



Lieb–Thirring Inequality for the 2D Pauli Operator

Rupert L. Frank^{1,2,3} , Hynek Kovařík⁴

¹ Mathematisches Institut, Ludwig-Maximilians-Universität München, Theresienstr. 39, 80333 Munich, Germany

² Mathematics, 253-37, Caltech, Pasadena, CA 91125, USA. E-mail: r.frank@lmu.de; rlfrank@caltech.edu

³ Munich Center for Quantum Science and Technology, Schellingstr. 4, 80799 München, Germany

⁴ DICATAM, Sezione di Matematica, Università degli studi di Brescia, Brescia, Italy

Received: 15 April 2024 / Accepted: 29 October 2024

© The Author(s) 2024

Abstract: By the Aharonov–Casher theorem, the Pauli operator P has no zero eigenvalue when the normalized magnetic flux α satisfies $|\alpha| < 1$, but it does have a zero energy resonance. We prove that in this case a Lieb–Thirring inequality for the γ -th moment of the eigenvalues of $P + V$ is valid under the optimal restrictions $\gamma \geq |\alpha|$ and $\gamma > 0$. Besides the usual semiclassical integral, the right side of our inequality involves an integral where the zero energy resonance state appears explicitly. Our inequality improves earlier works that were restricted to moments of order $\gamma \geq 1$.

1. Introduction and Main Result

1.1. Background. We are interested in quantitative information on the negative eigenvalues of the operator

$$P + V \quad \text{in } \mathsf{L}^2(\mathbb{R}^2, \mathbb{C}^2).$$

where P is the Pauli operator,

$$P = \begin{pmatrix} H^+ & 0 \\ 0 & H^- \end{pmatrix}, \quad H^\pm = (-i\nabla + A)^2 \pm B. \quad (1.1)$$

Here $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a vector field and the function $B : \mathbb{R}^2 \rightarrow \mathbb{R}$ is defined by

$$B = \operatorname{curl} A = \partial_1 A_2 - \partial_2 A_1.$$

© 2024 by the authors. This paper may be reproduced, in its entirety, for non-commercial purposes. Partial support through US National Science Foundation grant DMS-1954995 (R.L.F.), as well as through the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) through Germany's Excellence Strategy EXC-2111-390814868 and through TRR 352-Project-ID 470903074 (R.L.F.) is acknowledged.

For simplicity we restrict ourselves to the case where $V : \mathbb{R}^2 \rightarrow \mathbb{R}$ is scalar, that is, acts trivially on the \mathbb{C}^2 part of $L^2(\mathbb{R}^2, \mathbb{C}^2)$. Both B and V are assumed to be sufficiently regular and to decay in a suitable sense at infinity, as will be made precise later on.

Physically, the operator $P + V$ describes a quantum particle moving in a plane in the presence of a magnetic field of strength B pointing orthogonal to this plane and in the presence of an electric field with potential V . The matrix structure of P and the $\pm B$ term in P come from the interaction of the spin of the particle with the magnetic field. This spin-orbit coupling is neglected when considering the magnetic Schrödinger operator. This simplifies the model, but has the effect of destroying some of the structure of the Pauli operator. In particular, zero modes are removed and the bottom of the spectrum is stabilized. In our study we will *not* neglect the spin-orbit coupling and we will pay special attention to effects coming from the low energy part of the operator P .

When B and V are sufficiently regular and sufficiently fast decaying (we will be more precise later on), the differential expression $P + V$ can be realized as a self-adjoint, lower bounded operator in the Hilbert space $L^2(\mathbb{R}^2, \mathbb{C}^2)$ and the negative spectrum of this operator consists only of eigenvalues with finite multiplicities and with zero as their only possible accumulation point. Labelling these eigenvalues as E_j , where multiplicities are taken into account, we are interested in bounding sums

$$\sum_j |E_j|^\gamma = \text{Tr}(P + V)_-^\gamma$$

from above for different choices of the parameter $\gamma > 0$. These upper bounds shall involve integrals over \mathbb{R}^2 of powers of V and quantities defined in terms of the magnetic field B . The prototype of such bounds are the Lieb–Thirring inequalities, which in the nonmagnetic case state that for any $\gamma > 0$ there is a universal constant L_γ such that for all real $V \in L^1_{\text{loc}}(\mathbb{R}^2)$ one has

$$\text{Tr}(-\Delta + V)_-^\gamma \leq L_\gamma \int_{\mathbb{R}^2} V(x)_-^{\gamma+1} dx. \quad (1.2)$$

Here $a_\pm := \max\{\pm a, 0\}$, so that $a = a_+ - a_-$. The bound (1.2) goes back to the work of Lieb and Thirring [35] and has created a huge literature. For further reading on this topic we refer to the monograph [22], the review [20] and references therein.

One feature about (1.2) that will be relevant for our discussion is that the inequality gets stronger as γ gets smaller. This is formalized by the Aizenman–Lieb argument [3] (see also [22, Lemma 5.2]), which says that the validity of inequality (1.2) for some $\gamma = \gamma_0$ implies its validity for all $\gamma \geq \gamma_0$.

Turning our attention back to magnetic fields, it is not difficult to see that (1.2) remains valid when $-\Delta$ is replaced by $(-i\nabla + A)^2$; see [22, Theorem 4.61], [18] and references therein. More precisely, for any $\gamma > 0$ there is a constant \tilde{L}_γ such that for any real $V \in L^1_{\text{loc}}(\mathbb{R}^2)$ and any $A \in L^2_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ the analogue of (1.2) holds with constant \tilde{L}_γ . Note that the right side in the resulting inequality is independent of A .

The situation is quite a bit more complicated for the Pauli operator, that is, when the spin-orbit coupling is taken into account. There have been many works addressing this question and we will review them in some detail later in this introduction. For the present discussion the following two facts are important. First, there can be no bound of the form (1.2) with a right side that is independent of A . Second, previous works are restricted to the range $\gamma \geq 1$. Both phenomena are related to the existence of zero modes

of the Pauli operator. The existence of the latter and their structure is described by the Aharonov–Casher theorem [2].

What we shall show in the present paper is that if the normalized magnetic flux

$$\alpha := \frac{1}{2\pi} \int_{\mathbb{R}^2} B(x) dx < \infty \quad (1.3)$$

satisfies

$$|\alpha| < 1, \quad (1.4)$$

then a Lieb–Thirring inequality holds for $P + V$ whenever $\gamma \geq |\alpha|$ and $\gamma > 0$. Moreover, we shall show that this restriction on γ is optimal.

According to the Aharonov–Casher theorem [2] (see also [8]), assumption (1.4) implies that the Pauli operator P does not have a zero eigenvalue. Heuristically, this eliminates the reason for the restriction $\gamma \geq 1$ in earlier works. The Pauli operator P does, however, have a zero energy resonance, that is, there is a function ψ_0 with $P\psi_0 = 0$ that decays at infinity, but not fast enough to be square integrable. (We recall that the operator P has a zero resonance for any α .) The decay of this resonance function ψ_0 will be what dictates the optimal condition $\gamma \geq |\alpha|$ on the exponent in the Lieb–Thirring inequality. Our Lieb–Thirring inequality will have two terms on the right side, the first one being the standard term from (1.2) and the second one involving explicitly the resonance function ψ_0 .

1.2. Definitions and main result. We now turn to a precise formulation of our result, beginning with a careful definition of the Pauli operator P . The standard definition of P assumes that $A \in L^2_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ and proceeds from the quadratic forms

$$\int_{\mathbb{R}^2} |(\Pi_1 + i\Pi_2)\psi^+|^2 dx + \int_{\mathbb{R}^2} |(\Pi_1 - i\Pi_2)\psi^-|^2 dx$$

where $\Pi_j := -i\partial_j + A_j$ and where the form is defined for all $(\psi^+, \psi^-) \in L^2(\mathbb{R}^2, \mathbb{C}^2)$ for which the distributions $(\Pi_1 + i\Pi_2)\psi^+$ and $(\Pi_1 - i\Pi_2)\psi^-$ belong to $L^2(\mathbb{R}^2)$. We will *not* adapt this definition, although the one we choose is equivalent to this standard definition in situations with enough regularity. The reason is that our assumptions are more naturally formulated in terms of the scalar potential h (defined momentarily) rather than in terms of the vector potential A , on which the standard definition is based.

The approach that we follow was promoted by Erdős and Vugalter and investigated in detail in their paper [16]. To motivate it, we assume that there is a real function $h \in L^2_{\text{loc}}(\mathbb{R}^2)$ such that $A_1 = -\partial_2 h$ and $A_2 = \partial_1 h$. Then a computation shows that

$$\begin{aligned} \int_{\mathbb{R}^2} |(\Pi_1 + i\Pi_2)\psi^+|^2 dx &= \int_{\mathbb{R}^2} e^{2h} |(\partial_1 + i\partial_2)e^{-h}\psi^+|^2 dx, \\ \int_{\mathbb{R}^2} |(\Pi_1 - i\Pi_2)\psi^-|^2 dx &= \int_{\mathbb{R}^2} e^{-2h} |(\partial_1 - i\partial_2)e^h\psi^-|^2 dx. \end{aligned}$$

The basic idea is to use the right sides to *define* the Pauli operator. Note that if such a function h exists, then $\Delta h = \partial_1 A_2 - \partial_2 A_1 = B$.

We now proceed to the actual definition of P , following [16]. We assume that μ is a signed real regular Borel measure on \mathbb{R}^2 with $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}^2$ (which plays

the role of the magnetic field). Then, by [16, Theorem 2.7 and Corollary 2.3] there is a real-valued $h \in \bigcap_{p < 2} W_{\text{loc}}^{1,p}(\mathbb{R}^2)$ such that

$$\Delta h = \mu \quad \text{in } \mathbb{R}^2$$

and $e^{2h}, e^{-2h} \in L^1_{\text{loc}}(\mathbb{R}^2)$. Fixing any such h , the quadratic form

$$\int_{\mathbb{R}^2} e^{2h} |(\partial_1 + i\partial_2)e^{-h}\psi^+|^2 dx + \int_{\mathbb{R}^2} e^{-2h} |(\partial_1 - i\partial_2)e^h\psi^-|^2 dx, \quad (1.5)$$

defined for all $(\psi^+, \psi^-) \in L^2(\mathbb{R}^2, \mathbb{C}^2)$ for which the integrals are finite, is nonnegative and closed in $L^2(\mathbb{R}^2, \mathbb{C}^2)$ [16, Theorem 2.5] and therefore generates a selfadjoint, non-negative operator P in $L^2(\mathbb{R}^2, \mathbb{C}^2)$. This operator depends on the choice of the function h , but one can show that for two different choices of functions h corresponding to the same μ the resulting operators are unitarily equivalent by a gauge transformation [16, Theorem 2.5]. Clearly, for two functions h differing by an additive constant the corresponding operators coincide. Moreover, if there is an $A \in L^2_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ with $\partial_1 A_2 - \partial_2 A_1 = \mu$ (in the sense of distributions), then the operator P is unitarily equivalent to the Pauli operator defined via the standard approach outlined above [16, Proposition 2.10].

We can now formulate our assumptions on the magnetic field. It is formulated in terms of the auxiliary function h that appears in the definition of the Pauli operator.

Assumption 1.1. There is an $\alpha \in (-1, 1)$ and an $R > 0$ such that the two numbers

$$m^\pm := \text{ess sup}_{x \in \mathbb{R}^2} \frac{e^{\pm h(x)}}{(1 + |x|/R)^{\pm\alpha}} \quad (1.6)$$

are both finite.

We emphasize that, if μ is absolutely continuous with $B = \frac{d\mu}{dx} \in L^1(\mathbb{R}^2)$, then the validity of Assumption 1.1 implies that the number α is necessarily given by the expression (1.3). We provide a proof of this claim in Lemma A.1.

The number α plays an important role in what follows. In contrast, the number R will only play a minor role and is only introduced for dimensional consistency.

A simple case where Assumption 1.1 is satisfied is $\mu = (\alpha/R)\mathcal{H}_{\partial B(0,R)}^1$, with $\mathcal{H}_{\partial B(0,R)}^1$ denoting surface measure on the circle $\partial B(0, R)$ of radius R centered at the origin. In this case we can choose $h = \alpha \ln_+(|x|/R)$ and we see that Assumption 1.1 is satisfied with the given α and R .

We emphasize that, while we can treat μ that are not absolutely continuous, our main interest is in the absolutely continuous case with $B = \frac{d\mu}{dx} \in L^1(\mathbb{R}^2)$. For instance it is easy to see that if B satisfies

$$|B(x)| \leq CR^{-2}(1 + |x|/R)^{-\rho} \quad \text{with some } C > 0 \text{ and } \rho > 2, \quad (1.7)$$

then (Assumption 1.1) holds with α given by (1.3) and the given R . The numbers m^\pm are bounded in terms of C and ρ . In Lemma A.2 we show that Assumption 1.1 is satisfied under rather weak integrability assumptions on B .

We are now ready to formulate our main result.

(c) Concerning the condition $|\alpha| < 1$ in Assumption 1.1 we remark that our bound cannot hold for $|\alpha| > 1$. This follows again from weak coupling asymptotics in [25, 30], which state that $\text{Tr}(P + \lambda V)_-^\gamma$ behaves like λ^γ when $|\alpha| \geq 1$. Meanwhile, the second term on the right side of our bound behaves like $\lambda^{1+\gamma-|\alpha|}$, showing that the bound can only hold when $|\alpha| \leq 1$. This leaves open the case $|\alpha| = 1$ for which one might expect a bound for $\gamma \geq 1$. As discussed in the next subsection, under somewhat different assumptions on the magnetic field such a bound was indeed shown in [42], which is why we did not investigate it further.

(d) The function $e^{-(\text{sgn } \alpha)(h-h_0)}$ coincides, up to a phase factor, with the zero energy resonance function ψ_0 mentioned in Sect. 1.1. Also, since the operator P does not change if a constant is added to h , the difference $h - h_0$ that appears in our bound is indeed a natural quantity. When h is continuous at the origin, we clearly have $h_0 = h(0)$. The particular way of how to define h_0 in the discontinuous case is dictated mostly by technical convenience. We emphasize that h_0 is finite in view of Assumption 1.1. The fact that the point 0 is singled out in the definition of h_0 reflects that this point is singled out in (1.6). Clearly, in applications one is free to choose this singled-out point.

(e) Our bound depends on the ‘magnetic field’ μ only via the function h and this dependence is only via the quantities α , R and m^\pm from Assumption 1.1. In particular, note that

$$m^+ m^- = \text{ess sup}_{x \in \mathbb{R}^2} \frac{e^{h(x)}}{(1+|x|/R)^\alpha} \text{ess sup}_{x \in \mathbb{R}^2} \frac{e^{-h(x)}}{(1+|x|/R)^{-\alpha}} \geq 1.$$

In the weak field limit where B is replaced by λB and $\lambda \rightarrow 0$, the function h is replaced by λh and α by $\lambda \alpha$, while R remains unchanged. The product $m^+ m^-$ is replaced by $(m^+ m^-)^\lambda$, which tends to 1. Our proof will show that for fixed $\gamma > 0$, the constant $C(|\lambda \alpha|, \gamma)$ remains bounded as $\lambda \rightarrow 0$; see Remark 3.2. Thus, our bound is stable in the limit $\lambda \rightarrow 0$ and reproduces the nonmagnetic Lieb–Thirring inequality (1.2). This property is not shared, for instance, by the bound from [42] discussed in the next subsection.

(f) We have been somewhat cavalier about our assumptions on V . Here is a more precise statement: If $V \in L^1_{\text{loc}}(\mathbb{R}^2)$ is real and if the right side in the bound in the theorem is finite, then V_- is infinitesimally form bounded with respect to P and for the operator $P + V$, defined via quadratic forms, the stated bound holds. This follows by standard arguments from our proof. The same statement holds for all Lieb–Thirring-type inequalities in this paper and will not be repeated each time.

(g) Assume that \mathcal{V} is a locally integrable function on \mathbb{R}^2 taking values in the Hermitian 2×2 -matrices. Then the Lieb–Thirring inequality (1.8) holds for the operator $P + \mathcal{V}$, provided on the right side we replace $V(x)_-^p$ by $\text{Tr}_{\mathbb{C}^2}(\mathcal{V}(x)_-^p)$ for $p \in \{1 + \gamma, 1 + \gamma - |\alpha|\}$. This simply follows from the inequality $\mathcal{V}(x) \geq -\|\mathcal{V}(x)_-\|$ (with $\|\cdot\|$ the operator norm on \mathbb{C}^2), our bound in the scalar case and the bound $\|\mathcal{V}(x)_-\|^p \leq \text{Tr}_{\mathbb{C}^2}(\mathcal{V}(x)_-^p)$. For this reason we restrict ourselves to the case of a scalar electric potential.

(h) Our inequality comes with explicit values for the constants $C(|\alpha|, \gamma)$, but since they are far from optimal we do not state them explicitly.

1.3. Previous results. Let us review some previous works on Lieb–Thirring inequalities for Pauli operators and compare them with our new results. Throughout we focus on the

two-dimensional case and leave out many important advances in the three dimensional case, starting with Erdős’s foundational work [13] and reviewed in [6, 14].

Lieb, Solovej and Yngvason [34] showed that when B is constant, then

$$\mathrm{Tr}(P + V)_- \leq C \left(\int_{\mathbb{R}^2} V(x)_-^2 dx + |B| \int_{\mathbb{R}^2} V(x)_- dx \right).$$

This was generalized by Erdős and Solovej [15, Thm. 3.2] (based on the strategy in [13]), who showed that for any $\gamma \geq 1$ there is a constant C_γ such that

$$\mathrm{Tr}(P + V)_-^\gamma \leq C_\gamma \left(\int_{\mathbb{R}^2} V(x)_-^{\gamma+1} dx + \|B\|_\infty \int_{\mathbb{R}^2} V(x)_-^\gamma dx \right). \quad (1.11)$$

More relevant for us is an earlier work of Sobolev [42], where it was shown under fairly general conditions on B that for any $\gamma \geq 1$ there is a constant C_γ such that

$$\mathrm{Tr}(P + V)_-^\gamma \leq C_\gamma \left(\int_{\mathbb{R}^2} V(x)_-^{\gamma+1} dx + \int_{\mathbb{R}^2} b(x) V(x)_-^\gamma dx \right). \quad (1.12)$$

Here $b \geq 0$ denotes a “smeared” modification of $|B|$, see [42, Sec. 2] for details. It should be noted that b is not uniquely defined. The relevance of a smeared magnetic field was pointed out in [13]. A Lieb–Thirring inequality similar to (1.12), but with a different “smeared” modification of B can probably be obtained by adapting the proof of [41, Theorem 1.2] to the two-dimensional setting. This will again require the assumption $\gamma \geq 1$.

Returning our attention to (1.12), we note that the assumptions on B in [42] are somewhat implicit. In order to compare the results with ours, we assume that B satisfies the pointwise decay condition (1.7). Then it is easily verified that an effective magnetic field b in the sense of [42] can be constructed in such a way that

$$b(x) \leq C_B (1 + |x|)^{-2} \quad \forall x \in \mathbb{R}^2,$$

see [42, Eqs. (2.7)-(2.11)]. Inequality (1.12) then implies that for any $\gamma \geq 1$,

$$\mathrm{Tr}(P + V)_-^\gamma \leq C_{1,\gamma} \int_{\mathbb{R}^2} V(x)_-^{\gamma+1} dx + C_{2,\gamma}(B) \int_{\mathbb{R}^2} (1 + |x|)^{-2} V(x)_-^\gamma dx. \quad (1.13)$$

Let us compare (1.11) and (1.13) with our bound in Theorem 1.2. Importantly, (1.11) and (1.13) do not have a restriction on the normalized flux α of B . Meanwhile, they are restricted to values $\gamma \geq 1$. When $|\alpha| < 1$, Hölder’s inequality yields that

$$\int_{\mathbb{R}^2} (1 + |x|)^{-2\alpha} V(x)_-^{\gamma+1-|\alpha|} dx \leq \left(\int_{\mathbb{R}^2} (1 + |x|)^{-2} V(x)_-^\gamma dx \right)^{|\alpha|} \left(\int_{\mathbb{R}^2} V(x)_-^{\gamma+1} dx \right)^{1-|\alpha|}.$$

Since $e^{-2h} \leq (m^-)^2 (1 + |x|)^{-2\alpha}$, we see that our bound in Theorem 1.2 implies (1.13) for $|\alpha| < 1$.

In the strong coupling regime, where V is replaced by λV with $\lambda \rightarrow \infty$, the bounds (1.11), (1.13) and our bound all reproduce the optimal $\lambda^{\gamma+1}$ growth.

In the weak coupling regime, where V is replaced by λV with $\lambda \rightarrow 0_+$, $\mathrm{Tr}(P + \lambda V)_-^\gamma$ vanishes like λ^γ when $|\alpha| \geq 1$, so both (1.11) and (1.13) are order-sharp in this case.

However, $\text{Tr}(P + \lambda V)_-^\gamma$ vanishes like $\lambda_\alpha^{\frac{\gamma}{\alpha}}$ when $0 < |\alpha| < 1$, and in this regime (1.11) and (1.13) are no longer order-sharp, while the bound from Theorem 1.2 is.

Concerning the regime of a weak magnetic field, where B is replaced by λB with $\lambda \rightarrow 0$, we see that (1.11) turns into the ordinary Lieb–Thirring inequality (for $\gamma \geq 1$), as does the bound in Theorem 1.2 (for $\gamma > 0$). Meanwhile, as pointed out in [42, p. 614], inequalities (1.12) and (1.13) are not applicable in the regime of a weak magnetic field.

These observations indicate that Theorem 1.2 improves over both (1.11) and (1.13) in the small flux regime $|\alpha| < 1$. It displays not only the optimal behavior in the strong coupling limit, but also in the weak coupling limit. (The optimality in the latter limit occurs for the critical value $\gamma = |\alpha|$; see also Remark 1.3(b) above.) Moreover, Theorem 1.2 allows for a smooth passage to inequality (1.2) in the limit of a vanishing magnetic field. This is made possible by replacing the integrand $(1 + |x|)^{-2} V(x)_-^\gamma$ in (1.13) by $(1 + |x|)^{-2|\alpha|} V(x)_-^{\gamma+1-|\alpha|}$.

It is also interesting to view our results from the point of view of Lieb–Thirring inequalities in the presence of zero energy resonances (aka virtual levels). Lieb–Thirring inequalities when a critical Hardy weight is subtracted from the Laplacian were shown in [10, 11, 19, 24]. In these cases, like in the present one, there is an algebraically decaying zero energy resonance function. The resulting inequality, however, only has a single term in contrast to our bound for P when $\alpha \neq 0$, which has two terms. This is connected with the fact that eigenvalues for the Hardy operator are exponentially small in the limit of a vanishing coupling constant. Hardy–Lieb–Thirring inequalities for fractional Pauli operators in three dimensions were studied in [6].

Lieb–Thirring inequalities in the presence of a resonance function that is bounded from above and away from zero were studied in [26]. The resulting inequality has only a single term. This is relevant in the present case in the simplest case $\alpha = 0$.

In [12, 21] we investigated Lieb–Thirring inequalities in the context of Schrödinger operators on continuous graphs that are sparse in some sense. The Lieb–Thirring inequalities in this case have two terms, reflecting the different behavior in the strong and the weak coupling limit; for the latter see [29]. Compared with [21] the results in the present paper are substantially more precise, as we are able to prove the Lieb–Thirring inequality in the critical case $\gamma = |\alpha| > 0$, while the corresponding question is left open in [21]. Nevertheless some techniques from [21] will play a role in our analysis of subcritical cases; see Sect. 3.

Our result also shows similarities to the logarithmic Lieb–Thirring inequality for the two-dimensional Schrödinger operator [31], where again two terms appear on the right side. Our proof in the critical case uses some ideas from [31] (and [10]); see Sect. 4. An important conceptual difference, however, is that in the relevant bound on the lowest eigenvalue in Proposition 4.5 still two terms appear while there is only one term in the corresponding bound in [31, Lemma 1]. This leads to substantial technical difficulties that need to be overcome; see Appendix B for a proof of the fact that both terms are necessary.

We also mention the recent (non-magnetic) Lieb–Thirring-type inequalities in [4, 5], which, like our bound, involve the solution of a Schrödinger-type equation. The important difference, however, is that the equation in [4, 5] depends on V , whereas in our case the equation for the zero energy resonance function only depends on P and not on V . Therefore the bounds are of a rather different nature.

Finally, we mention the works [7, 17, 25, 30, 44] that quantify different aspects of the instability of the bottom of the spectrum of the Pauli operator. Some of the techniques developed there will be relevant for us here.

1.4. Strategy of the proof. Since the Pauli operator is block diagonal, Theorem 1.2 is an immediate consequence of two theorems concerning the individual blocks. These two operators are defined by the first and the second quadratic form on the right side of (1.5) and are denoted by H^+ and H^- , respectively. They are operators acting in the space $L^2(\mathbb{R}^2)$ of complex-valued functions. When μ is absolutely continuous with sufficiently regular density $B = \frac{d\mu}{dx}$, these operators coincide with those given by (1.1).

The block-diagonality of P and the spin-independence of V imply

$$\mathrm{Tr}_{L^2(\mathbb{R}^2, \mathbb{C}^2)}(P + V)_-^\gamma = \mathrm{Tr}_{L^2(\mathbb{R}^2)}(H^+ + V)_-^\gamma + \mathrm{Tr}_{L^2(\mathbb{R}^2)}(H^- + V)_-^\gamma. \quad (1.14)$$

In what follows we solely discuss the operators H^+ and H^- , rather than P .

It will turn out that for $\alpha \neq 0$ only one of the two operators H^+ and H^- is ‘critical’ while the other one is ‘subcritical’. (One could give a mathematical definition of what we mean by ‘critical’ and ‘subcritical’, but since we do not need anything from the corresponding theory, we will use these terms only in a colloquial sense and refer to [37, 44] for some background.) In order to discuss the distinction between H^+ and H^- , we shall assume that

$$\alpha \geq 0.$$

This is no loss of generality, since replacing μ by $-\mu$ can be compensated by replacing h by $-h$ and then replacing α by $-\alpha$ in (1.6). Of course this is also consistent with the expression (1.3) for α in the regular case. The product m^+m^- that appears in our bounds is invariant under this replacement.

With this convention in place, the operator H^- is ‘critical’, while H^+ is ‘subcritical’ for $\alpha > 0$. The following result says that for the subcritical operator a Lieb–Thirring inequality holds for arbitrarily small $\gamma > 0$.

Theorem 1.4. *Let Assumption 1.1 be satisfied with $\alpha \geq 0$. Then for any $\gamma > 0$ there is a constant $L(\gamma, \mu)$ such that for every real $V \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ one has*

$$\mathrm{Tr}(H^+ + V)_-^\gamma \leq L(\gamma, \mu) \int_{\mathbb{R}^2} V(x)_-^{\gamma+1} dx.$$

The constant $L(\gamma, \mu)$ can be chosen such that

$$L(\gamma, \mu) \leq C(\alpha, \gamma) (m^+m^-)^{2(\gamma+1)},$$

where $C(\alpha, \gamma)$ depends only on α and γ . The same assertion holds for the operator H^- if $\alpha = 0$.

Theorem 1.5. *Let Assumption 1.1 be satisfied with $\alpha > 0$. Then for any $\gamma \geq \alpha$ there are constants $L_1(\gamma, \mu)$ and $L_2(\gamma, \mu)$ such that for every real $V \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ one has*

$$\mathrm{Tr}(H^- + V)_-^\gamma \leq L_1(\gamma, \mu) \int_{\mathbb{R}^2} V(x)_-^{\gamma+1} dx + L_2(\gamma, \mu) \int_{\mathbb{R}^2} e^{-2(h(x)-h_0)} V(x)_-^{\gamma+1-\alpha} dx.$$

The constants $L_1(\gamma, \mu)$ and $L_2(\gamma, \mu)$ can be chosen such that

$$L_1(\gamma, \mu) \leq C(\alpha, \gamma) (m^+m^-)^{2(\gamma+1)},$$

$$L_2(\gamma, \mu) \leq C(\alpha, \gamma) R^{-2\alpha} (m^+m^-)^{2(\gamma-\alpha+2)},$$

where $C(\alpha, \gamma)$ depends only on α and γ .

As we already mentioned, in view of (1.14), Theorem 1.2 is an immediate consequence of Theorems 1.4 and 1.5. Most of the remarks following Theorem 1.2 have analogues for Theorems 1.4 and 1.5, showing in particular their optimality. We omit the details.

We will prove Theorems 1.4 and 1.5 only in the case

$$R = 1.$$

This is no loss of generality, according to the following simple scaling argument. If μ satisfies Assumption 1.1 for some R and α , then we can define a measure $\tilde{\mu}$ on \mathbb{R}^2 that satisfies Assumption 1.1 with the same α , but with $R = 1$. For absolutely continuous μ the corresponding densities B and \tilde{B} are related by $\tilde{B}(y) := R^2 B(Ry)$, and this relation is extended in the natural sense to measures. When passing from μ to $\tilde{\mu}$, the function h is replaced by the function $\tilde{h}(y) := h(Ry)$, which proves our claim about Assumption 1.1. Denoting by \tilde{H}^\pm the operators corresponding to $\tilde{\mu}$, we see that the operators $H^\pm + V$ are unitarily equivalent to the operators $R^{-2}(\tilde{H}^\pm + \tilde{V})$ with $\tilde{V}(y) := R^2 V(Ry)$. As a consequence, Theorems 1.4 and 1.5 for $\tilde{H}^\pm + \tilde{V}$ (with $R = 1$) imply the corresponding theorems for the original operator $H^\pm + V$ (with arbitrary $R > 0$).

2. Passage to Weighted Spaces

In this section we will show that Lieb–Thirring inequalities for H^\pm follow from corresponding Lieb–Thirring inequalities for certain operators \mathcal{H}^\pm that act in a weighted L^2 space and are defined through a weighted Dirichlet integral. For this argument it is crucial that $\alpha < 1$.

2.1. Lower bound on Q^\pm . Let

$$Q^\pm[\psi] := \int_{\mathbb{R}^2} e^{\pm 2h} |(\partial_1 \pm i \partial_2) e^{\mp h} \psi|^2 dx$$

denote the quadratic form of the operator H^\pm .

The simple pointwise bound $|(\partial_{x_1} \pm i \partial_{x_2})\varphi|^2 \leq 2|\nabla\varphi|^2$, together with (1.6) (recall our convention $R = 1$), shows that

$$Q^\pm[e^{\pm h}\varphi] \leq 2(m^\pm)^2 \int_{\mathbb{R}^2} (1 + |x|)^{\pm 2\alpha} |\nabla\varphi|^2 dx. \quad (2.1)$$

This holds irrespectively of the value of α , as long as Assumption 1.1 is valid. The following proposition shows that, under the assumption $|\alpha| < 1$, the reverse bound holds, up to changing the value of the constant. This will be one of the main technical tools in the proof of our results.

Proposition 2.1. *Let Assumption 1.1 be satisfied with $\alpha \geq 0$ and $R = 1$. Then for all $\varphi \in C_c^1(\mathbb{R}^2)$,*

$$Q^\pm[e^{\pm h}\varphi] \geq q_\alpha(m^\mp)^{-2} \int_{\mathbb{R}^2} (1 + |x|)^{\pm 2\alpha} |\nabla\varphi|^2 dx, \quad (2.2)$$

where

$$q_\alpha = \frac{2^{-2\alpha-1}(1-\alpha)^2}{2\alpha + 2^{-2\alpha-1}(1-\alpha)^2}. \quad (2.3)$$

To prove Proposition 2.1 we will need some classical results on doubly weighted one-dimensional Hardy inequalities. For the proof we refer to [36, 43], see also [23] and references therein.

Lemma 2.2. *Let U, W be nonnegative, measurable functions on $(0, \infty)$ and let f be a locally absolutely continuous function on $(0, \infty)$. Then the inequality*

$$\int_0^\infty W(t) |f(t)|^2 dt \leq C(U, W) \int_0^\infty U(t) |f'(t)|^2 dt \quad (2.4)$$

holds

(a) if $\liminf_{t \rightarrow \infty} |f(t)| = 0$ with

$$C(U, W) = 4 \sup_{s > 0} \left(\int_s^\infty U(t)^{-1} dt \right) \left(\int_0^s W(t) dt \right). \quad (2.5)$$

(b) if $\liminf_{t \rightarrow 0} |f(t)| = 0$ with

$$C(U, W) = 4 \sup_{s > 0} \left(\int_0^s U(t)^{-1} dt \right) \left(\int_s^\infty W(t) dt \right). \quad (2.6)$$

We now turn to the proof of the main result of this section. The argument has some similarities with one used in [17], but our focus is different.

Proof of Proposition 2.1. By the bounds (1.6), we have

$$Q^\pm[e^{\pm h} \varphi] \geq (m^\mp)^{-2} \int_{\mathbb{R}^2} (1 + |x|)^{\pm 2\alpha} |(\partial_1 \pm i\partial_2)\varphi|^2 dx.$$

For $\alpha = 0$, the assertion follows immediately from the fact that

$$\int_{\mathbb{R}^2} |(\partial_1 \pm i\partial_2)\varphi|^2 dx = \int_{\mathbb{R}^2} |\nabla \varphi|^2 dx.$$

For $\alpha > 0$ we introduce polar coordinates $x = (r \cos \theta, r \sin \theta)$ and expand $\varphi(r \cdot)$ into a Fourier series,

$$\varphi(x) = \sum_{m \in \mathbb{Z}} e^{im\theta} \varphi_m(r), \quad \varphi_m(r) = \frac{1}{2\pi} \int_0^{2\pi} e^{-im\theta} \varphi(r, \theta) d\theta.$$

A computation (see also [44, Section 10]) shows that

$$\int_0^{2\pi} |(\partial_1 \pm i\partial_2)\varphi|^2 d\theta = \sum_{m \in \mathbb{Z}} \left| \varphi'_m(r) \mp \frac{m \varphi_m(r)}{r} \right|^2 \quad (2.7)$$

and

$$\int_0^{2\pi} |\nabla \varphi|^2 d\theta = \sum_{m \in \mathbb{Z}} \left(|\varphi'_m(r)|^2 + \frac{m^2}{r^2} |\varphi_m(r)|^2 \right). \quad (2.8)$$

Thus, the assertion will follow if we can prove that for each $m \in \mathbb{Z}$,

$$\begin{aligned} & \int_0^\infty (1+r)^{\pm 2\alpha} \left| \varphi'_m(r) \mp \frac{m \varphi_m(r)}{r} \right|^2 r dr \\ & \geq q_\alpha \int_0^\infty (1+r)^{\pm 2\alpha} \left(|\varphi'_m(r)|^2 + \frac{m^2}{r^2} |\varphi_m(r)|^2 \right) r dr. \end{aligned} \quad (2.9)$$

Integrating by parts we find

$$\begin{aligned} & \int_0^\infty (1+r)^{\pm 2\alpha} \left| \varphi'_m(r) \mp \frac{m \varphi_m(r)}{r} \right|^2 r dr \\ & = \int_0^\infty (1+r)^{\pm 2\alpha} \left(|\varphi'_m|^2 + \frac{m^2 |\varphi_m|^2}{r^2} + 2\alpha m \frac{|\varphi_m|^2}{(1+r)r} \right) r dr. \end{aligned} \quad (2.10)$$

When $m \geq 0$ the last term on the right side is nonnegative and we arrive at (2.9), even with constant 1 instead of q_α . (Note that $q_\alpha \leq 1$.)

From now on we assume that $m \leq -1$. The basic idea is to prove a Hardy inequality that allows us to absorb the last term on the right side of (2.10) into the left side. We will apply Lemma 2.2 with

$$f(r) = r^{\mp m} \varphi_m(r), \quad U(r) = r^{\pm 2m+1} (1+r)^{\pm 2\alpha}, \quad W(r) = r^{\pm 2m-1} (1+r)^{\pm 2\alpha}.$$

Note that the left side of (2.10) is equal to $\int_0^\infty U(r) |f'(r)|^2 dr$. In order to bound the constant in Lemma 2.2, we distinguish two cases according to the sign.

Case of the upper sign. In this case we have $\liminf_{t \rightarrow 0} |f(t)| = 0$, so we aim at applying part (b) of Lemma 2.2. We have

$$\int_0^s U(t)^{-1} dt = \int_0^s t^{-2m-1} (1+t)^{-2\alpha} dt \leq \frac{s^{-2m}}{2|m|} \mathbb{1}_{(0,1]}(s) + \frac{s^{-2m-2\alpha}}{2(|m|-\alpha)} \mathbb{1}_{(1,\infty)}(s) \quad (2.11)$$

and

$$\begin{aligned} \int_s^\infty W(t) dt &= \int_s^\infty t^{-1+2m} (1+t)^{2\alpha} dt \leq 2^{2\alpha} \left[\frac{s^{2m}}{2|m|} + \frac{1}{2(|m|-\alpha)} \right] \\ &\quad \mathbb{1}_{(0,1]}(s) + 2^{2\alpha} \frac{s^{2m+2\alpha}}{2(|m|-\alpha)} \mathbb{1}_{(1,\infty)}(s). \end{aligned}$$

Hence

$$\sup_{0 < s \leq 1} \left(\int_0^s U(t)^{-1} dt \right) \left(\int_s^\infty W(t) dt \right) \leq \frac{2^{2\alpha}}{4m^2} + \frac{2^{2\alpha}}{4|m|(|m|-\alpha)} \leq \frac{2^{2\alpha+1}}{4m^2(1-\alpha)},$$

where we have used the elementary bound

$$(k-\alpha)^2 \geq k^2 (1-\alpha)^2 \quad 0 < \alpha < 1, \quad k \in \mathbb{Z}.$$

Similarly,

$$\sup_{1 < s < \infty} \left(\int_0^s U(t)^{-1} dt \right) \left(\int_s^\infty W(t) dt \right) \leq \frac{2^{2\alpha}}{4(|m|-\alpha)^2} \leq \frac{2^{2\alpha}}{4m^2(1-\alpha)^2}.$$

Remark 2.3. The assumption $\alpha < 1$ in Proposition 2.1 is optimal. Indeed, the inequality

$$\int_{\mathbb{R}^2} (1 + |x|)^{-2\alpha} |(\partial_1 - i\partial_2)\varphi|^2 dx \geq c \int_{\mathbb{R}^2} (1 + |x|)^{-2\alpha} |\nabla\varphi|^2 dx \quad \forall \varphi \in C_c^1(\mathbb{R}^2) \quad (2.14)$$

fails to hold, for any $c > 0$, as soon as $\alpha \geq 1$. Indeed, by density it would then also hold for the functions $\varphi^{(R)}$, $R > 0$, given by

$$\varphi^{(R)}(r \cos \theta, r \sin \theta) := \begin{cases} re^{-i\theta} & \text{if } r \leq R, \\ \frac{R^2}{r} e^{-i\theta} & \text{if } R < r. \end{cases}$$

A short calculation using (2.7) and (2.8), however, shows that

$$\lim_{R \rightarrow \infty} \frac{\int_{\mathbb{R}^2} (1 + |x|)^{-2\alpha} |(\partial_1 - i\partial_2)\varphi^{(R)}|^2 dx}{\int_{\mathbb{R}^2} (1 + |x|)^{-2\alpha} |\nabla\varphi^{(R)}|^2 dx} = 0,$$

which obviously contradicts (2.14).

Meanwhile, inequality (2.2) with the upper sign can be extended to all α such that $1 < \alpha \notin \mathbb{Z}$. Since we will not use this bound, we omit its proof.

2.2. Equivalence of quadratic forms. So far we have worked with the operators H^\pm in the space $L^2(\mathbb{R}^2)$. Now we pass to certain operators \mathcal{H}^\pm in the weighted spaces $L^2(\mathbb{R}^2, (1 + |x|)^{\pm 2\alpha} dx)$ and show that Lieb–Thirring inequalities for the new operators imply Lieb–Thirring inequalities for the original operators.

We consider the quadratic form

$$\int_{\mathbb{R}^2} (1 + |x|)^{\pm 2\alpha} |\nabla\varphi|^2 dx$$

in the Hilbert space $L^2(\mathbb{R}^2, (1 + |x|)^{\pm 2\alpha} dx)$. The form domain consists of functions $\varphi \in H_{loc}^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2, (1 + |x|)^{\pm 2\alpha} dx)$ for which the form is finite. It is easy to see that this form is closed in $L^2(\mathbb{R}^2, (1 + |x|)^{\pm 2\alpha} dx)$. We denote the resulting selfadjoint, nonnegative operator in $L^2(\mathbb{R}^2, (1 + |x|)^{\pm 2\alpha} dx)$ by \mathcal{H}^\pm .

From Proposition 2.1 we deduce the following upper bound on the Riesz means that we are interested in.

Corollary 2.4. *Let Assumption 1.1 be satisfied with $\alpha \geq 0$ and $R = 1$. Then for any $\gamma > 0$,*

$$\text{Tr}_{L^2(\mathbb{R}^2, dx)} (H^\pm + V)_-^\gamma \leq q_\alpha^\gamma \text{Tr}_{L^2(\mathbb{R}^2, (1 + |x|)^{\pm 2\alpha} dx)} (\mathcal{H}^\pm - q_\alpha^{-1} (m^\pm m^-)^2 V_-)_-^\gamma$$

with the constants q_α from Proposition 2.1 and m^\pm from (1.6).

Proof. According to Proposition 2.1 we have for all $\varphi \in C_c^1(\mathbb{R}^2)$

$$\begin{aligned} & Q^\pm[e^{\pm h}\varphi] + \int_{\mathbb{R}^2} V e^{\pm 2h} |\varphi|^2 dx \\ & \geq q_\alpha(m^\mp)^{-2} \left(\int_{\mathbb{R}^2} (1 + |x|)^{\pm 2\alpha} |\nabla\varphi|^2 dx - q_\alpha^{-1} (m^\pm m^\mp)^2 \int_{\mathbb{R}^2} (1 + |x|)^{\pm 2\alpha} V_- |\varphi|^2 dx \right) \end{aligned}$$

and

$$\int_{\mathbb{R}^2} e^{\pm 2h} |\varphi|^2 dx \geq (m^\mp)^{-2} \int_{\mathbb{R}^2} (1 + |x|)^{\pm 2\alpha} |\varphi|^2 dx.$$

We know from [16, Theorem 2.5] that the set $e^{\pm h} C_c^1(\mathbb{R}^2)$ is a form core for the operator H^\pm . It is also easy to see that $C_c^1(\mathbb{R}^2)$ is a form core for \mathcal{H}^\pm . Therefore these inequalities imply, by the variational principle

$$N(H^\pm + V + \tau) \leq N(\mathcal{H}^\pm - q_\alpha^{-1} (m^\pm m^\mp)^2 V_- + q_\alpha^{-1} \tau) \quad \text{for all } \tau \geq 0.$$

Here, $N(T)$ denotes the number of negative eigenvalues, counting multiplicities, of a selfadjoint operator T . Using the identity

$$\text{Tr } T_-^\gamma = \gamma \int_0^\infty N(T + \tau) \tau^{\gamma-1} d\tau,$$

we obtain the claimed inequality. \square

Remark 2.5. If instead of Proposition 2.1 one uses inequality (2.1), one can argue similarly to prove the ‘reverse’ inequality

$$\text{Tr}_{L^2(\mathbb{R}^2, (1+|x|)^{\pm 2\alpha} dx)} (\mathcal{H}^\pm + V)_-^\gamma \leq 2^\gamma \text{Tr}_{L^2(\mathbb{R}^2, dx)} (H^\pm - 2^{-1} (m^- m^+)^2 V_-)_-^\gamma$$

In this sense the problem of proving Lieb–Thirring inequalities for $H^\pm + V$ is equivalent, up to constants, to proving such inequalities for $\mathcal{H}^\pm + V$. From now on we will deal with the latter problem.

3. Proof of Theorem 1.4

In this section we prove the first one of our main results, Theorem 1.4. This is substantially simpler than the second one, Theorem 1.5, since either the operators are subcritical (H^+ with $\alpha > 0$), or they are critical, but the endpoint value of γ is excluded (H^\pm with $\alpha = 0$).

3.1. Proof of Theorem 1.4 for $\alpha = 0$. While the approach in the following subsection works for $\alpha = 0$ as well, one can already at this point finish easily the proof in this case by adapting the argument in [26].

Proof of Theorem 1.4 for $\alpha = 0$. By the argument at the end of Sect. 1.4 we may assume $R = 1$. For $\alpha = 0$, the operators \mathcal{H}^\pm coincide with the Laplacian $-\Delta$ in $L^2(\mathbb{R}^2)$. Therefore, Corollary 2.4, together with the usual Lieb–Thirring inequality in \mathbb{R}^2 , see (1.2), implies that for any $\gamma > 0$

$$\begin{aligned} \text{Tr}_{L^2(\mathbb{R}^2, dx)} (H^\pm + V)_-^\gamma &\leq \text{Tr}_{L^2(\mathbb{R}^2, dx)} (-\Delta - (m^+ m^-)^2 V_-)_-^\gamma \\ &\leq L_\gamma (m^+ m^-)^{2(\gamma+1)} \int_{\mathbb{R}^2} V(x)_-^{\gamma+1} dx. \end{aligned}$$

This is the claimed inequality. \square

3.2. Proof of Theorem 1.4 for general α . We will use a method of Lieb [32] of proving Lieb–Thirring inequalities, which is based on a pointwise upper bound on the heat kernel. Such pointwise bounds have been studied in great generality by Grigor'yan, Saloff-Coste and others; see [27, 28, 39, 40] and references therein. The usefulness of Grigor'yan–Saloff-Coste theory in the context of Lieb–Thirring inequalities was observed in [21].

In order to apply the results of Grigor'yan and Saloff-Coste it is convenient to exchange the weight $(1+|x|)^{\pm 2\alpha}$ with the smooth weight $(1+|x|^2)^{\pm\alpha}$. Strictly speaking, this replacement is not necessary, as one can verify that the relevant results of Grigor'yan–Saloff-Coste theory remain valid for our weight that this smooth away from a point and Lipschitz near that point. However, to shorten the presentation we will make this replacement at the expense of a further, controlled deterioration of the constant.

We consider the quadratic form

$$\int_{\mathbb{R}^2} (1+|x|^2)^{\pm\alpha} |\nabla \varphi|^2 dx$$

in the Hilbert space $L^2(\mathbb{R}^2, (1+|x|^2)^{\pm\alpha} dx)$. This form, with form domain consisting of functions $\varphi \in H_{\text{loc}}^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2, (1+|x|^2)^{\pm\alpha} dx)$ for which the form is finite, is nonnegative and closed. We denote the resulting selfadjoint, nonnegative operator in $L^2(\mathbb{R}^2, (1+|x|^2)^{\pm\alpha} dx)$ by \mathcal{K}^\pm .

Using the bounds

$$2^{-\frac{1}{2}} (1+|x|) \leq (1+|x|^2)^{\frac{1}{2}} \leq 1+|x|$$

and proceeding as in the proof of Corollary 2.4, we find that

$$\begin{aligned} \text{Tr}_{L^2(\mathbb{R}^2, (1+|x|)^{\pm 2\alpha} dx)} (\mathcal{H}^\pm + V)_-^\gamma &\leq \text{Tr}_{L^2(\mathbb{R}^2, (1+|x|^2)^{\pm\alpha} dx)} (\mathcal{K}^\pm - 2^\alpha V_-)_-^\gamma, \\ \text{Tr}_{L^2(\mathbb{R}^2, (1+|x|^2)^{\pm\alpha} dx)} (\mathcal{K}^\pm + V)_-^\gamma &\leq \text{Tr}_{L^2(\mathbb{R}^2, (1+|x|)^{\pm 2\alpha} dx)} (\mathcal{H}^\pm - 2^\alpha V_-)_-^\gamma. \end{aligned} \quad (3.1)$$

In view of these inequalities we will now prove Lieb–Thirring inequalities for the operator \mathcal{K}^\pm . Let

$$p^\pm(t; x, y) := e^{-t\mathcal{K}^\pm}(x, y)$$

denote the heat kernel generated by \mathcal{K}^\pm . According to [28, Equation (4.10)], for any $0 \leq \alpha < 1$ there is a constant C such that

$$p^\pm(t; x, x) \leq C t^{-1} (1+|x| + \sqrt{t})^{\mp 2\alpha} \quad \forall t > 0, \quad \forall x \in \mathbb{R}^2. \quad (3.2)$$

Let us comment on the bound (3.2). The result in [28] is much more general. It gives matching upper and lower bounds for $p^\pm(t; x, y)$ for general $x, y \in \mathbb{R}^2$. Also in the case of the upper sign the restriction $\alpha < 1$ is not necessary. When comparing (3.2) with [28, Equation (4.10)], note that our $\pm 2\alpha$ plays the role of their α . We also note that there is a typographical error in [28, Equation (4.10)], which we have corrected in (3.2). (Indeed, inserting the formula for $\mu_\alpha(B(x, r))$ before [28, Equation (4.10)] into [28, Theorem 2.7], we see that α there needs to be replaced by $\alpha/2$.)

Lieb's method [32] yields the upper bound

$$\text{Tr} (\mathcal{K}^\pm + V)_-^\gamma \leq K_{a,\gamma} \int_{\mathbb{R}^2} \int_0^\infty p^\pm(t; x, x) t^{-1-\gamma} (t V(x) + a)_- dt (1+|x|^2)^{\pm\alpha} dx, \quad (3.3)$$

valid for any parameter $a > 0$, with constant

$$K_{a,\gamma} = \Gamma(\gamma + 1) \left(e^{-a} - a \int_a^\infty s^{-1} e^{-s} ds \right)^{-1}. \quad (3.4)$$

The proof of (3.3) given (3.2) is well-known, but for the sake of completeness we provide some details. Similarly as before, for $\tau > 0$ let $N(\mathcal{K}^\pm + V + \tau)$ denote the number, counting multiplicities, of negative eigenvalues of $\mathcal{K}^\pm + V + \tau$. By the Birman–Schwinger principle, we have

$$\begin{aligned} N(\mathcal{K}^\pm + V + \tau) &\leq N(\mathcal{K}^\pm - V_- + \tau) = \text{Tr } \mathbb{1}(V_-^{\frac{1}{2}}(\mathcal{K}^\pm + \tau)^{-1}V_-^{\frac{1}{2}} > 1) \\ &\leq F(1)^{-1} \text{Tr } F(V_-^{\frac{1}{2}}(\mathcal{K}^\pm + \tau)^{-1}V_-^{\frac{1}{2}}) \end{aligned}$$

for any nondecreasing function F on $[0, \infty)$. Given a nonnegative, convex function f on $[0, \infty)$ with $f(0) = 0$, we see that

$$F(\kappa) := \int_0^\infty e^{-y} f(\kappa y) \frac{dy}{y}$$

is a nondecreasing function of κ and by Lieb's trace formula and Jensen's inequality we find as in [24]

$$\text{Tr } F(V_-^{\frac{1}{2}}(\mathcal{K}^\pm + \tau)^{-1}V_-^{\frac{1}{2}}) \leq \int_0^\infty \int_{\mathbb{R}^2} e^{-t\tau} p^\pm(t; x, x) f(tV(x)_-) (1 + |x|^2)^{\pm\alpha} dx \frac{dt}{t}.$$

Inserting this bound into the formula

$$\text{Tr } (\mathcal{K}^\pm + V)_-^\gamma = \gamma \int_0^\infty N(\mathcal{K}^\pm + V + \tau) \tau^{\gamma-1} d\tau,$$

carrying out the τ integration and choosing $f(x) = (x - a)_+$ we arrive at (3.3).

We now turn to the proof of our first main result.

Proof of Theorem 1.4. Inequality (3.2) implies

$$p^+(t; x, x) \leq \frac{C_\alpha}{t} (1 + |x|)^{-2\alpha} \quad \forall t > 0, \quad \forall x \in \mathbb{R}^2. \quad (3.5)$$

Inserting this into (3.3) we obtain, for any $\gamma > 0$,

$$\begin{aligned} \text{Tr}(\mathcal{K}^+ + V)_-^\gamma &\leq C_\alpha K_{a,\gamma} \int_{\mathbb{R}^2} \int_0^\infty t^{-2-\gamma} (tV(x) + a)_- dt dx \\ &= \frac{C_\alpha K_{a,\gamma}}{a^\gamma \gamma(\gamma + 1)} \int_{\mathbb{R}^2} V(x)_-^{\gamma+1} dx. \end{aligned}$$

Combining this with the bounds from Corollary 2.4 and from (3.1) we obtain (for $R = 1$, as we may assume)

$$\begin{aligned} \text{Tr}_{L^2(\mathbb{R}^2, dx)} (H^+ + V)_-^\gamma &\leq q_\alpha^\gamma \text{Tr}_{L^2(\mathbb{R}^2, (1+|x|)^{2\alpha} dx)} \left(\mathcal{H}^+ - q_\alpha^{-1} (m^+ m^-)^2 V_- \right)_-^\gamma \\ &\leq q_\alpha^\gamma \text{Tr}_{L^2(\mathbb{R}^2, (1+|x|)^2 dx)} \left(\mathcal{K}^+ - q_\alpha^{-1} 2^\alpha (m^+ m^-)^2 V_- \right)_-^\gamma \\ &= q_\alpha^{-1} 2^{\alpha(\gamma+1)} \frac{C_\alpha K_{a,\gamma}}{a^\gamma \gamma(\gamma + 1)} (m^+ m^-)^{2(\gamma+1)} \int_{\mathbb{R}^2} V(x)_-^{\gamma+1} dx, \end{aligned}$$

which is the claimed Lieb–Thirring inequality for $H^+ + V$. The proof for $H^- + V$ when $\alpha = 0$ is similar. \square

Remark 3.1. It is interesting to note that the inequality in Theorem 1.5 can be obtained by the same method for $\gamma > \alpha$. Indeed, inequality (3.2) implies

$$p^-(t; x, x) \leq C_\alpha (t^{-1} (1 + |x|)^{2\alpha} + t^{\alpha-1}) \quad \forall t > 0, \quad \forall x \in \mathbb{R}^2. \quad (3.6)$$

Inserting this into (3.3) we obtain, for any $\gamma > \alpha$,

$$\begin{aligned} \text{Tr}(\mathcal{H}^- + V)_-^\gamma &\leq \frac{C_\alpha K_{a,\gamma}}{a^\gamma \gamma(\gamma+1)} \int_{\mathbb{R}^2} V(x)_-^{\gamma+1} dx \\ &\quad + \frac{C_\alpha K_{a,\gamma}}{a^{\gamma-\alpha} (\gamma-\alpha)(\gamma-\alpha+1)} \int_{\mathbb{R}^2} (1 + |x|)^{-2\alpha} V(x)_-^{1+\gamma-\alpha} dx. \end{aligned}$$

Combining this with the bound from Corollary 2.4, we obtain Theorem 1.5 for $\gamma > \alpha$. Note that in the second term on the right side, we estimate

$$(1 + |x|)^{-2\alpha} \leq (m^+ m^-)^2 e^{-2(h(x) - h_0)}.$$

(Indeed, $e^{-h(x)} \leq m^- (1 + |x|)^{-\alpha}$, so $e^{\|h\|_{L^\infty(B(0,\varepsilon))}} \leq m^+ (1 + \varepsilon)^\alpha$, which according to our convention means that $e^{h_0} \leq m^+$.)

Remark 3.2. We claim that for any fixed $\gamma > 0$ the limsup of the constants in the Lieb–Thirring inequalities in Theorems 1.4 and 1.5 remains finite as $\alpha \rightarrow 0$. This follows from the proofs that we have just given, together with the fact that the constants C_α in (3.5) and (3.6) remain bounded as $\alpha \rightarrow 0$. The latter claim follows from the explicit nature of the bounds in the Grigor'yan–Saloff–Coste theory. The basic ingredients, namely the volume doubling property and the Poincaré inequality (see [28, Theorem 2.7]), hold with constants that remain bounded as $\alpha \rightarrow 0$.

4. Proof of Theorem 1.5

In this section we prove the second of our main results, Theorem 1.5. We will assume throughout that $0 < \alpha < 1$ and will prove this theorem only in the critical case $\gamma = \alpha$. This implies the result in the full regime $\gamma \geq \alpha$, either by the Aizenman–Lieb argument [3] or by Remark 3.1. Moreover, according to Corollary 2.4 it suffices to prove the corresponding inequality for \mathcal{H}^- rather than H^- .

4.1. Reduction to radial functions. For a function f on \mathbb{R}^2 let

$$\mathcal{P}\{f\} := (2\pi)^{-1} \int_0^{2\pi} f(|x| \cos \theta, |x| \sin \theta) d\theta$$

and $\mathcal{P}^\perp := \mathbb{1} - \mathcal{P}$. For any radial weight w on \mathbb{R}^2 , \mathcal{P} is the orthogonal projection onto radial functions in $L^2(\mathbb{R}^2, w(x)dx)$. The operator \mathcal{P} commutes with \mathcal{H}^- . Moreover, by the Schwarz inequality we have

$$V_- \leq 2\mathcal{P}V_- \mathcal{P} + 2\mathcal{P}^\perp V_- \mathcal{P}^\perp.$$

From this, we conclude that

$$\text{Tr}(\mathcal{H}^- + V)_-^\gamma \leq \text{Tr}(\mathcal{P}(\mathcal{H}^- - 2V_-)\mathcal{P})_-^\gamma + \text{Tr}(\mathcal{P}^\perp(\mathcal{H}^- - 2V_-)\mathcal{P}^\perp)_-^\gamma. \quad (4.1)$$

We will treat the two terms on the right side separately. In this subsection we will treat the second term. We note that the first term, which will be treated in the remaining subsections, corresponds essentially to an operator in one dimension.

Proposition 4.1. For any $\gamma > 0$,

$$\mathrm{Tr}(\mathcal{P}^\perp(\mathcal{H}^- + V)\mathcal{P}^\perp)_-^\gamma \leq \frac{9}{8} L_\gamma \int_{\mathbb{R}^2} V(x)_-^{\gamma+1} dx,$$

where L_γ is the constant in (1.2).

Proof. We shall show that

$$\mathrm{Tr}_{L^2(\mathbb{R}^2, (1+|x|)^{-2\alpha} dx)}(\mathcal{P}^\perp(\mathcal{H}^- - V)\mathcal{P}^\perp)_-^\gamma \leq \mathrm{Tr}_{L^2(\mathbb{R}^2, dx)} \left(\mathcal{P}^\perp \left(-\frac{8}{9} \Delta - V \right) \mathcal{P}^\perp \right)_-^\gamma,$$

where we make explicit the fact that the traces on the two sides are in different Hilbert spaces. Once we have shown this inequality, we can appeal to the standard Lieb–Thirring inequality (1.2) to deduce the bound in the proposition.

We consider the unitary operator $U : L^2(\mathbb{R}^2, dx) \rightarrow L^2(\mathbb{R}^2, (1+|x|)^{-2\alpha} dx)$, $\psi \mapsto (1+|x|)^\alpha \psi$. Since U commutes with \mathcal{P} and V , it suffices to show that

$$U^* \mathcal{P}^\perp \mathcal{H}^- \mathcal{P}^\perp U \geq \frac{8}{9} \mathcal{P}^\perp(-\Delta) \mathcal{P}^\perp.$$

That is, we need to show that, if $\mathcal{P}\psi = 0$, then

$$\int_{\mathbb{R}^2} (1+|x|)^{-2\alpha} |\nabla((1+|x|)^\alpha \psi)|^2 dx \geq \frac{8}{9} \int_{\mathbb{R}^2} |\nabla \psi|^2 dx.$$

We compute

$$\begin{aligned} (1+|x|)^{-2\alpha} |\nabla((1+|x|)^\alpha \psi)|^2 &= (1+|x|)^{-2\alpha} |(1+|x|)^\alpha \nabla \psi + \alpha(1+|x|)^{\alpha-1} \frac{x}{|x|} \psi|^2 \\ &= |\nabla \psi|^2 + \alpha^2 (1+|x|)^{-2} |\psi|^2 + \alpha(1+|x|)^{-1} \frac{x}{|x|} \cdot \nabla(|\psi|^2). \end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned} \int_{\mathbb{R}^2} (1+|x|)^{-2\alpha} |\nabla((1+|x|)^\alpha \psi)|^2 dx &= \int_{\mathbb{R}^2} (|\nabla \psi|^2 - (\alpha \nabla \cdot ((1+|x|)^{-1} \frac{x}{|x|}) \\ &\quad - \alpha^2 (1+|x|)^{-2} |\psi|^2) dx \\ &= \int_{\mathbb{R}^2} (|\nabla \psi|^2 - \alpha(1+|x|)^{-2} (|x|^{-1} - \alpha) |\psi|^2) dx. \end{aligned}$$

We introduce polar coordinates $x = (r \cos \theta, r \sin \theta)$. Since $\mathcal{P}^\perp(-\partial_\theta^2) \mathcal{P}^\perp \geq \mathcal{P}^\perp$, we have, if $\mathcal{P}\psi = 0$,

$$\begin{aligned} \int_{\mathbb{R}^2} (1+|x|)^{-2\alpha} |\nabla((1+|x|)^\alpha \psi)|^2 dx &= \int_{\mathbb{R}^2} \left((1+|x|)^{-2\alpha} |\partial_r(1+|x|)^\alpha \psi|^2 + |x|^{-2} |\partial_\theta \psi|^2 \right) dx \\ &\geq \int_{\mathbb{R}^2} |x|^{-2} |\psi|^2 dx. \end{aligned}$$

Combining the previous two equations we find, for any $\vartheta \in [0, 1]$,

$$\begin{aligned} \int_{\mathbb{R}^2} (1+|x|)^{-2\alpha} |\nabla((1+|x|)^\alpha \psi)|^2 dx &\geq \vartheta \int_{\mathbb{R}^2} |\nabla \psi|^2 dx \\ &\quad + \int_{\mathbb{R}^2} ((1-\vartheta)|x|^{-2} \\ &\quad - \vartheta \alpha(1+|x|)^{-2} (|x|^{-1} - \alpha)) |\psi|^2 dx. \end{aligned}$$

We can choose $\vartheta \in [0, 1]$ (depending on α) such that

$$(1 - \vartheta)|x|^{-2} - \vartheta\alpha(1 + |x|)^{-2}(|x|^{-1} - \alpha) \geq 0 \quad \text{for all } x \in \mathbb{R}^2.$$

More precisely, we choose

$$\vartheta := \left(\sup_{r>0} \left(1 + \alpha(1+r)^{-2}r(1-\alpha r) \right) \right)^{-1}.$$

The supremum is attained at $r = 1/(2\alpha + 1)$, which leads to

$$\vartheta = \frac{4(\alpha + 1)}{5\alpha + 4} \geq \frac{8}{9}.$$

This proves the claimed inequality. \square

We note that the inequality $\alpha < 1$ that we assume throughout this section was only used at the very end of the previous proof when we bounded ϑ from below. Thus, an analogue of Proposition 4.1 is valid even for $\alpha \geq 1$, but with a constant that depends on α .

4.2. Reduction to the lowest eigenvalue. In the previous subsection we have treated the second term on the right side of (4.1). In this subsection we treat the first term, that is, we deal with the operator $\mathcal{P}(\mathcal{H}^- + V)\mathcal{P}$.

We let \mathfrak{h}^- denote the operator in $L^2(\mathbb{R}_+, (1+r)^{-2\alpha}r dr)$ generated by the quadratic form

$$\int_0^\infty (1+r)^{-2\alpha} |\varphi'(r)|^2 r dr,$$

defined on locally absolutely continuous functions φ on \mathbb{R}_+ belonging to $L^2(\mathbb{R}_+, (1+r)^{-2\alpha}r dr)$ for which the integral is finite. If for a given function V on \mathbb{R}^2 we let

$$v(r) := \frac{1}{2\pi} \int_0^{2\pi} V(r \cos \theta, r \sin \theta) d\theta, \quad (4.2)$$

then the nontrivial part of the operator $\mathcal{P}(\mathcal{H}^- + V)\mathcal{P}$ is equal to $\mathfrak{h}^- + v$ and, in particular,

$$\text{Tr}_{L^2(\mathbb{R}^2, (1+|x|)^{-2\alpha} dx)} (\mathcal{P}(\mathcal{H}^- + V)\mathcal{P})_-^\gamma = \text{Tr}_{L^2(\mathbb{R}_+, (1+r)^{-2\alpha}r dr)} (\mathfrak{h}^- + v)_-^\gamma. \quad (4.3)$$

In the remainder of this section we will treat v as a given function on \mathbb{R}_+ , ignoring that there is an underlying function V on \mathbb{R}^2 .

Our strategy to bound the right side of (4.3) will be to impose a Dirichlet boundary condition at $r = 1$. This will result in two operators \mathfrak{h}_0^- and \mathfrak{h}_∞^- in $L^2((0, 1), (1+r)^{-2\alpha}r dr)$ and $L^2((1, \infty), (1+r)^{-2\alpha}r dr)$, respectively. These operators act in the same way as \mathfrak{h}^- , but functions in their form domain vanish at the point $r = 1$. Since imposing a Dirichlet boundary condition is a rank one perturbation of the resolvent, it follows that

$$\begin{aligned} \text{Tr}_{L^2(\mathbb{R}_+, (1+r)^{-2\alpha}r dr)} (\mathfrak{h}^- + v)_-^\gamma &\leq \text{Tr}_{L^2((0, 1), (1+r)^{-2\alpha}r dr)} (\mathfrak{h}_0^- + v)_-^\gamma \\ &\quad + \text{Tr}_{L^2((1, \infty), (1+r)^{-2\alpha}r dr)} (\mathfrak{h}_\infty^- + v)_-^\gamma \\ &\quad + (\inf \text{spec}(\mathfrak{h}^- + v))_-^\gamma. \end{aligned} \quad (4.4)$$

In the following two propositions we will treat the first two terms on the right side, respectively. The third term will be treated in the next subsection.

Proposition 4.2. For any $\gamma > 0$,

$$\mathrm{Tr}_{L^2((0,1),(1+r)^{-2\alpha}r\,dr)}(\mathfrak{h}_0^- + v)_-^\gamma \leq 2^{2\alpha(\gamma+1)} L_\gamma 2\pi \int_0^1 v(r)_-^{\gamma+1} r \, dr,$$

where L_γ is the constant in (1.2).

Proof. For functions φ in the form domain of \mathfrak{h}_0^- we bound

$$\begin{aligned} & \int_0^1 (1+r)^{-2\alpha} |\varphi'(r)|^2 r \, dr + \int_0^1 v(r) |\varphi(r)|^2 (1+r)^{-2\alpha} r \, dr \\ & \geq 2^{-2\alpha} \int_0^1 |\varphi'(r)|^2 r \, dr - \int_0^1 v(r)_- |\varphi(r)|^2 r \, dr \end{aligned}$$

and

$$\int_0^1 (1+r)^{-2\alpha} |\varphi(r)|^2 r \, dr \geq 2^{-2\alpha} \int_0^1 |\varphi(r)|^2 r \, dr.$$

By a similar argument as in the proof of Corollary 2.4, this implies

$$\mathrm{Tr}_{L^2((0,1),(1+r)^{-2\alpha}r\,dr)}(\mathfrak{h}_0^- + v)_-^\gamma \leq \mathrm{Tr}_{L^2((0,1),r\,dr)}\left(-r^{-1}\partial_r r \partial_r - 2^{2\alpha} v_-\right)_-^\gamma,$$

where the operator $-r^{-1}\partial_r r \partial_r - 2^{2\alpha} v_-$ is considered with a Dirichlet boundary condition at $r = 1$. This operator coincides with the nontrivial part of $\mathcal{P}(-\Delta - 2^{2\alpha}v(|\cdot|_-)\mathcal{P}$ acting in $L^2(B(0, 1), dx)$ with a Dirichlet boundary condition. Extending the operator to all of \mathbb{R}^2 and removing the projection \mathcal{P} does not decrease the Riesz means and therefore we have, by the standard Lieb–Thirring inequality (1.2),

$$\begin{aligned} \mathrm{Tr}_{L^2((0,1),r\,dr)}\left(-r^{-1}\partial_r r \partial_r - 2^{2\alpha} v_-\right)_-^\gamma & \leq \mathrm{Tr}_{L^2(\mathbb{R}^2,dx)}\left(-\Delta - 2^{2\alpha} \mathbb{1}_{B(0,1)} v(|\cdot|_-)\right)_-^\gamma \\ & \leq 2^{2\alpha(\gamma+1)} L_\gamma \int_{B(0,1)} v(|x|)_-^{\gamma+1} dx \\ & = 2^{2\alpha(\gamma+1)} L_\gamma 2\pi \int_0^1 v(r)_-^{\gamma+1} r \, dr. \end{aligned}$$

Combining this with the previous inequality yields the assertion. \square

Proposition 4.3. For any $\gamma > 0$,

$$\mathrm{Tr}_{L^2((1,\infty),(1+r)^{-2\alpha}r\,dr)}(\mathfrak{h}_\infty^- + v)_-^\gamma \leq C_{\alpha,\gamma} \int_1^\infty v(r)_-^{\gamma+1} r \, dr.$$

Proof. Arguing similarly as at the beginning of the previous proof we find that

$$\mathrm{Tr}_{L^2((1,\infty),(1+r)^{-2\alpha}r\,dr)}(\mathfrak{h}_0^- + v)_-^\gamma \leq \mathrm{Tr}_{L^2((1,\infty),r^{-2\alpha+1}dr)}\left(-r^{2\alpha-1}\partial_r r^{-2\alpha+1} \partial_r - 2^{2\alpha} v_-\right)_-^\gamma,$$

where the operator $-r^{2\alpha-1}\partial_r r^{-2\alpha+1} \partial_r - 2^{2\alpha} v_-$ is considered with a Dirichlet boundary condition at $r = 1$. Extending the operator to all of \mathbb{R}_+ we will consider

$$\mathrm{Tr}_{L^2(\mathbb{R}_+,r^{-2\alpha+1}dr)}\left(-r^{2\alpha-1}\partial_r r^{-2\alpha+1} \partial_r + \tilde{v}\right)_-^\gamma,$$

where $\tilde{v} = -2^{2\alpha} v_-$ on $(1, \infty)$ and $\tilde{v} = 0$ on $(0, 1)$. The operator $-r^{2\alpha-1} \partial_r r^{-2\alpha+1} \partial_r$ acts with a Dirichlet boundary condition at the origin. More precisely, it is defined as the closure of the quadratic form $\int_0^\infty r^{-2\alpha+1} |\varphi'(r)|^2 dr$ defined for $\varphi \in C_c^1(\mathbb{R}_+)$. (We emphasize that $\mathbb{R}_+ = (0, \infty)$, so functions in $C_c^1(\mathbb{R}_+)$ vanish in a neighborhood of the origin.)

We consider the unitary operator $U : L^2(\mathbb{R}_+, dr) \rightarrow L^2(\mathbb{R}_+, r^{-2\alpha+1} dr)$, $\eta \mapsto r^{\alpha-\frac{1}{2}} \eta$. Let us set $\varphi(r) = \mathcal{U}\eta(r) = r^{\alpha-\frac{1}{2}} \eta(r)$ and compute

$$\begin{aligned} |\varphi'(r)|^2 &= \left| r^{\alpha-\frac{1}{2}} \eta'(r) + (\alpha - \frac{1}{2}) r^{\alpha-\frac{3}{2}} \eta(r) \right|^2 \\ &= r^{2\alpha-1} |\eta'(r)|^2 + 2(\alpha - \frac{1}{2}) r^{2\alpha-2} \operatorname{Re} \overline{\eta'(r)} \eta(r) + (\alpha - \frac{1}{2})^2 r^{2\alpha-3} |\eta(r)|^2 \\ &= r^{2\alpha-1} |\eta'(r)|^2 + (\alpha - \frac{1}{2}) r^{2\alpha-2} \left(|\eta(r)|^2 \right)' + (\alpha - \frac{1}{2})^2 r^{2\alpha-3} |\eta(r)|^2, \end{aligned}$$

leading to

$$\int_0^\infty |\varphi'(r)|^2 r^{-2\alpha+1} dr = \int_0^\infty \left(|\eta'(r)|^2 + (\alpha^2 - \frac{1}{4}) r^{-2} |\eta(r)|^2 \right) dr.$$

Thus, we have shown that the operator $-r^{2\alpha-1} \partial_r r^{-2\alpha+1} \partial_r$ in $L^2(\mathbb{R}_+, r^{-2\alpha+1} dr)$ is unitarily equivalent to the operator $-\partial_r^2 + (\alpha^2 - \frac{1}{4}) r^{-2}$ in $L^2(\mathbb{R}_+, dr)$. It follows that

$$\operatorname{Tr}_{L^2(\mathbb{R}_+, r^{-2\alpha+1} dr)} \left(-r^{2\alpha-1} \partial_r r^{-2\alpha+1} \partial_r + \tilde{v} \right)_-^\gamma = \operatorname{Tr}_{L^2(\mathbb{R}_+, dr)} \left(-\partial_r^2 + (\alpha^2 - \frac{1}{4}) r^{-2} + \tilde{v} \right)_-^\gamma,$$

For a lower bound we drop the term $\alpha^2 r^{-2}$ and recognize the operator $-\partial_r^2 - \frac{1}{4} r^{-2}$ as being unitarily equivalent to the radial part of the Laplace operator in \mathbb{R}^2 . It follows that

$$\begin{aligned} \operatorname{Tr}_{L^2(\mathbb{R}_+, dr)} \left(-\partial_r^2 + (\alpha^2 - \frac{1}{4}) r^{-2} + \tilde{v} \right)_-^\gamma &\leq \operatorname{Tr}_{L^2(\mathbb{R}_+, dr)} \left(-\partial_r^2 - \frac{1}{4} r^{-2} + \tilde{v} \right)_-^\gamma \\ &= \operatorname{Tr}_{L^2(\mathbb{R}^2, dx)} (\mathcal{P}(-\Delta + \tilde{v}(|\cdot|)) \mathcal{P})_-^\gamma \\ &\leq \operatorname{Tr}_{L^2(\mathbb{R}^2, dx)} (-\Delta + \tilde{v}(|\cdot|))_-^\gamma \\ &\leq L_\gamma \int_{\mathbb{R}^2} \tilde{v}(|x|)_-^{\gamma+1} dx \\ &= L_\gamma 2\pi \int_0^\infty \tilde{v}(r)_-^{\gamma+1} r dr. \end{aligned}$$

Here we used the standard Lieb–Thirring inequality (1.2) on \mathbb{R}^2 . Combining the previous inequalities yields the assertion. \square

Remark 4.4. There is an alternative way of finishing the proof without appealing to the Lieb–Thirring inequality (1.2). Namely, when $\alpha \geq \frac{1}{2}$ one can drop the term $(\alpha^2 - \frac{1}{4}) r^{-2}$ for a lower bound and for $0 < \alpha < \frac{1}{2}$ one can drop this term at the expense of reducing the constant 1 in front of $-\partial_r^2$ by Hardy’s inequality. One arrives at having to bound $\operatorname{Tr}_{L^2(\mathbb{R}_+, dr)} (-\theta_\alpha \partial_r^2 + \tilde{v})_-^\gamma$ with $\theta_\alpha := \min\{1, 4\alpha^2\}$. This is possible in view of bounds by Egorov and Kondratiev [9, Sec. 8.8].

A drawback of the proof that we just sketched is that the constant diverges as $\alpha \rightarrow 0$ because of the presence of θ_α . This can be remedied by using more refined one-dimensional inequalities that take the Hardy term into account [11].

In this connection it is interesting to note that for $0 < \alpha \leq \frac{1}{2}$ and $\gamma \geq \alpha$, the above proof also gives the bound

$$\mathrm{Tr}_{L^2((1,\infty),(1+r)^{-2\alpha}r\,dr)} (\mathfrak{h}_\infty^- + v)_-^\gamma \leq \tilde{C}_{\alpha,\gamma} \int_1^\infty v(r)_-^{1+\gamma-\alpha} r^{-2\alpha+1} \, dr.$$

This follows from the fact, proved in [11], that the inequality

$$\mathrm{Tr}_{L^2(\mathbb{R}_+, dr)} (-\partial_r^2 - \frac{1}{4}r^{-2} + w)_-^\gamma \leq L_{\gamma,a} \int_0^\infty w(r)_-^{\gamma+\frac{1+a}{2}} r^a \, dr$$

is valid for $\gamma = \frac{1-a}{2}$ when $0 \leq a < 1$. We apply this inequality with $a = 1 - 2\alpha$.

4.3. Bound on the lowest eigenvalue. In the previous subsection we have bounded the first and second term on the right side of (4.4). In this subsection we discuss the third term, that is, we discuss a lower bound on the lowest eigenvalue for $\mathfrak{h}^- + v$. We shall prove the following bound

Proposition 4.5. *For any $0 < \alpha < 1$ there is a constant C_α such that*

$$(\inf \mathrm{spec} (\mathfrak{h}^- + v))_-^\alpha \leq C_\alpha \left(\int_0^\infty v(r)_-^{1+\alpha} r \, dr + \int_0^\infty v(r)_- (1+r)^{-2\alpha} r \, dr \right).$$

It is natural to wonder whether in the bound in the proposition a single term on the right side suffices. This is not the case, as will be discussed in Appendix B.

Proposition 4.5 is in some sense the main step in the proof of our main result. It is certainly the most technical step and, indeed, in this subsection we only show how to reduce the proof to a technical lemma that will be verified in the following section. This lemma is stated in terms of the operator T_α in $L^2(\mathbb{R}_+, dr)$ that is defined through the closure of the quadratic form

$$\int_0^\infty |\eta'(r)|^2 \, dr - \alpha |\eta(1)|^2 - \frac{1}{4} \int_0^1 \frac{|\eta(r)|^2}{r^2} \, dr + \left(\alpha^2 - \frac{1}{4} \right) \int_1^\infty \frac{|\eta(r)|^2}{r^2} \, dr \quad (4.5)$$

defined for $\eta \in C_c^1(\mathbb{R}_+)$. We denote by $(T_\alpha + \kappa^2)^{-1}(r, r')$, $r, r' \in \mathbb{R}_+$, the integral kernel of the operator $(T_\alpha + \kappa^2)^{-1}$. In the following lemma we bound the difference between these kernels at α and at 0.

Lemma 4.6. *For any $\alpha \in (0, 1)$ there is a constant C_α such that for all $\kappa > 0$ and all $r, r' \in \mathbb{R}_+$ one has*

$$\left| (T_\alpha + \kappa^2)^{-1}(r, r') - (T_0 + \kappa^2)^{-1}(r, r') \right| \leq C_\alpha \kappa^{-2\alpha} \sqrt{rr'} (1+r)^{-\alpha} (1+r')^{-\alpha}.$$

Accepting this lemma for the moment, let us prove the main result of this subsection.

Proof of Proposition 4.5. The proof will consist of two steps. In the first step we will prove the bound

$$(\inf \operatorname{spec} (T_\alpha + v))_-^\alpha \leq C'_\alpha \left(\int_0^\infty v(r)_-^{1+\alpha} r dr + \int_0^\infty v(r)_- (1+r)^{-2\alpha} r dr \right). \quad (4.6)$$

and in a second step we will show that this inequality implies that in Proposition 4.5.

Step 1. Let us denote

$$G_0(\kappa) := (T_0 + \kappa^2)^{-1} \quad \text{and} \quad \Gamma_\alpha(\kappa) := (T_\alpha + \kappa^2)^{-1} - (T_0 + \kappa^2)^{-1}. \quad (4.7)$$

By the variational principle, for the proof of (4.6) we may assume that $v \leq 0$. We denote

$$\kappa_* := (\inf \operatorname{spec} (T_\alpha + v))_-^{\frac{1}{2}}.$$

We may assume that $\kappa_* > 0$, for otherwise (4.6) is trivially true.

From the Birman–Schwinger principle we deduce that

$$1 = \left\| v_-^{\frac{1}{2}} (T_\alpha + \kappa_*^2)^{-1} v_-^{\frac{1}{2}} \right\| \leq \left\| v_-^{\frac{1}{2}} G_0(\kappa_*) v_-^{\frac{1}{2}} \right\| + \left\| v_-^{\frac{1}{2}} \Gamma_\alpha(\kappa_*) v_-^{\frac{1}{2}} \right\|, \quad (4.8)$$

where $\|\cdot\|$ denotes the operator norm in $L^2(\mathbb{R}_+)$. We distinguish two cases depending on the size of the first term on the right side of (4.8).

Assume first that $\left\| v_-^{\frac{1}{2}} G_0(\kappa_*) v_-^{\frac{1}{2}} \right\| \leq \frac{1}{2}$. Then, by (4.8),

$$\left\| v_-^{\frac{1}{2}} \Gamma_\alpha(\kappa_*) v_-^{\frac{1}{2}} \right\| \geq \frac{1}{2}. \quad (4.9)$$

Meanwhile, it follows from Lemma 4.6 that

$$\left\| v_-^{\frac{1}{2}} \Gamma_\alpha(\kappa) v_-^{\frac{1}{2}} \right\|_{\text{HS}} \leq C_\alpha \kappa^{-2\alpha} \int_0^\infty v(r)_- (1+r)^{-2\alpha} r dr,$$

for all $\kappa > 0$, where $\|\cdot\|_{\text{HS}}$ denotes the Hilbert–Schmidt norm in $L^2(\mathbb{R}_+)$. Estimating the Hilbert–Schmidt norm from below by the operator norm and setting $\kappa = \kappa_*$, we obtain

$$\left\| v_-^{\frac{1}{2}} \Gamma_\alpha(\kappa_*) v_-^{\frac{1}{2}} \right\| \leq C_\alpha \kappa_*^{-2\alpha} \int_0^\infty v(r)_- (1+r)^{-2\alpha} r dr.$$

Combining this with (4.9), we obtain

$$\kappa_*^{2\alpha} \leq 2C_\alpha \int_0^\infty v(r)_- (1+r)^{-2\alpha} r dr,$$

which implies (4.6).

If, on the contrary, $\left\| v_-^{\frac{1}{2}} G_0(\kappa_*) v_-^{\frac{1}{2}} \right\| > \frac{1}{2}$, then the Birman–Schwinger principle implies that

$$\inf \operatorname{spec} (T_0 - 2v_-) < -\kappa_*^2.$$

Since T_0 is unitarily equivalent to the radial part of the Laplace operator in \mathbb{R}^2 , cf. (4.5), so we infer that

$$\inf \text{spec}(-\Delta - 2v(|\cdot|)_-) < -\kappa_*^2.$$

Combining this with the usual Lieb–Thirring inequality (1.2), we obtain

$$\kappa_*^{2\alpha} \leq \text{Tr}(-\Delta - 2v(|\cdot|)_-)^{\alpha} \leq 2^{\alpha+1} L_{\alpha} 2\pi \int_0^{\infty} v(r)_-^{\alpha+1} r dr,$$

which implies (4.6). (Note that instead of the Lieb–Thirring inequality (1.2) the so-called one-particle Lieb–Thirring inequality, that is, a Sobolev interpolation inequality [22, Subsection 5.1.2], would suffice.) This completes the proof of (4.6).

Step 2. We now deduce the bound in the proposition from the bound (4.6). This is achieved by bringing the weight $(1+r)^{-2\alpha} r$ appearing for the operator \mathfrak{h}^- into a more canonical form and then applying a unitary transformation to remove this more canonical weight. We define

$$w(r) := \begin{cases} r & \text{if } 0 < r \leq 1, \\ r^{1-2\alpha} & \text{if } 1 < r < \infty. \end{cases} \quad (4.10)$$

We denote by h_{α} the operator in $L^2(\mathbb{R}_+, w(r)dr)$ associated with the quadratic form

$$\int_0^{\infty} |\varphi'(r)|^2 w(r) dr, \quad (4.11)$$

defined on functions $\varphi \in L^2(\mathbb{R}_+, w(r)dr)$ that are locally absolutely continuous on \mathbb{R}_+ and for which the quadratic form is finite. Using the bounds

$$2^{-2\alpha} w(r) \leq (1+r)^{-2\alpha} r \leq w(r),$$

we find, similarly as in the proof of Corollary 2.4,

$$\left(\inf \text{spec}_{L^2(\mathbb{R}_+, (1+r)^{-2\alpha} r dr)} (\mathfrak{h}^- + v) \right)_- \leq \left(\inf \text{spec}_{L^2(\mathbb{R}_+, w(r)dr)} (h_{\alpha} - 2^{2\alpha} v_-) \right)_-.$$

This reduces the proof of the bound in the proposition to the proof of the bound for $h_{\alpha} - 2^{2\alpha} v_-$.

The unitary mapping $\mathcal{U} : L^2(\mathbb{R}_+, dr) \rightarrow L^2(\mathbb{R}_+, w(r)dr)$ given by $\mathcal{U}\eta(r) := w(r)^{-\frac{1}{2}}\eta(r)$ satisfies

$$\begin{aligned} \int_0^{\infty} |(\mathcal{U}\eta)'|^2 w(r) dr &= \int_0^{\infty} |\eta'(r)|^2 dr - \alpha|\eta(1)|^2 - \frac{1}{4} \int_0^1 \frac{|\eta(r)|^2}{r^2} dr \\ &\quad + \left(\alpha^2 - \frac{1}{4} \right) \int_1^{\infty} \frac{|\eta(r)|^2}{r^2} dr. \end{aligned}$$

Clearly, the form core $C_c^1(\mathbb{R}_+)$ of T_{α} is mapped into the form domain of h_{α} . Conversely, arguing as in [22, Lemma 2.33] one can show that $C_c^1(\mathbb{R}_+)$ is a form core of h_{α} , and the image of it under \mathcal{U}^{-1} is in the form domain of T_{α} . These facts imply that

$$\mathcal{U}^{-1} h_{\alpha} \mathcal{U} = T_{\alpha}.$$

As a consequence, for any \tilde{v} (in particular, for $\tilde{v} = -2^{2\alpha} v_-$)

$$\inf \text{spec}_{L^2(\mathbb{R}_+, w(r)dr)} (h_{\alpha} + \tilde{v}) = \inf \text{spec}_{L^2(\mathbb{R}_+, dr)} (T_{\alpha} + \tilde{v}).$$

This concludes the proof of the proposition. \square

4.4. Proof of Theorem 1.5. We finally put all the ingredients from this section together and prove our second main result.

Proof of Theorem 1.5. Given a sufficiently regular real function V on \mathbb{R}^2 we define the function v on \mathbb{R}_+ by (4.2). Combining (4.3), (4.4) and Propositions 4.2, 4.3 and 4.5, we see that for each $0 < \alpha < 1$ there is a constant C_α such that

$$\begin{aligned} & \text{Tr}_{L^2(\mathbb{R}^2, (1+|x|)^{-2\alpha} dx)} (\mathcal{P}(\mathcal{H}^- + V)\mathcal{P})_-^\alpha \\ & \leq C_\alpha \left(\int_0^\infty v(r)_-^{1+\alpha} r dr + \int_0^\infty v(r)_- (1+r)^{-2\alpha} r dr \right) \\ & \leq \frac{C_\alpha}{2\pi} \left(\int_{\mathbb{R}^2} V(x)_-^{1+\alpha} dx + \int_{\mathbb{R}^2} V(x)_- (1+|x|)^{-2\alpha} dx \right). \end{aligned}$$

The last inequality comes from Hölder's inequality for the angular integration. Combining this inequality with (4.1) and Proposition 4.1, we see that for each $0 < \alpha < 1$ there is a constant C'_α such that

$$\text{Tr}_{L^2(\mathbb{R}^2, (1+|x|)^{-2\alpha} dx)} (\mathcal{H}^- + V)_-^\alpha \leq C'_\alpha \left(\int_{\mathbb{R}^2} V(x)_-^{1+\alpha} dx + \int_{\mathbb{R}^2} V(x)_- (1+|x|)^{-2\alpha} dx \right).$$

Under Assumption 1.1 with $R = 1$ (and, as everywhere in this section, $\alpha > 0$), we can use the same argument as in Remark 3.1 to replace the weight $(1+|x|)^{-2\alpha}$ in the second term by $(m^+m^-)^2 e^{-2(h(x)-h_0)}$. The claimed inequality in Theorem 1.5 for $\gamma = \alpha$ then follows from Corollary 2.4. Note that this yields, in particular, the claimed dependence of the constants on the product m^+m^- . As we have already mentioned at the beginning of this section, the claimed inequality for $\gamma > \alpha$ follows either from the inequality for $\gamma = \alpha$ by the Aizenman–Lieb argument [3] or by Remark 3.1. Finally, the case $R \neq 1$ can be reduced to the case $R = 1$ by scaling as we already observed. \square

5. Proof of Lemma 4.6

In this section we will give the proof of Lemma 4.6. We use the notation $\Gamma_\alpha(\kappa)$ from (4.7) for the resolvent difference. Thus, we are looking for a pointwise bound on the integral kernel of the operator $\Gamma_\alpha(\kappa)$.

Our first goal is to find an explicit formula for this integral kernel. Let

$$\begin{aligned} A_\alpha(\kappa) &:= \kappa I_0(\kappa) K_{\alpha+1}(\kappa) + \kappa I_1(\kappa) K_\alpha(\kappa) - 2\alpha I_0(\kappa) K_\alpha(\kappa), \\ B_\alpha(\kappa) &:= \kappa I_0(\kappa) I_{\alpha+1}(\kappa) - \kappa I_1(\kappa) I_\alpha(\kappa) + 2\alpha I_0(\kappa) I_\alpha(\kappa), \\ D_\alpha(\kappa) &:= \kappa K_1(\kappa) K_\alpha(\kappa) - \kappa K_0(\kappa) K_{\alpha+1}(\kappa) + 2\alpha K_0(\kappa) K_\alpha(\kappa), \end{aligned} \tag{5.1}$$

and put

$$f_\alpha(\kappa) := \frac{D_\alpha(\kappa)}{A_\alpha(\kappa)} \quad \text{and} \quad g_\alpha(\kappa) := \frac{B_\alpha(\kappa)}{A_\alpha(\kappa)}.$$

It will turn out that $A_\alpha(\kappa) \neq 0$, so $f_\alpha(\kappa)$ and $g_\alpha(\kappa)$ are well defined.

Lemma 5.1. *One has $A_\alpha(\kappa) \neq 0$ for all $\kappa > 0$. Moreover, for all $\kappa > 0$, $\alpha \in (0, 1]$ and $0 < r \leq r' < \infty$,*

$$\begin{aligned} \Gamma_\alpha(r, r'; \kappa) &= \sqrt{rr'} \times \begin{cases} f_\alpha(\kappa) I_0(\kappa r) I_0(\kappa r') & \text{if } 0 < r \leq r' \leq 1, \\ K_\alpha(\kappa r') I_\alpha(\kappa r) - I_0(\kappa r) K_0(\kappa r') + g_\alpha(\kappa) K_\alpha(\kappa r) K_\alpha(\kappa r') & \text{if } 1 < r \leq r', \\ I_0(\kappa r) (A_\alpha^{-1}(\kappa) K_\alpha(\kappa r') - K_0(\kappa r')) & \text{if } 0 < r \leq 1 \leq r'. \end{cases} \end{aligned} \quad (5.2)$$

The same formula is valid when $r > r'$, provided the variables r and r' are interchanged.

Proof. We begin by deriving a formula for the integral kernel of $(T_\alpha + \kappa^2)^{-1}$ for $\alpha \in [0, 1]$. By Sturm–Liouville theory it can be written in terms of two solutions v_1 and v_2 of the system

$$\begin{aligned} -v'' - \frac{1}{4} r^{-2} v &= -\kappa^2 v && \text{in } (0, 1), \\ -v'' + (\alpha^2 - \frac{1}{4}) r^{-2} v &= -\kappa^2 v && \text{in } (1, \infty), \\ v(1_-) &= v(1_+) \\ v'(1_-) &= v'(1_+) + \alpha v(1_+). \end{aligned}$$

(The jump condition at $r = 1$ comes from the term $\alpha|\eta(1)|^2$ in the quadratic form (4.5) of the operator T_α .) The solution v_1 is supposed to lie in the form domain of T_α near the origin and the solution v_2 is supposed to be square-integrable at infinity.

Using standard facts about Bessel's equation [1, Sec. 9], we find that these two solutions are given by

$$v_1(r) = \sqrt{r} \times \begin{cases} I_0(\kappa r) & \text{if } 0 < r \leq 1, \\ \tilde{A}_\alpha(\kappa) I_\alpha(\kappa r) + \tilde{B}_\alpha(\kappa) K_\alpha(\kappa r) & \text{if } 1 < r < \infty, \end{cases} \quad (5.3)$$

and

$$v_2(r) = \sqrt{r} \times \begin{cases} \tilde{D}_\alpha(\kappa) I_0(\kappa r) + \tilde{C}_\alpha(\kappa) K_0(\kappa r) & \text{if } 0 < r \leq 1, \\ K_\alpha(\kappa r) & \text{if } 1 < r < \infty, \end{cases} \quad (5.4)$$

with coefficients $\tilde{A}_\alpha(\kappa)$, $\tilde{B}_\alpha(\kappa)$, $\tilde{C}_\alpha(\kappa)$ and $\tilde{D}_\alpha(\kappa)$ that are determined by the continuity and jump conditions at $r = 1$. (We will give explicit expressions later in this proof.) Using the Wronski relation [1, Eq. 9.6.15] for the Bessel functions, viz.

$$W\{K_\nu(z), I_\nu(z)\} = I_\nu(z) K_{\nu+1}(z) + K_\nu(z) I_{\nu+1}(z) = \frac{1}{z}, \quad (5.5)$$

we obtain

$$W\{v_1, v_2\} = \tilde{C}_\alpha(\kappa) = \tilde{A}_\alpha(\kappa).$$

Let us show that $\tilde{A}_\alpha(\kappa) \neq 0$ for all $\kappa > 0$. Indeed, if we had $\tilde{A}_\alpha(\kappa_0) = 0$ for some $\kappa_0 > 0$, then v_1 would be an eigenfunction of T_α with eigenvalue $-\kappa_0^2$, but this contradicts the fact that T_α is a nonnegative operator. The latter fact follows from Step 2 in the proof of Proposition 4.5, where we showed that T_α is unitarily equivalent to the manifestly nonnegative operator h_α .

By Sturm–Liouville theory, it follows from the above facts that for any $\kappa > 0$, the integral kernel of $(T_\alpha + \kappa^2)^{-1}$ is given by

$$(T_\alpha + \kappa^2)^{-1}(r, r') = \sqrt{rr'} \times \begin{cases} K_0(\kappa r') I_0(\kappa r) + \tilde{f}_\alpha(\kappa) I_0(\kappa r) I_0(\kappa r') & \text{if } 0 < r \leq r' \leq 1, \\ K_\alpha(\kappa r') I_\alpha(\kappa r) + \tilde{g}_\alpha(\kappa) K_\alpha(\kappa r) K_\alpha(\kappa r') & \text{if } 1 < r \leq r', \\ \tilde{A}_\alpha^{-1}(\kappa) I_0(\kappa r) K_\alpha(\kappa r') & \text{if } 0 < r \leq 1 \leq r', \end{cases} \quad (5.6)$$

where we have denoted

$$\tilde{f}_\alpha(\kappa) := \frac{\tilde{D}_\alpha(\kappa)}{\tilde{A}_\alpha(\kappa)} \quad \text{and} \quad \tilde{g}_\alpha(\kappa) := \frac{\tilde{B}_\alpha(\kappa)}{\tilde{A}_\alpha(\kappa)}.$$

As usual, the formula for $r > r'$ follows by interchanging the variables.

Note also that $\tilde{A}_0(\kappa) = \tilde{C}_0(\kappa) = 1$ and $\tilde{D}_0(\kappa) = \tilde{B}_0(\kappa) = 0$, so, in particular,

$$\tilde{f}_0(\kappa) = \tilde{g}_0(\kappa) = 0.$$

Thus, recalling the definition (4.7), we see that (5.6) implies the formula in the lemma, except that the untilded quantities appear there rather than the tilded ones. Thus, to complete the proof we need to show that the former coincide with the latter. To do so, we replace the derivatives of Bessel functions appearing in the jump condition at $r = 1$ using [1, Eq. 9.6.26] in terms of Bessel functions without derivatives. Solving the corresponding system of four linear equations with four unknowns, we see that $A_\alpha(\kappa)$, $B_\alpha(\kappa)$, $D_\alpha(\kappa)$ are given by the expressions on the right side of (5.1), as claimed. (Note that we have already shown that $\tilde{C}_\alpha(\kappa) = \tilde{A}_\alpha(\kappa)$, which is confirmed by the solution of the system of linear equations.) \square

Next, we bound the quantities appearing in Lemma 5.1.

Lemma 5.2. *Let $\alpha \in (0, 1]$. The following bounds hold for all $\kappa > 0$ with an implicit constant depending possibly on α :*

$$\begin{aligned} |A_\alpha(\kappa)|^{-1} - 1 &\lesssim \kappa^{-2\alpha} \mathbb{1}_{(0,1)}(\kappa) + \kappa^{-1} \mathbb{1}_{[1,\infty)}(\kappa), \\ |f_\alpha(\kappa)| &\lesssim \kappa^{-2\alpha} \mathbb{1}_{(0,1)}(\kappa) + \kappa^{-1} e^{-2\kappa} \mathbb{1}_{[1,\infty)}(\kappa), \\ |g_\alpha(\kappa)| &\lesssim \mathbb{1}_{(0,1)}(\kappa) + \kappa^{-1} e^{2\kappa} \mathbb{1}_{[1,\infty)}(\kappa). \end{aligned}$$

In fact, in the following proof we will establish the asymptotic behavior of the three quantities in the lemma for $\kappa \rightarrow 0$ and $\kappa \rightarrow \infty$. The above bounds, however, are all that we need.

For the proof we make use of the following asymptotic facts about Bessel functions:

$$K_\nu(z) = \begin{cases} -\ln z + C + \mathcal{O}(z^2 |\ln z|) & \text{if } \nu = 0, \\ \left(\frac{z}{2}\right)^{-\nu} \frac{1}{2} \Gamma(\nu) - \left(\frac{z}{2}\right)^\nu \frac{1}{2\nu} \Gamma(1 - \nu) + \mathcal{O}(z^{2-\nu}) & \text{if } 0 < \nu < 1, \\ z^{-1} + \mathcal{O}(z |\ln z|) & \text{if } \nu = 1, \\ \left(\frac{z}{2}\right)^{-\nu} \frac{1}{2} \Gamma(\nu) + \mathcal{O}(z^{2-\nu}) & \text{if } \nu > 1, \end{cases} \quad \text{as } z \rightarrow 0, \quad (5.7)$$

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \Gamma(\nu + 1)^{-1} + \mathcal{O}(z^{2+\nu}) \quad \text{as } z \rightarrow 0,$$

see [1, (9.6.10) and (9.6.2)], and

$$\begin{aligned} K_\nu(z) &= \sqrt{\frac{\pi}{2z}} e^{-z} \left(1 + \frac{4\nu^2 - 1}{8z} + \mathcal{O}(z^{-2}) \right) \quad \text{as } z \rightarrow \infty, \\ I_\nu(z) &= \sqrt{\frac{1}{2\pi z}} e^z \left(1 - \frac{4\nu^2 - 1}{8z} + \mathcal{O}(z^{-2}) \right) \quad \text{as } z \rightarrow \infty. \end{aligned} \tag{5.8}$$

see [1, (9.7.1) and (9.7.2)]. The constant in (5.7) for $\nu = 0$ is known, but its value is irrelevant for our purposes.

In addition, we use the following global properties of Bessel functions: K_ν and I_ν are positive. Moreover, I_0 is decreasing. (In fact, I_ν is decreasing and K_ν is increasing for any $\nu \geq 0$, but we will not need this.)

Proof. *Step 1. Asymptotics at the origin.* From (5.7) we deduce that, as $\kappa \rightarrow 0$,

$$\begin{aligned} A_\alpha(\kappa) &= 2^{-\alpha} \Gamma(1 - \alpha) \kappa^\alpha + \mathcal{O}(\kappa^{2-\alpha}), \\ B_\alpha(\kappa) &= 2^{1-\alpha} \Gamma(\alpha)^{-1} \kappa^\alpha + \mathcal{O}(\kappa^{2+\alpha}), \\ D_\alpha(\kappa) &= 2^{-1-\alpha} \Gamma(\alpha) \kappa^{-\alpha} + \mathcal{O}(\kappa^\alpha \ln \kappa). \end{aligned}$$

We note in this computation there is a cancellation at order $\kappa^{-\alpha}$ for $A_\alpha(\kappa)$ and at order $\kappa^{-\alpha} \ln \kappa$ for $D_\alpha(\kappa)$.

These asymptotics imply that, as $\kappa \rightarrow 0$,

$$\begin{aligned} f_\alpha(\kappa) &= \frac{\Gamma(\alpha)}{2 \Gamma(1 - \alpha)} \kappa^{-2\alpha} + \mathcal{O}(|\ln \kappa| + \kappa^{2-4\alpha}), \\ g_\alpha(\kappa) &= \frac{2}{\Gamma(\alpha) \Gamma(1 - \alpha)} + \mathcal{O}(\kappa^{2-2\alpha}). \end{aligned}$$

Step 2. Asymptotics at infinity. From (5.8) we deduce that, as $\kappa \rightarrow \infty$,

$$\begin{aligned} A_\alpha(\kappa) &= 1 + \mathcal{O}(\kappa^{-1}), \\ B_\alpha(\kappa) &= \frac{\alpha}{2\pi\kappa} e^{2\kappa} (1 + \mathcal{O}(\kappa^{-1})), \\ D_\alpha(\kappa) &= \frac{\alpha\pi}{2\kappa} e^{-2\kappa} (1 + \mathcal{O}(\kappa^{-1})). \end{aligned}$$

These asymptotics imply that, as $\kappa \rightarrow \infty$,

$$\begin{aligned} f_\alpha(\kappa) &= \frac{\alpha\pi}{2\kappa} e^{-2\kappa} (1 + \mathcal{O}(\kappa^{-1})), \\ g_\alpha(\kappa) &= \frac{\alpha}{2\pi\kappa} e^{2\kappa} (1 + \mathcal{O}(\kappa^{-1})). \end{aligned}$$

Step 3. Uniform bounds. The asymptotics in Steps 1 and 2 imply that the claimed bounds in the lemma hold for all sufficiently small and all sufficiently large κ . Since $A_\alpha(\kappa)$, $B_\alpha(\kappa)$ and $D_\alpha(\kappa)$ are continuous functions of κ and since $A_\alpha(\kappa)$ does not vanish according to Lemma 5.1, we obtain the claimed bounds for all $\kappa > 0$. \square

After these preparations we are ready to prove the claimed pointwise bound on the integral kernel of $\Gamma_\alpha(\kappa)$.

Proof of Lemma 4.6. By selfadjointness, it suffices to prove the bound for $r \leq r'$, which we will assume throughout the proof. We split the integral kernel $\Gamma_\alpha(r, r'; \kappa)$ of $\Gamma_\alpha(\kappa)$ as

$$\Gamma_\alpha(r, r'; \kappa) = \sqrt{rr'} (\mathcal{S}(r, r'; \kappa) + \mathcal{R}(r, r'; \kappa)) \quad (5.9)$$

with

$$\mathcal{S}(r, r'; \kappa) = \begin{cases} f_\alpha(\kappa) I_0(\kappa r) I_0(\kappa r') & \text{if } 0 < r \leq r' \leq 1, \\ g_\alpha(\kappa) K_\alpha(\kappa r) K_\alpha(\kappa r') & \text{if } 1 < r \leq r', \\ I_0(\kappa r) (A_\alpha^{-1}(\kappa) K_\alpha(\kappa r') - K_0(\kappa r')) & \text{if } 0 < r \leq 1 \leq r', \end{cases} \quad (5.10)$$

and

$$\mathcal{R}(r, r'; \kappa) = \begin{cases} K_\alpha(\kappa r') I_\alpha(\kappa r) - I_0(\kappa r) K_0(\kappa r') & \text{if } 1 < r \leq r', \\ 0 & \text{elsewhere,} \end{cases} \quad (5.11)$$

and show that both pieces satisfy the bound claimed in the lemma. This is the content of the following two respective steps.

To simplify the notation, we shall use the symbol \lesssim to indicate the existence of a constant such that the inequality holds when the right side is multiplied by this constant. The constant may depend on $\alpha \in (0, 1]$, but is independent of $0 < r \leq r' < \infty$ and $\kappa > 0$.

Step 1. In this step we show that $|\mathcal{S}(r, r'; \kappa)| \lesssim \kappa^{-2\alpha} (1+r)^{-\alpha} (1+r')^{-\alpha}$ for all $0 < r \leq r' < \infty$.

We distinguish three cases.

First, let $0 < r \leq r' \leq 1$. We claim that

$$\sup_{\kappa > 0, \rho \leq 1} \kappa^{2\alpha} |f_\alpha(\kappa)| I_0^2(\kappa \rho) < \infty. \quad (5.12)$$

Once we have shown this, we deduce that

$$|\mathcal{S}(r, r'; \kappa)| = |f_\alpha(\kappa)| I_0(\kappa r) I_0(\kappa r') \lesssim \kappa^{-2\alpha} \lesssim \kappa^{-2\alpha} (1+r)^{-\alpha} (1+r')^{-\alpha},$$

which is the claimed bound.

To prove (5.12), we first assume $\kappa \leq 1$. Then, by (5.7), $I_0(\kappa \rho)^2 \lesssim 1$ for all $\rho \leq 1$ and, by Lemma 5.2, $|f_\alpha(\kappa)| \lesssim \kappa^{-2\alpha}$. This proves (5.12) for $\kappa \leq 1$. Now let $\kappa \geq 1$. Then, by (5.8) and the monotonicity of I_0 , $I_0(\kappa \rho)^2 \leq I_0(\kappa)^2 \lesssim \kappa^{-1} e^{2\kappa} \lesssim \kappa^{-2\alpha+1} e^{2\kappa}$ for all $\rho \leq 1$. Moreover, by Lemma 5.2, $|f_\alpha(\kappa)| \lesssim \kappa^{-1} e^{-2\kappa}$. This proves (5.12) for $\kappa \geq 1$.

Next, let $1 \leq r \leq r'$. We claim that

$$\sup_{\kappa > 0, \rho \geq 1} (\kappa \rho)^{2\alpha} |g_\alpha(\kappa)| K_\alpha^2(\kappa \rho) < \infty. \quad (5.13)$$

Once we have shown this, we deduce that

$$|\mathcal{S}(r, r'; \kappa)| = |g_\alpha(\kappa)| K_\alpha(\kappa r) K_\alpha(\kappa r') \lesssim (\kappa r)^{-2\alpha} (\kappa r')^{-2\alpha} \lesssim \kappa^{-2\alpha} (1+r)^{-\alpha} (1+r')^{-\alpha},$$

which is the claimed bound.

To prove (5.13), we first assume $\kappa \leq 1$. Then, by Lemma 5.2, $|g_\alpha(\kappa)| \lesssim 1$. Meanwhile, it follows from (5.7) and (5.8) that

$$\sup_{z>0} z^\alpha K_\alpha(z) < \infty.$$

(We emphasize that $\alpha > 0$.) This proves (5.13) for $\kappa \leq 1$. Now let $\kappa \geq 1$. Then, by Lemma 5.2, $|g_\alpha(\kappa)| \lesssim \kappa^{-1} e^{2\kappa}$. Moreover, by (5.8), $K_\alpha(\kappa\rho)^2 \lesssim (\kappa\rho)^{-1} e^{-2\kappa\rho}$ for all $\rho \geq 1$. Thus,

$$(\kappa\rho)^{2\alpha} |g_\alpha(\kappa)| K_\alpha^2(\kappa\rho) \lesssim (\kappa\rho)^{2\alpha-1} \kappa^{-1} e^{2\kappa} e^{-2\kappa\rho}.$$

The function $z \mapsto z^{2\alpha-1} e^{-z}$ is decreasing on $((2\alpha-1)_+, \infty)$. Since $\kappa\rho \geq 1 > (2\alpha-1)_+$, we deduce that

$$(\kappa\rho)^{2\alpha-1} \kappa^{-1} e^{2\kappa} e^{-2\kappa\rho} \leq \kappa^{2\alpha-2} \leq 1.$$

This proves (5.13) for $\kappa \geq 1$.

Finally, let $r \leq 1 \leq r'$. It follows from (5.7) and (5.8) that

$$\sup_{z>0} \max\{z^\alpha, z^2\} I_0(z) |K_\alpha(z) - K_0(z)| < \infty \quad (5.14)$$

and

$$\sup_{z>0} \max\{z^\alpha, z\} I_0(z) K_\alpha(z) < \infty. \quad (5.15)$$

We use the monotonicity of I_0 , together with (5.14) and (5.15), to bound

$$\begin{aligned} |\mathcal{S}(r, r'; \kappa)| &= I_0(\kappa r) |A_\alpha^{-1}(\kappa) K_\alpha(\kappa r') - K_0(\kappa r')| \\ &\leq I_0(\kappa r') |A_\alpha^{-1}(\kappa) K_\alpha(\kappa r') - K_0(\kappa r')| \\ &\leq I_0(\kappa r') |K_\alpha(\kappa r') - K_0(\kappa r')| + |1 - A_\alpha^{-1}(\kappa)| I_0(\kappa r') K_\alpha(\kappa r') \\ &\lesssim \min\{(\kappa r')^{-\alpha}, (\kappa r')^{-2}\} \\ &\quad + \min\{(\kappa r')^{-\alpha}, (\kappa r')^{-1}\} [\kappa^{-\alpha} \mathbb{1}_{(0,1)}(\kappa) + \kappa^{-1} \mathbb{1}_{[1,\infty)}(\kappa)]. \end{aligned}$$

We claim that the right side is bounded by $\kappa^{-2\alpha} (r')^{-\alpha}$. Since $(r')^{-\alpha} \lesssim (1+r')^{-\alpha} (1+r)^{-\alpha}$ for $r \leq 1 \leq r'$, this implies the claimed bound.

For the first term on the right side, we bound $\min\{(\kappa r')^{-\alpha}, (\kappa r')^{-2}\} \leq (\kappa r')^{-2\alpha} \leq \kappa^{-2\alpha} (r')^{-\alpha}$, as desired. For the second term we distinguish according to the size of κ . For $\kappa < 1$ we bound

$$\min\{(\kappa r')^{-\alpha}, (\kappa r')^{-1}\} \kappa^{-\alpha} \leq \kappa^{-2\alpha} (r')^{-\alpha},$$

and for $\kappa \geq 1$ we bound, using $\kappa r' \geq 1$,

$$\min\{(\kappa r')^{-\alpha}, (\kappa r')^{-1}\} \kappa^{-1} \leq (\kappa r')^{-1} \kappa^{-1} \leq (\kappa r')^{-\alpha} \kappa^{-\alpha} = \kappa^{-2\alpha} (r')^{-\alpha}.$$

This completes the proof of the bound that we claimed at the beginning of this step.

Step 2. In this step we show that $|\mathcal{R}(r, r'; \kappa)| \lesssim (\kappa r')^{-2\alpha}$ for all $1 \leq r \leq r'$. Since $(r')^{-2\alpha} \leq (1+r)^{-\alpha} (1+r')^{-\alpha}$ for all such r, r' , we obtain the claimed bound for $\mathcal{R}(r, r'; \kappa)$.

To prove this claim, we decompose $\mathcal{R}(r, r'; \kappa)$ with $1 \leq r \leq r'$ further as

$$\mathcal{R}(r, r'; \kappa) = \mathcal{R}_a(r, r'; \kappa) - \mathcal{R}_b(r, r'; \kappa)$$

with

$$\mathcal{R}_a(r, r'; \kappa) := I_0(\kappa r) (K_\alpha(\kappa r') - K_0(\kappa r'))$$

and

$$\mathcal{R}_b(r, r'; \kappa) := K_\alpha(\kappa r') (I_0(\kappa r) - I_\alpha(\kappa r)).$$

By (5.14) (together with $\max\{z^\alpha, z^2\} \geq z^{2\alpha}$) and monotonicity of I_0 it follows that, if $1 \leq r \leq r'$, then

$$|\mathcal{R}_a(r, r'; \kappa)| = I_0(\kappa r) |K_\alpha(\kappa r') - K_0(\kappa r')| \leq I_0(\kappa r') |K_\alpha(\kappa r') - K_0(\kappa r')| \lesssim (\kappa r')^{-2\alpha},$$

which is the claimed bound.

To estimate $\mathcal{R}_b(r, r'; \kappa)$ we will distinguish two cases.

If $\kappa r \geq 1$, then we use the upper bound (see (5.8))

$$K_\alpha(z') |I_0(z) - I_\alpha(z)| \lesssim \frac{e^{z-z'}}{\sqrt{z'} z^{3/2}} \quad \forall z, z' \geq 1, \quad (5.16)$$

which implies that for $z' \geq 1$ we have

$$\sup_{1 \leq z \leq z'} K_\alpha(z') |I_0(z) - I_\alpha(z)| \lesssim (z')^{-2} \leq (z')^{-2\alpha},$$

which is the claimed bound.

If $\kappa r \leq 1$, then we use the bound

$$\sup_{z' > 0} (z')^{2\alpha} K_\alpha(z') < \infty,$$

which follows from (5.7) and (5.8), as well as the bound

$$\sup_{0 < z \leq 1} |I_0(z) - I_\alpha(z)| < \infty, \quad (5.17)$$

which follows from (5.7). Combining these two inequalities yields the claimed bound. \square

The previous proof relies heavily on the fact that the leading terms in the asymptotic expansion of $K_v(z)$ for $z \rightarrow \infty$ coincide for different v ; see (5.8). This is used in (5.14) and (5.16). This shows that subtracting the integral kernel of $(T_0 + \kappa^2)^{-1}$ from $(T_\alpha + \kappa^2)^{-1}$ leads to certain cancelations in the integral kernel of $\Gamma_\alpha(\kappa)$ in terms of $\kappa^{-2\alpha} \sqrt{rr'} (1+r)^{-\alpha} (1+r')^{-\alpha}$. Notice in particular that the bound

$$(T_\alpha + \kappa^2)^{-1}(r, r') \lesssim \kappa^{-2\alpha} \sqrt{rr'} (1+r)^{-\alpha} (1+r')^{-\alpha}$$

cannot hold. Indeed, if this bound held, we could follow the reasoning in the proof of Proposition 4.5 and prove the inequality

$$(\inf \text{spec}(\mathfrak{h}^- + v))_-^\alpha \lesssim \int_0^\infty v(r)_- (1+r)^{-2\alpha} r dr.$$

This, however, would contradict the results of Appendix B.

Funding Open Access funding enabled and organized by Projekt DEAL.

Data availability No data was used for the research described in the article.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Appendix A. On Assumption 1.1

Let us show that the parameter α defined in (1.3) and the parameter α appearing in Assumption 1.1 coincide when μ is absolutely continuous with integrable density $B = \frac{d\mu}{dx}$. This follows from the following more general result.

Lemma A.1. *Let μ be a signed real Borel measure on \mathbb{R}^2 of finite total variation and assume that there is an $h \in W_{\text{loc}}^{1,1}(\mathbb{R}^2)$ such that $\Delta h = \mu$ in the sense of distributions. Assume that there is an $\alpha \in \mathbb{R}$ such that both numbers m^\pm in (1.6) are finite. Then $\alpha = (2\pi)^{-1}\mu(\mathbb{R}^2)$.*

Proof. For two parameters $\rho > 0$ and $\sigma > 1$ to be specified later, we introduce the function χ on \mathbb{R}^2 by

$$\chi(x) := \begin{cases} 1 & \text{if } |x| \leq \rho, \\ 1 - (\ln \sigma)^{-1} \ln(|x|/\rho) & \text{if } \rho < |x| \leq \sigma\rho, \\ 0 & \text{if } |x| > \sigma\rho. \end{cases}$$

This function is Lipschitz and has compact support. Since $h \in W_{\text{loc}}^{1,1}(\mathbb{R}^2)$, we can test the equation $\Delta h = \mu$ against χ and obtain

$$\int_{\mathbb{R}^2} \chi(x) d\mu(x) = - \int_{\mathbb{R}^2} \nabla \chi(x) \cdot \nabla h(x) dx.$$

(Strictly speaking, we test the equation against the convolution of χ with a C_c^∞ mollifier and pass to the limit in the mollification parameter.) Since μ has finite total variation, dominated convergence implies that the left side tends to $\mu(\mathbb{R}^2)$ as $\rho \rightarrow \infty$ for any fixed σ . Since χ is harmonic in $\{\rho < |x| < \sigma\rho\}$, we see that the right side is equal to

$$\begin{aligned} - \int_{\mathbb{R}^2} \nabla \chi \cdot \nabla h dx &= - \int_{\rho < |x| < \sigma\rho} \nabla \chi \cdot \nabla h dx = - \int_{|x|=\sigma\rho} \frac{x}{|x|} \cdot (\nabla \chi) h ds(x) \\ &\quad + \int_{|x|=\rho} \frac{x}{|x|} \cdot (\nabla \chi) h ds(x) \\ &= (\ln \sigma)^{-1} (\rho\sigma)^{-1} \int_{|x|=\sigma\rho} h ds(x) - (\ln \sigma)^{-1} \rho^{-1} \int_{|x|=\rho} h ds(x). \end{aligned}$$

Here ds denotes integration with respect to the surface measure. Assumption 1.1 means that for all $x \in \mathbb{R}^2$,

$$-\ln m^- \leq h(x) - \alpha \ln(1 + |x|/R) \leq \ln m^+.$$

This implies that for $r \in \{\rho, \sigma\rho\}$, we have

$$-2\pi r \ln m^- \leq \int_{|x|=r} h \, ds(x) - 2\pi\alpha r \ln(1 + r/R) \leq 2\pi r \ln m^+$$

and therefore

$$\left| - \int_{\mathbb{R}^2} \nabla \chi \cdot \nabla h \, dx - (\ln \sigma)^{-1} 2\pi\alpha \ln \frac{1 + \sigma\rho/R}{1 + \rho/R} \right| \leq (\ln \sigma)^{-1} 2\pi \ln(m^+ m^-).$$

Letting first $\rho \rightarrow \infty$ and then $\sigma \rightarrow \infty$ we easily find that

$$- \int_{\mathbb{R}^2} \nabla \chi \cdot \nabla h \, dx \rightarrow 2\pi\alpha.$$

This proves the claimed identity $\mu(\mathbb{R}^2) = 2\pi\alpha$. \square

Next, we show that Assumption 1.1 is satisfied for a large class of absolutely continuous measures.

Lemma A.2. *Let $B \in L^1(\mathbb{R}^2, (1 + \ln_+ |x|)dx)$ such that, for some $p > 1$,*

$$\sup_{x \in \mathbb{R}^2} \int_{|y-x|<1} |B(y)|^p \, dy < \infty.$$

Then

$$h(x) := \frac{1}{2\pi} \int_{\mathbb{R}^2} B(y) \ln |x - y| \, dy$$

belongs to $W_{\text{loc}}^{1,r}(\mathbb{R}^2)$ for $r = \frac{2p}{2-p}$ if $p < 2$, any $r < \infty$ if $p = 2$ and $r = \infty$ if $p > 2$. It solves $\Delta h = B$ and satisfies Assumption 1.1 with α given by (1.3).

Proof. The facts that $h \in W_{\text{loc}}^{1,1}(\mathbb{R}^2)$ with

$$\nabla h(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} B(y) \frac{x - y}{|x - y|^2} \, dy$$

and that $\Delta h = B$ in the sense of distributions are in [33, Theorem 6.21]. For a given $a \in \mathbb{R}^2$ we decompose the integral giving ∇h into the outside of $B(a, 1)$ and its inside. The part from the outside is bounded in $B(a, \frac{1}{2})$ since $B \in L^1(\mathbb{R}^2)$. The part from the inside defines a function in $L^r(\mathbb{R}^2)$ with $r = \frac{2p}{2-p}$ if $p < 2$ by the weak Young inequality [33, Theorem 4.3] and the fact that $B \in L^p(B(a, 1))$. This also implies the assertion for $p = 2$. The fact that the part from the inside is bounded for $p > 2$ follows from Hölder's inequality.

It remains to show that $x \mapsto h(x) - \alpha \ln(1 + |x|)$ is bounded. We begin by noting that, with $\frac{1}{p} + \frac{1}{p'} = 1$,

$$\left| \int_{|y-x|<1} B(y) \ln |x - y| \, dy \right| \leq \left(\int_{|y-x|<1} |B(y)|^p \, dy \right)^{\frac{1}{p}} \left(\int_{|z|<1} |\ln |z||^{p'} \, dz \right)^{\frac{1}{p'}}.$$

By assumption the right side is bounded with respect to x . Next, we bound

$$\left| \int_{|y| < \frac{1}{2}|x|} B(y) \ln |x - y| dy - \ln |x| \int_{|y| < \frac{1}{2}|x|} B(y) dy \right| \leq \int_{|y| < \frac{1}{2}|x|} |B(y)| \ln \frac{|x - y|}{|x|} dy.$$

For $|y| < \frac{1}{2}|x|$ we have $\frac{1}{2} < \frac{|x-y|}{|x|} < \frac{3}{2}$, so the integral on the right side is bounded with respect to x . We also note that if $|x| > 2$, then

$$\left| \int_{|y| < \frac{1}{2}|x|} B(y) dy - 2\pi\alpha \right| \leq \int_{|y| \geq \frac{1}{2}|x|} |B(y)| dy \leq \frac{1}{\ln(|x|/2)} \int_{\mathbb{R}^2} |B(y)| \ln_+ |y| dy.$$

This, when multiplied by $\ln |x|$ is bounded for $|x| \geq e$, say. Finally, we note that when $|x-y| \geq 1$ and $|y| \geq \frac{1}{2}|x|$, then $0 \leq \ln |x-y| \leq \ln(|x|+|y|) \leq \ln(\frac{3}{2}|y|) = \ln \frac{3}{2} + \ln_+ |y|$. Therefore

$$\left| \int_{|y-x| \geq 1, |y| \geq \frac{1}{2}|x|} B(y) \ln |x - y| dy \right| \leq \int_{\mathbb{R}^2} (\ln \frac{3}{2} + \ln_+ |y|) |B(y)| dy.$$

To summarize, we have shown that $h(x) - \alpha \ln |x|$ is bounded for $|x| \geq e$. Since $\alpha \ln |x| - \alpha \ln(1 + |x|)$ is bounded for $|x| \geq e$ as well, we have proved the assertion for $|x| \geq e$. The boundedness for $|x| < e$ proceeds along the same lines and we omit the details. \square

Appendix B. Failure of a One-Term Bound on $\inf \text{spec}(\mathfrak{h}^- + v)$

In this section we discuss the optimality of the bound in Proposition 4.5. More precisely, we shall show that for any given $0 < \alpha < 1$, neither

$$(\inf \text{spec}(\mathfrak{h}^- + v))_-^\alpha \leq C_\alpha \int_0^\infty v(r)_-^{1+\alpha} r dr \quad (\text{B.1})$$

nor

$$(\inf \text{spec}(\mathfrak{h}^- + v))_-^\alpha \leq C_\alpha \int_0^\infty v(r)_- (1+r)^{-2\alpha} r dr \quad (\text{B.2})$$

can hold with some constant C_α for all real $v \in L^1_{\text{loc}}(\mathbb{R}_+)$. We recall that the operator \mathfrak{h}^- is defined in Sect. 4.2.

This failure of (B.1) and (B.2) is in contrast to what happens in other related situations, for instance in [31, Lemma 1], where the ‘weak coupling term’ is able to control the lowest eigenvalue uniformly. In view of this failure it is perhaps less surprising that the proof of Theorem 1.5 in the case $\gamma = \alpha$ is rather involved.

To show the claimed failure we apply the duality argument of Lieb and Thirring [35] (see also [22, Proposition 5.3]). This shows that (B.1) and (B.2) are equivalent to the Sobolev interpolation inequalities

$$\begin{aligned} & \left(\int_0^\infty (1+r)^{-2\alpha} |\varphi'(r)|^2 r dr \right)^{\frac{1}{1+\alpha}} \left(\int_0^\infty (1+r)^{-2\alpha} |\varphi(r)|^2 r dr \right)^{\frac{\alpha}{1+\alpha}} \\ & \geq S_\alpha \left(\int_0^\infty (1+r)^{-2(1+\alpha)} |\varphi(r)|^{\frac{2(1+\alpha)}{\alpha}} r dr \right)^{\frac{\alpha}{1+\alpha}} \end{aligned} \quad (\text{B.3})$$

and

$$\left(\int_0^\infty (1+r)^{-2\alpha} |\varphi'(r)|^2 r dr \right)^{1-\alpha} \left(\int_0^\infty (1+r)^{-2\alpha} |\varphi(r)|^2 r dr \right)^\alpha \geq S_\alpha \operatorname{ess\,sup}_{\mathbb{R}_+} |\varphi|^2, \quad (\text{B.4})$$

respectively, with some $S_\alpha > 0$ and for all locally absolutely continuous functions φ on \mathbb{R}_+ for which the two integrals on the left sides are finite. We shall show that neither (B.3) nor (B.4) holds.

To see that (B.3) fails, consider $\varphi(r) = v(\varepsilon r)$ with a fixed nice function v and a parameter $\varepsilon \ll 1$. As $\varepsilon \rightarrow 0$, we compute

$$\begin{aligned} \int_0^\infty |\varphi'(r)|^2 \frac{r dr}{(1+r)^{2\alpha}} &= \varepsilon^{2\alpha} \int_0^\infty |v'(s)|^2 \frac{s ds}{(\varepsilon+s)^{2\alpha}} \sim \varepsilon^{2\alpha} \int_0^\infty |v'(s)|^2 s^{1-2\alpha} ds, \\ \int_0^\infty |\varphi(r)|^2 \frac{r dr}{(1+r)^{2\alpha}} &= \varepsilon^{-2(1-\alpha)} \int_0^\infty |v(s)|^2 \frac{s ds}{(\varepsilon+s)^{2\alpha}} \sim \varepsilon^{-2(1-\alpha)} \int_0^\infty |v(s)|^2 s^{1-2\alpha} ds, \\ \int_0^\infty |\varphi(r)|^{\frac{2(1+\alpha)}{\alpha}} \frac{r dr}{(1+r)^{2(1+\alpha)}} &= \varepsilon^{2\alpha} \int_0^\infty |v(s)|^{\frac{2(1+\alpha)}{\alpha}} \frac{s ds}{(\varepsilon+s)^{2(1+\alpha)}} \rightarrow \text{const} |v(0)|^{\frac{2(1+\alpha)}{\alpha}}, \end{aligned}$$

where the constant on the right side of the final relation is positive. (In fact, it is equal to $\int_0^\infty (1+t)^{-2(1+\alpha)} t dt$.) Thus, the left side of (B.3) behaves like $\varepsilon^{2\alpha^2/(1+\alpha)}$, while the right side remains positive (if $v(0) \neq 0$). Since $\alpha > 0$, we arrive at a contradiction.

To see that (B.4) fails, consider $\varphi(r) = v(Mr)$ with a fixed nice function v and a parameter $M \gg 1$. As $M \rightarrow \infty$, we compute

$$\begin{aligned} \int_0^\infty |\varphi'(r)|^2 \frac{r dr}{(1+r)^{2\alpha}} &= \int_0^\infty |v'(s)|^2 \frac{s ds}{(1+s/M)^{2\alpha}} \rightarrow \int_0^\infty |v'(s)|^2 s ds, \\ \int_0^\infty |\varphi(r)|^2 \frac{r dr}{(1+r)^{2\alpha}} &= M^{-2} \int_0^\infty |v(s)|^2 \frac{s ds}{(1+s/M)^{2\alpha}} \sim M^{-2} \int_0^\infty |v(s)|^2 s ds, \\ \operatorname{ess\,sup}_{\mathbb{R}_+} |\varphi|^2 &= \operatorname{ess\,sup}_{\mathbb{R}_+} |v|^2. \end{aligned}$$

Thus, the left side of (B.4) behaves like $M^{-2\alpha}$, while the right side is independent of M . Since $\alpha > 0$, we arrive at a contradiction.

References

1. Abramowitz, M., Stegun, I.: Handbook of Mathematical Functions. National Bureau of Standards (1964)
2. Aharonov, Y., Casher, A.: Ground state of a spin- $\frac{1}{2}$ charged particle in a two-dimensional magnetic field. Phys. Rev. A **19**(6), 2461–2462 (1979)
3. Aizenman, M., Lieb, E.H.: On semiclassical bounds for eigenvalues of Schrödinger operators. Phys. Lett. A **66**, 427–429 (1978)
4. Bachmann, S., Froese, R., Schraven, S.: Counting eigenvalues of Schrödinger operators using the landscape function. J. Spec. Theory, to appear
5. Bachmann, S., Froese, R., Schraven, S.: Two-sided Lieb–Thirring bounds. Preprint (2024). [arXiv:2403.19023](https://arxiv.org/abs/2403.19023)
6. Bley, G.A., Fournais, S.: Hardy–Lieb–Thirring inequalities for fractional Pauli operators. Commun. Math. Phys. **365**(2), 651–683 (2019)
7. Breuer, J., Kovařík, H.: Resonances at the threshold for Pauli operators in dimension two. Ann. Henri Poincaré, **65**(6), 2839–2875
8. Cycon, H.L., Froese, F.G., Kirsch, W., Simon, B.: Schrödinger Operators with Application to Quantum Mechanics and Global Geometry. Springer (1987)
9. Egorov, Y., Kondratiev, V.: On Spectral Theory of Elliptic Operators. Birkhäuser (1996)

10. Ekholm, T., Frank, R.L.: On Lieb–Thirring inequalities for Schrödinger operators with virtual level. *Commun. Math. Phys.* **264**(3), 725–740 (2006)
11. Ekholm, T., Frank, R.L.: Lieb–Thirring inequalities on the half-line with critical exponent. *J. Eur. Math. Soc.* **10**, 739–755 (2008)
12. Ekholm, T., Frank, R.L., Kovářík, H.: Eigenvalue estimates for Schrödinger operators on metric trees. *Adv. Math.* **226**(6), 5165–5197 (2011)
13. Erdős, L.: Magnetic Lieb–Thirring inequalities. *Commun. Math. Phys.* **170**(3), 629–668 (1995)
14. Erdős, L.: Recent developments in quantum mechanics with magnetic fields. *Proc. Sympos. Pure Math.*, 76, Part 1, Amer. Math. Soc., Providence, RI, 401–428 (2007)
15. Erdős, L., Solovej, J.P.: Semiclassical eigenvalue estimates for the Pauli operator with strong non-homogeneous magnetic fields. *Commun. Math. Phys.* **188**, 599–656 (1997)
16. Erdős, L., Vugalter, V.: Pauli operator and Aharonov–Casher theorem for measure valued magnetic fields. *Commun. Math. Phys.* **225**(2), 399–421 (2002)
17. Fanelli, L., Kovářík, H.: Quantitative Hardy inequality for magnetic Hamiltonians. *Commun. PDE* (2024). <https://doi.org/10.1080/03605302.2024.2403010>
18. Frank, R.L.: Remarks on eigenvalue estimates and semigroup domination. *Contemp. Math.*, 500, Amer. Math. Soc., Providence, RI, 63–86 (2009)
19. Frank, R.L.: A simple proof of Hardy–Lieb–Thirring inequalities. *Commun. Math. Phys.* **290**(2), 789–800 (2009)
20. Frank, R.L.: The Lieb–Thirring inequalities: recent results and open problems. *Proc. Sympos. Pure Math.*, 104, Amer. Math. Soc., Providence, RI, 45–86 (2021)
21. Frank, R.L., Kovářík, H.: Heat kernels of metric trees and applications. *SIAM J. Math. Anal.* **45**(3), 1027–1046 (2013)
22. Frank, R.L., Laptev, A., Weidl, T.: Schrödinger operators: eigenvalues and Lieb–Thirring inequalities. Cambridge Studies in Advanced Mathematics 200, Cambridge University Press, Cambridge (2023)
23. Frank, R.L., Laptev, A., Weidl, T.: An improved one-dimensional Hardy inequality. *J. Math. Sci.* **268**, 323–342 (2022)
24. Frank, R.L., Lieb, E.H., Seiringer, R.: Hardy–Lieb–Thirring inequalities for fractional Schrödinger operators. *J. Am. Math. Soc.* **21**(4), 925–950 (2008)
25. Frank, R.L., Morozov, S., Vugalter, S.: Weakly coupled bound states of Pauli operators. *Calc. Var. Partial Differ. Equ.* **40**, 253–271 (2011)
26. Frank, R.L., Simon, B., Weidl, T.: Eigenvalue bounds for perturbations of Schrödinger operators and Jacobi matrices with regular ground states. *Commun. Math. Phys.* **282**(1), 199–208 (2008)
27. Grigor'yan, A.: Heat kernels on weighted manifolds and applications. In: *The Ubiquitous Heat Kernel*, Contemp. Math., 398, Amer. Math. Soc., Providence, RI, 93–191 (2006)
28. Grigor'yan, A., Saloff-Coste, L.: Stability results for Harnack inequalities. *Ann. Inst. Fourier* **55**, 825–890 (2005)
29. Kovářík, H.: Weakly coupled Schrödinger operators on regular metric trees. *SIAM J. Math. Anal.* **39**, 1135–1149 (2007)
30. Kovářík, H.: Spectral properties and time decay of the wave functions of Pauli and Dirac operators in dimension two. *Adv. Math.* **398**, 108244 (2022)
31. Kovářík, H., Vugalter, S., Weidl, T.: Spectral estimates for two-dimensional Schrödinger operators with application to quantum layers. *Commun. Math. Phys.* **275**(3), 827–838 (2007)
32. Lieb, E.: Bound states of the Laplace and Schrödinger operators. *Bull. Am. Math. Soc.* **82**, 751–753 (1976)
33. Lieb, E.H., Loss, M.: Analysis. Second edition. Graduate Studies in Mathematics 14, Amer. Math. Soc., Providence, RI (2010)
34. Lieb, E.H., Solovej, J.P., Yngvason, J.: Ground states of large quantum dots in magnetic fields. *Phys. Rev. B* **51**, 10646–10665 (1995)
35. Lieb, E.H., Thirring, W.E.: Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities. *Studies in Mathematical Physics*, Princeton University Press, 269–303 (1976)
36. Muckenhoupt, B.: Hardy's inequality with weights. *Stud. Math.* **44**, 31–38 (1972)
37. Pinchover, Y.: Topics in the theory of positive solutions of second-order elliptic and parabolic partial differential equations. *Proc. Sympos. Pure Math.*, 76, Part 1, Amer. Math. Soc., Providence, RI, 329–355 (2007)
38. Raikov, G.D.: Strong Electric Field Eigenvalue Asymptotics for the Schrödinger Operator with Electro-magnetic Potential. *Lett. Math. Phys.* **21**, 41–49 (1991)
39. Saloff-Coste, L.: Aspects of Sobolev-type inequalities. London Math. Soc. Lecture Note Ser. 289, Cambridge University Press, Cambridge (2002)
40. Saloff-Coste, L.: The heat kernel and its estimates, *Advanced Studies in Mathematics* (2009)
41. Shen, Z.: On moments of negative eigenvalues for the Pauli operator. *J. Diff. Eq.* **149**, 292–327 (1998)

42. Sobolev, A.: On the Lieb-Thirring estimates for the Pauli operator. *Duke Math. J.* **82**, 607–635 (1996)
43. Tomaselli, G.: A class of inequalities. *Boll. Unione Mat. Ital.*, IV **2**, 622–663 (1969)
44. Weidl, T.: Remarks on virtual bound states for semi-bounded operators. *Commun. PDE*. **124**, 25–60 (1999)

Communicated by R. Seiringer