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RESEARCH ARTICLE

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Approximation properties of torsion classes

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Abstract

We strengthen a result of Bagaria and Magidor (*Trans. Amer. Math. Soc.* **366** (2014), no. 4, 1857–1877) about the relationship between large cardinals and torsion classes of abelian groups, and prove that

(1) the *Maximum Deconstructibility* principle introduced in Cox (*J. Pure Appl. Algebra* **226** (2022), no. 5) requires large cardinals; it sits, implication-wise, between Vopěnka's Principle and the existence of an ω_1-strongly compact cardinal.

(2) While deconstructibility of a class of modules always implies the precovering property by Saorín and Šťovíček (*Adv. Math.* **228** (2011), no. 2, 968–1007), the concepts are (consistently) nonequivalent, even for classes of abelian groups closed under extensions, homomorphic images, and colimits.

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Contents

1 INTRODUCTION

Approximation theory of modules provides tools for studying general modules by choosing special classes of modules and finding approximations (*precovers* and *preenvelopes)*, and minimal approximations (*covers* and *envelopes*), of the general modules by modules in the chosen classes.

A model case is the special class $\mathcal{P}_0$ of *projective modules* that yields projective precovers, and projective resolutions of modules. Similarly, the class $\mathcal{F}_0$ of *flat modules* yields flat covers, and minimal flat resolutions of modules, [5]. Dually, the class $\mathcal{I}_0$ of *injective modules* yields injective envelopes, and minimal injective coresolutions of modules.

In the case when minimal approximations exist, they provide invariants of modules, the model examples being Bass' invariants and dual Bass' invariants of modules over commutative Noetherian rings [11].

Approximation theory makes it possible to extend classic homological algebra to the setting of relative homological algebra [10]. Applications of the extended setting are numerous: in the theory of finitely generated modules over artin algebras [2], tilting theory of commutative Noetherian rings [14], and Gorenstein homological algebra [15], to name a few.

Classic structure theory of modules is based on decompositions into (possibly infinite) direct sums of indecomposable modules. This is a strong tool in the study of modules of finite length and injective modules over commutative Noetherian rings [1]. However, decomposability fails for many other classes of interest. In contrast, the weaker notion of *deconstructibility* (namely, decomposition into a transfinite extension of modules from a given *set* of modules) is almost omnipresent.

The key point is that deconstructible classes of modules always provide precovers, that is, each deconstructible class is precovering [18]. The opposite question is more subtle.

Question 1. Suppose $\mathcal{C}$ is a class of modules that is precovering and closed under transfinite extensions. Is $\mathcal{C}$ deconstructible?

In this paper, we use some results of Bagaria and Magidor from [3] to provide a negative answer to Question 1, at least in the absence of large cardinals:

Theorem. *If there are no ω_1-strongly compact cardinals, then there is a precovering class of abelian groups that is closed under transfinite extensions, homomorphic images, and colimits, but is* not *deconstructible.*

In fact, the torsion class

$$^{\perp_0}\mathbb{Z} := \{A \in \mathbf{Ab} \ : \ \mathrm{Hom}(A, \mathbb{Z}) = 0\}$$

is always a covering class that is closed under transfinite extensions, homomorphic images, and colimits. In this respect we prove:

Theorem 2. *$^{\perp_0}\mathbb{Z}$ is deconstructible if and only if there is an ω_1-strongly compact cardinal.*

Theorem 2 strengthens a result of Bagaria and Magidor, who got the same equivalence with "deconstructible" weakened to "bounded" (we show in Lemma 11 that these concepts are equivalent, for any class of modules closed under transfinite extensions, homomorphic images, and colimits).

In [6], the first author proved that any deconstructible class is:

(a) closed under transfinite extensions (by definition), and
(b) "eventually almost everywhere closed under quotients". This is an extremely weak version of saying that if $A \subset B$ are both in the class, then so is B/A. In particular, it holds if the class is closed under homomorphic images.

He introduced the *Maximum Deconstructiblity* principle, which asserts that any class satisfying both (a) and (b) is deconstructible, and proved that

Vopěnka's Principle (VP) implies Maximum Deconstructibility.

Maximum Deconstructibility appeared very powerful, since [6] showed that it implied deconstructibility of many classes in Gorenstein homological algebra that (so far) are not known to be deconstructible in ZFC alone. But it was unclear whether Maximum Deconstructibility had any large cardinal strength at all. We again use the results of Bagaria and Magidor to show that it does:

Theorem. *Maximum Deconstructibility implies the existence of an ω_1-strongly compact cardinal.*

So, Maximum Deconstructibility lies, implication-wise, between VP and the existence of an ω_1-strongly compact cardinal. The first author still conjectures that it is equivalent to VP.

Section 2 has preliminaries. Section 3 proves the approximation properties of torsion classes that are provable in ZFC alone. Section 4 discusses some results of Bagaria–Magidor [3]. Section 5 proves the main theorems mentioned above, and Section 6 includes some questions.

2 | PRELIMINARIES

Our notation and conventions follow Kanamori [16] (for set theory and large cardinals) and Göbel–Trlifaj [14] (for module theory). By *ring* we will mean a unital and not necessarily commutative ring. If R is a ring, the class of left R-modules will be denoted by R-Mod. We will say that M is a *module* rather than a left R-module whenever R is clear from the context. For a regular cardinal κ and a module M, the collection of all submodules N of M that are $<\kappa$-generated will be denoted by $[M]^{<\kappa}$; if $|R| < \kappa$ and κ is regular and uncountable, then "$<\kappa$-generated" is equivalent to "of cardinality less than κ". A cardinal κ is *strongly compact* if every κ-complete filter (on any set whatsoever) can be extended to a κ-complete ultrafilter. Bagaria and Magidor considered a weakening of strong compactness:

Definition 3 (Bagaria and Magidor [3]). An uncountable cardinal κ is called ω_1-*strongly compact* if every κ-complete filter extends to an ω_1-complete ultrafilter.

Note that if κ is ω_1-strong compact then so is any cardinal $\lambda \geqslant \kappa$. Unlike strongly compact cardinals, an ω_1-strong compact cardinal may fail to be regular [3, section 6].

3 | APPROXIMATION PROPERTIES AND TORSION CLASSES

Given a class $\mathcal{K}$ of R-modules, a *$\mathcal{K}$-filtration* is a $\subseteq$-increasing and $\subseteq$-continuous sequence $\langle M_\xi \mid \xi < \eta\rangle$ of modules such that $M_0 = 0$ and for all $\xi < \eta$ such that $\xi + 1 < \eta$, M_ξ is a submodule of $M_{\xi+1}$ and $M_{\xi+1}/M_\xi$ is isomorphic to a member of $\mathcal{K}$. A module M is *$\mathcal{K}$-filtered* whenever there is a $\mathcal{K}$-filtration $\langle M_\xi \mid \xi < \eta\rangle$ whose union is M. We shall denote the class of all $\mathcal{K}$-filtered modules by $\mathrm{Filt}(\mathcal{K})$. $\mathcal{K}$ is *closed under transfinite extensions* whenever $\mathrm{Filt}(\mathcal{K}) \subseteq \mathcal{K}$. Finally, $\mathcal{K}^{<\kappa}$ denotes the class of all $<\kappa$-presented members of $\mathcal{K}$.

Definition 4. Let $\mathcal{K}$ be a class of modules and $\lambda \leqslant \kappa$ be regular cardinals.

(1) $\mathcal{K}$ is *κ-deconstructible* whenever $\mathcal{K} = \mathrm{Filt}\left(\mathcal{K}^{<\kappa}\right)$; equivalently, $\mathcal{K}$ is closed under transfinite extensions, and every $M \in \mathcal{K}$ admits a $\mathcal{K}^{<\kappa}$-filtration.[†]
(2) $\mathcal{K}$ is *(κ, λ)-cofinal* if $\mathcal{K} \cap [M]^{<\kappa}$ is $\subseteq$-cofinal in $[M]^{<\lambda}$ for every module $M \in \mathcal{K}$ (i.e., whenever every $N \in [M]^{<\lambda}$ is contained in some $<\kappa$-generated submodule of M that lies in $\mathcal{K}$).
(3) $\mathcal{K}$ is *bounded by κ* whenever $M = \sum\left(\mathcal{K} \cap [M]^{<\kappa}\right)$ for all $M \in \mathcal{K}$ (i.e., if every $x \in M$ is contained in some $<\kappa$-generated submodule of M that lies in $\mathcal{K}$).[‡]
(4) $\mathcal{K}$ is *κ-decomposable* if every module in $\mathcal{K}$ is a direct sum of $<\kappa$-presented modules from $\mathcal{K}$.

$\mathcal{K}$ is said to be *deconstructible* provided it is κ-deconstructible for a regular cardinal κ. The same convention is applied to the rest of above-mentioned properties.

Clearly, any (κ, λ)-cofinal class is bounded by κ. In the forthcoming Lemma 11, we will argue that for certain classes $\mathcal{K}$, "bounded by κ" and "(κ, κ)-cofinal" are equivalent concepts.

The following definitions are due to Enochs (generalizing earlier work of Auslander):

Definition 5. Let $\mathcal{K}$ be a class of modules. We say that $\mathcal{K}$ is:

(1) *Precovering*: if every module M posseses a *$\mathcal{K}$-precover*; namely, a morphism $f : C \to M$ with $C \in \mathcal{K}$ such that $\mathrm{Hom}(D, f)$ is surjective for all $D \in \mathcal{K}$;
(2) *Covering*: If every module M posseses a *$\mathcal{K}$-cover*; namely, a $\mathcal{K}$-precover $f : C \to M$ such that for each $g \in \mathrm{End}(C)$ if $fg = f$ then g is an automorphism of C.

Saorín and Šťovíček [18] proved that if a class $\mathcal{K}$ (of, say, modules) is deconstructible, then it is precovering. In Section 5, we argue that there are nicely behaved covering classes which are yet, at least consistently, not deconstructible. The concrete example we have in mind is the torsion class of the abelian group $\mathbb{Z}$ in a context where the set-theoretic universe does not have any ω_1-strongly compact cardinals (see Definition 3).

Dickson [8] introduced the concept of *torsion pairs*, which are pairs $(\mathcal{A}, \mathcal{B})$ such that

$$\mathcal{A} = {}^{\perp_0}\mathcal{B} := \{X \;:\; \mathrm{Hom}(X, B) = 0 \text{ for all } B \in \mathcal{B}\}$$

and

$$\mathcal{B} = \mathcal{A}^{\perp_0} := \{Y \;:\; \mathrm{Hom}(A, Y) = 0 \text{ for all } A \in \mathcal{A}\}.$$

[†] This is the definition of deconstructibility in most newer references, such as [19], since it is the version that (by [18]) implies the precovering property. Some other sources (e.g., Cox [6], Göbel–Trlifaj [14]) only require that $\mathcal{K} \subseteq \mathrm{Filt}\left(\mathcal{K}^{<\kappa}\right)$ (i.e., with $\subseteq$ rather than equality).

[‡] This property was introduced by Gardner in [13] and later investigated by Dugas in [9].

A class of modules is called a *torsion class* if it is of the form ${}^{\perp_0}\mathcal{B}$ for some class $\mathcal{B}$; notice that in this situation,

$$\left({}^{\perp_0}\mathcal{B}, \left({}^{\perp_0}\mathcal{B}\right)^{\perp_0}\right)$$

is easily seen to be a torsion pair (the torsion pair *cogenerated by* $\mathcal{B}$). So, torsion classes are exactly those classes that are the left part of some torsion pair. Whenever $\mathcal{X}$ is a singleton $\{X\}$, the convention is to write ${}^{\perp_0}X$ rather than ${}^{\perp_0}\{X\}$.

By [8], torsion classes are exactly those classes that are closed under arbitrary direct sums, extensions, and homomorphic images [20, Proposition VI.2.1]. They have many other desirable features; we list the ones that are relevant for this paper.

Lemma 6. *Torsion classes are:*

(1) *closed under homomorphic images,*
(2) *closed under colimits,*
(3) *closed under transfinite extensions,*
(4) *covering classes.*

(1) is immediate from the definition of torsion class, and (2) follows from closure under direct sums and homomorphic images. Closure under transfinite extensions follows easily from closure under extensions and colimits. Finally, given any module M and any torsion class $\mathcal{T} := {}^{\perp_0}\mathcal{X}$, the *trace of* $\mathcal{T}$ in M is defined as

$$r(M) := \sum_{T\in\mathcal{T}} \{\operatorname{im}\pi \ : \ \pi \in \operatorname{Hom}(T, M)\}.$$

Then it is easily seen that $r(M) \in \mathcal{T}$ and that the inclusion $r(M) \to M$ is a $\mathcal{T}$-cover of M.

4 | ON SOME RESULTS OF BAGARIA AND MAGIDOR

In Bagaria–Magidor [3, section 5], a torsion class ${}^{\perp_0}\mathcal{X}$ is called "κ-generated" whenever every $A \in {}^{\perp_0}\mathcal{X}$ is a *direct* sum of subgroups in ${}^{\perp_0}\mathcal{X}$, each of cardinality $<\kappa$; this is what is usually called κ-decomposable (see Definition 4). The use of the word "direct" on [3, p. 1867] appears to be a misprint,[†] since the arguments there never use (or conclude) directness of the relevant sums. The arguments in [3, section 5], and the citation they provide for the concept (Dugas [9]), appear to use the weaker property that every group in the class is a sum of subgroups in the class of cardinality $<\kappa$. The latter condition is what we called *bounded by* κ in Section 2 (this terminology was used by Gardner [13] and Dugas [9]). So, we shall use the *bounded by* κ terminology to state the Bagaria–Magidor results. They proved the following.

Theorem 7 [3, Theorem 5.1]. *If κ is a δ-strongly compact cardinal and X is an abelian group of cardinality less than δ, then ${}^{\perp_0}X$ is bounded by κ. In fact, they prove the stronger property of ${}^{\perp_0}X$ being (κ, ω_1)-cofinal.*[‡]

[†] The anonymous referee has suggested that the authors of [3] might have intended to say *directed sum* instead of *direct sum*.

[‡] Recall that this stands for the following property: For each $G \in {}^{\perp_0}\mathbb{Z}$, every countable subgroup $H \leqslant G$ is included in a member of ${}^{\perp_0}\mathbb{Z} \cap [G]^{<\kappa}$ (see Definition 4).

Theorem 8 [3, Theorem 5.3]. *If* ${}^{\perp_0}\mathbb{Z}$ *is bounded in* κ*, then* κ *is* ω_1*-strongly compact.*

Corollary 9 [3, Corollary 5.4]. ${}^{\perp_0}\mathbb{Z}$ *is bounded in* κ *if and only if* κ *is* ω_1*-strongly compact.*

The following ZFC theorem (due to the third author) shows that, while ${}^{\perp_0}\mathbb{Z}$ can be bounded (by the Bagaria–Magidor results), it can never be decomposable:

Theorem 10. ${}^{\perp_0}\mathbb{Z}$ *is not decomposable.*

Proof. Suppose toward a contradiction that ${}^{\perp_0}\mathbb{Z}$ is κ-decomposable, with κ (without loss of generality) a regular uncountable cardinal. Fix any prime p. Then all p-groups[†] are in ${}^{\perp_0}\mathbb{Z}$. So, in particular, the κ-decomposability assumption of ${}^{\perp_0}\mathbb{Z}$ implies

$$\text{Every } p\text{-group is a direct sum of } <\kappa\text{-sized subgroups.} \tag{1}$$

For a p-group G, we can consider its p-length, which is the least ordinal σ such that $G_\sigma = G_{\sigma+1}$, where G_σ is defined recursively by $G_0 := G$, $G_{\sigma+1} = \{pg \ : \ g \in G_\sigma\}$, and $G_\sigma = \bigcap_{\xi<\sigma} G_\xi$ for limit ordinals σ. By a construction of Walker [21] (see also Bazzoni–Šťovíček [4]), there exists a p-group, denoted P_{κ^+}, whose p-length is exactly $\kappa^+ + 1$.[‡] By (1),

$$P_{\kappa^+} = \bigoplus_{i\in I} Q_i$$

for some collection $(Q_i)_{i\in I}$ of $<\kappa$-sized subgroups. Since subgroups of p-groups are also p-groups, each Q_i has a p-length, which is $<\kappa$ because $|Q_i| < \kappa$. And it is easy to check that the p-length of a direct sum is at most the supremum of the p-lengths of the direct summands, so $\bigoplus_{i\in I} Q_i$ has p-length at most κ, contradicting that the p-length of P_{κ^+} is $\kappa^+ + 1$. □

5 | MAXIMUM DECONSTRUCTIBILITY, AND DECONSTRUCTIBILITY OF TORSION CLASSES, REQUIRE LARGE CARDINALS

Through this section, κ will denote a regular uncountable cardinal. Our main goal is to clarify, for a given class of modules $\mathcal{K}$, the relationship between $\mathcal{K}$ being κ-deconstructible and $\mathcal{K}$ being bounded in κ. As noted in Section 2, any (κ,κ)-cofinal class $\mathcal{K}$ is bounded in κ. In fact the former property seems to be strictly stronger than the latter, and they both follow from κ-deconstructibility. Lemma 11 provides some important scenarios where these concepts are equivalent. Combining this lemma with the results of Bagaria and Magidor from Section 4 allows us to prove the main results mentioned in the introduction, namely,

(1) that the *Maximum Deconstructibility* principle introduced in [6] entails the existence of large cardinals;
(2) that in the absence of ω_1-strongly compact cardinals, the class ${}^{\perp_0}\mathbb{Z}$ is a covering class (cf. Lemma 6) that is *not* deconstructible.

† Abelian groups whose elements all have order a power of p.

‡ The group is indexed by strictly decreasing finite sequences of ordinals $\kappa^+ > \beta_1 > ... > \beta_k$, with relations $p(\kappa^+\beta_1...\beta_{k+1}) = \kappa^+\beta_1...\beta_k$ and $p(\kappa^+) = 0$.

Lemma 11. *Suppose κ is a regular uncountable cardinal and R is a ring of size less than κ, and $\mathcal{K}$ is a class of R-modules. Consider the following statements:*

(a) *$\mathcal{K}$ is κ-deconstructible;*
(b) *$\mathcal{K}$ is (κ, κ)-cofinal and closed under transfinite extensions;*
(c) *$\mathcal{K}$ is bounded by κ and closed under transfinite extensions.*

Then:

(1) *(a) $\Rightarrow$ (b) $\Rightarrow$ (c);*
(2) *if $\mathcal{K}$ is closed under quotients, that is, if*

$$(A \subset B,\ A \in \mathcal{K},\ \text{and } B \in \mathcal{K}) \Rightarrow\ B/A \in \mathcal{K}$$

for all modules A and B, then (a) and (b) are equivalent;
(3) *if $\mathcal{K}$ is closed under homomorphic images and colimits, then all three statements are equivalent.*

Proof. The (a) $\Rightarrow$ (b) direction follows from the Hill Lemma (cf. Göbel–Trlifaj [14, Theorem 7.10]). The (b) $\Rightarrow$ (c) implication is immediate from the definitions.

Now suppose $\mathcal{K}$ is closed under transfinite extensions and quotients. We prove (b) $\Rightarrow$ (a). Without losing any generality, we may assume that (the universe of) every module $M \in \mathcal{K}$ is a cardinal. Since $\mathcal{K}$ is closed both under transfinite extensions and quotients, and $|R| < \kappa$, then by [6, Theorem 6.3] in order to prove κ-deconstructibility of $\mathcal{K}$ it suffices to show that if $M \in \mathcal{K}$, then $\mathcal{K} \cap \wp^*_\kappa(M)$ is stationary in $\wp^*_\kappa(M)$, where $\wp^*_\kappa(M)$ denotes the $<\kappa$-sized submodules of M whose intersection with κ is transitive (i.e., those $< \kappa$-sized submodules N of M such that $N \cap \kappa \in \kappa$).

The set $\mathcal{K} \cap \wp^*_\kappa(M)$ being stationary in $\wp^*_\kappa(M)$ is equivalent to the following statement ([12]): for each function F from finite subsets of M into M, there is an $X \in \mathcal{K} \cap \wp^*_\kappa(M)$ such that X is closed under the function F. We prove this next.

Fix an arbitrary function F from *finite* subsets of M into M; notice that since F is finitary, for any infinite $D \subset M$, the closure of D under the function F has the same cardinality as D. Using the assumption that $\mathcal{K}$ is $\subseteq$-cofinal in $[M]^{<\kappa}$, fix some $C_0 \in \mathcal{K} \cap [M]^{<\kappa}$. Recursively build a $\subseteq$-increasing ω-chain

$$C_0 \subseteq X_0 \subseteq C_1 \subseteq X_1 \subseteq C_2 \subseteq X_2 \subseteq \ldots$$

such that for each $n \in \omega$:

- X_n is the closure of $C_n \cup \sup(C_n \cap \kappa)$ under the function F. This closure has cardinality $< \kappa$, because $|C_n| < \kappa$ and hence (by regularity of κ) $|C_n \cup \sup(C_n \cap \kappa)| < \kappa$;
- (for $n \geqslant 1$) $C_n \in \mathcal{K}$, $C_n \supseteq X_{n-1}$, and $|C_n| < \kappa$. This is possible by the assumption that $\mathcal{K}$ is $\subseteq$-cofinal in $[M]^{<\kappa}$ (but note that $C_n \cap \kappa$ might fail to be transitive).

Let $X := \bigcup_{n<\omega} X_n$. Then X is closed under F, $|X| < \kappa$, and $X \cap \kappa = \sup_{n<\omega} \sup(C_n \cap \kappa)$; in particular, $X \cap \kappa$ is transitive. Also, note that $X = \bigcup_{n<\omega} C_n$. Since $\mathcal{K}$ is closed under quotients, each C_{n+1}/C_n is in $\mathcal{K}$. So, since $\mathcal{K}$ is closed under transfinite extensions, and each C_{n+1}/C_n is in $\mathcal{K}$, we conclude that $X \in \mathcal{K}$.

Now suppose that $\mathcal{K}$ is closed under homomorphic images and colimits; we show that (c) implies (b) (and hence (a) will hold too, since closure under homomorphic images trivially implies closure under quotients in the sense of 2). Suppose $\mathcal{K}$ is bounded in κ. Suppose $M \in \mathcal{K}$, and fix any $X \in [M]^{<\kappa}$. By κ-boundedness of $\mathcal{K}$, for each $x \in X$ there is a $Y_x \in \mathcal{K} \cap [M]^{<\kappa}$ with $x \in Y_x$.

Then $S := \sum_{x\in X} Y_x$ is a $<\kappa$-sized submodule of M and contains X. And S is a homomorphic image of the colimit of the (possibly nondirected) diagram of inclusions

$$\{Y_x \to Y_z \ : \ Y_x \subseteq Y_z \text{ and } x, z \in X\}.$$

So, by closure of $\mathcal{K}$ under homomorphic images and colimits, $S \in \mathcal{K}$. □

We now focus on abelian groups. Recall that by Lemma 6 any torsion class ${}^{\perp_0}\mathcal{X}$ is closed under homomorphic images, transfinite extensions, and colimits. So by Lemma 11, a class of the form ${}^{\perp_0}\mathcal{X}$ is deconstructible if and only if it is bounded in some cardinal. Furthermore, as noted in the introduction, although an ω_1-strongly compact cardinal might be singular, all cardinals above it are also ω_1-strongly compact. In particular (by considering its successor if necessary), there exists an ω_1-strongly compact cardinal if and only if there exists a *regular* ω_1-strongly compact cardinal. Then Lemma 6, Lemma 11, and the Corollary 9 of Bagaria and Magidor immediately imply Theorem 2 from the introduction, which asserted that ${}^{\perp_0}\mathbb{Z}$ is deconstructible if and only if there is an ω_1-strongly compact cardinal. In fact, making use of the Theorem 7 of Bagaria and Magidor, we have:

Corollary 12. *The following are equivalent.*

(1) *There exists an ω_1-strongly compact cardinal.*
(2) ${}^{\perp_0}\mathbb{Z}$ *is deconstructible.*
(3) *For all countable abelian groups X, ${}^{\perp_0}X$ is deconstructible.*

Since ${}^{\perp_0}\mathbb{Z}$ is closed under transfinite extensions and homomorphic images, we solve part of [6, Question 8.2] in the affirmative:

Corollary 13. Maximum Deconstructibility *of [6] has large cardinal consistency strength. Specifically, it lies implication-wise, in between the existence of an ω_1-strongly compact and VP.*

Recall that deconstructible classes of modules are always precovering ([14, Theorem 7.21]). Thus, another interesting corollary of Corollary 12 and Lemma 6 is the existence of precovering classes (even covering classes) which are not deconstructible:

Corollary 14. *If there are no ω_1-strongly compact cardinals, then there is a covering class in* ***Ab*** *that is closed under colimits, transfinite extensions, and homomorphic images, but is not deconstructible (namely, the class ${}^{\perp_0}\mathbb{Z}$). Thus,* ZFC *cannot prove that all subclasses of* ***Ab*** *satisfying the clauses in Lemma 6 are deconstructible.* □

Remark 15. The following statement has recently been proven in [17, Remark 4 and Corollary 3.4]: Assume VP holds. Then for any ring R, each class of R-modules that is closed under finite direct sums, extensions, and direct limits is deconstructible.

6 | QUESTIONS

Question 16. Suppose that $\kappa \geqslant \omega_2$ is a regular cardinal such that ${}^{\perp_0}X$ is deconstructible for all abelian groups of size $<\kappa$. Must κ be strongly compact?

Much of the literature focuses on deconstructibility and precovering properties of "roots of Ext" classes, that is, classes of the form

$$^{\perp}\mathcal{B} := \left\{ A \ : \ \forall B \in \mathcal{B}\ \mathrm{Ext}^1(A, B) = 0 \right\}$$

for some fixed class $\mathcal{B}$ of modules. Our results in Section 5 consistently separates deconstructibility from precovering for $^{\perp_0}\mathbb{Z}$, but this is very far from being a root of Ext; it does not even contain the ring $\mathbb{Z}$.

This suggests asking whether deconstructibility is equivalent to precovering for classes of the form $^{\perp}\mathcal{B}$, but there is a caveat: the first author proved in [7] that it is consistent, relative to consistency of VP, that over every *hereditary* ring (such as $\mathbb{Z}$), *every* class of the form $^{\perp}\mathcal{B}$ is deconstructible. So, deconstructibility is trivially equivalent to precovering for roots of Ext (over hereditary rings) in that model. But the following questions are open:

Question 17. Is it consistent with ZFC that there is a class $\mathcal{B}$ of abelian groups such that $^{\perp}\mathcal{B}$ is precovering, but not deconstructible?

Question 18. Does ZFC prove the existence of a ring R and a class $\mathcal{B}$ of R-modules, such that $^{\perp}\mathcal{B}$ is precovering, but not deconstructible? By the remarks above, such a ring could not be provably hereditary (unless VP is inconsistent).

Both questions are even open if we replace "classes of the form $^{\perp}\mathcal{B}$" with "classes that are closed under transfinite extensions and contain the ring" (such classes would contain all free modules).

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