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ON THE GROWTH OF THE FLOER BARCODE

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ABSTRACT. This paper is a follow-up to the authors' recent work on barcode entropy. We study the growth of the barcode of the Floer complex for the iterates of a compactly supported Hamiltonian diffeomorphism. In particular, we introduce sequential barcode entropy which has properties similar to barcode entropy, bounds it from above and is more sensitive to the barcode growth. In the same vein, we explore another variant of barcode entropy based on the total persistence growth and revisit the relation between the growth of periodic orbits and topological entropy. We also study the behavior of the spectral norm, aka the γ-norm, under iterations. We show that the γ-norm of the iterates is separated from zero when the map has sufficiently many hyperbolic periodic points and, as a consequence, it is separated from zero C^∞-generically in dimension two. We also touch upon properties of the barcode entropy of pseudo-rotations and, more generally, γ-almost periodic maps.

1. INTRODUCTION

The main theme of the paper is the growth of the Floer complex for the iterates of a compactly supported Hamiltonian diffeomorphism. In particular, we focus on the exponential growth rate of the barcode of the Floer complex and the behavior of the spectral norm.

There is a wide range of interpretations of the question of the growth of the Floer complex for the iterates φ^k and there seems to be no obvious way to make the question precise fitting all or most of the aspects of the problem. When trying to articulate this question, it is useful to keep two facts in mind.

First of all, in most cases the "effective" diameter of the action spectrum $\mathcal{S}(\varphi^k)$ grows at most polynomially with the order of iteration. For instance, as a simple consequence of the isoperimetric inequality, the diameter

$$\operatorname{diam}\mathcal{S}(\varphi^k) := \max\mathcal{S}(\varphi^k) - \min\mathcal{S}(\varphi^k)$$

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of this set for contractible orbits grows at most linearly in k when the underlying symplectic manifold M is a surface of genus $g \geq 2$ and it grows at most quadratically when $M = \mathbb{T}^{2n}$; cf. [37]. In fact, we are not aware of any example where the diameter has been shown to grow faster than linearly. When M is not aspherical or non-contractible periodic orbits are included, the notion of the diameter is more involved and more ambiguous, but even then the effective diameter grows polynomially in most cases; see [11, Rmk. 3.4]. Similarly, the index spectrum defined as in, e.g., [21] can be shown to grow at most polynomially in many situations.

On the other hand, the number of k-periodic points of φ can grow arbitrarily fast, e.g., superexponentially. Moreover, in dimension two this behavior is in some sense common, i.e., occurs for a dense subset of an open set of Hamiltonian diffeomorphisms in the analytic topology; see [2] and references therein.

With this in mind, one useful way to measure the growth of the Floer complex of φ^k is by counting the number of bars $b_\epsilon(\varphi^k)$ of length greater than $\epsilon > 0$ in its barcode. The limit $\hbar(\varphi)$ as $\epsilon \searrow 0$ of the exponential growth rate of $b_\epsilon(\varphi^k)$ is called the *barcode entropy* and closely related to the topological entropy $\mathrm{h}_{\mathrm{top}}(\varphi)$ of φ; [11]. In particular, $\hbar(\varphi) \leq \mathrm{h}_{\mathrm{top}}(\varphi)$ and hence $b_\epsilon(\varphi^k)$ grows at most exponentially in k, and $\hbar(\varphi) = \mathrm{h}_{\mathrm{top}}(\varphi)$ in dimension two.

Replacing the fixed threshold ϵ by $\epsilon_k > 0$ which depends on k gives rise to a somewhat different way to measure the size of the Floer complex, which we explore in this paper. The exponential growth rate of $b_{\epsilon_k}(\varphi^k)$, where $\epsilon_k \searrow 0$ subexponentially, is encoded by the *sequential barcode entropy* $\hat{\hbar}(\varphi) \geq \hbar(\varphi)$ of φ. We show that sequential barcode entropy has properties similar to barcode entropy. For instance, we still have that $\hat{\hbar}(\varphi) \leq \mathrm{h}_{\mathrm{top}}(\varphi)$, and hence $b_{\epsilon_k}(\varphi^k)$ grows at most exponentially when ϵ_k is subexponential, and again $\hat{\hbar}(\varphi) = \hbar(\varphi) = \mathrm{h}_{\mathrm{top}}(\varphi)$ in dimension two. (Hypothetically, it is possible that $\hat{\hbar}(\varphi) = \hbar(\varphi)$ in all dimensions. However, neither $\hat{\hbar}(\varphi)$ nor $\hbar(\varphi)$ is equal to $\mathrm{h}_{\mathrm{top}}(\varphi)$ in general when $\dim M \geq 6$; see [9].)

A different perspective on the barcode growth and a variant of barcode entropy is explored in Section 3.3. There we introduce and prove some basic properties of a version of entropy based on the growth of the total persistence of the barcode of φ^k. In that section, we also revisit the problem of relating the growth of periodic orbits and topological entropy. Namely, while in general beyond dimension two these two invariants are known not to be connected (see, e.g., [2, 27]) even for Hamiltonian diffeomorphisms, we show that the exponential growth rate of periodic orbits does give a lower bound for the topological entropy, albeit with a correction term coming from the decay of the shortest bar.

The spectral norm, a.k.a. the γ-norm, $\gamma(\varphi) \leq \operatorname{diam} \mathscr{S}(\varphi)$ is roughly speaking the difference between the homological maximum and the homological minimum of the action functional (see, e.g., [15, 34, 35, 39, 48]), giving another and more robust way to measure the "diameter" of $\mathscr{S}(\varphi)$. (We will recall the definition of the γ-norm in Section 2.3.) Upper bounds on the γ-norm have been extensively studied — see the references above and also, e.g., [30, 41, 42]. Here we are interested in lower bounds on the sequence $\gamma(\varphi^k)$ and more specifically in the question if this sequence can get arbitrarily close to zero. The connection with the barcode growth comes from the fact that $\gamma(\varphi)$ gives an upper bound on the boundary depth, i.e., the length of the largest finite bar, $\beta_{\max}(\varphi)$; see [44, 45] and [30]. From this perspective, very little seems to be known about the behavior of the sequence $\gamma(\varphi^k)$ outside the case of pseudo-rotations; see [22, 26]. Here, refining [11, Prop. 6.5], we show that the sequence $\gamma(\varphi^k)$ is bounded away from zero when φ has sufficiently many hyperbolic periodic points. This is the case, for instance, when $\mathrm{h}_{\mathrm{top}}(\varphi) > 0$ and $\dim M = 2$; or when φ is a strongly non-degenerate Hamiltonian diffeomorphism of a positive genus surface. In particular, in dimension two, this sequence is bounded away from zero C^∞-generically; cf. [31].

Finally, we also look at another extreme and examine *γ-approximate identities*, the maps whose iterates approximate the identity arbitrarily well with respect to the γ-norm, and *γ-almost periodic maps*, i.e., the maps such that $\gamma(\varphi^k)$ becomes arbitrarily small with positive frequency; see [23]. The main, and to date the only, source of such maps are Hamiltonian pseudo-rotations; cf. [22, 26]. In particular, we show that for a γ-almost periodic map φ the sequence $b_\epsilon(\varphi^k)$ is bounded for all $\epsilon > 0$ and hence $\hbar(\varphi) = 0$. This is a step toward the proof of the conjecture that $\mathrm{h}_{\mathrm{top}}(\varphi) = 0$ for Hamiltonian pseudo-rotations.

This paper is formally independent from [11], but conceptually it is a follow-up to that work and perhaps should be better read with that work in mind.

2. DEFINITIONS AND RESULTS

2.1. **Floer homology and barcode entropy.**

Throughout the paper we use conventions and notation from [11]. Referring the reader to [11, Sect. 3] and references therein for a much more detailed discussion, here we only touch upon several key points.

2.1.1. *Floer homology and barcodes.*

In this paper all Lagrangian submanifolds $L \subset M$ are assumed to be closed and monotone with minimal Chern number at least 2. Hamiltonian diffeomorphisms φ are always required to have compact support, and when M is not compact we assume that it is sufficiently well-behaved at infinity (e.g., convex, or wide in the sense of [25, Defn. 3.1]) so that the filtered Floer complex and homology for the pair $(L, \varphi(L))$ or the map φ itself can be defined; cf. [11, Rmk. 2.8].

For the sake of simplicity Floer complexes and homology and also the ordinary homology are taken over the ground field $\mathbb{F} = \mathbb{F}_2$. When L and L' are Hamiltonian isotopic and intersect transversely, we denote by $\mathrm{CF}(L, L')$ the Floer complex of the pair (L, L'). This complex is generated by the intersections $L \cap L'$ over the *universal Novikov field* Λ. This is the field of formal sums

$$\lambda = \sum_{j \geq 0} f_j T^{a_j},$$

where $f_j \in \mathbb{F}$ and $a_j \in \mathbb{R}$ and the sequence a_j (with $f_j \neq 0$) is either finite or $a_j \to \infty$.

Due to our choice of the Novikov field, the complex $\mathrm{CF}(L, L')$ is not graded. However, fixing a Hamiltonian isotopy from L to L' and "cappings" of intersections, we obtain a filtration on $\mathrm{CF}(L, L')$ by the Hamiltonian action. The differential on the complex is defined in the standard way. Note that the complex breaks down into a direct sum of subcomplexes over homotopy classes of paths from L to L'. Then, to define the action filtration on $\mathrm{CF}(L, L')$, we also need to pick a reference path in every homotopy class.

The barcode $\mathscr{B}(L, L')$ of the Floer complex $\mathrm{CF}(L, L')$, in the most refined form, is a collection of finite or semi-infinite intervals defined, in general, up to some shift ambiguity. For our purposes, it is convenient to forgo the location of the intervals and treat $\mathscr{B}(L, L')$ as a collection (i.e., a multiset) of positive numbers including ∞. A construction of barcodes suitable for our purposes is introduced and worked out in detail in [47] and also discussed in [11]. Below we briefly go over it.

Set $\mathscr{C} = \mathrm{CF}(L, L')$ and fix a filtration $\mathscr{A} : \mathscr{C} \to \mathbb{R} \cup \{-\infty\}$ via Hamiltonian action where $\mathscr{A}(0)$ is set to $-\infty$. A finite set of vectors $\xi_i \in \mathscr{C}$ is called *orthogonal* if for any collection $\lambda_i \in \Lambda$ we have $\mathscr{A}(\sum \lambda_i \xi_i) = \max \mathscr{A}(\lambda_i \xi_i)$. A Λ-basis $\{\alpha_i, \eta_j, \gamma_j\}$ of $\mathscr{C}$ is called a *singular decomposition* if it is orthogonal and $\partial_{Fl} \alpha_i = 0$, $\partial_{Fl} \gamma_j = \eta_j$. It is shown in [47, Sects. 2 and 3] that $\mathscr{C}$ admits a singular decomposition. Ordering the pairs (η_j, γ_j) by the action difference, we obtain

$$\mathscr{A}(\gamma_1) - \mathscr{A}(\eta_1) \leq \mathscr{A}(\gamma_2) - \mathscr{A}(\eta_2) \leq \cdots.$$

In this paper we refer to the multiset formed by the differences $\mathscr{A}(\gamma_i) - \mathscr{A}(\eta_i)$ together with $\dim_\Lambda \mathrm{HF}(L, L')$ many ∞'s (corresponding to the basis elements α_i) as the *barcode* of $\mathscr{C} = \mathrm{CF}(L, L')$ and denote it by $\mathscr{B}(L, L')$. Moreover, abusing notation, we call these numbers *finite/infinite bars*. In the original definition [47, Def. 6.3], barcode also contains information about the location of these bars, i.e., the bars are pinned, whereas our version only keeps the length of the bars. For our purposes the length data suffices; hence, for simplicity, we forgo the locations. The barcode $\mathscr{B}(L, L')$ is independent of the choice of a singular decomposition and other auxiliary data involved in the construction of $\mathrm{CF}(L, L')$; see [47, Thm. 7.1] and [45, Prop. 6.2]. Also note that $\mathscr{B}(L, L') = \mathscr{B}(L', L)$ as was shown in [47, Prop. 2.20]; see also [11, Sec. 3.3.1].

Recall that the *Hofer norm* of a Hamiltonian diffeomorphism $\varphi\colon M \to M$ is defined as

$$\|\varphi\|_H = \inf_H \int_{S^1} \big(\max_M H_t - \min_M H_t\big)\, dt,$$

where the infimum is taken over all 1-periodic in time Hamiltonians H generating φ, i.e., $\varphi = \varphi_H$. We refer the reader to, e.g., [36] and references therein for a very detailed discussion of the Hofer norm. The *Hofer distance* between two Hamiltonian isotopic Lagrangian submanifolds L and L' is

$$d_H(L, L') = \inf\big\{\|\varphi\|_H \mid \varphi(L) = L'\big\};$$

see [8] and, e.g., [46] for further references.

The most important feature of the barcode, at least for our purposes, is that it is continuous in the Lagrangian with respect to the C^∞-norm and even the Hofer norm (or the γ-norm; see Section 2.3). In order to state the continuity property, let us first recall the relevant metric on the space of barcodes. We say that the barcodes $\mathscr{B}_1$, $\mathscr{B}_2$ are δ*-matched* if, after deleting as needed bars of length $< 2\delta$, one can find a bijection $\mathscr{B}_1 \to \mathscr{B}_2$; $\beta_1^i \mapsto \beta_2^i$ such that $|\beta_1^i - \beta_2^i| < 2\delta$. The infimum of all such δ's is called the *bottleneck distance* between $\mathscr{B}_i$. This is indeed a distance on the space of un-pinned barcodes, which is bounded from above by the bottleneck distance on the space of pinned barcodes under the natural forgetful map between these spaces. The continuity property can be stated as follows; see [47, Sect. 12] for details. Assume that Lagrangian submanifolds L, L' and L'' are Hamiltonian isotopic such that $L \pitchfork L'$ and $L \pitchfork L''$. Then the bottleneck distance between $\mathscr{B}(L, L')$ and $\mathscr{B}(L, L'')$ is bounded from above by the Hofer distance $d_H(L', L'')$. This property allows one to extend the definition of the barcode "by continuity" to the case where the manifolds are not transverse.

2.1.2. *Barcode entropy.* In this section we review the definition of barcode entropy introduced in [11]. Let L, L' be two transverse Lagrangians as in the previous section and let $\mathscr{B}(L, L')$ be the barcode of the Floer complex $\mathrm{CF}(L, L')$. Set

$$b_\epsilon(L, L') := \left| \left\{ \beta \in \mathscr{B}(L, L') \mid \beta > \epsilon \right\} \right|,$$

and denote the total number of bars in the barcode by

$$b(L, L') := |\mathscr{B}(L, L')| \geq b_\epsilon(L, L').$$

Omitting the definition of the barcode in the non-transverse case, we extend the barcode counting function $b_\epsilon(L, L')$ to the situation where L and L' need not be transverse by setting

$$b_\epsilon(L, L') := \liminf_{\tilde{L} \to L'} b_\epsilon(L, \tilde{L}) \in \mathbb{Z}. \tag{2.1}$$

Here the limit is taken over all Lagrangian submanifolds $\tilde{L} \pitchfork L$ which are Hamiltonian isotopic to L' and converge to L' in the C^∞-topology (or at least in the C^1-topology). As a consequence, $d_H(\tilde{L}, L') \to 0$, where d_H is the Hofer distance. Alternatively, we could have required $\tilde{L}$ be Hamiltonian isotopic to L, transverse to L' and converge to L. Since $b_\epsilon(L, L') \in \mathbb{Z}$, the limit in (2.1) is necessarily attained, i.e., there exists $\tilde{L}$ arbitrarily close to L' such that $b_\epsilon(L, L') = b_\epsilon(L, \tilde{L})$. Observe that definition (2.1) extends to the transverse case. Namely, one direction is a consequence of the "C^∞-stability" of essentially all the data related to $\mathrm{CF}(L, L')$; alternatively, though unnecessary, one can use the continuity of barcodes. As for the other direction, one can, for instance, take the constant sequence.

REMARK 2.1. In this paper we use the barcode counting function as a stable lower bound for the number of intersections. Namely, first of all, note that

$$|L \cap L'| = \dim_\Lambda \mathrm{CF}(L, L') = 2b(L, L') - \dim_\Lambda \mathrm{HF}(L, L') \geq b(L, L') \geq b_\epsilon(L, L')$$

whenever $L \pitchfork L'$. This directly follows from the definition of the barcode. In particular, in the transverse case, $b_\epsilon(L, L')$ gives a lower bound for the number of intersections:

$$|L \cap L'| \geq b(L, L') \geq b_\epsilon(L, L'). \tag{2.2}$$

Assume now that Lagrangians L, L' and L'' are Hamiltonian isotopic, $L'' \pitchfork L$ and $d_H(L', L'') < \delta/2$. Then, whether or not L and L' are transverse, we have

$$|L \cap L''| \geq b_\epsilon(L, L'') \geq b_{\epsilon+\delta}(L, L'). \tag{2.3}$$

Here the first inequality is just (2.2) and the second inequality, which holds regardless of $L \pitchfork L''$ or not, is the extension of the continuity property from the previous section via the limit (2.1) to the non-transverse case.

DEFINITION 2.2 (Relative Barcode Entropy)**.** The *barcode entropy of φ relative to* (L, L') is

$$\hbar(\varphi; L, L') := \lim_{\epsilon \searrow 0} \hbar_\epsilon(\varphi; L, L') \in [0, \infty],$$

where

$$\hbar_\epsilon(\varphi; L, L') := \limsup_{k \to \infty} \frac{\log^+ b_\epsilon\big(L, L^k\big)}{k} \text{ and } L^k := \varphi^k(L').$$

Here and throughout the paper, the logarithm is taken base 2 and $\log^+ = \log$ except that $\log^+ 0 = 0$. Note that $\hbar_\epsilon(\varphi; L, L')$ is increasing as $\epsilon \searrow 0$, and hence the limit exists, although *a priori* it could be infinite.

Next, we discuss the absolute barcode entropy. Let M be a closed monotone symplectic manifold and again let $\varphi\colon M \to M$ be a Hamiltonian diffeomorphism. Then we can apply the above constructions to $L = \Delta = L'$, the diagonal in the symplectic square $\big(M \times M, (-\omega, \omega)\big)$, with φ replaced by $id \times \varphi$, or directly to the Floer complex $\mathrm{CF}(\varphi)$ of φ *for all free homotopy classes of loops in M*. In the latter case we denote by $\mathscr{B}(\varphi)$ the resulting barcode. For instance, we have

$$\begin{aligned} b_\epsilon\big(\varphi^k\big) &:= b_\epsilon\big(L, L^k\big) \\ &= \Big|\Big\{\text{bars of length greater than } \epsilon \text{ in the barcode } \mathscr{B}\big(\varphi^k\big)\Big\}\Big|, \end{aligned}$$

where $L = \Delta$ and L^k is the graph of φ^k, and in the second equality we tacitly assumed that φ^k is non-degenerate. We emphasize that we include the 1-periodic orbits in all free homotopy classes of loops in M as generators of $\mathrm{CF}(\varphi)$ in contrast with a more common definition involving only contractible 1-periodic orbits. This is absolutely essential for the definition of barcode entropy.

DEFINITION 2.3 (Absolute Barcode Entropy)**.** The *ϵ-barcode entropy* of φ is

$$\hbar_\epsilon(\varphi) := \limsup_{k \to \infty} \frac{\log^+ b_\epsilon\big(\varphi^k\big)}{k}$$

and the *(absolute) barcode entropy* of φ is

$$\hbar(\varphi) := \lim_{\epsilon \searrow 0} \hbar_\epsilon(\varphi) \in [0, \infty]$$

or, in other words,

$$\hbar(\varphi) := \hbar(id \times \varphi; \Delta, \Delta).$$

Again, the limit in the definition of $\hbar(\varphi)$ exists since $\hbar_\epsilon(\varphi)$ is increasing as $\epsilon \searrow 0$.

By [11, Thm. 5.1], $\hbar(\varphi; L, L') \le \mathrm{h}_{\mathrm{top}}(\varphi) < \infty$, and hence $b_\epsilon(L, L^k)$ grows at most exponentially:

$$b_\epsilon\big(L, L^k\big) \le 2^{ck} \text{ for large } k, \tag{2.4}$$

where we can take any $c > \mathrm{h}_{\mathrm{top}}(\varphi)$. In particular, $\hbar(\varphi) \le \mathrm{h}_{\mathrm{top}}(\varphi) < \infty$ and again

$$b_\epsilon\big(\varphi^k\big) \le 2^{ck} \tag{2.5}$$

for large k. At the same time, the number of periodic points of φ, and hence $b\big(\varphi^k\big)$, can grow arbitrarily fast and, as a consequence, the shortest bar can also go to zero arbitrarily fast; see [2] and Section 3.3. One of our goals in this paper is to refine (2.4), (2.5) and [11, Thm. 5.1].

REMARK 2.4. Although Definition 2.3 closely resembles the definition of topological entropy, the similarity is rather deceiving. For instance, the ϵ-entropy family $\hbar_\epsilon(\varphi)$ is well-defined for every $\epsilon > 0$, while its counterpart for topological entropy depends on the background metric. Note that as a consequence when $\hbar(\varphi) > 0$ we obtain a new numerical invariant of φ: the threshold value of ϵ for which the entropy is positive, $\sup\{\epsilon > 0 \mid \hbar_\epsilon(\varphi) > 0\}$.

2.2. **Barcode growth and sequential entropy.** In this section we consider the growth of $b_{\epsilon_k}\big(L, L^k\big)$ for a certain class of sequences $\epsilon_k > 0$. To be more precise, a bounded sequence $\epsilon_k > 0$ is said to be *subexponential* if

$$\epsilon_k 2^{\eta k} \to \infty \text{ for all } \eta > 0$$

or, equivalently,

$$\lim_{k\to\infty} \frac{\log^+ \epsilon_k}{k} = 0. \tag{2.6}$$

For instance, a constant sequence or a polynomially decaying sequence is subexponential. For a sequence $\epsilon_k > 0$, define the *relative sequential* $\{\epsilon_k\}$*-barcode entropy* to be

$$\hat{\hbar}_{\{\epsilon_k\}}(\varphi; L, L') := \limsup_{k\to\infty} \frac{\log^+ b_{\epsilon_k}\big(L, L^k\big)}{k} \in [0, \infty] \text{ with } L^k := \varphi^k(L').$$

Furthermore, let us partially order positive sequences by $\{\epsilon'_k\} \preceq \{\epsilon_k\}$ whenever $\epsilon'_k \le \epsilon_k$ for all large $k \in \mathbb{N}$. Clearly,

$$\hat{\hbar}_{\{\epsilon'_k\}}(\varphi; L, L') \ge \hat{\hbar}_{\{\epsilon_k\}}(\varphi; L, L') \text{ when } \{\epsilon'_k\} \preceq \{\epsilon_k\}. \tag{2.7}$$

Then, in the setting of Section 2.1, we define the *relative sequential barcode entropy* as

$$\hat{\hbar}(\varphi; L, L') := \sup_{\{\epsilon_k\}} \hat{\hbar}_{\{\epsilon_k\}}(\varphi; L, L') \in [0, \infty],$$

where the supremum is taken over all subexponential sequences $\{\epsilon_k\}$. (In this definition one can replace the supremum by the direct limit with respect to the reversed partial ordering, i.e., as the sequences get closer and closer to zero.) The absolute sequential entropy $\hat{\hbar}(\varphi)$ and its $\{\epsilon_k\}$-counterpart $\hat{\hbar}_{\{\epsilon_k\}}(\varphi)$ are defined in a similar fashion. By (2.7), for every subexponential sequence $\epsilon_k \to 0$, we have

$$\hbar(\varphi; L, L') \leq \hat{\hbar}_{\{\epsilon_k\}}(\varphi; L, L') \leq \hat{\hbar}(\varphi; L, L') \text{ and } \hbar(\varphi) \leq \hat{\hbar}_{\{\epsilon_k\}}(\varphi) \leq \hat{\hbar}(\varphi). \tag{2.8}$$

(We are not aware of any examples where the inequality between $\hat{\hbar}$ and $\hbar$ is strict and hypothetically it is possible that the sequential barcode entropy is always equal to the barcode entropy.) Our key result is the following refinement of (2.4), (2.5) and [11, Thm. 5.1].

THEOREM 2.5. *Let $\{\epsilon_k\}$ be a subexponential sequence. Then $b_{\epsilon_k}\big(L, L^k\big)$ grows at most exponentially. Furthermore,*

$$\hat{\hbar}(\varphi; L, L') \leq \mathrm{h}_{\mathrm{top}}(\varphi).$$

Note that, as a consequence, $\hat{\hbar}_{\{\epsilon_k\}}(\varphi; L, L') < \infty$ and $\hat{\hbar}(\varphi; L, L') < \infty$ which is *a priori* not obvious. We prove Theorem 2.5 in Section 3. Here we only point out that the proof of Theorem 2.5 ultimately relies on Yomdin's theorem, [49], and in the theorem and throughout the paper all maps and submanifolds are assumed to be C^∞-smooth.

Applying Theorem 2.5 to $id \times \varphi$ and the diagonal, we arrive at a refinement of [11, Thm. A]:

COROLLARY 2.6. *Let $\{\epsilon_k\}$ be a subexponential sequence. Then $b_{\epsilon_k}\big(\varphi^k\big)$ grows at most exponentially, and*

$$\hat{\hbar}(\varphi) \leq \mathrm{h}_{\mathrm{top}}(\varphi).$$

As a consequence of Theorem 2.5 and Corollary 2.6, $b_{\epsilon_k}(L, L^k)$ and $b_{\epsilon_k}\big(\varphi^k\big)$ grow at most exponentially whenever the sequence $\{\epsilon_k\}$ is subexponential:

$$b_{\epsilon_k}\big(L, L^k\big) \leq 2^{ck} \text{ and } b_{\epsilon_k}\big(\varphi^k\big) \leq 2^{ck} \text{ for large } k,$$

where we can take any $c > \mathrm{h}_{\mathrm{top}}(\varphi)$. These inequalities refine (2.4) and (2.5).

Recall that $\hbar(\varphi) \geq \mathrm{h}_{\mathrm{top}}(\varphi|_K)$ for any (closed) hyperbolic subset K [11, Thm. B]. By (2.8), this lower bound, like any lower bound on $\hbar(\varphi)$, also holds for $\hat{\hbar}(\varphi)$. Furthermore, by [11, Thm. C], $\hbar(\varphi) = \mathrm{h}_{\mathrm{top}}(\varphi)$ when M is a surface, and thus we have the following result.

COROLLARY 2.7. *Assume that φ is a compactly supported Hamiltonian diffeomorphism of a surface. Then*

$$\hbar(\varphi) = \hat{\hbar}_{\{\epsilon_k\}}(\varphi) = \hat{\hbar}(\varphi) = \mathrm{h}_{\mathrm{top}}(\varphi), \tag{2.9}$$

whenever $\{\epsilon_k\}$ is subexponential and $\epsilon_k \to 0$.

To summarize, sequential barcode entropy has essentially the same key properties as the barcode entropy originally defined in [11]. For instance, properties (i)-(iv) from Prop. 4.4 therein holds for sequential barcode entropy too. A possible exception is the Hofer lower semi-continuity of the relative barcode entropy in the Lagrangian (part (v) of Prop. 4.4). Unlike the properties (i)–(iv), the proof of (v) from [11] does not carry over to the sequential case. On the other hand, as we mentioned above, it is entirely possible that in general the two entropies are equal. Yet, Corollary 2.7 in its full form certainly does not generalize to higher dimensions. Namely, there are examples of Hamiltonian diffeomorphisms where $\dim M \geq 6$ and the last equality in (2.9) turn into a strict inequality with $\mathrm{h_{top}}(\varphi) > 0$ and $\hbar(\varphi) = \hat{\hbar}_{\{\epsilon_k\}}(\varphi) = \hat{\hbar}(\varphi) = 0$; see [9].

REMARK 2.8 (Topological sequential entropy)**.** One could also modify the definition of topological entropy in a way similar to sequential barcode entropy. The resulting "sequential" topological entropy is equal to the topological entropy for C^∞-maps of compact manifolds M. This is a consequence of Yomdin's theory [7, Prop. 3.10]. (We are grateful to David Burguet for explaining to us the connection and a proof of the equality.) On the other hand, it is not hard to construct a C^0-map with zero topological entropy and, for instance, infinite "sequential" topological entropy.

Namely, let (M, g) be a closed Riemannian manifold of $\dim M \geq 2$ and $B_j \subset M$ be a sequence of disjoint balls of subexponentially decreasing radius $\delta_j \to 0$. We can take any such sequence δ_j. For each $j \in \mathbb{N}$, there is a diffeomorphism φ_j of M (which can be taken to be a Hamiltonian diffeomorphism if the manifold is symplectic) supported in B_j with $\mathrm{h_{top}}(\varphi_j) = 0$ and such that the maximal number of δ_j/j-separated points with respect to the metric $d_k^{\varphi_j}(x, y) := \max_{0\leq i\leq k-1} d_g(\varphi_j^i(x), \varphi_j^i(y))$ is greater than j^j for $k = j$, and hence all $k \geq j$. To construct such a diffeomorphism, we first show that there exist j^j disjoint finite sequences of j points $x_0^s, \dots, x_{j-1}^s$, $s = 1, \dots, j^j$, in B_j which are δ_j/j-separated, i.e., $\max_i d_g\big(x_i^s, x_i^{s'}\big) > \delta_j/j$ whenever, $s \neq s'$. Then we define $\varphi_j \colon B_j \to B_j$ on disjoint path-connected neighborhoods of these sequences, turning the sequences into orbits and making sure that the resulting map φ_j has zero topological entropy. Let $\varphi\colon M \to M$ be the map given by composing all φ_j or, equivalently, taking their "disjoint union". By construction, the "sequential" topological entropy of φ is infinite but $\mathrm{h_{top}}(\varphi) = 0$. (If, instead of j^j, we took 2^j sequences, we would get a map with positive, but possibly finite, "sequential" topological entropy and zero topological entropy.)

REMARK 2.9 (Lower bounds on the growth of $b_\epsilon(\varphi^k)$)**.** A related question is that of a lower bound on $b_\epsilon(\varphi^k)$ or $b_\epsilon(L, L^k)$, although it is not entirely clear how to pose this question in a meaningful way. The difficulty is that one cannot expect any particular growth behavior without additional conditions on φ or M. For instance, obviously $b_\epsilon(\varphi^k) = n+1$ for all $\epsilon > 0$ when φ is a non-degenerate pseudo-rotation of $\mathbb{CP}^n$; see also Proposition 2.17. We refer the reader to Section 2.4 for more details on pseudo-rotations. Furthermore, when $\dim M = 2$ and φ is autonomous, or even integrable, we conjecture that $b_\epsilon(\varphi^k)$ grows at most polynomially with k. On the other hand, in all dimensions, when φ has a locally maximal hyperbolic subset K with $\mathrm{h_{top}}(\varphi|_K) > 0$, the sequence $k \mapsto b_\epsilon(\varphi^{kN})$ grows exponentially for some $N \in \mathbb{N}$ and $\epsilon > 0$. This is a consequence of [29, Thm. 18.5.6] and [11, Prop. 3.8 and 6.2]. Therefore, by [31], $b_\epsilon\big(\varphi^{kN}\big)$ grows exponentially for C^∞-generic φ when $\dim M = 2$. (To be more precise, the set of φ such that this sequence grows exponentially for some $\epsilon > 0$ and N depending on φ is C^∞-residual.) When the genus of M is positive we do not have any example of a strongly non-degenerate φ with $\mathrm{h_{top}}(\varphi) = 0$, although we believe that such Hamiltonian diffeomorphisms exist. It is not clear what growth of $b_\epsilon\big(\varphi^k\big)$ one should expect in this case.

2.3. **Lower bounds on the γ-norm.** As observed in [11, Sect. 6.1.5] our results on barcode and topological entropy and their proofs yield as a byproduct lower bounds on the spectral norm a.k.a. γ-norm of the iterates of φ. One of such results is Proposition 2.10, stated below, which refines [11, Prop. 6.5].

Recall that when M^{2n} is a closed symplectic manifold the γ-norm of a Hamiltonian diffeomorphism φ is defined as

$$\gamma(\varphi) = \inf\left\{ \mathrm{c}(H) + \mathrm{c}\left(H^{\mathrm{inv}}\right) \mid \varphi = \varphi_H \right\},$$

where H^{inv} is the Hamiltonian generating the flow $\left(\varphi_H^t\right)^{-1}$ and c is the spectral invariant associated with the fundamental class $[M] \in \mathrm{H}_{2n}(M)$. Then

$$\gamma(\varphi) \le \|\varphi\|_H.$$

We refer the reader to, e.g., [15, 34, 35, 39, 48] for the proof and a further discussion of the γ-norm.

PROPOSITION 2.10. *Let $\varphi\colon M \to M$ be a Hamiltonian diffeomorphism of a closed weakly monotone symplectic manifold with more than $\dim \mathrm{H}_*(M)$ hyperbolic periodic points. Then the sequence $\gamma\big(\varphi^k\big)$, $k \in \mathbb{N}$, is bounded away from zero.*

We prove this proposition in Section 3.4. Note that having more than $\dim \mathrm{H}_*(M)$ hyperbolic periodic points, or more than any fixed number of hyperbolic periodic points, is an open property in C^1-topology. By the Conley conjecture [19, 20, 38] every Hamiltonian diffeomorphism φ of a positive genus surface $\Sigma_{g\ge 1}$ has infinitely many periodic points. By the Lefschetz formula, roughly speaking, at least half of these periodic points are hyperbolic if φ is strongly non-degenerate, which is the case C^∞-generically (recall that, in dimension two, an elliptic or a negative hyperbolic fix point has Lefschetz index 1 and a positive hyperbolic one has index -1). Hence we have:

COROLLARY 2.11. *There exists a C^1-open neighborhood $U \subset Ham(\Sigma_{g\ge1}, \omega)$ of the set of strongly non-degenerate Hamiltonian diffeomorphisms of $\Sigma_{g\ge1}$ such that the sequence $\gamma\big(\varphi^k\big)$, $k \in \mathbb{N}$, is bounded away from zero for every $\varphi \in U$. In particular, this sequence is bounded away from zero C^∞-generically.*

REMARK 2.12. For a strongly non-degenerate Hamiltonian diffeomorphism φ of $M = S^2$, the sequence $\gamma\big(\varphi^k\big)$ is not bounded away from zero if and only if φ is a pseudo-rotation. Indeed, this sequence contains a subsequence converging to zero for all, not necessarily non-degenerate, pseudo-rotations of $\mathbb{CP}^n$; see [22]. In the opposite direction, when $M = S^2$, the existence of one hyperbolic periodic point is enough to bound the sequence $\gamma(\varphi^k)$ away from zero. Strictly speaking, we need a positive hyperbolic periodic point but the positivity can always be achieved up to passing to an even iteration. Hence, more generally, without any non-degeneracy assumption, if this is not the case for φ, then all periodic points of φ are elliptic (by our conventions 1 is an elliptic eigenvalue, hence in dimension two, all degenerate fixed points are elliptic). For strongly non-degenerate Hamiltonian diffeomorphisms φ, by the Lefschetz formula, this forces φ to be a pseudo-rotation.

The conditions of the proposition are satisfied by an even wider margin when φ has a hyperbolic invariant set with positive topological entropy. (We refer the reader to, e.g., [29, Sect. 6] for the definition and a detailed discussion of *hyperbolic* invariant sets. Here, all such sets are required to be compact by definition.) To be more precise, recall that a compact invariant set K of φ is said to be *locally maximal* if there exists a neighborhood $U \supset K$ such that K is the maximal invariant subset of U or, in other words, $x \in K$ whenever the entire orbit $\{\varphi^k(x) \mid k \in \mathbb{Z}\}$ through x is contained in U. For instance, the orbit $K = \{\varphi^k(x)\}$ of a hyperbolic periodic point x is locally maximal. By [3, Thm. 3.3], whenever φ has a hyperbolic invariant set K it also has a locally maximal hyperbolic invariant set K' with $\mathrm{h_{top}}\left(\varphi|_{K'}\right)$ arbitrarily close to $\mathrm{h_{top}}\left(\varphi|_{K}\right)$. In particular, $\mathrm{h_{top}}\left(\varphi|_{K'}\right) > 0$ if $\mathrm{h_{top}}\left(\varphi|_{K}\right) > 0$, and $\varphi|_{K'}$ has infinitely many (hyperbolic) periodic orbits; cf. [29, Thm. 18.5.6]. Thus we have proved the following.

COROLLARY 2.13. *Assume that a Hamiltonian diffeomorphism $\varphi\colon M \to M$ has a hyperbolic invariant set K with $\mathrm{h_{top}}\left(\varphi|_{K}\right) > 0$. Then the sequence $\gamma\left(\varphi^k\right)$, $k \in \mathbb{N}$, is bounded away from zero.*

By the results from [28], φ always has a hyperbolic invariant set when $\dim M = 2$ and $\mathrm{h_{top}}(\varphi) > 0$. Furthermore, C^∞-generically $\mathrm{h_{top}}(\varphi) > 0$ in dimension two as is proved in [31]. Therefore, we have the following (cf. Corollary 2.11 and Remark 2.12).

COROLLARY 2.14. *Let φ be a Hamiltonian diffeomorphism of a surface M with $\mathrm{h_{top}}(\varphi) > 0$. Then the sequence $\gamma\left(\varphi^k\right)$, $k \in \mathbb{N}$, is bounded away from zero. In particular, again, this sequence is bounded away from zero C^∞-generically in dimension two.*

REMARK 2.15. Actually, to derive from Proposition 2.10 the fact that the sequence $\gamma\left(\varphi^k\right)$ is bounded away from zero C^∞-generically in dimension two, we do not need to invoke results from [19, 20, 38] or [31]. Indeed, note that whenever φ has an elliptic periodic point one can create a horseshoe and hence infinitely many hyperbolic periodic points by a C^∞-small perturbation. This is a consequence of the Birkhoff–Lewis theorem. Then, in the hypothetical situation where φ does not have elliptic periodic points, a standard index argument shows that it must have infinitely many hyperbolic periodic points.

REMARK 2.16. The converse of Corollary 2.14 (or Proposition 2.10) is not true in general. Even in dimension two, the sequence $\gamma\left(\varphi^k\right)$, $k \in \mathbb{N}$, can be bounded away from zero when $\mathrm{h}_{\mathrm{top}}(\varphi) = 0$. It is easy to construct an autonomous Hamiltonian diffeomorphism φ of a surface of positive genus with no hyperbolic periodic points and $\gamma(\varphi^k) \to \infty$. For instance, as in [39, Ex. 5.6], one can take $H = \sin(2\pi\theta)$ where θ is the first angular coordinate on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. This is of course impossible for $M = S^2$ because the γ-norm in this case is bounded from above; see [15] and also [41, 42]. Interestingly, other than this fact and Remark 2.12 and Corollary 2.14 essentially nothing seems to be known about the behavior of the γ-norm under iterations when $M = S^2$. For instance, we do not know whether the converse of Corollary 2.14 (or Proposition 2.10) is true for $M = S^2$, or even if the sequence $\gamma(\varphi^k)$ is bounded away from zero when the Hamiltonian is autonomous and a convex or concave function of the latitude.

The reader can also find further results, based on [13], on the generic growth of the γ-norm in [12].

2.4. **Approximate identities and Hamiltonian pseudo-rotations.** In this section, looking at the results from Section 2.3 from a different perspective, we focus on two classes of maps with subexponential growth of b_ϵ: γ-approximate identities and Hamiltonian pseudo-rotations of $\mathbb{CP}^n$.

Defining approximate identities, it is useful to work in a greater generality than needed for our purposes. Consider a class of compactly supported diffeomorphisms φ of a smooth manifold M (e.g., all such diffeomorphisms or, as above, compactly supported Hamiltonian diffeomorphisms, etc.), equipped with some metric, e.g., the C^0- or C^1- or C^r-metric or the γ-metric in the Hamiltonian case which we are interested in here. The norm $\|\varphi\|$ is by definition the distance from φ to the identity. Following [23], we will call φ a $\|\cdot\|$-*approximate identity*, or a $\|\cdot\|$-*a.i.* for the sake of brevity, if $\varphi^{k_i} \to id$ with respect to $\|\cdot\|$ for some sequence $k_i \to \infty$. We will often suppress the norm in the notation. In dynamics, a.i.'s are usually referred to as *rigid* maps which sometimes clashes with the same term used for structural stability. (We believe that a confusion with approximate identities in analysis is unlikely.) Approximate identities have been extensively studied, although usually from a perspective different than ours; see, e.g., [4, 17, 18] and also [23] for further references.

The definition can be refined or modified in several ways and one such refinement is of particular interest to us. Namely, for a given $\epsilon > 0$, consider the iterations φ^{k_i} such that

$$\left\|\varphi^{k_i}\right\| < \epsilon.$$

Thus $k_i = k_i(\epsilon)$ is a strictly increasing sequence. Then φ is said to be $\|\cdot\|$-*almost periodic* if for every $\epsilon > 0$ the sequence k_i is *quasi-arithmetic*, i.e., the difference between any two consecutive terms is bounded by a constant, possibly depending on ϵ. Almost periodic maps are closely related to compact group actions on M: φ is C^0-almost periodic if and only if the family $\{\varphi^k\}$ is equicontinuous and thus generates a compact abelian group of (compactly supported) homeomorphisms, [24].

Almost periodicity and rigidity (the C^0-a.i. condition) impose strong restrictions on the dynamics of φ. For instance, a C^0-a.i. clearly cannot be topologically mixing and every point of M belongs to its ω- and α-limit sets. In particular, a C^0-a.i. cannot have hyperbolic invariant sets. (See, e.g., [4, 23] for more details, examples and references.) Furthermore, as pointed out in [4, p. 681], any C^0-a.i. has zero topological entropy, although this fact is not immediately obvious.

In the Hamiltonian setting, it is natural to consider a.i.'s and almost periodic maps φ with respect to the γ-norm and we are concerned here with the effect of these conditions on the dynamics of φ. For instance, by Corollaries 2.13 and 2.14, a γ-a.i. φ cannot have locally maximal hyperbolic invariant sets with positive entropy and in dimension two we necessarily have $\mathrm{h}_{\mathrm{top}}(\varphi) = 0$. Here we focus on barcode entropy and the barcode growth.

PROPOSITION 2.17. *Let φ be a γ-almost periodic Hamiltonian diffeomorphism of a closed monotone symplectic manifold M and let L and L' be closed Lagrangian submanifolds of M as in Section 2.1. Then for every $\epsilon > 0$ the sequences $b_\epsilon\big(L, L^k\big)$ and $b_\epsilon\big(\varphi^k\big)$ are bounded. In particular, $\hbar(\varphi; L, L') = 0$ and $\hbar(\varphi) = 0$.*

We prove this proposition, which is an easy consequence of a result from [30], in Section 3.5 where we also show that $\hbar(\varphi; L, L') = 0$ and $\hbar(\varphi) = 0$ for γ-a.i.'s under a certain growth condition on the sequence k_i; see Proposition 3.9. We conjecture that this is true for all γ-a.i.'s.

A C^0-almost periodic Hamiltonian diffeomorphism or C^0-a.i. is automatically γ-almost periodic or, respectively, γ-a.i. when M is symplectically aspherical, [6], and also for some other classes of monotone symplectic manifolds M including $\mathbb{CP}^n$, [41]. However, we are not aware of any example of Hamiltonian C^0-a.i.'s on a symplectically aspherical manifold and hypothetically such maps do not exist. (See [37] for the proof in the C^1-case and [23] for a further discussion.)

To date, the only known examples of Hamiltonian γ-a.i.'s or γ-almost periodic Hamiltonian diffeomorphisms (beyond those coming from torus actions) are *Hamiltonian pseudo-rotations* φ of $\mathbb{CP}^n$ (see [22, 23] and also [26]), although one can expect the same to be true for Hamiltonian pseudo-rotations of many other manifolds. A Hamiltonian pseudo-rotation is a Hamiltonian diffeomorphism with the minimal possible number of periodic points, where minimality is interpreted in the spirit of Arnold's conjecture. The actual definitions vary in general (see [10, 22, 40]), but for $M = \mathbb{CP}^n$ the requirement is that φ has exactly $n+1$ periodic points, which are then necessarily the fixed points. Moreover, by [22] and [40], all periodic points have one-dimensional local Floer homology, in particular, $b_\epsilon\big(\varphi^k\big) = n+1$ for any $\epsilon > 0$.

The simplest example of a Hamiltonian pseudo-rotation is a generic element in a Hamiltonian circle or torus action with isolated fixed points. However, in general, Hamiltonian pseudo-rotations can have very interesting dynamics. For instance, Hamiltonian (a.k.a. area-preserving, in this case) pseudo-rotations of $S^2 = \mathbb{CP}^1$ with exactly three invariant measures, which are then the two fixed points and the area form, were constructed in [1]; see also [17]. This construction was extended to symplectic toric manifolds of any dimension in [32].

As an immediate consequence of Proposition 2.17, we have the following.

COROLLARY 2.18. *Let φ be a Hamiltonian pseudo-rotation of $\mathbb{CP}^n$ and let L and L' be Lagrangian submanifolds of $\mathbb{CP}^n$ as in Section 2.1. Then for every $\epsilon > 0$ the sequence $b_\epsilon\big(L, L^k\big)$ is bounded. In particular, $\hbar(\varphi; L, L') = 0$.*

We do not know if we can replace barcode entropy by sequential barcode entropy in Proposition 2.17 and Corollary 2.18. Note also that the absolute counterpart of Corollary 2.18 is obvious in contrast with Proposition 2.17: $b_\epsilon\big(\varphi^k\big) = n+1$ for any $\epsilon > 0$, and hence $\hat{\hbar}(\varphi) = \hbar(\varphi) = 0$, for a pseudo-rotation φ of $\mathbb{CP}^n$.

We conjecture that $\mathrm{h_{top}}(\varphi) = 0$ for any Hamiltonian pseudo-rotation of $\mathbb{CP}^n$, and Corollary 2.18 provides some indirect evidence supporting this conjecture. In dimension two, the conjecture follows immediately from, e.g., the results in [28] asserting that any area-preserving positive-entropy $C^{1+\alpha}$-diffeomorphism of a compact surface must have a horseshoe, and hence has infinitely many periodic points. Furthermore, Hamiltonian pseudo-rotations φ of D^2 or $\mathbb{CP}^n$ satisfying a certain additional condition on the rotation number or the rotation vector are known to be C^0-a.i.'s; see [5] and also [4] for D^2 and [22] for $\mathbb{CP}^n$ and [26] for Anosov-Katok pseudo-rotations. Thus, in this case, $\mathrm{h_{top}}(\varphi) = 0$ by the observation from [4] mentioned above.

3. PROOFS AND REFINEMENTS

In this section we prove Theorem 2.5, Proposition 2.10 and also refine and prove Proposition 2.17. Along the way we discuss some other ways to measure barcode growth. The proof of Theorem 2.5 hinges on a construction from [11], which we call a Lagrangian tomograph and describe next.

3.1. **Lagrangian tomograph.**

Let L be a closed Lagrangian submanifold of a symplectic manifold M^{2n}. A *Lagrangian tomograph* is a map $\Psi\colon B \times L \to M$, where $B = B^d$ is a ball of possibly very large dimension d, which satisfies the following properties:

(i) The map Ψ is a submersion onto its image, the maps $\Psi_s := \Psi|_{\{s\}\times L}$ are smooth embeddings for all $s \in B$ and $\Psi_0 = \iota_L$ where $\iota_L\colon L \to M$ is the inclusion map;

(ii) The images $L_s = \Psi\big(\{s\} \times L\big)$ are Lagrangian submanifolds of M Hamiltonian isotopic to L.

Thus a Lagrangian tomograph is a family of Lagrangian submanifolds L_s which are parametrized by a ball B and meet some additional requirements. We call $d = \dim B$ the *dimension of the tomograph.* Note that we have $L_0 = L$ by (i). A Lagrangian tomograph always exists for any closed Lagrangian submanifold L. In fact, a Lagrangian tomograph of dimension d exists if and only if L admits an immersion into $\mathbb{R}^d$; see [11, Lemma 5.6]. We will need the following lemma.

LEMMA 3.1. *For some $C_H > 0$ depending only on the tomograph, we have*

$$d_H(L_0, L_s) \le C_H \|s\|. \tag{3.1}$$

Proof. Since B is compact it suffices to show that (3.1) holds when $\|s\|$ is small. Fix a Weinstein neighborhood of $L = L_0$. Then, near $s = 0$, each Lagrangian L_s is given by the graph of some exact form α_s. Let $\alpha_s = df_s$ be a smooth family of primitives; see [16]. (Note that, since Ψ is smooth, the family α_s is smooth in $s \in B$.) The claim (3.1) follows from the following two inequalities:

$$d_H(L_0, L_s) \le \max_L f_s - \min_L f_s \le C_H \|s\|$$

for some constant $C_H > 0$. Here, in the first inequality one can take $\pi^* f_s \colon T^*L \to \mathbb{R}$, where $\pi\colon T^*L \to L$ is the projection map, as the generating Hamiltonian. (The first inequality turns into equality when $\|s\|$ is sufficiently small, [33], but we do not need this fact.) The second inequality follows from the smoothness of the family f_s. □

Next, let $\tilde{L}$ be a closed n-dimensional submanifold of M. Set

$$N(s) := |L_s \cap \tilde{L}| \in [0, \infty].$$

Since Ψ is a submersion, $\Psi_s \pitchfork \tilde{L}$ for almost all $s \in B$. Hence $N(s) < \infty$ almost everywhere and N is an integrable function on B.

Fix an auxiliary Riemannian metric on M and let ds be a smooth measure on B, e.g., the standard Lebesgue measure. The key to the proof of Theorem 2.5 is the following observation.

LEMMA 3.2 (Crofton's inequality; Lemma 5.3 in [11])**.** *We have*

$$\int_B N(s)\, ds \le C_{CR} \cdot \mathrm{vol}(\tilde{L}),$$

where the constant C_{CR} depends on ds, Ψ and the metric on M, but not on $\tilde{L}$.

Of course, the lemma holds without the requirement that the submanifolds L_s are Lagrangian – Condition (i) is sufficient. However, Condition (ii) is essential for the rest of the proof and hence we included it in the definition of a Lagrangian tomograph.

3.2. **Proof of Theorem 2.5.** As are many arguments of this type, the proof is ultimately based on Yomdin's theorem, [49], and quite similar to the proof of [11, Thm. 5.1]. To prove the theorem, it suffices to show that

$$\mathrm{h_{top}}(\varphi) \ge \hat{\hbar}_{\{\epsilon_k\}}(\varphi; L, L') \tag{3.2}$$

for every subexponential sequence $\{\epsilon_k\}$ with $\hat{\hbar}_{\{\epsilon_k\}}(\varphi; L, L') > 0$. We will further assume that $b_{\epsilon_k}\big(L_0, L^k\big) > 0$ for all $\epsilon_k \in \{\epsilon_k\}$. This can be always achieved without changing the growth rate via passing to a subsequence.

Set $L^k := \varphi^k(L')$. By Lemma 3.2, we have a sequence of integrable functions N_k on B such that

$$\int_B N_k(s)\, ds \le C_{CR} \cdot \mathrm{vol}\big(L^k\big),$$

where the constant C_{CR} is independent of k.

For any sequence of balls $B_k \subset B$ of radius δ_k centered at the origin, we have the following chain of inequalities:

$$C_{CR}\,\mathrm{vol}\left(L^k\right) \geq \int_B N_k\,ds \geq \int_{B_k} N_k\,ds \geq \int_{B_k} b_{\epsilon_k/2}\left(L_s, L^k\right)ds,$$

where in the last inequality we used (2.2). Let C_H be the constant from (3.1). By (2.3),

$$b_{\epsilon_k/2}\left(L_s, L^k\right) \geq b_{\eta_k}\left(L_0, L^k\right)$$

with $\eta_k = \epsilon_k/2 + 2C_H\delta_k$. Then, setting $\delta_k = \epsilon_k/4C_H$, we obtain the inequality

$$b_{\epsilon_k/2}\left(L_s, L^k\right) \geq b_{\epsilon_k}\left(L_0, L^k\right)$$

as long as $s \in B_k$. Therefore,

$$\int_{B_k} b_{\epsilon_k/2}\left(L_s, L^k\right)ds \geq \int_{B_k} b_{\epsilon_k}\left(L_0, L^k\right)ds = \mathrm{vol}\left(B_k\right) b_{\epsilon_k}\left(L_0, L^k\right),$$

where we took ds to be the Lebesgue measure. To summarize,

$$\mathrm{vol}\left(L^k\right) \geq C_{CR}^{-1}\,\mathrm{vol}\left(B_k\right) b_{\epsilon_k}\left(L_0, L^k\right). \tag{3.3}$$

Taking $\log^+$ of both sides and dividing by k, we have

$$\frac{\log^+ \mathrm{vol}\left(L^k\right)}{k} \geq \frac{\log^+ b_{\epsilon_k}\left(L_0, L^k\right)}{k} + d\cdot\frac{\log^+ \epsilon_k}{k} + O(1/k), \tag{3.4}$$

where $d = \dim B$. Due to the condition that $\{\epsilon_k\}$ is subexponential, i.e., (2.6), the second term on the right goes to zero as $k \to \infty$. Thus, passing to the limit, we have

$$\limsup_{k\to\infty} \frac{\log^+ \mathrm{vol}\left(L^k\right)}{k} \geq \hbar_{\{\epsilon_k\}}(\varphi; L, L').$$

By Yomdin's theorem, [49], the left hand side is bounded from above by $\mathrm{h}_{\mathrm{top}}(\varphi)$, and (3.2) follows, which concludes the proof of the theorem. □

REMARK 3.3. This argument actually tells us a little bit more than Theorem 2.5. Focusing on the case of absolute entropy for the sake of simplicity, observe that (3.3) holds whenever ϵ_k is sufficiently small. The threshold for ϵ_k is determined by the tomograph. Then, by (3.4), for any sequence $\{\epsilon_k\}$ which eventually becomes small enough, for instance whenever $\epsilon_k \to 0$, subexponential or not, we have

$$\mathrm{h}_{\mathrm{top}}(\varphi) + d\cdot\limsup \frac{|\log^+ \epsilon_k|}{k} \geq \hbar_{\{\epsilon_k\}}(\varphi).$$

3.3. **The shortest bar and total persistence.**

In this section we briefly discuss some other ways to measure the size of a barcode and relevant notions of barcode entropy. The first one is centered around the shortest bar.

PROPOSITION 3.4. *Let φ be a strongly non-degenerate Hamiltonian diffeomorphism $\varphi\colon M \to M$, where M is closed and weakly monotone; cf. Section 2.1. Denote by $\beta_k^{\min}$ the shortest bar for φ^k and by $p(k)$ the number of k-periodic points of φ. Then*

$$\mathrm{h}_{\mathrm{top}}(\varphi) \geq \limsup \frac{\log^+ p(k) - d \cdot \left|\log^+ \beta_k^{\min}\right|}{k}, \tag{3.5}$$

where we can take as d the minimal dimension of the Euclidean space which M can be immersed into.

To put this result in perspective, recall that when $\dim M > 2$ there is no clear-cut connection between the growth of $p(k)$ and topological entropy in either direction; see [27]. Proposition 3.4 resolves the problem to a certain extent by providing a lower bound for topological entropy in terms of the exponential growth rate of $p(k)$, but with a correction term coming from the decay of $\beta_k^{\min}$. Furthermore, even in dimension two, $p(k)$ can grow arbitrarily fast, even when φ is strongly non-degenerate; see [2, Thm. 1.2]. Then, by (3.5), $\beta_k^{\min}$ must go to zero superexponentially and (3.5) still provides some information.

Proof. Observe that
$$p(k) = 2b_\epsilon\left(\varphi^k\right) - \dim M$$
for all $\epsilon < \beta_k^{\min}$. Then, as in Remark 3.3, it is not hard to see from (3.4) with $\epsilon_k = \min\{c, \beta_k^{\min}\}$ that

$$k\,\mathrm{h}_{\mathrm{top}}(\varphi) + d \cdot \left|\log^+ \min\left\{c, \beta_k^{\min}\right\}\right| \geq \log^+ p(k) + o(k), \tag{3.6}$$

where $c > 0$ is the threshold mentioned in the remark. On the other hand, we have $\beta_k^{\min} \leq \|\varphi^k\|_H \leq k\|\varphi\|_H$; see [45]. It follows that

$$\left|\log^+ \min\left\{c, \beta_k^{\min}\right\}\right| = \left|\log^+ \beta_k^{\min}\right| + O(\log k). \tag{3.7}$$

Now, (3.5) follows from (3.6) and (3.7). □

Another way to measure the size of a barcode is by looking at the total persistence, i.e., the sum of the finite bar lengths taken to some power $\alpha > 0$. To be more specific, set
$$\sigma_\alpha(\varphi) = \sum \beta_i(\varphi)^\alpha \in [0, \infty],$$
where $\alpha > 0$ is fixed and the sum is over all finite bars in $\mathscr{B}(\varphi)$. This is a Floer theoretic variant of the total persistence; see, e.g., [14, 43] and references therein. Clearly, the above sum is finite when φ is strongly non-degenerate and this is the case we will focus on here. Then we introduce a family of total persistence barcode entropies

$$\hbar(\alpha, \varphi) := \limsup_{k \to \infty} \frac{\log^+ \sigma_\alpha\left(\varphi^k\right)}{k} \in [0, \infty].$$

In the strongly non-degenerate case, similarly to above, one can regard $\hbar(\alpha, \varphi)$ as the exponential growth rate of periodic points $p(k)$ counted with certain weights coming from the barcode.

PROPOSITION 3.5. *Let φ be a strongly non-degenerate Hamiltonian diffeomorphism of a closed and weakly monotone symplectic manifold M. Then, for $\hbar(\alpha,\varphi)$ defined as above:*

(i) *The function $\alpha \mapsto \hbar(\alpha,\varphi)$ is (non-strictly) decreasing.*
(ii) *We have $\hbar(\alpha,\varphi) \geq \hbar_\epsilon(\varphi)$ for all $\epsilon > 0$. As a consequence, $\hbar(\alpha,\varphi) \geq \hbar(\varphi)$.*
(iii) *For $\alpha \geq d$, where d is the dimension of a tomograph,*

$$\hbar(\alpha,\varphi) \leq \mathrm{h}_{\mathrm{top}}(\varphi) < \infty.$$

In the same vein, a variant of total persistence barcode entropy can be defined for a Lagrangian or a pair of Lagrangians, and a similar result holds in this setting.

COROLLARY 3.6. *Assume that M is a closed surface. Then, for $\alpha \geq 3$ and any strongly non-degenerate Hamiltonian diffeomorphism $\varphi\colon M \to M$, we have*

$$\hbar(\alpha,\varphi) = \hbar(\varphi) = \mathrm{h}_{\mathrm{top}}(\varphi).$$

Proof. Any closed surface can be immersed into $\mathbb{R}^3$ and even embedded when M is orientable. Hence, when M is such a surface, there exists a Lagrangian tomograph of dimension $d = 3$; see [11]. Now, for $\alpha \geq 3$, in dimension two we have the chain of (in)equalities

$$\hbar(\varphi) \leq \hbar(\alpha,\varphi) \leq \mathrm{h}_{\mathrm{top}}(\varphi) = \hbar(\varphi),$$

where the first two inequalities follow from the proposition and in the last equality we use [11, Thm. C]. Therefore, all three invariants are equal. □

REMARK 3.7. We do not know if it can happen that $\hbar(\alpha,\varphi) = \infty$ for some $0 < \alpha < d$, e.g., for $\alpha = 1$ which corresponds to the total bar length growth. But if it can, the infimum

$$\overline{\alpha}_\varphi := \inf\{\alpha > 0 \mid \hbar(\alpha,\varphi) < \infty\}$$

would be a new Hausdorff dimension–type invariant of φ associated with the barcodes $\mathcal{B}(\varphi^k)$.

Proof of Proposition 3.5. We start by introducing a "truncated" version of the invariant $\hbar(\alpha,\varphi)$. For $b > 0$, set

$$\sigma_\alpha^b(\varphi^k) = \sum \min\left\{b, \beta_i\left(\varphi^k\right)\right\}^\alpha$$

and let $\hbar^b(\alpha,\varphi)$ be the exponential growth rate (in k) of $\sigma_\alpha^b(\varphi^k)$. More precisely,

$$\hbar^b(\alpha,\varphi) := \limsup_{k\to\infty} \frac{\log^+ \sigma_\alpha^b\left(\varphi^k\right)}{k} \in [0,\infty].$$

Clearly,

$$\sigma_\alpha^b(\varphi) \leq \sigma_\alpha(\varphi),$$

and hence

$$\hbar^b(\alpha,\varphi) \leq \hbar(\alpha,\varphi).$$

We claim that in fact

$$\hbar^b(\alpha,\varphi) = \hbar(\alpha,\varphi) \tag{3.8}$$

for all $b > 0$ and $\alpha > 0$. Deferring the proof of (3.8) to the end, let us first prove the proposition.

For Part (i), we set $b = 1$ and observe that $\sigma^1_\alpha(\varphi^k)$ is decreasing in $\alpha > 0$. It follows that the growth rate $\hbar^1(\alpha,\varphi)$, and hence $\hbar(\alpha,\varphi)$ by (3.8), is also a decreasing function of $\alpha > 0$.

Furthermore,

$$\epsilon^\alpha b_\epsilon(\varphi) \le \sigma_\alpha(\varphi)$$

for all φ and $\epsilon > 0$. Applying this inequality to φ^k, taking $\log^+$ and passing to the limit we obtain Part (ii). (This argument is independent of (3.8).)

Next, let us focus on Part (iii). By Part (i), we may assume that $\alpha = d$. Let L^k be the graph of φ^k and L_0 the diagonal in $M \times M$. Thus for any $\epsilon > 0$, $b_\epsilon(\varphi^k) = b_\epsilon(L_0, L^k)$.

Fix a tomograph L_s about L_0. Then as in Section 3.2 we have

$$N_k(s) \ge b\big(L_s, L^k\big) \ge b_{C_H\|s\|}\big(L_s, L^k\big) \ge b_{3C_H\|s\|}\big(L_0, L^k\big) = b_{3C_H\|s\|}\big(\varphi^k\big)$$

whenever $L_s \pitchfork L^k$; see (2.3) and Lemma 3.1. Integrating both sides and using Lemma 3.2, we obtain

$$C_{CR}\,\mathrm{vol}\big(L^k\big) \ge \int_B N_k(s)\,ds \ge \int_B b_{3C_H\|s\|}\big(\varphi^k\big)\,ds. \tag{3.9}$$

A change of variables with $r = 3C_H\|s\|$ yields

$$\int_B b_{3C_H\|s\|}\big(\varphi^k\big)\,ds = C\int_0^{r_0} b_r\big(\varphi^k\big) r^{d-1}\,dr \tag{3.10}$$

for some $C > 0$ and $r_0 > 0$ independent of k. On the other hand, the truncated sum $\sigma^{r_0}_d\big(\varphi^k\big)$ can also be written as

$$\sigma^{r_0}_d\big(\varphi^k\big) = d\cdot\sum\int_0^{r_0} \mathbb{1}_{[0,\beta_i(\varphi^k))}(r)\,r^{d-1}\,dr = d\cdot\int_0^{r_0} b_r\big(\varphi^k\big) r^{d-1}\,dr, \tag{3.11}$$

where $\mathbb{1}_{[a,b)}(r)$ is the characteristic function of $[a,b)$. Here the second equality is a consequence of the identity

$$\sum \mathbb{1}_{[0,\beta_i(\varphi^k))}(r) = b_r(\varphi^k).$$

We combine (3.9), (3.10) and (3.11) to infer as in the proof of Theorem 2.5 that

$$\mathrm{h}_{\mathrm{top}}(\varphi) \ge \hbar^{r_0}(d,\varphi) = \hbar(d,\varphi).$$

It remains to prove the claim (3.8). As in (3.7), the claim relies on the linear upper bound $\beta_i\big(\varphi^k\big) \le k\|\varphi\|_H$; see [45]. More precisely, we have

$$\sum_{\beta_i(\varphi^k)>b} b^\alpha \le \sum_{\beta_i\left(\varphi^k\right)>b} \beta_i\big(\varphi^k\big)^\alpha \le \sum_{\beta_i(\varphi^k)>b} (k\|\varphi\|_H)^\alpha. \tag{3.12}$$

Here all three terms and, in particular, the first two have the same exponential growth rate. Now, (3.8) is a consequence of (3.12) and the general fact that

$$\limsup \frac{\log^+(p_k+q_k)}{k} = \max\left\{\limsup \frac{\log^+ p_k}{k}, \limsup \frac{\log^+ q_k}{k}\right\}, \tag{3.13}$$

which holds for any real sequences $p_k \geq 0$ and $q_k \geq 0$. Namely, set

$$p_k = \sum_{\beta_i(\varphi^k)>b} b^\alpha \quad \text{and} \quad p'_k = \sum_{\beta_i(\varphi^k)>b} \beta_i(\varphi^k)^\alpha,$$

and also

$$q_k = \sum_{\beta_i(\varphi^k)\leq b} \beta_i(\varphi^k)^\alpha.$$

Then

$$\sigma_\alpha^b(\varphi^k) = p_k + q_k \quad \text{and} \quad \sigma_\alpha(\varphi^k) = p'_k + q_k.$$

By (3.12), p_k and p'_k have the same exponential growth rate. Then it follows from (3.13) that $\sigma_\alpha^b(\varphi^k)$ and $\sigma_\alpha(\varphi^k)$ have the same exponential growth rate, too. □

REMARK 3.8. Just as in Theorem 2.5, a similar construction can also be carried out in the relative setting for a pair of Lagrangians and an analogue of Proposition 3.5 also holds in this case, with the same proof.

3.4. **Proof of Proposition 2.10.** For some $N \in \mathbb{N}$, φ has more than $\dim \mathrm{H}_*(M)$ hyperbolic N-periodic points. We denote the set of such points by $\mathcal{K}$. Thus $|\mathcal{K}| > \dim \mathrm{H}_*(M)$ and clearly $\mathcal{K}$ is a locally maximal hyperbolic set. Furthermore, every point in $\mathcal{K}$ is also ℓN-periodic for all $\ell \in \mathbb{N}$. Then, arguing as in the proof of [11, Thm. B] and using [11, Prop. 3.8 and 6.2], we conclude that for a sufficiently small $\epsilon > 0$ and any $\ell \in \mathbb{N}$,

$$b_\epsilon(\varphi^{\ell N}) > \dim \mathrm{H}_*(M),$$

and hence $\varphi^{\ell N}$ has a finite bar of length greater than $\epsilon > 0$.

Also recall that as is proved in [30, Thm. A], for any φ,

$$\beta_{\max}(\varphi) \leq \gamma(\varphi), \tag{3.14}$$

where the left-hand side is the *boundary depth*, i.e., the longest finite bar in the barcode of φ. Thus, for a sufficiently small $\epsilon > 0$,

$$\epsilon < \beta_{\max}(\varphi^{\ell N}) \leq \gamma(\varphi^{\ell N}). \tag{3.15}$$

Next, arguing by contradiction, assume that there exists a sequence $k_i \to \infty$ such that $\gamma(\varphi^{k_i}) \to 0$. We claim that when $k_i < k_j$ are large enough, the difference $k_j - k_i$ is not divisible by N. This will imply the proposition; for then the sequence k_i would contain an infinite subsequence with the difference between any two terms not divisible by N. This is impossible because there are only finitely many residues modulo N.

To prove the claim, assume the contrary:

$$k_j - k_i = \ell N.$$

Then by the triangle inequality for γ, we have

$$\gamma(\varphi^{\ell N}) \leq \gamma(\varphi^{k_j}) + \gamma(\varphi^{-k_i}).$$

Here, the right hand side becomes arbitrarily small when k_i and k_j are large, but the left hand side is bounded from below by $\epsilon > 0$ by (3.15). This contradiction concludes the proof of the proposition. □

3.5. **Proof and a refinement of Proposition 2.17.** Although we do not have any examples of γ-a.i.'s which are not pseudo-rotations, and hence γ-almost periodic when $M = \mathbb{CP}^n$, it is interesting to see that the zero-entropy part of the statement of the proposition holds under a less restrictive condition than γ-almost periodicity.

To state the result, let us assume that φ is a γ-a.i. and denote by $k_i = k_i(\epsilon)$ a strictly increasing sequence of integers such that $\gamma\big(\varphi^{k_i}\big) < \epsilon$. Furthermore, assume that there exists $\alpha < 1$ independent of ϵ and such that

$$k_{i+1} - k_i \le \alpha k_i, \tag{3.16}$$

when k_i is sufficiently large (depending on ϵ). For instance, all γ-almost periodic Hamiltonian diffeomorphisms meet this requirement.

PROPOSITION 3.9. *Let φ be a γ-a.i. satisfying* (3.16). *Then $\hbar(\varphi; L, L') = 0$ and $\hbar(\varphi) = 0$.*

Proof of Propositions 2.17 and 3.9. The absolute case of the propositions concerning $b_\epsilon\big(\varphi^k\big)$ and $\hbar(\varphi)$ follows from the relative case by setting $L = L'$ to be the diagonal in $M \times M$ and replacing φ by $id \times \varphi$. Hence we will focus on the relative case.

The key to the proof is [30, Thm. B] which asserts, roughly speaking, that one can replace the Hofer norm in (2.3) by the γ-norm. Namely, recall that for any two Lagrangian submanifolds L' and L'' Hamiltonian isotopic to each other the γ-distance between L' and L'' is defined as

$$\gamma(L', L'') := \inf\big\{\gamma(\psi) \mid \psi(L') = L''\big\} \le d_H(L', L'').$$

Then, as a consequence of [30, Thm. B], we have the following refinement of (2.3):

$$b_{\epsilon+\delta}(L, L') \le b_\epsilon(L, L'') \text{ when } \gamma(L', L'') < \delta/2. \tag{3.17}$$

In the setting of Proposition 2.17, fix $\epsilon > 0$. We claim that there exist $N \in \mathbb{N}$ such that

$$b_\epsilon\big(L, L^k\big) \le \max_{0 \le \ell \le N} b_{\epsilon/2}\big(L, L^\ell\big) \tag{3.18}$$

for all sufficiently large $k \in \mathbb{N}$. Indeed, since φ is γ-almost periodic, there exists a sequence of positive integers

$$k_1 < k_2 < k_3 < \dots$$

such that

$$\gamma\big(\varphi^{k_i}\big) < \epsilon/4 \text{ and } k_{i+1} - k_i \le N$$

for some N. Let $k \ge k_1$ and write $k = k_i + \ell$ with $0 \le \ell \le N$. Then

$$L^k = \varphi^{k_i}\big(L^\ell\big) \text{ with } \gamma\big(\varphi^{k_i}\big) < \epsilon/4.$$

Hence, by (3.17),

$$b_\epsilon\big(L, L^k\big) \le b_{\epsilon/2}\big(L, L^\ell\big)$$

for all $k \ge k_1$. This proves (3.18) and completes the proof of Proposition 2.17.

Turning to the proof of Proposition 3.9, for the sake of brevity, set

$$b_\epsilon(k) := b_\epsilon\big(L, L^k\big) \text{ and } \hbar_\epsilon := \hbar_\epsilon(\varphi; L, L').$$

Let k_i be a strictly increasing sequence of positive integers such that

$$\gamma\big(\varphi^{k_i}\big) < \epsilon/4 \text{ and } k_{i+1} - k_i \le \alpha k_i \text{ with } \alpha < 1.$$

As above, for $k \ge k_1$, write $k = k_i + \ell$ where now $0 \le \ell \le \alpha k_i$. Then

$$b_\epsilon(k) = b_\epsilon(k_i + \ell) \le b_{\epsilon/2}(\ell).$$

Furthermore, for any $\eta > \hbar_{\epsilon/2}$ and some constant C,

$$\log^+ b_{\epsilon/2}(\ell) \le \eta\ell + C \le \eta\alpha k_i + C \le \eta\alpha k + C.$$

Combining these inequalities, we have

$$\log^+ b_\epsilon(k) \le \eta\alpha k + C.$$

Dividing by k and passing to the upper limit as $k \to \infty$, we see that $\hbar_\epsilon \le \eta\alpha$ for all $\eta > \hbar_{\epsilon/2}$, and hence $\hbar_\epsilon \le \alpha\hbar_{\epsilon/2}$. Equivalently, $\hbar_{\epsilon/2} \ge \alpha^{-1}\hbar_\epsilon$ with $0 < \alpha < 1$. Iterating this argument, we conclude that

$$\hbar \ge \hbar_{2^{-i}\epsilon} \ge \alpha^{-i}\hbar_\epsilon \to \infty \text{ as } i \to \infty$$

unless $\hbar_\epsilon = 0$. Since $\hbar < \infty$, we must have $\hbar_\epsilon = 0$, and hence $\hbar = 0$. □

REMARK 3.10. As we have pointed out in Section 2.4, we do not know if we can replace barcode entropy by sequential barcode entropy in Propositions 2.17 and 3.9.

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