{X,n}$ (resp. $T_{Y,n}$) be the endofunctor of $D^b_{\mathrm{coh}}(X)$ (resp. $D^b_{\mathrm{coh}}(Y)$) given by tensoring with $\mathcal{C}_{X,n}$ (resp. $\mathcal{C}_{Y,n}$). Then

$$\Phi \circ T_{X,n} \circ \Phi^{-1} \simeq T_{Y,n}.$$

Proof. Let $p : X \times Y \to X$ and $q : X \times Y \to Y$ be the projections. The proof of Lemma 3.3 implies that there exist isomorphisms

$$\sigma_n : p^* K_X^{\otimes n} \otimes^{\mathbf{L}} P \simeq P \otimes^{\mathbf{L}} q^* K_Y^{\otimes n}$$

in $D^b_{\mathrm{coh}}(X \times Y)$. To prove the proposition it suffices to extend these isomorphisms to an isomorphism of complexes

$$\lambda : p^* \mathcal{C}_{X,n} \otimes^{\mathbf{L}} P \simeq q^* \mathcal{C}_{Y,n} \otimes^{\mathbf{L}} P. \tag{3.9.1}$$

Indeed such an isomorphism defines an isomorphism

$$\Phi \circ T_{X,n} \simeq T_{Y,n} \circ \Phi.$$

For an integer s define $\mathscr{C}^{\leq s}_{\overline{X},n}$ to be the complex which in degrees $i \leq s$ is the same as $\mathscr{C}_{X,n}$ but which has zero terms in degree $> s$. There is then a distinguished triangle for each s

$$\mathscr{C}^{s}_{X,n}[-s] \longrightarrow \mathscr{C}^{\leq s}_{\overline{X},n} \longrightarrow \mathscr{C}^{\leq s-1}_{\overline{X},n} \longrightarrow \mathscr{C}^{s}_{X,n}[-s+1]. \tag{3.9.2}$$

To prove the proposition we construct for each s an isomorphism in $D^b_{\mathrm{coh}}(X \times Y)$

$$\lambda^{\leq s} : p^*\mathscr{C}^{\leq s}_{\overline{X},n} \otimes^{\mathbf{L}} P \simeq q^*\mathscr{C}^{\leq s}_{\overline{Y},n} \otimes^{\mathbf{L}} P,$$

such that the diagram

$$\begin{array}{ccc} p^*\mathscr{C}^{s}_{X,n}[-s] \otimes^{\mathbf{L}} P & \xrightarrow{\wedge^{-s+1}(\tilde{\tau})\otimes\sigma_{ns}} & q^*\mathscr{C}^{s}_{Y,n}[-s] \otimes^{\mathbf{L}} P \\ \downarrow & & \downarrow \\ p^*\mathscr{C}^{\leq s}_{\overline{X},n} \otimes^{\mathbf{L}} P & \xrightarrow{\lambda^{\leq s}} & q^*\mathscr{C}^{\leq s}_{\overline{Y},n} \otimes^{\mathbf{L}} P \end{array} \tag{3.9.3}$$

commutes.

Lemma 3.10. *Let s, i, and j be integers with $j > s$.*

(i) *We have*

$$\mathrm{Hom}_{D^b_{\mathrm{coh}}(X\times Y)}(p^*\mathscr{C}^{\leq s}_{\overline{X},n} \otimes^{\mathbf{L}} P, q^*\mathscr{C}^{i}_{Y,n} \otimes^{\mathbf{L}} P[-j]) = 0.$$

(ii) *The restriction map*

$$\begin{array}{c} \mathrm{Hom}_{D^b_{\mathrm{coh}}(X\times Y)}(p^*\mathscr{C}^{\leq s}_{\overline{X},n} \otimes^{\mathbf{L}} P, q^*\mathscr{C}^{i}_{Y,n} \otimes^{\mathbf{L}} P[-s]) \\ \downarrow \\ \mathrm{Hom}_{D^b_{\mathrm{coh}}(X\times Y)}(p^*\mathscr{C}^{s}_{X,n} \otimes^{\mathbf{L}} P[-s], q^*\mathscr{C}^{i}_{Y,n} \otimes^{\mathbf{L}} P[-s]) \end{array}$$

is injective.

Proof. By considering the distinguished triangles (3.9.2) the proof of (i) is reduced to showing that for all integers s, i, and $j > s$ we have

$$\mathrm{Hom}_{D^b_{\mathrm{coh}}(X\times Y)}(p^*\mathscr{C}^{s}_{X,n} \otimes^{\mathbf{L}} P[-s], q^*\mathscr{C}^{i}_{Y,n} \otimes^{\mathbf{L}} P[-j]) = 0.$$

This follows from noting that elements of this group correspond to morphisms of functors

$$\Phi \circ \Phi^{\mathscr{C}^{s}_{X,n}[-s]} \to \Phi^{\mathscr{C}^{i}_{Y,n}[-j]} \circ \Phi$$

which can be lifted to the dg-categories of complexes of coherent sheaves. Using the isomorphism

$$\Phi^{\mathscr{C}^i_{Y,n}[-j]} \circ \Phi \simeq \Phi \circ \Phi^{\mathscr{C}^i_{X,n}[-j]}$$

and applying Φ^{-1} we see that we have to show that there are no nonzero morphisms of functors

$$\Phi^{\mathscr{C}^s_{X,n}[-s]} \to \Phi^{\mathscr{C}^i_{X,n}[-j]}$$

which can be lifted to the dg-category. Here for a complex $K \in D^b_{\mathrm{coh}}(X)$ we write Φ^K for the endofunctor given by tensoring with K, and similarly for complexes on Y. Equivalently, we need to show that there are no nonzero morphisms in $D^b_{\mathrm{coh}}(X \times X)$

$$\Delta_{X*}\mathscr{C}^s_{X,n}[-s] \to \Delta_{X*}\mathscr{C}^i_{X,n}[-j],$$

which follows from the fact that $j > s$.

Statement (ii) follows from (i) and consideration of the triangles (3.9.2). □

We now construct $\lambda^{\leq s}$ inductively. For s sufficiently negative we have $\mathscr{C}^{\leq s}_{X,n} = 0$ so there is nothing to show. So we assume that $\lambda^{\leq s}$ has been defined and construct $\lambda^{\leq(s+1)}$. For this consider the diagram of distinguished triangles

$$\begin{array}{ccc}
p^*\mathscr{C}^{s+1}_{X,n}[-(s+1)] \otimes^{\mathbf{L}} P & \xrightarrow{\tau\otimes\sigma} & q^*\mathscr{C}^{s+1}_{Y,n}[-(s+1)] \otimes^{\mathbf{L}} P \\
\downarrow & & \downarrow \\
p^*\mathscr{C}^{\leq(s+1)}_{X,n} \otimes^{\mathbf{L}} P & \dashrightarrow & q^*\mathscr{C}^{\leq(s+1)}_{Y,n} \otimes^{\mathbf{L}} P \\
\downarrow & & \downarrow \\
p^*\mathscr{C}^{\leq s}_{X,n} \otimes^{\mathbf{L}} P & \xrightarrow{\lambda^{\leq s}_n} & q^*\mathscr{C}^{\leq s}_{Y,n} \otimes^{\mathbf{L}} P \\
\downarrow & & \downarrow \\
p^*\mathscr{C}^{s+1}_{X,n}[-s] \otimes^{\mathbf{L}} P & \xrightarrow{\tau\otimes\sigma} & q^*\mathscr{C}^{s+1}_{Y,n}[-s] \otimes^{\mathbf{L}} P,
\end{array}$$

where the bottom-most inner square commutes by Lemma 3.10 (ii). Now define $\lambda^{\leq(s+1)}$ to be a morphism as indicated by the dotted arrow, which exists by axiom TR 3 of triangulated categories [15, Tag 0145]. This completes the proof of Proposition 3.9. □

3.11. Proof of 3.4 (a)

If $x \in U_{X,n}$ then the skyscraper sheaf $\kappa(x) \in D^b_{\mathrm{coh}}(X)$ has the property that there exists an element $\alpha \in H_{X,n}$ such that

$$\alpha : \kappa(x) \to \kappa(x) \otimes^{\mathbf{L}} K_X^{\otimes n}$$

is an isomorphism. It follows that P_x has the property that there exists an element $\alpha' \in H_{Y,n}$ for which the map

$$\alpha' : P_x \to P_x \otimes^{\mathbf{L}} K_Y^{\otimes n}$$

is an isomorphism. The statement for $P|_{U_{X,n} \times Y}$ follows from this and the following 3.12.

To get the statement for $W_{X,n}$ note that if $x \in W_{X,n}$ is a point then from the equation (3.9.1) we find that

$$P_x \simeq P_x \otimes^{\mathbf{L}} \mathscr{C}_{Y,n}.$$

We get the statement for $P_{W_{x,n} \times Y}$ from this and the following 3.13.

Lemma 3.12. *Let $Q \in D^b_{\mathrm{coh}}(Y)$ be a complex such that there exists an element $\alpha \in H_{Y,n}$ for which the induced map*

$$\alpha : Q \to Q \otimes^{\mathbf{L}} K_Y^{\otimes n}$$

is an isomorphism. Then the support of Q is contained in $U_{Y,n}$.

Proof. Indeed the assumptions imply that for a point $z \in Y$ in the support of Q the fiber $\alpha(z) \in K_Y^{\otimes n}(z)$ is nonzero, and therefore $z \in U_{Y,n}$. □

Lemma 3.13. *Let $Q \in D^b_{\mathrm{coh}}(Y)$ be a complex such that*

$$Q \otimes^{\mathbf{L}} \mathscr{C}_{Y,n} \simeq Q.$$

Then the support of Q is contained in $W_{Y,n}$.

Proof. Let $z \in \mathrm{Supp}(Q)$ be a point in the support. Let t be the largest integer for which $\mathscr{H}^t(Q)_z \neq 0$. Since $\mathscr{C}_{Y,n} \in \mathrm{D}^{\leq 0}(Y)$ we then have

$$\mathscr{H}^t(Q \otimes^{\mathbf{L}} \mathscr{C}_{Y,n})_z \simeq \mathscr{H}^t(Q)_z \otimes_{\mathscr{O}_{Y,z}} \mathscr{H}^0(\mathscr{C}_{Y,n})_z.$$

We therefore find that

$$\mathscr{H}^t(Q)_z \otimes_{\mathscr{O}_{Y,z}} \mathscr{H}^0(\mathscr{C}_{Y,n})_z \simeq \mathscr{H}^t(Q)_z.$$

Since we assume that $\mathscr{H}^t(Q)_z$ is nonzero, this implies, by Nakayama's lemma, that $\mathscr{H}^0(\mathscr{C}_{Y,n})_z$ is generated by a single element. It follows that the subsheaf

$$\Sigma_{Y,n} \subset \pi_n^* \mathscr{O}_{\mathbf{P}H_{Y,n}}(1)$$

generated by the image of $H_{Y,n}$ is locally free of rank 1 at z, which implies that $z \in W_{Y,n}$. □

3.14. Set-theoretic support

In order to prove 3.4 (b) we will first need a set-theoretic statement.

Lemma 3.15. *Let $f : Z \to W_{Y,n}$ be a morphism, with Z proper, and let $\mathscr{F}$ be a coherent sheaf on Z such that $\mathscr{F} \otimes f^*\mathscr{C}_{Y,n} \simeq \mathscr{F}$. Then $f(\mathrm{Supp}(\mathscr{F})) \subset W_{Y,n}$ is contained in a finite union of fibers of π_n.*

Proof. It suffices to prove the lemma after making a base change to an algebraic closure of k. Replacing Z by an alteration if necessary we may assume that Z is smooth and proper over k and that $\mathscr{F}$ is supported on all of Z.

Note that over $W_{Y,n}$ we have $\mathscr{C}_{Y,n} \simeq \mathscr{O}_Y(n)$ so $f^*\mathscr{C}_{Y,n} \simeq f^*\pi_n^*\mathscr{O}_{\mathbf{P}H_{Y,n}}(1)$.

Let r be the generic rank of $\mathscr{F}$. Then taking determinants we find that

$$\det(\mathscr{F}) \simeq \det(\mathscr{F}) \otimes f^*\pi_n^*\mathscr{O}_{\mathbf{P}H_{Y,n}}(r).$$

Therefore $f^*\pi_n^*\mathscr{O}_{\mathbf{P}H_{Y,n}}(1))$ is a torsion line bundle on Z, which implies that the image of Z in $\mathbf{P}H_{Y,n}$ is a zero-dimensional subscheme. □

3.16. For a point $x \in W_{X,n}$ the skyscraper sheaf $\kappa(x)$ has the property that

$$\kappa(x) \otimes^{\mathbf{L}} \mathscr{C}_{X,n} \simeq \kappa(x).$$

It follows that we also have

$$P_x \otimes^{\mathbf{L}} \mathscr{C}_{Y,n} \simeq P_x$$

in $D^b_{\mathrm{coh}}(Y)$ (Note: we already showed that the support of these complexes lies in $W_{Y,n}$). By the lemma we conclude that the image of the support of P_x in $\mathbf{P}H_{Y,n}$ lies in a finite number of fibers of π_n. And since $\mathrm{End}_{D^b_{\mathrm{coh}}(Y)}(P_x) = k$ the support is, in fact, connected. We have shown:

Corollary 3.17. *The set-theoretic support of $P|_{W_{X,n}\times Y}$ is contained in*

$$W_{X,n} \times_{\mathbf{P}H_{Y,n}} W_{Y,n},$$

where the map $W_{X,n} \to \mathbf{P}H_{Y,n}$ is the composition of $\pi_n : W_{X,n} \to \mathbf{P}H_{X,n}$ and the isomorphism (3.2.1).

3.18. Though not used in what follows, we also observe that P induces derived equivalences of open varieties as follows. Note that since

$$P_{U,n} := P|_{U_{X,n}\times U_{Y,n}} \text{ (resp. } P_{W,n} := P|_{W_{X,n}\times W_{Y,n}})$$

has proper support over both $U_{X,n}$ and $U_{Y,n}$ (resp. $W_{X,n}$ and $W_{Y,n}$) the complex P induces functors

$$\Phi_{U,n} : D^b_{\mathrm{coh}}(U_{X,n}) \to D^b_{\mathrm{coh}}(U_{Y,n}), \quad \Phi_{W,n} : D^b_{\mathrm{coh}}(W_{X,n}) \to D^b_{\mathrm{coh}}(W_{Y,n}).$$

Proposition 3.19. *The functors $\Phi_{U,n}$ and $\Phi_{W,n}$ are equivalences of triangulated categories.*

Proof. That $\Phi_{U,n}$ is an equivalence can be seen as follows. Let $P^\vee \in D^b_{\mathrm{coh}}(Y \times X)$ be the complex defining $\Phi^{-1} : D^b_{\mathrm{coh}}(Y) \to D^b_{\mathrm{coh}}(X)$, and let $P^\vee_{U,n}$ be the restriction of $P^\vee$ to $U_{Y,n} \times U_{X,n}$, which defines

$$\Phi^\vee_{U,n} : D^b_{\mathrm{coh}}(U_{Y,n}) \to D^b_{\mathrm{coh}}(U_{X,n})$$

We claim that $\Phi_{U,n} \circ \Phi^\vee_{U,n} \simeq \mathrm{id}_{D^b_{\mathrm{coh}}(U_{Y,n})}$ and $\Phi^\vee_{U,n} \circ \Phi_{U,n} \simeq \mathrm{id}_{D^b_{\mathrm{coh}}(U_{X,n})}$.

To see this observe that the restriction of P to $U_{X,n} \times Y$ is equal to the pushforward of $P_{U,n}$ by 3.4 (a), and similarly for $P^\vee$. Since the diagram

$$\begin{array}{ccc} X \times Y \times X & \hookleftarrow & U_{X,n} \times Y \times U_{X,n} \\ \downarrow{\scriptstyle \mathrm{pr}_{13}} & & \downarrow{\scriptstyle \mathrm{pr}_{13}} \\ X \times X & \hookleftarrow & U_{X,n} \times U_{X,n} \end{array}$$

is cartesian we conclude that the pushforward of $p_{12}^* P_{U,n} \otimes p_{23}^* P_{U,n}^\vee$ along the map

$$p_{13} : U_{X,n} \times U_{Y,n} \times U_{X,n} \to U_{X,n} \times U_{X,n}$$

is isomorphic to $\Delta_{U_{X,n}*}\mathcal{O}_{U_{X,n}}$. It follows that $\Phi_{U,n}^\vee \circ \Phi_{U,n} \simeq \mathrm{id}_{D^b_{\mathrm{coh}}(U_{X,n})}$. The isomorphism

$$\Phi_{U,n} \circ \Phi_{U,n}^\vee \simeq \mathrm{id}_{D^b_{\mathrm{coh}}(U_{Y,n})}$$

is shown similarly.

The proof that $\Phi_{W,n}$ is an equivalence follows verbatim from the preceding argument replacing "U" by "W" everywhere. □

3.20. Proof of 3.4 (b)

3.21. First recall Beilinson's resolution of the diagonal on a projective space $\mathbf{P}(V)$ [2]. This resolution takes the form (let d denote the dimension of $\mathbf{P}(V)$)

$$\begin{aligned} p_1^*\mathcal{O}_{\mathbf{P}(V)}(-d) \otimes p_2^*\Omega^d_{\mathbf{P}(V)}(d) \to \cdots \\ \to p_1^*\mathcal{O}_{\mathbf{P}(V)}(-1) \otimes p_2^*\Omega^1_{\mathbf{P}(V)}(1) \to \mathcal{O}_{\mathbf{P}(V)\times\mathbf{P}(V)} \to \mathcal{O}_\Delta. \end{aligned}$$

The transition maps are obtained as follow. We have

$$\begin{aligned} &\mathrm{Hom}(p_1^*\mathcal{O}_{\mathbf{P}(V)}(-i) \otimes p_2^*\Omega^i_{\mathbf{P}(V)}(i), p_1^*\mathcal{O}_{\mathbf{P}(V)}(-i+1) \otimes p_2^*\Omega^{i-1}_{\mathbf{P}(V)}(i-1)) \\ &\simeq \mathrm{Hom}_{\mathbf{P}(V)}(\mathcal{O}_{\mathbf{P}(V)}(-i), \mathcal{O}_{\mathbf{P}(V)}(-i+1)) \otimes \mathrm{Hom}_{\mathbf{P}(V)}(\Omega^i_{\mathbf{P}(V)}(i), \Omega^{i-1}_{\mathbf{P}(V)}(i-1)) \\ &\simeq V \otimes \mathrm{Hom}_{\mathbf{P}(V)}(\Omega^i_{\mathbf{P}(V)}(i), \Omega^{i-1}_{\mathbf{P}(V)}(i-1)). \end{aligned}$$

From the twisted Euler sequence

$$0 \to \mathcal{O}_{\mathbf{P}(V)}(-1) \to V^\vee \otimes_k \mathcal{O} \to T_{\mathbf{P}(V)}(-1) \to 0$$

we obtain an isomorphism

$$V^\vee \simeq H^0(\mathbf{P}(V), T_{\mathbf{P}(V)}(-1)).$$

Together with the natural map

$$V^\vee \simeq H^0(\mathbf{P}(V), T_{\mathbf{P}(V)}(-1)) \to \mathrm{Hom}_{\mathbf{P}(V)}(\Omega^i_{\mathbf{P}(V)}(i), \Omega^{i-1}_{\mathbf{P}(V)}(i-1)) \tag{3.21.1}$$

we then get a map

$$V \otimes V^\vee \to \mathrm{Hom}(p_1^*\mathcal{O}_{\mathbf{P}(V)}(-i) \otimes p_2^*\Omega^i_{\mathbf{P}(V)}(i), p_1^*\mathcal{O}_{\mathbf{P}(V)}(-i+1) \otimes p_2^*\Omega^{i-1}_{\mathbf{P}(V)}(i-1)).$$

The image of the identity class in $V \otimes V^\vee$ defines under this map the differential in the Beilinson resolution.

3.22. Returning to the proof of 3.4 (b), let $\mathbf{P} \subset \mathbf{P}H_{X,n}$ be the closure of the image of $W_{X,n}$, viewed as a scheme with the reduced-induced structure, and let

$$f : W_{X,n} \to \mathbf{P}, \quad g : W_{Y,n} \to \mathbf{P}$$

be the natural maps.

3.23. Consider first the case when k is infinite.

In this case, for a suitable subspace $V \subset H_{X,n}$ the induced rational map

$$\mathbf{P} \dashrightarrow \mathbf{P}(V)$$

is everywhere defined, finite, and generically étale (see for example [5, 2.11]).

Let

$$f' : W_{X,n} \to \mathbf{P}(V), \quad g' : W_{Y,n} \to \mathbf{P}(V)$$

be the induced maps, and let $E(f', g')^\bullet$ be the associated cosimplicial scheme. We write

$$d'_i : W_{X,n} \times W_{Y,n} \to E(f', g')^1 = W_{X,n} \times \mathbf{P}(V) \times W_{Y,n}$$

for the structure maps in this cosimplicial scheme (and similarly for other maps occurring in the cosimplicial structure).

With the notation of Paragraph 3.8 we have

$$f'^*\mathcal{O}_{\mathbf{P}(V)}(1) = \Sigma_{X,n}|_{W_{X,n}}, \quad g'^*\mathcal{O}_{\mathbf{P}(V)}(1) = \Sigma_{Y,n}|_{W_{Y,n}}.$$

To ease notation we will write $\Sigma_{W_{X,n}}$ and $\Sigma_{W_{Y,n}}$ for these sheaves in the remainder of the proof. Note that these are line bundles on $W_{X,n}$ and $W_{Y,n}$.

For an open set $B \subset \mathbf{P}(V)$ we can also consider the restrictions

$$f'_B : W_{X,n,B} \to B, \quad g'_B : W_{Y,n,B} \to B,$$

and the associated cosimplicial scheme

$$E(f'_B, g'_B) \hookrightarrow E(f', g').$$

3.24. We have a cartesian diagram

$$\begin{array}{ccc} W_{X,n} & \longrightarrow & W_{X,n} \times \mathbf{P}(V) \\ \downarrow & & \downarrow \\ \mathbf{P}(V) & \xrightarrow{\Delta} & \mathbf{P}(V) \times \mathbf{P}(V). \end{array}$$

Pulling back the Beilinson resolution of the diagonal of $\mathbf{P}(V)$ we obtain a complex on $W_{X,n} \times \mathbf{P}(V)$ of the form

$$\begin{aligned} (3.24.1) \quad & p_1^* \Sigma_{W_{X,n}}^{\otimes(-d)} \otimes p_2^* \Omega^d_{\mathbf{P}(V)}(d) \to \cdots \\ & \to p_1^* \Sigma_{W_{X,n}}^{\otimes(-1)} \otimes p_2^* \Omega^1_{\mathbf{P}(V)}(1) \to \mathcal{O}_{W_{X,n} \times \mathbf{P}(V)}. \end{aligned}$$

Over the locus in $W_{X,n}$ where the map f' is flat this is a resolution of $\mathcal{O}_{\gamma_f}$, where $\gamma_f : W_{X,n} \to W_{X,n} \times \mathbf{P}(V)$ is the graph of f. Pulling the complex (3.24.1) back to $W_{X,n} \times \mathbf{P}(V) \times W_{Y,n}$ along the first two projections we get a complex on $E(f', g')^1$, which over the preimage of the flat locus of f' is a resolution of $d_{0*}\mathcal{O}_{W_{X,n} \times W_{Y,n}}$.

3.25. For $s \leq 0$ set

$$(3.25.1) \qquad P_X^s := p_{13}^*(p_1^* \Sigma_{W_{X,n}}^{\otimes s} \otimes P) \otimes p_2^* \Omega^{-s}_{\mathbf{P}(V)}(-s),$$

an object of $D^b_{\mathrm{coh}}(W_{X,n} \times \mathbf{P}(V) \times Y)$, and let $P_X^s \to P_X^{s+1}$ be the maps induced by the maps in (3.24.1).

Projecting along $W_{X,n} \times \mathbf{P}(V) \times Y \to W_{X,n}$ we find

$$\begin{aligned} & \mathbf{R}\mathrm{Hom}_{W_{X,n} \times \mathbf{P}(V) \times Y}(P_X^i, P_X^j) \\ & \simeq \mathbf{R}\Gamma(W_{X,n} \times \mathbf{P}(V) \times Y, p_1^* \Sigma_{W_{X,n}}^{\otimes(j-i)} \otimes p_2^*[\Omega^{-i}_{\mathbf{P}(V)}(-i), \Omega^{-j}_{\mathbf{P}(V)}(-j)] \otimes p_{13}^*[P, P]) \\ & \simeq \mathbf{R}\Gamma(W_{X,n}, \Sigma_{W_{X,n}}^{\otimes(j-i)} \otimes \mathbf{R}p_{1*}[P, P] \otimes_k \mathbf{R}\mathrm{Hom}(\Omega^{-i}_{\mathbf{P}(V)}(-i), \Omega^{-j}_{\mathbf{P}(V)}(-j)), \end{aligned}$$

where to ease the notation we write $[-,-]$ for $\mathbf{R}\mathscr{H}om(-,-)$. Now we have

$$\mathbf{R}p_{1*}\,\mathbf{R}\mathscr{H}om(P,P) \in D^{\geq 0}(W_{X,n})$$

and the natural map

$$\mathcal{O}_{W_{X,n}} \to R^0 p_{1*}\,\mathbf{R}\mathscr{H}om(P,P)$$

is an isomorphism (see for example the discussion in [8, Remark 5.1]). We conclude that

$$\mathrm{Ext}^s(P_X^i, P_X^j) = 0$$

for $s < 0$. Moreover, we have

$$\begin{aligned}\mathrm{Hom}(P_X^s, P_X^{s+1}) &\simeq \Gamma(W_{X,n}, \Sigma_{W_{X,n}}) \otimes \mathrm{Hom}_{\mathbf{P}(V)}(\Omega^{-s}(-s), \Omega^{-(s+1)}(-(s+1))) \\ &\hookrightarrow \Gamma(X, K_X^{\otimes n}) \otimes \mathrm{Hom}_{\mathbf{P}(V)}(\Omega^{-s}(-s), \Omega^{-(s+1)}(-(s+1))).\end{aligned}$$

Using the map (3.21.1) we get a map

$$\Gamma(X, K_X^{\otimes n}) \otimes V^\vee \to \Gamma(X, K_X^{\otimes n}) \otimes \mathrm{Hom}_{\mathbf{P}(V)}(\Omega^{-s}(-s), \Omega^{-(s+1)}(-(s+1))).$$

The image of the class $\lambda_X \in \Gamma(X, K_X^{\otimes n}) \otimes V^\vee$ adjoint to the inclusion $V \to \Gamma(X, K_X^{\otimes n})$ then equals the class of the differential $P_X^s \to P_X^{s+1}$.

By [10, 1.4] the complex $P_X^\bullet$ in $D^b_{\mathrm{coh}}(W_{X,n} \times \mathbf{P}(V) \times Y)$ is induced by a unique object $\mathscr{P}_X \in DF(W_{X,n} \times \mathbf{P}(V) \times Y)$ of the filtered derived category.

Note that by 3.17 this complex is supported on $W_{X,n} \times \mathbf{P}(V) \times W_{Y,n}$, and we view $\mathscr{P}_X$ more symmetrically as an object of $DF(W_{X,n} \times \mathbf{P}(V) \times W_{Y,n})$.

3.26. We can also interchange X and Y and define

$$P_Y^s := p_{13}^*(P \otimes p_2^* \Sigma_{W_{Y,n}}^{\otimes s}) \otimes p_2^* \Omega^{-s} \Omega_{\mathbf{P}(V)}(-s).$$

Using the isomorphism (3.2.1) we view P_Y^s as an object in $D^b_{\mathrm{coh}}(X \times \mathbf{P}(V) \times W_{Y,n})$. As above we then get an object $\mathscr{P}_Y \in DF(W_{X,n} \times \mathbf{P}(V) \times W_{Y,n})$.

The isomorphism constructed in the proof of Proposition 3.9

$$P \otimes p_1^* \Sigma_{X,n} \simeq P \otimes p_2^* \Sigma_{Y,n}$$

induces isomorphisms

$$\lambda_s : P_X^s \to P_Y^s.$$

These isomorphisms are compatible with the transition maps (this follows from the construction of the isomorphism (3.2.1)) and therefore we get an

isomorphism of complexes

$$\lambda_\bullet : P_X^\bullet \to P_Y^\bullet.$$

By [10, 1.3] this is induced by a unique isomorphism

$$\lambda : \mathscr{P}_X \to \mathscr{P}_Y$$

in $DF(W_{X,n} \times \mathbf{P}(V) \times W_{Y,n})$. If $B \subset \mathbf{P}(V)$ is an open set over which f'_B and g'_B are flat this induces an isomorphism

$$\lambda : d'^*_0 P|_{W_{X,n,B} \times W_{Y,n,B}} \to d'^*_1 P|_{W_{X,n,B} \times W_{Y,n,B}}$$

in $D^b_{\mathrm{coh}}(W_{X,n,B} \times B \times W_{Y,n,B})$.

Lemma 3.27. *If $B \subset \mathbf{P}(V)$ is an open set over which f' and g' are flat, then the following hold (let $\mathrm{pr}_3 : W_{X,n,B} \times B \times W_{Y,n,B} \to W_{Y,n,B}$ denote the projection to the third factor):*

(i) $\mathbf{R}^i \mathrm{pr}_{3*}\, \mathbf{R}\mathscr{H}om(d'^*_0 P, d'^*_1 P) = 0$ *for* $i < 0$.

(ii) *The natural maps*

$$\mathbf{R}^0 \mathrm{pr}_{3*}\, \mathbf{R}\mathscr{H}om(d'^*_0 P, d'^*_1 P) \longrightarrow \mathbf{R}^0 \mathscr{H}om(\mathbf{R}\mathrm{pr}_{3*} d'^*_0 P, \mathbf{R}\mathrm{pr}_{3*} d'^*_1 P) \longleftarrow \mathscr{O}_{W_{Y,n,B}}$$

are isomorphisms, where the second map is obtained from the identification $\mathrm{pr}_3 \circ d'_1 \simeq \mathrm{pr}_3 \circ d'_0$.

(iii) *The isomorphism λ satisfies the cocycle condition on $E(f_B, g_B)^2$.*

Proof. Since $d'^*_0 P \simeq d'^*_1 P$, to prove (i) and (ii) it suffices to prove the analogous statements with $d'^*_0 P$ replaced by $d'^*_1 P$.

Consider the diagram

$$\begin{array}{ccccc}
W_{X,n,B} \times W_{Y,n,B} & \xrightarrow{d'_1} & W_{X,n,B} \times B \times W_{Y,n,B} & & \\
\downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle p_{23}} & \searrow^{p_3} & \\
W_{Y,n,B} & \xrightarrow{\quad j \quad} & B \times W_{Y,n,B} & & \\
 & & \downarrow{\scriptstyle p_2} & & \\
 & & W_{Y,n,B}, & &
\end{array}$$

where j is the graph of g'_B. Note that we have an isomorphism

$$\mathbf{L}p_{13}^*P \otimes^{\mathbf{L}} d'_{1*}\mathcal{O}_{W_{X,n,B}\times W_{Y,n,B}} \simeq d'_{1*}P$$

which upon applying $\mathbf{L}d_1'^*$ yields an isomorphism

$$P \otimes^{\mathbf{L}} \mathbf{L}d_1'^* d'_{1*}\mathcal{O}_{W_{X,n,B}\times W_{Y,n,B}} \simeq \mathbf{L}d_1'^* d'_{1*}P.$$

This defines isomorphisms

$$\begin{aligned}\mathbf{R}\mathcal{H}om(d'_{1*}P, d'_{1*}P) &\simeq d'_{1*}\,\mathbf{R}\mathcal{H}om(\mathbf{L}d_1'^* d'_{1*}P, P)\\ &\simeq d'_{1*}\,\mathbf{R}\mathcal{H}om(P \otimes^{\mathbf{L}} \mathbf{L}d_1'^* d'_{1*}\mathcal{O}_{W_{X,n,B}\times W_{Y,n,B}}, P)\\ &\simeq d'_{1*}(\mathbf{R}\mathcal{H}om(P,P) \otimes^{\mathbf{L}} (\mathbf{L}d_1'^* d'_{1*}\mathcal{O}_{W_{X,n,B}\times W_{Y,n,B}})^\vee).\end{aligned}$$

We then have

$$\begin{aligned}&\mathbf{R}p_{23*}\,\mathbf{R}\mathcal{H}om(d'_{1*}P, d'_{1*}P)\\ &\simeq \mathbf{R}p_{23*}(d'_{1*}(\mathbf{R}\mathcal{H}om(P,P) \otimes^{\mathbf{L}} (\mathbf{L}d_1'^* d'_{1*}\mathcal{O}_{W_{X,n,B}\times W_{Y,n,B}})^\vee))\\ &\simeq j_*((\mathbf{R}p_{2*}\,\mathbf{R}\mathcal{H}om(P,P)) \otimes^{\mathbf{L}} (\mathbf{L}j^* j_*\mathcal{O}_{W_{Y,n,B}})^\vee)\end{aligned}$$

and the natural map

$$j_*\mathcal{O}_{W_{Y,n,B}} \to j_*((\mathbf{R}p_{2*}\,\mathbf{R}\mathcal{H}om(P,P)) \otimes^{\mathbf{L}} (\mathbf{L}j^* j_*\mathcal{O}_{W_{Y,n,B}})^\vee)$$

is an isomorphism in degrees ≤ 0. Pushing forward to $W_{Y,n,B}$ we get statements (i) and (ii). Note that under these identifications the element $1 \in k$ corresponds to the previously constructed isomorphism λ.

To complete the proof of it remains to show that the map λ satisfies the cocycle condition on $E(f'_B, g'_B)^2$. For this note that the preceding argument shows that the map

$$\mathrm{Hom}_{D^b_{\mathrm{coh}}(E(f'_B,g'_B)^2)}(\tau_{2*}P, \tau_{0*}P) \to \mathrm{Hom}_{D^b_{\mathrm{coh}}(E(f'_B,g'_B)^0)}(P,P)$$

induced by pushing forward along the map $E(f'_B, g'_B)^2 \to E(f'_B, g'_B)^0$ given by the unique map $[2] \to [0]$ is an isomorphism. Since the pushforward of the map λ is the identity on P this implies that the cocycle condition holds. □

3.28. From this and 1.1 we conclude that there exists a dense open subset $B \subset \mathbf{P}(V)$ such that the restriction of P to $W_{X,n,B} \times W_{Y,n,B}$ is induced by pushforward from $W_{X,n,B} \times_B W_{Y,n,B}$. Let $A \subset \mathbf{P}$ be the preimage of B and

assume further that B is chosen such that $\mathbf{P} \to \mathbf{P}(V)$ is étale over B. Then (note that with our notation we have $W_{X,n,A} = W_{X,n,B}$)

$$W_{X,n,A} \times_A W_{Y,n,A} \hookrightarrow W_{X,n,B} \times_B W_{Y,n,B}$$

is open and closed and a complex on $W_{X,n,B} \times_B W_{Y,n,B}$ inducing P is necessarily supported on $W_{X,n,A} \times_A W_{Y,n,A}$ (since we know that $P|_{W_{X,n,A} \times W_{Y,n,A}}$ is set-theoretically supported on $W_{X,n,A} \times_A W_{Y,n,A}$). It follows that $P|_{W_{X,n,A} \times W_{Y,n,A}}$ is, in fact, the pushforward of a complex on $W_{X,n,A} \times_A W_{Y,n,A}$.

This completes the proof of 3.4 (b) in the case of infinite k.

3.29. To handle the case of finite k, note the following variant of 3.27 above. For an open subset $A \subset \mathbf{P}$ let $E(f_A, g_A)^\bullet$ be the cosimplicial scheme associated to the maps

$$f_A : W_{X,n,A} \to A, \quad g_A : W_{Y,n,A} \to A.$$

Lemma 3.30. *There exists a dense open subset $A \subset \mathbf{P}$ such that the following hold (let $\mathrm{pr}_3 : W_{X,A} \times A \times W_{Y,n,A} \to W_{Y,n,A}$ denote the projection to the third factor)*

(i) $\mathbf{R}^i \mathrm{pr}_{3*}\, \mathbf{R}\mathscr{H}om(d_{0*}P, d_{1*}P) = 0$ *for* $i < 0$.

(ii) *The natural maps*

$$\mathbf{R}^0 \mathrm{pr}_{3*}\, \mathbf{R}\mathscr{H}om(d_{0*}P, d_{1*}P) \longrightarrow \mathbf{R}^0\mathscr{H}om(\mathbf{R}\mathrm{pr}_{3*}d_{0*}P, \mathbf{R}\mathrm{pr}_{3*}d_{1*}P) \longleftarrow \mathcal{O}_{W_{Y,n,A}}$$

are isomorphisms, where the second map is obtained from the identification $\mathrm{pr}_3 \circ d_1 \simeq \mathrm{pr}_3 \circ d_0$.

(iii) *The map $\lambda : d_{0*}P \to d_{1*}P$, obtained from the isomorphisms in (ii) and the section $1 \in \Gamma(W_{Y,n,A}, \mathcal{O}_{W_{Y,n,A}})$ is an isomorphism and satisfies the cocycle condition on $E(f_A, g_A)^2$.*

Proof. It suffices to verify the lemma after passing to a field extension of k. By the case of an infinite field we may therefore assume that there exists an open subset A such that $P|_{W_{X,n,A} \times W_{Y,n,A}}$ is the pushforward of a complex on $W_{X,n,A} \times_A W_{Y,n,A}$. In particular, we may assume that we have an isomorphism $d_{0*}P \simeq d_{1*}P$. The proof now proceeds as in the proof of 3.27. □

Combining this with 1.1 we then obtain 3.4 (b) in the case of finite k as well. □

4. Rouquier functors

In this section we explain how Rouquier's work [14] can be combined with our main result on support of complexes to obtain restrictions on kernels of derived equivalences. This is also related to work of Lombardi [9].

4.1. The Albanese torsor

4.2. Let k be a perfect field and let X/k be a smooth projective variety. Let $\mathscr{P}ic_X$ denote the $\mathbf{G}_m$-gerbe over the Picard scheme Pic_X classifying line bundles on X, and set

$$\mathscr{P}ic^0_X := \mathrm{Pic}^0_X \times_{\mathrm{Pic}_X} \mathscr{P}ic_X.$$

We assume that Pic^0_X is a smooth scheme (this is automatic in characteristic 0), and therefore an abelian variety, and write Alb_X for the dual abelian scheme.

4.3. For a smooth projective variety X/k let $\mathbf{T}^0_X$ denote the functor which to any k-scheme T associates the set of isomorphism classes of morphisms of Picard stacks

$$s : \mathrm{Pic}^0_{X,T} \to \mathscr{P}ic^0_{X,T}$$

over the identity. Observe that any two such sections differ by a morphism of Picard stacks

$$\rho : \mathrm{Pic}^0_{X,T} \to B\mathbf{G}_{m,T}.$$

Considering the commutative diagram

$$\begin{array}{ccc} \mathrm{Pic}^0_{X,T} \times \mathrm{Pic}^0_{X,T} & \xrightarrow{m} & \mathrm{Pic}^0_{X,T} \\ \downarrow{\scriptstyle \rho\times\rho} & & \downarrow{\scriptstyle \rho} \\ B\mathbf{G}_{m,T} \times B\mathbf{G}_{m,T} & \xrightarrow{m_{B\mathbf{G}_m}} & B\mathbf{G}_{m,T} \end{array}$$

and the fact that for the line bundle $\mathscr{M}$ on $B\mathbf{G}_{m,T}$ corresponding to the standard character of $\mathbf{G}_m$ we have

$$m^*_{B\mathbf{G}_m}(\mathscr{M}) \simeq \mathscr{M} \boxtimes \mathscr{M}$$

it follows that ρ corresponds to a line bundle $\mathscr{L}$ on $\mathrm{Pic}^0_{X,T}$ which is translation invariant; that is, a point of

$$\mathrm{Alb}_X := \mathrm{Pic}^0_{\mathrm{Pic}^0_X}.$$

Note also that a point $x \in X(k)$ yields a section s. Indeed given x we can interpret Pic^0_X as classifying pairs $(\mathscr{L}, \sigma)$ consisting of a line bundle $\mathscr{L}$ on X and a trivialization $\sigma : \mathscr{L}(x) \simeq \kappa(x)$. From this it follows that $\mathbf{T}^0_X$ is a torsor under Alb_X and there is a natural morphism

$$c_X : X \to \mathbf{T}^0_X.$$

If we trivialize $\mathbf{T}^0_X$ using a point of X then this is identified with the usual map from X to its Albanese.

Note also that we have a canonical isomorphism (this amounts to the fact that the translation action of an abelian variety A on $\mathrm{Pic}^0(A)$ is trivial)

$$\mathrm{Pic}^0_{\mathbf{T}^0_X} \simeq \mathrm{Pic}^0_{\mathrm{Alb}_X}$$

and therefore an isomorphism

$$\mathrm{Pic}^0_{\mathbf{T}^0_X} \simeq \mathrm{Pic}^0_X.$$

Chasing through these identifications one finds that this is simply given by

$$c_X^* : \mathrm{Pic}^0_{\mathbf{T}^0_X} \to \mathrm{Pic}^0_X.$$

4.4. We say that an autoequivalence

$$\alpha : D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(X)$$

satisfies the *Rouquier condition* R_X if the complex $Q_\alpha \in D^b_{\mathrm{coh}}(X \times X)$ defining α is isomorphic to $\Gamma_{\sigma *}\mathscr{L}$, where $\Gamma_\sigma : X \to X \times X$ is the graph $x \mapsto (x, \sigma(x))$ of an automorphism σ of X and $\mathscr{L}$ is an invertible sheaf on X numerically equivalent to 0.

Let $\mathscr{R}^0_X$ be the fibered category which to any k-scheme T associates the groupoid of objects $Q \in D^b_{\mathrm{coh}}((X \times X)_T)$ of T-perfect complexes such that for all geometric points $\bar{t} \to T$ the fiber $Q_{\bar{t}} \in D^b_{\mathrm{coh}}((X \times X)_{k(\bar{t})})$ defines an equivalence $D^b_{\mathrm{coh}}(X_{\bar{t}}) \to D^b_{\mathrm{coh}}(X_{\bar{t}})$ satisfying R_X and whose associated automorphism $X_{\bar{t}} \to X_{\bar{t}}$ lies in the connected component of the identity in $\mathrm{Aut}(X)$. Let $\mathbf{R}^0_X$ denote the group scheme

$$\mathbf{R}^0_X := \mathrm{Pic}^0_X \times \mathrm{Aut}^0_X.$$

Then $\mathscr{R}^0_X$ is a $\mathbf{G}_m$-gerbe over $\mathbf{R}^0_X$.

The key result of Rouquier that we will need is the following:

Theorem 4.5 (Rouquier). *Let Y/k be a second smooth projective variety related to X by an equivalence $\Phi : D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(Y)$.*

(i) For any T/k and $Q \in D^b_{\mathrm{coh}}((X \times X)_T)$ in $\mathscr{R}^0_X(T)$ the complex in $D^b_{\mathrm{coh}}((Y \times Y)_T)$ associated to the auto-equivalence $\Phi \circ \Phi_Q \circ \Phi^{-1}$ of $D^b_{\mathrm{coh}}(Y)$ is in $\mathscr{R}^0_Y(T)$.

(ii) The induced functor

$$\tilde{\tau} : \mathscr{R}^0_X \to \mathscr{R}^0_Y \tag{4.5.1}$$

is an equivalence of gerbes.

Proof. See [14, 4.18]. □

By passing to coarse moduli spaces the equivalence (4.5.1) induces an isomorphism

$$\tau : \mathbf{R}^0_X \to \mathbf{R}^0_Y.$$

Assumption 4.6. *We assume for the rest of this section that* Pic^0_X *is reduced and that* τ *takes* Pic^0_X *to* Pic^0_Y.

Remark 4.7. This assumption holds in many instances of interest.

(i) If k has characteristic 0 then the assumption that Pic^0_X is reduced is automatic.

(ii) The assumption that Pic^0_X is reduced implies that it is an abelian variety. If this holds and furthermore Aut^0_Y is affine, then automatically Pic^0_X is mapped to Pic^0_Y.

(iii) In characteristic 0 the condition that Pic^0_X is taken to Pic^0_Y can be checked on Hochschild cohomology. The map on tangent spaces at the identity of the morphism τ is a map

$$T\tau : H^1(X, \mathcal{O}_X) \oplus H^0(X, T_X) \to H^1(Y, \mathcal{O}_Y) \oplus H^0(Y, T_Y).$$

Using the HKR isomorphism this map is identified with the map on Hochschild cohomology

$$HH^1(X) \simeq HH^1(Y).$$

4.8. Under this assumption the map $\tilde{\tau}$ induces an isomorphism of Picard stacks

$$\tilde{\gamma} : \mathcal{P}ic^0_X \to \mathcal{P}ic^0_Y$$

over an isomorphism of abelian varieties

$$\gamma : \mathrm{Pic}^0_X \to \mathrm{Pic}^0_Y.$$

It therefore also induces an isomorphism of torsors of sections

$$\rho : \mathbf{T}^0_X \to \mathbf{T}^0_Y$$

compatible with the isomorphism

$$\gamma^t : \mathrm{Alb}_X \to \mathrm{Alb}_Y.$$

The main result of this section is the following:

Theorem 4.9. *Let P denote a kernel for the equivalence Φ. Then there exists a dense open subset $A \subset \mathbf{T}^0_Y$ such that the restriction of P to $c_X^{-1}\rho^{-1}(A) \times c_Y^{-1}(A) \subset X \times Y$ is isomorphic to the image of an object in*

$$D^b_{\mathrm{coh}}(c_X^{-1}\rho^{-1}(A) \times_A c_Y^{-1}(A)) \to D^b_{\mathrm{coh}}(c_X^{-1}\rho^{-1}(A) \times c_Y^{-1}(A)).$$

The proof occupies the remainder of the section.

Lemma 4.10. *Let $\mathcal{L}^u_X$ (resp. $\mathcal{L}^u_Y$) be the universal line bundle on $X \times \mathcal{P}ic^0_X$ (resp. $\mathcal{P}ic^0_Y \times Y$). Then we have a canonical isomorphism*

$$p_{12}^*\mathcal{L}^u_X \otimes^{\mathbf{L}} p_{13}^*P \simeq (1_X \times \tilde{\gamma} \times 1_Y)^*(p_{23}^*\mathcal{L}^u_Y \otimes^{\mathbf{L}} p_{13}^*P) \tag{4.10.1}$$

in $D^b_{\mathrm{coh}}(X \times \mathcal{P}ic^0_X \times Y)$.

Proof. To ease notation let us write $\mathcal{P}_X$ (resp. $\mathcal{P}_Y$) for $\mathcal{P}ic^0_X$ (resp. $\mathcal{P}ic^0_Y$). The isomorphism $\tilde{\gamma}$ is characterized by the condition that the complex in

$D^b_{\mathrm{coh}}((Y \times Y)_{\mathscr{P}_X})$ representing the autoequivalence

$$\Phi \circ (\otimes \mathscr{L}^u_X) \circ \Phi^{-1} \tag{4.10.2}$$

is isomorphic to $\Delta_{Y*}(1_Y \times \tilde{\gamma})^* \mathscr{L}^u_Y$. Consider the cartesian square

$$\begin{array}{ccc} (X \times Y)_{\mathscr{P}_X} & \xrightarrow{(x,y)\mapsto(y,x,y)} & (Y \times X \times Y)_{\mathscr{P}_X} \\ \downarrow{\scriptstyle q_2} & & \downarrow{\scriptstyle q_{13}} \\ Y_{\mathscr{P}_X} & \xrightarrow{\Delta_Y} & (Y \times Y)_{\mathscr{P}_X}, \end{array}$$

where we write q_- for the various projections (to distinguish from the projections p_- from $X \times \mathscr{P}ic^0_X \times Y$ in the statement of the lemma). Then (4.10.2) is represented by the complex

$$\mathbf{R}q_{13*}(q_2^* \mathscr{L}^u_X \otimes^{\mathbf{L}} q_{12}^* P \otimes^{\mathbf{L}} q_{23}^* P^\vee \otimes q_3^* \omega_Y[\dim(Y)]) \in D^b_{\mathrm{coh}}((Y \times Y)_{\mathscr{P}_X}).$$

The characterizing isomorphism

$$\mathbf{R}q_{13*}(q_2^* \mathscr{L}^u_X \otimes^{\mathbf{L}} q_{12}^* P \otimes^{\mathbf{L}} q_{23}^* P^\vee \otimes q_3^* \omega_Y[\dim(Y)]) \simeq \Delta_{Y*}(1_Y \times \tilde{\gamma})^* \mathscr{L}^u_Y \tag{4.10.3}$$

corresponds by adjunction to a map

$$(q_2^* \mathscr{L}^u_X \otimes^{\mathbf{L}} q_{12}^* P \otimes^{\mathbf{L}} q_{23}^* P^\vee \otimes q_3^* \omega_Y[\dim(Y)]) \to q_{13}^! (\Delta_{Y*}(1_Y \times \tilde{\gamma})^* \mathscr{L}^u_Y).$$

Using the isomorphism

$$q_{13}^!(-) \simeq q_{13}^*(-) \otimes^{\mathbf{L}} q_2^* \omega_X[\dim(X)]$$

we find that the characterizing isomorphism (4.10.3) corresponds by adjunction to a morphism

$$(q_2^* \mathscr{L}^u_X \otimes^{\mathbf{L}} q_{12}^* P \otimes^{\mathbf{L}} q_{23}^* P^\vee \otimes q_3^* \omega_Y[\dim(Y)]) \to q_{13}^* (\Delta_{Y*}(1_Y \times \tilde{\gamma})^* \mathscr{L}^u_Y) \otimes^{\mathbf{L}} q_2^* \omega_X[\dim(X)]$$

in $D^b_{\mathrm{coh}}((Y \times X \times Y)_{\mathscr{P}_X})$. Using the isomorphism

$$P \otimes p_2^* \omega_Y[\dim(Y)] \simeq P \otimes p_1^* \omega_X[\dim(X)]$$

and adjunction this, in turn, corresponds to a map

$$p_{12}^*\mathscr{L}_X^u \otimes^{\mathbf{L}} p_{13}^*P \to (1_X \times \tilde{\gamma} \times 1_Y)^*(p_{23}^*\mathscr{L}_Y^u \otimes^{\mathbf{L}} p_{13}^*P) \tag{4.10.4}$$

in $D^b_{\mathrm{coh}}(X \times \mathscr{P}ic^0_X \times Y)$. If we restrict this to a fiber over a point $[\mathscr{L}] \in \mathscr{P}ic^0_X$ corresponding to a line bundle $\mathscr{L}$ on X then the left side of (4.10.4) defines the functor $\Phi \circ (\otimes\mathscr{L}) : D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(Y)$ and the right side defines the functor $(\Phi \circ (\otimes\mathscr{L}) \circ \Phi^{-1}) \circ \Phi$ and our map is the natural isomorphism between them. By Orlov's theorem it follows that (4.10.4) defines an isomorphism in all fibers over $\mathscr{P}ic^0_X$ and therefore is an isomorphism. □

Lemma 4.11. *Let S be a noetherian scheme and let $\mathscr{F} \in D^b_{\mathrm{coh}}(\mathbf{T}^0_{X,S})$ be a complex with associated complex $\mathscr{F}^\rho \in D^b_{\mathrm{coh}}(\mathbf{T}^0_{Y,S})$. Then we have*

$$p_{12}^*c_{X,S}^*\mathscr{F} \otimes^{\mathbf{L}} p_{13}^*P \simeq p_{23}^*c_{Y,S}^*\mathscr{F}^\rho \otimes^{\mathbf{L}} p_{13}^*P \tag{4.11.1}$$

in $D^b_{\mathrm{coh}}(X \times S \times Y)$.

Proof. Note that the diagram

$$\begin{array}{ccc} \mathscr{P}ic^0_{\mathbf{T}^0_X} & \xrightarrow{c_X^*} & \mathscr{P}ic^0_X \\ \downarrow & & \downarrow \\ \mathrm{Pic}^0_{\mathbf{T}^0_X} & \xrightarrow{c_X^*} & \mathrm{Pic}^0_X \end{array}$$

is cartesian, and identifies $\mathbf{T}^0_X$ with the $\mathbf{G}_m$-torsor of sections of

$$\mathscr{P}ic^0_{\mathbf{T}^0_X} \to \mathrm{Pic}^0_{\mathbf{T}^0_X}.$$

In particular, there is a universal line bundle $\mathscr{M}_X^u$ on $\mathrm{Pic}^0_{\mathbf{T}^0_X} \times \mathbf{T}^0_X$. Similarly there is a universal line bundle $\mathscr{M}_Y^u$ over $\mathrm{Pic}^0_{\mathbf{T}^0_Y} \times \mathbf{T}^0_Y$, and the isomorphism

$$((\rho^*)^{-1} \times \rho) : \mathrm{Pic}^0_{\mathbf{T}^0_X} \times \mathbf{T}^0_X \to \mathrm{Pic}^0_{\mathbf{T}^0_Y} \times \mathbf{T}^0_Y$$

comes equipped with an isomorphism

$$((\rho^*)^{-1} \times \rho)^*\mathscr{M}_Y^u \simeq \mathscr{M}_X^u.$$

The functor

$$D^b_{\mathrm{coh}}(\mathrm{Pic}^0_{X,S}) \to D^b_{\mathrm{coh}}(\mathbf{T}^0_{X,S}), \quad \mathscr{G} \mapsto \mathbf{R}p_{2*}(p_1^*\mathscr{G} \otimes^{\mathbf{L}} \mathscr{M}^u_{X,S})$$

is an equivalence of categories, where we write $\mathscr{M}^u_{X,S}$ for the pullback of $\mathscr{M}^u_X$ to $(\mathrm{Pic}^0_{\mathbf{T}^0_X} \times \mathbf{T}^0_X)_S$. Indeed this can be verified after making a field extension, where it reduces to the standard derived equivalence between an abelian variety and its dual. In particular, we can write

$$\mathscr{F} = \mathbf{R}p_{2*}(p_1^*\mathscr{G} \otimes^{\mathbf{L}} \mathscr{M}^u_{X,S})$$

for a unique object $\mathscr{G} \in D^b_{\mathrm{coh}}(\mathrm{Pic}^0_{X,S})$. Note also that if $\mathscr{G}^\rho \in D^b_{\mathrm{coh}}(\mathrm{Pic}^0_{Y,S}))$ is the complex corresponding to $\mathscr{G}$ under the isomorphism

$$\gamma : \mathrm{Pic}^0_X \to \mathrm{Pic}^0_Y$$

then $\mathscr{G}^\rho$ transforms to $\mathscr{F}^\rho$ on $\mathbf{T}^0_Y$ under the equivalence defined by $\mathscr{M}^u_Y$.

Consider the diagram

$$\begin{array}{ccccc}
X \times \mathrm{Pic}^0_{\mathbf{T}^0_X} \times S \times Y & \longrightarrow & X \times S \times Y & \longrightarrow & X \times Y \\
\downarrow & & \downarrow & & \\
X \times \mathrm{Pic}^0_{\mathbf{T}^0_X} \times S & \longrightarrow & X \times S & & \\
\downarrow & & \downarrow & & \\
\mathbf{T}^0_X \times \mathrm{Pic}^0_{\mathbf{T}^0_X} \times S & \longrightarrow & \mathbf{T}^0_X \times S & & \\
\downarrow & & & & \\
\mathrm{Pic}^0_{\mathbf{T}^0_X} \times S. & & & &
\end{array}$$

From this we see that the complex on the left side of (4.11.1) is isomorphic to the complex

$$\mathbf{R}p_{134*}(p_{14}^*P \otimes^{\mathbf{L}} p_{12}^*\mathscr{M}^u_X \otimes^{\mathbf{L}} p_{23}^*\mathscr{G}).$$

Using the isomorphism (4.10.1) we find that the image of

$$p_{14}^*P \otimes^{\mathbf{L}} p_{12}^*\mathscr{M}^u_X \otimes^{\mathbf{L}} p_{23}^*\mathscr{G}$$

in

$$D^b_{\mathrm{coh}}(X \times \mathrm{Pic}^0(\mathbf{T}^0_Y) \times S \times Y)$$

is equal to

$$p_{14}^*P \otimes^{\mathbf{L}} p_{24}^*\mathscr{M}^u_Y \otimes^{\mathbf{L}} p_{23}^*\mathscr{G}.$$

From this the result follows. □

Lemma 4.12. *Let $x \in X$ be a point with image $z \in \mathbf{T}^0_X$. Then the complex $P_x \in Y_{\kappa(x)}$ is set-theoretically supported on $c_Y^{-1}(\rho(z))$.*

Proof. After making the field extension from k to $\kappa(x)$, we may assume that x is a k-rational point.

The support of P_x in $X \times Y$ is contained in the support of $P_{c_X^{-1}(z)} = P|_{c_X^{-1}(z)\times Y}$, so it suffices to show that the support of $P_{c_X^{-1}(z)}$ is contained in $X \times c_Y^{-1}(\rho(z))$.

For this apply Lemma 4.11 with $S = \mathrm{Spec}(k)$ and $\mathscr{F}$ the skyscraper sheaf $\kappa(z)$ on $\mathbf{T}^0_X$. We then find that the support of $P_{c_X^{-1}(z)}$ is equal to the support of $P|_{X\times c_Y^{-1}(\rho(z))}$. □

4.13. Let $W_X \subset \mathbf{T}^0_X$ (resp. $W_Y \subset \mathbf{T}^0_Y$) be the scheme-theoretic image of c_X (resp. c_Y). If $x \in X$ is a point then it follows from Lemma 4.12 that $c_Y^{-1}(\rho(c_X(x)))$ is nonempty; that is, $\rho(c_X(x)) \in W_Y$. Since W_X and W_Y are integral it follows that ρ restricts to a morphism

$$W_X \to W_Y, \tag{4.13.1}$$

which we again denote by ρ. By considering the inverse transform we see that this map is an isomorphism.

4.14. Let $f : X \to W_X$ be the map induced by c_X, and let

$$g : Y \to W_X$$

denote the composition of $c_Y : Y \to W_Y$ with the inverse of (4.13.1), and let $E^\bullet$ be the associated cosimplicial scheme as in 2.2. Applying 4.11 with $S = W_X$ and $\mathscr{F}$ the sheaf $u_*\mathcal{O}_{W_X}$, where $u : W_X \to \mathbf{T}^0_X \times W_X$ is the graph of the inclusion, we find that on

$$E^1 = X \times W_X \times Y$$

we have

$$d_{0*}P \simeq d_{1*}P.$$

4.15. Having established the existence of this isomorphism we can proceed as in the case of the canonical fibration. Namely if $A \subset W_X$ is an open subset

over which f and g are flat, and

$$f_A : f^{-1}(A) \to A, \quad g_A : g^{-1}(A) \to A$$

are the restrictions, then the same argument shows that the map

$$\mathrm{Hom}_{E(f_A,g_A)^2}(\tau_{2*}P, \tau_{0*}P) \to \mathrm{Hom}_{f^{-1}(A)\times g^{-1}(A)}(P, P),$$

induced by the surjection $[2] \to [0]$, is an isomorphism. From this it follows that the isomorphism $d_{0*}P \simeq d_{1*}P$ satisfies the cocycle condition, after restriction to A. Theorem 4.9 then follows using 1.1. □

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