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Tutorial Article

An offender–defender safety game☆

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ABSTRACT

In this tutorial we study a safety analog of the classical zero-sum differential game with positive definite penalties on the state and the two inputs. Consider a nonlinear system affine in two inputs, which are called "offender" and "defender." Let the inputs have the opposing objectives in relation to an infinite-time cost which, in addition to penalizing the inputs of both agents, incorporates a safety index of the system (a barrier function), with the defender aiming to maximize the system safety and the offender aiming to minimize it. If there is a pair of (offender, defender) non-Nash feedback policies of the $L_g h$ form with a safe outcome, namely, where the defender maintains safety while the offender fails to violate safety, then there exists an inverse optimal pair of policies that attain a Nash equilibrium relative to the safety minimax objective. In the tutorial we study both deterministic and stochastic offenders. The deterministic offender applies its feedback through its deterministic input value, while the stochastic offender applies its feedback through its incremental covariance. In addition to Nash policies for a minimax offender–defender formulation, we provide feedback laws for the defender, in the scenario where the offender action is unrestricted by optimality, and where the defender ensures input-to-state safety in the deterministic and stochastic senses. This tutorial is derived from our recent article on inverse optimal safety filters, by setting the nominal control to zero and declaring the disturbance to be the offender agent.

Among several illustrative examples, one is particularly interesting and unconventional. We consider a safety game played on a unicycle vehicle between its two inputs: the angular velocity and the linear velocity, as the opposing players. We consider two scenarios. In the first, the angular velocity, acting as an offender, attempts to run the vehicle into an obstacle by steering, while the linear velocity, acting as a defender, drives the vehicle forward or in reverse to prevent the vehicle being run into the obstacle. In the second scenario, the linear velocity acts as an offender and angular velocity acts as a defender (in the deterministic case by varying the heading rate; in the stochastic case by varying the variance of a white noise driving the heading rate). A "wind" towards the obstacle advantages the offender in both scenarios. The input policies derived are optimal in the sense of their opposite objectives, under the best possible policy of the opponent, under meaningful costs on their actions. The linear velocity input prevails, whether acting in the role of a defender, in which case the collision with the obstacle is prevented, or in the role of an offender, in which case the collision with the obstacle is achieved.

1. Introduction

1.1. Goals of the tutorial

Even with as few agents as two, acting on a common system (rather than on two distinct systems), the problem of guaranteeing safety of a nonlinear dynamical system by feedback is a rich one.

In this tutorial, to keep things as clear and simple as possible, we eliminate any performance objectives, and hence, a "nominal control" and the associated safety filters, and focus on a problem in which two agents are engaged purely in a competition over a system's safety.

We consider two agents – an "offender" and a "defender" – whose goals regarding safety are opposite. The offender aims to violate the system's safety, while the defender's task is to maintain safety. In engaging in such a manner, both agents' goals are related not only to the same system but also to the same control barrier function (CBF) $h(x)$, where the defender's goal is to keep h positive and the offender's to make it negative, at least for a portion of the system's infinite-horizon operation.

If the offender is unrestricted in its action, the most that the defender can expect to achieve is that the safety degrades gracefully with the intensity of the offender's action. We provide defender feedback

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laws to ensure that. However, to make the offender–defender competition more "fair", and mathematically well posed, we formulate a zero-sum two-player non-cooperative game between the offender and defender, where the running cost for both agents is related to the CBF h, and where the input magnitudes of both agents are penalized over the infinite horizon. We seek optimal control policies for both the defender and the offender and find policies that constitute a Nash equilibrium. Since, for arbitrary costs, the determination of the Nash policies would require a solution of a Hamilton–Jacobi–Isaacs partial differential equation (PDE), we formulate the problem as an inverse optimal control problem, where the CBF h is given and the Nash policies are found, which correspond to a meaningful running cost that rewards the defender for enhancing safety and rewards the offender for eroding safety.

We consider both deterministic and stochastic offenders. In the stochastic case, the offender's action is white but its intensity (covariance) is a feedback law based on the safety game.

The tutorial contains theorems, proofs, and examples. Theorems are stated for the sake of clarity and precision. The proofs are given in the appendix, so as not to interfere with the flow of the tutorial exposition. The examples chosen to illustrate the results are the simplest possible. They are all scalar. Due to the relative complexity of the feedback policies that the defender and offender need to employ in the scenarios considered, scalar examples serve best the function of the illustration of the concepts presented.

1.2. The CBF framework for safety

Our study of a two-agent competition over the safety of a single system is pursued using a common CBF. Let us review some of the principal literature on CBFs first.

The paper Ames, Grizzle, and Tabuada (2014) and Ames, Xu, Grizzle, and Tabuada (2017) marked a watershed in the study of nonlinear control systems under state constraints. By advancing the CBF notion proposed in Wieland and Allgöwer (2007), it laid the foundation for a Lyapunov-like alternative to constraint-handling by MPC (Rawlings, Mayne, & Diehl, 2017) or barrier Lyapunov functions (BLF) (Tee, Ge, & Tay, 2009). Following the min-norm inspiration from Freeman and Kokotovic (1996), the authors of Ames et al. (2014, 2017) proposed to mitigate the safety-liveness tradeoff using a quadratic program (QP). Virtually all the work on CBF-based safety maintenance today employs QP-based redesigns of the nominal control, referred to as "safety filters". CBFs have since been used in a range of domains, including multi-agent robotics (Glotfelter, Corts, & Egerstedt, 2017; Santillo & Jankovic, 2021; Wang, Ames, & Egerstedt, 2017), automotive systems (Ames et al., 2014; Rahman, Jankovic, & Santillo, 2021; Xu, Grizzle, Tabuada, & Ames, 2018), robust safety (Jankovic, 2018; Kolathaya & Ames, 2019; Xu, Tabuada, Grizzle, & Ames, 2015), delay systems (Abel, Janković, & Krstić, 2020; Janković, 2018; Molnár, Singletary, Orosz, & Ames, 2021; Prajna & Jadbabaie, 2005), and stochastic systems (Clark, 2021; Prajna, Jadbabaie, & Pappas, 2007; Santoyo, Dutreix, & Coogan, 2021).

Since CBFs define constraints and, as such, represent system outputs, when paired with system inputs they have relative degrees. CBFs of high relative degree, under that name, were first studied in the 2015 articles (Hsu, Xu, & Ames, 2015; Wu & Sreenath, 2015) with progress following in Breeden and Panagou (2021), Nguyen and Sreenath (2016), Xiao and Belta (2019), Xu (2018) and continuing. However, control designs for specific CBFs of arbitrarily high relative degrees first appeared a decade earlier, in the 2006 article (Krstic & Bement, 2006), which presents backstepping designs for regulation to the boundary of the safe set, referred to, at that time, as 'non-overshooting control.' The backstepping design of CBFs originated in Krstic and Bement (2006) is currently experiencing a revival, from its use for stochastic nonlinear systems (Li & Krstic, 2020), to safety for PDEs (Koga & Krstic, 2023).

To reduce the conservativeness of classical asymptotic or exponential safety, the notion of prescribed-time safety was introduced in Abel, Steeves, Krstic, and Jankovic (2022), applied to robotics experiments (Bertino, Naseradinmousavi, & Krstic, 2023) and source seeking (Koga & Krstic, 2023), and extended to fixed-time safety with homogeneous/nonsmooth feedback (Polyakov & Krstic, 2022).

Safe extremum seeking with CBFs that are measured but not known analytically was introduced in Williams, Krstic, and Scheinker (2022).

1.3. $L_g h$ Safety filters and inverse optimality

Let $g(x)$ denote the control-affine system's input vector field. Then $L_g h$ denotes $\frac{\partial h}{\partial x} g$. CBF-QPs have $L_g h$ as a factor multiplying a nonnegative quantity. A factor of $L_g h$ is a tell-tale sign of potential optimality. The so-called "$L_g V$ controllers", where V is a CBF, have a storied history in nonlinear stabilization. Sontag's 'universal formula' (Sontag, 1989b) is an $L_g V$ controller. Sepulchre, Janković, and Kokotović (1997) produced a collection of results with such "damping controllers" and showed that every $L_g V$ controller is optimal with respect to a meaningful cost functional if multiplied by a factor of two or more, which, in particular, indicates the controller's infinite gain margin. Such properties of $L_g V$ controllers inspired their further development under uncertainties. In Krstic and Li (1998), for systems affine in control and disturbances, inverse optimal controllers were designed that solve a zero-sum game problem, in which the disturbance maximizes and the control minimizes a meaningful cost. In Ito and Freeman (2002) and Pan, Ezal, Krener, and Kokotovic (2001), global inverse optimality was augmented with local direct optimality. In Deng and Krstic (1997) and Deng, Krstic, and Williams (2001) stochastic inverse optimal designs were introduced: $L_g V$ controllers for inverse optimal stabilization in probability in Deng and Krstic (1997) and controllers that are inverse optimal for a zero-sum game relative to the unknown covariance acting as the opposing player in Deng et al. (2001). Finally, in Li and Krstic (1997), *adaptive* $L_g V$ controllers were designed that minimize a penalty not only on the plant's state and the input, but also on the parameter estimation error—thus far the only pairings of controllers and parameter estimators which are not merely optimal 'asymptotically' but over the entire time horizon.

The CBF-QP safety filters are only "pointwise optimal". Infinite-horizon optimality has been pursued in Almubarak, Sadegh, and Theodorou (2022), Almubarak, Theodorou, and Sadegh (2021), Chen, Ahmadi, and Ames (2020) and Cohen and Belta (2020) but only towards achieving optimal stabilization, not *optimal safety*. In Krstic (2023) we introduced a large variety of additional families of safety filters, of which some have infinite-horizon optimality properties with respect to safety. These safety filters maximize safety while minimizing the deviation of the control applied from the nominal control. The contents of this tutorial article are deduced from Krstic (2023) by setting the nominal control to zero, declaring the disturbance to be the offending agent, and treating the control in Krstic (2023) as the defending agent. The results of Krstic (2023) are probably more easily understood in the stripped-down format in this tutorial, with the issue of "liveness" eliminated from consideration and the control and disturbance inputs put on the same footing, as the offender and defender agents.

1.4. Disturbances as offender agents: Deterministic and stochastic

Under deterministic disturbances, two main ideas have emerged. Robust CBFs (Jankovic, 2018) ensure safety under a disturbance with a known bound. In Input-to-State Safety (ISSf) (Kolathaya & Ames, 2019), which mirrors input-to-state stability (ISS) (Sontag, 1989a), the disturbance is bounded but potentially arbitrarily large and, being also unvanishing, may take the system outside of the safe set, in proportion to the size of the disturbance. Controllers that render the safety violation proportional to the disturbance are introduced in the

2006 work on non-overshooting control (Krstic & Bement, 2006) with a backstepping design for a high relative degree CBF.

In the stochastic case, a general CBF-based safety analysis is presented in Clark (2021). A mean-non-overshooting tracking design for stochastic strict-feedback systems is given in Li and Krstic (2020).

1.5. Organization of tutorial

Sections 2–5 deal with the case where the offender agent is deterministic. The main result, which solves an offender–defender game over the system's safety, is in Section 5, following an introduction of min-norm safety designs for a defender countering an unrestricted offender in Section 4. Stochastic offenders are dealt with in Section 7, where noise-to-state safety subject to unrestricted incremental covariance of the offender is presented, and in 8, where a minimax game, played between a deterministic offender and a stochastic offender employing feedback through incremental covariance, is solved. Section 9 recapitulates the results and shows a side-by-side summary of the deterministic results versus stochastic results and min-norm versus Sontag-like designs.

A (unicycle) vehicle example plays a particularly important role in our presentation. However, rather than considering a two-agent example with two vehicles being the offender and defender agents, which would be a safety game of the conventional pursuit-evasion type, in Section 6 we consider an unconventional, and in our opinion intellectually intriguing safety game involving a single vehicle whose two inputs – linear velocity and angular velocity – are the two opposing players. In other words, the "agency", as offender and defender, is assigned to the two inputs on the same vehicle. In colloquial terms, one should imagine one actor having control over the steering wheel and the other actor over the accelerator and brake pedals, as well as the forward/reverse stick shift. In such a safety game, in which either input can have either the offender or defender role, can the defender always foil the objective of the offender to cause a collision with an obstacle? Precise and not necessarily obvious answers to this question are given in Examples 4, 5, 6, and 8. The bottom line is that steering has less influence than forward/reverse driving over the collision outcome in a game with a meaningful performance index in which the defender is rewarded for increasing safety, offender for decreasing safety, and the actions of both are penalized.

1.6. Notation

Let $a < 0 < b$. A continuous function $\gamma : (a,b) \to \mathbb{R}$ with $\gamma(0) = 0$ is of extended class $\mathcal{K}_{(a,b)}$ if it is strictly increasing. In particular, $\mathcal{K} = \mathcal{K}_{[0,+\infty)}$. A continuous function $\beta : (a,b) \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ is of class $\mathcal{KL}_{(a,b)}$ if it is of class $\mathcal{K}_{(a,b)}$ in its first argument and has a zero limit as its second argument goes to infinity.

2. Input-to-state safety, under offender agent input

We start with definitions of a barrier function and safe set.

Definition 1 (*Barrier Function*). The scalar-valued differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ with $\inf_{x\in\mathbb{R}^n} h(x) < 0$ and $\sup_{x\in\mathbb{R}^n} h(x) > 0$ is referred to as a *barrier function candidate*. The set $C = \{\, x \in \mathbb{R}^n \mid h(x) \geq 0\}$ without isolated points is referred to as a *safe set*.

Consider first a system driven input u_1,

$$\dot{x} = f(x) + g_1(x)u_1 , \quad u_1 \in \mathbb{R}^{m_1} , \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n, g_1 : \mathbb{R}^n \to \mathbb{R}^{n\times m_1}$, with an initial condition $x_0 = x(0)$. The sole (vector) input to this system, u_1, is a disturbance. This input's effect on the system may be to drive the state out of the safe set C. Hence, anticipating that in the sequel there will be an additional input tasked with maintaining safety, and counteracting the effect of the disturbance in a two-player game theoretic setting, we refer to the disturbance u_1 as an *offender* agent. The other input, to be labeled u_2 will be referred to as a *defender* agent.

Definition 2 (*ISSf*). The system (1) is said to be *input-to-state safe* (ISSf) on the set C if there exist $\rho \in \mathcal{K}$ and $\beta \in \mathcal{KL}_{(\inf h(\xi), \sup h(\xi))} =: \mathcal{KL}_h$ such that, for all initial conditions $x_0 \in \mathbb{R}^n$ and all locally bounded functions $u_1 : [0,+\infty) \to \mathbb{R}^{m_1}$, the resulting solution satisfies

$$h(x(t)) \geq \beta(h(x_0), t) - \rho\left(\sup_{0\leq\tau\leq t} |u_1(\tau)|\right) , \quad \forall t \geq 0 , \tag{2}$$

where the function ρ is referred to as the *ISSf gain function*.

This property is not new. Controller design ensuring ISSf, using backstepping for non-overshooting control, goes as far back as 2006 in the paper Krstic and Bement (2006)—see the safety bound (61) of Theorem 3 with a disturbance of unlimited unknown bound $\bar{d}$, as well as the safety bound (90) of Proposition 1 with an observer-based non-overshooting controller.

The following definition is a very slightly adjusted version of Lyu, Xu, and Hong (2022, Definition 4).

Definition 3 (*ISSf Barrier Function*). The function h is called an *ISSf barrier function* (ISSf-BF) if there exists a function $\rho : [0,+\infty) \to [0, -\inf h(\xi))$ of class $\mathcal{K}$ and a function α in $\mathcal{K}_{(\inf h(\xi), \sup h(\xi))}$ such that, for all $x \in \mathbb{R}^n, u_1 \in \mathbb{R}^{m_1}$,

$$\min\{0, h(x)\} \leq -\rho(|u_1|) \quad \Rightarrow \quad L_f h + L_{g_1} h u_1 \geq -\alpha(h) . \tag{3}$$

The following result is a variation on Lyu et al. (2022, Theorem 1), proved by adapting Krstić and Deng (2000, Theorem 2.2) and Kolathaya and Ames (2019, Theorem 1).

Lemma 1. *For system (1), if there exists a ISSf-BF h, then the system is ISSf with $\beta(r,t)$ in (2) defined by the solution to $\dot{\underline{h}} = -\alpha(\underline{h})$, $\underline{h}(0) = r$.*

3. ISSf-CBFs and safety ensured by defender agent

Consider now, with locally Lipschitz f, g_1, g_2, the system

$$\dot{x} \;\; = \;\; f(x) + g_1(x)u_1 + g_2(x)u_2 , \quad u_1 \in \mathbb{R}^{m_1}, u_2 \in \mathbb{R}^{m_2} , \tag{4}$$

where u_1 is an offender and u_2 is a defender.

Definition 4 (*ISSf-CBF*). A scalar differentiable function h is called a *ISSf-control barrier function (ISSf-CBF)* for (4) if there exists a class $\mathcal{K}$ function $\rho : \mathbb{R}_{\geq 0} \to [0, -\inf h(\xi))$ and $\alpha \in \mathcal{K}_{(\inf h(\xi), \sup h(\xi))} =: \mathcal{K}_h$ such that, for all $x \in \mathbb{R}^n, u_1 \in \mathbb{R}^{m_1}$,

$$\min\{0, h(x)\} \leq -\rho(|u_1|) \;\Rightarrow\; \sup_{u_2\in\mathbb{R}^{m_2}} \left\{ L_f h + L_{g_1} h u_1 + L_{g_2} h u_2 \right\} \geq -\alpha(h). \tag{5}$$

The following result for CBFs is obtained by adapting our CLF result (Krstic & Li, 1998, Lemma 2.1).

Lemma 2. *A pair (h, ρ) satisfies (5) if and only if*

$$L_{g_2} h(x) = 0 \quad \Rightarrow \quad \omega(x) \geq 0 \tag{6}$$

where

$$\omega(x) = L_f h - \left|L_{g_1} h\right| \rho^{-1}(\max\{0, -h(x)\}) + \alpha(h(x)) . \tag{7}$$

ISSf-CBFs, which do not require the disturbance to be in a known compact set, are different from Robust CBFs (Choi, Lee, Sreenath, Tomlin, & Herbert, 2021; Jankovic, 2018). That is the very purpose of the antecedent in the implication (5) and the term $\rho^{-1}(\max\{0, -h(x)\})$ in (7).

In the next section, our defender will employ an ISSf variant of the classical CBF-QP feedback law to guarantee ISSf in the presence of an offender. However, before we proceed to this conventional approach, we point out that safety can be ensured, by the defender, with a number of other feedback choices, one of which is inspired by Sontag's formula (Sontag, 1989b) and converted from stabilization to safety.

Theorem 1 (*Defender Uses a Variant of Sontag's Formula; Input-to-state Safety Ensured*). *If there exists a ISSf-CBF, the system (4) is rendered ISSf using the following Sontag-type control law:*[1]

$$u_2 = u_S(x) = (L_{g_2}h)^T \begin{cases} \kappa(x), & (L_{g_2}h)^T \neq 0 \\ 0, & (L_{g_2}h)^T = 0\,, \end{cases} \tag{8}$$

where, with $\omega(x)$ defined in (7),

$$\kappa(x) = \frac{-\omega + \sqrt{\omega^2 + \left(L_{g_2}h(L_{g_2}h)^T\right)^2}}{L_{g_2}h(L_{g_2}h)^T} = \frac{L_{g_2}h(L_{g_2}h)^T}{\omega + \sqrt{\omega^2 + \left(L_{g_2}h(L_{g_2}h)^T\right)^2}}\,. \tag{9}$$

4. Min-norm defender feedback laws that ensure ISSf

Now we turn our attention to designing a defender control u_2 that achieves safety, in the presence of an offender input u_1, using QP/min-norm control. Let an ISSf-CBF $h(x)$ be available, with associated (ρ, α). Then, consider the QP problem for the defender agent, given by

$$\bar{u}_{QP} = \arg\min_{v\in\mathbb{R}^{m_2}} |v|^2 \quad \text{subject to} \tag{10}$$

$$\omega(x) + L_{g_2}h(x)v \geq 0\,. \tag{11}$$

The well-known explicit solution to this problem is (Freeman & Kokotovic, 1996)

$$\bar{u}_{QP} = \begin{cases} 0, & \omega(x) \geq 0 \\ -\dfrac{\omega}{|L_{g_2}h|^2}(L_{g_2}h)^T, & \omega(x) < 0\,. \end{cases} \tag{12}$$

Remark 1. Regarding the possible division by $L_{g_2}h = 0$ in the second case of (12), we recall that, by Lemma 2, every ISSf-CBF satisfies the implication $L_{g_2}h = 0 \Rightarrow \omega(x) \geq 0$, which is equivalent to the implication $\omega(x) < 0 \Rightarrow L_{g_2}h \neq 0$, and this precludes $L_{g_2}h$ being zero in the second case of (12), i.e., a division by zero is not possible.

Since controls like (12) appear in our paper at least half a dozen times, for the sake of compactness we write it as

$$\bar{u}_{QP} = (L_{g_2}h)^T \frac{\max\{0, -\omega\}}{|L_{g_2}h|^2}\,, \tag{13}$$

with a recollection from (6) that $\omega < 0 \Rightarrow L_{g_2}h \neq 0$ and with a notational convention that $0/0 = 0$.

With the QP feedback (12) for the defender agent, we have the following result.

Theorem 2 (*Defender Using QP-ISSf-CBF Formula Ensures ISSf*). *The control law*

$$u_2 = \bar{u}_{QP}(x) \tag{14}$$

with $\bar{u}_{QP}(x)$ defined in (12) and $\omega(x)$ defined in (7) renders system (4) ISSf with respect to the ISSf-CBF $h(x)$, with a gain function ρ, i.e., there exists $\beta \in \mathcal{KL}_h$ such that, for all $t \geq 0$,

$$h(x(t)) \geq \beta(h(x_0), t) - \rho\left(\sup_{0\leq\tau\leq t} |u_1(\tau)|\right)\,. \tag{15}$$

Example 1 (*Integrator with Offender Having Higher Control Authority*). Consider the system

$$\dot{x} = (1 + x^2)u_1 + u_2 \tag{16}$$

with a ISSf-CBF

$$h(x) = -x\,. \tag{17}$$

One can consider (16) a "pursuit-evasion problem on a line", where x is the relative position between a pursuer/offender and an evader/defender, u_1 is the pursuer input (aiming to drive h positive), u_2 is the evader input (aiming to keep h negative), and the pursuer is "kinematically advantaged" at larger distances through the input coefficient $1 + x^2$. For some $\rho \in \mathcal{K}_\infty$, (7) is

$$\omega = -(1 + x^2)\rho^{-1}(\max\{0, x\}) + \alpha(h(x)) \tag{18}$$

and the QP formula (12) gives

$$\bar{u}_{QP} = \min\left\{0, -(1 + x^2)\rho^{-1}(\max\{0, x\}) + \alpha(h(x))\right\}\,. \tag{19}$$

Taking, e.g.,

$$\alpha(h) = h\,, \tag{20}$$

the overall feedback (14), given by

$$u_2 = \min\left\{0, -(1 + x^2)\rho^{-1}(\max\{0, x\}) - x\right\}\,, \tag{21}$$

guarantees, $\forall \rho \in \mathcal{K}_\infty$,

$$x(t) \leq e^{-t}x_0 + \rho\left(\sup_{0\leq\tau\leq t} |u_1(\tau)|\right)\,, \quad \forall t \geq 0\,. \tag{22}$$

In the pursuit-evader interpretation, capture by the kinematically advantaged pursuer is possible but the degree of violation $\rho\left(\sup_{0\leq\tau\leq t} |u_1(\tau)|\right)$ is in proportion to the magnitude of the pursuing input u_1. □

A 'half-Sontag' formula ($u_S/2$) also generates min-norm control.

Theorem 3 (*"Half-Sontag" Formula Also has a Pointwise Min-norm Property*). *The feedback*

$$u_2 = \frac{1}{2}u_S\,, \tag{23}$$

with u_S defined in (8), (9) and ω defined in (7) renders system (4) ISSf and is the pointwise minimizer of $|v|^2$ subject to the following constraint more conservative than (11):

$$\frac{1}{2}\left(\omega - \sqrt{\omega^2 + \left(L_{g_2}h(L_{g_2}h)^T\right)^2}\right) + L_{g_2}hv \geq 0\,. \tag{24}$$

5. Inverse optimal input-to-state safety game

Let us re-examine system (4) with its to agents: the offender u_1 and defender u_2. The presence of two agents, with exactly opposing goals in relation to safety, leads us to formulate the problem of feedback control design for the opposing agents as a differential game (Başar & Bernhard, 1998; Başar & Olsder, 1998) of the zero-sum type.

In this zero-sum game, the objective for both the offender and the defender is for their respective control efforts, u_1 and u_2, to remain small. However, their objectives contradict one another regarding safety: defender u_2 desires to keep $h(x(t))$ from becoming too small, while the offender u_1 desires to make $h(x(t))$ small and, in fact, negative.

We pursue the following zero-sum two-player minimax (supinf, to be precise) optimization problem:

$$\sup_{u_2\in\mathcal{U}_2} \inf_{u_1\in\mathcal{U}_1} \left\{ \lim_{t\to\infty} \left[2\beta h(x(t)) + \int_0^t \left(l(x) - u_2^T R_2(x)u_2 + \beta\lambda\gamma\left(\frac{|u_1|}{\lambda}\right)\right)d\tau\right]\right\}, \tag{25}$$

where $\mathcal{U}_1, \mathcal{U}_2$ are sets of locally bounded functions of x. In this problem, $R_2(x) = R_2(x)^T > 0$ for all x and u_0, γ and γ' are in class $\mathcal{K}_\infty$, the constants β and λ are positive, and $l(x)$ is a weight on the state, upper bounded by a class $\mathcal{K}_\infty$ function of h.

[1] See also the proof of Theorem 3.2 in Krstic and Li (1998) and Remark 5 of Kolathaya and Ames (2019).

We do not approach the game (25) as a problem of direct determination of a Nash equilibrium but as an *inverse* problem: both the Nash control laws u_1^* and u_2^*, as well as the weights $l(x), R_2(x), \gamma(\cdot)$, are up to the offender and the defender, respectively, to choose. Even $h(x)$ is available for design (by the defender), for a given safe set C.

Before we continue, let us introduce the following notation: For a class $\mathcal{K}_\infty$ function γ whose derivative exists and is also a class $\mathcal{K}_\infty$ function, $\ell\gamma$ denotes the Legendre–Fenchel transform

$$\ell\gamma(r) = \int_0^r (\gamma')^{-1}(s)ds \tag{26}$$

$$=r(\gamma')^{-1}(r) - \gamma\left((\gamma')^{-1}(r)\right), \quad \text{(by Lemma 3.a)} \tag{27}$$

where $(\gamma')^{-1}(r)$ stands for the inverse function of $\frac{d\gamma(r)}{dr}$.

In the next theorem is this tutorial's main result: a pair of defender–offender feedback choices, which settle to a Nash equilibrium for a game in which the defender is rewarded for enhancing safety and the offender is rewarded for reducing safety.

Theorem 4 (*A "defender-offender" Policy Pair that Attains a Nash Equilibrium in an Inverse Optimal Sense*). *Consider the auxiliary system of (4),*

$$\dot{x} = f(x) - g_1(x)\ell\gamma(2|L_{g_1}h(x)|)\frac{(L_{g_1}h(x))^{\mathrm{T}}}{|L_{g_1}h(x)|^2} + g_2(x)u_2 \tag{28}$$

in which the offender agent employs the feedback

$$u_1 = \bar{u}_1(x) := -\ell\gamma(2|L_{g_1}h(x)|)\frac{(L_{g_1}h(x))^{\mathrm{T}}}{|L_{g_1}h(x)|^2} \tag{29}$$

and where γ is a class $\mathcal{K}_\infty$ function whose derivative γ' is also a class $\mathcal{K}_\infty$ function. Suppose that there exists a matrix-valued function $R_2(x) = R_2(x)^{\mathrm{T}} > 0$ such that the defender feedback of the form

$$u_2 = \bar{u}_2(x) := R_2(x)^{-1}\left(L_{g_2}h(x)\right)^{\mathrm{T}} \tag{30}$$

ensures safety of the system (28) with respect to CBF candidate $h(x)$, namely, ensures that

$$L_f h - \ell\gamma(2|L_{g_1}h|) + L_{g_2}hR_2^{-1}\left(L_{g_2}h\right)^{\mathrm{T}} \geq -\alpha(h) \tag{31}$$

for some $\alpha \in \mathcal{K}_h$. Then the defender feedback

$$u_2 = \bar{u}_2^*(x) := \beta\bar{u}_2(x) = \beta R_2^{-1}\left(L_{g_2}h(x)\right)^{\mathrm{T}}, \quad \beta \geq 2 \tag{32}$$

applied to (4) maximizes the cost functional

$$J(u_2) = \inf_{u_1\in\mathcal{U}_1}\left\{\lim_{t\to\infty}\left[2\beta h(x(t))+\int_0^t\left(l(x)-u_2^{\mathrm{T}}R_2(x)u_2+\beta\lambda\gamma\left(\frac{|u_1|}{\lambda}\right)\right)d\tau\right]\right\} \tag{33}$$

for any $\lambda \in (0,2]$, where

$$l(x) = -2\beta\left[L_f h - \ell\gamma(2|L_{g_1}h|) + L_{g_2}hR_2^{-1}\left(L_{g_2}h\right)^{\mathrm{T}}\right]$$
$$-\beta(2-\lambda)\ell\gamma(2|L_{g_1}h|) - \beta(\beta-2)L_{g_2}hR_2^{-1}\left(L_{g_2}h\right)^{\mathrm{T}} \tag{34}$$
$$\leq 2\beta\alpha(h) \tag{35}$$

is decrescent in the CBF h on the interval $(\inf h, \sup h)$*, and where the Nash feedback for the offender is given by*

$$u_1 = u_1^*(x) := -\lambda(\gamma')^{-1}(2|L_{g_1}h|)\frac{(L_{g_1}h)^{\mathrm{T}}}{|L_{g_1}h|}. \tag{36}$$

The parameter $\beta \geq 2$ in the statement of Theorem 4 represents a design degree of freedom for the defender. The presence of the offender's design parameter λ (note that it parameterizes not only the penalty on the offender but also the penalty on the state's proximity to the boundary, i.e., the reward for the state's distance from the boundary, $l(x)$) indicates that the defender's family of feedback laws is inverse optimal with respect to an entire family of different cost functionals, and an entire family of offender feedback laws.

Remark 2. Even though not explicit in the proof of Theorem 4, the CBF $h(x)$ solves the following family of *Hamilton–Jacobi–Isaacs* (HJI) PDEs:

$$L_f h - \frac{\lambda}{2}\ell\gamma(2|L_{g_1}h|) + \frac{\beta}{2}L_{g_2}hR_2^{-1}\left(L_{g_2}h\right)^{\mathrm{T}} + \frac{l}{2\beta} = 0, \tag{37}$$

parameterized by $(\beta,\lambda) \in [2,\infty)\times(0,2]$. ☐

Remark 3. It is also easily seen from the proof of Theorem 4 that, even for initial conditions on the boundary, the level of attenuation of the offender's action that the defender achieves is

$$2\beta h(x(t)) + 2\beta\int_0^\infty \alpha(h(x))\,dt$$
$$\geq 2\beta h(x(t)) + \int_0^\infty l(x)\,dt$$
$$\geq \int_0^\infty u_2^{\mathrm{T}}R_2(x)u_2\,dt - \beta\lambda\int_0^\infty \gamma\left(\frac{|u_1|}{\lambda}\right)dt$$
$$\geq -\beta\lambda\int_0^\infty \gamma\left(\frac{|u_1|}{\lambda}\right)dt. \tag{38}$$

Summarizing, we refer to the property

$$h(x(t)) + \int_0^\infty \alpha(h(x))\,dt \geq -\frac{\lambda}{2}\int_0^\infty \gamma\left(\frac{|u_1|}{\lambda}\right)dt, \tag{39}$$

as *integral input-to-state safety* (iISSf). ☐

Example 2 (*Integrator Game Where Offender has a Higher Control Authority*). Consider the system from Example 1. Take

$$\gamma(r) = \ell\gamma(2r) = r^2. \tag{40}$$

With

$$R_2 = \frac{1}{\max\{0, -\alpha(h(x))\} + (1+x^2)^2} > 0, \tag{41}$$

condition (31) is satisfied. The defender's feedback policy (32) is given by

$$u_2 = -\frac{\beta}{R_2} = \beta\left[-(1+x^2)^2 + \min\{0,-x\}\right] \tag{42}$$

and, for all $\beta \geq 2$, is the maximizer of

$$J(u_2) = \inf_{u_1\in\mathcal{U}_1}\left\{\lim_{t\to\infty}\left[-2\beta x(t) + \int_0^t\left(l(x) - R_2u_2^2 + \frac{\beta}{\lambda}u_1^2\right)d\tau\right]\right\} \tag{43}$$

for any $\lambda \in (0,2]$, with $l(x) \leq -2\beta x$. The defender feedback (42) achieves

$$x(+\infty) + \int_0^\infty x(t)\,dt \leq \frac{1}{2\lambda}\int_0^\infty u_1^2(t)\,dt. \tag{44}$$

Additionally, for all $\beta \geq 1$, the defender feedback (42) with $\alpha(h) = h$ guarantees

$$x(t) \leq e^{-t}x_0 + \frac{1}{4}\left(\sup_{0\leq\tau\leq t}|u_1(\tau)|\right)^2, \quad \forall t \geq 0. \tag{45}$$

Now we turn our attention to the Nash policy of the offender, which is given by

$$u_1^* = \lambda(1+x^2). \tag{46}$$

Let us examine the offender–defender Nash policy pair (46), (42), for an initial condition in the safe set $x < 0$. This pair is given by, simply,

$$u_2^* = -\beta(1+x^2)^2, \quad \beta \geq 2 \tag{47}$$

$$u_1^* = \lambda(1+x^2), \quad 0 < \lambda \leq 2. \tag{48}$$

For the plant (16), the closed-loop system with the Nash feedback policies is

$$\dot{x} = (\lambda-\beta)(1+x^2)^2, \quad \lambda-\beta < 0. \tag{49}$$

With an initial condition $x(0) < 0$ in the safe set, and $\dot{x} < 0$, the forward invariance of the system in the set $x < 0$ is ensured. In other words, the defender succeeds at maintaining safety when the offender's action is penalized with u_1^2 in the cost function. The value of the game is $J^*(x(0)) = -2\beta x(0) > 0$, which is finite, in spite of the fact that $x(t), u_1(x(t)), u_2(x(t))$ all grow without bound when $0 < \lambda < \beta$. The pursuit-evasion meaning of the outcome $x(t) \to -\infty$ for (49) when $x(0) < 0$ is that the evader succeeds at the evasion. □

For the general result in Theorem 4, a natural question arises: Is the ISSf QP feedback law of the defender, (12), (7), inverse optimal? The following theorem, proven similarly to Theorem 4, answers the question in the affirmative.

Theorem 5 (*A Fortified QP-ISSf-CBF Policy by Defender Ensures a Nash Outcome*). *Consider system (4) with associated ISSf-CBF h and a gain function ρ. For any $\beta \geq 2$, the defender feedback*

$$u_2 = \bar{u}^*_{\rm QP}(x, u_0) = \beta \bar{u}_{\rm QP}(x), \tag{50}$$

with $\bar{u}_{\rm QP}$ defined in

$$\bar{u}_{\rm QP} = (L_{g_2}h)^T \frac{\max\{0, -\omega\}}{|L_{g_2}h|^2} \tag{51}$$

$$\omega(x) = L_f h - \left|L_{g_1}h\right| \rho^{-1}(\max\{0, -h\}) + \alpha(h), \tag{52}$$

maximizes

$$J(u_2) = \inf_{u_1 \in \mathcal{U}_1} \left\{ \lim_{t\to\infty} \left[2\beta h(x(t)) + \int_0^t \left(l(x) - R_2(x)|u_2|^2 + \frac{\beta}{\lambda} R_1(x)|u_1|^2 \right) d\tau \right] \right\} \tag{53}$$

for all $\lambda \in (0, 2]$, where

$$R_1(x) = \frac{1}{\rho^{-1}(\max\{0, -h\})} > 0 \tag{54}$$

$$R_2(x) = \frac{|L_{g_2}h|^2}{\max\{0, -\omega\}} > 0 \tag{55}$$

$$l(x) \leq 2\beta\alpha(h(x)), \tag{56}$$

and the Nash feedback law of the offender is given by

$$u_1 = u_1^*(x) := -\lambda \rho^{-1}(\max\{0, -h(x)\}) \frac{(L_{g_1}h(x))^{\rm T}}{|L_{g_1}h(x)|}. \tag{57}$$

The weight R_1 in (54) is infinite in the safe set $h(x) \geq 0$ where the Nash feedback (57) spends no effort. Likewise, R_2 in (55) is infinite when $\omega \geq 0$ since control (12) puts in no effort when the system is safe on its own. We also recall from Remark 1 that (12) precludes $L_{g_2}h$ from being zero when $\omega < 0$, so R_2 can, in fact, never be zero, namely, u_2 is penalized for all x.

Example 3. Back to Example 1, the defender feedback law $u_2 = \beta \bar{u}_{\rm QP}$, $\beta \geq 2$, with

$$\bar{u}_{\rm QP} = \min\left\{0, -(1 + x^2)\rho^{-1}(\max\{0, x\}) - x\right\}, \tag{58}$$

results in

$$x(+\infty) + \int_0^\infty x(t)\,dt \leq \frac{1}{2\lambda} \int_0^\infty \frac{u_1^2(t)}{\rho^{-1}(\max\{0, x(t)\})}\,dt, \tag{59}$$

which, unlike control (42) in Example 2, fails to achieve a finite integral gain in the safe set $x \leq 0$ like (44). □

6. Safety games for a unicycle vehicle

We return to the general result on inverse optimal zero-sum/mini-max/offender–defender safety games in Theorem 4 and illustrate it with two examples on a well-studied nonlinear system with two inputs—the nonholonomic unicycle. The game is played not between two (offender and defender) vehicles but between the two inputs (steering and forward "driving") on the same vehicle.

Example 4 (*Unicycle with Steering Offender*). We first consider the unicycle system with the state vector $\bar{x} = [x, y, \theta]^{\rm T}$,

$$\dot{x} = u_2 \cos\theta + v \tag{60}$$

$$\dot{y} = u_2 \sin\theta \tag{61}$$

$$\dot{\theta} = u_1, \tag{62}$$

where $v \in \mathbb{R}$ is a constant drift (wind) disturbance in the x direction. In this example we pose, for pedagogical purposes, a problem that is unconventional and, arguably, artificial: we treat the two unicycle inputs as opponents in a game. The steering (heading rate) input u_1 is treated as the offender, trying to violate safety, while the propulsive (linear velocity) input u_2 is treated as the defender, trying to maintain safety. As for the uncommon labeling of the linear input as u_2 and the angular input as u_1, this is simply for consistency with our previous presentation where the offender is labeled as agent 1 and the defender as agent 2.

While for unicycles the obstacle is often taken as a circular "safety bubble", in this example we carefully select the simplest possible obstacle that is consistent with the presence of a disturbance that advantages the offender. Given the wind v in the x-direction, which we will take to have a negative (leftward) velocity, we take the safe set to be the right half-plane $\{x > 0\}$, namely, we take the safety boundary to be the "wall" $x = 0$. Such a safe set would normally give rise to a simple position-based CBF of the form

$$h(\bar{x}) = x. \tag{63}$$

But this CBF is "asymmetric" in terms of the relative degrees of the two agents: it is relative degree one with respect to the defender u_2 but relative degree two with respect to the offender u_1. In other words, $h = x$ advantages the defender since the offender u_1 cannot directly influence dh/dt. After a careful consideration, we take the CBF

$$h(\bar{x}) = \frac{x}{\cos^2\theta}, \tag{64}$$

which is of relative degree one with respect to both u_1 and u_2. There is additional logic behind choosing the CBF (64), apart from "equalizing" the relative degrees. For $\theta = \pm\pi/2$, the vehicle is headed in parallel to the boundary, which is inherently safe. Finally, the vehicle cannot be actuated in its "sideways" direction, and for this reason the CBF $h = x$ fails to satisfy the CBF condition at headings $\theta = \pm\pi/2$, making the defense task impossible for the defender. Such a singularity for the defender is removed with the CBF $h = x/\cos^2\theta$ (see Fig. 1).

Let us now proceed with working out the details of the application of Theorem 4. For the Lie derivatives of (64), one obtains

$$L_f h(\bar{x}) = \frac{v}{\cos^2\theta}, \tag{65}$$

$$L_{g_1}h(\bar{x}) = \frac{2x\tan\theta}{\cos^2\theta}, \tag{66}$$

$$L_{g_2}h(\bar{x}) = \frac{1}{\cos\theta}. \tag{67}$$

Taking

$$\gamma(r) = \ell\gamma(2r) = r^2, \qquad \alpha(h) = h, \tag{68}$$

in order to satisfy condition (31) one gets

$$R_2 = \frac{1}{\max\{0, -(x + v)\} + 4x^2 \dfrac{\tan^2\theta}{\cos^2\theta}}. \tag{69}$$

The defender's Nash feedback policy is then

$$u_2^* = \frac{\beta}{\cos\theta}\left(\max\{0, -(x + v)\} + 4x^2 \frac{\tan^2\theta}{\cos^2\theta}\right), \quad \beta \geq 2. \tag{70}$$

while the offender's Nash feedback policy is

$$u_1^* = -\lambda \frac{2x\tan\theta}{\cos^2\theta}, \quad 0 < \lambda \leq 2. \tag{71}$$

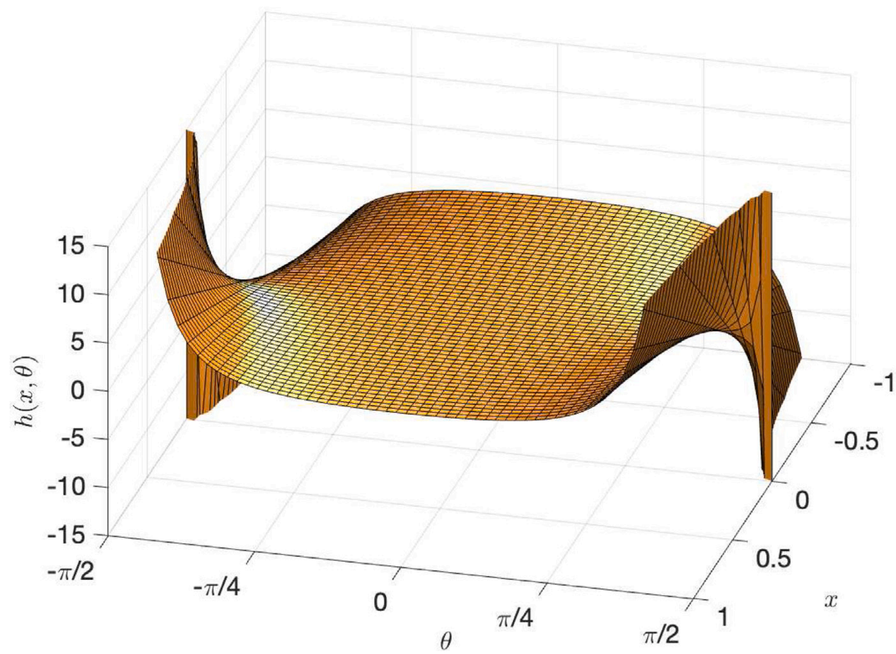

Fig. 1. Plot of CBF $h = x/\cos^2\theta$ over the interval $(-\pi/2, \pi/2)$ for θ. This CBF is of the same relative degree – one – relative to both inputs, angular velocity (offender) and linear velocity (defender). To equalize the relative degrees for the two agents, the CBF treats headings parallel with the wall as safe, even when the vehicle is close to the wall, which is natural since the nonholonomic unicycle cannot be actuated sideways.

Both of the policies blow up when $\cos\theta = 0$, namely, at the heading angles $\theta = \pm\pi/2$, which are parallel with the wall. Let us see that this singularity is actually never encountered, unless the vehicle starts parallel with the wall. We will see this by showing that the offender policy (71) guarantees that the heading that starts in the interval $(-\pi/2, \pi/2)$ remains in that interval, and likewise, the heading starting in $(\pi/2, 3\pi/2)$ is maintained in that interval by (71). The closed-loop dynamics of θ are

$$\dot{\theta} = -2\lambda x \frac{\tan\theta}{\cos^2\theta}\,. \tag{72}$$

Consider the Lyapunov function

$$V = \frac{1}{\cos^2\theta} - 1 = \tan^2\theta\,. \tag{73}$$

This is system (72)'s Lyapunov function both for the equilibrium $\theta = 0$ on the interval $(-\pi/2, \pi/2)$ and for the equilibrium $\theta = \pi$ on the interval $(\pi/2, 3\pi/2)$. Its derivative is $\dot{V} = -4\lambda x^2 \tan^2\theta/\cos^4\theta = -4\lambda x^2 \left(1/\cos^2\theta - 1\right)/\cos^4\theta$, namely,

$$\dot{V} = -4\lambda x^2 (1+V)^2 V\,, \tag{74}$$

which is negative definite in V in the safe set $x > 0$. Hence, the equilibria $\theta = 0$ and $\theta = \pi$ are asymptotically stable with regions of attraction $(-\pi/2, \pi/2)$ and $(\pi/2, 3\pi/2)$, respectively, and, more importantly, each of these two intervals is forward invariant. In conclusion, the heading values $\theta = \pm\pi/2$, at which the Nash feedback laws u_1^* and u_2^* are singular, will never be attained, unless the initial heading has one of these two singular values. In fact, the heading will monotonically converge to the headings orthogonal to the wall.

Let us now turn our attention to the game between the angular and linear velocity inputs. What do their Nash feedback laws optimize? The defender maximizes and the offender minimizes the cost functional

$$\lim_{t\to\infty}\left[\frac{x(t)}{\cos^2\theta(t)} + \int_0^t \left(\eta(x(\tau), \theta(\tau)) - \frac{1}{2\beta}\,\frac{u_2^2(\tau)}{\max\{0, -(x(\tau)+\upsilon)\} + 4x^2\frac{\tan^2\theta(\tau)}{\cos^2\theta(\tau)}}\right.\right.$$
$$\left.\left. + \frac{1}{2\lambda}u_1^2(\tau)\right)\mathrm{d}\tau\right], \tag{75}$$

where

$$\eta(x,\theta) \le \frac{x}{\cos^2\theta}\,. \tag{76}$$

In more specific terms, (75) indicates that at minimal (weighted) costs to the agents' respective efforts, the defender agent maximizes the vehicle's distance from the wall and its parallelism with the wall, while the offender agent minimizes the vehicle's distance from the wall and maximizes its orthogonality to the wall. The defender's cost to act is infinite when the vehicle is orthogonal to the wall and far from the wall, and indeed the defender is inactive in those conditions. The game value, under the Nash feedback laws of the offender and defender, is $x(0)/\cos^2\theta(0)$.

It is without a doubt difficult to comprehend a situation where the steering is used to run a vehicle into a wall, while acceleration, braking, in both forward and reverse directions, is used to prevent such a collision from happening. To gain better understanding, it is helpful to rewrite (70) as

$$u_2^* = \beta\left(\frac{\max\{0, -(x+\upsilon)\}}{\cos\theta} + \cos\theta\,\frac{1}{\lambda^2}(u_1^*)^2\right), \quad \beta \ge 2. \tag{77}$$

The first term within the parentheses is simply the linear velocity action under the CBF $h = x$, namely, when steering is not employed to interfere with safety. When steering becomes a threat to safety, linear velocity employs an additional action, in proportion to the square of the harmful steering action.

If the vehicle heading is eastward, the linear velocity augmentation is positive, namely, away from the wall. When the heading is nearly parallel to the wall, namely, when $\cos\theta$ is small in absolute value, the action of steering becomes harmless and the augmentation $\cos\theta \frac{1}{\lambda^2} u_1^*$ is negligible. In other words, for a steering action (71) optimized to attempt a collision with the wall by orienting the vehicle towards the wall, and doing so faster when the vehicle is father from the wall (in proportion to x), the best defense by linear velocity is simply to drive in the direction away from the wall, and at a speed suitably adjusted to the steering rate towards the wall and the heading towards the wall.

Next we present simulation results with the Nash policies (70), (71) applied to the unicycle (60)–(62), with a leftward wind, $\upsilon < 0$. Fig. 2 shows two selected trajectories $y(x)$ of the vehicle, with vehicle heading superimposed as directed arrows, whereas Fig. 3 shows the time responses of the heading $\theta(t)$, offender input $u_1(t)$, and defender input $u_2(t)$. The figure captions provide observations and interpretations. □

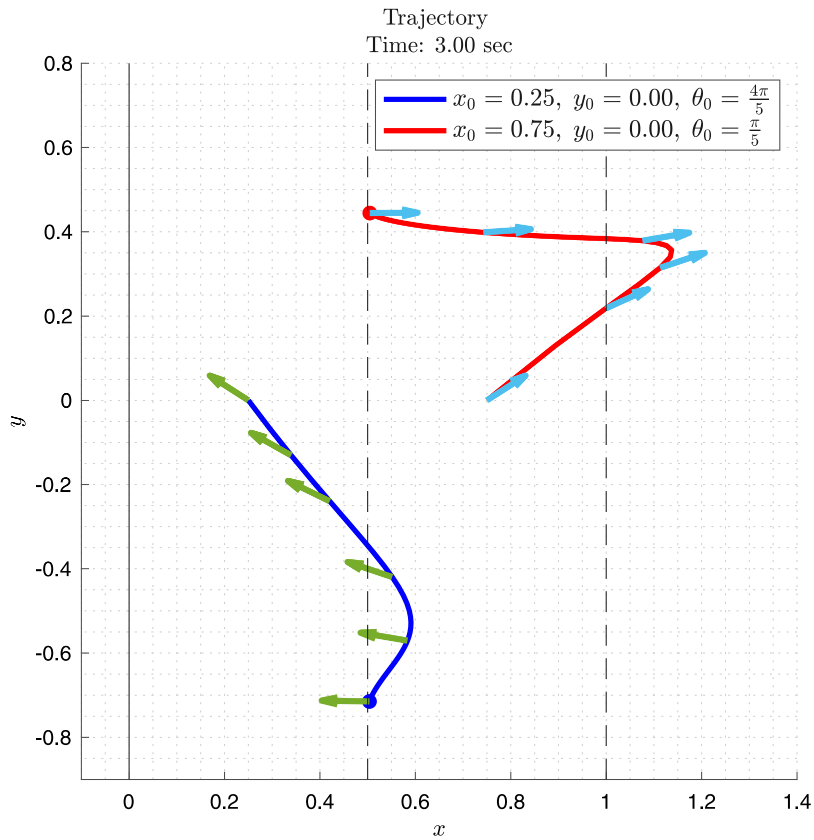

Fig. 2. Under leftward wind, all the trajectories starting on the safe (right) side of the wall obstacle at $x = 0$ converge to a vertical equilibrium manifold $x = 0.5$, to the right of the wall, with terminal headings orthogonal to the wall in both trajectories. The dots represent the trajectories' terminal points. The green arrows represent the heading directions of the vehicle. Note that they are not tangential to the trajectories because of the wind v: based on (60), (61), $\arctan(x/y) \neq \theta$. The blue trajectory starts close to the wall, pointing towards it but, while the steering attempts to turn the vehicle towards the wall, the defending linear velocity input acts negatively and drives the vehicle in reverse, away from the wall. The red trajectory starts at a safer distance from the wall and, similar to the red trajectory, the defender successfully counteracts the offender's attempt. In both trajectories, the linear velocity input converges to a value that balances out with the wind velocity, settling at the equilibrium manifold $x = 0.5$, with a heading orthogonal to the wall in both cases. The dashed vertical line at $x = 1$ represents the x-position to the left of which $\max\{0, -(x+v)\}$ becomes positive and the linear velocity input actively counteracts the wind $v < 0$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Example 5 (*Unicycle with Steering Defender*)**.** In Example 4, one would be legitimate to ask: why is the angular velocity chosen as the offender and the linear velocity as the defender? Why not flip the roles? Indeed, such a flipping of the roles is of interest. Skipping the details of the calculations, for the unicycle model

$$\dot{x} = u_1 \cos\theta + v \tag{78}$$

$$\dot{y} = u_1 \sin\theta \tag{79}$$

$$\dot{\theta} = u_2\,, \tag{80}$$

we obtain the Nash policies for the linear velocity offender u_1 and for the angular velocity defender u_2 as, respectively,

$$u_1^* = -\frac{\lambda}{\cos\theta} \tag{81}$$

$$u_2^* = \frac{\beta}{2x}\frac{\cos\theta}{\sin\theta}\left(1 + \max\{0, -(x+v)\}\right)\,. \tag{82}$$

From (81) it is evident that the linear velocity acts towards the wall and blows up when the vehicle is parallel to the wall, whereas from (82) it is clear that the angular velocity is high near the wall and shuts off in parallel with the wall. From these two observations it is clear that the offender is highly advantaged and the defender stands practically no chance of preventing the vehicle from being run into the wall. Plugging (81) into (78), one gets

$$\dot{x} = -\lambda + v\,, \tag{83}$$

which shows that, for $v < \lambda$, the angular velocity defender is powerless in trying to prevent the vehicle from being run into the wall by the linear velocity offender, since

$$x(t) = x_0 - (\lambda - v)t\,, \tag{84}$$

namely, the collision happens at time $t = x_0/(\lambda - v)$. It is interesting to also see what is happening with the heading in the meantime. Plugging (82) into (79), one gets

$$\frac{\mathrm{d}}{\mathrm{d}t}\cos\theta = -\frac{\beta}{2x}\left(1 + \max\{0, -(x+v)\}\right)\cos\theta\,, \tag{85}$$

namely,

$$\frac{\mathrm{d}}{\mathrm{d}t}\cos^2\theta \leq -\frac{\beta}{x}\cos^2\theta = -\frac{\beta}{x_0 - (\lambda - v)t}\cos^2\theta\,, \tag{86}$$

from which one gets

$$\cos^2\theta(t) \leq \left(1 - \frac{\lambda - v}{x_0}t\right)^{\frac{\beta}{\lambda - v}}\cos^2\theta_0\,. \tag{87}$$

Hence, no later than time $t = x_0/(\lambda - v)$ the vehicle's heading becomes parallel with the wall. We do not show simulation results for the feedback pair (81), (82) since the outcome of running into the wall is obvious from the above discussion.

The vehicle being run into the wall by the linear velocity offender should not be regarded as a defeat of the angular velocity defender because the defender's task is not to keep $x(t)$ positive but to maximize the following performance index under a minimizing policy of the offender:

$$\lim_{t\to\infty}\left[\frac{x(t)}{\cos^2\theta(t)} + \int_0^t \left(\eta(x(\tau), \theta(\tau))\right.\right.$$

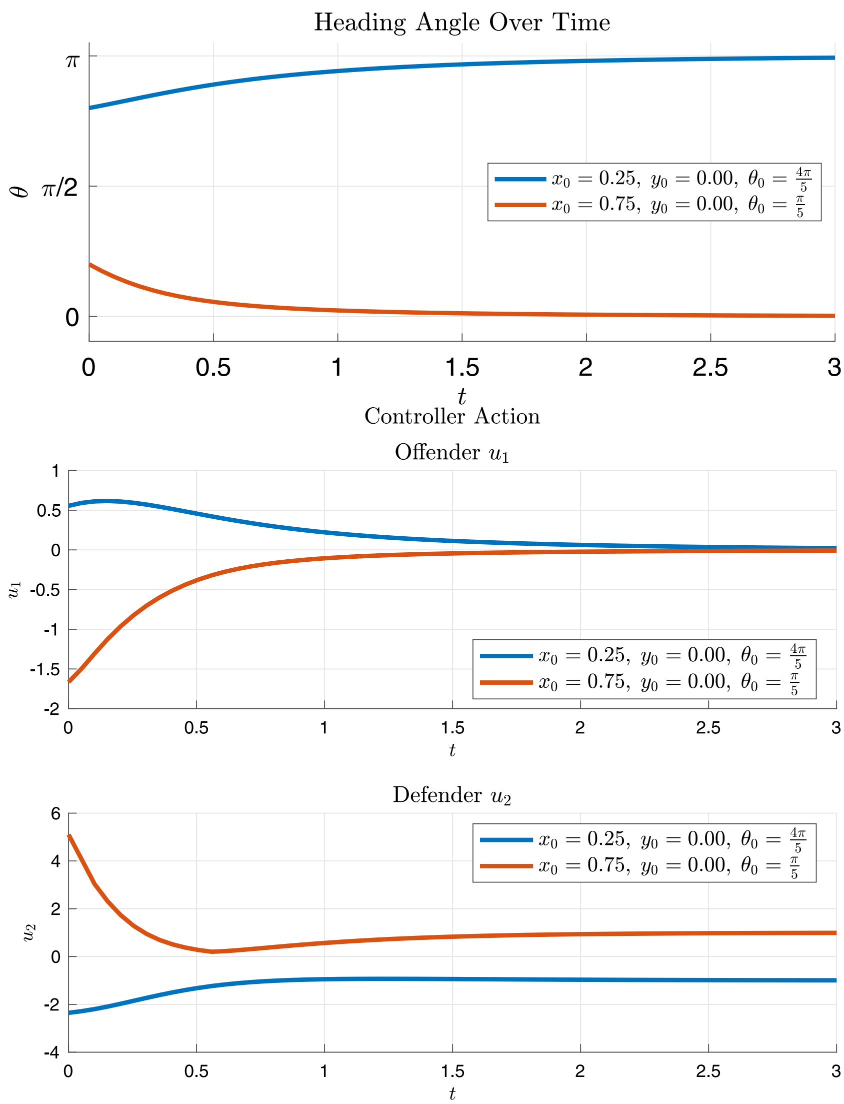

Fig. 3. The heading transient of the vehicle, controlled by the offender u_1, is monotonic and orients the vehicle orthogonally to the wall, with the intent of running the vehicle into the wall. The Nash policy inputs of the offender and defender are smooth and lead to the vehicle trajectories in Fig. 2.

$$-\frac{2}{\beta}x^2\frac{\tan^2\theta}{\cos^2\theta}\frac{1}{1+\max\{0,-(x+v)\}}u_2^2(\tau)+\frac{1}{2\lambda}u_1^2(\tau)\Bigg)\mathrm{d}\tau\Bigg], \tag{88}$$

where η satisfies (76). Furthermore, attaining $x(t)\to 0$ as, at the same time, $\theta(t)\to\pm\pi/2$, is a Pyrrhic victory for the offender, since its input $|u_1^*(t)|\to\infty$.

The simultaneous occurrence of $x(t)\to 0$ and $\theta(t)\to\pm\pi/2$ is the consequence of the safety being defined through the CBF $h = x/\cos^2\theta$, which allows the vehicle to reach the wall, as long as it does so with a heading parallel to the wall.

The x-component of the velocity at the time of reaching the wall, while heading parallel to the wall, is $-\lambda+v<0$. This "sideways" motion of the unicycle does not contradict its nonholonomic character for two reasons: wind of velocity v is acting on the unicycle and the linear velocity input is infinite at that time.

By examining (85) and (87), it is intriguing that the Nash policy (82) regulates the heading to be parallel with the wall. This outcome is a consequence of optimality. Not only is the feedback (82) zero when the heading is parallel with the wall, but it is also infinite when the heading is orthogonal to the wall. It is easy to understand why this is optimal: at any heading other than parallel with the wall, the linear velocity input can "shift to reverse" and direct the vehicle towards the wall, with the highest effect exhibited when the heading is orthogonal to the wall. For this reason, enforcing parallel heading is the angular velocity defender's best (Nash) option. □

Example 6 (*Summary of Unicycle Games from Examples 4 and 5*). In order to draw additional insight and some clear conclusions, we set the wind velocity to $v=0$ and focus our attention only on the safe set $x>0$, in which case $\max\{0,-(x+v)\}=0$ and the expressions for the Nash policies simplify. For enhancing conceptual insight, we clean up the expressions by setting $\beta=\lambda=2$. We also remind the reader that the CBF is given by

$$h(x,\theta)=\frac{x}{\cos^2\theta}. \tag{89}$$

For the game in which the steering is used for offense, we obtain simple expressions for the angular velocity and linear velocity feedback laws, respectively, as

$$\text{ang. vel. offender:}\quad u_1^* = -4h\tan\theta \tag{90}$$

$$\text{lin. vel. defender:}\quad u_2^* = \frac{\cos\theta}{2}u_1^{*2}, \tag{91}$$

or, if θ is eliminated, the linear velocity defender is acting in response to the angular velocity offender and the safety measure h as

$$u_2^{*2} = 4h^2\frac{u_1^{*4}}{u_1^{*2}+16h^2}. \tag{92}$$

In (90), when the vehicle is far from the wall and heading (nearly) in parallel with the wall, the steering offender turns vigorously towards the wall. In response, in (91) the defender speeds in the direction away from the wall and in proportion to the orthogonality to the wall and the square of the turning rate of the offender. In plain terms, the defender drives the vehicle away from the wall as the offender tries to steer it into the wall.

On the other hand, when steering is used for defense, the expressions for the angular velocity and linear velocity feedback laws, respectively, are

$$\text{lin. vel. offender:} \quad u_1^* = -\frac{2}{\cos\theta} \tag{93}$$

$$\text{ang. vel. defender:} \quad u_2^* = \frac{u_1^{*2}}{4h\tan\theta}, \tag{94}$$

or, if θ is eliminated, the angular velocity defender is acting in response to the linear velocity offender and the safety measure h as

$$u_2^{*2} = \frac{1}{4h^2}\frac{u_1^{*4}}{u_1^{*2}-4}. \tag{95}$$

In (93), when the vehicle heads (nearly) in parallel with the wall, the driving offender directs the vehicle towards the wall, at high velocity. In response, in (94) the defender turns in a manner that counteracts the offender's speed and direction relative to the wall, and does so in proportion to the proximity and orthogonality to the wall. In plain terms, the defender turns away from the wall, as the offender always drives towards the wall.

It is also important to examine the optimality of these policies, based on the payoff/cost functions (75) and (88). While the offender is always penalized quadratically, with a weight that is independent of the state, the weight of the quadratic penalty on the defender is given, in the two respective cases, as

$$R_2^{\text{lin.vel.def.}} = \frac{1}{h^2\sin^2\theta} \qquad \text{and} \qquad R_2^{\text{ang.vel.def.}} = h^2\sin^2\theta. \tag{96}$$

In other words, when the offense is by steering, and when the vehicle is close to the wall and (nearly) orthogonal to it, the weight on the driving control is high and the defender should slow down the vehicle. In contrast, when the defense is by steering, and when the vehicle is close to the wall and (nearly) orthogonal to it, the weight on the steering control is low and the defender should turn vigorously (towards heading parallel to the wall).

To summarize, the Nash policies of the opposing actors follow common sense of their opposing goals relative to the collision with the wall. Additionally, the specific quantitative formulae for these Nash policies go beyond the mere common sense and actually guarantee, in a precise manner, optimality of the non-cooperative kind.

Finally, we are able to provide the explicit expressions for the system's closed-loop trajectories. Calculations reveal that the CBF $h(x,\theta)$ given in (89) actually remains constant when $v = 0$ and $\lambda = 2$, in both offender–defender scenarios. In other words, $h(x(t),\theta(t)) = h_0 = x_0/\cos^2\theta_0$. This result is then used to obtain the explicit trajectories $y(\theta)$ and $y(x)$, getting, at the end, the following expressions in the closed loop for both offender–defender scenarios:

$$x(\theta) = h_0\cos^2\theta \tag{97}$$

$$y(\theta) = \frac{h_0}{2}\left[\sin(2\theta) - 2\theta\right] \tag{98}$$

$$y(x) = \pm\left[x\sqrt{\frac{h_0}{x}-1} - h_0\arctan\sqrt{\frac{h_0}{x}-1}\right]. \tag{99}$$

The expression (99) is arguably the most important, as it gives the closed-loop trajectories of the vehicles, parameterized in the initial $h_0 = x_0/\cos^2\theta_0$. The vehicle paths $y(x)$ are always monotonic – increasing or decreasing in x – depending on the value of h_0. The direction of the trajectories depends on the offender–defender scenario. With steering as the offender, the trajectories move eastward in the x-y plane, away from the wall, at an *exponential* rate $x(t) = h_0\left(1-\tan^2\theta_0\,\mathrm{e}^{-8h_0(t-t_0)}\right)$. With steering as the defender, the trajectories move westward, towards the wall, at a *linear* (finite-time) rate of $x(t) = x_0 - 2(t-t_0)$ (see Fig. 4).

It is perhaps fascinating that with two entirely different pairs of feedback laws, for the two offender–defender scenarios, the vehicle paths are the same for such distinct control laws. This has to be attributed to the control laws originating from non-cooperative game problems, in which there is in both cases one offender and one defender, as well as a common safety metric for optimization. However, the vehicle paths being the same is far from expected. The two offender–defender scenarios do not differ merely in the roles of the offender and defender being reversed. The linear and angular velocity inputs do not enter the unicycle in a "symmetric" manner and their defender action is penalized differently in the two scenarios, as indicated in (96). □

7. Noise-to-state safety (NSSf) and defender feedback laws that ensures NSSf

Now we turn our attention to systems in which the offender input w is stochastic, namely,

$$\mathrm{d}x = f(x)\,\mathrm{d}t + g_1(x)\,\mathrm{d}w + g_2(x)u_2\,\mathrm{d}t, \tag{100}$$

where w is an r-dimensional standard Wiener process and f, g_1, g_2 are locally Lipschitz. However, a stochastic input w is not a policy—it is an open-loop signal. To model a stochastic offender who is "strategic", we allow it to have a time-varying incremental covariance $\Sigma_1(t)\Sigma_1(t)^{\mathrm{T}}\,\mathrm{d}t$, which is unknown to the defender. In other words, we consider stochastic inputs w that satisfy

$$E\left\{\mathrm{d}w\,\mathrm{d}w^{\mathrm{T}}\right\} = \Sigma_1(t)\Sigma_1(t)^{\mathrm{T}}\,\mathrm{d}t \tag{101}$$

where $\Sigma_1(t)$ is a bounded function taking values in the set of nonnegative definite matrices. For matrices $X = [x_1, x_2, \dots, x_n]$, we use the Frobenius norm

$$|X|_{\mathcal{F}} \triangleq \left(\mathrm{Tr}\left\{X^{\mathrm{T}}X\right\}\right)^{1/2} = \left(\mathrm{Tr}\left\{XX^{\mathrm{T}}\right\}\right)^{1/2} \tag{102}$$

and note that $|X|_{\mathcal{F}} = |\mathrm{col}(X)|$, where $\mathrm{col}(X) = [x_1^{\mathrm{T}}, x_2^{\mathrm{T}}, \dots, x_n^{\mathrm{T}}]^{\mathrm{T}}$.

For a barrier function candidate $h(x)$, we recall that Itô's lemma states that, along the solutions of (100), the following holds,

$$\mathrm{d}h = \mathcal{L}h\,\mathrm{d}t + L_{g_1}h\,\mathrm{d}w, \tag{103}$$

where

$$\mathcal{L}h = L_{f+g_2u_2}h + \frac{1}{2}\mathrm{Tr}\left\{\Sigma_1^{\mathrm{T}}g_1^{\mathrm{T}}\frac{\partial^2 h}{\partial x^2}g_1\Sigma_1\right\} \tag{104}$$

is referred to as the *infinitesimal generator* of h.

When the covariance is unknown and time-varying, it needs to be treated as a deterministic disturbance in Sections 2–4. Accordingly, only a graceful degradation of safety in the presence of the offender action $\Sigma_1(t)$ can be expected, as in (2). We refer to such a stochastic property as *noise-to-state safety* (NSSf).

From here on we proceed formally, with systems and controllers that satisfy a certain barrier function inequality, without going a step further to establish safety in probability, or at least in the mean, which would be done by employing the techniques as in the proof of Theorem 3.2 in Krstić and Deng (2000), the techniques in Theorem 3 in Clark (2021), or the technique in the proof of Lemma 1 in Li and Krstic (2020).

Hence, we pursue the attainment of the following barrier function condition

$$\begin{gathered}\min\{0, h(x)\} \le -\rho\left(\left|\Sigma_1\Sigma_1^{\mathrm{T}}\right|_{\mathcal{F}}\right)\\ \Downarrow \\ L_f h + L_{g_2}hu_2 + \tfrac{1}{2}\mathrm{Tr}\left\{\Sigma_1^{\mathrm{T}}g^{\mathrm{T}}\tfrac{\partial^2 h}{\partial x^2}g\Sigma_1\right\} \ge -\alpha(h),\end{gathered} \tag{105}$$

for system (100) with (101), by defender feedback $u_2 = \bar{u}_2(x)$, and call this condition the *noise-to state barrier function condition* (NSBFc).

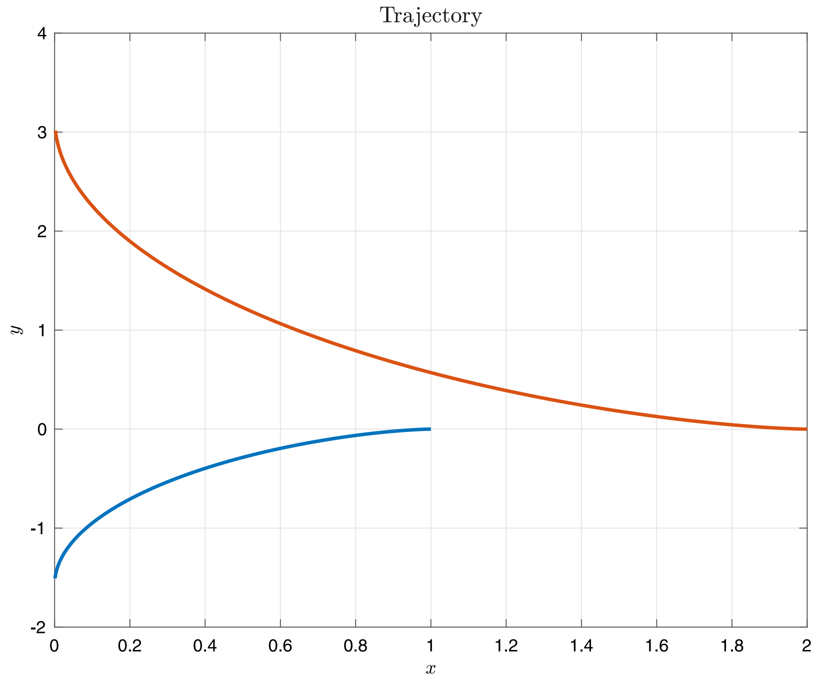

Fig. 4. Two examples of vehicle paths given by the formula (99) for $h_0 = 1$ (blue) and $h_0 = 2$ (red). Each path has one end on the x-axis ($y = 0$) and the other end on the wall $x = 0$. For each initial condition, in either of the offender–defender scenarios, the vehicle travels along one such path. If the offender input is angular velocity, the motion is away from the wall. If the offender input is linear velocity, the motion is towards the wall. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Heretofore, we have dealt with CBF and ISSf-CBFs. We say that a function h is a *noise-to-state safety control barrier function* (NSSf-CBF) if, in addition to its usual conditions, it satisfies the implication

$$L_{g_2} h = 0 \quad \Rightarrow \quad \omega \geq 0\,, \tag{106}$$

where

$$\omega(x) = L_f h(x) + \alpha(h) - \frac{1}{2}\left|g_1^{\mathrm{T}} \frac{\partial^2 h}{\partial x^2} g_1\right|_{\mathcal{F}} \rho^{-1}(\max\{0, -h(x)\}) \tag{107}$$

for a class $\mathcal{K}$ $\rho : [0, +\infty) \to [0, -\inf h(\xi))$ and $\alpha \in \mathcal{K}_h$.

Theorem 6 (*Defender Uses Either Sontag or QP-NSSf-CBF Formula; Noise-to-state Safety Ensured*). *Under a defender feedback of either the Sontag type or of the QP type, namely,*

$$u_2 = u_{\mathrm{S}}(x) \;:=\; (L_{g_2} h)^{\mathrm{T}} \frac{|L_{g_2} h|^2}{\omega + \sqrt{\omega^2 + \left(|L_{g_2} h|^2\right)^2}}\,, \tag{108}$$

$$u_2 = \bar{u}_{\mathrm{QP}}(x) \;:=\; (L_{g_2} h)^T \frac{\max\{0, -\omega(x)\}}{|L_{g_2} h|^2}\,, \tag{109}$$

along with $\omega(x)$ defined in (107), system (100) with (101) satisfies the NSBFc in (105).

Example 7. To illustrate a design for NSSf, we return to Example 1 but with the offender input u_1 replaced by offender white noise of unknown variance $\Sigma_1(t)$, namely, to

$$\mathrm{d}x = (1 + x^2)\Sigma_1(t)\,\mathrm{d}w + u_2\,\mathrm{d}t\,. \tag{110}$$

To vary the design a bit but still keep it simple, we choose

$$h(x) = -x^3 \quad \text{and} \quad \alpha(h) = 3h\,. \tag{111}$$

Conducting the calculations with (107), with arbitrary $\rho \in \mathcal{K}_\infty$, we arrive at the QP feedback for the defender,

$$\bar{u}_{\mathrm{QP}} = \min\left\{0, -(1 + x^2)^2 \rho^{-1}\left(\max\{0, |x|x\}\right) - x\right\}\,. \quad \square \tag{112}$$

8. Inverse optimal noise-to-state safety game

Next, we give a pair of offender–defender feedback laws which ensure the attainment of a safety-based Nash equilibrium in a stochastic inverse optimal sense.

Theorem 7 (*A "Defender-Offender" Policy Pair that Attains a Nash Equilibrium in a Stochastic Inverse Optimal Sense*). *Consider the defender feedback law*

$$u_2 = \bar{u}_2(x) = R_2^{-1}\left(L_{g_2} h\right)^{\mathrm{T}} \frac{\ell\gamma_2\left(\left|L_{g_2} h R_2^{-1/2}\right|\right)}{\left|L_{g_2} h R_2^{-1/2}\right|^2}, \tag{113}$$

where $h(x)$ is a barrier function candidate, γ_1 and γ_2 are class $\mathcal{K}_\infty$ functions whose derivatives are also class $\mathcal{K}_\infty$ functions, and $R_2(x, u_0)$ is a matrix-valued function such that $R_2(x) = R_2(x)^{\mathrm{T}} > 0$. If the defender feedback (113) makes the system

$$\mathrm{d}x = f(x)\,\mathrm{d}t + g_1(x)d\bar{w} + g_2(x)u_2\,\mathrm{d}t \tag{114}$$

satisfy the NSBFc with respect to an NSSf-CBF candidate $h(x)$, where $\bar{w}$ is an offender input represented by an r-dimensional stochastic process with offender incremental covariance given by the feedback law

$$\bar{\Sigma}_1\bar{\Sigma}_1^{\mathrm{T}} = -2g_1^{\mathrm{T}}\frac{\partial^2 h}{\partial x^2}g_1 \frac{\ell\gamma_1\left(\left|g_1^{\mathrm{T}}\frac{\partial^2 h}{\partial x^2}g_1\right|_{\mathcal{F}}\right)}{\left|g_1^{\mathrm{T}}\frac{\partial^2 h}{\partial x^2}g_1\right|_{\mathcal{F}}^2}, \tag{115}$$

namely, if the condition

$$L_f h - \ell\gamma_1\left(\left|g_1^{\mathrm{T}}\frac{\partial^2 h}{\partial x^2}g_1\right|_{\mathcal{F}}\right) + \ell\gamma_2\left(\left|L_{g_2} h R_2^{-1/2}\right|\right) \geq -\alpha(h) \tag{116}$$

is satisfied, then the defender feedback

$$u_2 = \bar{u}_2^*(x) = \frac{\beta}{2} R_2^{-1}\left(L_{g_2} h\right)^{\mathrm{T}} \frac{(\gamma_2')^{-1}\left(\left|L_{g_2} h R_2^{-1/2}\right|\right)}{\left|L_{g_2} h R_2^{-1/2}\right|}, \quad \beta \geq 2 \tag{117}$$

maximizes the cost functional

$$J(u_2) = \inf_{\Sigma_1 \in \mathcal{U}_1} \left\{ \lim_{t\to\infty} E\left[2\beta h(x(t)) + \int_0^t \left(l(x) - \beta^2 \gamma_2 \left(\frac{2}{\beta} \left| R_2^{1/2} u_2 \right| \right) + \beta\lambda\gamma_1 \left(\frac{\left| \Sigma_1 \Sigma_1^{\mathrm{T}} \right|_{\mathcal{F}}}{\lambda} \right) \right) d\tau \right] \right\}, \tag{118}$$

where $\lambda \in (0, 2]$ *and*

$$\begin{aligned} l(x) &= -2\beta \left[L_f h - \ell\gamma_1 \left(\left| g_1^{\mathrm{T}} \frac{\partial^2 h}{\partial x^2} g_1 \right|_{\mathcal{F}} \right) + \ell\gamma_2 \left(\left| L_{g_2} h R_2^{-1/2} \right| \right) \right] \\ &\quad - \beta(\beta - 2)\ell\gamma_2 \left(\left| L_{g_2} h R_2^{-1/2} \right| \right) - \beta(2-\lambda)\ell\gamma_1 \left(\left| g_1^{\mathrm{T}} \frac{\partial^2 h}{\partial x^2} g_1 \right|_{\mathcal{F}} \right) \\ &\le 2\beta\alpha(h), \end{aligned} \tag{119}$$

and where the offender incremental covariance Nash feedback law is given by

$$\Sigma_1 \Sigma_1^{\mathrm{T}} = \left(\Sigma_1 \Sigma_1^{\mathrm{T}} \right)^* := -\lambda(\gamma_1')^{-1} \left(\left| g_1^{\mathrm{T}} \frac{\partial^2 h}{\partial x^2} g_1 \right|_{\mathcal{F}} \right) \frac{g_1^{\mathrm{T}} \frac{\partial^2 h}{\partial x^2} g_1}{\left| g_1^{\mathrm{T}} \frac{\partial^2 h}{\partial x^2} g_1 \right|_{\mathcal{F}}}. \tag{120}$$

Remark 4. Similar to Remark 2, even though not explicit in the statement of Theorem 7, $h(x)$ solves the following family of *Hamilton–Jacobi–Isaacs* equations parameterized by $\beta \in [2, \infty)$ and $\lambda \in (0, 2]$:

$$L_f h - \frac{\lambda}{2}\ell\gamma_1 \left(\left| g_1^{\mathrm{T}} \frac{\partial^2 h}{\partial x^2} g_1 \right|_{\mathcal{F}} \right) + \frac{\beta}{2}\ell\gamma_2 \left(\left| L_{g_2} h R_2^{-1/2} \right| \right) + \frac{l(x)}{2\beta} = 0. \tag{121}$$

This equation, which depends only on known quantities, helps explain why we are formulating a defender policy through a differential game for safety, with Σ_1 as an offender agent. □

Remark 5. Similar to Remark 3, we refer to the property

$$\lim_{t\to\infty} E\left\{ h(x(t)) + \int_0^t \alpha(h(x))\, \mathrm{d}t \right\} \ge -\frac{\lambda}{2} \int_0^t \gamma_1 \left(\frac{\left| \Sigma_1 \Sigma_1^{\mathrm{T}} \right|_{\mathcal{F}}}{\lambda} \right) \mathrm{d}t \tag{122}$$

as *integral noise-to-state safety* (iNSSf). □

Example 8. In Examples 4 and 5 we illustrated the results of that section on a unicycle, with the angular and linear velocity inputs being opponents in a deterministic safety game. Following the results of the present section, one could ask why at least one of the two agents would not be made stochastic? This is indeed of interest, and possible. One can make the angular velocity either stochastic or deterministic and either the offender or defender. Likewise, one can consider all these four possibilities for the linear velocity. In this example we focus on one of these combinations, which we consider particularly interesting, challenging, and perhaps natural. We make the linear velocity to be a deterministic offender and the angular velocity to be a stochastic defender, namely, we consider the unicycle system

$$\dot{x} = u_1 \cos\theta + v \tag{123}$$

$$\dot{y} = u_1 \sin\theta \tag{124}$$

$$\mathrm{d}\theta = \mathrm{d}w, \tag{125}$$

where the variance of the Brownian motion acting on the angular velocity is the defender input,

$$E\{\mathrm{d}w^2\} = u_2 \mathrm{d}t. \tag{126}$$

One can note from (125) that the heading is

$$\theta(t) = \theta_0 + w(t), \tag{127}$$

namely, the heading performs a “random walk” of controlled intensity $u_2(t)$ around the initial heading θ_0. Clearly, the angular velocity input u_2 is highly limited in how much it can influence safety since the heading $\theta(t)$ is a random walk around the initial heading. Note also that this example is not covered by Theorem 7, in which the offender is stochastic and the defender deterministic. In this example it is the opposite: the offender is deterministic and the defender stochastic. We make this switch (which requires a straightforward reformulating of Theorem 7) for two reasons. One reason is to illustrate that there is nothing limiting in Theorem 7 about attributing the stochastic character to the offender. The other reason is that it turns out that it is only in the case where the stochastic agent is angular velocity that the Nash policy, which has the meaning of variance, is nonnegative.

Skipping the details of the calculations, with the CBF $h = x/\cos^2\theta$ we obtain the Nash policies of the linear velocity deterministic offender and of the angular velocity stochastic defender as, respectively,

$$u_1^* = -\frac{\lambda}{\cos\theta} \tag{128}$$

$$u_2^* = \beta \frac{\cos^2\theta}{x} \frac{1 + \max\{0, -(x+v)\}}{2(1 + 2\sin^2\theta)}. \tag{129}$$

From (128) it is evident that the linear velocity acts towards the wall and blows up when the vehicle is parallel to the wall, whereas from (129) it is clear that the variance of angular velocity is high near the wall and shuts off in parallel with the wall. From these two observations it is clear that the offender is highly advantaged and the defender stands practically no chance of preventing the vehicle from being run into the wall since the heading does not really steer the vehicle but simply makes its heading a Brownian motion of intensity (129), while (128) ensures that the direction is always towards the wall. Nevertheless, the two Nash policies attain a Nash equilibrium by having the offender minimize and the defender maximize the performance index

$$E\left\{ \lim_{t\to\infty} \left[\frac{x(t)}{\cos^2\theta(t)} + \int_0^t \left(\eta(x(\tau), \theta(\tau)) - \frac{2}{\beta} \frac{x^2}{\cos^6\theta} \frac{(1 + 2\sin^2\theta)^2}{1 + \max\{0, -(x+v)\}} u_2^2(\tau) + \frac{1}{2\lambda} u_1^2(\tau) \right) \mathrm{d}\tau \right] \right\}, \tag{130}$$

where η satisfies (76).

Figs. 5 and 6 show the simulation results under the feedback laws (128) and (129). The details of the observations are in the figures' captions. In particular, while in both trajectories in Fig. 5 the vehicle runs into the wall, which should not be regarded as a defeat of the defender because the defender's task in the performance index (130) is not to keep the $x(t)$ positive but to maximize the performance index (130) under a minimizing policy of the offender. □

9. Conclusions

In this tutorial we have presented feedback control policies in a two-agent scenario, where the offending and defending agents have opposing objectives in relation to the system safety. Two types of offenders have been considered: deterministic and stochastic. The deterministic offender applies its feedback through its deterministic input value, while the stochastic offender applies its feedback through its incremental covariance.

Two types of results were presented, in both the deterministic and stochastic case. One set of results where the defender pursues input-to-state/noise-to-state safety for an arbitrary unknown offender action. The other set of results were of the zero-sum game type, where the offenders and the defenders settle into a Nash equilibrium for a game that is played over the system's safety.

It will be of value to the reader of this tutorial that we recapitulate some of its results, especially contrasting the results that come in pairs.

First, the basic ingredients in the design of the feedback for the defender, in both the deterministic and stochastic case, where the offender is unconstrained by a cost on his action, are the CBF-QP feedback and the ‘half-Sontag’ feedback, given, respectively, by

$$\bar{u}_{\mathrm{QP}}(x) = (L_{g_2} h)^T \frac{\max\{0, -\omega(x)\}}{|L_{g_2} h|^2}, \tag{131}$$

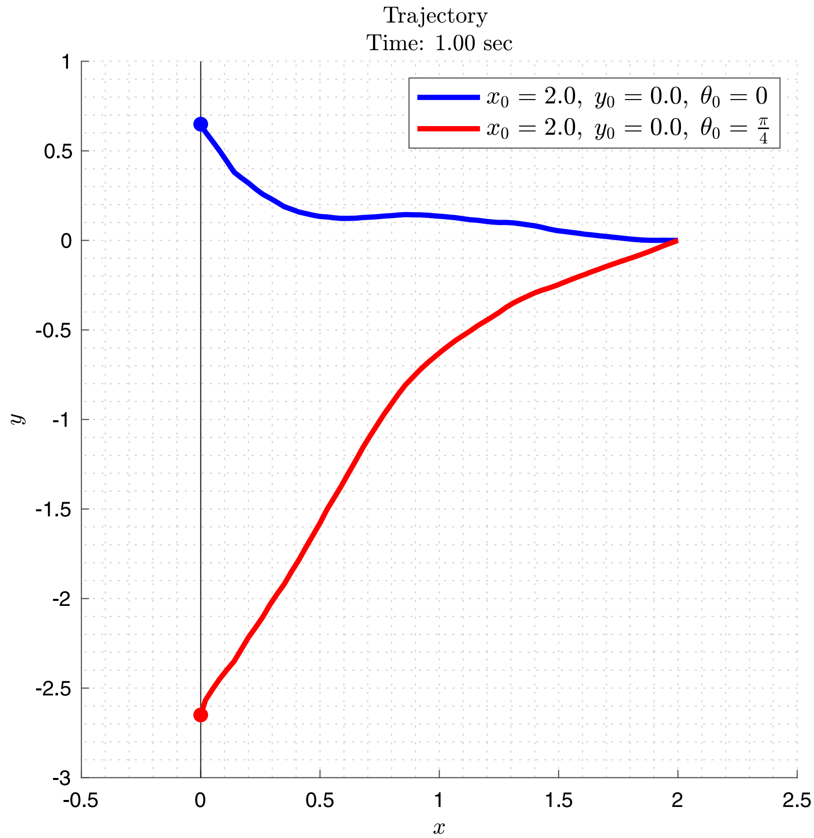

Fig. 5. Two trajectories in which the dots represent the trajectories' terminal points. The offender input, applied through a deterministic linear velocity, which always directs the vehicle towards the wall, is advantaged over the defender input, applied through the variance of a stochastic angular velocity, which makes the heading a Brownian motion of varying intensity around the initial heading. The result of this inequity is that, while in Fig. 2 the defender always prevents the safety violation, in the present figure the offender always succeeds in violating safety, namely, in running the vehicle into the wall. This should not be regarded as a "win" for the offender and a "loss" for the defender because the defender's task in the performance index (130) is not to keep the $x(t)$ positive but to maximize the performance index (130) under a minimizing policy of the offender.

$$u_S(x)/2 = (L_{g_2}h)^T \frac{-\omega + \sqrt{\omega^2 + |L_{g_2}h|^4}}{2|L_{g_2}h|^2} \tag{132}$$

There is barely any difference between these two feedback laws when $|L_{g_2}h|$ is small relative to $|\omega|$.

Second, the Nash policy of the offender, in the deterministic and stochastic cases, is given, respectively, by

$$u_1^* = -\lambda(\gamma_1')^{-1}(2|L_{g_1}h|)\frac{(L_{g_1}h)^T}{|L_{g_1}h|} \tag{133}$$

$$\left(\Sigma_1\Sigma_1^T\right)^* = -\lambda(\gamma_1')^{-1}\left(\left|g_1^T\frac{\partial^2 h}{\partial x^2}g_1\right|_F\right)\frac{g_1^T\frac{\partial^2 h}{\partial x^2}g_1}{\left|g_1^T\frac{\partial^2 h}{\partial x^2}g_1\right|_F} \tag{134}$$

Evidently, the deterministic and stochastic policies are highly consistent among one another, with the only difference arising out of Ito's calculus and the effect of the Hessian (rather than gradient) of the CBF.

Third, the resulting integral ISSf/NSSf properties, under the Nash policies for the defender but under an arbitrary policy for the offender are given by the inequalities, in the deterministic and stochastic cases, respectively, by

$$h(x(t)) + \int_0^\infty \alpha(h(x))\,\mathrm{d}t \geq -\frac{\lambda}{2}\int_0^\infty \gamma\left(\frac{|u_1|}{\lambda}\right)\mathrm{d}t \tag{135}$$

$$\lim_{t\to\infty} E\left\{h(x(t)) + \int_0^t \alpha(h(x))\,\mathrm{d}t\right\} \geq -\frac{\lambda}{2}\int_0^t \gamma_1\left(\frac{\left|\Sigma_1\Sigma_1^T\right|_F}{\lambda}\right)\mathrm{d}t \tag{136}$$

It is evident from these integral inequalities that the safety loss, incurred over the infinite horizon, may increase, at worst, in (class $\mathcal{K}$) proportion with the intensity of the offender's action.

Fourth, the underlying Hamilton–Jacobi–Isaacs PDEs solved by the CBF $h(x)$ under the inverse optimal actions of the offender and defender are given, in the deterministic and stochastic case, respectively, by

$$L_f h - \frac{\lambda}{2}\ell\gamma_1(2|L_{g_1}h|) + \frac{\beta}{2}L_{g_2}hR_2(x)^{-1}\left(L_{g_2}h\right)^T + \frac{l(x)}{2\beta} = 0 \tag{137}$$

$$L_f h - \frac{\lambda}{2}\ell\gamma_1\left(\left|g_1^T\frac{\partial^2 h}{\partial x^2}g_1\right|_F\right) + \frac{\beta}{2}\ell\gamma_2\left(\left|L_{g_2}hR_2^{-1/2}\right|\right) + \frac{l(x)}{2\beta} = 0 \tag{138}$$

Finally, as evident throughout the tutorial, the defender and offender policies are highly similar under the safety game scenario.

Looking towards future research, it is of interest to expand the consideration from the unicycle example towards (1) more general vehicle models, incorporating dynamics, and (2) offender inputs which are of environmental nature (wind, gravity, etc.).

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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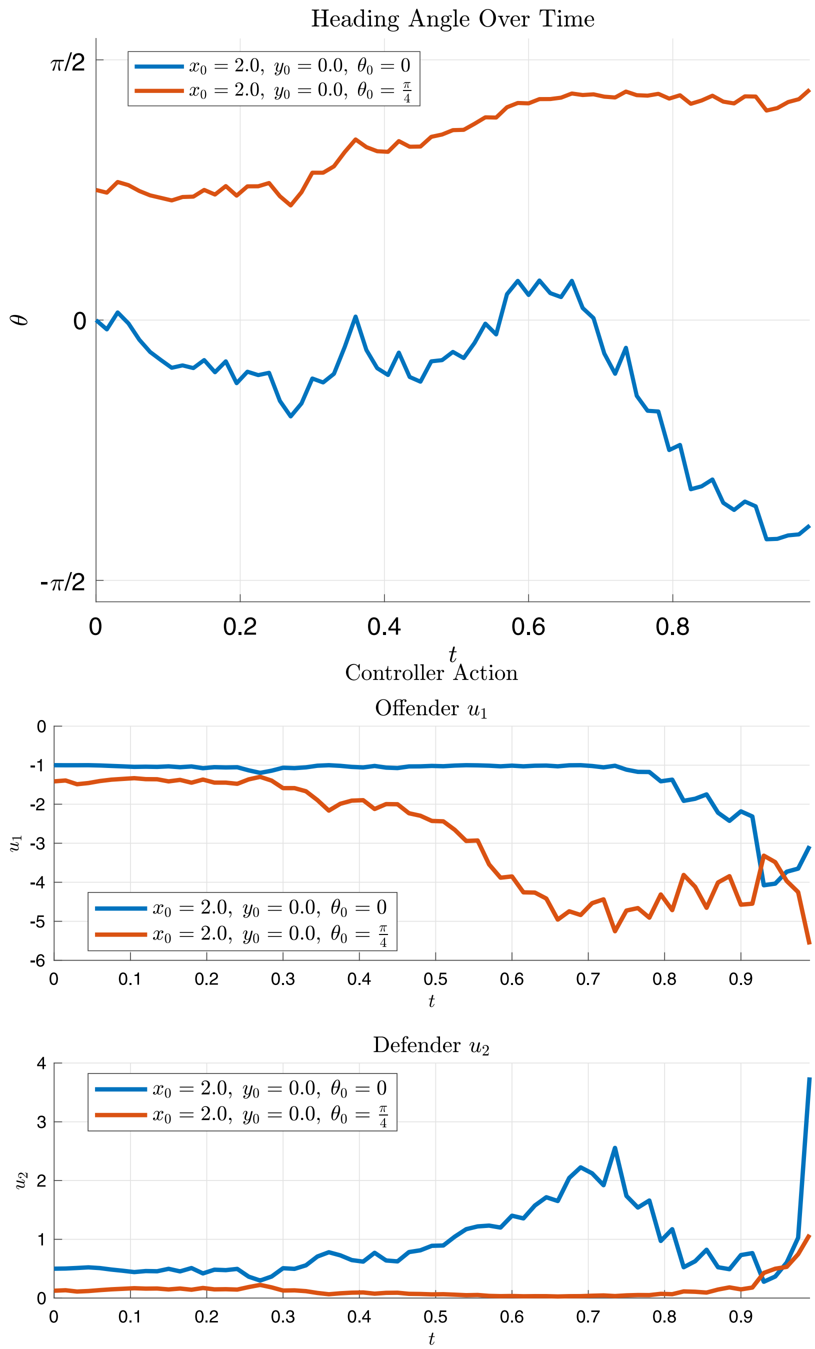

Fig. 6. The two heading transients of the vehicle perform a random walk, but with a variance $u_2^*(t)$ given by (129) changing in time and going large (theoretically, towards infinity) as $x(t) \to 0^+$, as the vehicle runs into the wall.

Appendix

Legendre–Fenchel transform and Young's inequality

Lemma 3 (Krstic & Li, 1998, *Lemma A.1*). *If γ and its derivative γ' are class $\mathcal{K}_\infty$, then the Legendre–Fenchel transform satisfies the following properties:*

$$(a)\quad \ell\gamma(r) = r(\gamma')^{-1}(r) - \gamma\left((\gamma')^{-1}(r)\right) = \int_0^r (\gamma')^{-1}(s)ds \tag{139}$$

$$(b)\quad \ell\ell\gamma = \gamma \tag{140}$$

$$(c)\quad \ell\gamma \text{ is a class } \mathcal{K}_\infty \text{ function} \tag{141}$$

$$(d)\quad \ell\gamma(\gamma'(r)) = r\gamma'(r) - \gamma(r)\,. \tag{142}$$

Lemma 4 (*Young's Inequality, Hardy, Littlewood, & Polya, 1989, Theorem 156*). *For any $x, y \in \mathbb{R}^n$, and for any $\gamma \in \mathcal{K}_\infty$ whose derivative is also in $\mathcal{K}_\infty$,*

$$x^{\mathrm{T}}y \le \gamma(|x|) + \ell\gamma(|y|)\,, \tag{143}$$

and the equality is achieved if and only if

$$y = \gamma'(|x|)\frac{x}{|x|}\,,\quad \text{that is, for } x = (\gamma')^{-1}(|y|)\frac{y}{|y|}\,. \tag{144}$$

Proofs of theorems

Proof of Theorem 1 (*Defender Uses Sontag $\Longrightarrow$ ISSf*)**.** We substitute (8) into (4) and get

$$\dot h = L_f h + L_{g_1} h u_1 - \omega + \sqrt{\omega^2 + \left(L_{g_2} h (L_{g_2} h)^{\mathrm T}\right)^2} \geq -\alpha(h(x)) + |L_{g_1} h| \left[\rho^{-1}(\max\{0, -h(x)\}) - |u_1|\right] . \tag{145}$$

For $\min\{0, h(x)\} \leq -\rho(|u_1|)$ we thus have

$$\dot h = L_{f+g_2\alpha_S} + L_{g_1} h u_1 \geq -\alpha(h(x)) , \tag{146}$$

which, thanks to Lemma 1, completes the proof of ISSf. □

Proof of Theorem 2 (*Defender Uses QP-ISSf-CBF $\Longrightarrow$ ISSf*)**.** We substitute (14) and (12) into (4), get

$$\begin{aligned}\dot h &= L_f h + L_{g_1} h u_1 + L_{g_2} h \bar u_{\mathrm{QP}} \\ &= -\alpha(h(x)) + \omega + \max\{0, -\omega\} + |L_{g_1} h| \rho^{-1}(\max\{0, -h(x)\}) + L_{g_1} h u_1 \\ &\geq -\alpha(h(x)) + \max\{\omega, 0\} + |L_{g_1} h| \left[\rho^{-1}(\max\{0, -h(x)\}) - |u_1|\right] \\ &\geq -\alpha(h(x)) + |L_{g_1} h| \left[\rho^{-1}(\max\{0, -h(x)\}) - |u_1|\right] ,\end{aligned} \tag{147}$$

and invoke Lemma 1. □

Proof of Theorem 3 (*"Half-Sontag" Formula is Also Min-norm*)**.** The pointwise minimization result is immediate from (10)–(12). For (4), (23) ISSf follows from

$$\dot h = -\alpha(h(x)) + \frac{1}{2}\left(\omega + \sqrt{\omega^2 + \left(L_{g_2} h (L_{g_2} h)^{\mathrm T}\right)^2}\right) + |L_{g_1} h| \rho^{-1}(\max\{0, -h(x)\}) + L_{g_1} h u_1 . \quad \square \tag{148}$$

Proof of Theorem 4 (*Inverse Optimally Nash "Defender-Offender" Policy Pair*)**.** Thanks to (31), (34), we get (35). Substituting $l(x)$ into (33), it follows that

$$\begin{aligned} J(u_2) &= \inf_{u_1 \in \mathcal U_1} \Bigg\{ \lim_{t\to\infty} \Bigg[2\beta h(x(t)) + \int_0^t \Bigg(-2\beta L_f h + \beta\lambda\ell\gamma(2|L_{g_1} h|) \\ &\quad - \beta^2 L_{g_2} h R_2^{-1} \left(L_{g_2} h\right)^{\mathrm T} - u_2^{\mathrm T} R_2 u_2 + \beta\lambda\gamma\left(\frac{|u_1|}{\lambda}\right)\Bigg) d\tau \Bigg] \Bigg\} \\ &= \inf_{u_1 \in \mathcal U_1} \Bigg\{ \lim_{t\to\infty} \Bigg[2\beta h(x(t)) - 2\beta \int_0^t \left(L_f h + L_{g_1} h d + L_{g_2} h u_2 \right) d\tau \\ &\quad - \int_0^t \left(u_2^{\mathrm T} R_2 u_2 - 2\beta L_{g_2} h u_2 + \beta^2 L_{g_2} h R_2^{-1} \left(L_{g_2} h\right)^{\mathrm T} \right) d\tau \\ &\quad + \int_0^t \left(\beta\lambda\gamma\left(\frac{|u_1|}{\lambda}\right) + 2\beta L_{g_1} h u_1 + \beta\lambda\ell\gamma(2|L_{g_1} h|) \right) d\tau \Bigg] \Bigg\} \\ &= \inf_{u_1 \in \mathcal U_1} \Bigg\{ \lim_{t\to\infty} \Bigg[2\beta h(x(t)) - 2\beta \int_0^t dh - \int_0^t (u_2 - \bar u_2^*)^{\mathrm T} R_2 (u_2 - \bar u_2^*) d\tau \\ &\quad + \beta \int_0^t \Bigg[\lambda\gamma\left(\frac{|u_1|}{\lambda}\right) - \lambda\gamma\left((\gamma')^{-1}(2|L_{g_1} h|)\right) \\ &\quad + 2\left(\lambda |L_{g_1} h| (\gamma')^{-1}(2|L_{g_1} h|) + L_{g_1} h u_1\right)\Bigg] d\tau \Bigg] \Bigg\} \qquad \text{(by (27))} \\ &= 2\beta h(x(0)) + \beta\lambda \inf_{u_1 \in \mathcal U_1} \int_0^\infty \Pi(u_1, u_1^*)\, dt \\ &\quad - \int_0^\infty (u_2 - \bar u_2^*)^{\mathrm T} R_2 (u_2 - \bar u_2^*) d\tau\, dt , \end{aligned} \tag{149}$$

where

$$\Pi(u_1, u_1^*) = \gamma\left(\frac{|u_1|}{\lambda}\right) - \gamma\left(\frac{|u_1^*|}{\lambda}\right) - \gamma'\left(\frac{|u_1^*|}{\lambda}\right) \frac{(u_1^*)^{\mathrm T}}{\lambda |u_1^*|} (u_1^* - u_1) \tag{150}$$

and

$$u_1^*(x) = -\lambda (\gamma')^{-1}(2|L_{g_1} h|) \frac{(L_{g_1} h)^{\mathrm T}}{|L_{g_1} h|} . \tag{151}$$

By Lemma 3.d, $\Pi(u_1, u_1^*)$ can be rewritten as

$$\Pi(u_1, u_1^*) = \gamma\left(\frac{|u_1|}{\lambda}\right) + \ell\gamma\left(\gamma'\left(\frac{|u_1^*|}{\lambda}\right)\right) + \gamma'\left(\frac{|u_1^*|}{\lambda}\right) \frac{(u_1^*)^{\mathrm T}}{|u_1^*|} \frac{u_1}{\lambda} . \tag{152}$$

Then by Lemma 4 we have

$$\begin{aligned}\Pi(u_1, u_1^*) &\geq \gamma\left(\frac{|u_1|}{\lambda}\right) + \ell\gamma\left(\gamma'\left(\frac{|u_1^*|}{\lambda}\right)\right) - \gamma\left(\frac{|u_1|}{\lambda}\right) - \ell\gamma\left(\gamma'\left(\frac{|u_1^*|}{\lambda}\right)\right) \\ &= 0 ,\end{aligned} \tag{153}$$

and $\Pi(u_1, u_1^*) = 0$ if and only if $\frac{u_1}{\lambda} = (\gamma')^{-1}\left(\gamma'\left(\frac{|u_1^*|}{\lambda}\right)\right) \frac{u_1^*}{|u_1^*|}$, that is,

$$\Pi(u_1, u_1^*) = 0 \qquad \text{iff} \qquad u_1 = u_1^* . \tag{154}$$

Thus

$$\inf_{u_1 \in \mathcal U_1} \int_0^\infty \Pi(u_1, u_1^*)\, dt = 0 , \tag{155}$$

and the "worst case" disturbance is given by (151). The maximum of (149) is reached with

$$u_2 = \bar u_2^* . \tag{156}$$

Hence the control law (32) maximizes the cost functional (33). The value function of (33) is

$$J^*(x) = 2\beta h(x) . \quad \square \tag{157}$$

Proof of Theorem 6 (*NSSf Under Sontag or QP-NSSf-CBF Formula*)**.** For both control laws, a direct substitution yields

$$\mathcal L h \geq -\alpha(h(x)) \tag{158}$$

whenever

$$\min\{0, h(x)\} \leq -\rho\left(\left|\Sigma_1 \Sigma_1^{\mathrm T}\right|_{\mathcal F}\right) . \quad \square \tag{159}$$

Proof of Theorem 7 (*Stochastic Inverse Optimally Nash "Defender-Offender" Policy Pair*)**.** According to Dynkin's formula and by substituting $l(x)$ into $J(u_2)$, we have

$$\begin{aligned} J(u_2) &= \inf_{\Sigma_1 \in \mathcal U_1} \Bigg\{ \lim_{t\to\infty} E\Bigg[2\beta h(x(t)) \\ &\quad + \int_0^t \Bigg(l(x) - \beta^2 \gamma_2\left(\frac{2}{\beta}\left|R_2^{1/2} u_2\right|\right) + \beta\lambda\gamma_1\left(\frac{\left|\Sigma_1 \Sigma_1^{\mathrm T}\right|_{\mathcal F}}{\lambda}\right)\Bigg) d\tau \Bigg] \Bigg\} \\ &= \inf_{\Sigma_1 \in \mathcal U_1} \Bigg\{ \lim_{t\to\infty} E\Bigg[2\beta h(x(0)) \\ &\quad + \int_0^t \Bigg(2\beta \mathcal L h \,|_{(100)} + l(x) - \beta^2 \gamma_2\left(\frac{2}{\beta}\left|R_2^{1/2} u_2\right|\right) \\ &\quad + \beta\lambda\gamma_1\left(\frac{\left|\Sigma_1 \Sigma_1^{\mathrm T}\right|_{\mathcal F}}{\lambda}\right)\Bigg) d\tau \Bigg] \Bigg\} \\ &= \inf_{\Sigma_1 \in \mathcal U_1} \{ 2\beta E\{V(x(0))\} \\ &\quad + \lim_{t\to\infty} E \int_0^t \Bigg[-\beta^2 \gamma_2\left(\frac{2}{\beta}\left|R_2^{1/2} u_2\right|\right) - \beta^2 \ell\gamma_2\left(\left|L_{g_2} h R_2^{-1/2}\right|\right) \\ &\quad + 2\beta L_{g_2} h u_2 \\ &\quad + \beta\lambda\gamma_1\left(\frac{\left|\Sigma_1 \Sigma_1^{\mathrm T}\right|_{\mathcal F}}{\lambda}\right) + \beta\lambda\ell\gamma_1\left(\left|g_1^{\mathrm T} \frac{\partial^2 h}{\partial x^2} g_1\right|_{\mathcal F}\right) \\ &\quad + \beta \mathrm{Tr}\left\{\Sigma_1^{\mathrm T} g_1^{\mathrm T} \frac{\partial^2 h}{\partial x^2} g_1 \Sigma_1\right\}\Bigg] d\tau \Bigg\} . \end{aligned} \tag{160}$$

Using Lemma 4 we have

$$-2\beta L_{g_2} h u_2 = \beta^2 \left(\frac{2}{\beta} R_2^{1/2} u_2\right)^{\mathrm T} \left(-R_2^{-1/2} \left(L_{g_2} h\right)^{\mathrm T}\right)$$

$$\leq \beta^2\gamma_2\left(\frac{2}{\beta}\left|R_2^{1/2}u_2\right|\right)+\beta^2\ell\gamma_2\left(\left|L_{g_2}hR_2^{-1/2}\right|\right) \tag{161}$$

and

$$\beta\mathrm{Tr}\left\{\Sigma_1^{\mathrm{T}}g_1^{\mathrm{T}}\frac{\partial^2 h}{\partial x^2}g_1\Sigma_1\right\}=\beta\left(\mathrm{col}\left(\Sigma_1\Sigma_1^{\mathrm{T}}\right)\right)^{\mathrm{T}}\left(\mathrm{col}\left(g_1^{\mathrm{T}}\frac{\partial^2 h}{\partial x^2}g_1\right)\right)$$
$$\leq \beta\lambda\gamma_1\left(\frac{\left|\Sigma_1\Sigma_1^{\mathrm{T}}\right|_{\mathcal{F}}}{\lambda}\right)+\beta\lambda\ell\gamma_1\left(\left|g_1^{\mathrm{T}}\frac{\partial^2 h}{\partial x^2}g_1\right|_{\mathcal{F}}\right) \tag{162}$$

and the equalities hold when (117) and

$$\left(\Sigma_1\Sigma_1^{\mathrm{T}}\right)^*=-\lambda(\gamma_1')^{-1}\left(\left|g_1^{\mathrm{T}}\frac{\partial^2 h}{\partial x^2}g_1\right|_{\mathcal{F}}\right)\frac{g_1^{\mathrm{T}}\frac{\partial^2 h}{\partial x^2}g_1}{\left|g_1^{\mathrm{T}}\frac{\partial^2 h}{\partial x^2}g_1\right|_{\mathcal{F}}}. \tag{163}$$

So the “worst case” unknown covariance is given by (163), the minimum of (160) is reached with

$$u_2=\bar{u}_2^* \tag{164}$$

and

$$\min_{u_2} J(u_2)=2\beta E\left\{h(x(0))\right\}. \quad \square \tag{165}$$

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