$\mathcal{L}_2$-Suboptimal Control for Nonlinear Systems via Convex Optimization

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***Abstract*—Analogous to $\mathcal{H}_\infty$-methods for linear time invariant systems, $\mathcal{L}_2$-gain is an important input-output characterization of robustness for nonlinear systems. The Hamilton-Jacobi Inequality can be used to establish $\mathcal{L}_2$-gain if an appropriate storage function can be identified. Continuous piecewise affine storage functions for small-signal $\mathcal{L}_2$-stability analysis of nonlinear systems have previously been applied to open-loop analysis. Here, they are used to develop a suboptimal controller synthesis method for nonlinear systems that minimizes the local, closed-loop small-signal $\mathcal{L}_2$-gain. The method selects a piecewise affine state feedback controller using convex optimization.**

***Index Terms*—Optimal control, robust control, LMIs, stability of nonlinear systems, computational methods.**

I. Introduction

WHEN ensuring that stability is robust to error in linear time invariant (LTI) models, $\mathcal{H}_\infty$ optimal control is a popular tool. The stability margin of a feedback interconnection between an uncertainty block and an LTI system is usually characterized by the closed-loop $\mathcal{H}_\infty$-norm. However, if no LTI model adequately captures the system dynamics, an $\mathcal{H}_\infty$ optimal controller can have poor performance in practice. Though the $\mathcal{H}_\infty$-norm does not directly generalize to nonlinear systems, it is equivalent to the $\mathcal{L}_2$-induced norm from input to output signals in the time domain (with zero initial state) [1]. Hence, the $\mathcal{L}_2$-gain minimization problem is the analogue to the $\mathcal{H}_\infty$ optimal control, minimizing the response to disturbances not captured by a nonlinear model. While the resulting optimization problem can be non-convex and extremely challenging to solve, this letter presents a practical method to find piecewise affine state-feedback controllers that are locally $\mathcal{L}_2$-suboptimal using convex optimization.

The Hamilton-Jacobi Inequality (HJI) can establish the $\mathcal{L}_2$-gain of a nonlinear system, provided an appropriate storage function can be found [2, Ch. 8], but there is no systematic method to perform this search for general nonlinear systems. While sum-of-squares (SOS) programming has been applied to polynomial dynamics with relaxed HJI conditions, it has also revealed non-existence of a uniform bound, as well as lack of monotonicity, on the degree of polynomial Lyapunov function in asymptotic stability analysis for certain nonlinear autonomous systems [3], [4]. Alternatively, the construction of continuous piecewise affine (CPA) Lyapunov functions for asymptotic stability analysis [5], [6], [7] has recently inspired small-signal $\mathcal{L}_2$-gain analysis using CPA storage functions [8], [9]. Here, we leverage the previous, CPA-based small-signal $\mathcal{L}_2$-gain analysis to inspire a new, $\mathcal{L}_2$-suboptimal controller synthesis method for constrained nonlinear systems.

The proposed method selects a piecewise affine state-feedback controller, as well as a CPA storage function as a HJI solution candidate, for the closed-loop dynamics over a subset of the state space. Previously, [8], [9] derived error bounds so that solving the HJI at a finite number of states established an *open-loop* small-signal $\mathcal{L}_2$-gain bound. In comparison, this letter defines a controller synthesis problem by deriving an extra error bound that includes variable state-feedback gains. Finally, a convex optimization problem is posed to find a piecewise affine state-feedback controller that minimizes the closed-loop small-signal $\mathcal{L}_2$-gain over a subset of the state space. The results are demonstrated in a numerical example.

II. Preliminaries

All vectors are defined over the Euclidean n-space. The inner product of two vectors is denoted by $\langle\cdot,\cdot\rangle$. The non-negative real numbers are denoted $\mathbb{R}^+$. A closed (open) interval of integers between a and b is denoted by $\mathbb{Z}_a^b$ ($\mathbb{Z}_{\underline{a}}^{\underline{b}}$). Bold $\mathbf{I}$ and $\mathbf{0}$ denote the identity matrix and zero matrix, respectively. The set of all real, symmetric, $n\times n$ matrices is denoted by $\mathbb{S}^n$, and $\star$ replaces transpose entries in elements of $\mathbb{S}^n$. Negative semi-definiteness of $M\in\mathbb{S}^n$ is denoted $M\preceq 0$. For $v\in\mathbb{R}^n$, v^a denotes its a^{th} element. The k^{th} column of $g\in\mathbb{R}^{n\times m}$ is denoted by g_k and the $(j,k)^{th}$ element is g_{jk}.

The p-norm of a vector is denoted as $\|\cdot\|_p$, where $1\le p\le\infty$ and $p=2$ when the subscript is omitted. All matrix norms considered here are induced by vector p-norms. A function $f\in\mathcal{L}_2^n$ if $\|f\|_{\mathcal{L}_2}^2 := \int_0^\infty f(t)^{\mathrm{T}}f(t)\,dt<\infty$. It is in the *extended* $\mathcal{L}_2$ *space*, $f\in\mathcal{L}_{2e}^n$, if its *truncation*, $f_T(t)$ is in $\mathcal{L}_2^n$, where $f_T(t)=f(t)$, $0\le t<T$ and $f_T(t)=0$, $t\ge T$. The set of functions that are k-times continuously differentiable over their domains is $\mathcal{C}^k$. The gradient of $V:\mathbb{R}^n\to\mathbb{R}$ is $\nabla V(x)=[\frac{\partial}{\partial x_1}V(x),\ldots,\frac{\partial}{\partial x_n}V(x)]$, and its Hessian matrix is denoted by $\nabla^2 V(x)$.

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A. $\mathcal{L}_2$-Stability

The following definitions introduce $\mathcal{L}_2$-stability and the Hamilton-Jacobi Inequality that form the basis of the main results. To begin, dissipativity is key to stability analysis.

Definition 1 (Dissipativity [10]): Consider a nonlinear system $\mathcal{G} : \mathbb{R}^n \times \mathcal{L}_{2e}^m \to \mathcal{L}_{2e}^p$, with state-space realization

$$\begin{cases} \dot{x}(t) = f\big(x(t)\big) + g\big(x(t)\big)u(t), \quad x(0) \in \mathbb{R}^n \\ y(t) = h\big(x(t)\big), \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^p$ are the state, input, and output of $\mathcal{G}$, respectively. The system $\mathcal{G}$ is *dissipative* with respect to the *supply rate* $s : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ if there exists a locally bounded *storage function*, $V : \mathbb{R}^n \to \mathbb{R}^+$, satisfying

$$V(x(0)) \geq \sup_{t\geq 0, u\in\mathcal{U}} \left\{ V\big(x(t)\big) - \int_0^t s\big(u(\tau), y(\tau)\big)d\tau \right\}, \tag{2}$$

where $\mathcal{U} = \{u \in \mathcal{L}_{2e}^m \mid \int_0^t |s(u(\tau), y(\tau))|d\tau < \infty \ \forall t \geq 0\}$.

When Eq. (1) is specified with an initial state $x(0) = x_0$, it corresponds to mapping $\mathcal{G}_{x_0}:\mathcal{L}_{2e}^m \to \mathcal{L}_{2e}^p$. Here we specify $x(0) \in \mathbb{R}^n$ to $\mathcal{G}$, which is a set of mappings $\mathcal{G}_{x_0}$, each with an immutable x_0 value. Note that local boundedness of the storage function was not required in Willem's original definition, but this additional property is crucial for defining the storage functions as solutions to partial differential inequalities (PDIs) [10].

Systems are $\mathcal{L}_2$-stable if they have finite $\mathcal{L}_2$-gain, which can be defined in terms of dissipativity. The dependence on t of $x(t), u(t), y(t)$ is dropped hereafter for brevity.

Definition 2 ($\mathcal{L}_2$-Gain [2, Ch. 3]): A state space system $\mathcal{G}$ (Eq. (1)) has finite $\mathcal{L}_2$-gain if for some finite $\gamma \geq 0$ it is dissipative with respect to the supply rate

$$s(u, y) = \frac{1}{2}\gamma^2\|u\|^2 - \frac{1}{2}\|y\|^2. \tag{3}$$

The $\mathcal{L}_2$*-gain* of $\mathcal{G}$, $\hat{\gamma}(\mathcal{G})$, is the infimum of all such γ.

The next theorem provides sufficient conditions to establish gain of a state-space realization. It is much like [1, Th. 2], but the more general setting of [10, Th. 3.1] accommodates nonzero initial conditions. Most importantly, it admits lower semi-continuous storage functions, because its PDI is understood in the weak sense. Here, [10, Th. 3.1] with the supply rate from Eq. (3) implies the second inequality, and the first follows identically to the proof of [1, Th. 2].

Theorem 1 (HJI [10, Th. 3.1], [1, Th. 2]): Consider a state space system $\mathcal{G}$ (Eq. (1)), and let $\gamma > 0$. If there exists a locally bounded solution, $V : \mathbb{R}^n \to \mathbb{R}^+$, to the *HJI*,

$$\nabla V(x)f(x) + \frac{1}{2\gamma^2}\nabla V(x)g(x)g^{\mathrm{T}}(x)\nabla^{\mathrm{T}}V(x) + \frac{1}{2}h^{\mathrm{T}}(x)h(x) \leq 0, \ \forall x \in \mathbb{R}^n, \tag{4}$$

then V and γ also satisfy

$$\nabla V(x)f(x) + \nabla V(x)g(x)u \leq \frac{1}{2}\gamma^2\|u\|^2 - \frac{1}{2}\|y\|^2, \ \forall x \in \mathbb{R}^n, u \in \mathbb{R}^m, \tag{5}$$

implying that system $\mathcal{G}$ is $\mathcal{L}_2$-stable and that $\hat{\gamma}(\mathcal{G}) \leq \gamma$.

When solving the HJI (4) with $V(x)$ and γ^2 as design variables, their product makes the inequality nonlinear. Applying the Schur complement [11, Ch. 2] to (4) results in an equivalent linear matrix inequality (LMI) [9, Th. 1],

$$H(x) = \begin{bmatrix} \nabla V(x)f(x) & \star & \star \\ g^{\mathrm{T}}(x)\nabla^{\mathrm{T}}V(x) & -2\gamma^2\mathbf{I} & \star \\ h(x) & 0 & -2\mathbf{I} \end{bmatrix} \preceq 0, \ \forall x \in \mathbb{R}^n. \tag{6}$$

Ineq. (6) represents an infinite number of LMIs, one for each x in a region. However, triangulation of the region can limit the number of LMI constraints.

B. Continuous Piecewise Affine Storage Function

This subsection clarifies notation related to triangulations, which will be used to define storage functions and controllers.

A set of vectors $\{x_j\}_{j=0}^N \subset \mathbb{R}^n$ is *affinely independent* if $\sum_{j=1}^N \lambda_j(x_j - x_0) = 0$ implies $\lambda_j = 0$ for each $j \in \mathbb{Z}_1^N$. An *n-simplex* is the convex hull of a set of $n+1$ affinely independent vectors. A *triangulation* is the union of a finite set of n-simplexes whose intersections are either a face or empty [6].

In this letter, we search for a CPA storage function when solving Ineq. (6). Because a CPA function is uniquely defined by its values on the vertexes of its triangulation, we move from searching over an infinite dimensional space of potential storage functions, $V(\cdot)$, to a finite dimensional vector space of design variables, $V(x_{i,j})$.

Definition 3 (CPA Function [6]): Consider a triangulation $\mathcal{T} = \cup_{i=1}^{m_\mathcal{T}} \Delta_i$, where $m_\mathcal{T} \in \mathbb{Z}_1^\infty$ and Δ_i is the i^{th} n-simplex. A function, $V:\mathcal{T} \to \mathbb{R}$, is *continuous piecewise affine (CPA)* if for each $i \in \mathbb{Z}_1^{m_\mathcal{T}}$ there exists $w_i \in \mathbb{R}^n$ and $a_i \in \mathbb{R}$ such that $V(x) = w_i^{\mathrm{T}}x + a_i$ for every $x \in \Delta_i$.

CPA storage functions have piecewise constant gradients that greatly simplify computation. The gradient of a CPA function is ill-defined on simplex boundaries, but computing it on simplex interiors is adequate, since [10, Th. 3.1] implies that Ineq. (6) can be understood in the weak sense.

Lemma 1 (Gradient of CPA Function [6]): The local gradient of a CPA function, $\nabla V(x) = \nabla V_i = w_i$ for all $x \in$ interior(Δ_i), can be computed as

$$\nabla^{\mathrm{T}}V_i = \begin{bmatrix} \big(x_{i,1} - x_{i,0}\big)^{\mathrm{T}} \\ \vdots \\ \big(x_{i,n} - x_{i,0}\big)^{\mathrm{T}} \end{bmatrix}^{-1} \begin{bmatrix} V\big(x_{i,1}\big) - V\big(x_{i,0}\big) \\ \vdots \\ V\big(x_{i,n}\big) - V\big(x_{i,0}\big) \end{bmatrix}. \tag{7}$$

With the choice of CPA storage functions, Eq. (6) still represents an infinite number of LMIs, but the work of this letter will surmount this hurdle by adding error bounds to a finite number of constraints.

C. Error Bound

The following lemma bounds the difference between a nonlinear function and its affine approximation over an n-simplex. It was presented implicitly in [9, Th. 4], but part of the result was initially established by [5, Proposition 2.2].

Lemma 2 (Vector Error Bound [5], [9]): Let $\Delta \subset \mathbb{R}^n$ be an n-simplex with vertices $\{x_j\}_{j=0}^n$ that include $x_0 = 0$ if $0 \in \Delta$. Consider a vector function $f : \Delta \to \mathbb{R}^n$, where $f \in \mathcal{C}^2$. Let $\beta \in \mathbb{R}^n$ and $E_f(\cdot) \in \mathbb{R}^n$ have respectively elements,

$$\beta^a := \max_{j,k\in\mathbb{Z}_1^n, x\in\Delta} \left| \left[\nabla^2 f^a(x)\right]_{jk} \right|, \tag{8}$$

$$E_f^a(x_j) := \begin{cases} \phi^a(x_j), \text{ if } 0 \notin \Delta, \text{ and} \\ \theta^a(x_j), \text{ if } 0 \in \Delta, \text{ where} \end{cases} \tag{9a}$$

$$\phi^a(x_j) = \langle -\frac{1}{2}\nabla^2 f^a(z_a)(x_j - x), x_j - x\rangle, \tag{9b}$$

$$\theta^a(x_j) = \langle -\frac{1}{2}\left(\nabla^2 f^a(z_a)x - \nabla^2 f^a(z_{a,j})x_j\right), x_j\rangle, \tag{9c}$$

for some z_a on the line segment between x and x_j in Eq. (9b); for some z_a on the line segment between x and 0, and some $z_{a,j}$ on the line segment between x_j and 0 in Eq. (9c). Set

$$c_j := \begin{cases} n \max_{k\in\mathbb{Z}_0^n} \|x_j - x_k\|^2, & \text{if } 0 \notin \Delta, \text{ and} \\ n\|x_j\|\left(\max_{k\in\mathbb{Z}_1^n}\|x_k\| + \|x_j\|\right) & \text{if } 0 \in \Delta. \end{cases} \tag{10}$$

Then for all $x \in \Delta$,

$$f(x) - \sum_{j=0}^{n} \lambda_j f(x_j) = \sum_{j=0}^{n} \lambda_j E_f(x_j) \le \frac{1}{2}\beta \sum_{j=0}^{n} \lambda_j c_j, \tag{11}$$

where values λ_j satisfy $x = \sum_{j=0}^{n} \lambda_j x_j$, $0 \le \lambda_j \le 1$. Consider also an affine function $V{:}\Delta \to \mathbb{R}$. Let $\bar{\beta} := \|\beta\|_\infty$. If $l \ge \|\nabla V(x)\|_1$ for all $x \in \Delta$, then on Δ,

$$\nabla V(x)f(x) - \sum_{j=0}^{n} \lambda_j \nabla V(x_j) f(x_j)$$
$$= \sum_{j=0}^{n} \lambda_j \nabla V(x_j) E_f(x_j) \le \frac{1}{2} l\bar{\beta} \sum_{j=0}^{n} \lambda_j c_j. \tag{12}$$

D. Local, Small-Signal $\mathcal{L}_2$-Gain

While $\mathcal{L}_2$-gain applies to all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, local, small-signal $\mathcal{L}_2$-gain constrains input signals so that the states remain in a subset of the state-space. The following definition is adapted from [12, Ch. 5]. In it, a subset of the state-space of Eq. (1), $\mathcal{A} \subset \mathbb{R}^n$, is robustly positive invariant (RPI) to a constrained input set, $\mathcal{U} \subset \mathcal{L}_{2e}^m$. If $x(0) \in \mathcal{A}$ and $u \in \mathcal{U}$ then $x(t) \in \mathcal{A}$ for all $t \ge 0$.

Definition 4 (Local, Small-Signal $\mathcal{L}_2$-Gain): The mapping $\mathcal{G} : \mathbb{R}^n \times \mathcal{L}_{2e}^m \to \mathcal{L}_{2e}^p$, defined by Eq. (1), has finite local, small-signal $\mathcal{L}_2$-gain on $\mathcal{A} \subset \mathbb{R}^n$ if: $\mathcal{A}$ is RPI to $\mathcal{U} = \{u \in \mathcal{L}_{2e}^m \mid \sup_{t\ge0}\|u_T(t)\|_\infty \le r_u\}$ for some $r_u > 0$ and all $T \ge 0$; and $V : \mathbb{R}^n \to \mathbb{R}^+$ satisfies Eq. (2) with $s(u, y)$ from Eq. (3) for some finite $\gamma \ge 0$, whenever $x(t) \in \mathcal{A}$. The *local, small-signal $\mathcal{L}_2$-gain* of $\mathcal{G}$, $\hat{\gamma}(\mathcal{G})$, is the infimum of all such γ.

Theorems 2 and 3 of this letter rely on the HJI to bound the local, small-signal $\mathcal{L}_2$-gain of systems. The next corollary connects these concepts.

Corollary 1: Consider state space system $\mathcal{G} : \mathbb{R}^n \times \mathcal{U} \to \mathcal{L}_{2e}^p$ (Eq. (1)), where $\mathcal{U} = \{u \in \mathcal{L}_{2e}^m \mid \sup_{t\ge0}\|u_T(t)\|_\infty \le r_u\}$ for some $r_u > 0$ and all $T \ge 0$. Suppose that $\mathcal{A} \subset \mathbb{R}^n$ is RPI to $\mathcal{U}$ and that all assumptions from Theorem 1 hold for all $x \in \mathcal{A}$. Then $\mathcal{G}$ has local, small-signal $\mathcal{L}_2$-gain of at most γ on $\mathcal{A}$.

Proof: Since $\mathcal{A}$ is RPI to $\mathcal{U}$, trajectories with $x(0) \in \mathcal{A}$ remain in $\mathcal{A}$ for all $t \ge 0$. Following the logic of Theorem 1, Eq. (5) is satisfied for all $x \in \mathcal{A}$, and therefore all $x(t)$. By Definition 4, γ bounds the local, small-signal $\mathcal{L}_2$-gain of $\mathcal{G}$ on $\mathcal{A}$. ■

This letter focuses on closed-loop control design to satisfy HJI (6) on a subset of the state space, Ω. After that, [8, Th. 5] provides a convex optimization method that can be used to identify $\mathcal{A} \subseteq \Omega$ that is RPI to $\mathcal{U}$. Co-optimization of the gain bound and invariant set is left to future work.

III. Main Results

This section extends the CPA storage function approach from small-signal $\mathcal{L}_2$-stability analysis in [8], [9] to controller synthesis. By deriving Taylor expansion error bounds for the feedback term, we develop a two-step optimization process that minimizes closed-loop $\mathcal{L}_2$-gain bounds. The process uses a CPA storage function combined with a piecewise affine state-feedback controller over each n-simplex.

A. Theoretical Gain Criteria

The following theorem establishes criteria to bound a closed-loop system's gain, separating the contributions of a plant and controller.

Theorem 2: Consider state space system $\mathcal{G}{:}\mathbb{R}^n \times \mathcal{L}_{2e}^m \to \mathcal{L}_{2e}^p$ defined by Eq. (1), where $f, g, h \in \mathcal{C}^2$, $f(\mathbf{0}) = \mathbf{0}$, $g(\mathbf{0}) = \mathbf{0}$, $h(\mathbf{0}) = \mathbf{0}$, and $x(t)$, $u(t)$, and $y(t)$ are respectively the state, input, and output of $\mathcal{G}$. Let $\mathcal{G}_{cl}$ with $y = \mathcal{G}_{cl}(r)$ be composed of $\mathcal{G}$ in closed-loop with a state-feedback policy $u(x, r) = \mathfrak{K}(x)+r$ with exogenous disturbances $r(t)$. Suppose that $\Omega \subset \mathbb{R}^n$ is a connected, compact, constrained subset of the state space of Eq. (1) and has triangulation $\mathcal{T} = \cup_{i=1}^{m_\mathcal{T}} \Delta_i$, where $m_\mathcal{T} \in \mathbb{Z}_1^\infty$, Δ_i is the i^{th} n-simplex, and $\{x_{i,j}\}_{j=0}^n \subset \mathbb{R}^n$ are the vertexes of Δ_i. Further, $\mathfrak{K}(x) = K_i x$ for all $x \in \Delta_i$ with matrices $\{K_i\}_{i=1}^{m_\mathcal{T}} \subset \mathbb{R}^{m\times n}$. Suppose there exists a CPA function $V{:}\mathcal{T} \to \mathbb{R}^+$ and constants $\psi \in \mathbb{R}$, $\mathcal{L} = \{l_i\}_{i=1}^{m_\mathcal{T}} \subset \mathbb{R}^+$ and $\mathcal{K} = \{\kappa_i\}_{i=1}^{m_\mathcal{T}} \subset \mathbb{R}^+$ satisfying for all $(i, j) \in \mathbb{Z}_1^{m_\mathcal{T}} \times \mathbb{Z}_0^n$,

$$\psi > 0, \tag{13a}$$
$$V(x_{i,j}) \ge 0, \tag{13b}$$
$$\|\nabla V_i\|_1 \le l_i, \tag{13c}$$
$$\|K_i\|_1 \le \kappa_i, \text{ and} \tag{13d}$$
$$M_{i,j} \preceq 0, \tag{13e}$$

where

$$M_{i,j} = \begin{bmatrix} M_{i,j}^{11} & \star & \star \\ g(x_{i,j})^{\mathrm{T}} \nabla^{\mathrm{T}} V_i & -2\psi\mathbf{I} + \mathbf{I} & \star \\ h(x_{i,j}) & \mathbf{0} & -\mathbf{I} \end{bmatrix}, \tag{14a}$$

$$M_{i,j}^{11} = \nabla V_i f(x_{i,j}) + \nabla V_i g(x_{i,j}) K_i x_{i,j} + \frac{1}{2} l_i \beta_i c_{i,j}$$
$$+ l_i \nu_i \kappa_i c_{i,j} + \frac{1}{2} l_i \mu_i \bar{x}_i \kappa_i c_{i,j} + \frac{1}{4} l_i^2 c_{i,j}^2 \sum_{k=1}^{m} \mu_{i,k}^2$$
$$+ \frac{1}{4} c_{i,j}^2 \sum_{a=1}^{p} \rho_{i,a}^2, \tag{14b}$$

∇V_i is computed by Lemma 1, $c_{i,j}$ is defined by Eq. (10), and for each $k \in \mathbb{Z}_1^m$, $a \in \mathbb{Z}_1^p$,

$$\beta_i := \max_{p,q,r\in\mathbb{Z}_1^n, \xi\in\Delta_i} \left| \frac{\partial^2 f^p(x)}{\partial x_q \partial x_r}\bigg|_{x=\xi} \right|, \tag{15a}$$

$$\nu_{i,k} := \max_{p,q\in\mathbb{Z}_1^n, \xi\in\Delta_i} \left| \frac{\partial g_k^p(x)}{\partial x_q}\bigg|_{x=\xi} \right|, \quad \nu_i := \max_{k\in\mathbb{Z}_1^m} \nu_{i,k}, \tag{15b}$$

$$\mu_{i,k} := \max_{p,q,r\in\mathbb{Z}_1^n, \xi\in\Delta_i} \left| \frac{\partial^2 g_k^p(x)}{\partial x_q \partial x_r}\bigg|_{x=\xi} \right|, \quad \mu_i := \max_{k\in\mathbb{Z}_1^m} \mu_{i,k}, \tag{15c}$$

$$\rho_{i,a} := \max_{q,r\in\mathbb{Z}_1^n, \xi\in\Delta_i} \left| \frac{\partial^2 h^a(x)}{\partial x_q \partial x_r}\bigg|_{x=\xi} \right|, \text{ and} \tag{15d}$$

$$\bar{x}_i := \max_{j\in\mathbb{Z}_0^n} \|x_{i,j}\|_1, \tag{15e}$$

Then V is a CPA storage function satisfying the HJI (6) for $\mathcal{G}_{cl}$ on Ω. Suppose there exists $\mathcal{A} \subseteq \Omega$ that is RPI with respect to the dynamics $\mathcal{G}_{cl}$ with exogenous disturbances satisfying $r(t) \in \mathcal{U} = \{r(t) \in \mathcal{L}_{2e}^m | \sup_{t\geq 0} \|r_T(t)\|_\infty \leq r_u\}$ for some $r_u > 0$ and all $T \geq 0$. Then $\mathcal{G}_{cl}$ has local, small-signal $\mathcal{L}_2$-gain of at most $\gamma = \sqrt{\psi}$ on $\mathcal{A}$.

Proof: In this proof, the HJI (6) is shown to hold on each simplex for the closed-loop system. This follows in the vein of [9, Th. 4], but some adaptation is needed for control design of closed-loop systems. The term $g(x)\mathfrak{K}(x)$ introduces new design variables. There is also a small difference in the completion of squares. In the proof, Lemma 2 is used to relate $H(x) \preceq 0, x \in \text{interior}(\Delta_i)$ to $H(x_{i,j})$. Next, we derive error bounds of $g(x)\mathfrak{K}(x)$. Lastly, in addition to the HJI condition, by Corollary 1, the assumption of a RPI set to the signal $r \in \mathcal{U}$ implies that the system is local, small-signal $\mathcal{L}_2$-stable.

We begin by substituting $f_{cl}(x) := f(x) + g(x)\mathfrak{K}(x)$ for $f(x)$ into Ineq. (6) and apply Lemma 2 over each i^{th} n-simplex, where any $x \in \Delta_i$ is a convex combination of the vertexes $x = \sum_j \lambda_j x_{i,j}$ with $0 \leq \lambda_j \leq 1$, to obtain

$$H(x) - \sum_{j=0}^n \lambda_j H(x_{i,j}) = \sum_{j=0}^n \lambda_j \begin{bmatrix} E_{i,j}^{11} & E_{i,j}^{21\mathrm{T}} \\ E_{i,j}^{21} & \mathbf{0} \end{bmatrix} \equiv \sum_{j=0}^n \lambda_j E_{i,j}, \tag{16}$$

where

$$E_{i,j}^{11} = \nabla V(x) f_{cl}(x) - \nabla V(x_{i,j}) f_{cl}(x_{i,j}),$$
$$E_{i,j}^{21\mathrm{T}} = \left[\nabla V(x) E_{g_1}(x_{i,j}) \ , \ldots \ \nabla V(x) E_{g_m}(x_{i,j}) \ \ E_h^{\mathrm{T}}(x_{i,j})\right],$$

and $E_{gk}(x_{i,j})$, for all $k \in \mathbb{Z}_1^m$, have components $E_{gk}^a(x_{i,j})$, for all $a \in \mathbb{Z}_1^n$, by substituting all symbols f with g_k, as well as x_j with $x_{i,j}$ in Eq. (9). Similarly for $E_h(x_{i,j})$.

To remove the dependence on x in $H(x) \preceq 0$, we first bound the indefinite matrix $E_{i,j}$ by completing the square, as

$$E_{i,j} \preceq \begin{bmatrix} E_{i,j}^{11} + E_{i,j}^{21\mathrm{T}} E_{i,j}^{21} & \star \\ \mathbf{0} & \mathbf{I} \end{bmatrix}. \tag{17}$$

Using a subscript i to distinguish values that differ across n-simplexes, Eq. (12), (11) in Lemma 2 yield $E_{i,j}^{21\mathrm{T}} E_{21}^{i,j} \leq \frac{1}{4} l_i^2 c_{i,j}^2 \sum_{k=1}^m \mu_{i,k}^2 + \frac{1}{4} c_{i,j}^2 \sum_{a=1}^p \rho_{i,a}^2$.

We bound $E_{i,j}^{11}$ next. Only the proof for $0 \notin \Delta_i$ case is shown because the same steps follow for $0 \in \Delta_i$. By Eq. (7), $\nabla V(x) = \nabla V_i$ for all $x \in \Delta_i$, so it can be factored out. Let $U^a(x) := [g(x)Kx]^a$. Grouping the f, U terms yields $E_{i,j}^{11} = \nabla V_i (E_{i,j}^f + E_{i,j}^U)$, where for each $a \in \mathbb{Z}_1^n$,

$$\left[E_{i,j}^f\right]^a = \langle -\frac{1}{2} \nabla^2 f^a(z_a)(x_{i,j} - x), x_{i,j} - x\rangle, \text{ and} \tag{18}$$

$$\left[E_{i,j}^U\right]^a = \langle -\frac{1}{2} \nabla^2 U^a(z_a)(x_{i,j} - x), x_{i,j} - x\rangle. \tag{19}$$

Eq. (18), (19) follow from Eq. (9b). Lemma 2 can be applied directly to bound $\nabla V_i E_{i,j}^f$ by Eq. (12), but more work is required before applying Holder's inequality, $\nabla V_i E_{i,j}^U \leq \|\nabla V_i\|_1 \left\|E_{i,j}^U\right\|_\infty$. Apply Cauchy-Schwarz inequality on $[E_{i,j}^U]^a$, again using subscript i to index n-simplexes, and with $c_{i,j}$ defined by Eq. (10), we obtain

$$\left\|E_{i,j}^U\right\|_\infty = \max_{a\in\mathbb{Z}_1^n} \left|\left[E_{i,j}^U\right]^a\right| \leq \frac{1}{2} \max_{z_a\in\Delta_i} \left\|\nabla^2 U^a(z_a)\right\|_2 \frac{1}{n} c_{i,j}. \tag{20}$$

We examine next the components of Hessian $\nabla^2 U^a(x)$, which has elements $[\nabla^2 U^a(x)]_{uv} = [A(x)]_{uv} + [A^{\mathrm{T}}(x)]_{uv} + [B(x)]_{uv}$, where

$$[A(x)]_{uv} = \sum_{j=1}^m \frac{\partial}{\partial x_v} g_{aj}(x) K_{ju}, \text{ and} \tag{21}$$

$$[B(x)]_{uv} = \sum_{j=1}^m \frac{\partial^2}{\partial x_v \partial x_u} g_{aj}(x) \sum_{k=1}^n K_{jk} x_k. \tag{22}$$

The induced 2-norm of $Q \in \mathbb{R}^{n\times n}$ is bounded by $\|Q\|_2 \leq \sqrt{\|Q\|_1 \|Q\|_\infty}$ [13, Ch. 5]. Since

$$\|Q\|_1 = \max_{j\in\mathbb{Z}_1^n} \sum_{i=1}^n |Q_{ij}| \leq n \max_{i,j\in\mathbb{Z}_1^n} |Q_{ij}|, \text{ and}$$

$$\|Q\|_\infty = \max_{i\in\mathbb{Z}_1^n} \sum_{j=1}^n |Q_{ij}| \leq n \max_{i,j\in\mathbb{Z}_1^n} |Q_{ij}|,$$

each 2-norm of $A(x)$ and $B(x)$ are bounded above by n times maximum absolute-valued element, which by applying Holder's inequality on Eqs. (21), and (22) yields $\|A(x)\|_2 \leq n\nu_i\kappa_i$ and $\|B(x)\|_2 \leq n\mu_i\bar{x}_i\kappa_i$. Apply Holder's inequality and by the triangle inequality on $\left\|\nabla^2 U^a(x)\right\|_2$, from Eq. (20) we find

$$\nabla V_i E_{i,j}^U \leq l_i c_{i,j} \left(\nu_i \kappa_i + \frac{1}{2} \mu_i \bar{x}_i \kappa_i\right). \tag{23}$$

Applying this to bound $E_{i,j}^{11} + E_{i,j}^{21\mathrm{T}} E_{i,j}^{21}$ in Eq. (17) and substituting in Eq. (16) results in

$$H(x) - \sum_{j=0}^n \lambda_j H(x_{i,j}) \preceq \sum_{j=0}^n \lambda_j \begin{bmatrix} \bar{E}_{i,j}^{11} & \star \\ \mathbf{0} & \mathbf{I} \end{bmatrix} =: \sum_{j=0}^n \lambda_j \bar{E}_{i,j},$$

$$\text{where} \bar{E}_{i,j}^{11} = \frac{1}{2} l_i \beta_i c_{i,j} + l_i \nu_i \kappa_i c_{i,j} + \frac{1}{2} l_i \mu_i \bar{x}_i \kappa_i c_{i,j} + \frac{1}{4} l_i^2 c_{i,j}^2 \sum_{k=1}^m \mu_{i,k}^2 + \frac{1}{4} c_{i,j}^2 \sum_{a=1}^p \rho_{i,a}^2. \tag{24}$$

Rearranging, $H(x) \preceq \sum_{j=0}^n \lambda_j (H(x_{i,j}) + \bar{E}_{i,j})$. Therefore, with $0 \leq \lambda_j \leq 1$, imposing $H(x_{i,j}) + \bar{E}_{i,j} \preceq 0$ for all $j \in \mathbb{Z}_0^n$ implies $H(x) \preceq 0$ for all $x \in \Delta_i$. Additionally, to impose also the HJI (6) as in $H(x_{i,j}) \preceq 0$ for all vertexes in $\mathcal{T}$, S-procedure can be applied because each $\bar{E}_{i,j} \succeq 0$. Thus, by imposing the constraints (13e) at the vertexes $x_{i,j}$ of each n-simplex Δ_i, by Corollary 1, $\mathcal{G}_{cl}$ with $u(t) \in \mathcal{U}$ has local, small-signal $\mathcal{L}_2$-gain $\leq \gamma$ on Ω. ■

B. Computable Gain Criteria

In Theorem 2, terms involving the product of variables V_i and K_i or l_i and κ_i create bilinear and non-convex constraints. However, if an initial feasible solution is provided, an iterative convex overbounding (ICO) technique is guaranteed to find solutions under LMI constraints. The convex optimization problem is presented in the following theorem.

Theorem 3: Let Eq. (1), $\Omega, \mathcal{A}, \mathcal{T}$, and $V, \mathfrak{K}, \psi$ all satisfy the assumptions of Theorem 2. Further, define new CPA function $V + \delta V$, controller $(\mathfrak{K} + \delta\mathfrak{K})(x)$, and bound $\psi + \delta\psi$. Then the solutions of the following problem satisfy Theorem 2 with $\psi + \delta\psi^* \leq \psi$.

$$\delta V^* \delta\psi^*, \delta\mathcal{L}^*, \delta\mathcal{K}^*, \delta\mathfrak{K}^* = \underset{\delta V, \delta\psi, \delta\mathcal{L}, \delta\mathcal{K}, \delta\mathfrak{K}}{\operatorname{argmin}} \ \delta\psi$$

subject to

$$\psi + \delta\psi > 0, \tag{25a}$$
$$V(x_{i,j}) + \delta V(x_{i,j}) \geq 0 \;\; \forall i \in \mathbb{Z}_1^{m\mathcal{T}}, \forall j \in \mathbb{Z}_0^n, \tag{25b}$$
$$\|\nabla \delta V_i\|_1 \leq \delta l_i \;\; \forall i \in \mathbb{Z}_1^{m\mathcal{T}}, \tag{25c}$$
$$\|\delta K_i\|_1 \leq \delta\kappa_i \;\; \forall i \in \mathbb{Z}_1^{m\mathcal{T}}, \tag{25d}$$
$$M_{i,j} \preceq 0 \;\; \forall i \in \mathbb{Z}_1^{m\mathcal{T}}, \forall j \in \mathbb{Z}_0^n, \tag{25e}$$

where $\delta\mathcal{L} = \{\delta l_i\}_{i=1}^{m\mathcal{T}} \subset \mathbb{R}^+$, $\delta\mathcal{K} = \{\delta\kappa_i\}_{i=1}^{m\mathcal{T}} \subset \mathbb{R}^+$, $\delta\mathfrak{K}=\{\delta K_i\}_{i=1}^{m\mathcal{T}} \subset \mathbb{R}^{m\times n}$, $\nabla\delta V_i := \nabla\delta V(x) \,\forall x \in \Delta_i$ by replacing V_i with δV_i in Eq. (7), and

$$M_{i,j} = \begin{bmatrix} M_{i,j}^{11} & \star & \star & \star & \star & \star \\ M_{i,j}^{21} & -2(\psi+\delta\psi)\mathbf{I}+\mathbf{I} & \star & \star & \star & \star \\ h(x_{i,j}) & \mathbf{0} & -\mathbf{I} & \star & \star & \star \\ \delta l_i & \mathbf{0} & \mathbf{0} & t_{i,j}^l & \star & \star \\ \delta\kappa_i & \mathbf{0} & \mathbf{0} & 0 & t_{i,j}^\kappa & \star \\ M_{i,j}^{61} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -2\mathbf{I} \end{bmatrix}, \tag{26a}$$

$$\begin{aligned} M_{i,j}^{11} &= (\nabla V_i + \nabla\delta V_i) f(x_{i,j}) + \nabla V_i g(x_{i,j})(K_i + \delta K_i) x_{i,j} \\ &+ \nabla\delta V_i g(x_{i,j}) K_i x_{i,j} + \frac{1}{2}(l_i + \delta l_i)\beta_i c_{i,j} \\ &+ \delta l_i \nu_i \kappa_i c_{i,j} + l_i \nu_i (\kappa_i + \delta\kappa_i) c_{i,j} + \frac{1}{4} c_{i,j}^2 \sum_{a=1}^{p} \rho_{i,a}^2 \\ &+ \frac{1}{2} l_i \mu_i \bar{x}_i (\kappa_i + \delta\kappa_i) c_{i,j} + \frac{1}{2}\delta l_i \mu_i \bar{x}_i \kappa_i c_{i,j} \\ &+ \frac{1}{4} l_i^2 c_{i,j}^2 \sum_{k=1}^{m} \mu_{i,k}^2 + \frac{1}{2} l_i \delta l_i c_{i,j}^2 \sum_{k=1}^{m} \mu_{i,k}^2, \end{aligned} \tag{26b}$$

$$M_{i,j}^{21} = g(x_{i,j})^{\mathrm{T}} (\nabla^{\mathrm{T}} V_i + \nabla^{\mathrm{T}} \delta V_i), \tag{26c}$$

$$M_{i,j}^{61} = \nabla^{\mathrm{T}}\delta V_i + \frac{1}{2} g(x_{i,j}) \delta K_i x_{i,j}, \tag{26d}$$

$$t_{i,j}^l = \frac{-4}{c_{i,j}^2 \sum_{k=1}^m \mu_{i,k}^2 + \bar{x}_i \mu_i c_{i,j} + 2\nu_i c_{i,j}}, \tag{26e}$$

$$t_{i,j}^\kappa = \frac{-4}{\bar{x}_i \mu_i c_{i,j} + 2\nu_i c_{i,j}}, \tag{26f}$$

with $l_i = \|\nabla V_i\|_1, \kappa_i = \|K_i\|_1$, $c_{i,j}$ defined by Eq. (10), and other constants defined in Eq. (15) for each $k \in \mathbb{Z}_1^m, a \in \mathbb{Z}_1^p$.

Proof: We must have $\psi + \delta\psi^* \leq \psi$ because the minimization problem stated in Eq. (25) inherits feasibility from Theorem 2 with $\delta(\cdot) = 0$ for each l_i, κ_i, ψ, V, and $\delta\mathfrak{K} = \mathbf{0}$. The remainder of this proof is devoted to formulate Eq. (26). This is done through a sequence of completing the squares on cross-terms between design variables.

Following the proof of Theorem 2, substitute first $(\cdot)$ with $(\cdot) + \delta(\cdot)$ in Eq. (14b) for each $l_i, K_i, \kappa_i, V, \psi$, noting that the $\delta(\cdot)$ terms are design variables. Since gradient is a linear operator, $\nabla(V_i + \delta V_i) = \nabla V_i + \nabla\delta V_i$. Bound next $\|\nabla V_i + \nabla\delta V_i\|_1$ and $\|K_i + \delta K_i\|_1$ with the triangle inequality. Lastly, Young's relation can be applied on terms involving the product of δl_i and $\delta\kappa_i$, as well as $\nabla\delta V_i$ and δK_i, followed by Schur complements to obtain LMI constraints.

Complete the squares to obtain

$$\delta l_i \delta\kappa_i \left(\nu_i c_{i,j} + \frac{1}{2}\mu_i \bar{x}_i c_{i,j}\right) \leq \left(\frac{1}{2}\nu_i c_{i,j} + \frac{1}{4}\mu_i \bar{x}_i c_{i,j}\right)\left(\delta l_i^2 + \delta\kappa_i^2\right).$$

To eliminate the product $\nabla\delta V_i g(x_{i,j})\delta K(x_{i,j})$, apply Young's relation [14, Eq. (5)],

$$\begin{aligned} &\mathrm{He}\big(\nabla\delta V_i \tfrac{1}{2} g(x_{i,j}) \delta K x_{i,j}\big) \preceq \\ &\frac{1}{2}\big(\nabla\delta V_i + \frac{1}{2} x_{i,j}^{\mathrm{T}} \delta K_i^{\mathrm{T}} g^{\mathrm{T}}(x_{i,j})\big)\big(\nabla^{\mathrm{T}}\delta V_i + \frac{1}{2} g(x_{i,j}) \delta K x_{i,j}\big), \end{aligned}$$

where $\mathrm{He}(A) = A + A^{\mathrm{T}}$ for $A \in \mathbb{R}^{n\times n}$.

Finally, perform Schur complement [11, Ch. 2] to all three inner products or squared terms and obtain the definition of $M_{i,j}$ in Eq. (26). ■

C. Implementation Notes

Theorem 3 provides an optimization problem to simultaneously search for a storage function that bounds closed-loop gain, while searching for a controller that minimizes that gain. However, it requires an initial controller and storage function. If the open-loop system $\mathcal{G}$ is $\mathcal{L}_2$-stable, then solutions V, and γ of $\mathcal{G}$ via [9, Th. 4], along with $\mathfrak{K} = \{K_i = \mathbf{0}\}_{i=1}^{m\mathcal{T}}$ can serve this purpose, as demonstrated in the numerical example.

If $\mathcal{G}$ is not $\mathcal{L}_2$-stable, initialization is challenging, and a stabilizing controller may not even exist. One heuristic would be to fix any random set of control gains, $\mathfrak{K} = \{K_i\}_{i=1}^{m\mathcal{T}}$, and solve for the other design variables in Theorem 2, while minimizing the violations of the constraints in Eq. (13). Then, perturbations on the control gains and other variables can be iteratively selected to minimize the violation of Eq. (25) rather than $\delta\psi$. Once a feasible solution is found, perturbations can be selected following Eq. (25) directly.

Finally, to establish local, small-signal $\mathcal{L}_2$-gain, a RPI subset of Ω must be found. This can be done after selecting a controller via [8, Th. 5], but co-optimizing the gain and RPI set on which it is established is left to future work.

IV. Numerical Example

The effectiveness of Theorem 3 for controller synthesis was demonstrated by comparing closed-loop and open-loop small-signal $\mathcal{L}_2$-gain bounds (γ_{cl} and γ_{ol}), where γ_{ol} was computed from [9, Th. 4].

Consider the dynamical system from [8], [9],

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} x_2 \\ -\sin x_1 - x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \end{bmatrix} u, \quad x(0) \in \Omega, \quad y = x_2, \tag{27}$$

with $x \in \mathbb{R}^2, u \in \mathbb{R}, y \in \mathbb{R}$, and $\Omega := \{(x_1, x_2) \mid |x_1| \leq 0.8, |x_2| \leq 0.8, -0.8 \leq x_1 + x_2 \leq 0.8\}$. A piecewise affine state-feedback controller was designed to minimize closed-loop $\mathcal{L}_2$-gain bound on Ω of system Eq. (27) using Theorem 3.

While any triangulation suffices to set up constraints, a Delaunay triangulation was used to avoid slivers. Methods to compute Delaunay triangulations in $\mathbb{R}^n$ are well-established [15, Ch. 9], and multiple software packages are available for triangulations in $\mathbb{R}^2$. Using the mesh generation procedure from [16], we constructed three triangulations of Ω containing 266, 530, and 1,806 2-simplexes, respectively. The size of each 2-simplex decreases with an increasing total number of 2-simplexes.

Using [9, Th. 4], we obtained the open-loop gain bound γ_{ol} on Ω and the CPA storage function V. These solutions, along with controller $\mathfrak{K} = \{K_i = \mathbf{0}\}_{i=1}^{m\mathcal{T}}$, served as the initial feasible solution for the control design in Theorem 3, yielding the closed-loop gain bound γ_{cl} on Ω and the final controller

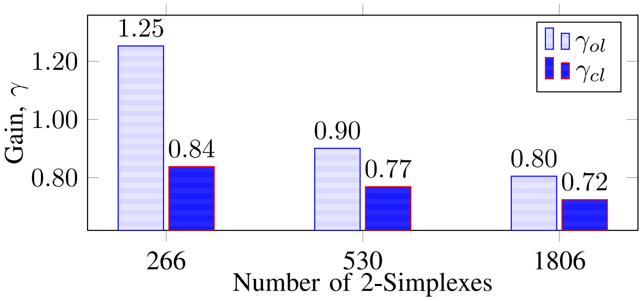

Fig. 1. Gains γ_{ol} and γ_{cl} for 3 triangulation schemes.

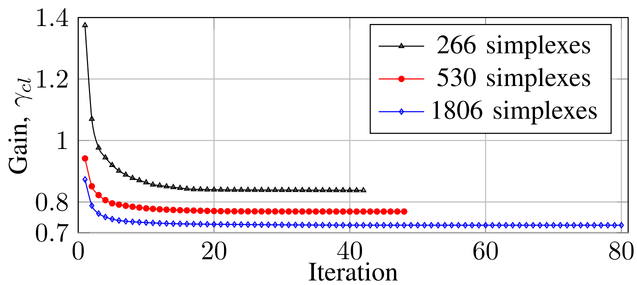

Fig. 2. Gain γ_{cl} after each iteration for 3 triangulation schemes.

$\mathfrak{K}(x)$. We implemented the optimization problem in MATLAB using the YALMIP toolbox [17] with the MOSEK semi-definite programming solver [18]. For the ICO algorithm, we set the termination condition as $|\delta\psi| < 10^{-6}$, balancing computational efficiency with the need to avoid premature convergence to local minima.

The closed-loop gain γ_{cl} remained smaller than open-loop gain γ_{ol} for all triangulation schemes (Fig. 1). The Taylor expansion error bounds reduced with the decreasing sizes of 2-simplexes (due to $c_{i,j}$ by Eq. (10)), which reduced conservativeness in the HJI constraints. As a result, both $\mathcal{L}_2$-gain bounds decreased asymptotically. The γ_{cl} for triangulation of 266 2-simplexes already achieved result close to the numerical limit, whereas γ_{ol} did not. Therefore, this triangulation appears to have the largest improvement.

Among all three triangulation schemes, γ_{cl} improved most within the first 10 iterations (Fig. 2), demonstrating the effectiveness of Theorem 3. The refinement of triangulation improved the $\mathcal{L}_2$-gain bound but led to an $\mathcal{O}(m_{\mathcal{T}})$ increase in both variables and constraints, resulting in an $\mathcal{O}(m_{\mathcal{T}}^3)$ increase in computational cost per iteration of Eq. (25) when using interior point methods for this 2D example [19, Ch. 6]. The current method will not scale well if a dense triangulation is needed, especially for systems with many states. An adaptive triangulation scheme of varying n-simplex density can be considered to reduce $m_{\mathcal{T}}$ for future work.

The small-signal bound r_u was not sought in this example because it can be done by applying [8, Th. 5] to the closed-loop system.

V. Conclusion

Based on the HJI, we developed two theorems for computing piecewise affine state-feedback controllers that minimize the local, closed-loop small-signal $\mathcal{L}_2$-gain bound of nonlinear systems. Theorem 3 establishes that this can be accomplished using semi-definite programming. Our numerical results demonstrate improved closed-loop performance compared to open-loop bounds on the region of interest. The approach benefits from the computational efficiency of semi-definite programming and the implementation simplicity of piecewise affine state-feedback.

The local, small-signal $\mathcal{L}_2$-gain bound from Theorem 2, 3 apply only to the search space. Nonetheless, local analysis of nonlinear systems is sufficient for the design requirements in many practical applications. While local, small-signal $\mathcal{L}_2$-stability requires input constraints to maintain states within the analyzed region, future work will address relaxing the condition $g(0) = 0$ and co-optimizing the closed-loop gain bound with input signal bound.

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