The Annals of Probability
2024, Vol. 52, No. 5, 1954–1991
https://doi.org/10.1214/24-AOP1692

CAPACITY OF THE RANGE OF RANDOM WALK: THE LAW OF THE ITERATED LOGARITHM

BY AMIR DEMBO[1,a] AND IZUMI OKADA[2,b]

[1]*Mathematics Department and Statistics Department, Stanford University,* [a]*adembo@stanford.edu*
[2]*Department of Mathematics and Informatics, Faculty of Science, Chiba University,* [b]*iokada@math.s.chiba-u.ac.jp*

We establish both the lim sup and the lim inf law of the iterated logarithm (LIL) for the capacity of the range of a simple random walk in any dimension $d \geq 3$. While for $d \geq 4$, the order of growth in n of such LIL at dimension d matches that for the volume of the random walk range in dimension $d-2$, somewhat surprisingly this correspondence breaks down for the capacity of the range at $d = 3$. We further establish such LIL for the Brownian capacity of a *three*-dimensional Brownian sample path and novel, sharp moderate deviations bounds for the capacity of the range of a *four*-dimensional simple random walk.

1. Introduction and main results. Let τ_A denote the first positive hitting time of a finite set A by a simple random walk (SRW) on $\mathbb{Z}^d$, denoted hereafter $(S_m)_{m\geq 0}$. Recall that the corresponding (Newtonian) capacity is given, for $d \geq 3$, by

$$\mathrm{Cap}(A) := \sum_{x \in A} P^x(\tau_A = \infty) = \lim_{|z|\to\infty} \frac{P^z(\tau_A < \infty)}{G(0, z)}$$

(where $G(x, y)$ denotes the Green's function of the walk). The asymptotics of the capacity $R_n := \mathrm{Cap}(\mathcal{R}_n)$ of the random walk range $\mathcal{R}_n := \{S_1, \dots, S_n\}$ is relatively trivial for $d = 2$ (for then $R_n = \frac{2+o(1)}{\pi} \log(\operatorname{diam} \mathcal{R}_n)$, see [24], Lemma 2.3.5). In contrast, for $d \geq 3$, such asymptotics is of an on going interest. Indeed, the strong law

$$\lim_{n\to\infty} \frac{R_n}{n} = \alpha_d \quad \text{a.s., for all } d \geq 3,$$

is an immediate consequence of the subadditive ergodic theorem, with $\alpha_d > 0$ iff $d \geq 5$ (as shown in [19]). Recall Green's function for the d-dimensional Brownian motion

$$(1.1) \qquad G_B(x, y) := \int_0^\infty (2\pi t)^{-d/2} e^{-|x-y|^2/(2t)}\, dt = \begin{cases} \dfrac{1}{2\pi}|x-y|^{-1} & d = 3, \\ \dfrac{1}{2\pi^2}|x-y|^{-2} & d = 4, \end{cases}$$

and the corresponding Brownian capacity of $D \subset \mathbb{R}^d$,

$$\mathrm{Cap}_B(D)^{-1} := \inf\left\{ \int\int G_B(x, y)\mu(dx)\mu(dy) : \mu(D) = 1 \right\}.$$

More recently, Chang [12] showed that, for $d = 3$,

$$\frac{R_n}{\sqrt{n}} \overset{\mathcal{D}}{\Longrightarrow} \frac{1}{3\sqrt{3}} \mathrm{Cap}_B\big(B[0, 1]\big),$$

Received August 2022; revised January 2024.
MSC2020 subject classifications. Primary 60F15; secondary 60G50.
Key words and phrases. Capacity, random walk, Brownian motion, law of iterated logarithm.

whereas Asselah et al [5] showed that in this case further, for some C finite,

$$C^{-1}\sqrt{n} \le E[R_n] \le C\sqrt{n}.$$

We denote throughout by $\overline{X}$ the centering $\overline{X} = X - E[X]$ of a generic random variable X. In higher dimensions $d \ge 4$, the centered capacity $\overline{R}_n$ converges after proper scaling to a nondegenerate limit law, which is Gaussian iff $d \ge 5$ (see [6] for $d = 4$ and [5, 31] for $d \ge 5$). For $d \ge 5$, estimates of the corresponding large and moderate deviations are provided in [2] (but they are not sharp enough to imply a LIL), while the central limit theorem (CLT) is further established in [16] for R_n and a class of symmetric α-stable walks, provided $d > 5\alpha/2$. We note in passing that similar questions for critical branching random walk on $\mathbb{Z}^d$, conditioned to have total population n, have also been studied in [7–9].

In view of these works, a natural question, which we fully resolve here, is to determine the almost sure fluctuations of $n \mapsto R_n$ for the SRW, in the form of some LIL (possibly after centering R_n when $d \ge 4$). Specifically, using hereafter $\log_k a = \log(\log_{k-1} a)$ for $k \ge 2$, with $\log_1 a$ for the usual logarithm, here is our first main result about the SRW in $\mathbb{Z}^3$.

THEOREM 1.1. *For $d = 3$, almost surely,*

$$\limsup_{n\to\infty} \frac{R_n}{h_3(n)} = 1, \qquad \liminf_{n\to\infty} \frac{R_n}{\hat{h}_3(n)} = 1, \tag{1.2}$$

where

$$h_3(n) := \frac{\sqrt{6}\pi}{9}(\log_3 n)^{-1}\sqrt{n \log_2 n}, \qquad \hat{h}_3(n) := \frac{\sqrt{6}\pi^2}{9}\sqrt{n(\log_2 n)^{-1}}. \tag{1.3}$$

Utilizing (3.21), we also get from Theorem 1.1 the following consequence about the Brownian capacity of the 3-dimensional (Brownian) sample path.

COROLLARY 1.2. *For $d = 3$, almost surely,*

$$\limsup_{n\to\infty} \frac{\mathrm{Cap}_B(B[0,n])}{3\sqrt{3}h_3(n)} = 1, \qquad \liminf_{n\to\infty} \frac{\mathrm{Cap}_B(B[0,n])}{3\sqrt{3}\hat{h}_3(n)} = 1.$$

REMARK. From the variational characterization of $\mathrm{Cap}_B(B[0,n])$ and with $\mu(\cdot)$ the push-forward of the uniform law on $[0,n]$ by the Brownian path $t \mapsto B_t$, we get that

$$\frac{\pi n^2}{\mathrm{Cap}_B(B[0,n])} \le \int_0^n dt \int_0^t |B_t - B_s|^{-1}\, ds =: \eta\big([0,n]^2_<\big).$$

It thus follows from the lim sup-LIL of [13], Theorem. 1.2, for $\eta([0,n]^2_<)$, that almost surely,

$$\liminf_{n\to\infty} \frac{\mathrm{Cap}_B(B[0,n])}{3\sqrt{3}\hat{h}_3(n)} \ge \frac{3\sqrt{3}}{8\sqrt{2}\pi\rho},$$

where ρ is given by [13], formula (1.15) for $d = 3$, $\sigma = 1$, $\psi(\lambda) = \lambda^2/2$.

We next provide the LIL for the centered capacity $\overline{R}_n$ of the range, first in case of the SRW on $\mathbb{Z}^4$ and then for SRW on $\mathbb{Z}^d$, $d \ge 5$.

THEOREM 1.3. *For $d = 4$, almost surely,*

$$\limsup_{n\to\infty} \frac{\overline{R}_n}{h_d(n)} = 1, \qquad \liminf_{n\to\infty} \frac{\overline{R}_n}{\hat{h}_d(n)} = -1, \tag{1.4}$$

where for some nonrandom $0 < c_\star < \infty$,

$$h_4(n) := \frac{\pi^2}{8}\frac{n\log_3 n}{(\log n)^2}, \qquad \hat{h}_4(n) := c_\star \frac{n\log_2 n}{(\log n)^2}. \tag{1.5}$$

THEOREM 1.4. *For any* $d \geq 5$, *the* LIL-*s* (1.4) *hold almost surely, now with*

$$h_d(n) = \hat{h}_d(n) := \sigma_d\sqrt{2n(1+1_{\{d=5\}}\log n)\log_2 n}, \quad d \geq 5, \tag{1.6}$$

where the nonrandom, finite $\sigma_d^2 > 0$ *are given by the leading asymptotic of* $\mathrm{var}(R_n)$ (*c.f.* [31], *Theorem A, for* σ_5 *and* [5], *Theorem* 1.1, *for* σ_d, $d \geq 6$).

REMARK 1.5. Our proof of Theorem 1.4 via Skorokhod embedding also yields Strassen's LIL for the a.s. set of limit points in $C([0,1])$ of the functions $\{t \mapsto h_d(n)^{-1}\overline{R}_{tn}\}$, for any $d \geq 5$.

REMARK 1.6. The moment generating function of the limit in law of $-((\log n)^2/n)\overline{R}_n$, for the SRW on $\mathbb{Z}^4$, blows-up at a finite, positive λ. The value of λ is identified in [13], Theorem 1.3. In Lemma 4.6 we establish the uniform in n boundedness of the moment generating function of $-((\log n)^2/n)\overline{R}_n$ for a small enough argument.

We note that $R_n \approx nE[\hat{P}^{S_{n/2}}(\hat{\tau}_{\mathcal{R}_n} = \infty)]$ at any fixed $n \gg 1$ and $d \geq 3$, where $\hat{P}$ and $\hat{\tau}_A$ denote the law and the first hitting time by an i.i.d. copy of the SRW. Similarly, the volume of $\mathcal{R}_n$ in any dimension ($d \geq 1$) is approximately $nP(\tau_0 > n)$. It has been observed before (see, e.g., [5], Section 6) that the typical order of growth of $E[\hat{P}^{S_{n/2}}(\hat{\tau}_{\mathcal{R}_n} = \infty)]$ at any $d \geq 3$ matches that of $P(\tau_0 > n)$ at $d' = d-2$, yielding the same order of growth in n for R_n at $d \geq 3$ and for the volume of $\mathcal{R}_n$ at $d' = d-2$. In Theorems 1.3 and 1.4, our LIL for $d \geq 4$ adheres to such a match with the scale for the LIL of the volume of $\mathcal{R}_n$ at $d' = d-2$ (see [10, 11, 21] for the latter LIL at any $d' \geq 2$ as well as the limit distribution results for the volume of the Wiener sausage at $d' \geq 2$, and the corresponding LIL at $d' \geq 3$, in [25] and [15, 34], respectively). In contrast, this relation *breaks down at the* lim sup LIL *for* $d = 3$, with the appearance of the novel factor $(\log_3 n)^{-1}$ in Theorem 1.1. Nevertheless, even at $d = 3$, the relevant deviations of R_n are due to those in the diameter of $\mathcal{R}_n$, except that the upper tails for these two variables differ in their growth rates. Specifically, our proofs in Sections 3.1 and 3.3 yield the following (sharper) result.

PROPOSITION 1.7. *Let* $M_n := \max_{1\leq i\leq n}|S_i|$. *For* SRW *of* $\mathbb{Z}^3$ *and any* $\epsilon > 0$,

$$P(\{R_n \geq (1-\epsilon)h_3(n)\} \cap \{M_n \geq (1-\epsilon)\psi(n)\} \text{ i.o.}) = 1,$$

$$P(\{R_n \leq (1+\epsilon)\hat{h}_3(n)\} \cap \{M_n \leq (1+\epsilon)\hat{\psi}(n)\} \text{ i.o.}) = 1,$$

where $\psi(n) := \sqrt{(2/3)n\log_2 n}$ *and* $\hat{\psi}(n) := \pi\sqrt{(1/6)n(\log_2 n)^{-1}}$.

We note in passing that Proposition 1.7 is a rotation-invariant result, and in particular, it applies also under any (fixed) rotation of the SRW lattice $\mathbb{Z}^3$. Further, Proposition 1.7 implies that almost surely, the lim sup (resp., lim inf) of R_n are essentially attained simultaneously with those for M_n, since for $d = 3$, almost surely,

$$\limsup_{n\to\infty}\frac{M_n}{\psi(n)} = 1, \qquad \liminf_{n\to\infty}\frac{M_n}{\hat{\psi}(n)} = 1. \tag{1.7}$$

Indeed, by the invariance principle it suffices for proving (1.7) to show the equivalent a.s. statement for *three*-dimensional Bessel process, and the latter follows by mimicking the proof of Chung's one-dimensional LIL (see [14]), starting with the estimate (3.24). We note in passing that while the scaling $\psi(n)$ of the upper fluctuation of M_n is the same as that for a single coordinate of our SRW, this is not true about the scaling $\hat{\psi}(n)$, which is about the tail probability of confining the walk to stay within an Euclidean ball in $\mathbb{R}^3$.

In a follow-up work, [1], Corollary 1.2, determines the value of $c_\star$ of Theorem 1.3 in terms of the best constant in a generalized Gagliardo–Nirenberg inequality. In contrast, the following analog of Proposition 1.7 in case $d \geq 4$ is still open.

OPEN PROBLEM 1.8. *Consider the* SRW $S_i = (S_i^1, \dots, S_i^d) \in \mathbb{Z}^d$, $d \geq 4$. *For* $d' = d-2$, *let* $\hat{S}_i^{d'} = (S_i^1, \dots, S_i^{d'}, 0, 0)$ *and* $V_{d'}(n) = |\{\hat{S}_1^{d'}, \dots, \hat{S}_n^{d'}\}|$. *Pick nonrandom* $\psi_{d'}(n)$ *such that a.s.*

$$\limsup_{n\to\infty} \frac{\overline{V}_{d'}(n)}{\psi_{d'}(n)} = 1.$$

We then conjecture that for $h_d(n)$ *of Theorems* 1.3–1.4 *and any* $\epsilon > 0$,

$$P\left(\{\overline{R}_n \geq (1-\epsilon)h_d(n)\} \cap \{\overline{V}_{d'}(n) \geq (1-\epsilon)\psi_{d'}(n)\}\ i.o.\right) = 1.$$

While we consider throughout only the discrete time SRW whose increments are the $2d$ neighbors of the origin in $\mathbb{Z}^d$, due to sharp concentration of Poisson variables, all our results apply also for the continuous time SRW with i.i.d. Exponential(1) clocks and up to the scaling $n \mapsto (1-\rho)n$, also to the ρ-lazy discrete time SRW. By definition of R_n, our results apply to any random walk on a group with a finite symmetric set of generators, whose words are isomorphic to those of the SRW (e.g., an invariance of our results under any nonrandom, invertible affine transformation of the walk). We note in passing the recent work [27] on the strong law for any symmetric random walk on a group of growth index d and the corresponding CLT in case $d \geq 6$, suggesting the possibility of a future extension of our LIL-s in this context.

Beyond the intrinsic interest in R_n, its asymptotic is also relevant for the study of intersections between two independent random walks (e.g., see [24], Chapter 3). Similarly, [3, 4] utilize bounds on R_n to gain insights about the so-called Swiss cheese picture for $d = 3$. Further, to understand Sznitman's [32] random interlacement model, one may use moment estimates for the capacity of the union of ranges (c.f. [12] and the references therein). Finally, the capacity equals the summation of all entries of the inverse of the (positive definite) Green's function matrix (see (2.2)), a point of view which [28] uses, for $d = 2$, to estimate the geometry of late points of the walk.

As for the organization of this paper, we prove Theorem 1.1 in Section 3, relying on certain relations between the capacity and Green's function which we explore in Section 2. Our proof of Theorem 1.1 further indicates that the lim sup-LIL is due to exceptional time where $\mathcal{R}_n$ has a cylinder-like shape, with one dimension being about $\psi(n)$ while the other two are $O(\psi(n)/(\log_2 n))$ (see Lemma 2.1 and Section 3.2). In contrast, the lim inf-LIL seems to be due to times where the shape of $\mathcal{R}_n$ is close enough to a ball of radius $\hat{\psi}(n)$ to approximately match the capacity of such a ball (see (3.31) and (3.32)).

Sections 4 and 5 are devoted to the proofs of Theorems 1.3 and 1.4, respectively. Our proofs rely on the decomposition (4.2) of R_{n_k} as the sum of k independent variables $\{U_j\}$ which are the capacities of the walk restricted to the k parts of a partition of $[1, n_k]$, minus some random $\Delta_{n_k,k} \geq 0$ (which ties all these parts together). For any $d \geq 5$, the effect of $\Delta_{n_k,k}$ on the LIL is negligible, so upon coupling R_{n_k} with a one-dimensional Brownian motion, we immediately get the LIL for the former out of the standard LIL for the latter. As seen in Section 4, the situation is *way more delicate for* $d = 4$, where $E\Delta_{n,k} \approx h_4(n)$ dominates for a suitable slowly growing $k = k_n$ the fluctuations of the i.i.d. $\{U_j\}$. The lim sup-LIL is then due to the exceptional (random) sequence $\{n_k\}$ where $\Delta_{n_k,k} = o(h_4(n_k))$, while the lim inf-LIL is due to the exceptional $\{n_k\}$ for which $\Delta_{n_k,k} \approx \hat{h}_4(n_k) \gg E\Delta_{n_k,k}$. Indeed, whereas Theorem 1.3 is proved via the framework developed in [11], Section 4, for the LIL for the volume of $\mathcal{R}_n$ in the planar case ($d' = 2$), special care is needed here in order to establish tight control on the moderate deviations of $\Delta_{n,k}$ and U_j in case $d = 4$ (c.f. Lemmas 4.1, 4.3, 4.6 and 4.7, which may be of independent interest).

2. Capacity geometry and Green's function.

The following asymptotic for the *three*-dimensional capacity of cylinder-like domains (which we prove at the end of this section) is behind the factor $(\log_3 n)^{-1}$ in the lim sup-LIL of (1.2).

LEMMA 2.1. *For $m \geq 1$ and $r \geq k \in \mathbb{N}$, let*

$$\mathcal{C}_m(\ell, r) := (\ell\mathbb{Z})^3 \cap \{(x_1, x_2, x_3) : x_1^2 + x_2^2 \leq r^2, 1 \leq x_3 \leq m\}.$$

Fix $b < 2/3$, $r_m = o(m)$, $r_m \uparrow \infty$. If $\mathcal{C}_m(1, r_m) \supseteq \mathcal{C}_m \supseteq \mathcal{C}_m(\ell, r_m)$ for some $\ell \leq r_m^b$, then

$$\lim_{m\to\infty} \frac{\mathrm{Cap}(\mathcal{C}_m)}{m(\log(m/r_m))^{-1}} = \frac{\pi}{3}. \tag{2.1}$$

REMARK 2.2. In the sequel we prove a stronger result, namely, that the upper bound in (2.1) holds as soon as $\mathcal{C}_m$ is contained in a union $\mathcal{C}_m^\star(r_m)$ of at most m/r_m balls $\mathbb{B}(z_i, r_m)$ of radius r_m in $\mathbb{Z}^3$, of centers such that $|z_{i+1} - z_i| \leq r_m$ for $1 \leq i < m/r_m$ (where $\mathcal{C}_m(1, r)$ is merely one possible choice for $\mathcal{C}_m^\star(r)$).

Indeed, in Section 3 we will see that lim sup of R_n is roughly attained on the event $\{S_n^i \geq \psi(n)\}$ for $\psi(n)$ of Proposition 1.7, with $\mathcal{R}_n$ then having approximately the shape of such $\mathcal{C}_m$ for $m = \psi(n)$, and $r_m = cm/\log_2 n$; hence, from Lemma 2.1 we find that

$$R_n \approx \mathrm{Cap}(\mathcal{C}_m) \approx \frac{\pi}{3} m\big(\log(m/r_m)\big)^{-1} \approx \frac{\pi}{3}\psi(n)(\log_3 n)^{-1},$$

which is precisely $h_3(n)$ of Theorem 1.1.

We proceed with two lemmas relating the capacity of SRW with its Green's function,

$$G(x, y) = \sum_{i=0}^{\infty} P^x(S_i = y).$$

To this end, partition Ω by the last time the walk visits $X = \{x_1 \neq x_2 \cdots \neq x_j\}$, to get that

$$1 = \sum_{\ell=1}^{j} G(x_i, x_\ell) P^{x_\ell}(\tau_X = \infty) \quad \forall 1 \leq i \leq j. \tag{2.2}$$

LEMMA 2.3. *For any set $X = \{x_1, \dots, x_j\}$ and with $\{x_j\}$ not necessarily distinct,*

$$\frac{j}{\max_{1\leq\ell\leq j}\{\sum_{i=1}^{j} G(x_i, x_\ell)\}} \leq \mathrm{Cap}(X) \leq \frac{j}{\min_{1\leq\ell\leq j}\{\sum_{i=1}^{j} G(x_i, x_\ell)\}}. \tag{2.3}$$

PROOF. The set X of size $|X| = k \leq j$ consists WLOG of distinct points $\hat{X} = \{\hat{x}_1 \neq \hat{x}_2 \cdots \neq \hat{x}_k\}$, where $\hat{x}_v$ appears $m_v \geq 1$ times in X (and $\sum_{v\leq k} m_v = j$). Though (2.3) follows from the characterization of $\mathrm{Cap}(\hat{X})$, as in [20], Lemma 2.2(i), we proceed instead with a direct, short and elementary proof of these bounds. Specifically, setting $v(\ell)$ for the index of x_ℓ in $\hat{X}$ and $q_v := (m_v)^{-1} P^{\hat{x}_v}(\tau_{\hat{X}} = \infty)$, we see that

$$\mathrm{Cap}(X) = \mathrm{Cap}(\hat{X}) = \sum_{v=1}^{k} P^{\hat{x}_v}(\tau_{\hat{X}} = \infty) = \sum_{\ell=1}^{j} q_{v(\ell)}, \tag{2.4}$$

and moreover, summing (2.2) over $i \le j$, we get that

$$j = \sum_{i=1}^{j}\sum_{v=1}^{k} G(x_i, \hat{x}_v) P^{\hat{x}_v}(\tau_{\hat{X}} = \infty) = \sum_{\ell=1}^{j} q_{v(\ell)} \sum_{i=1}^{j} G(x_i, x_\ell). \tag{2.5}$$

The bounds of (2.3) are an immediate consequence of (2.4) and (2.5). □

LEMMA 2.4. *For* $Z_1 = \{x_1, \dots, x_{j_1}\}$, $Z_2 = \{x_{j_1+1}, \dots, x_{j_1+j_2}\}$ *with* $\{x_i\}$ *not necessarily distinct*,

$$\mathrm{Cap}(Z_1 \cup Z_2) \le \mathrm{Cap}(Z_2) + \frac{j_1 + j_2}{\min_{x \in Z_1 \setminus Z_2}\{\sum_{i=1}^{j_1+j_2} G(x_i, x)\}}.$$

PROOF. Since $\tau_{Z_1 \cup Z_2} \le \tau_{Z_2}$, it follows that

$$\mathrm{Cap}(Z_1 \cup Z_2) \le \mathrm{Cap}(Z_2) + \sum_{x \in Z_1 \setminus Z_2} P^x(\tau_{Z_1 \cup Z_2} = \infty). \tag{2.6}$$

For $\hat{X}$ enumerating the *distinct* points in $Z_1 \cup Z_2$, $v(\ell)$, q_v as in Lemma 2.3, we have that

$$\sum_{x \in Z_1 \setminus Z_2} P^x(\tau_{Z_1 \cup Z_2} = \infty) = \sum_{\ell=1}^{j_1+j_2} q_{v(\ell)} 1_{\{\hat{x}_{v(\ell)} \in Z_1 \setminus Z_2\}},$$

$$j_1 + j_2 = \sum_{\ell=1}^{j_1+j_2} q_{v(\ell)} \sum_{i=1}^{j_1+j_2} G(x_i, x_\ell).$$

Combining these identities with (2.6) yields the stated upper bound. □

REMARK 2.5. In particular, applying Lemma 2.4 for

$$Z_1 = \bigcup_{i \in (j, J-j]} \hat{Z}_i, \qquad Z_2 = \bigcup_{i \in [1,j] \cup (J-j,J]} \hat{Z}_i,$$

we have that, for any $\hat{Z}_i \subset \mathbb{Z}^d$, $2j < J$,

$$\mathrm{Cap}(Z_1 \cup Z_2) \le \mathrm{Cap}(Z_2) + \frac{\sum_{i=1}^{J} |\hat{Z}_i|}{\min_{x \in Z_1} \sum_{i=1}^{J} \sum_{y \in \hat{Z}_i} G(x, y)}.$$

PROOF OF LEMMA 2.1. By the monotonicity of $A \mapsto \mathrm{Cap}(A)$, it suffices to provide a uniform in $\ell \le r_m^b$ lower bound on $\mathrm{Cap}(\mathcal{C}_m(\ell, r_m))$ and a matching upper bound on $\mathrm{Cap}(C_m^\star(r_m))$, valid for any union $\mathcal{C}_m^\star(r_m)$ of at most m/r_m balls $\mathbb{B}(z_i, r_m)$ of radius r_m in $\mathbb{Z}^3$ and centers such that $|z_{i+1} - z_i| \le r_m$ for $1 \le i < m/r_m$. With $|\mathcal{C}_m(\ell, r_m)| = (1 + o(1))\pi m r_m^2 \ell^{-3}$, we get such a lower bound from Lemma 2.3, upon showing that for SRW on $\mathbb{Z}^3$,

$$\sum_{y \in \mathcal{C}_m(\ell, r_m)} G(x, y) \le 3\big(1 + o(1)\big) r_m^2 \ell^{-3} \log(m/r_m), \forall \ell \le r_m^b, \forall x \in \mathcal{C}_m(\ell, r_m). \tag{2.7}$$

Fixing $b < 2/3$, since the right side of (2.7) diverges in m, uniformly in $\ell \le r_m^b$, we can ignore any bounded contribution to its left side. In particular, with $m/r_m \uparrow \infty$ and $G(x, y)$ bounded, it suffices to sum in (2.7) only over $y \in \mathcal{C}_m(\ell, r_m)$ with $|x - y| \ge r_m^{2/3} \gg \ell$ and use the asymptotics

$$G(x, y) = \frac{3 + o(1)}{2\pi} |x - y|^{-1} \tag{2.8}$$

(e.g., see [24], Theorem 1.5.4). Setting $u_m = m$, we have, for any $v_m \uparrow \infty$ and $r \in [r_m^{2/3}, v_m r_m]$, at most $Cr^2\ell^{-3}$ points $y \in \mathcal{C}_m(\ell, r_m)$ with $|x - y| \in [r, r + 1]$; while for each $r \in [v_m r_m, u_m]$, there are at most $(2\pi + o(1))r_m^2 \ell^{-3}$ such points in $\mathcal{C}_m(\ell, r_m)$. Thus, taking $v_m^2 \ll \log(m/r_m)$ yields

$$\begin{aligned} \sum_{y \in \mathcal{C}_m(\ell, r_m)} G(x, y) &\le \frac{3 + o(1)}{2\pi \ell^3} \left[C \int_{r_m^{2/3}}^{v_m r_m} r \, dr + 2\pi r_m^2 \int_{v_m r_m}^{u_m} r^{-1} \, dr \right] \\ &= (3 + o(1)) r_m^2 \ell^{-3} \log(u_m / (r_m v_m)) \end{aligned} \tag{2.9}$$

from which (2.7) immediately follows. Turning to upper bound on $\mathrm{Cap}(\mathcal{C}_m^\star(r_m))$, take now $u_m := (m/r_m)^{1-\epsilon_m}$ and $v_m := (m/r_m)^{\epsilon_m} \uparrow \infty$ for some $\epsilon_m \to 0$, splitting $\mathcal{C}_m^\star(r_m)$ to $Q_1 \cup Q_2$, where

$$Q_1 := \bigcup_{i \in (u_m, (m/r_m) - u_m)} \mathbb{B}(z_i, r_m), \qquad Q_2 := \bigcup_{i \notin (u_m, (m/r_m) - u_m)} \mathbb{B}(z_i, r_m).$$

Note that $\mathcal{C}_m^\star(r_m)$ has at most $(4\pi/3 + o(1))r_m^2 m$, possibly overlapping, points. Thus, combining Lemma 2.4 with the upper bound of Lemma 2.3, we get the upper bound of (2.1), once we show that, for some $\delta_m \to 0$,

$$\sum_{y \in \mathcal{C}_m^\star(r_m)} G(x, y) \ge (4 + \delta_m) r_m^2 \log(m/r_m) \quad \forall x \in Q_1, \tag{2.10}$$

$$\sum_{y \in Q_2} G(x, y) \ge \frac{|Q_2|}{m\delta_m} \log(m/r_m) \quad \forall x \in Q_2. \tag{2.11}$$

Fixing $x \in \mathbb{B}(z_i, r_m) \subset Q_1$, consider only the contribution to the LHS of (2.10) from all points $y \in B(z_j, r_m)$ with $|j - i| \in [v_m, u_m]$. For such a pair $|y - x| \le (|j - i| + 3)r_m$, hence by (2.8),

$$G(x, y) \ge \frac{3 + o(1)}{2\pi} r_m^{-1} |j - i|^{-1},$$

resulting with

$$\sum_{y \in \mathcal{C}_m^\star(r_m)} G(x, y) \ge 2 \frac{3 + o(1)}{2\pi} \frac{|\mathbb{B}(0, r_m)|}{r_m} \sum_{j = v_m}^{u_m} j^{-1} = (4 + o(1)) r_m^2 \log(u_m / v_m),$$

which for our choices of u_m and v_m is as stated in (2.10) (for some $\delta_m \to 0$, uniformly over $x \in Q_1$). Further, Q_2 consists of two sets with an equal number of elements, each of diameter at most $(1 + o(1))u_m r_m$. Thus, we get by (2.8) that, for some $c > 0$,

$$\sum_{y \in Q_2} G(x, y) \ge c|Q_2|(u_m r_m)^{-1} \quad \forall x \in Q_2,$$

and (2.11) follows upon choosing $\epsilon_m \to 0$ slow enough so that $(m/r_m)^{\epsilon_m} \gg \log(m/r_m)$. □

3. LIL for SRW on $\mathbb{Z}^3$: Proof of Theorem 1.1. To ease the presentation, we omit hereafter the integer-part symbol $\lceil\cdot\rceil$ and divide the section to four parts, establishing the lower and then upper bounds, first for the lim sup-LIL of (1.2) and then for the lim inf-LIL of (1.2). Our proofs for the lim sup-LIL are *discrete* in nature, relying on Remark 2.2, Lemma 2.3 and the direct evaluation of certain sharp tail probabilities for SRW. In contrast, we prove the lim inf-LIL by first replacing R_n by the corresponding quantity about the Brownian capacity of the range of the 3D Brownian motion.

3.1. *The lower bound in the limsup-*LIL. Recall

$$\psi(t) := \sqrt{(2/3)t\log_2 t}, \qquad h_3(t) := \frac{\pi}{3}\psi(t)(\log_3 t)^{-1},$$

of Proposition 1.7 and Theorem 1.1, respectively, and for the SRW (S_m) on $\mathbb{Z}^3$, set

$$A_t := \{S_t^1 \geq \psi(t)\}, \qquad V_I := 1_{A_t}\sum_{\ell\in I} G(0,S_\ell), \qquad I \subseteq [0,t]\cap\mathbb{Z}.$$

The lower bound in our lim sup-LIL is attained via partial sums on disjoint intervals I_n of suitably growing length t_n and controllable Green function values (utilizing the LHS of (2.3)). The key for this is our next lemma (whose proof is deferred to the end of this subsection), showing that A_{t_n} yields the appropriate bound on the sum $V_{[0,t_n]}$ of Green function values.

LEMMA 3.1. *Fixing $\delta \in (0,1)$, for $\gamma_t := t(\log_2 t)^{-1}(\log_3 t)^{3/2}$ and some $\zeta_t \to 0$ when $t\to\infty$,*

$$P\left(V_{[0,\gamma_t]} \geq \frac{\delta t}{h_3(t)}\right) \leq \zeta_t P(A_t), \tag{3.1}$$

$$P\left(V_{(\gamma_t,t]} \geq \frac{(1+2\delta)t}{2h_3(t)}\right) \leq \zeta_t P(A_t). \tag{3.2}$$

Indeed, from (3.1)–(3.2) we see that, for any $\delta > 0$, there exists $\zeta_t \to 0$ as $t\to\infty$ such that

$$P(A_t^\star) \leq 2\zeta_t P(A_t), \qquad A_t^\star := \left\{V_{[0,t]} \geq \frac{(1+4\delta)}{2}\frac{t}{h_3(t)}\right\}. \tag{3.3}$$

Proceeding to deduce the lower bound in the limsup-LIL out of (3.3), given $\epsilon > 0$, we choose $q > 1$ large and $\delta > 0$ small; so for $t_n := q^n - q^{n-1}$ and all large n,

$$\frac{1-\delta}{1+4\delta}h_3(t_n) \geq (1-\epsilon)h_3(q^n). \tag{3.4}$$

We then partition $\mathbb{Z}_+$ to disjoint intervals $I_n := (q^{n-1}, q^n]\cap\mathbb{Z}$ of length t_n and set the events

$$\hat{A}_n := \{S^1_{q^n} - S^1_{q^{n-1}} \geq \psi(t_n)\}, \qquad \hat{H}_n(i) := \left\{\hat{V}_n(i) \geq (1+4\delta)\frac{t_n}{h_3(t_n)}\right\},$$

where

$$\hat{V}_n(i) := \sum_{\ell\in I_n} G(S_i, S_\ell).$$

Setting

$$H_t(i) := \left\{\sum_{\ell=1}^{t} G(S_i,S_\ell) \geq (1+4\delta)\frac{t}{h_3(t)}\right\}, \quad i\in[1,t],$$

note that by the stationarity of the SRW increments $(\hat{A}_n, \hat{H}_n(q^{n-1}+i)) \stackrel{d}{=} (A_{t_n}, H_{t_n}(i))$. Further, the event A_t is invariant to any permutation of the SRW increments $\{X_j, j \le t\}$, whereas, given A_t, the event $H_t(i)$ depends only on $\{S_\ell - S_i, \ell \in [1,t]\}$. Further, the permuted increments $\hat{X}_j = X_{(i+j)\,\mathrm{mod}(t)}$ result with $\hat{S}_\ell = S_{\ell+i} - S_i$ for all $\ell \in [1, t-i]$, whereas $\hat{X}_j = X_{(i+1-j)\,\mathrm{mod}(t)}$ result with $\hat{S}_\ell = S_i - S_{i-\ell}$ for all $\ell \in [1,i]$. Since $G(x,y) = G(-x,-y) \ge 0$, it thus follows that, conditional on A_t the random sum in each of the events $H_t(i)$ is stochastically dominated by twice the random sum in the event $A_t^\star$ of (3.3). Consequently, (3.3) yields that

$$\max_{i \in I_n}\{P(\hat{H}_n(i)|\hat{A}_n)\} \le 4\zeta_{t_n} \to 0. \tag{3.5}$$

Next, consider the independent events $G_n := \{|\Lambda_n| \ge (1-\delta)t_n\}$, where Λ_n is the subset of all those $i \in I_n$ for which $\hat{H}_n(i)$ does not hold. From Markov's inequality and (3.5), it follows that

$$P(G_n^c|\hat{A}_n) = P(|I_n| - |\Lambda_n| \ge \delta t_n|\hat{A}_n) \le \frac{1}{\delta t_n}\sum_{i \in I_n} P(\hat{H}_n(i)|\hat{A}_n) \le \frac{4\zeta_{t_n}}{\delta} \to 0. \tag{3.6}$$

Setting $x(t) := \sqrt{2\log_2 t}$, note that (S_m^1) is the partial sum of $\{-1,0,1\}$-valued, zero-mean, i.i.d. variables of variance $1/3$. By the asymptotic normality of the moderate deviations for such partial sums (see [29], Theorem VIII.2.1), we have that, for some $o_{\mathsf{t}}(1) \to 0$ as $\mathsf{t} \to \infty$, uniformly over $\mathsf{x}/\mathsf{t}^{1/6}$ small,

$$P((\mathsf{t}/3)^{-1/2} S_{\mathsf{t}}^1 \ge \mathsf{x}) = (1 + o_{\mathsf{t}}(1))\overline{\Phi}(\mathsf{x}), \quad \overline{\Phi}(\mathsf{x}) := \int_{\mathsf{x}}^\infty \frac{e^{-u^2/2}}{\sqrt{2\pi}}\,du. \tag{3.7}$$

In particular,

$$\begin{aligned} P(\hat{A}_n) = P(A_{t_n}) &= P((t_n/3)^{-1/2} S_{t_n}^1 \ge x(t_n)) \\ &= (1+o(1))\overline{\Phi}(x(t_n)) \ge c_1 x(t_n)^{-1} e^{-x(t_n)^2/2} \ge \frac{c_2}{n\sqrt{\log n}} \end{aligned}$$

for some positive c_1 and $c_2 = c_2(q)$. Hence, by (3.6) and for all n large enough,

$$P(G_n) \ge P(\hat{A}_n \cap G_n) \ge \frac{1}{2}P(\hat{A}_n) \ge \frac{c_2}{2n\sqrt{\log n}}.$$

Having $\{G_n\}$ independent with $\sum_n P(G_n) = \infty$, we deduce by the second Borel–Cantelli lemma that a.s. the events G_n hold for infinitely many values of n. Since

$$\hat{\mathcal{R}}_{q^n} := \{S_i\}_{i \in \Lambda_n} \subseteq \mathcal{R}_{q^n},$$

we have by the monotonicity of $A \mapsto \mathrm{Cap}(A)$, the nonnegativity of $G(x,y)$, Lemma 2.3 and the definition of $\Lambda_n \subseteq I_n$ that

$$R_{q^n} \ge \mathrm{Cap}(\hat{\mathcal{R}}_{q^n}) \ge \frac{|\Lambda_n|}{\max_{i \in \Lambda_n}\{\hat{V}_n(i)\}} \ge \frac{|\Lambda_n| h_3(t_n)}{(1+4\delta)t_n}.$$

Consequently, in view of (3.4), we have on the event G_n that

$$R_{q^n} \ge \frac{1-\delta}{1+4\delta} h_3(t_n) \ge (1-\epsilon) h_3(q^n),$$

which since G_n holds infinity often, yields the lower bound in the lim sup-LIL (along the sub-sequence q^n and with $\mathcal{R}_{q^n} \supseteq \hat{\mathcal{R}}_{q^n}$ of roughly the shape of $\mathcal{C}_{\psi(t_n)}$ of Lemma 2.1).

Turning to the task of proving Lemma 3.1, we give two sharp tail estimates for the path of the *one-dimensional walk* (S_m^1) that we will use later for proving (3.1) and (3.2), respectively.

LEMMA 3.2. *Fixing $\delta \in (0,1)$, for some $C < \infty$ and all t large enough,*

$$\sup_{\ell \le \gamma_t} \{P(S^1_{t-\ell} \ge \psi(t) - \sqrt{\ell})\} \le C P(A_t), \tag{3.8}$$

$$P(L_t^c | A_t) \le 4\zeta_t \to 0, \quad L_t := \bigcap_{\ell \in (\gamma_t, t]} \left\{ S^1_\ell \ge \frac{\psi(t)\ell}{(1+\delta)t} \right\}. \tag{3.9}$$

REMARK 3.3. In (3.8) we claim that the decay $t \mapsto P(A_t)$ of our moderate deviations upper-tail event A_t is within a universal factor of that for such an event with a granted (free) upper fluctuation of $\sqrt{\ell}$ in the first $\ell \le \gamma_t$ steps of S^1_ℓ. The event L_t requires S^1_ℓ to stay above a linear slope which is $1/(1+\delta) < 1$ of the slope $\psi(t)/t$ of A_t. Thus, if $\ell \mapsto S^1_\ell$ was a SRW on $\mathbb{Z}$, we could have applied en-route to (3.9) a ballot theorem (after conditioning on $S^1_{\gamma_t}$ and S^1_t). It is not so here, due to the additional randomness in number of steps of the SRW $S_\ell \in \mathbb{Z}^3$ along the other two coordinate axis. We thus resort to proving (3.9) via Gaussian approximations.

PROOF OF LEMMA 3.2. Let $\eta_t := \gamma_t/t = (\log_3 t)^{3/2}/(\log_2 t)$, setting $x(t,r) := (x(t) - \sqrt{3r})/\sqrt{1-r}$ for $r \le \eta_t$ and $x(t) = x(t,0) := \sqrt{2\log_2 t}$. Then in view of the uniform Gaussian approximation of (3.7), we get (3.8), once we show that uniformly in $r \le \eta_t$ the standard Gaussian measure of $[x(t,r),\infty)$ is at most C times the Gaussian measure of $[x(t),\infty)$. Note that $\eta_t \to 0$ and $x(t) \to \infty$ as $t \to \infty$, hence $x(t,r)/x(t) \to 1$ uniformly in $r \le \eta_t$. It thus remains only to show that, for some $C < \infty$ and all t large enough,

$$\inf_{r \le \eta_t} \{x(t,r)^2 - x(t)^2\} \ge -2\log C. \tag{3.10}$$

Next, our expression for $x(t,r)$ is such

$$(1-r)[x(t,r)^2 - x(t)^2] = (x(t)\sqrt{r} - \sqrt{3})^2 + 3r - 3 \ge -3,$$

yielding (3.10) and thereby also (3.8).

Next, setting $s_j := jt/(\log_2 t)$, we partition $(\gamma_t, t]$ into the disjoint intervals $J_j = (s_j, s_{j+1}]$, $j \in [(\log_3 t)^{3/2}, \log_2 t)$. We likewise partition the events $A_t \cap L_t^c$ according to whether the stopping time $\tau_t := \inf\{\ell \ge \gamma_t : S^1_\ell < \frac{\psi(t)\ell}{(1+\delta)t}\}$ equals γ_t or, alternatively, which interval J_j contains τ_t. Note that if $\tau_t > s_j$, then $S^1_{s_j} \ge \frac{\psi(t)s_j}{(1+\delta)t}$, and conditioning on the SRW filtration at $\tau_t \in J_j$, we get by the strong Markov property (and i.i.d. increments) of the SRW that

$$P(A_t \cap \{\tau_t \in J_j\}) \le q_t(s_j, 0) \sup_{s \in J_j} p_t(t-s, 0),$$

where

$$q_t(s_j, y) := P\left((S^1_{s_j})_+ \ge \frac{\psi(t)s_j}{(1+\delta)t} - y\right), \qquad p_t(r,y) := P\left(S^1_r \ge \frac{\psi(t)(\delta t + r)}{(1+\delta)t} + y\right)$$

and $(S^1_{s_j})_+ := S^1_{s_j} \vee 0$. The only other way for the event L_t^c to occur is by having

$$S^1_{\gamma_t} < \frac{\psi(t)\eta_t}{1+\delta} =: \Delta_t \log_2 t.$$

Partitioning to $\{S^1_{\gamma_t} \in I_i\}$, for $I_i := [(\log_2 t - i)\Delta_t - \Delta_t, (\log_2 t - i)\Delta_t)$ when $0 \le i < \log_2 t$ and $I_{\log_2 t} := (-\infty, 0)$ yields that

$$P(A_t \cap \{S^1_{\gamma_t} \in I_i\}) \le q_t(\gamma_t, i\Delta_t + \Delta_t) p_t(t - \gamma_t, i\Delta_t)$$

and, consequently,

$$P(A_t \cap L_t^c) \le \sum_i P(A_t \cap \{S^1_{\gamma_t} \in I_i\}) + \sum_j P(A_t \cap \{\tau_t \in J_j\})$$
$$\le (1+\log_2 t)\Big[\sup_{y\in[0,\Delta_t \log_2 t]} \{q_t(\gamma_t, y+\Delta_t) p_t(t-\gamma_t, y)\}$$
$$+ \sup_{s\in J_j, j\ge (\log_3 t)^{3/2}} q_t(s_j, 0) p_t(t-s, 0)\Big].$$

As $P(A_t) = p_t(t, 0)$, we thus get (3.9), if for some $\zeta_t \to 0$

(3.11) $$q_t(s_j, 0) p_t(t-s, 0) \le \frac{\zeta_t}{\log_2 t} p_t(t, 0) \quad \forall j \ge (\log_3 t)^{3/2}, s \in J_j,$$

(3.12) $$q_t(\gamma_t, y+\Delta_t) p_t(t-\gamma_t, y) \le \frac{2\zeta_t}{\log_2 t} p_t(t, 0) \quad \forall y \in [0, \Delta_t \log_2 t].$$

Proceeding to verify (3.11) and (3.12), we rely on (3.7) to replace both $q_t(\cdot)$ and $p_t(\cdot)$ by the Gaussian measure of the corresponding intervals. We claim that when doing so, all partial sums appearing in (3.11) and (3.12) be at time index $\mathrm{t} \ge \delta\psi(t)/2 \to \infty$. Indeed, note that $p_t(r, y) = 0$ for $y \ge 0$, unless the time index r is at least $\delta\psi(t)/2$; whereas in all the terms $q_t(\cdot,\cdot)$ that appear there, such time indices are $s_j \ge \gamma_t \ge \delta\psi(t)/2$. Further, the argument x of $\overline{\Phi}(\cdot)$ in such Gaussian approximations of the probabilities $q_t(\cdot)$ and $p_t(\cdot)$ that appear in (3.11) is $\frac{x(t)}{1+\delta}\sqrt{s_j/t}$ and $\frac{x(t)}{1+\delta}\frac{1+\delta-s/t}{\sqrt{1-s/t+\xi_t}}$ at $\xi_t = 0$, respectively (where $x(t) = \sqrt{3}\psi(t)/\sqrt{t}$). Taking instead $\xi_t := (\log_2 t)t^{-1/6}$ guarantees that all such space arguments x be uniformly of $o(\mathrm{t}^{1/6})$. The Gaussian approximations then hold, as in (3.7), with the same $o(1)$ *relative* error for all the terms, which we thus ignore hereafter. Note also that $s/t \in [\eta_t, 1)$ with $s_j/t \ge s/t - \varepsilon_t$ for $\varepsilon_t := (\log_2 t)^{-1}$. In conclusion, by the preceding it suffices for (3.11) to show that

(3.13) $$\sup_{u\in[\eta_t,1)} \left\{\overline{\Phi}\left(\frac{x(t)}{1+\delta}\sqrt{u-\varepsilon_t}\right)\overline{\Phi}\left(\frac{x(t)(1+\delta-u)}{(1+\delta)\sqrt{1-u+\xi_t}}\right)\right\} \le \frac{\zeta_t}{\log_2 t}\overline{\Phi}(x(t)).$$

The arguments of $\overline{\Phi}(\cdot)$ in (3.13) grow to infinity with t, uniformly over $u \in [\eta_t, 1)$. Thus, recalling that $|\log\overline{\Phi}(y) + \log y + y^2/2|$ is bounded at $y \to \infty$, upon taking the logarithm of both sides of (3.13), noting that $\eta_t \ge 2\varepsilon_t$ and ignoring all the uniformly bounded terms, such as $x(t)^2\varepsilon_t$ and $\log x(t) - \frac{1}{2}\log_3 t$, it suffices to show that, for some $\tilde{\zeta}_t \to 0$ as $t \to \infty$,

$$\sup_{u\in[\eta_t,1)} \left\{\frac{1}{2}x(t)^2 - \frac{1}{2}\frac{x(t)^2 u}{(1+\delta)^2} - \frac{1}{2}\frac{x(t)^2(1+\delta-u)^2}{(1+\delta)^2(1-u+\xi_t)} - \frac{1}{2}\log u\right\} \le \log\tilde{\zeta}_t - \frac{1}{2}\log_3 t.$$

With $\eta_t \gg \xi_t$, it is easy to verify that, for any $u \ge \eta_t$

$$\frac{u}{(1+\delta)^2} + \frac{(1+\delta-u)^2}{(1+\delta)^2(1-u+\xi_t)} - 1 \ge \frac{u}{(1-u+\xi_t)}\frac{\delta^2}{2(1+\delta)^2} =: \theta_t(u).$$

Hence, substituting $x(t)^2 = 2\log_2 t$ in the preceding, we arrive (after some algebra) at

(3.14) $$\sup_{u\in[\eta_t,1)} \left\{-(\log_2 t)\theta_t(u) - \frac{1}{2}\log u\right\} \le \log\tilde{\zeta}_t - \frac{1}{2}\log_3 t.$$

Since $u \mapsto \theta_t(u)$ is nondecreasing, the supremum on the left side of (3.14) is attained at $u = \eta_t$, where for large t it is at most $-\frac{\delta^2}{10}(\log_3 t)^{3/2}$. This is more than enough for (3.14) to hold, thereby establishing (3.11). We next turn to (3.12), where the Gaussian approximation of (3.7) again applies for all three probabilities, with uniform, hence negligible, $o(1)$ relative errors. Thus, analogously to (3.13), the bound (3.12) is a consequence of having

$$\sup_{v\in[0,1]}\left\{\overline{\Phi}\left(\frac{x(t)}{1+\delta}\sqrt{\eta_t}(v-\varepsilon_t)\right)\overline{\Phi}\left(\frac{x(t)(1+\delta-v\eta_t)}{(1+\delta)\sqrt{1-\eta_t}}\right)\right\}\le\frac{\zeta_t}{\log_2 t}\overline{\Phi}\bigl(x(t)\bigr) \tag{3.15}$$

(temporarily setting $\overline{\Phi}(\mathsf{x})=1$ wherever $\mathsf{x}<0$). Considering first $v\le 1/2$, we bound the left-most term of (3.15) by one, and as before, take the logarithm of both sides, replace $\log\overline{\Phi}(y)$ by $-\log y-y^2/2$ and eliminate all uniformly bounded terms to find that (3.15) holds because

$$\inf_{v\in[0,\frac{1}{2}]}\left\{\frac{(1+\delta-v\eta_t)^2}{(1+\delta)^2(1-\eta_t)}\right\}-1\ge\frac{\eta_t\delta}{1+\delta}$$

and $x(t)^2\eta_t=2(\log_3 t)^{3/2}\gg\log_3 t$. To complete the proof of (3.15), it thus suffices (similarly to (3.14)) to have for some $\tilde{\zeta}_t\to 0$ that

$$\sup_{v\in[1/2,1]}\{-(\log_2 t)\theta_t(v)\}\le\log\tilde{\zeta}_t-\log_3 t, \tag{3.16}$$

where, recalling that $\eta_t\gg\varepsilon_t$, it is easy to check that, for any $v\in[\frac{1}{2},1]$,

$$\theta_t(v):=\frac{\eta_t(v-\varepsilon_t)^2}{(1+\delta)^2}+\frac{(1+\delta-v\eta_t)^2}{(1+\delta)^2(1-\eta_t)}-1\ge\theta_t(1)\ge\frac{\eta_t\delta^2}{(1+\delta)^2}.$$

The preceding suffices for (3.16) and thereby for (3.12), thus completing the proof. □

PROOF OF LEMMA 3.1. Starting with (3.1), note that for some $C_1<\infty$ and any $\ell\ge 0$,

$$E\bigl[G(0,S_\ell)\bigr]=\sum_{i=\ell}^{\infty}P^0(S_i=0)\le C_1(1+\sqrt{\ell})^{-1}.$$

Further, recall from (2.8) that $G(0,y)\le C_2/(1+|y|)$ for some $C_2<\infty$ and all $y\in\mathbb{Z}^3$. Hence,

$$E\bigl[G(0,S_\ell)1_{A_t}\bigr]\le C_2(1+\sqrt{\ell})^{-1}P(A_t)+\sum_{|y|\le\sqrt{\ell}}G(0,y)P\bigl(\{S_\ell=y\}\cap A_t\bigr). \tag{3.17}$$

With $(S_\ell,S_t-S_\ell)\overset{d}{=}(S_\ell,\hat{S}_{t-\ell})$ for SRW $(\hat{S}_m)$, which is independent of (S_m), if $y\in\mathbb{Z}^3$ is such that $|y^1|\le|y|\le\sqrt{\ell}$, then

$$P\bigl(\{S_\ell=y\}\cap A_t\bigr)\le P(S_\ell=y)P\bigl(\hat{S}^1_{t-\ell}\ge\psi(t)-\sqrt{\ell}\bigr).$$

Therefore, thanks to (3.8), for any $\ell\le\gamma_t$ the right-most term in (3.17) is at most

$$E\bigl[G(0,S_\ell)\bigr]P\bigl(\hat{S}^1_{t-\ell}\ge\psi(t)-\sqrt{\ell}\bigr)\le C_1(1+\sqrt{\ell})^{-1}CP(A_t).$$

We thus deduce from (3.17) that, for some $C_3,C_4<\infty$,

$$E[V_{[0,\gamma_t]}]=\sum_{\ell=0}^{\gamma_t}E\bigl[G(0,S_\ell)1_{A_t}\bigr]\le C_3P(A_t)\sum_{\ell=0}^{\gamma_t}(1+\sqrt{\ell})^{-1}\le C_4P(A_t)\sqrt{\gamma_t}.$$

Consequently, by Markov's inequality and our choice of γ_t,

$$P\left(V_{[0,\gamma_t]}\ge\frac{\delta t}{h_3(t)}\right)\le\zeta_tP(A_t),$$

where our choice of γ_t results with $\zeta_t:=C_4h_3(t)\sqrt{\gamma_t}/(\delta t)\to 0$ as $t\to\infty$.

Turning to (3.2), recall our choice of γ_t implying that

$$\sum_{\ell\in(\gamma_t,t]}\frac{1}{\ell}\le\log(t/\gamma_t)\le\log_3 t.$$

Further, with $h_3(t)=(\pi/3)\psi(t)(\log_3 t)^{-1}$, it follows that $t/(2h_3(t))=\frac{3}{2\pi}(t/\psi(t))(\log_3 t)$. Consequently, for all t large enough, the event L_t of (3.9) implies, thanks to (2.8), that

$$V_{(\gamma_t,t]}\le\sum_{\ell\in(\gamma_t,t]}G(0,S_\ell)\le\frac{3+o(1)}{2\pi}\sum_{\ell\in(\gamma_t,t]}|S_\ell^1|^{-1}\le\frac{3+o(1)}{2\pi}\frac{(1+\delta)t}{\psi(t)}(\log_3 t)\le\frac{(1+2\delta)t}{2h_3(t)}.$$

Since the same conclusion applies when A_t^c holds (in which case $V_{(\gamma_t,t]}=0$), we see that (3.2) is an immediate consequence of (3.9). □

3.2. *The upper bound in the limsup*-LIL. Recall that R_n is nondecreasing. Further, for any $t_n=q^n$, $q>1$, we have that, eventually, $h_3(t_n)/h_3(t_{n-1})\le q$. It thus suffices to prove the upper bound in our lim sup-LIL only along each such sequence t_n (thereafter taking $q\downarrow 1$ to complete the proof). To this end, fix $q>1$ and $\delta,\eta>0$, and set $m:=(1+\delta)^3\psi(t_n)$, with $r_m=m/b_m\uparrow\infty$ for $b_m:=2\eta(1+\delta)^6(\log_2 t_n)$ (so $r_m:=\psi(t_n)/(2\eta(1+\delta)^3\log_2 t_n)$). We aim to cover $\mathcal{R}_{t_n}$ by the union $\mathcal{C}_m^\star(r_m)$ of b_m balls of radius r_m each, with the centers of consecutive balls at most r_m apart. Indeed, as shown in Section 2 (see Remark 2.2), this would yield $R_{t_n}\le(1+\delta+o(1))^3h_3(t_n)$, so we then conclude by taking $\delta\downarrow 0$ and $q\downarrow 1$.

Specifically, starting at $T_0=0$, set the increasing stopping times

$$T_i:=\inf\{k>T_{i-1}:|S_k-S_{T_{i-1}}|>r_m-1\}\quad\forall i\ge 1,$$

noting that the event $\{T_{b_m}\ge t_n\}$ implies the aforementioned containment $\mathcal{R}_{t_n}\subseteq\mathcal{C}_m^\star(r_m)$. Further, with $\exp(-(1+\delta)\log_2 t_n)\le Cn^{-(1+\delta)}$ summable, upon employing the first Borel–Cantelli lemma, it remains only to establish the following key lemma.

LEMMA 3.4. *For any $q>1$ and small $\delta>0$, there exist $\eta>0$ and $C<\infty$ such that*

$$P(T_{b_m}<t_n)\le C\exp\bigl(-(1+\delta)\log_2 t_n\bigr)\quad\forall n. \tag{3.18}$$

PROOF. By the strong Markov property and the independence of increments of the walk, we see that T_{b_m} is the sum of b_m i.i.d. copies of the first exit time T_1 of the (discrete) ball $\mathbb{B}(0,r_m-1)$, by the 3D-SRW. As $r_m\uparrow\infty$, Skorokhod's embedding implies (see [23], Lemma 3.2) for some $m_o,c<\infty$ and $\varepsilon>0$ (depending only on $\delta>0$), all $m\ge m_o$ and $u\ge 0$,

$$P(T_1<u)\le ce^{-r_m^\varepsilon}+P(3\underline{T}<u)=ce^{-r_m^\varepsilon}+P\bigl(3(1+\delta)^{-1}r_m^2\hat{T}_1<u\bigr),$$

with $\hat{T}_1:=\inf\{t\ge 0:|B_t|\ge 1\}$ the Brownian hitting time of the unit sphere $\mathbb{S}^2$ and $\underline{T}:=\inf\{t\ge 0:|B_t|\ge r_m/\sqrt{1+\delta}\}$ (so the identity above is merely Brownian scaling). Further, here $b_me^{-r_m^\varepsilon}\ll\exp(-2\log_2 t_n)$ and $t_n=3r_m^2\eta b_m$ with $b_m/(2\eta)=3m^2/(2t_n)=(1+\delta)^6(\log_2 t_n)$. It thus suffices to show that, for some $\eta=\eta(\delta)>0$ and all m,

$$P\left(\frac{1}{b_m}\sum_{i=1}^{b_m}\hat{T}_i<\eta\right)\le e^{-(1-\delta)^3b_m/(2\eta)}, \tag{3.19}$$

where $\hat{T}_i$ are i.i.d. copies of $\hat{T}_1$. To this end, covering $\mathbb{S}^2$ by c_δ balls of radius δ each, centered at some $\theta_i\in\mathbb{S}^2$, we have by the triangle inequality that

$$\max_i\{\langle\theta_i,B_{\hat{T}_1}\rangle\}\ge 1-\delta.$$

Hence, fixing $\lambda > 0$, we get upon applying Doob's optional stopping theorem for the martingale $M_t = \sum_i \exp(\lambda\langle\theta_i, B_t\rangle - \lambda^2 t/2)$ at the stopping time $\hat{T}_1$ that

$$c_\delta = M_0 = E[M_{\hat{T}_1}] \geq e^{\lambda(1-\delta)} E\big[e^{-\lambda^2 \hat{T}_1/2}\big].$$

Consequently, by Markov's inequality we have for any $\eta, \lambda, \delta > 0$ and integer $b \geq 1$ that

$$P\left(\frac{1}{b}\sum_{i=1}^{b} \hat{T}_i < \eta\right) \leq e^{\lambda^2 b\eta/2} E\big[e^{-\lambda^2 \hat{T}_1/2}\big]^b \leq \big(c_\delta e^{\lambda^2\eta/2 - \lambda(1-\delta)}\big)^b. \tag{3.20}$$

Taking the optimal $\lambda = (1-\delta)/\eta$, it is easy to check that, for $\eta \leq \eta(\delta) = \delta(1-\delta)^2/(2\log c_\delta)$, the LHS of (3.20) is at most $\exp(-(1-\delta)^3 b/(2\eta))$. We thus got (3.19) for any $\delta > 0$, provided $\eta \leq \eta(\delta)$, thereby completing the proof. □

REMARK 3.5. One has for any $\delta > 0$ small and all t large enough, the classical bound

$$P\Big(\max_{1\leq k\leq t} |S_k| \geq (1+\delta)^3 \psi(t)\Big) \leq C e^{-(1+\delta)\log_2 t}.$$

We need in (3.18) a stronger result, since for any b_m and $r_m \gg 1$,

$$\Big\{\max_{1\leq k\leq t_n} |S_k| \geq b_m r_m\Big\} \subset \{T_{b_m} < t_n\},$$

and while $b_m r_m = (1+\delta)^3 \psi(t_n)$, our crude use of δ-cover of $\mathbb{S}^2$ in proving Lemma 3.4 requires us to also have $b_m/(\log_2 t_n) \to 0$ as $\delta \downarrow 0$.

3.3. *The upper bound in the liminf*-LIL. For any $A \subset \mathbb{R}^3$ and $r > 0$, let

$$\mathrm{Nbd}(A, r) := \bigcup_{x\in A} \mathbb{B}(x, r)$$

denote the r-blowup of A. Utilizing [12], we first relate R_n with a suitable Brownian capacity, as stated next.

LEMMA 3.6. *We can couple the* SRW *with a* 3D *Brownian motion* $(B_t, t \geq 0)$ *such that*

$$\lim_{n\to\infty} \frac{R_n}{\mathrm{Cap}_B(B[0, n/3])} = \frac{1}{3} \quad a.s., \tag{3.21}$$

and for any $\delta \in (0, 1/2)$,

$$\lim_{n\to\infty} \frac{R_n}{\mathrm{Cap}_B(\mathrm{Nbd}(B[0, n/3], n^{1/2-\delta}))} = \frac{1}{3} \quad a.s. \tag{3.22}$$

PROOF. The results were essentially shown in [12]. Indeed, [12], (4.15), shows that (3.21) holds when each ratio is restricted to the events E_n, while it is also shown that a.s. E_n holds for all sufficiently large n (combine [12], (4.2), with Borel–Cantelli). Hence, (3.21) also holds without such a restriction. Turning to show (3.22), let $\tilde{P}$ denote the probability of an independent Brownian motion $(\tilde{B}_t)$. Fixing $\delta \in (0, 1/2)$ and some $y_n \in \mathbb{Z}^3$ such that $|y_n| = n^{1/2+\delta}$, we similarly obtain (after dispensing of events E_n) that by the same argument as in [12], (4.4), a.s. one has, for all large n,

$$\begin{aligned}&\mathrm{Cap}_B\big(\mathrm{Nbd}\big(B[0, n/3], n^{1/2-\delta}\big)\big)\\ &\quad = \big(2\pi + o(1)\big) n^{1/2+\delta} \tilde{P}\big(\mathrm{Nbd}\big(B[0, n/3], n^{1/2-\delta}\big) \cap \big(y_n + \tilde{B}[0, \infty)\big) \neq \varnothing | B[0, n/3]\big).\end{aligned}$$

By [12], (4.13) and (4.4), a.s., the latter expression is for all n large $(1+o(1))\mathrm{Cap}_B(B[0, n/3])$. Thus,

$$\lim_{n\to\infty} \frac{\mathrm{Cap}_B(B[0,n/3])}{\mathrm{Cap}_B(\mathrm{Nbd}(B[0,n/3],n^{1/2-\delta}))} = 1 \quad \text{a.s.},$$

which in view of (3.21) completes the proof of (3.22). □

Proceeding to show the upper bound in the lim inf-LIL, let $r(n) := \pi\sqrt{n/(2\log_2 n)}$, that is, $r(n) = \sqrt{3}\hat{\psi}(n)$ for $\hat{\psi}(\cdot)$ of Proposition 1.7. Recall that by [17], Lemma 1.1, or [22] and Brownian scaling, for some $c > 0$ and all $t, r > 0$,

$$P\Big(\sup_{s\in[0,t)} \{|B_s|\} \le r\Big) \ge 2ce^{-\frac{\pi^2 t}{2r^2}}. \tag{3.23}$$

We have used in (3.23) also that the largest eigenvalue of the Dirichlet Laplacian in the unit ball in $\mathbb{R}^d$ is $-j^2$, where $j = j_{(d-2)/2,1}$ denotes the first positive zero of the Bessel function of the first kind with index $(d-2)/2$ and, in particular, that $j_{1/2,1} = \pi$ (see [35], p. 490). Considering (3.23) for $r = r(s_n)$, $s_n = n^n$ and the Brownian increments in the disjoint intervals $[s_{n-1}, s_n)$ of length $s_n - s_{n-1}$, result with

$$P\Big(\sup_{t\in[s_{n-1},s_n)} \{|B_t - B_{s_{n-1}}|\} \le r(s_n)\Big) \ge c\exp(-\log_2 s_n) = \frac{c}{n\log n}. \tag{3.24}$$

Thus, thanks to the independence of Brownian increments on these disjoint intervals, we get from the second Borel–Cantelli lemma that

$$P\Big(\liminf_{n\to\infty}\Big(r(s_n)^{-1} \sup_{t\in[s_{n-1},s_n)} \{|B_t - B_{s_{n-1}}|\}\Big) \le 1\Big) = 1. \tag{3.25}$$

Further, as $\sqrt{s_{n-1}(\log_2 s_{n-1})} = o(r(s_n))$, by Kinchin's LIL for the Brownian motion,

$$P\Big(\limsup_{n\to\infty}\Big(r(s_n)^{-1} \sup_{t\le s_{n-1}} |B_t|\Big) = 0\Big) = 1. \tag{3.26}$$

Combining (3.25) and (3.26), we deduce that

$$P\Big(\liminf_{n\to\infty}\Big(r(s_n)^{-1} \sup_{t<s_n} |B_t|\Big) \le 1\Big) = 1.$$

This, of course, implies that also

$$P\Big(\liminf_{n\to\infty}\Big(r(n)^{-1} \sup_{t<n} |B_t|\Big) \le 1\Big) = 1. \tag{3.27}$$

Recall that for any $r > 0$ one has that $r^{-1}\mathrm{Cap}_B(\mathbb{B}(0,r)) = \mathrm{Cap}_B(\mathbb{B}(0,1)) = 2\pi$ $(= \kappa_1$ on [33], p. 356). By (3.27) and for any $\epsilon > 0$, a.s. $B[0,n] \subset \mathbb{B}(0,(1+\epsilon)r(n))$ for infinitely many values of n in which case also $\mathrm{Cap}_B(B[0,n]) \le 2\pi(1+\epsilon)r(n)$. That is,

$$P\big(\mathrm{Cap}_B(B[0,n]) \le 2\pi(1+\epsilon)r(n) \text{ i.o.}\big) = 1. \tag{3.28}$$

By Brownian scaling the sequence $\{\sqrt{3}\mathrm{Cap}_B(B[0,n/3])\}$ has the same law as the sequence $\{\mathrm{Cap}_B(B[0,n])\}$. Thus, in view of (3.21), we can also construct a coupling so that, for any $\epsilon > 0$, we have that a.s.

$$R_n \le \frac{(1+\epsilon)}{3\sqrt{3}}\mathrm{Cap}_B(B[0,n])$$

for all n large enough. With $r(n) = \frac{3\sqrt{3}}{2\pi}\hat{h}_3(n)$, it thus follows from (3.28) that

$$P\big(R_n \le (1+\epsilon)^2\hat{h}_3(n) \text{ i.o.}\big) = 1,$$

and taking $\epsilon \downarrow 0$ establishes the stated upper bound in our lim inf-LIL.

3.4. *The lower bound of the liminf-*LIL. Fixing $a > 0$, we have by [33], (1.4), that, for any $f(t) \uparrow \infty$ such that $f(t) = o(t^{2/3})$,

$$\lim_{t\to\infty} \frac{f(t)}{t} \log P\left(\left|\mathrm{Nbd}(B[0,t],a)\right| \le \left(\frac{\pi}{\sqrt{2}}\right)^3 f(t)^{3/2}\omega_3\right) = -1, \tag{3.29}$$

where ω_3 denotes the volume of the unit ball (using here that the largest eigenvalue of the Dirichlet Laplacian in the *unit volume ball* in $\mathbb{R}^3$ is $-\omega_3^{2/3}\pi^2$). Fixing $\delta \in (0, 1/2)$ as in (3.22), Brownian scaling by time factor $3n^{2\delta-1}$ yields equality in distribution between the sequences

$$\left|\mathrm{Nbd}(B[0,n/3], n^{1/2-\delta})\right| \stackrel{d}{=} 3^{-3/2}n^{3/2-3\delta}\left|\mathrm{Nbd}(B[0,n^{2\delta}], 3^{-1/2})\right|.$$

Thus, considering (3.29) for $a = 3^{-1/2}$, $t = n^{2\delta}$ and $f(t) = (1-\epsilon)^3 n^{2\delta}(\log_2 n)^{-1}$, we arrive at

$$\begin{aligned} &P\left(\left|\mathrm{Nbd}(B[0,n/3], n^{1/2-\delta})\right| \le (1-\epsilon)^2\hat{\psi}(n)^3\omega_3\right) \\ &\quad = P\left(\left|\mathrm{Nbd}(B[0,n^{2\delta}], 3^{-1/2})\right| \le (1-\epsilon)^2\left(\frac{\pi}{\sqrt{2}}\right)^3 \left(n^{2\delta}(\log_2 n)^{-1}\right)^{3/2}\omega_3\right) \\ &\quad \le C\exp\left(-(1-\epsilon)^{-2}(\log_2 n)\right). \end{aligned}$$

Considering $n_k = q^k$, we get by the first Borel–Cantelli lemma that, for fixed $q > 1$ and $\epsilon > 0$,

$$\liminf_{k\to\infty} \frac{|\mathrm{Nbd}(B[0,n_k/3], n_k^{1/2-\delta})|}{\omega_3\hat{\psi}(n_k)^3} \ge (1-\epsilon)^2 \quad \text{a.s.} \tag{3.30}$$

With $n \mapsto |\mathrm{Nbd}(B[0,n/3], n^{1/2-\delta})|$ monotone increasing and $\hat{\psi}(q^k)/\hat{\psi}(q^{k-1}) \to 1$ as $k \to \infty$ followed by $q \downarrow 1$, we deduce from (3.30) that

$$\liminf_{n\to\infty} \frac{|\mathrm{Nbd}(B[0,n/3], n^{1/2-\delta})|}{\omega_3\hat{\psi}(n)^3} \ge 1 \quad \text{a.s.} \tag{3.31}$$

Next, recall the Poincaré–Carleman–Szegö theorem [30] that, for any $r > 0$,

$$\inf_{|A|=\omega_3 r^3}\{\mathrm{Cap}_B(A)\} = \mathrm{Cap}_B(\mathbb{B}(0,r)) = r\mathrm{Cap}_B(\mathbb{B}(0,1)) = 2\pi r. \tag{3.32}$$

Recall that $\hat{h}_3(n) = \frac{2\pi}{3}\hat{\psi}(n)$. Hence, by (3.32) for $A = \mathrm{Nbd}(B[0,n/3], n^{1/2-\delta})$, we have in view of (3.30) that

$$\liminf_{n\to\infty} \frac{\mathrm{Cap}_B(\mathrm{Nbd}(B[0,n/3], n^{1/2-\delta}))}{3\hat{h}_3(n)} \ge 1 \quad \text{a.s.,}$$

which together with (3.22) yields the stated lower bound for the lim inf-LIL of R_n in $\mathbb{Z}^3$.

4. LIL for SRW on $\mathbb{Z}^4$: Proof of Theorem 1.3.

Hereafter we consider, for integers $0 \le a \le b \le c$, the random variables

$$R_{a,b} := \mathrm{Cap}(\mathcal{R}(a,b]), \qquad V_{a,b,c} := R_{a,b} + R_{b,c} - R_{a,c} \ge 0. \tag{4.1}$$

Note that by shift invariance $R_{a,b} \stackrel{d}{=} R_{0,b-a} = R_{b-a}$ and $R_{a,b}$ is independent of $R_{b,c}$ (due to the independence of increments). In particular, for any increasing $\{n_k\}$ starting at $n_0 = 0$, one has the decomposition

$$R_{n_k} := \sum_{j=1}^{k} U_j - \Delta_{n_k,k} \tag{4.2}$$

in terms of the independent variables $U_j := R_{n_{j-1},n_j}$ and the sum of nonnegative variables

$$\Delta_{n_k,k} = \sum_{j=1}^{k-1} V_{n_{j-1},n_j,n_k} = \sum_{j=1}^{k-1} V_{0,n_j,n_{j+1}}. \tag{4.3}$$

We use different nonrandom subsequences (n_k) for $d = 4$, for $d = 5$ and for $d \geq 6$. Also, the subsequences used for the lim sup-LIL and for the lim inf LIL-s be different. As we shall see, for such suitable n_k, the fluctuation in $\sum_j U_j$ is negligible for the LIL-s of Theorem 1.3, where $E\Delta_{n_k,k} \approx h_4(n_k)$, the lim sup-LIL being due to the exceptional times with $\Delta_{n_k,k} = o(E\Delta_{n_k,k})$, while the lim inf-LIL is due to the exceptional times where $\Delta_{n_k,k} \approx \hat{h}_4(n_k) \gg E\Delta_{n_k,k}$. In contrast, we show in Section 5 that $\Delta_{n_k,k}$ has a negligible effect when $d \geq 5$, where the LIL follows the usual pattern for sums of independent variables (namely, that of the LIL for a Brownian motion). We take a relatively small $k = O(\log_2 n_k)$ for the lim sup-LIL and $d = 4$, with larger $k = O((\log n_k)^\alpha)$ for the lim inf-LIL and for any $d \geq 5$.

4.1. *The lower bound in the limsup*-LIL. We start with the statement of a key lemma about the SRW on $\mathbb{Z}^4$ (which is related to [13], Theorem 2.2, in the $4D$ Brownian motion case).

LEMMA 4.1. *Suppose (S_m) and $(\tilde{S}_m)$ are two independent* SRW *on $\mathbb{Z}^4$. Let*

$$X_n := \frac{1}{n} \sum_{i,\ell \in [1,n]} G(S_i, \tilde{S}_\ell).$$

Then for some $C < \infty$ and any $p, n \in \mathbb{N}$,

$$E[X_n^p] \leq C^p p!. \tag{4.4}$$

One immediate consequence of (4.4) is that, for any $c < 1/C$,

$$\sup_n E[e^{cX_n}] < \infty. \tag{4.5}$$

The proof of Lemma 4.1 follows the same scheme as that of [26], Lemma 2. However, [26] crucially relies on an explicit representation of the moments of the Brownian self-intersection local time via variances of linear combinations of Brownian increments. Lacking any such tool here, our more involved proof of Lemma 4.1 relies instead on the following elementary bounds, where (4.6)–(4.8) implies also that (4.4)–(4.5) hold for $\int_0^1 \int_0^1 |\beta_s - \tilde{\beta}_t|^{-2}\, ds\, dt$ and the independent, standard *four*-dimensional Brownian motions $(\beta_s, s \geq 0)$, $(\tilde{\beta}_t, t \geq 0)$ (which is an improvement over the upper bound of [6], Prop. 4.1).

LEMMA 4.2. *There exists $C < \infty$ such that, for any $t > 0$ and $x, y \in \mathbb{R}^4$,*

$$E[|\beta_t - x|^{-2}] \leq C \min\{t^{-1}, |x|^{-2}\} \leq Ct^{-1/2}|x|^{-1}, \tag{4.6}$$

$$E[|\beta_t - x|^{-1}|\beta_t - y|^{-1}] \leq Ct^{-1/2}(|x| \vee |y|)^{-1} \leq 2Ct^{-1/2}|y - x|^{-1}, \tag{4.7}$$

$$E[|\beta_t - x|^{-2}|\beta_t - y|^{-1}] \leq 2Ct^{-1/2}|x|^{-1}|y - x|^{-1}. \tag{4.8}$$

Similarly, for $|\cdot|_+ = |\cdot| \vee 1$, any $i \geq 0$ and $x, y \in \mathbb{Z}^4$,

$$E[|S_i - x|_+^{-2}] \leq C \min\{|i|_+^{-1}, |x|_+^{-2}\} \leq C|i|_+^{-1/2}|x|_+^{-1}, \tag{4.9}$$

$$E[|S_i - x|_+^{-1}|S_i - y|_+^{-1}] \leq C|i|_+^{-1/2}(|x|_+ \vee |y|_+)^{-1} \leq 2C|i|_+^{-1/2}|y - x|_+^{-1}, \tag{4.10}$$

$$E[|S_i - x|_+^{-2}|S_i - y|_+^{-1}] \leq 2C|i|_+^{-1/2}|x|_+^{-1}|y - x|_+^{-1}. \tag{4.11}$$

PROOF. Denoting by $\phi_s(x) := (2\pi s)^{-2}\exp(-\frac{|x|^2}{2s})$ the density at x of the Gaussian law of β_s, we get from (1.1) after change of variables that

$$E[|\beta_t - x|^{-2}] = 2\pi^2 E[G_B(\beta_t, x)] = 2\pi^2 \int_t^\infty \phi_s(x)\,ds = t^{-1}\varphi_1(|x|^2/t) = |x|^{-2}\varphi_2(t/|x|^2),$$

for the finite decreasing functions $\varphi_1(r) := \frac{1}{2}\int_1^\infty u^{-2}e^{-r/(2u)}\,du$, $\varphi_2(r) := \frac{1}{2}\int_r^\infty u^{-2} \times e^{-1/(2u)}\,du$. Thus, (4.6) holds for any $C \geq \varphi_1(0) \vee \varphi_2(0)$. Next, by Cauchy–Schwarz and (4.6),

$$E(|\beta_t - x|^{-1}|\beta_t - y|^{-1}) \leq (E[|\beta_t - x|^{-2}])^{1/2}(E[|\beta_t - y|^{-2}])^{1/2} \leq Ct^{-1/2}|y|^{-1}.$$

Exchanging x with y yields the first inequality in (4.7), whereby the second inequality follows (as $|x - y| \leq |x| + |y| \leq 2|x| \vee |y|$). Now, by the triangle inequality, for $\beta_t \neq x \neq y$,

$$|y - x||\beta_t - x|^{-2}|\beta_t - y|^{-1} \leq |\beta_t - x|^{-1}(|\beta_t - x|^{-1} + |\beta_t - y|^{-1}),$$

so taking the expectation and using (4.6) and (4.7) to bound the RHS, results with (4.8).

With $S_0 = 0$, clearly (4.9) holds at $i = 0$, while for $i \geq 1$, recall [24], Theorem 1.2.1 and Theorem 1.5.4, that for some C finite and any $x \in \mathbb{Z}^4$,

$$P(S_i = x) \leq Ci^{-2}[e^{-2|x|^2/i} + (|x|^2 \vee i)^{-1}], \tag{4.12}$$

$$C^{-1}|x|_+^{-2} \leq G(0, x) \leq C|x|_+^{-2}. \tag{4.13}$$

By (4.13)

$$E[|S_i - x|_+^{-2}] \leq CE[G(S_i, x)] = C\sum_{\ell=i}^\infty P(S_\ell = x).$$

Further, for some C finite and all $i \geq 1$, $x \in \mathbb{Z}^4$,

$$\sum_{\ell=i}^\infty \ell^{-2}(|x|^2 \vee \ell)^{-1} \leq Ci^{-1}(|x|^2 \vee i)^{-1} \leq C\min\{i^{-1}, |x|_+^{-2}\}.$$

Thus, in view of (4.12), the same computation as for (4.6) yields the first inequality of (4.9). The second inequality of (4.9) follows (as $a \wedge b \leq \sqrt{ab}$ for any $a, b > 0$), and since the strictly positive $|\cdot|_+$ satisfies the triangle inequality on $\mathbb{Z}^4$, we get first (4.10) and then (4.11) by the same reasoning that led to (4.7) and (4.8), respectively. $\square$

PROOF OF LEMMA 4.1. From (4.11), for any $p \geq 2$, $1 \leq s_1 \leq \cdots \leq s_p$ and $\{y_1, \ldots, y_p\} \subset \mathbb{Z}^4$,

$$\begin{aligned}
&E\left[\prod_{i=1}^{p-1}|S_{s_i} - y_i|_+^{-2}|S_{s_{p-1}} - y_p|_+^{-1}\right] \\
&\quad = E\left[\prod_{i=1}^{p-2}|S_{s_i} - y_i|_+^{-2}|S_{s_{p-1}} - S_{s_{p-2}} - (y_{p-1} - S_{s_{p-2}})|_+^{-2}\right. \\
&\qquad\qquad \left.\times |S_{s_{p-1}} - S_{s_{p-2}} - (y_p - S_{s_{p-2}})|_+^{-1}\right] \\
&\quad \leq C|s_{p-1} - s_{p-2}|_+^{-1/2}|y_p - y_{p-1}|_+^{-1}E\left[\prod_{i=1}^{p-2}|S_{s_i} - y_i|_+^{-2}|S_{s_{p-2}} - y_{p-1}|_+^{-1}\right],
\end{aligned}$$

where we set throughout $s_0 = 0$ and $y_0 = 0$. Thus, by induction on $p \geq 1$,

$$E\left[\prod_{i=1}^{p-1} |S_{s_i} - y_i|_+^{-2} |S_{s_{p-1}} - y_p|_+^{-1}\right] \leq C^{p-1} \prod_{i=1}^{p-1} |s_i - s_{i-1}|_+^{-1/2} \prod_{i=1}^{p} |y_i - y_{i-1}|_+^{-1}. \tag{4.14}$$

Next, note that by (4.9), for any $p \geq 1$,

$$\begin{aligned} E\left[\prod_{i=1}^{p} |S_{s_i} - y_i|_+^{-2}\right] &= E\left[\prod_{i=1}^{p-1} |S_{s_i} - y_i|_+^{-2} |S_{s_p} - S_{s_{p-1}} - (y_p - S_{s_{p-1}})|_+^{-2}\right] \\ &\leq C|s_p - s_{p-1}|_+^{-1/2} E\left[\prod_{i=1}^{p-1} |S_{s_i} - y_i|_+^{-2} |S_{s_{p-1}} - y_p|_+^{-1}\right]. \end{aligned}$$

Hence, setting $t_0 = 0$, in view of (4.14) and the independence of (S_i) and $(\tilde{S}_i)$,

$$E\left[\prod_{i=1}^{p} |S_{s_i} - \tilde{S}_{t_i}|_+^{-2}\right] \leq C^p \prod_{i=1}^{p} |s_i - s_{i-1}|_+^{-1/2} E\left[\prod_{i=1}^{p} |\tilde{S}_{t_i} - \tilde{S}_{t_{i-1}}|_+^{-1}\right]. \tag{4.15}$$

Suppose that $t_{\sigma(1)} \leq \cdots \leq t_{\sigma(p)}$ for some permutation σ of $\{1, \ldots, p\}$. Then conditioning on $(\tilde{S}_i, i \leq t_{\sigma(p-1)})$, we get by (4.10) and the independence of increments that, when $\sigma(p) = \ell$,

$$\begin{aligned} &E\left[\prod_{i=1}^{p} |\tilde{S}_{t_i} - \tilde{S}_{t_{i-1}}|_+^{-1}\right] \\ &\quad \leq C|t_{\sigma(p)} - t_{\sigma(p-1)}|_+^{-1/2} E\left[\prod_{i=1}^{\ell-1} |\tilde{S}_{t_i} - \tilde{S}_{t_{i-1}}|_+^{-1} |\tilde{S}_{t_{\ell+1}} - \tilde{S}_{t_{\ell-1}}|_+^{-1} \prod_{i=\ell+2}^{p} |\tilde{S}_{t_i} - \tilde{S}_{t_{i-1}}|_+^{-1}\right]. \end{aligned}$$

Any permutation σ of $\{1, \ldots, p\}$ with $\sigma(p) = \ell$ must be a bijection from $\{1, \ldots, p-1\}$ to $\{1, \ldots, \ell-1, \ell+1, \ldots, p\}$. We can thus further bound the right side of the preceding inequality, inductively according to the values of $\sigma(j)$, for $j = p-1, \ldots, 1$, and thereby arrive at

$$E\left[\prod_{i=1}^{p} |\tilde{S}_{t_i} - \tilde{S}_{t_{i-1}}|_+^{-1}\right] \leq C^p \prod_{j=1}^{p} |t_{\sigma(j)} - t_{\sigma(j-1)}|_+^{-1/2}, \quad \sigma(0) := 0. \tag{4.16}$$

Combining (4.15) and (4.16), we conclude that, for any nondecreasing (s_i) and $(t_{\sigma(j)})$,

$$E\left[\prod_{i=1}^{p} |S_{s_i} - \tilde{S}_{t_i}|_+^{-2}\right] \leq C^{2p} \prod_{i=1}^{p} |s_i - s_{i-1}|_+^{-1/2} \prod_{j=1}^{p} |t_{\sigma(j)} - t_{\sigma(j-1)}|_+^{-1/2}. \tag{4.17}$$

We next bound $E[X_n^p]$ by (4.13) and enumerate over all words (s_i) and (t_i) of length p with symbols from $[1, n]$, according to the numbers k and k' of distinct symbols in each word. Recalling that $|0|_+ = 1$ and having at most k^p words of length p composed of given (fixed) k distinguished symbols, we thus deduce from (4.17) that, for any $n, p \in \mathbb{N}$,

$$\begin{aligned} E[X_n^p] &\leq \frac{1}{n^p} C^p E\left[\sum_{s_i, t_i \in [1,n]} \prod_{i=1}^{p} |S_{s_i} - \tilde{S}_{t_i}|_+^{-2}\right] \leq C^{2p}\left[\sum_{k=1}^{p} k^p n^{-(p-k)/2} J_{k,n}\right]^2, \\ J_{k,n} &:= \frac{1}{n^k} \sum_{1 \leq s_1 < \cdots < s_k \leq n} \prod_{i=1}^{k} \left(\frac{s_i}{n} - \frac{s_{i-1}}{n}\right)^{-1/2}. \end{aligned} \tag{4.18}$$

Considering $\theta_i = s_i/n$, we see that $\{J_{k,n}\}$ are, for each $k \in \mathbb{N}$, the Riemann sums of

$$J_k := \int_{0=\theta_0<\theta_1<\cdots<\theta_k\le 1} \prod_{i=1}^{k} (\theta_i - \theta_{i-1})^{-1/2}\, d\theta_1 \cdots d\theta_k \le C^k (k!)^{-1/2} \tag{4.19}$$

(for the inequality, see, e.g., [26], proof of Lemma 2). Note that $J_k = (Q^k \mathbf{1})(0)$ for the positive linear operator $(Qf)(x) = \int_0^{1-x} y^{-1/2} f(x+y)\,dy$ on $C([0,1])$. Setting $(y)_n = \lceil yn \rceil/n$, we have that $J_{k,n} = (Q_n^k \mathbf{1})(0)$ for the positive linear operators $(Q_n f)(x) = \int_0^{1-x} (y)_n^{-1/2} f((x)_n + (y)_n)\,dy$. It is easy to see that $(Q_n f) \le (Qf)$ are both nonincreasing whenever $f(\cdot)$ is nonincreasing. By induction on $k \ge 0$, we thus have that $Q_n^k \mathbf{1} \le Q^k \mathbf{1}$, pointwise, so in particular, $J_{k,n} \le J_k$ for any $k, n \in \mathbb{N}$. Further, $k! \ge (k/e)^k$ and $J_{k,n} = 0$, unless $k \le n$. Hence, in view of (4.18) and (4.19), we find that

$$E[X_n^p] \le C^{2p} \left[\sum_{k=1}^{p} k^p k^{-(p-k)/2} J_k \right]^2 \le C^{2p} \left[\sum_{k=1}^{p} k^{(p+k)/2} C^k (k/e)^{-k/2} \right]^2 \le p^2 C^{4p} e^p p^p,$$

and the uniform moment bounds (4.4) on X_n follow. □

For any interval I, consider the range $\mathcal{R}_I = \{S_i\}_{i\in I}$ and $\tilde{\mathcal{R}}_I = \{\tilde{S}_i\}_{i \in I}$ of independent SRW-s. Fixing $\alpha > 0$, let $n_\alpha := n(\log n)^{-\alpha}$ and denoting by $\hat{P}$ and $\hat{\tau}_A$ the probability and the hitting time to a set A by another independent SRW $(\hat{S}_i)$, set for $i \in [n_{2\alpha}, n - n_{2\alpha}]$,

$$g_{n,\alpha}(i) := 1_{\{S_i \notin \mathcal{R}(i, i+n_{2\alpha}]\}} \hat{P}^{S_i}(\hat{\tau}_{\mathcal{R}(i-n_{2\alpha}, i+n_{2\alpha}]} = \infty), \tag{4.20}$$

with $g_{n,\alpha}(i) = 1$ for $i \in [0, n_{2\alpha}) \cup (n - n_{2\alpha}, n]$ and $\tilde{g}_{n,\alpha}(i)$ defined analogously for the SRW $(\tilde{S}_i)$. In particular, for any $i \in [n_{2\alpha}, n - n_{2\alpha}]$,

$$\hat{g}_{n,\alpha} := E\big[1_{\{S_{n_{2\alpha}} \notin \mathcal{R}(n_{2\alpha}, 2n_{2\alpha}]\}} \hat{P}^{S_{n_{2\alpha}}}(\hat{\tau}_{\mathcal{R}_{2n_{2\alpha}}} = \infty)\big] = E\big[g_{n,\alpha}(i)\big]. \tag{4.21}$$

The next lemma, whose proof is deferred to the end of this section, allows us to complete the proof of the limsup-LIL lower bound for SRW on $\mathbb{Z}^4$ by comparing

$$Y_{n,m} := \sum_{\substack{i \in [1,n],\\ \ell \in [1,m]}} g_{n,\alpha}(i) G(S_i, \tilde{S}_\ell) \tilde{g}_{m,\alpha}(\ell),$$

with the simpler to analyze

$$\underline{Y}_{n,m} := \hat{g}_{n,\alpha} \hat{g}_{m,\alpha} \sum_{\substack{i \in [1,n],\\ \ell \in [1,m]}} G(S_i, \tilde{S}_\ell),$$

whose moments we bound in Lemma 4.1 (for $m = n$, see also (4.37) for the decay of $\hat{g}_{n,\alpha}$).

LEMMA 4.3. *We have that $EY_{n,m} \le C\sqrt{nm}$ for some C finite and all $m, n \in \mathbb{N}$, and if $n^\epsilon \le m \le n$, then for some $C = C(\epsilon, \alpha) < \infty$ and any $\epsilon > 0$, $\alpha > 4$,*

$$EY_{n,m} \le C\sqrt{nm}(\log n)^{-2}, \tag{4.22}$$

$$E\big[(Y_{n,m} - \underline{Y}_{n,m})^2\big] \le Cnm(\log n)^{-\alpha/2}. \tag{4.23}$$

PROOF OF THE LIMSUP-LIL LOWER BOUND. Equipped with Lemma 4.1 and Lemma 4.3, we derive the limsup-LIL lower bound for SRW on $\mathbb{Z}^4$, and with $n \mapsto R_n$ having a similar structure as $|\mathcal{R}_n|$ for the SRW on $\mathbb{Z}^2$, we adapt the proof in [11], Prop. 4.4, of the limsup-LIL lower-bound for the latter sequence. Specifically, set $p = [(-\kappa + \log_3 n)/(\log 2)]$ with $\kappa < \infty$ large and $k = 2^p$ (so $k = \gamma \log_2 n$ for small $\gamma \le e^{-\kappa}$). Centering both sides of

(4.2) for $n_j = jm$, $m = n/k$ (assumed for simplicity to be integer), we have that, for i.i.d. $U_j := R_{(j-1)m,jm}$ and the nonrandom $\varphi_n := ER_n$,

$$\overline{R}_n = k\varphi_{n/k} - \varphi_n + \sum_{j=1}^{k} \overline{U}_j - \sum_{j=1}^{k-1} V_{0,jm,(j+1)m}. \tag{4.24}$$

Further, denoting by θ the time shift $S_i \mapsto S_{i+1}$, we set

$$\chi(A,B) := \sum_{y\in A}\sum_{z\in B} P^y(\tau_{A\cup B} = \infty) G(y,z) P^z(\tau_B = \infty),$$

and for all $i \in (0,n]$,

$$h_n(i) := 1_{\{S_i \notin \mathcal{R}(i,n]\}} \hat{P}^{S_i}(\hat{\tau}_{\mathcal{R}_n} = \infty) \le g_{n,\alpha}(i) \tag{4.25}$$

(see (4.20)), recalling from (4.1) and [6], Prop. 1.6, that

$$\begin{aligned} 0 &\le V_{0,jm,(j+1)m} \le \chi(\mathcal{R}_{jm}, \mathcal{R}(jm,(j+1)m]) + \chi(\mathcal{R}(jm,(j+1)m], \mathcal{R}_{jm}) \\ &\le 2 \sum_{y\in\mathcal{R}_{jm}} \sum_{z\in\mathcal{R}(jm,(j+1)m]} P^y(\tau_{\mathcal{R}_{jm}} = \infty) G(y,z) P^z(\tau_{\mathcal{R}(jm,(j+1)m]} = \infty) \\ &= 2\sum_{i=1}^{jm}\sum_{\ell=1}^{m} h_{jm}(i) G(S_i, S_{jm+\ell}) h_m(\ell) \circ \theta_{jm} \\ &\le 2\sum_{i=1}^{jm}\sum_{\ell=1}^{m} g_{jm,\alpha}(i) G(S_i, S_{jm+\ell}) g_{m,\alpha}(\ell) \circ \theta_{jm} := 2W_j. \end{aligned} \tag{4.26}$$

Setting in addition

$$\underline{W}_j := \hat{g}_{jm,\alpha}\hat{g}_{m,\alpha} \sum_{i=0}^{m-1}\sum_{\ell=1}^{m} G(S_{jm-i}, S_{jm+\ell}), \tag{4.27}$$

$$\underline{\hat{W}}_j := \hat{g}_{jm,\alpha}\hat{g}_{m,\alpha} \sum_{i=m}^{jm-1}\sum_{\ell=1}^{m} G(S_{jm-i}, S_{jm+\ell}), \tag{4.28}$$

we see that, for any fixed j,

$$W_j - \underline{W}_j - \underline{\hat{W}}_j \overset{d}{=} Y_{jm,m} - \underline{Y}_{jm,m}. \tag{4.29}$$

Next, following [6], we let

$$\begin{aligned} \chi_n(i,j) &:= \chi(\mathcal{R}_n^{(i,2j-1)}, \mathcal{R}_n^{(i,2j)}) + \chi(\mathcal{R}_n^{(i,2j)}, \mathcal{R}_n^{(i,2j-1)}) \\ &\text{for } \mathcal{R}_n^{(i,j)} := \mathcal{R}[(j-1)2^{-i}n, j2^{-i}n] \end{aligned}$$

and take the expected value in [6], Prop. 2.3, to arrive at

$$k\varphi_{n/k} - \varphi_n = \sum_{i=1}^{p}\sum_{j=1}^{2^{i-1}} E[\chi_n(i,j)] - \sum_{i=1}^{p}\sum_{j=1}^{2^{i-1}} E[\epsilon_n(i,j)]. \tag{4.30}$$

In [6], Prop. 2.3, it is shown that, for p fixed, the nonnegative right-most sum is at most $C(\log n)^2$. The same applies for our choice of growing $p = p(n)$. Indeed, as each $\epsilon_n(i,j)$ is bounded by the intersection of the ranges of two independent SRW-s of length $n/2^i$, we have that $\max_{i,j} E[\epsilon_n(i,j)] \le \log n$ (see [24], Section 3.4), and with at most $2^p \le C \log n$ such terms, we conclude that

$$\sum_{i=1}^{p}\sum_{j=1}^{2^{i-1}} E\big[\chi_n(i,j)\big] - C(\log n)^2 \le k\varphi_{n/k} - \varphi_n \le \sum_{i=1}^{p}\sum_{j=1}^{2^{i-1}} E\big[\chi_n(i,j)\big]. \tag{4.31}$$

Recall [6], Prop. 6.1, that

$$\lim_{n\to\infty} \frac{2(\log n)^2}{\pi^4 n} E\big[\chi_n(1,1)\big] = \int_{A_1^1} E\big[G_\beta(\beta_s,\beta_t)\big]\,ds\,dt \quad \text{for } A_1^1 = [0,2^{-1}) \times (2^{-1},1].$$

Now, using (1.1) (at $d=4$), with $\int_{A_1^1} |t-s|^{-1}\,ds\,dt = \log 2$ and $E|\beta_1|^{-2} = \frac{1}{2}$, we see that

$$\int_{A_1^1} E\big[G_\beta(\beta_s,\beta_t)\big]\,ds\,dt = \frac{1}{2\pi^2} E\big[|\beta_1|^{-2}\big] \int_{A_1^1} |t-s|^{-1}\,dt\,ds = \frac{\log 2}{4\pi^2}.$$

By definition it follows that $E[\chi_n(i,j)] = E[\chi_{n_i'}(1,1)]$ for $n_i' := n2^{1-i}$ and all i, j. So in view of the preceding, we deduce that, for any $p = o(\log n)$, as $n \to \infty$,

$$\max_{i\le p}\left\{\left|\frac{2(\log n_i')^2}{\pi^4 n_i'} E\big[\chi_{n_i'}(1,1)\big] - \frac{\log 2}{4\pi^2}\right|\right\} \to 0.$$

Recall also (see (1.5)) that

$$p = \big(1+o(1)\big)\frac{\log_3 n}{\log 2}, \qquad h_4(n) = \frac{\pi^2}{8}\frac{n \log_3 n}{(\log n)^2}.$$

It thus follows that

$$\sum_{i=1}^{p}\sum_{j=1}^{2^{i-1}} E\big[\chi_n(i,j)\big] = \big(1+o(1)\big) p \frac{\log 2}{4\pi^2}\frac{\pi^4 n}{2(\log n)^2} = \big(1+o(1)\big) h_4(n), \tag{4.32}$$

and combining (4.31) and (4.32), we arrive at

$$k\varphi_{n/k} - \varphi_n = \big(1+o(1)\big) h_4(n). \tag{4.33}$$

In view of (4.24), (4.26) and (4.33), we get our limsup-LIL lower bound, precisely as in the proof of [11], Prop. 4.4, once we find for any $\varepsilon > 0$, constants $c_1 < \infty$, $c_2 > 0$ and for any $k = 2^p$, $m = n/k$, p as above, some events G_k such that $P(G_k) \ge \frac{1}{4} c_2^k$ and

$$G_k \subseteq \bigcap_{j=1}^{k}\left\{\overline{U}_j \ge -\frac{c_1 m}{(\log m)^2}\right\} =: \bigcap_{j=1}^{k} B_j, \tag{4.34}$$

$$G_k \subseteq \left\{\sum_{j=1}^{k-1} W_j \le 3\varepsilon \frac{n \log_3 n}{(\log m)^2}\right\}. \tag{4.35}$$

To this end, it suffices to construct events F_k such, that for some $c_3 < \infty$,

$$P(F_k) \ge c_2^k, \qquad F_k \subseteq \left\{\max_{j<k}\{\underline{\hat{W}}_j\} \le \frac{c_3 m}{(\log m)^2}\right\} \bigcap_{j=1}^{k} B_j. \tag{4.36}$$

Indeed, we shall see that $P(\mathcal{C}_i) \le \frac{1}{4} c_2^k$ for $k \le \gamma \log_2 n$, $\gamma > 0$ small and $n \to \infty$, where

$$\mathcal{C}_1 := \left\{\sum_{j \text{ odd}} \underline{W}_j > \varepsilon \frac{n \log_3 n}{(\log m)^2}\right\}, \qquad \mathcal{C}_2 := \left\{\sum_{j \text{ even}} \underline{W}_j > \varepsilon \frac{n \log_3 n}{(\log m)^2}\right\},$$

$$\mathcal{C}_3 := \left\{\max_{j<k}\{W_j - \underline{W}_j - \underline{\hat{W}}_j\} > \frac{m}{(\log m)^2}\right\}.$$

Taking $G_k := F_k \bigcap_{i=1}^3 \mathcal{C}_i^c$, this would imply that $P(G_k) \ge \frac{1}{4} c_2^k$ for large k, and it is easy to check, as stated, that both (4.34) and (4.35) then hold for such G_k.

Next, utilizing the union bound, (4.29), Markov's inequality and (4.23), we get that

$$P(\mathcal{C}_3) \le \sum_{j=1}^{k-1} P\Big(W_j - \underline{W}_j - \underline{\hat{W}}_j \ge \frac{m}{(\log m)^2}\Big) = \sum_{j=1}^{k-1} P\Big(Y_{jm,m} - \underline{Y}_{jm,m} \ge \frac{m}{(\log m)^2}\Big)$$
$$\le \frac{(\log m)^4}{m^2} \sum_{j=1}^{k-1} E\big[(Y_{jm,m} - \underline{Y}_{jm,m})^2\big] \le C(\log m)^{4-\alpha/2} \sum_{j=1}^{k-1} j \le Ck^2(\log m)^{4-\alpha/2}.$$

In particular, for $\alpha > 8 + 2\gamma \log(1/c_2)$, k as above and $n = mk \ge n_0$, the preceding bound implies that $P(\mathcal{C}_3) \le \frac{1}{4} c_2^k$. Turning to deal with $\underline{W}_j$ and $\underline{\hat{W}}_j$, upon expressing (4.21) via the independent SRW-s $\hat{S}_i$, $S_i^+ := S_{n'+i} - S_{n'}$ and $S_i^- := S_{n'-i} - S_{n'}$, at $n' = n_{2\alpha}$, it follows from [5], (1.4), that

$$\hat{g}_{n,\alpha} = E\big[1_{\{0 \notin \mathcal{R}_{n'}^+\}} \hat{P}^0(\hat{\tau}_{\mathcal{R}_{n'}^+ \cup \mathcal{R}^-([0,n'-1])} = \infty)\big]$$
(4.37)
$$= P\big(0 \notin \mathcal{R}_{n'}^+, \hat{R}_\infty \cap \big(\mathcal{R}_{n'}^+ \cup \mathcal{R}^-([0,n'-1])\big) = \varnothing\big) = \big(1+o(1)\big)\frac{\pi^2}{8}(\log n)^{-1}$$

(note that $\log n' = (1+o(1))\log n$). In view of (4.27), we note that $\{m^{-1}(\hat{g}_{m,\alpha})^{-2}\underline{W}_j\}$ are, for odd j, independent copies of X_m of Lemma 4.1 (except for now including also $\ell = 0$ in X_m). It thus follows from (4.5) and (4.37) that for some $c > 0$, and all k, m,

$$E\Big[\exp\Big(cm^{-1}(\log m)^2 \sum_{j \text{ odd}} \underline{W}_j\Big)\Big] \le \exp(k/c).$$

Hence, for $n = mk$, one has by Markov's inequality that

$$P(\mathcal{C}_1) \le \exp(-\varepsilon ck \log_3 n)\exp(k/c)$$

decays as $n \to \infty$, faster than $\frac{1}{4}c_2^k$. By the same reasoning, this applies also for $\mathcal{C}_2$.

Finally, in view of (4.13), (4.28) and (4.37), for some $C < \infty$ and any m, j,

$$\underline{\hat{W}}_j \le \frac{Cm^2}{(\log m)^2} \sum_{s=1}^{j-1} \operatorname{dist}(\mathcal{R}((s-1)m, sm]), \mathcal{R}((jm,(j+1)m]))^{-2}. \tag{4.38}$$

As in the proof of [11], Prop. 4.4, fixing a unit vector $\mathbf{u}$, we let $F_j := \bigcap_{i=1}^j (A_i \cap B_i)$, while taking here B_i of (4.34), and

$$A_i := \Big\{S_{im} \subset \mathbb{B}(i\sqrt{m}\mathbf{u}, \sqrt{m}/8), \mathcal{R}((i-1)m, im]) \subset \mathbb{B}\Big(\Big(i - \frac{1}{2}\Big)\sqrt{m}\mathbf{u}, \frac{3}{4}\sqrt{m}\Big)\Big\}.$$

The event F_k guarantees that, for any $s < j$, the distance of $\mathcal{R}((s-1)m, sm])$ from $\mathcal{R}((jm,(j+1)m])$ be at least $(j-s-1/2)\sqrt{m}$, so (4.38) results with the RHS of (4.36) (for $c_3 = C\sum(r-1/2)^{-2}$ finite). As for the LHS of (4.36), recall [6], Theorem 1.2, that $\{\frac{(\log m)^2}{m}\overline{R}_m\}$ converges in law, hence is a uniformly tight sequence. In particular, for any $\delta > 0$, there exists $c_1 = c_1(\delta)$ finite such that $P(B_1^c) \le \delta$ for B_1 of (4.34), uniformly in m. Further, $\{B_j, j \ge 1\}$ are i.i.d., and by the invariance principle, there exists $c_2 > 0$ such that

$$\lim_{m\to\infty} \inf_{S_0 \in \mathbb{B}(0,\sqrt{m}/8)} \{P^{S_0}(A_1)\} = \inf_{\beta_0 \in \mathbb{B}(0,\frac{1}{4})} \Big\{P\Big(|\beta_1 - 2\mathbf{u}| < \frac{1}{4}, \sup_{t\in[0,1]} |\beta_t - \mathbf{u}| < \frac{3}{2}\Big)\Big\} \ge 2c_2.$$

As F_j is measurable on $\sigma(S_i, i \le jm)$, by the Markov property of the SRW and its independent, stationary increments, for any $j \ge 1$,

$$P(A_j \cap B_j | F_{j-1}) \ge \inf_{S_0 \in \mathbb{B}(0,\sqrt{m}/8)} \{P^{S_0}(A_1)\} - P(B_1^c) \ge c_2,$$

provided $\delta > 0$ is small enough and $m \ge m_0$ finite, thereby establishing the LHS of (4.36). □

We conclude this subsection by proving Lemma 4.3. To this end, note that

$$E\big[(Y_{n,m} - \underline{Y}_{n,m})^2\big] = \frac{1}{2} \sum_{\pi} \sum_{\substack{(i_1,i_2)\in[1,n]^2,\\ (\ell_1,\ell_2)\in[1,m]^2}} E[g \cdot \mathcal{G}], \tag{4.39}$$

where the sum is over the two permutations π of $\{1, 2\}$ and

$$g := \big(g_{n,\alpha}(i_1)\tilde{g}_{m,\alpha}(\ell_{\pi_1}) - \hat{g}_{n,\alpha}\hat{g}_{m,\alpha}\big)\big(g_{n,\alpha}(i_2)\tilde{g}_{m,\alpha}(\ell_{\pi_2}) - \hat{g}_{n,\alpha}\hat{g}_{m,\alpha}\big), \tag{4.40}$$

$$\mathcal{G} := G(S_{i_1}, \tilde{S}_{\ell_{\pi_1}})G(S_{i_2}, \tilde{S}_{\ell_{\pi_2}}). \tag{4.41}$$

Setting for $\alpha > 0$,

$$I_\alpha(n) := [n_\alpha, n - n_\alpha]^2 \cap \{(i, j) : j - i \ge n_\alpha\},$$

the key to (4.23) is to bound $|E[g \cdot \mathcal{G}]|/E[\mathcal{G}]$ uniformly over $(i_1, i_2) \in I_\alpha(n)$ and $(\ell_1, \ell_2) \in I_\alpha(m)$. For $(i_1, i_2) \in I_\alpha(n)$, $n' = n_{2\alpha}$, we will show that the contribution from the complement of

$$H_{i_1}^{(n)} := \{|S_{i_1} - S_{i_1-n'}| \le \sqrt{n'}\log n'\},$$

$$H_{i_1,i_2}^{(n)} := H_{i_1}^{(n)} \cap \{|S_{i_2} - S_{i_2-n'} + S_{i_1+n'} - S_{i_1}| \le \sqrt{n'}\log n'\}$$

is negligible and the same applies for the analogous events $\widetilde{H}_{\ell_1}^{(m)}$, $\widetilde{H}_{\ell_1,\ell_2}^{(m)}$ defined in terms of the SRW $(\tilde{S}_\ell)$, $(\ell_1, \ell_2) \in I_\alpha(m)$ and $m' = m_{2\alpha}$. Further, from (4.21) it follows that $E[g] = 0$ for such (i_1, i_2) and (ℓ_1, ℓ_2), allowing us to instead bound (in terms of $E[\mathcal{G}]$), the value of

$$\big|E[g\mathcal{G}1_{H_{i_1,i_2}^{(n)} \cap \widetilde{H}_{\ell_1,\ell_2}^{(m)}}] - E[g1_{H_{i_1,i_2}^{(n)} \cap \widetilde{H}_{\ell_1,\ell_2}^{(m)}}]E[\mathcal{G}]\big|.$$

Decomposing the events $H_{i_1}^{(n)}$ and $H_{i_1,i_2}^{(n)}$ as

$$H_{i_1}^{(n)} = \bigcup_{|u|\le\sqrt{n'}\log n'} H_{i_1}^{(n')}(u), \qquad H_{i_1,i_2}^{(n)} = \bigcup_{|u|,|v|\le\sqrt{n'}\log n'} H_{i_1,i_2}^{(n')}(u, v),$$

$$H_{i_1}^{(n')}(u) := \{S_{i_1} - S_{i_1-n'} = u\},$$

$$H_{i_1,i_2}^{(n')}(u, v) := H_{i_1}^{(n')}(u) \cap \{S_{i_2} - S_{i_2-n'} + S_{i_1+n'} - S_{i_1} = v\},$$

and such decomposition for $\widetilde{H}_{\ell_1,\ell_2}^{(m)}$, we show in the sequel that given $H_{i_1,i_2}^{(n')}(u, v) \cap \widetilde{H}_{\ell_1,\ell_2}^{(m')}(\tilde{u}, \tilde{v})$, makes $\mathcal{G}$ independent of g, whereby the following estimates shall be utilized.

LEMMA 4.4. *Fix $\alpha > 2$, $\epsilon > 0$ and a permutation π of $\{1, 2\}$. Then for $n^\epsilon \le m \le n$,*

$$\begin{aligned} F_1(u, \tilde{u}) &:= E\big[G(S_{i_1}, \tilde{S}_{\ell_1}) \big| H_{i_1}^{(n')}(u) \cap \widetilde{H}_{\ell_1}^{(m')}(\tilde{u})\big] \\ &= \big(1 + O\big((\log n)^{2-\alpha}\big)\big)E\big[G(S_{i_1}, \tilde{S}_{\ell_1})\big], \end{aligned} \tag{4.42}$$

$$\begin{aligned} F_2(u, v, \tilde{u}, \tilde{v}) &:= E\big[G(S_{i_1}, \tilde{S}_{\ell_{\pi_1}})G(S_{i_2}, \tilde{S}_{\ell_{\pi_2}}) \big| H_{i_1,i_2}^{(n')}(u, v) \cap \widetilde{H}_{\ell_1,\ell_2}^{(m')}(\tilde{u}, \tilde{v})\big] \\ &= \big(1 + O\big((\log n)^{2-\alpha}\big)\big)E\big[G(S_{i_1}, \tilde{S}_{\ell_{\pi_1}})G(S_{i_2}, \tilde{S}_{\ell_{\pi_2}})\big], \end{aligned} \tag{4.43}$$

uniformly over $(i_1, i_2) \in I_\alpha(n)$, $(\ell_1, \ell_2) \in I_\alpha(m)$, $|u|, |v| \le \sqrt{n'}\log n'$ *and* $|\tilde{u}|, |\tilde{v}| \le \sqrt{m'}\log m'$.

PROOF. For $(i_1, i_2) \in I_\alpha(n)$, the law of (S_{i_1}, S_{i_2}), given $H^{(n')}_{i_1,i_2}(u, v)$, is as $(u + S^{(1)}_{i_1-n'}, u + v + S^{(1)}_{i_2-3n'})$ for an independent SRW $S^{(1)}_i$. Similarly, when $(\ell_1, \ell_2) \in I_\alpha(m)$, the law of $(\tilde{S}_{\ell_1}, \tilde{S}_{\ell_2})$, given $\widetilde{H}^{(m')}_{\ell_1,\ell_2}(\tilde{u}, \tilde{v})$, is as $(\tilde{u} + \tilde{S}^{(1)}_{\ell_1-m'}, \tilde{u} + \tilde{v} + \tilde{S}^{(1)}_{\ell_2-3m'})$. Consequently,

$$F_1(u, \tilde{u}) = E\big[G(u + S_{i_1-n'}, \tilde{u} + \tilde{S}_{\ell_1-m'})\big],$$

$$F_2(u, v, \tilde{u}, \tilde{v}) = \begin{cases} E\big[G(u + S_{i_1-n'}, \tilde{u} + \tilde{S}_{\ell_1-m'}) \\ \qquad \times G(u + v + S_{i_2-3n'}, \tilde{u} + \tilde{v} + \tilde{S}_{\ell_2-3m'})\big] & \text{if } \pi_1 = 1, \\ E\big[G(u + S_{i_1-n'}, \tilde{u} + \tilde{v} + \tilde{S}_{\ell_2-3m'}) \\ \qquad \times G(u + v + S_{i_2-3n'}, \tilde{u} + \tilde{S}_{\ell_1-m'})\big] & \text{if } \pi_1 = 2. \end{cases}$$

Note that for some $C = C(\epsilon)$ finite, $c = c(\epsilon) > 0$, any $m \ge n^\epsilon$ and (ℓ_1, ℓ_2),

$$P\big((H^{(m)}_{\ell_1})^c\big) \le P\big((H^{(m)}_{\ell_1,\ell_2})^c\big) \le 2P\big(|S_{2m'}| > \sqrt{m'}\log m'\big) \le Ce^{-c(\log n)^2} \tag{4.44}$$

with the same bound applying also for $P((H^{(n)}_{i_1,i_2})^c)$. Now, by (4.13) and (4.17) (at $p = 1, 2$),

$$E\big[G(S_i, \tilde{S}_\ell)\big] \le Ci^{-1/2}\ell^{-1/2}, \tag{4.45}$$

$$E\big[G(S_{i_1}, \tilde{S}_{\ell_{\pi_1}})G(S_{i_2}, \tilde{S}_{\ell_{\pi_2}})\big] \le Ci_1^{-1/2}(i_2 - i_1)^{-1/2}\ell_1^{-1/2}(\ell_2 - \ell_1)^{-1/2} \tag{4.46}$$

with the LHS of (4.45) and (4.46) being the expected values of $F_1(\cdot)$ and $F_2(\cdot)$ according to the joint law of the corresponding SRW increments (for independent SRW S_i and $\tilde{S}_\ell$). In view of (4.44), it thus suffices to bound the maximum fluctuation of $F_s(\cdot)$, $s = 1, 2$ over $|u|, |v| \le \sqrt{n'}\log n'$ and $|\tilde{u}|, |\tilde{v}| \le \sqrt{m'}\log m'$ by $C(\log n)^{2-\alpha}$ times the RHS of (4.45) and (4.46), respectively. To this end, since $F_1(\cdot)$ depends only on $u - \tilde{u}$ and $F_2(\cdot)$ depends only on $u - \tilde{u}$ and $v - \tilde{v}$ if $\pi_1 = 1$ or on $u + v - \tilde{u}$ and $v + \tilde{v}$ if $\pi_1 = 2$, we may WLOG fix $\tilde{u} = \tilde{v} = 0$ and consider the maximum fluctuation of

$$F_1(u) = E\big[G(u + S_{i_1-n'}, \tilde{S}_{\ell_1-m'})\big],$$

$$F_2(u, v) = \begin{cases} E\big[G(u + S_{i_1-n'}, \tilde{S}_{\ell_1-m'})G(v + S_{i_2-3n'}, \tilde{S}_{\ell_2-3m'})\big] & \text{if } \pi_1 = 1, \\ E\big[G(u + S_{i_1-n'}, \tilde{S}_{\ell_2-3m'})G(v + S_{i_2-3n'}, \tilde{S}_{\ell_1-m'})\big] & \text{if } \pi_1 = 2, \end{cases}$$

over $|u|, |v| \le 3\sqrt{n'}\log n'$. Further, with both n_α/n' and m_α/m' diverging (as $(\log n)^\alpha$), it follows that uniformly over $(i_1, i_2) \in I_\alpha(n)$, $(\ell_1, \ell_2) \in I_\alpha(m)$ and $m \ge n^\epsilon$, the RHS of (4.45) and (4.46) also bound $F_1(0)$ and $F_2(0, 0)$, respectively. Consequently, it suffices to show that, for some C finite and all $|u|, |v| \le 3\sqrt{n'}\log n'$,

$$\big|F_1(u) - F_1(0)\big| \le C(\log n)^{2-\alpha}F_1(0), \tag{4.47}$$

$$\big|F_2(u, v) - F_2(0, 0)\big| \le C(\log n)^{2-\alpha}F_2(0, 0). \tag{4.48}$$

Turning to this task, since $t_2 := \ell_2 - \ell_1 - 2m' \ge 0$, $t_3 := i_2 - i_1 - 2n' \ge 0$, $G(x, y) = G(y - x, 0)$ and $S_i \overset{(d)}{=} -S_i$, we can further simplify the functions $F_s(\cdot)$ to be

$$F_1(u) = E\big[G(S_{t_1}, u)\big],$$

$$F_2(u, v) = \begin{cases} E\big[G(S_{t_1}, u)G(S_{t_1+t_2+t_3}, v)\big] & \text{if } \pi_1 = 1, \\ E\big[G(S_{t_1+t_2}, u)G(S_{t_1+t_3}, v)\big] & \text{if } \pi_1 = 2, \end{cases}$$

where $t_1 = i_1 + \ell_1 - n' - m'$. Denoting by $p_j(u) := P(S_j = u)$, it is easy to check that

$$F_1(u) = \sum_{j_0 > t_1} p_{j_0}(u), \qquad F_2(u, v) = \sum_{j_1, j_2} p_{j_1}(u) p_{j_2}(v), \tag{4.49}$$

where the sum is over $j_1 > t_1$ and $j_2 > t_1 + t_2 + t_3$ in case $\pi = 1$, and over $j_1 > t_1 + t_2$, $j_2 > t_1 + t_3$ when $\pi_1 = 2$. By the local CLT for the SRW on $\mathbb{Z}^4$, we have for some $C < \infty$ that

$$\left| \frac{p_j(u) + p_{j+1}(u)}{p_j(0) + p_{j+1}(0)} - 1 \right| \le \frac{C|u|^2}{9j} \le Cn'(\log n')^2 t_1^{-1} \le 2C(\log n)^{2-\alpha}$$

throughout the range of parameters considered here (utilizing the fact that $j_0, j_1, j_2 \ge t_1 \ge n_\alpha/2$). The same bound applies with v instead of u, and plugging these bounds in (4.49) results with (4.47)–(4.48), thereby completing the proof of the lemma. □

PROOF OF LEMMA 4.3. First, observe that g of (4.40) can be written also as

$$\begin{aligned} g &= g_0 - g_1 - g_2 + \hat{g}_{n,\alpha}^2 \hat{g}_{m,\alpha}^2, \\ g_0 &:= g_{n,\alpha}(i_1)\tilde{g}_{m,\alpha}(\ell_{\pi_1}) g_{n,\alpha}(i_2)\tilde{g}_{m,\alpha}(\ell_{\pi_2}), \\ g_1 &:= g_{n,\alpha}(i_1)\tilde{g}_{m,\alpha}(\ell_{\pi_1})\hat{g}_{n,\alpha}\hat{g}_{m,\alpha}, \\ g_2 &:= \hat{g}_{n,\alpha}\hat{g}_{m,\alpha} g_{n,\alpha}(i_2)\tilde{g}_{m,\alpha}(\ell_{\pi_2}). \end{aligned}$$

Proceeding to show (4.23), note that $g_{n,\alpha}(i)$ and $\tilde{g}_{m,\alpha}(\ell)$ are measurable on $\mathcal{F}_i := (S_{i+j} - S_i, j \in (-n', n'])$, $n' := n_{2\alpha}$, and $\tilde{\mathcal{F}}_\ell := (\tilde{S}_{\ell+j} - \tilde{S}_\ell, j \in (-m', m'])$, $m' = m_{2\alpha}$, respectively. Further, when $(i_1, i_2) \in I_\alpha(n)$ and $(\ell_1, \ell_2) \in I_\alpha(m)$, under the event $H_{i_1,i_2}^{(n')}(u, v) \cap \widetilde{H}_{\ell_1,\ell_2}^{(m')}(\tilde{u}, \tilde{v})$ the law of $\mathcal{G}$ of (4.41), given $\mathcal{F}_{i_1}$, $\mathcal{F}_{i_2}$, $\tilde{\mathcal{F}}_{\ell_1}$ and $\tilde{\mathcal{F}}_{\ell_2}$, is determined by $H_{i_1,i_2}^{(n')}(u, v) \cap \widetilde{H}_{\ell_1,\ell_2}^{(m')}(\tilde{u}, \tilde{v})$. Thus, for $s = 0, 1, 2$, and any such (i_1, i_2), (ℓ_1, ℓ_2),

$$E[g_s \mathcal{G} 1_{H_{i_1,i_2}^{(n')}(u,v) \cap \widetilde{H}_{\ell_1,\ell_2}^{(m')}(\tilde{u},\tilde{v})}] = E\big[g_s 1_{H_{i_1,i_2}^{(n')}(u,v) \cap \widetilde{H}_{\ell_1,\ell_2}^{(m')}(\tilde{u},\tilde{v})} E\big[\mathcal{G} | H_{i_1,i_2}^{(n')}(u, v) \cap \widetilde{H}_{\ell_1,\ell_2}^{(m')}(\tilde{u}, \tilde{v})\big]\big].$$

With $g_s \ge 0$ and $E[g_s] = \hat{g}_{n,\alpha}^2 \hat{g}_{m,\alpha}^2$, whenever $(i_1, i_2) \in I_\alpha(n)$ and $(\ell_1, \ell_2) \in I_\alpha(m)$, we get, from (4.43) (of Lemma 4.4), that, for some universal $C < \infty$,

$$\begin{aligned} &\big|E[g_s \mathcal{G} 1_{H_{i_1,i_2}^{(n)} \cap \widetilde{H}_{\ell_1,\ell_2}^{(m)}}] - E[g_s 1_{H_{i_1,i_2}^{(n)} \cap \widetilde{H}_{\ell_1,\ell_2}^{(m)}}] E[\mathcal{G}]\big| \\ &\quad\le \sum_{\substack{|u|,|v| \le \sqrt{n'} \log n', \\ |\tilde{u}|,|\tilde{v}| \le \sqrt{m'} \log m'}} E[g_s 1_{H_{i_1,i_2}^{(n')}(u,v) \cap \widetilde{H}_{\ell_1,\ell_2}^{(m')}(\tilde{u},\tilde{v})}] \\ &\qquad \times \big|E\big[\mathcal{G} | H_{i_1,i_2}^{(n')}(u, v) \cap \widetilde{H}_{\ell_1,\ell_2}^{(m')}(\tilde{u}, \tilde{v})\big] - E[\mathcal{G}]\big| \\ &\quad\le C(\log n)^{2-\alpha} \hat{g}_{n,\alpha}^2 \hat{g}_{m,\alpha}^2 E[\mathcal{G}]. \end{aligned} \tag{4.50}$$

In addition, with $g_s \in [0, 1]$, $\mathcal{G}$ uniformly bounded and $(\log m)/(\log n) \ge \epsilon$, we have from (4.44) that

$$\begin{aligned} E[g_s \mathcal{G} 1_{(H_{i_1,i_2}^{(n)} \cap \widetilde{H}_{\ell_1,\ell_2}^{(m)})^c}] + E[g_s 1_{(H_{i_1,i_2}^{(n)} \cap \widetilde{H}_{\ell_1,\ell_2}^{(m)})^c}] E[\mathcal{G}] &\le C_1\big(P\big((H_{i_1,i_2}^{(n)})^c\big) + P\big((H_{\ell_1,\ell_2}^{(m)})^c\big)\big) \\ &\le 2Ce^{-c(\log n)^2}. \end{aligned} \tag{4.51}$$

Combining (4.50) and (4.51) for $s = 0, 1, 2$, we thus find for the zero-mean

$$g = g_0 - g_1 - g_2 + \hat{g}_{n,\alpha}^2 \hat{g}_{m,\alpha}^2$$

that, uniformly over π, $(i_1, i_2) \in I_\alpha(n)$ and $(\ell_1, \ell_2) \in I_\alpha(m)$,

$$
\begin{aligned}
\big|E[g \cdot \mathcal{G}]\big| &\le \sum_{s=0}^{2} \big|E[g_s \mathcal{G}] - E[g_s]E[\mathcal{G}]\big| \\
&\le 3C(\log n)^{2-\alpha} \hat{g}_{n,\alpha}^2 \hat{g}_{m,\alpha}^2 E[\mathcal{G}] + 6Ce^{-c(\log n)^2}. \qquad (4.52)
\end{aligned}
$$

Further, as $|g| \le 1$, we have from (4.13) and (4.17) at $p = 2$ that, for some $C_2 < \infty$ and uniformly over all $(i_1, i_2) \in [1, n]^2$, $(\ell_1, \ell_2) \in [1, m]^2$,

$$
\big|E[g \cdot \mathcal{G}]\big| \le E[\mathcal{G}] \le C_2 (i_1 \wedge i_2)^{-1/2} |i_2 - i_1|_+^{-1/2} (\ell_1 \wedge \ell_2)^{-1/2} |\ell_2 - \ell_1|_+^{-1/2}. \qquad (4.53)
$$

Next, note that from (4.37) we have, for some $C_3, C < \infty$,

$$
\begin{aligned}
&\hat{g}_{n,\alpha}^2 \hat{g}_{m,\alpha}^2 \sum_{\pi} \sum_{\substack{(i_1,i_2)\in[1,n]^2, \\ (\ell_1,\ell_2)\in[1,m]^2}} E[\mathcal{G}] \\
&\quad \le \frac{C_3}{(\log n)^2 (\log m)^2} \sum_{\substack{1 \le i_1 \le i_2 \le n \\ 1 \le \ell_1 \le \ell_2 \le m}} i_1^{-1/2} |i_2 - i_1|_+^{-1/2} \ell_1^{-1/2} |\ell_2 - \ell_1|_+^{-1/2} \\
&\quad \le \frac{Cnm}{(\log n)^2 (\log m)^2}.
\end{aligned}
$$

With the right-most term of (4.52) being $o(n^{-5})$, it follows that the overall contribution to the right side of (4.39) from $i_1, i_2 \in [n_\alpha, n - n_\alpha]$ with $|i_2 - i_1| \ge n_\alpha$ and $\ell_1, \ell_2 \in [m_\alpha, m - m_\alpha]$ with $|\ell_2 - \ell_1| \ge m_\alpha$ is at most $O(nm(\log n)^{-2-\alpha})$, as specified in (4.23). Further, the sum over the RHS of (4.53) under any of the following three restrictions:

$$
|i_2 - i_1| < n_\alpha, \qquad i_1 \wedge i_2 < n_\alpha, \qquad i_1 \vee i_2 > n - n_\alpha,
$$

is at most $O(nm\sqrt{n_\alpha/n}) = O(nm(\log n)^{-\alpha/2})$. With $(\log m)/(\log n) \ge \epsilon$, this applies also when summing the RHS of (4.53) under each of the analogous restrictions $|\ell_2 - \ell_1| < m_\alpha$, $\ell_1 \wedge \ell_2 < m_\alpha$ or $\ell_1 \vee \ell_2 > m - m_\alpha$. As $\alpha/2 < 2 + \alpha$, we have thus established (4.23).

Turning to (4.22), we similarly have from (4.42) of Lemma 4.4 that for some $C < \infty$ and $c > 0$, uniformly over $m \in [n^\epsilon, n]$, $i \in [n_\alpha, n - n_\alpha]$ and $\ell \in [m_\alpha, m - m_\alpha]$,

$$
\begin{aligned}
E\big[g_{n,\alpha}(i)\tilde{g}_{m,\alpha}(\ell) G(S_i, \tilde{S}_\ell)\big] &\le \big[1 + C(\log n)^{2-\alpha}\big] \hat{g}_{n,\alpha} \hat{g}_{m,\alpha} E\big[G(S_i, \tilde{S}_\ell)\big] + 2Ce^{-c(\log m)^2} \\
&\le C(\log n)^{-2} i^{-1/2} \ell^{-1/2} + 2Ce^{-c(\log m)^2} \qquad (4.54)
\end{aligned}
$$

(using in the latter inequality also (4.37), (4.13) and (4.17) at $p = 1$). As $\log m \ge \epsilon \log n$, the sum of the RHS of (4.54) over $i \le n$ and $\ell \le m$ is at most as specified (i.e., $O(\sqrt{nm}(\log n)^{-2})$. Further, even when $i < n_\alpha$ or $\ell < m_\alpha$ or $i > n - n_\alpha$ or $\ell > m - m_\alpha$, we still get the bound $Ci^{-1/2}\ell^{-1/2}$ on the LHS of (4.54). The sum of $i^{-1/2}\ell^{-1/2}$ subject to any one of the latter four restrictions is at most $O(\sqrt{m_\alpha n}) = O(\sqrt{nm}(\log m)^{-\alpha/2})$, which is as required (for $\alpha > 4$).

Finally, recall that, for some C, C_3 finite and all $m, n \in \mathbb{N}$,

$$
EY_{m,n} \le \sum_{i=1}^{n} \sum_{\ell=1}^{m} E\big[G(S_i, \tilde{S}_\ell)\big] \le C_3 \sum_{i=1}^{n} \sum_{\ell=1}^{m} i^{-1/2} \ell^{-1/2} \le C\sqrt{nm},
$$

as claimed. $\square$

4.2. *The upper bound in the limsup-*LIL. As in case of the capacity limsup-LIL lower bound, we adapt here the relevant element from the proof of the limsup-LIL of $|\mathcal{R}_n|$ and SRW $\mathbb{Z}^2$, namely [11], Prop. 4.1. To this end, we first establish a key approximate additivity for $\varphi_n := ER_n$.

LEMMA 4.5. *There exists c' finite, such that, for any $a, b \in \mathbb{N}$,*

$$0 \le \varphi_a + \varphi_b - \varphi_{a+b} \le c' \frac{\alpha^{1/4}(a+b)}{(\log(a+b))^2}, \tag{4.55}$$

where $\varphi_n := ER_n$ and $\alpha := \min(a,b)/(a+b)$.

PROOF. Starting at the expected value of (4.1), we get by the same reasoning we have used in deriving (4.26) that

$$0 \le E[V_{0,a,a+b}] = \varphi_a + \varphi_b - \varphi_{a+b} \le 2E \sum_{i=1}^{a} \sum_{\ell=1}^{b} g_{a,\alpha}(i) G(S_i, \tilde{S}_\ell) \tilde{g}_{b,\alpha}(\ell) = 2EY_{a,b}.$$

Assuming WLOG that $a \le b$, it thus suffices to verify that $EY_{a,b} \le Ca^{1/4}b^{3/4}/(\log b)^2$ (yielding (4.55) for some $c'(C) < \infty$). Indeed, for $a \ge \sqrt{b}$, this follows from (4.22), whereas if $a < \sqrt{b}$, then even the bound $EY_{a,b} \le C\sqrt{ab}$, which we have from Lemma 4.3, suffices. □

Recall (4.1) that $R_{a+b} - R_a - R_b \circ \theta_a = -V_{0,a,a+b} \le 0$ for any $a, b \ge 0$. This implies a *nonrandom bound* on the difference of such centered variables, yielding in terms of the nonrandom c' of Lemma 4.5 the upper bound

$$\overline{R}_{a+b} - \overline{R}_a - \overline{R}_b \circ \theta_a \le c' \left(\frac{\min(a,b)}{a+b} \right)^{1/4} \frac{(a+b)}{(\log(a+b))^2}. \tag{4.56}$$

Utilizing (4.56), we next establish sharp tail estimates for $\max_{j \le n}\{\overline{R}_j\}$ (in particular, improving upon [18], Lemma 2.5).

LEMMA 4.6. *For some $c > 0$, $C < \infty$ and all n,*

$$E\big[e^{cD^{(n)}}\big] \le C, \qquad D^{(n)} := \frac{(\log n)^2}{n} \max_{0 \le j \le n} \{\overline{R}_j\}. \tag{4.57}$$

PROOF. Note that (4.57) matches the statement of [11], (4.4), for $G_j := G_j^n := \frac{(\log n)^2}{n} \overline{R}_j$, $j \le n$. It is easy to check that the proof of [11], (4.3) and (4.4), applies verbatim for any variables $\{G_j\}$ that satisfy [11], (4.5) and (4.6), and, furthermore, that their argument applies, even if the power $\alpha^{1/2}$ on the right-most term in [11], (4.5), is replaced by $\alpha^{1/4}$. Indeed, this is what we have here, with (4.56) yielding that, for some nonrandom $c_1 < \infty$ and all $a \le j \le n$,

$$G_j^n - G_a^n \le G_{j-a}^n \circ \theta_a + c_1 \left(\frac{a}{j} \wedge \frac{j-a}{j} \right)^{1/4}.$$

To finish the proof note that, for some $c_2 < \infty$ and all $a, b \ge 0$, we get from [6], Cor. 1.5, that

$$E\big[(\overline{R}_b \circ \theta_a)^2\big] \le \frac{c_2 b^2}{(\log b)^4} \quad \text{hence } E\big[(G_j^n \circ \theta_a)^2\big] \le c_2 \frac{j^2 (\log n)^4}{n^2 (\log j)^4}, \tag{4.58}$$

which is precisely [11], (4.6). □

For the limsup-LIL upper bound, by Borel–Cantelli it suffices to show that, for any $q > 1$, $\gamma > 0$ and $\varepsilon > 0$,

$$\sum_i P\biggl(\frac{(\log m)^2}{m} \max_{r_{i-1}<\ell\le r_i} \{\overline{R}_\ell\} \ge \biggl(\frac{\pi^2}{8}+2\varepsilon\biggr) k \log k\biggr) < \infty, \tag{4.59}$$

where $r_i = q^i$, $k = 2^p$ for $p = [(\log\gamma + \log_3 r_i)/\log 2]$ and $m = \lceil r_i/k \rceil$. Now, considering (4.2) for $n_j = jm$, $j < k'$ and $n_{k'} = \ell$, it follows from (4.55) that

$$\frac{(\log m)^2}{m} \max_{(k'-1)m<\ell\le k'm} \{\overline{R}_\ell\} \le \sum_{j=1}^{k'} D_j^{(m)} + \frac{(\log m)^2}{m}\bigl(k'\varphi_m - \varphi_{k'm}\bigr) + c', \tag{4.60}$$

where $D_j^{(m)}$ are i.i.d. copies of $D^{(m)}$ of Lemma 4.6. With $k' \mapsto (k'\varphi_m - \varphi_{k'm})$ nondecreasing and $D^{(m)} \ge 0$, the maximum over $k' \le k$ of the RHS of (4.60) is attained at $k' = k$. Further, en route to (4.33), we showed that, as $p = o(\log m) \to \infty$,

$$\frac{1}{k\log k}\frac{(\log m)^2}{m}(k\varphi_m - \varphi_{km}) \to \frac{\pi^2}{8}. \tag{4.61}$$

Thus, noting that (4.57) results with

$$P\biggl(\sum_{j=1}^{k} D_j^{(m)} \ge \varepsilon k \log k\biggr) \le C^k e^{-\varepsilon c k(\log k)},$$

which is summable over i for our choice of $k = \gamma \log i$, we have established (4.59) and thereby completed the proof of the limsup-LIL.

4.3. *Nonrandom and positive liminf*-LIL. Setting hereafter $\tilde{h}_4(n) := n(\log_2 n)/(\log n)^2$, we first show that the $[-\infty,\infty]$-valued,

$$c_\star := -\liminf_{r\to\infty}\biggl\{\frac{\overline{R}_r}{\tilde{h}_4(r)}\biggr\},$$

is nonrandom. Indeed, recall (2.6) that $\mathrm{Cap}(\mathcal{R}_r) - \mathrm{Cap}(\mathcal{R}[k,r]) \in [0,k]$. Thus, for any k finite, changing $\mathcal{R}_k$ without altering S_k yields at most a difference of k in the value of $\mathrm{Cap}(\mathcal{R}_r)$, implying by the Hewitt–Savage zero-one law that $c_\star$ is nonrandom.

Turning next to show that $c_\star > 0$, it suffices to establish this for the subsequence $r_{j+1} = r_j + 2n_j$, where $n_j := 2^{j^2}$, $r_0 := 0$ and we proceed to show that infinitely many $-\overline{R}_{r_{j+1}}$ are at least of $O(\tilde{h}_4(n_j))$, due to heavy tails of the nonnegative variables $V_{0,n,2n}/\tilde{h}_4(n)$. Specifically, setting $Q_n := |\mathcal{R}(0,n] \cap \mathcal{R}(n,2n]|$, we have from [6], Prop. 1.6, that

$$V_{0,n,2n} \ge 2 \inf_{j,\ell\in[1,2n]}\{G(S_j,S_\ell)\} R_n R_{n,2n} - Q_n. \tag{4.62}$$

Recall from [24], Section 3.4, that $EQ_n \le C_0 \log n$ for some $C_0 < \infty$ and all n. Hence, by Markov's inequality

$$P(\hat{A}_n^c) := P\bigl(Q_n \ge (\log n)^3\bigr) \le C_0(\log n)^{-2}. \tag{4.63}$$

Further, recall [6], (1.4) and Cor. 1.5, that $ER_n \ge n/(\log n)$ and $\mathrm{Var}(R_n) \le C_1 n^2/(\log n)^4$ for some $C_1 < \infty$ and all n large, in which case, by Markov's inequality,

$$P\biggl(R_n \le \frac{n}{2\log n}\biggr) \le \biggl(\frac{2\log n}{n}\biggr)^2 \mathrm{Var}(R_n) \le \frac{4C_1}{(\log n)^2}.$$

Consequently, by the union bound,

$$P(A_n^c) := P\Big(\min(R_n, R_{n,2n}) \le \frac{n}{2\log n}\Big) \le 8C_1(\log n)^{-2}. \tag{4.64}$$

Next, from (4.13) we have that

$$F_{k,m} := \Big\{\max_{j\le 2km} |S_j| \le \sqrt{m}\Big\} \quad \Longrightarrow \quad \inf_{j,\ell\in[1,2km]}\{G(S_j, S_\ell)\} \ge (4Cm)^{-1}.$$

Setting $c = 1/(10C)$, it thus follows from (4.62) that for all $n = km \ge n'(C)$, on the event $G_n := F_{k,m} \cap \hat{A}_n \cap A_n$,

$$V_{0,n,2n} \ge \frac{2}{4Cm}\Big(\frac{n}{2\log n}\Big)^2 - (\log n)^3 \ge \frac{cnk}{(\log n)^2}.$$

Similarly to our derivation of the LHS of (4.36), it follows from the invariance principle that $P(F_{k,m}) \ge c_2^k$ for some $c_2 > 0$ and any $k, m \ge 1$. By (4.63)–(4.64) this implies in turn that $P(G_n) \ge \frac{1}{3}c_2^k$ for $k = 2^p = [\gamma \log_2 n]$, provided $\gamma' := \gamma \log(1/c_2) < 2$ and $n \ge n'$. To summarize, we have that, for $c' = c\gamma > 0$ and all $n \ge n'$,

$$P\big(V_{0,n,2n} \ge c'\tilde{h}_4(n)\big) \ge (\log n)^{-\gamma'}.$$

The same applies for the mutually independent $\{V_{r_j,r_j+n_j,r_{j+1}}\}$; hence, upon fixing $\gamma' < 1/2$, we get by the second Borel–Cantelli lemma that a.s.,

$$\limsup_{j\to\infty}\{\tilde{h}_4(n_j)^{-1} V_{r_j,r_j+n_j,r_{j+1}}\} \ge c' > 0. \tag{4.65}$$

Now, as in (4.24), for any $r \ge 0$, $n \ge 1$,

$$\overline{R}_{r,r+2n} = 2\varphi_n - \varphi_{2n} + \overline{R}_{r,r+n} + \overline{R}_{r+n,r+2n} - V_{r,r+n,r+2n}. \tag{4.66}$$

Considering (4.31) for $p = 1$ (i.e., $k = 2$), we see that, as $n \to \infty$,

$$\frac{(\log n)^2}{n}[2\varphi_n - \varphi_{2n}] \le \frac{(\log n)^2}{n} E\big[\chi_{2n}(1,1)\big] \to \frac{\pi^2 \log 2}{4}. \tag{4.67}$$

Further, recall (4.59) that, for any $\delta > 0$,

$$\sum_j P\big(\overline{R}_{n_j} \ge (1+\delta)h_4(n_j)\big) < \infty. \tag{4.68}$$

The same applies, of course, also for $\overline{R}_{n_j} \circ \theta^{r_j}$ and $\overline{R}_{n_j} \circ \theta^{r_j+n_j}$, so with $h_4(n)/\tilde{h}_4(n) \to 0$, we deduce from (4.65)–(4.68) (at $n = n_j$ and $r = r_j$) that a.s.,

$$\liminf_{j\to\infty}\Big\{\frac{\overline{R}_{r_j,r_{j+1}}}{\tilde{h}_4(n_j)}\Big\} \le -\limsup_{j\to\infty}\{\tilde{h}_4(n_j)^{-1} V_{r_j,r_j+n_j,r_{j+1}}\} \le -c'.$$

Now, from (4.56) (at $a = r_j$, $b = 2n_j$),

$$\overline{R}_{r_{j+1}} \le \overline{R}_{r_j} + \overline{R}_{r_j,r_{j+1}} + \frac{c_1 r_{j+1}}{(\log r_{j+1})^2},$$

and since $r_{j+1} \le 3n_j$, dividing by $\tilde{h}_4(n_j)$ and taking limits yields that

$$-3c_\star \le \liminf_{j\to\infty}\{\tilde{h}_4(n_j)^{-1}\overline{R}_{r_{j+1}}\} \le -c' + \limsup_{j\to\infty}\{\tilde{h}_4(n_j)^{-1}\overline{R}_{r_j}\}.$$

The last term is a.s. zero (as (4.68) applies also for $\{r_j\}$ instead of $\{n_j\}$ and $h_4(n)/\tilde{h}_4(n) \to 0$), so we conclude that $c_\star \ge c'/3 > 0$.

4.4. *Finiteness of the liminf*-LIL. We show that $c_\star \le c_o < \infty$ by following the proof of the upper bound of [10], Theorem 1.7, (on the liminf LIL of $|\mathcal{R}_n|$ in $\mathbb{Z}^2$), while replacing [10], Theorem 1.5, and [10], Lemma 10.3, respectively, by

$$\sup_n \{(\log n)(\log_2 n)^2 P(-\overline{R}_n > c_o \tilde{h}_4(n))\} < \infty, \tag{4.69}$$

$$\sup_n \Big\{(\log n)^2 P\Big(\max_{n/q_0 \le k \le n} (\overline{R}_n - \overline{R}_k) > \epsilon \tilde{h}_4(n)\Big)\Big\} < \infty, \tag{4.70}$$

holding for some $c_o < \infty$, any $\epsilon > 0$ and some $q_0(\epsilon) > 1$.

Similarly to $|\mathcal{R}_n|$, the capacity is subadditive (see (4.1)) and upon centering satisfies (4.56), which is the analog of [10], (10.2). Thus, the bound (4.70) follows, as in the proof of [10], Lemma 10.3, now using (4.57) to arrive at [10], (10.14), and to bound the RHS of [10], (10.15)).

Since $|\overline{R}_n - \overline{R}_{n'}| \le |n - n'|$, it suffices to prove (4.69) only for some $\{n_i\}$ such that $n_{i+1} - n_i = o(\tilde{h}_4(n_i))$. We take here all integers of the form $n = mk$, $k = 2^p$, $p = [(\log_2 n)/\log 2]$ (thus with gaps of size $k = O(\log n) = o(\tilde{h}_4(n))$). Setting such values and $n'_u := 2^{-u} n$ for $1 \le u \le p$, we have as in (4.24), now using an alternative expression for $\Delta_{n,k}$ of (4.2), that

$$-\overline{R}_n = \overline{\Delta}_{n,k} - \sum_{j=1}^k \overline{U}_j, \qquad \Delta_{n,k} = \sum_{u=1}^p \sum_{j=1}^{2^{u-1}} V_{(2j-2)n'_u,(2j-1)n'_u,2jn'_u}, \tag{4.71}$$

with k i.i.d. copies $\{\overline{U}_j\}$ of $\overline{R}_m$ and the i.i.d. variables $\{V_{(2j-2)n'_u,(2j-1)n'_u,2jn'_u}\}$ per fixed $u \ge 1$. Since $\mathrm{Var}(\overline{R}_m) \le C_1 m^2/(\log m)^4$, it follows by Markov's inequality that

$$P\Big(-\sum_{j=1}^k \overline{U}_j \ge \epsilon c_o \tilde{h}_4(n)\Big) \le (\epsilon c_o \tilde{h}_4(n))^{-2} \frac{C_1 k m^2}{(\log m)^4} \le \frac{2C_1}{(\epsilon c_o)^2 k (\log_2 n)^2}. \tag{4.72}$$

Setting $c_o > (1-\epsilon)^{-1} c_1^{-1}$, we arrive at (4.69) out of (4.71), (4.72) and the following lemma.

LEMMA 4.7. *For some $c_1 > 0$ and any $\lambda > 0$,*

$$\limsup_{n\to\infty} \frac{1}{\log_2 n} \log P(\overline{\Delta}_{n,k} \ge \lambda \tilde{h}_4(n)) \le -c_1 \lambda, \tag{4.73}$$

where $\Delta_{n,k}$ are as in (4.3) for $k = 2^p$ and $p = [(\log_2 n)/\log 2]$.

We note in passing [18], Lemma 2.6, which is somewhat related to Lemma 4.7. The proof of Lemma 4.7 relies in turn on our next result.

LEMMA 4.8. *Set $p = [(\log_2 n)/\log 2]$ and for any $r \in \mathbb{N}$, the partition $I_i^{(r)} := ((i-1)r, ir]$ of $\mathbb{N}$. Consider for $n'_u := 2^{-u} n$ and each $0 \le u \le p$, the i.i.d variables*

$$\alpha_j^{(n'_u)} := \frac{1}{n} \sum_{i \in I_{2j-1}^{(n'_u)}} \sum_{\ell \in I_{2j}^{(n'_u)}} G(S_i, S_\ell), \qquad 1 \le j \le 2^{u-1}.$$

Then for some $c_2 > 0$,

$$\sup_{n\in\mathbb{N}} E[e^{c_2 \overline{\Theta}_n}] < \infty, \qquad \Theta_n := \sum_{u=1}^p \sum_{j=1}^{2^{u-1}} \alpha_j^{(n'_u)}. \tag{4.74}$$

PROOF. With the SRW having independent and symmetric increments, one easily verifies that $\alpha_1^{(n'_u)} \overset{(d)}{=} 2^{-u} X_{n'_u}$ for X_n of Lemma 4.1. Consequently, from (4.4) and (4.5), we know that

$$\varphi(\lambda) := \sup_{u,n\in\mathbb{N}} E\big[\exp(\lambda 2^u \overline{\alpha}_1^{(n'_u)})\big] \le 1 + c\lambda^2 < \infty \tag{4.75}$$

for some $c < \infty$ and all $\lambda > 0$ small enough. The uniform MGF bound of (4.74) then follows as in the proof of [26], Theorem 1 (see Page 177 of [26]), upon setting $\alpha_0 = \alpha_1^{(n'_0)}$, $c_2 = b_\infty > 0$ of [26] and noting that (4.75) suffices in lieu of both [26], Lemma 2, and the scale invariance of [26], property (ii). □

PROOF OF LEMMA 4.7. For $u \le p$, consider the i.i.d. variables $W_{u,j} := W^{(n'_u)} \circ \theta_{(2j-2)n'_u}$, with

$$W^{(m)} := \sum_{i,\ell\in[1,m]} g_{m,\alpha}(i) G(S_i, S_{\ell+m}) g_{m,\alpha}(\ell) \circ \theta_m$$

having the law of W_1 of (4.26) and their (i.i.d.) approximations $\underline{W}_{u,j} := n\hat{g}^2_{n'_u,\alpha} \alpha_j^{(n'_u)}$ (for $g_{m,\alpha}(\cdot)$ and $\hat{g}_{m,\alpha}$ of Lemma 4.3). Setting

$$Z_u := \sum_{j=1}^{2^{u-1}} W_{u,j}, \qquad \underline{Z}_u := \sum_{j=1}^{2^{u-1}} \underline{W}_{u,j},$$

it follows by Cauchy–Schwarz and (4.23) that, for some $C < \infty$, any $u \le p$ and all n,

$$E\big[(Z_u - \underline{Z}_u)^2\big] \le 2^{2(u-1)} E\big[(W_{u,1} - \underline{W}_{u,1})^2\big] \le Cn^2 (\log n)^{-\alpha/2}. \tag{4.76}$$

In particular, taking $\alpha > 8 + 2c_1\lambda$ yields, by the union bound and Markov's inequality that for any $\epsilon > 0$,

$$\begin{aligned} P\Big(\sum_{u=1}^{p} (Z_u - \underline{Z}_u) \ge \epsilon \tilde{h}_4(n)\Big) &\le \sum_{u=1}^{p} P\big(Z_u - \underline{Z}_u \ge \epsilon p^{-1} \tilde{h}_4(n)\big) \\ &\le Cp^3 \big(\epsilon \tilde{h}_4(n)\big)^{-2} n^2 (\log n)^{-\alpha/2} \le C' \epsilon^{-2} (\log n)^{-c_1\lambda}. \end{aligned} \tag{4.77}$$

We also find for our choice of k (see (4.61)) that

$$E\Delta_{n,k} = \frac{\pi^2}{8} \tilde{h}_4(n)\big(1 + o(1)\big).$$

Similarly to (4.26), we have that $V_{(2j-2)n'_u,(2j-1)n'_u,2jn'_u} \le 2W_{u,j}$ for any u, j. In view of (4.71) and (4.77), it thus suffices to establish (4.73) with $\overline{\Delta}_{n,k}$ replaced by

$$2\sum_{u=1}^{p} \underline{Z}_u - \frac{\pi^2}{8} \tilde{h}_4(n),$$

which in view of (4.37) and the definition of Θ_n can be further replaced by

$$\frac{\pi^2}{8} \frac{n}{(\log n)^2} \bigg(\Big(\frac{\pi^2}{4} + o(1)\Big)\Theta_n - \log_2 n\bigg).$$

Moreover, by (4.76)

$$\bigg| E\sum_{u=1}^{p} Z_u - E\sum_{u=1}^{p} \underline{Z}_u \bigg| \le O\Big(\frac{pn}{(\log n)^{\alpha/4}}\Big) = o\big(\tilde{h}_4(n)\big)$$

and combining [6], Lemma 6.5, with the considerations as in (4.44) and after (4.54), we deduce that, for large m,

$$2E[W^{(m)}] = (1+o(1))E[V_{0,m,2m}].$$

It then follows that

$$2\sum_{u=1}^{p} E\underline{Z}_u = (1+o(1))\frac{\pi^2}{8}\tilde{h}_4(n),$$

and consequently,

$$\frac{\pi^2}{4}E\Theta_n = (1+o(1))\log_2 n.$$

Thus, it suffices to establish (4.73) with $\overline{\Delta}_{n,k}$ replaced by

$$\frac{\pi^4}{32}\frac{n}{(\log n)^2}\overline{\Theta}_n,$$

which in turns follows from (4.74), upon setting $c_1 = 32\pi^{-4}c_2 > 0$. $\square$

5. LIL for SRW on $\mathbb{Z}^d$, $d \geq 5$: Proof of Theorem 1.4. Since R_n for the SRW on $\mathbb{Z}^d$, $d \geq 5$, has similar structural properties to the size of the range of the SRW on $\mathbb{Z}^{d-2}$, we establish Theorem 1.4 by adapting the proof in [11], Section 3, for the LIL of the latter sequence. Specifically, setting $\rho_n := \sqrt{n \log n}$ when $d = 5$ and otherwise $\rho_n := \sqrt{n}$, we have from [31], Theorem A, in case $d = 5$, and from [5], Lemma 3.3, when $d \geq 6$, that

$$\lim_{n\to\infty}\frac{1}{\rho_n}\|\overline{R}_n\|_2 = \sigma_d. \tag{5.1}$$

Next, recalling for integers $0 \leq a \leq b \leq c$, the notations of (4.1),

$$R_{a,b} := \mathrm{Cap}(\mathcal{R}(a,b]), \qquad V_{a,b,c} := R_{a,b} + R_{b,c} - R_{a,c} \geq 0, \tag{5.2}$$

we proceed with the following variant of [5], Lemma 3.2.

LEMMA 5.1. *For any $0 \leq a < b$, set*

$$\widetilde{V}_{a,b} := \sup_{t\geq b}\{V_{a,b,t}\}, \qquad \hat{V}_{a,b} := \sup_{s\leq a}\{V_{s,a,b}\}.$$

Then for some $C_{d,\ell}$ finite, any $\ell \geq 1$ and all $a < b$,

$$E[\widetilde{V}_{a,b}^{\ell}] \leq C_{d,\ell} f_d(b-a)^{\ell}, \qquad E[\hat{V}_{a,b}^{\ell}] \leq C_{d,\ell} f_d(b-a)^{\ell}, \tag{5.3}$$

where

$$f_5(n) = \sqrt{n}, \qquad f_6(n) = \log n, \qquad f_d(n) = 1 \quad \forall d \geq 7.$$

PROOF. By the shift invariance of the SRW, we may WLOG set $a = 0$. Further, in view of [6], (2.9) and (2.11), for a fixed set A, the function

$$B \mapsto \mathrm{Cap}(A) + \mathrm{Cap}(B) - \mathrm{Cap}(A \cup B)$$

is nondecreasing (and bounded above by $\mathrm{Cap}(A)$). In particular, the value of $\widetilde{V}_{0,n}$ is attained for $t \to \infty$. Thus, from [5], Prop. 1.2, we arrive at

$$\widetilde{V}_{0,n} \leq 2\sum_{x\in\mathcal{R}_n}\sum_{y\in\mathcal{R}(n,\infty)} G(x,y) \stackrel{d}{=} 2\sum_{x\in\mathcal{R}_n}\sum_{y\in\hat{\mathcal{R}}_\infty} G(x,y), \tag{5.4}$$

where $\hat{\mathcal{R}}_\infty$, denotes the range of an independent SRW. Similarly, the value of $\hat{V}_{0,n}$ is attained at $s \to -\infty$, with the right side of (5.4) also bounding $\hat{V}_{0,n}$ (we then have $\mathcal{R}(-\infty, 0]$ instead of $\mathcal{R}(n,\infty)$ in (5.4)). Thereafter, adapting [5], Section 3.1, yields (5.3). Indeed, with $p_{2k}(x,y) := P^x(S_{2k} = y)$ the square of a transition probability, we have as in the proof of [5], Lemma 3.1, that, for *even* $k \ge 0$ and any $a \in \mathbb{Z}^d$,

$$s_a := \sum_{x,y\in\mathbb{Z}^d} G(0,x)G(0,y)p_k(x,y+a) \le s_0.$$

In case of a lazy SRW, this applies for any $k \ge 0$, so summing over $k \le n$ yields that

$$\begin{aligned}
&\max_{a\in\mathbb{Z}^d}\Big\{\sum_{x,y\in\mathbb{Z}^d} G(0,x)G(0,y)G_n(x,y+a)\Big\}\\
&\quad= \sum_{x,y\in\mathbb{Z}^d} G(0,x)G(0,y)G_n(x,y) \qquad (5.5)\\
&\quad= \sum_{x,y\in\mathbb{Z}^d} G_n(0,x)G(0,y)G(x,y) \le C_d f_d(n),
\end{aligned}$$

where we have utilized [5], (3.4), for the latter inequality. Further, as in [5], up to an increase of C_d value, (5.5) extends to the original SRW. Now, similarly to [5], (3.5), it follows from (5.4) that

$$E[\widetilde{V}_{0,n}^\ell] \le 2^\ell \sum_{\underline{x},\underline{y}} E\Bigg[\prod_{i=1}^\ell L_n(x_i)\Bigg] E\Bigg[\prod_{i=1}^\ell L_\infty(y_i)\Bigg] \prod_{i=1}^\ell G(x_i,y_i),$$

where $L_n(x)$ denotes the total SRW local time at $x \in \mathbb{Z}^d$, during time interval $[1,n]$ (and the same bound applies for $E[\hat{V}_{0,n}^\ell]$). For $\ell = 1$, we thus get (5.3) out of (5.5) (as $E[L_n(x)] = G_n(0,x)$ and $E[L_\infty(y)] = G(0,y)$). The general case then follows by an inductive argument, as in the proof of [5], Lemma 3.2, utilizing also that $a = 0$ is optimal in (5.5). □

Utilizing Lemma 5.1, we next establish the analog of [11], Lemma 3.3, for R_n.

LEMMA 5.2. *For any $d \ge 5$, $m \ge 3$, there exists c_m finite such that, for all $b > a \ge 0$,*

$$\|\overline{R}_b - \overline{R}_a\|_m \le c_m \rho_{b-a}. \tag{5.6}$$

Further, for some $\bar{c}_m$ finite and any $\lambda > 0$, $b > a \ge 0$,

$$P\Big(\max_{n\in[a,b]}\{|\overline{R}_n - \overline{R}_a|\} > \lambda\rho_{b-a}\Big) \le \bar{c}_m \lambda^{-m}. \tag{5.7}$$

PROOF. From (5.2) we see that

$$V_{0,a,b} = R_a + R_{a,b} - R_b \in [0, \hat{V}_{a,b}]$$

for $\hat{V}_{a,b}$ of Lemma 5.1. In particular, for any $m \ge 3$, $b > a$,

$$0 \le E[V_{0,a,b}]^m \le E[V_{0,a,b}^m] \le E[\hat{V}_{a,b}^m] \le C_{d,m} f_d(b-a)^m = o(\rho_{b-a}^m).$$

We can thus replace $R_b - R_a$ in (5.6) by $R_{a,b}$ and thereby, due to the shift invariance of the increments, set WLOG $a = 0$ (whereupon $R_a = 0$). Hence, analogously to [11], (3.34), it suffices for (5.6) to show inductively over $\ell \ge 1$ that $\sup_n\{L_{n,2\ell}\}$ is finite for

$$L_{n,\ell} := \frac{1}{\rho_n}\|\overline{R}_n\|_\ell.$$

The induction basis $\ell = 1$ is merely (5.1). Further, with

$$\lim_{n\to\infty} \rho_{2n}^{-1} \sup_{a<2n} \|\overline{V}_{0,a,2n}\|_{2\ell} = 0,$$

by the preceding decomposition, we can and shall replace R_{2n} in the induction step by

$$R_n + R_{n,2n} \overset{d}{=} R_n + \hat{R}_n,$$

where $\hat{R}_n$ denotes the capacity of the range of an independent second SRW. For any $\ell \geq 2$, by the induction hypothesis $\sup_n\{L_{n,k}\}$ are finite for all $k \leq 2(\ell - 1)$; hence,

$$\sup_{n,n'} \sum_{k=2}^{2\ell-2} \binom{2\ell}{k} L_{n,k}^k L_{n',2\ell-k}^{2\ell-k} := c_\ell < \infty.$$

Recalling that $\rho_n \leq 2^{-1/2}\rho_{2n}$, we thus get similarly to [11], (3.37) and (3.38), that

$$L_{2n,2\ell} \leq o_n(1) + \bigl(2^{-(\ell-1)} L_{n,2\ell}^{2\ell} + 2^{-\ell} c_\ell\bigr)^{1/(2\ell)}$$

from which it follows as in [11] that $\sup_j\{L_{2^j,2\ell}\}$ is finite. Finally, for any $n \in [2^{j-1}, 2^j)$, $j \geq 2$, we have as in the preceding that

$$R_{2^j} \overset{d}{=} R_n + \hat{R}_{2^j-n} - V_{0,n,2^j}.$$

Upon centering, taking the 2ℓth power and isolating the 2ℓth power of $\overline{R}_n$, the preceding identity results with

$$\bigl[L_{2^j,2\ell} + o_j(1)\bigr]^{2\ell} + c_\ell \geq (\rho_n/\rho_{2^j})^{2\ell} L_{n,2\ell}^{2\ell} \geq 4^{-\ell} L_{n,2\ell}^{2\ell}.$$

Thus, $\sup_n\{L_{n,2\ell}\}$ is finite as well, completing the induction step and thereby establishing (5.6). Finally, we get (5.7) out of (5.6) precisely as in deriving [11], (3.39), out of [11], (3.40). □

Recall the decomposition (4.2)–(4.3), for the independent variables $U_j := R_{n_{j-1},n_j}$ and any increasing $\{n_k\}$ starting at $n_0 = 0$,

$$R_{n_k} = \sum_{j=1}^{k} U_j - \Delta_{n_k,k}, \qquad \Delta_{n_k,k} := \sum_{j=1}^{k-1} V_{n_{j-1},n_j,n_k}. \tag{5.8}$$

Centering the random variables of the preceding identity, we arrive at

$$\overline{R}_{n_k} = \sum_{j=1}^{k} \overline{U}_j - \overline{\Delta}_{n_k,k} \tag{5.9}$$

with zero-mean, independent variables $\overline{U}_j$. Proceeding to show that $\overline{\Delta}_{n_k,k}$ has a negligible effect on $\overline{R}_{n_k}$, first recall from (5.1) that

$$\lim_{j\to\infty} \frac{E[\overline{U}_j^2]}{\rho_{n_j-n_{j-1}}^2} = \sigma_d^2, \tag{5.10}$$

whereas (5.6) at $a = n_{j-1}$, $b = n_j$ amounts to

$$E\bigl[|\overline{U}_j|^m\bigr] \leq (c_m \rho_{n_j-n_{j-1}})^m. \tag{5.11}$$

In case $d = 5$, we take the same values of α, $\beta < 1/2$ and $\{n_k\}$ as in the proof of [11], Theorem 2.1. Lemma 5.1 at $\ell = 4$ is then the analog of [11], (3.2), and utilizing it at $a = n_{j-1}$,

$b = n_j$, $j < k$, we find by following verbatim, the derivation of [11], (3.9), that for some c finite

$$\limsup_{k\to\infty} \frac{|\overline{\Delta}_{n_k,k}|}{\sqrt{n_k}(\log n_k)^{\beta}} \le c \quad \text{a.s.} \tag{5.12}$$

Thereafter, substituting (5.12) for [11], (3.9), and (5.11) to get [11], (3.16), by the same reasoning as in the proof of [11], Theorem 2.1, we find that a.s.,

$$\lim_{k\to\infty} h_d(n_k)^{-1}[\overline{R}_{n_k} - \sigma_d B_{\rho^2_{n_k}}] = 0 \tag{5.13}$$

for some one-dimensional standard Brownian motion $(B_t, t \ge 0)$. As shown after [11], (3.17), (apart from replacing [11], Lemma 3.3(b), by (5.7)), the stated LIL is then a direct consequence of Kinchin's LIL for the latter Brownian motion.

In case $d \ge 6$, we take $\{n_k\}$ again, as in the proof of [11], Theorem 2.1, except that now this is done for the choice of $\alpha = 1$. Then by Lemma 5.1, for $C = C_{d,2}$ and any $1 \le j < k$,

$$\mathrm{Var}(V_{n_{j-1},n_j,n_k}) \le C\big(\log(n_j - n_{j-1})\big)^2 \le C(\log n_k)^2.$$

Thus, for any $\beta > 0$ and all k, by Markov's inequality and the definition of $\Delta_{n_k,k}$ (see (5.8)),

$$P\big(|\overline{\Delta}_{n_k,k}| \ge n_k^{\beta}\big) \le n_k^{-2\beta}\mathrm{Var}(\Delta_{n_k,k}) \le Cn_k^{-2\beta}k^2(\log n_k)^2.$$

Since $|\{n_k\} \cap [2^{\ell}, 2^{\ell+1})| \le \ell$ for any $\ell \ge 1$, eventually $k \le (\log n_k)^2$. Hence, by the first Borel–Cantelli lemma, we have that, for any $\beta > 0$,

$$\limsup_{k\to\infty} n_k^{-\beta}|\overline{\Delta}_{n_k,k}| \le 1 \quad \text{a.s.} \tag{5.14}$$

We then get (5.13) by following, as for $d = 5$, the proof of [11], Theorem 2.1, utilizing again (5.10)–(5.11), while having now, via (5.14) at $\beta < 1/2$, a negligible contribution at scale $\rho_n = \sqrt{n}$ (instead of (5.12) and the scale $\sqrt{n \log n}$ throughout [11], (3.13)–(3.17)). Finally, recall that $n_{k+1} - n_k \le n_k/\ell$ whenever $n_k \in [2^{\ell}, 2^{\ell+1})$. Hence, in view of (5.7) at $m = 6$ and $\lambda = \varepsilon h_d(n_k)/\sqrt{n_{k+1} - n_k}$, we have that, for some c_ε finite, any $\varepsilon > 0$ and $n_k \in [2^{\ell}, 2^{\ell+1})$,

$$P\Big(\max_{n\in(n_k,n_{k+1})}\{|\overline{R}_n - \overline{R}_{n_k}|\} > \varepsilon h_d(n_k)\Big) \le c_\varepsilon \ell^{-3}.$$

With at most ℓ values of such n_k, by the first Borel–Cantelli lemma, the events on the LHS a.s. occur only for finitely many values of k and the stated LIL thus follows, as before, from (5.13).

Acknowledgments. We thank the anonymous referees for their detailed feedback, which greatly improved the exposition of this work.

Funding. This research was supported in part by NSF Grant DMS-1954337 (A.D.), by JSPS KAKENHI grant-in-aid for early career scientists JP20K14329 (I.O.) and by a JSPS overseas research fellowship (I.O.).

REFERENCES

[1] ADHIKARI, A. and OKADA, I. (2023). Moderate deviations for the capacity of the random walk range in dimension four. Preprint. Available at arXiv:2310.07685.

[2] ASSELAH, A. and SCHAPIRA, B. (2020). Deviations for the capacity of the range of a random walk. *Electron. J. Probab.* **25** Paper No. 154, 28. MR4193895 https://doi.org/10.1214/20-ejp560

[3] ASSELAH, A. and SCHAPIRA, B. (2020). On the nature of the Swiss cheese in dimension 3. *Ann. Probab.* **48** 1002–1013. MR4089500 https://doi.org/10.1214/19-AOP1380

[4] ASSELAH, A. and SCHAPIRA, B. (2021). The two regimes of moderate deviations for the range of a transient walk. *Probab. Theory Related Fields* **180** 439–465. MR4265025 https://doi.org/10.1007/s00440-021-01063-3

[5] ASSELAH, A., SCHAPIRA, B. and SOUSI, P. (2018). Capacity of the range of random walk on $\mathbb{Z}^d$. *Trans. Amer. Math. Soc.* **370** 7627–7645. MR3852443 https://doi.org/10.1090/tran/7247

[6] ASSELAH, A., SCHAPIRA, B. and SOUSI, P. (2019). Capacity of the range of random walk on $\mathbb{Z}^4$. *Ann. Probab.* **47** 1447–1497. MR3945751 https://doi.org/10.1214/18-AOP1288

[7] BAI, T. and HU, Y. (2022). Capacity of the range of branching random walks in low dimensions. *Tr. Mat. Inst. Steklova* **316** 32–46. MR4461469 https://doi.org/10.4213/tm4217

[8] BAI, T. and HU, Y. (2023). Convergence in law for the capacity of the range of a critical branching random walk. *Ann. Appl. Probab.* **33** 4964–4994. MR4674069 https://doi.org/10.1214/23-aap1938

[9] BAI, T. and WAN, Y. (2022). Capacity of the range of tree-indexed random walk. *Ann. Appl. Probab.* **32** 1557–1589. MR4429995 https://doi.org/10.1214/21-aap1689

[10] BASS, R. F., CHEN, X. and ROSEN, J. (2009). Moderate deviations for the range of planar random walks. *Mem. Amer. Math. Soc.* **198** viii+82. MR2493313 https://doi.org/10.1090/memo/0929

[11] BASS, R. F. and KUMAGAI, T. (2002). Laws of the iterated logarithm for the range of random walks in two and three dimensions. *Ann. Probab.* **30** 1369–1396. MR1920111 https://doi.org/10.1214/aop/1029867131

[12] CHANG, Y. (2017). Two observations on the capacity of the range of simple random walks on $\mathbb{Z}^3$ and $\mathbb{Z}^4$. *Electron. Commun. Probab.* **22** Paper No. 25, 9. MR3652038 https://doi.org/10.1214/17-ECP55

[13] CHEN, X. and ROSEN, J. (2010). Large deviations and renormalization for Riesz potentials of stable intersection measures. *Stochastic Process. Appl.* **120** 1837–1878. MR2673977 https://doi.org/10.1016/j.spa.2010.05.006

[14] CHUNG, K. L. (1948). On the maximum partial sums of sequences of independent random variables. *Trans. Amer. Math. Soc.* **64** 205–233. MR0026274 https://doi.org/10.2307/1990499

[15] CSÁKI, E. and HU, Y. (2007). Strong approximations of three-dimensional Wiener sausages. *Acta Math. Hungar.* **114** 205–226. MR2296543 https://doi.org/10.1007/s10474-006-0525-7

[16] CYGAN, W., SANDRIĆ, N. and ŠEBEK, S. (2022). Central limit theorem for the capacity of the range of stable random walks. *Stochastics* **94** 226–247. MR4372656 https://doi.org/10.1080/17442508.2021.1920941

[17] HARIYA, Y. and HASEGAWA, K. (2016). On divergence of expectations of the Feynman–Kac type with singular potentials. *J. Math. Soc. Japan* **68** 1271–1296. MR3523547 https://doi.org/10.2969/jmsj/06831271

[18] HUTCHCROFT, T. and SOUSI, P. (2023). Logarithmic corrections to scaling in the four-dimensional uniform spanning tree. *Comm. Math. Phys.* **401** 2115–2191. MR4610293 https://doi.org/10.1007/s00220-023-04686-w

[19] JAIN, N. and OREY, S. (1968). On the range of random walk. *Israel J. Math.* **6** 373–380. MR0243623 https://doi.org/10.1007/BF02771217

[20] JAIN, N. C. and OREY, S. (1973). Some properties of random walk paths. *J. Math. Anal. Appl.* **43** 795–815. MR0359004 https://doi.org/10.1016/0022-247X(73)90293-X

[21] JAIN, N. C. and PRUITT, W. E. (1972). The law of the iterated logarithm for the range of random walk. *Ann. Math. Stat.* **43** 1692–1697. MR0345216 https://doi.org/10.1214/aoms/1177692404

[22] KENT, J. T. (1980). Eigenvalue expansions for diffusion hitting times. *Z. Wahrsch. Verw. Gebiete* **52** 309–319. MR0576891 https://doi.org/10.1007/BF00538895

[23] LAWLER, G. F. (1996). Cut times for simple random walk. *Electron. J. Probab.* **1** no. 13, approx. 24 pp. MR1423466 https://doi.org/10.1214/EJP.v1-13

[24] LAWLER, G. F. (2013). *Intersections of Random Walks. Modern Birkhäuser Classics*. Birkhäuser, New York. Reprint of the 1996 edition. MR2985195 https://doi.org/10.1007/978-1-4614-5972-9

[25] LE GALL, J.-F. (1988). Fluctuation results for the Wiener sausage. *Ann. Probab.* **16** 991–1018. MR0942751

[26] LE GALL, J.-F. (1994). Exponential moments for the renormalized self-intersection local time of planar Brownian motion. In *Séminaire de Probabilités, XXVIII. Lecture Notes in Math.* **1583** 172–180. Springer, Berlin. MR1329112 https://doi.org/10.1007/BFb0073845

[27] MRAZOVIĆ, R., SANDRIĆ, N. and ŠEBEK, S. (2023). Capacity of the range of random walks on groups. *Kyoto J. Math.* **63** 783–805. MR4643004 https://doi.org/10.1215/21562261-2023-0003

[28] OKADA, I. (2019). Geometric structures of late points of a two-dimensional simple random walk. *Ann. Probab.* **47** 2869–2893. MR4021239 https://doi.org/10.1214/18-AOP1325

[29] PETROV, V. V. (1975). *Sums of Independent Random Variables. Ergebnisse der Mathematik und Ihrer Grenzgebiete* [*Results in Mathematics and Related Areas*], *Band* 82. Springer, New York. Translated from the Russian by A. A. Brown. MR0388499

[30] PÓLYA, G. and SZEGÖ, G. (1951). *Isoperimetric Inequalities in Mathematical Physics*. *Annals of Mathematics Studies*, *No*. 27. Princeton Univ. Press, Princeton, NJ. MR0043486

[31] SCHAPIRA, B. (2020). Capacity of the range in dimension 5. *Ann*. *Probab*. **48** 2988–3040. MR4164459 https://doi.org/10.1214/20-AOP1442

[32] SZNITMAN, A.-S. (2010). Vacant set of random interlacements and percolation. *Ann*. *of Math*. (2) **171** 2039–2087. MR2680403 https://doi.org/10.4007/annals.2010.171.2039

[33] VAN DEN BERG, M., BOLTHAUSEN, E. and DEN HOLLANDER, F. (2001). Moderate deviations for the volume of the Wiener sausage. *Ann*. *of Math*. (2) **153** 355–406. MR1829754 https://doi.org/10.2307/2661345

[34] WANG, Y. Q. and GAO, F. Q. (2011). Laws of the iterated logarithm for high-dimensional Wiener sausage. *Acta Math*. *Sin*. (*Engl*. *Ser*.) **27** 1599–1610. MR2822832 https://doi.org/10.1007/s10114-011-8529-9

[35] WATSON, G. N. (1944). *A Treatise on the Theory of Bessel Functions*. Cambridge Univ. Press, Cambridge. MR0010746