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Rayleigh-Taylor instability for nonhomogeneous incompressible geophysical fluid with partial viscosity

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Abstract

In this article, we investigate the Rayleigh-Taylor instability in a system of two-dimensional nonhomogeneous incompressible fluid equations with Coriolis force and partial viscosity. First, we employ variational methods to construct linear unstable solutions to the corresponding linearized equations of the system. Second, we utilize the classical Osgood lemma to derive nonlinear energy estimates for the perturbed equations. The local existence of solutions to the perturbed equations is established by using the semi-Galerkin method and the expanding domain method. Finally, we prove the nonlinear instability by combining the properties of the linear unstable solutions and the nonlinear energy estimates.
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Keywords: Rayleigh-Taylor instability; Variational method; Fluid model; Coriolis force; Partial viscosity

Contents

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1. Introduction

Fluid stability is a significant area of nonlinear sciences. The aim of studying fluid stability is to gain a deeper understanding of the behavior of fluid flows and to develop tools and techniques to control and manipulate these flows for practical applications. There is a vast amount of literature dedicated to the mathematical analysis of fluid stability and instability, cf. [3,4,13–16,21,30,35,43,44] among many others. A prime example is Rayleigh-Taylor instability when heavier fluid lies above lighter one [11,27,34], see the latest review article [49] for its many applications.

The purpose of this article is to study the Rayleigh-Taylor instability of nonhomogeneous rotating incompressible viscous fluids in the presence of a uniform gravitational field from the perspective of nonlinear instability. The motion of the fluid in $\mathbb{R}^3$ is governed by the following equations

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \mathbf{u}\cdot\nabla_3\rho = 0,\\
&\rho\frac{\partial \mathbf{u}}{\partial t} + \rho\mathbf{u}\cdot\nabla_3\mathbf{u} + f\mathbf{e}\times\mathbf{u} = N(\mathbf{u}) - \nabla_3 p - g\rho\mathbf{e},\\
&\nabla_3\cdot\mathbf{u} = 0,
\end{aligned} \tag{1.1}$$

where the unknown functions ρ, $\mathbf{u} = (u_1, u_2, u_3)$ and p denote the density, the velocity and the pressure, respectively. $N(\mathbf{u})$ is the viscosity term, ∇_3 is the 3D gradient operator. Throughout μ and g denote dynamical viscosity and the gravitational constant, respectively. f represents the speed of rotation around the vertical unit vector $\mathbf{e} = (0, 0, 1)$.

In the case of full viscosity $N(\mathbf{u}) = \mu\Delta\mathbf{u}$ and $f = 0$, the equations (1.1) reduce to the standard nonhomogeneous Navier-Stokes equations. The Rayleigh-Taylor instability and well-posedness have been thoroughly investigated, see [10,11,27,29]. Because of its importance in geophysics, fluid models of anisotropic viscosity with or without rotation have been intensively studied in recent years. Examples of partial viscosity include the formulation $N(\mathbf{u}) = \mu\Delta_h\mathbf{u} := \mu\left(\frac{\partial^2\mathbf{u}}{\partial x^2} + \frac{\partial^2\mathbf{u}}{\partial y^2}\right)$ [8] and the two-dimensional case $N(\mathbf{u}) = (\mu\frac{\partial^2 u}{\partial x^2}, 0)$ [41]. Existing research primarily focuses on the well-posedness [1,5,6,8,31,48], stability and the rate of asymptotic decay

[40,41] of these fluid models with partial viscosity. The study of the Rayleigh-Taylor instability in fluid models with partial viscosity is still limited and requires further investigation.

In this article, we study the Rayleigh-Taylor instability in the fluid model (1.1) with partial viscosity. Specifically, we consider the case where the fluid is uniform in the x direction and the partial viscosity term takes the form $N(\mathbf{u}) = (0, \mu\Delta\tilde{\mathbf{u}})$. This type of partial viscosity satisfies orthogonal invariance. The system (1.1) is reduced to the following equations:

$$\begin{aligned} &\frac{\partial\rho}{\partial t} + \tilde{\mathbf{u}}\cdot\nabla\rho = 0,\\ &\rho\frac{\partial u_1}{\partial t} + \rho\tilde{\mathbf{u}}\cdot\nabla u_1 - fu_2 = 0,\\ &\rho\frac{\partial\tilde{\mathbf{u}}}{\partial t} + \rho\tilde{\mathbf{u}}\cdot\nabla\tilde{\mathbf{u}} + fu_1\mathbf{e}_1 = \mu\Delta\tilde{\mathbf{u}} - \nabla p - g\rho\mathbf{e}_2,\ (\mathbf{x},t)\in\mathbb{R}^2\times(0,+\infty),\\ &\nabla\cdot\tilde{\mathbf{u}} = 0, \end{aligned} \tag{1.2}$$

where $\tilde{\mathbf{u}} = (u_2, u_3)$ is the components of the velocity on the $\mathbf{x} = (y, z)$ plane, ∇ is the 2D gradient operator $\nabla = (\frac{\partial}{\partial y}, \frac{\partial}{\partial z})$, Δ is the 2D Laplacian operator $\Delta = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$.

Let $\bar{u}_0$ be a fixed constant. The steady state solutions under investigation are characterized by

$$\mathbf{u}_s = (y\bar{u}_0, 0, 0),\ \rho_s = \rho_s(z),\ p_s = p_s(y, z), \tag{1.3}$$

where the pressure p_s and the density ρ_s satisfy the following geostrophic balance

$$-\frac{\partial p_s}{\partial y} = yf\bar{u}_0,\ -\frac{\partial p_s}{\partial z} = g\rho_s. \tag{1.4}$$

Furthermore we assume that the steady-state density satisfies the conditions

$$\rho_s' \in C_0^\infty(\mathbb{R}),\ \rho_s'(z_0) > 0,\ \text{for some } z_0\in\mathbb{R},\ \inf_{z\in\mathbb{R}}\rho_s > 0. \tag{1.5}$$

Here (1.5) indicates the presence of at least one region where the density is monotonically increasing, leading to the occurrence of Rayleigh-Taylor instability. We introduce the following perturbations

$$\mathbf{v} = \mathbf{u} - \mathbf{u}_s,\ \varrho = \rho - \rho_s,\ q = p - p_s. \tag{1.6}$$

Substituting (1.6) into (1.2), one finds the perturbations satisfy the following equations

$$\begin{aligned} &\frac{\partial\varrho}{\partial t} + \tilde{\mathbf{v}}\cdot\nabla(\varrho+\rho_s) = 0,\\ &(\varrho+\rho_s)\frac{\partial v_1}{\partial t} + (\varrho+\rho_s)\tilde{\mathbf{v}}\cdot\nabla v_1 + \bar{u}_0(\varrho+\rho_s)v_2 - fv_2 = 0,\\ &(\varrho+\rho_s)\frac{\partial\tilde{\mathbf{v}}}{\partial t} + (\varrho+\rho_s)\tilde{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}} + fv_1\mathbf{e}_1 = \mu\Delta\tilde{\mathbf{v}} - \nabla q - g\varrho\mathbf{e}_2,\\ &\nabla\cdot\tilde{\mathbf{v}} = 0, \end{aligned} \tag{1.7}$$

with the initial and far-field conditions

$$(\varrho, \mathbf{v})|_{t=0} = (\varrho_0, \mathbf{v}_0), \tag{1.8}$$

$$\lim_{|\mathbf{x}|\to+\infty} \mathbf{v}(t, \mathbf{x}) = 0, \ \forall t > 0. \tag{1.9}$$

Throughout the paper, we denote $\tilde{\mathbf{v}} = (v_2, v_3)$ for convenience.

The linear part of the nonlinear system is as follows

$$\begin{aligned} &\frac{\partial \varrho}{\partial t} + v_3 \varrho_s' = 0, \\ &\rho_s \frac{\partial v_1}{\partial t} + [\rho_s \bar{u}_0 - f] v_2 = 0, \\ &\rho_s \frac{\partial \tilde{\mathbf{v}}}{\partial t} + f v_1 \mathbf{e}_1 = \mu \Delta \tilde{\mathbf{v}} - \nabla q - g \varrho \mathbf{e}_2, \\ &\nabla \cdot \tilde{\mathbf{v}} = 0. \end{aligned} \tag{1.10}$$

Initially introduced by Rayleigh [38] and further explored by Taylor [17], the Rayleigh-Taylor (RT instability) instability is well known as hydrodynamic instability that occurs in fluid systems where a denser fluid is positioned above a lighter fluid. The fundamental characteristic of RT instability lies in its nonlinear nature, whereby small perturbations are amplified through nonlinear interactions, leading to a change in the system's behavior. Consequently, predicting and controlling the system becomes challenging, as even minor errors or alterations can result in a completely different evolutionary trajectory. Given its broad applicability in science and engineering, the RT instability has been extensively studied in theory and in numerical simulations [9,33,39,45,49]. Guo and Tice [19] investigated linear RT instability of a two-layer compressible viscous fluid model, see also [20] for extensions of the study to an inviscid fluid model. Subsequently, Jiang et al. [27] studied the nonlinear RT instability of a three dimensional viscous fluid model. Recently, Jiang et al. [28] showed the existence of unstable strong solutions to an abstract RT problem. We refer to [22–26,32,37,42,46,47] for more studies on various aspects of RT instability.

The influence of Coriolis force and partial viscosity is key for studying large-scale atmospheric and oceanic flows. It is widely acknowledged that rotational effects play a dominant role in such phenomena. Consequently, nearly all oceanographic and meteorological models that address large-scale phenomena incorporate the inclusion of Coriolis forces [36]. For instance, the circulation patterns in the ocean, particularly those associated with hurricanes, are primarily driven by substantial rotations. Other physical factors, including salinity and natural boundary conditions, also hold significant importance. In addition, the effect of partial viscosity (or partial dissipation) is of great significance in geophysics and fluid mechanics, especially in the simulation of large-scale atmospheric and oceanic flows. In order to gain a comprehensive understanding of the behavior of rotating fluids with partial viscosity, it is imperative to investigate the RT instability of the Navier-Stokes equations, taking into account both partial viscous effects and Coriolis forces.

Before stating the main results of this paper, we introduce a definition of nonlinear instability, more introduction on nonlinear instability of fluid equations can be see in [18].

Definition 1.1. We say that the steady state solution $\mathbf{u}_s$ is a nonlinear unstable solution of (1.2) if there exists σ and constant C_k such that for every k arbitrarily large and every δ arbitrarily small there exists a solution $\mathbf{u}$ of (1.2) satisfying

$$||\mathbf{u}(0,\mathbf{x})-\mathbf{u}_s||_{H^k}\le\delta,$$

and

$$||\mathbf{u}(T^\delta,\mathbf{x})-\mathbf{u}_s||_{L^2}\ge\sigma,$$

for some times T^δ where

$$T^\delta\le C_k\log(1+\delta^{-1})+C_k.$$

The main results of the article are summarized in the following three theorems.

Theorem 1.1 *(Linear instability). Suppose that the parameter f satisfies $f\ge\bar{u}_0\sup_{z\in\mathbb{R}}\rho_s$, and that the steady density profile ρ_s satisfies (1.5). Then the steady state $(\rho_s,\mathbf{u}_s,p_s)$ is linearly unstable. Namely, there exists a unique unstable solution $(\varrho,\mathbf{v},q)$ to (1.8)-(1.10) with the constant growth rate Λ defined by*

$$\Lambda=\sup_{|\xi|\in[R_1,R_2]}\lambda(|\xi|)\le\left\{\left|\left|\sqrt{\frac{f(f-\rho_s\bar{u}_0)}{\rho_s^2}}\right|\right|_{L^\infty(\mathbb{R})}+\sqrt{g}\left|\left|\sqrt{\frac{\rho_s'}{\rho_s}}\right|\right|_{L^\infty(\mathbb{R})}\right\},\tag{1.11}$$

where R_1,R_2 satisfy $\sqrt{\beta}<R_1<R_2<\infty$ and β is given by (2.23).

Theorem 1.1 holds true from Theorem 2.3 and Theorem 2.4.

Theorem 1.2 *(Existence). Assume that $||\varrho_0||^2_{H^1(\mathbb{R}^2)}+||v_{10}||^2_{H^1(\mathbb{R}^2)}+||\tilde{\mathbf{v}}_0||^2_{H^2(\mathbb{R}^2)}\le\sigma^2$. Then there exists a strong solution $(\varrho,v_1,\tilde{\mathbf{v}})$ to (1.7)-(1.9) such that*

$$\begin{aligned}&\varrho\in L^\infty(0,T;H^1(\mathbb{R}^2)),\, v_1\in L^\infty(0,T;H^1(\mathbb{R}^2)),\ v_{1t}\in L^2(\mathbb{R}^2\times(0,T)),\\ &\tilde{\mathbf{v}}\in L^\infty(0,T;H^2(\mathbb{R}^2)),\ \tilde{\mathbf{v}}_t\in L^\infty(0,T;L^2(\mathbb{R}^2)),\ \nabla\tilde{\mathbf{v}}_t\in L^2(\mathbb{R}^2\times(0,T)),\end{aligned}$$

where $0<T<T^*$ and T^* is the maximal time of existence of the solution.

Theorem 1.3 *(Nonlinear instability). Under the same conditions as Theorem 1.1, the steady state* $(0,0,0,0)$ *of (1.7)-(1.9) is unstable under the Lipschitz structure. That is, for any $k\ge2$, $\sigma>0$, there exists a constant $i_0:=i_0(k)>0$ and smooth initial data*

$$(\varrho_0,\mathbf{v}_0)\in(H^\infty(\mathbb{R}^2))^4,\ \textit{with}\ ||\varrho_0||^2_{H^k(\mathbb{R}^2)}+||\mathbf{v}_0||^2_{H^k(\mathbb{R}^2)}\le\sigma^2,\tag{1.12}$$

such that for some $0<K<\frac{i_0}{2}e^{\frac{\Lambda T^*}{2}}$ as well as F satisfying

$$F(y)\le Ky,\ \ \textit{for any}\ y\in[0,\infty),\tag{1.13}$$

there exists a strong solution $(\varrho, \mathbf{v})$ *of* (1.7)-(1.9)*, emanating from the initial data* (ϱ_0, v_0)*, satisfying*

$$||v_3(t_K)||_{L^2(\mathbb{R}^2)} > F(||(\varrho_0, \mathbf{v}_0)||_{H^k(\mathbb{R}^2)}) \\ \textit{for some } t_k \in (0, \frac{2}{\Lambda} \ln \frac{2K}{i_0}) \subset (0, T^*), \tag{1.14}$$

where the constant Λ *is given by* (1.11)*,* $H^\infty(\mathbb{R}^2) = \cap_{k=1}^{\infty} H^k(\mathbb{R}^2)$.

Our approach to the proof of Theorem 1.3 can be outlined as follows, cf. [2]. In Step 1, presented in Section 2, we construct linearly unstable solutions to the perturbed linear equations (1.10). We look for solutions with an exponential growth factor $e^{\lambda(\xi)t}$ where $\xi \in \mathbb{R}$ represents the horizontal spatial frequency. By the Fourier transform, the linear equations are reduced to an ordinary differential equation, and is solved via the classical variational method, leading to the establishment of the continuous function $\lambda(|\xi|) > 0$ defined on $(\sqrt{\beta}, +\infty)$ with β given by (2.23). In Step 2, discussed in Section 3, we obtain nonlinear energy estimates for the perturbed equations (1.7)-(1.9) with small initial data. These estimates are crucial for the subsequent proof of nonlinear instability. Furthermore, with the aid of the nonlinear energy estimates and the semi-Galerkin method, we are able to show the local existence of strong solutions to (1.7)-(1.9), as presented in Appendix A. Lastly, in Step 3, detailed in Section 4, we utilize the results obtained in Sections 2 and section 3, combined with the Lipschitz structure of F, to establish the instability of the nonlinear problem (1.14).

Now let us comment on the difficulty and methods in this study. Because of the Coriolis force, the regularity of v_1 and $\tilde{\mathbf{v}}$ in the given context are mutually dependent. Furthermore, the absence of the dissipative term in equation $(1.7)_2$ renders the regularity theory of elliptic equations inapplicable. Consequently, the method proposed in [27] cannot be employed to establish the regularity of v_1 and $\tilde{\mathbf{v}}$ using the regularity theory of Stokes equations and the Gronwall's lemma. To overcome this obstacle, we first provide estimates for $||\mathbf{v}||_{L^2(\mathbb{R}^2)}$. Subsequently, these results enable us to obtain estimates for $||\nabla\tilde{\mathbf{v}}||_{L^2(\mathbb{R}^2)}$, utilizing the regularity theory of Stokes equations. By utilizing these estimates, a differential inequality encompassing $||\nabla\varrho||^2_{L^2(\mathbb{R}^2)} + ||\nabla v_1||^2_{L^2(\mathbb{R}^2)} + ||\sqrt{\rho}\tilde{\mathbf{v}}_t||^2_{L^2(\mathbb{R}^2)} + 1$ can be derived. Finally, with the application of the classical Osgood lemma (Lemma 2.3 in [7]), we are able to establish the estimates for $||\nabla\varrho||^2_{L^2(\mathbb{R}^2)} + ||\nabla v_1||^2_{L^2(\mathbb{R}^2)} + ||\sqrt{\rho}\tilde{\mathbf{v}}_t||^2_{L^2(\mathbb{R}^2)}$. Hence, using the regularity theory of Stokes equation, the higher regularity of v_1 and $\tilde{\mathbf{v}}$ is obtained.

The rest of this paper is organized as follows. In section 2, we discuss the linear instability problem and obtain an exponential growing solution. In section 3, we deduce some nonlinear energy estimates of (1.7)-(1.9). In section 4, we give the proof of Theorem 1.3. In Appendix A, we investigate the local existence of strong solutions of (1.7)-(1.9).

2. Construction of unstable solutions to the linearized problem

Introducing the following growing mode ansatz of solutions

$$\varrho(\mathbf{x}, t) = \tilde{\varrho}(\mathbf{x})e^{\lambda t}, \ \mathbf{v}(\mathbf{x}, t) = \tilde{\mathbf{v}}(\mathbf{x})e^{\lambda t}, \ q(\mathbf{x}, t) = \tilde{q}(\mathbf{x})e^{\lambda t} \tag{2.1}$$

and substituting (2.1) into (1.10) as well as omitting the tilde, one can obtain

$$\begin{aligned}
&\lambda\varrho + \rho_s' v_3 = 0,\\
&\lambda\rho_s v_1 + [\rho_s \bar{u}_0 - f]v_2 = 0,\\
&\lambda\rho_s \tilde{\mathbf{v}} + f v_1 \mathbf{e}_1 = \mu\Delta\tilde{\mathbf{v}} - \nabla q - g\varrho\mathbf{e}_2,\\
&\mathrm{div}\tilde{\mathbf{v}} = 0.
\end{aligned} \tag{2.2}$$

Eliminating ϱ by using $(2.2)_1$, we have

$$\begin{aligned}
&\lambda\rho_s v_1 + [\rho_s \bar{u}_0 - f]v_2 = 0,\\
&\lambda\rho_s v_2 + f v_1 = \mu\Delta v_2 - \frac{\partial q}{\partial y},\\
&\lambda^2\rho_s v_3 = \lambda\mu\Delta v_3 - \lambda\frac{\partial q}{\partial z} + g v_3 \rho_s',\\
&\frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = 0.
\end{aligned} \tag{2.3}$$

Fixing a spatial frequency $\xi \in \mathbb{R}$ and taking the horizontal Fourier transform of $\mathbf{v}$ in (2.3), which we denote $\hat{\cdot}$ or $\mathcal{F}$. Namely,

$$\hat{f}(\xi, z) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(y, z) e^{-iy\xi}\, dy,$$

we derive that

$$\begin{aligned}
&\lambda\rho_s \hat{v}_1 + [\rho_s \bar{u}_0 - f]\hat{v}_2 = 0,\\
&\lambda\rho_s \hat{v}_2 + f\hat{v}_1 = \mu(-|\xi|^2 \hat{v}_2 + \hat{v}_2'') - i\xi\hat{q},\\
&\lambda^2\rho_s \hat{v}_3 = \lambda\mu(-|\xi|^2\hat{v}_3 + \hat{v}_3'') - \lambda\hat{q}' + g\hat{v}_3\rho_s',\\
&i\xi\hat{v}_2 + \hat{v}_3' = 0.
\end{aligned} \tag{2.4}$$

Denote $\varphi = i\hat{v}_1$, $\psi = i\hat{v}_2$, $\omega = \hat{v}_3$, $\eta = \hat{q}$, then (2.4) can be expressed as

$$\begin{aligned}
&\lambda\rho_s\varphi + (\rho_s\bar{u}_0 - f)\psi = 0,\\
&\lambda\rho_s\psi + f\varphi = \mu(-|\xi|^2\psi + \psi'') + \xi\eta,\\
&\lambda^2\rho_s\omega = \lambda\mu(-|\xi|^2\omega + \omega'') - \lambda\eta' + g\rho_s'\omega,\\
&\xi\psi + \omega' = 0,
\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}
&\varphi(-\infty) = \varphi(+\infty) = 0,\\
&\psi(-\infty) = \psi(+\infty) = \psi'(-\infty) = \psi'(+\infty) = 0,\\
&\omega(-\infty) = \omega(+\infty) = \omega'(-\infty) = \omega'(+\infty) = 0.
\end{aligned} \tag{2.6}$$

Eliminating η in (2.5), we obtain

$$\begin{aligned} &\lambda\rho_s\varphi = \frac{1}{\xi}(\rho_s\bar{u}_0 - f)\omega', \\ &-\lambda^2[|\xi|^2(\rho_s\omega) - (\rho_s\omega')'] = \lambda\mu[|\xi|^4\omega - 2|\xi|^2\omega'' + \omega''''] + \lambda\xi f\varphi' - |\xi|^2 g\rho_s'\omega, \end{aligned} \tag{2.7}$$

with

$$\begin{aligned} &\varphi(-\infty) = \varphi(+\infty) = 0, \\ &\omega(-\infty) = \omega(+\infty) = \omega'(-\infty) = \omega'(+\infty) = 0. \end{aligned} \tag{2.8}$$

Inspired by [27], we apply the variational method to construct the solutions of (2.7)-(2.8). Fixing the non-zero $\xi \in \mathbb{R}$ and $s > 0$. From (2.7)-(2.8), we obtain a family of the modified problems

$$\begin{aligned} -\lambda^2[|\xi|^2(\rho_s\omega) - (\rho_s\omega')'] = s\mu[|\xi|^4\omega - 2|\xi|^2\omega'' + \omega''''] \\ - \left[\frac{f(f-\rho_s\bar{u}_0)}{\rho_s}\omega'\right]' - |\xi|^2 g\rho_s'\omega, \end{aligned} \tag{2.9}$$

with (2.8). We define the energy functional of (2.9) as follows

$$\begin{aligned} E(\omega, s) = &\int_{\mathbb{R}} s\mu[|\xi|^4\omega^2 + 2|\xi|^2\omega'^2 + \omega''^2]dz + \int_{\mathbb{R}} \left[\frac{f(f-\rho_s\bar{u}_0)}{\rho_s}\right]|\omega'|^2 dz \\ &- \int_{\mathbb{R}} |\xi|^2 g\rho_s'\omega^2 dz, \end{aligned} \tag{2.10}$$

with a associated admissible set

$$\mathcal{A} = \{\omega \in H^2(\mathbb{R}) | J(\omega) := \int_{\mathbb{R}} \rho_s(|\xi|^2|\omega|^2 + |\omega'|^2)dz = 1\}. \tag{2.11}$$

Thus, one can find $-\lambda^2$ by minimizing

$$-\lambda^2(|\xi|) = \alpha(|\xi|, s) := \inf_{\omega\in\mathcal{A}} E(\omega, s). \tag{2.12}$$

In the following, we need to show that a minimizer of (2.11) exists for $\inf_{\omega\in\mathcal{A}} E(\omega, s) > -\infty$ and that the corresponding Euler-Lagrange equations are equivalent to (2.7) and (2.8).

Lemma 2.1. *For any fixed ξ with $|\xi| \neq 0$, $\inf_{\omega\in\mathcal{A}} E(\omega, s) > -\infty$. Assume that the parameter $f \geq \bar{u}_0 \sup_{z\in\mathbb{R}} \rho_s$ and there exists a $\bar{\omega} \in \mathcal{A}$, such that $E(\bar{\omega}, s) < 0$, then $E(\omega, s)$ achieves its minimum on $\mathcal{A}$. Additionally, let ω be a minimizer and $-\lambda^2 := E(\omega)$, then (ω, λ^2) satisfies (2.7) and (2.8). Furthermore, $\omega \in H^k(\mathbb{R})$ for any positive integer k.*

Proof. The proof is divided into two steps. Step1: Existence of minimizer. It is clear that

$$\int_{\mathbb{R}} \left[\frac{f(f-\rho_s\bar{u}_0)}{\rho_s}\right]|\omega'|^2 dz \geq -\left\|\frac{f(f-\rho_s\bar{u}_0)}{\rho_s^2}\right\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} \rho_s|\omega'|^2 dz$$
$$\geq -\left\|\frac{f(f-\rho_s\bar{u}_0)}{\rho_s^2}\right\|_{L^\infty(\mathbb{R})}. \tag{2.13}$$

Clearly, we find

$$-\int_{\mathbb{R}} |\xi|^2 g\rho_s'\omega^2 dz \geq -g\left\|\frac{\rho_s'}{\rho_s}\right\|_{L^\infty(\mathbb{R})}. \tag{2.14}$$

Consequently, from (2.13)-(2.14), one gets

$$E(\omega,s) \geq -\left[\left\|\frac{f(f-\rho_s\bar{u}_0)}{\rho_s^2}\right\|_{L^\infty(\mathbb{R})} + g\left\|\frac{\rho_s'}{\rho_s}\right\|_{L^\infty(\mathbb{R})}\right]. \tag{2.15}$$

Assume that $\omega_n \in \mathcal{A}$ is a minimizing sequence, then $E(\omega_n, s)$ is bounded. From (2.10) and (2.11), we obtain ω_n is bounded in $H^2(\mathbb{R})$. As a result, there exists a $\omega \in H^2(\mathbb{R})$ and subsequence of $\{\omega_n\}$ which is also denoted by $\{\omega_n\}$, such that ω_n converges weakly to ω in $H^2(\mathbb{R})$ and converges strongly to ω in $H^1_{Loc}(\mathbb{R})$. Therefore, with the help of the lower semi-continuity, locally strongly convergence and $E(\bar{\omega}, s) < 0$ for some $\bar{\omega} \in \mathcal{A}$, we obtain

$$E(\omega,s) \leq \lim_{n\to\infty} \inf E(\omega_n,s) = \inf_{\bar{\omega}\in\mathcal{A}} E(\bar{\omega},s) < 0, \text{ and } 0 < J(\omega) \leq 1.$$

Next, we show that $J(\omega) = 1$ by using the method of contradiction. Suppose that $J(\omega) < 1$. By the homogeneity of J we derive that there exists $\alpha \geq 1$ such that $J(\alpha\omega) = 1$. Namely, we obtain $\alpha\omega \in \mathcal{A}$. Thus, we have

$$E(\alpha\omega,s) = \alpha^2 E(\omega,s) \leq \alpha^2 \inf_{\mathcal{A}} E < \inf_{\mathcal{A}} E < 0,$$

which is a contradiction since $\alpha\omega \in \mathcal{A}$. Consequently, $J(\omega) = 1$. Namely, $E(\omega, s)$ achieves minimum on $\mathcal{A}$.

Step 2: The minimizer $\omega \in H^k$. Clearly, (2.12) is equivalent to

$$-\lambda^2(|\xi|) = \inf_{\omega\in H^2(\mathbb{R})} \frac{E(\omega,s)}{J(\omega)}. \tag{2.16}$$

Taking

$$\omega(\tau) = \omega + \tau\omega_0, \ \forall\omega_0 \in H^2(\mathbb{R}) \text{ and } \forall\tau \in \mathbb{R}.$$

Therefore, from (2.16), we have $I(\tau) := E(\omega(\tau), s) + \lambda^2(|\xi|)J(\omega(\tau)) \geq 0$, which implies that $I(\tau) \geq 0, \ \forall\tau \in \mathbb{R}$ and $I(0) = 0$. Thus, $I'(0) = 0$. With the help of (2.10), we have

$$I'(\tau)=2\int_{\mathbb{R}} s\mu[|\xi|^4(\omega+\tau\omega_0)\omega_0+2|\xi|^2(\omega'+\tau\omega_0')\omega_0'+(\omega''+\tau\omega_0'')\omega_0'']dz$$
$$+2\int_{\mathbb{R}}\left[\frac{f(f-\bar{u}_0)}{\rho_s}\right](\omega'+\tau\omega_0')\omega_0'dz-2\int_{\mathbb{R}}|\xi|^2 g\rho_s'(\omega+\tau\omega_0)\omega_0 dz$$
$$+2\lambda^2\int_{\mathbb{R}}\rho_s[|\xi|^2(\omega+\tau\omega_0)\omega_0+(\omega'+\tau\omega_0')\omega_0']dz.$$

Hence, it gets from $0=I'(0)$ that

$$\begin{aligned}&s\mu\int_{\mathbb{R}}(|\xi|^4\omega\omega_0+2|\xi|^2\omega'\omega_0'+\omega''\omega_0'')dz\\&=\int_{\mathbb{R}}[g|\xi|^2\rho_s'\omega\omega_0-\frac{f(f-\rho_s\bar{u}_0)}{\rho_s}\omega'\omega_0'-\lambda^2(\rho_s|\xi|^2\omega\omega_0+\rho_s\omega'\omega_0')]dz.\end{aligned}\tag{2.17}$$

Suppose that ω_0 is compactly supported in $\mathbb{R}$, thus, we find that ω satisfies (2.9) in the weak sense. Additionally, from (2.17), we obtain

$$\begin{aligned}\int_{\mathbb{R}}\omega''\omega_0''dz&=\frac{1}{s\mu}\int_{\mathbb{R}}\left\{g|\xi|^2\rho_s'\omega+\left[\frac{f(f-\rho_s\bar{u}_0)}{\rho_s}\omega'\right]'-\lambda^2(\rho_s|\xi|^2\omega-(\rho_s\omega')')\right.\\&\qquad\left.+s\mu(-|\xi|^4\omega+2|\xi|^2\omega'')\right\}\omega_0dz\\&:=\int_{\mathbb{R}}h\omega_0dz.\end{aligned}\tag{2.18}$$

For any $n\geq 1$, choosing $\omega_{1,n}\in C_0^\infty(\mathbb{R})$ satisfying $\omega_{1,n}(z)\equiv 1$ for $|z|\leq n$ and taking $\omega_0=\omega_{1,n}\int_{-\infty}^z\omega_2 d\tau$ with $\omega_2\in C_0^\infty(\mathbb{R})$. Then we obtain

$$\begin{aligned}\int_{\mathbb{R}}(\omega''\omega_{1,n})\omega_2'dz&=\int_{\mathbb{R}}[h\omega_{1,n}(\int_{-\infty}^{z}\omega_2)d\tau-\omega''\omega_{1,n}''(\int_{-\infty}^{z}\omega_2d\tau)-2\omega''\omega_{1,n}'\omega_2]dz\\&=\int_{\mathbb{R}}[\int_{z}^{+\infty}(h\omega_{1,n}-\omega''\omega_{1,n}'')d\tau-2\omega''\omega_{1,n}']\omega_2dz,\end{aligned}\tag{2.19}$$

which leads to $\omega''\in H^1_{Loc}(\mathbb{R})$, and

$$\omega'''=(\omega_{1,n}\omega'')'=\int_{z}^{+\infty}(h\omega_{1,n}-\omega''\omega_{1,n}'')d\tau-2\omega''\omega_{1,n}'.\tag{2.20}$$

Furthermore, through integration by parts, (2.18) can be rewritten as

$$-\int_{\mathbb{R}} \omega'''\omega_0' dz = \frac{1}{s\mu}\int_{\mathbb{R}} \left\{ g|\xi|^2\rho_s'\omega - \left[\frac{f(f-\rho_s\bar{u}_0)}{\rho_s}\omega'\right]' - \lambda^2(\rho_s|\xi|^2\omega - (\rho_s\omega')') + s\mu(-|\xi|^4\omega + 2|\xi|^2\omega'') \right\}\omega_0 dz, \tag{2.21}$$

which implies that $\omega'''' \in L^2(\mathbb{R})$. Thus, $\omega \in H^4_{Loc}(\mathbb{R}) \cap C^{3,\frac{1}{2}}_{Loc}(\mathbb{R})$, and $(2.8)_2$ is valid as well as $\omega''(\infty) = \omega'''(\infty) = 0$ holds true. Utilizing these facts and Hölder's inequality and integrations by parts, we obtain

$$\int_{\mathbb{R}} |\omega'''|^2 dz = -\int_{\mathbb{R}} \omega''\omega'''' dz \le ||\omega''||_{L^2(\mathbb{R})}||\omega''''||_{L^2(\mathbb{R})}, \tag{2.22}$$

which implies that $\omega'''' \in L^2(\mathbb{R})$. As a result, $\omega \in H^4(\mathbb{R})$ and solves (2.7)-(2.8). Furthermore, we obtain $\omega \in H^k(\mathbb{R}), k \in Z^+$. □

In the following, we need to show that there exists a fixed point such that $s = \mu$ by using the intermediate value theorem. To this end, we first define

$$\beta = \inf_{\omega\in\mathcal{B}} \int_{\mathbb{R}} \frac{f(f-\bar{u}_0\rho_s)}{\rho_s}|\omega'|^2 dz, \tag{2.23}$$

where $\mathcal{B} = \{\omega \in H^2(\mathbb{R}) | \int_{\mathbb{R}} g\rho_s'|\omega|^2 dz = 1\}$. It is worth noting that (1.5) ensures that $\mathcal{B} \neq \emptyset$.

Then, we first give some properties of $\alpha(s)$ as a function of s.

Lemma 2.2. *For each fixed $|\xi| \in (\sqrt{\beta}, +\infty)$, the function $\alpha(|\xi|, s),\ s \in (0, +\infty)$ has the following properties:*

(1) *Assume that the parameter f satisfies $f \ge \bar{u}_0 \sup_{z\in\mathbb{R}} \rho_s$, then, for any $a, b \in (\sqrt{\beta}, +\infty)$ with $a < b$, there exist constants $c_1, c_2 > 0$ depending on ρ_s, f, μ, g, a, b, such that*

$$\alpha(|\xi|, s) \le -c_1 + sc_2,\ \forall |\xi| \in [a, b]. \tag{2.24}$$

(2) *$\alpha(|\xi|, s) \in C^{0,1}_{Loc}(0, +\infty)$ is strictly increasing.*

Proof. If $|\xi|^2 > \beta$, then, there exists $\tilde{\omega} \in \mathcal{B}$, such that

$$0 < \frac{\int_{\mathbb{R}} \frac{f(f-\bar{u}_0\rho_s)}{\rho_s}|\tilde{\omega}'|^2 dz}{\int_{\mathbb{R}} g\rho_s'|\tilde{\omega}|^2 dz} < |\xi|^2. \tag{2.25}$$

Indeed, suppose that

$$\frac{\int_{\mathbb{R}} \frac{f(f-\bar{u}_0\rho_s)}{\rho_s}|\tilde{\omega}'|^2 dz}{\int_{\mathbb{R}} g\rho_s'|\tilde{\omega}|^2 dz} \geq |\xi|^2, \ \forall \tilde{\omega} \in \mathcal{B}. \tag{2.26}$$

Then, from (2.26), we deduce that

$$\beta = \inf_{\omega \in \mathcal{B}} \int_{\mathbb{R}} \frac{f(f-\bar{u}_0\rho_s)}{\rho_s}|\omega'|^2 dz = \inf_{\tilde{\omega}\in\mathcal{B}} \frac{\int_{\mathbb{R}} \frac{f(f-\bar{u}_0\rho_s)}{\rho_s}|\tilde{\omega}'|^2 dz}{\int_{\mathbb{R}} g\rho_s'|\tilde{\omega}|^2 dz} \geq |\xi|^2,$$

which is a contradiction since $|\xi|^2 > \beta$. Therefore, from (2.25), we have

$$-\int_{\mathbb{R}} \frac{f(f-\bar{u}_0\rho_s)}{\rho_s}|\tilde{\omega}'|^2 dz + \int_{\mathbb{R}} |\xi|^2 g\rho_s'|\tilde{\omega}|^2 dz > 0, \ \text{for } |\xi|^2 > \beta. \tag{2.27}$$

With the help of (1.5), we obtain

$$\left\{ \frac{-\int_{\mathbb{R}} [\frac{f(f-\rho_s\bar{u}_0)}{\rho_s}]|\tilde{\omega}'|^2 dz}{\int_{\mathbb{R}} \rho_s(|\xi|^2|\tilde{\omega}|^2 + |\tilde{\omega}'|^2) dz} + \frac{\int_{\mathbb{R}} |\xi|^2 g\rho_s'\tilde{\omega}^2 dz}{\int_{\mathbb{R}} \rho_s(|\xi|^2|\tilde{\omega}|^2 + |\tilde{\omega}'|^2) dz} \right\} := c_1 > 0 \tag{2.28}$$

and

$$\mu \frac{\int_{\mathbb{R}} [|\xi|^4\tilde{\omega}^2 + 2|\xi|^2\tilde{\omega}'^2 + \tilde{\omega}''^2] dz}{\int_{\mathbb{R}} \rho_s(|\xi|^2|\tilde{\omega}|^2 + |\tilde{\omega}'|^2) dz} := c_2 > 0. \tag{2.29}$$

Using (2.12) and (2.16), we deduce that

$$\begin{aligned}
\alpha(|\xi|, s) = & \inf_{\omega \in H^2(\mathbb{R})} \frac{E(\omega, s)}{J(\omega)} \\
& \leq \frac{\int_{\mathbb{R}} s\mu[|\xi|^4\tilde{\omega}^2 + 2|\xi|^2\tilde{\omega}'^2 + \tilde{\omega}''^2] dz + \int_{\mathbb{R}} [\frac{f(f-\rho_s\bar{u}_0)}{\rho_s}]|\tilde{\omega}'|^2 dz - \int_{\mathbb{R}} |\xi|^2 g\rho_s'\tilde{\omega}^2 dz}{\int_{\mathbb{R}} \rho_s(|\xi|^2|\tilde{\omega}|^2 + |\tilde{\omega}'|^2) dz} \\
& \leq s\mu \frac{\int_{\mathbb{R}} [|\xi|^4\tilde{\omega}^2 + 2|\xi|^2\tilde{\omega}'^2 + \tilde{\omega}''^2] dz}{\int_{\mathbb{R}} \rho_s(|\xi|^2|\tilde{\omega}|^2 + |\tilde{\omega}'|^2) dz} - \left\{ \frac{\int_{\mathbb{R}} [\frac{f(\rho_s\bar{u}_0 - f)}{\rho_s}]|\tilde{\omega}'|^2 dz}{\int_{\mathbb{R}} \rho_s(|\xi|^2|\tilde{\omega}|^2 + |\tilde{\omega}'|^2) dz} \right. \\
& \quad \left. + \frac{\int_{\mathbb{R}} |\xi|^2 g\rho_s'\tilde{\omega}^2 dz}{\int_{\mathbb{R}} \rho_s(|\xi|^2|\tilde{\omega}|^2 + |\tilde{\omega}'|^2) dz} \right\} \\
& := -c_1 + sc_2,
\end{aligned} \tag{2.30}$$

where the positive constant c_1, c_2 depending on ρ_s, f, μ, g, a, b.

Next, we shall show that the assertion (2) is valid. Let $I := [a, b] \subset (\sqrt{\beta}, +\infty)$ be a bounded interval and

$$E_1(\omega) = \mu \int_{\mathbb{R}} [|\xi|^4\omega^2 + 2|\xi|^2\omega'^2 + \omega''^2]dz. \tag{2.31}$$

For any $s \in I$, there exists a minimizing sequence $\{\omega_s^n\} \subset \mathcal{A}$ of $\inf_{\omega\in\mathcal{A}} E(\omega, s)$ such that

$$|\alpha(|\xi|, s) - E(\omega_s^n, s)| < 1. \tag{2.32}$$

With the help of (2.10), (2.15), (2.24), (2.32), we obtain

$$E_1(\omega) = \frac{1}{s}\left[E(\omega, s) + \int_{\mathbb{R}} \frac{f(\rho_s\bar{u}_0 - f)}{\rho_s}|\omega'|^2 dz + g\int_{\mathbb{R}} |\xi|^2\rho_s'\omega dz\right] \le L, \tag{2.33}$$

where

$$L = \frac{1}{a}\left[\max\left\{\left|\left|\frac{f(\rho_s\bar{u}_0 - f)}{\rho_s}\right|\right|_{L^\infty(\mathbb{R})} + g\left|\left|\frac{\rho_s'}{\rho_s}\right|\right|_{L^\infty(\mathbb{R})}, 1 + |-c_1 + bc_2|\right\}\right.$$
$$\left. + \left|\left|\frac{f(\rho_s\bar{u}_0 - f)}{\rho_s}\right|\right|_{L^\infty(\mathbb{R})} + g||\frac{\rho_s'}{\rho_s}||_{L^\infty(\mathbb{R})}\right].$$

For any $s_i \in I(i = 1, 2)$, let $\{\omega_{s_i}^n\} \subset \mathcal{A}$ be minimizing sequences of $\inf_{\omega\in\mathcal{A}} E(\omega, s_i)$. From (2.10) and (2.31), we drive

$$\begin{aligned} E(\omega_{s_2}^n, s_1) &= \int_{\mathbb{R}} s_1\mu[|\xi|^4|\omega_{s_2}^n|^2 + 2|\xi|^2|\omega_{s_2}^{n'}|^2 + |\omega_{s_2}^{n''}|^2]dz + \int_{\mathbb{R}} \left[\frac{f(f - \rho_s\bar{u}_0)}{\rho_s}\right]|\omega_{s_2}^n|^2 dz \\ &\quad - \int_{\mathbb{R}} |\xi|^2 g\rho_s'|\omega_{s_2}^n|^2 dz \\ &= E(\omega_{s_2}^n, s_2) + (s_1 - s_2)E_1(\omega_{s_2}^n), \end{aligned} \tag{2.34}$$

which implies that

$$\begin{aligned} \alpha(|\xi|, s_1) &\le \limsup_{n\to\infty} E(\omega_{s_2}^n, s_1) \le \limsup_{n\to\infty} E(\omega_{s_2}^n, s_2) + |s_1 - s_2|\limsup_{n\to\infty} E_1(\omega_{s_2}^n) \\ &\le \alpha(|\xi|, s_2) + L|s_1 - s_2|. \end{aligned} \tag{2.35}$$

Reversing the role of the subscript 1 and 2 of (2.35), we find that

$$|\alpha(|\xi|, s_1) - \alpha(|\xi|, s_2)| \le L|s_1 - s_2|,$$

which yields $\alpha(|\xi|, s) \in C_{Loc}^{0,1}(0, \infty)$.

In addition, from (2.34), due to $s_1 \le s_2$ and $E_1(\omega_{s_2}^n) \ge 0$, we obtain

$$E(\omega_{s_2}^n, s_1) \le E(\omega_{s_2}^n, s_2).$$

Hence, one can get that

$$\alpha(|\xi|, s_1) = \inf_{\omega\in\mathcal{A}} E(\omega, s_1) \le \limsup_{n\to\infty} E(\omega^n_{s_2}, s_1) \le \limsup_{n\to\infty} E(\omega^n_{s_2}, s_2) = \alpha(|\xi|, s_2), \quad (2.36)$$

which implies that $\alpha(|\xi|, s)$ is non-decreasing on $(0, +\infty)$. Suppose by way of contradiction that $\alpha(|\xi|, s_1) = \alpha(|\xi|, s_2)$. From (2.36), we obtain

$$s_1 E_1(\omega_{s_2}) = s_2 E_1(\omega_{s_2}),$$

which leads to $\omega_{s_2} = 0$. Consequently, the conclusion (2) holds true. □

Given $|\xi| \in (\sqrt{\beta}, +\infty)$, from (2.24), there exists a $s_0 > 0$ depending on the parameters $\rho_s, f, \mu, g, |\xi|$, such that for any $s \le s_0, \alpha(|\xi|, s) < 0$. Let

$$\mathcal{G}_{|\xi|} := \sup\{s|\alpha(|\xi|, \tau) < 0,\ \forall \tau \in (0, s)\} > 0, \quad (2.37)$$

which allows us to define $\lambda(|\xi|, s) = \sqrt{-\alpha(|\xi|, s)}, \forall s \in (0, \mathcal{G}_{|\xi|})$. According to Lemma 2.1 and Lemma 2.2, we obtain the following existence of (2.9).

Lemma 2.3. *Assume that the parameter f satisfies $f \ge \bar{u}_0 \sup_{z\in\mathbb{R}} \rho_s$, then, for each $|\xi| \in (\sqrt{\beta}, +\infty)$ and $s \in (0, \mathcal{G}_{|\xi|})$, there exists a solution $\omega(|\xi|, z) \ne 0$ with $\lambda(|\xi|, s) > 0$ for the (2.8)-(2.9). In addition, $\lambda(|\xi|, s) \in C^{0,1}_{Loc}(0, \mathcal{G}_{|\xi|})$ is strictly decreasing. Moreover, $\omega(|\xi|, s) \in H^k(\mathbb{R})$ for any positive integer.*

By the intermediate value theorem, we have the following result.

Lemma 2.4. *Suppose that the parameter f satisfies $f \ge \bar{u}_0 \sup_{z\in\mathbb{R}} \rho_s$ and $|\xi| \in (\sqrt{\beta}, +\infty)$, then there is a unique $s \in (0, \mathcal{G}_{|\xi|})$, such that $\lambda(|\xi|, s) = \sqrt{-\alpha(|\xi|, s)} > 0$ and $s = \lambda(|\xi|, s)$.*

Proof. The function $\phi(s) := \frac{s}{\lambda(|\xi|,s)}$ is continuous and strictly increasing on $(0, \mathcal{G}_{|\xi|})$ since $\lambda(|\xi|, s) \in C^{0,1}_{Loc}(0, \mathcal{G}_{|\xi|})$ and $\alpha(|\xi|, s)$ is strictly increasing. Additionally, with the help of (2.24) and (2.37), one can see that $\lambda(|\xi|, s)$ is bounded as $s \in (0, s_0)$ and $\lim_{s\to 0^+} \phi(s) = 0$. In addition, by the definition of $\mathcal{G}_{|\xi|}$, we obtain $\lim_{s\to\mathcal{G}^-_{|\xi|}} \lambda(|\xi|, s) = 0^+$ and $\lim_{s\to\mathcal{G}^-_{|\xi|}} \phi(s) = +\infty$. As a result, the intermediate value theorem implies that there exists a unique $s \in (0, \mathcal{G}_s)$ such that $\phi(s) = 1$. □

From Lemma 2.3 and Lemma 2.4, we conclude the existence of the problem (2.7)-(2.8).

Theorem 2.1. *Suppose that the parameter f satisfies $f \ge \bar{u}_0 \sup_{z\in\mathbb{R}} \rho_s$ and $|\xi| \in (\sqrt{\beta}, +\infty)$, there exist $\omega = \omega(|\xi|, z) \ne 0$, $\varphi = \varphi(|\xi|, z) \ne 0$ satisfies linearized equations (2.7)-(2.8). Further $(\omega, \varphi) \in H^k(\mathbb{R}) \times H^{k-1}(\mathbb{R})$ for any positive integer $k \ge 1$.*

2.1. Behavior of the solutions with respect to ξ

In this subsection, we investigate the behavior of the solutions from Theorem 2.1 in terms of ξ.

Lemma 2.5. *The function* $\lambda : (\sqrt{\beta}, \infty) \to (0, \infty)$ *is continuous and satisfies*

$$\sup_{\sqrt{\beta}<|\xi|<+\infty} \lambda(|\xi|) \le \left\{ \left\| \sqrt{\frac{f(f-\rho_s\bar{u}_0)}{\rho_s^2}} \right\|_{L^\infty(\mathbb{R})} + \sqrt{g} \left\| \sqrt{\frac{\rho_s'}{\rho_s}} \right\|_{L^\infty(\mathbb{R})} \right\}. \tag{2.38}$$

Proof. From (2.15), we obtain

$$\alpha(|\xi|, s) = \inf_{\omega\in\mathcal{A}} E(\omega, s) \ge -\left[\left\| \frac{f(f-\rho_s\bar{u}_0)}{\rho_s^2} \right\|_{L^\infty(\mathbb{R})} + g \left\| \frac{\rho_s'}{\rho_s} \right\|_{L^\infty(\mathbb{R})} \right].$$

Thus, we deduce

$$\lambda(|\xi|, s) = \sqrt{-\alpha(|\xi|, s)} \le \left[\sqrt{\left\| \frac{f(f-\rho_s\bar{u}_0)}{\rho_s^2} \right\|_{L^\infty(\mathbb{R})}} + \sqrt{g \left\| \frac{\rho_s'}{\rho_s} \right\|_{L^\infty(\mathbb{R})}} \right],$$

which implies that (2.38) is valid.

In the following, we show the continuity of λ with respect to ξ. It suffices to show the continuity of $\alpha(|\xi|, s)$ since $\lambda(|\xi|, s) = \sqrt{-\alpha(|\xi|, s)}$. For any fixed $\xi_0 \in (\sqrt{\beta}, +\infty)$, there exists an interval $[a, b] \subset (\sqrt{\beta}, +\infty)$ such that $|\xi_0| \in (a, b)$. Thus, we assume $|\xi| \to |\xi_0|$ with $|\xi| \in (a, b)$. Denote $|\xi|^2 = \delta + |\xi_0|^2$, this leads to $\delta \to 0$ as $|\xi| \to |\xi_0|$.

According to Lemma 2.1, for any $|\xi| \in (a, b)$, there exists $\omega_{|\xi|} \in \mathcal{A}$ such that

$$\begin{aligned} \alpha(|\xi|, s) = &\int_{\mathbb{R}} s\mu[|\xi|^4\omega_{|\xi|}^2 + 2|\xi|^2\omega_{|\xi|}'^2 + \omega_{|\xi|}''^2]dz + \int_{\mathbb{R}} \left[\frac{f(f-\rho_s\bar{u}_0)}{\rho_s}\right]|\omega_{|\xi|}'|^2 dz \\ &- \int_{\mathbb{R}} |\xi|^2 g\rho_s'\omega_{|\xi|}^2 dz. \end{aligned} \tag{2.39}$$

From (2.24), we get

$$s\mu \int_{\mathbb{R}} |\omega_{|\xi|}''|^2 dz \le \alpha(|\xi|, s) + \int_{\mathbb{R}} \left[\frac{f(\rho_s\bar{u}_0 - f)}{\rho_s}\right]|\omega_{|\xi|}'|^2 dz + \int_{\mathbb{R}} |\xi|^2 g\rho_s'\omega_{|\xi|}^2 dz,$$

which leads to

$$\int_{\mathbb{R}} |\omega_{|\xi|}''|^2 dz \le \frac{-c_1 + c_2 s}{s\mu} + \frac{1}{s\mu}\left[\left\| \frac{f(f-\rho_s\bar{u}_0)}{\rho_s^2} \right\|_{L^\infty(\mathbb{R})} + g \left\| \frac{\rho_s'}{\rho_s} \right\|_{L^\infty(\mathbb{R})} \right]. \tag{2.40}$$

Additionally, from (2.11), we find that

$$\int_{\mathbb{R}} |\omega_{|\xi|}'|^2 dz \le \frac{1}{\inf_{z\in\mathbb{R}} \rho_s}, \quad \int_{\mathbb{R}} |\omega_{|\xi|}|^2 dz \le \frac{1}{a^2 \inf_{z\in\mathbb{R}} \rho_s}. \tag{2.41}$$

Consequently, we have

$$||\omega_{|\xi|}||_{H^2(\mathbb{R})} \le c_3,$$

where c_3 depends on $\rho_s, f, \mu, s, g, a, b$.

Substituting $|\xi|^2 = |\xi_0|^2 + \delta$ into (2.39), we derive that

$$\begin{aligned}\alpha(|\xi|, s) = &\int_{\mathbb{R}} s\mu[|\xi_0|^4\omega^2_{|\xi|} + 2|\xi_0|^2\omega'^2_{|\xi|} + |\omega''|^2_{|\xi|}]dz + \int_{\mathbb{R}} \left[\frac{f(f-\rho_s\bar{u}_0)}{\rho_s}\right]|\omega'_{|\xi|}|^2 dz \\ &- \int_{\mathbb{R}} |\xi_0|^2 g\rho_s'\omega^2_{|\xi|}dz + \delta K(\delta, \omega_{|\xi|}) \\ &\ge \alpha(|\xi_0|, s) + \delta K(\delta, \omega_{|\xi|}),\end{aligned} \tag{2.42}$$

where

$$K(\delta, \omega_{|\xi|}) = \int_{\mathbb{R}} s\mu[(2|\xi_0| + \delta)|\omega_{|\xi|}|^2 + 2|\omega'_{|\xi|}|^2]dz - \int_{\mathbb{R}} g\rho_s'\omega^2_{|\xi|}dz.$$

Similarly, according to Lemma 2.1, for any $|\xi_0| \in (a, b)$, there exists $\omega_{|\xi_0|} \in \mathcal{A}$ such that

$$\begin{aligned}\alpha(|\xi_0|, s) = &\int_{\mathbb{R}} s\mu[|\xi_0|^4\omega^2_{|\xi_0|} + 2|\xi_0|^2\omega'^2_{|\xi_0|} + |\omega''|^2_{|\xi_0|}]dz + \int_{\mathbb{R}} \left[\frac{f(f-\rho_s\bar{u}_0)}{\rho_s}\right]|\omega'_{|\xi_0|}|^2 dz \\ &- \int_{\mathbb{R}} |\xi_0|^2 g\rho_s'\omega^2_{|\xi_0|}dz.\end{aligned} \tag{2.43}$$

Substituting $|\xi_0|^2 = |\xi|^2 - \delta$ in (2.43), we also obtain

$$\alpha(|\xi_0|, s) \ge \alpha(|\xi|, s) - \delta K(-\delta, \omega_{|\xi_0|}). \tag{2.44}$$

Combining (2.42) and (2.44), we have

$$\delta K(\delta, \omega_{|\xi|}) \le \alpha(|\xi|, s) - \alpha(|\xi_0|, s) \le \delta K(-\delta, \omega_{|\xi_0|}),$$

which implies that

$$\lambda(|\xi|, s) \to \lambda(|\xi_0|, s),\ \forall s \in (0, \mathcal{G}_{|\xi|}) \text{ as } \xi \to \xi_0. \tag{2.45}$$

With the help of (2.45) and Lemma 2.4, we find that for any $\epsilon > 0$, there exists $\delta > 0$ such that $|\lambda(|\xi|, s_{|\xi_0|}) - \lambda(|\xi_0|, s_{|\xi_0|})| < \epsilon$ and $s_{|\xi_0|} = \lambda(|\xi_0|, s_{|\xi_0|}) = \sqrt{-\alpha(|\xi_0|, s_{|\xi_0|})}$ as $||\xi| - |\xi_0|| < \delta$. Additionally, for each fixed $\xi \ne 0$, $\lambda(|\xi|, s)$ is strictly decreasing and continuous on $(0, \mathcal{G}_{|\xi|})$, and there exists a unique $s_{|\xi|} \in (0, \mathcal{G}_{|\xi|})$ such that $\lambda(|\xi|, s_{|\xi|}) = s_{|\xi|}$. As a result, we obtain $|\lambda(|\xi|, s_{|\xi|}) - \lambda(|\xi_0|, s_{|\xi_0|})| \le |\lambda(|\xi|, s_{|\xi_0|}) - \lambda(|\xi_0|, s_{|\xi_0|})| < \epsilon$ (See Fig. 1). Namely, $\lambda(|\xi|)$ is continuous. □

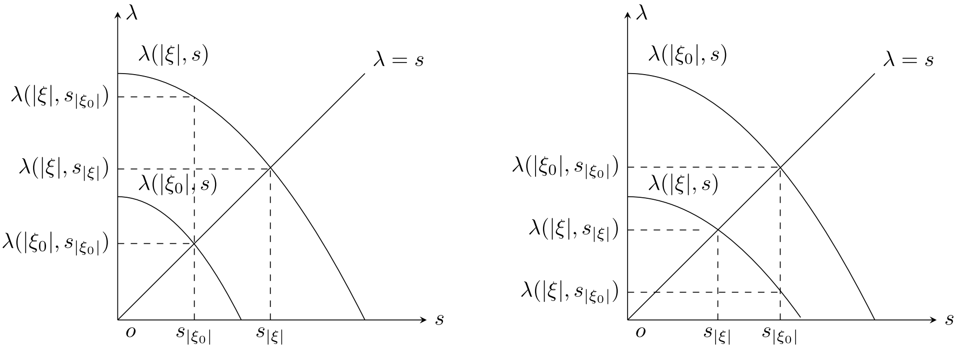

Fig. 1. Sketch of proof in Lemma 2.5: $|\lambda(|\xi|, s_{|\xi|}) - \lambda(|\xi_0|, s_{|\xi_0|})| \le |\lambda(|\xi|, s_{|\xi_0|}) - \lambda(|\xi_0|, s_{|\xi_0|})|$.

2.2. *Construction of a solution to the system* (2.5)-(2.6)

From $(2.7)_1$, we infer that the component φ of the solution to the system (2.7)-(2.8) is given by

$$\varphi(\xi, z) = (\lambda\rho_s\xi)^{-1}(\rho_s\bar{u}_0 - f)\omega'. \tag{2.46}$$

In this subsection, we shall apply the solution of (2.7)-(2.8) to construct a solution of the system (2.5)-(2.6). Namely, we have the following result.

Theorem 2.2. *Suppose that the parameter f satisfies $f \ge \bar{u}_0 \sup_{z\in\mathbb{R}} \rho_s$. Then for each $|\xi| \in (\sqrt{\beta}, +\infty)$, there exists a solution $(\varphi, \psi, \omega, \eta) = (\varphi(\xi, z), \psi(\xi, z), \omega(|\xi|, z), \eta(|\xi|, z))$ with $\lambda = \lambda(|\xi|) > 0$ to (2.5)-(2.6). Furthermore, the solution $(\varphi, \psi, \omega, \eta) \in H^{k-1}(\mathbb{R}) \times H^{k-1}(\mathbb{R}) \times H^k(\mathbb{R}) \times H^{k-3}(\mathbb{R})$ for any positive integer $k \ge 3$.*

Proof. Using Theorem 2.1 and (2.46), we can construct a solution $(\varphi, \omega, \lambda) = (\varphi(\xi, z), \omega(\xi, z), \lambda(|\xi|))$ satisfying (2.7)-(2.8). For $\lambda > 0$, $\omega \in \mathcal{A} \cap H^k(\mathbb{R})$ and $\varphi \in H^{k-1}(\mathbb{R})$, multiplying $(2.5)_2 \times \xi$ and utilizing $(2.5)_4$, we find η is given by

$$\eta = \eta(|\xi|, z) = [\mu\omega''' - (\mu|\xi|^2 + \lambda\rho_s)\omega' + f\xi\varphi]|\xi|^{-2}. \tag{2.47}$$

Thus,

$$(\varphi, \psi, \omega, \eta) = (\varphi, -\frac{1}{\xi}\omega', \omega, \eta)$$

is a solution of (2.5)-(2.6). □

Remark 2.1. For each fixed z, the solution $(\varphi(\xi, z), \psi(\xi, z), \omega(|\xi|, z), \eta(|\xi|, z))$ constructed in Theorem 2.2 possesses the following properties:

(1) $\lambda(|\xi|)$, $\omega(|\xi|, z)$ and $\eta(|\xi|, z)$ are even on ξ.
(2) $\varphi(\xi, z)$ and $\psi(\xi, z)$ are old on ξ.

Next, we shall provide an estimate of the solution $(\varphi, \psi, \omega, \eta)$ with ξ varying. To illustrate the dependence on ξ, we denote $\varphi(\xi) = \varphi(\xi, z)$, $\psi(\xi) = \psi(\xi, z)$, $\omega(\xi) = \omega(\xi, z)$, $\eta(\xi) = \eta(\xi, z)$.

Lemma 2.6. *Suppose that the parameter f satisfies $f \geq \bar{u}_0 \sup_{z\in\mathbb{R}} \rho_s$. Let $\sqrt{\beta} < R_1 \leq |\xi| \leq R_2$, $\varphi(\xi)$, $\psi(\xi)$, $\omega(\xi)$, $\eta(\xi)$ be a solution constructed as in Theorem 2.2. Then, for any positive integer $k \geq 3$, there exist positive constants A_k, B_k, C_k and D_k such that*

$$||\omega(\xi)||_{H^k(\mathbb{R})} \leq A_k,\ ||\eta(\xi)||_{H^{k-3}(\mathbb{R})} \leq B_k, \tag{2.48}$$

$$||\varphi(\xi)||_{H^{k-1}(\mathbb{R})} \leq C_k,\ ||\psi(\xi)||_{H^{k-1}(\mathbb{R})} \leq D_k. \tag{2.49}$$

In addition

$$||\omega||_{L^2(\mathbb{R})} > 0, \tag{2.50}$$

where the constants A_k, B_k, C_k and D_k depend on R_1, R_2, ρ_s, μ, g and f.

Proof. Throughout the proof, we denote by C by a positive constant which may vary from line to line, and depend on R_1, R_2, ρ_s, μ, g and f.

(2.50) is clear since $\omega \in \mathcal{A}$. With the help of Lemma 2.5, we get

$$m \leq \lambda(|\xi|) \leq M,\ \forall|\xi| \in [R_1, R_2], \tag{2.51}$$

where the constants m, M depend on R_1, R_2, ρ_s, μ, g and f. Therefore, combining (2.40) and (2.41), we deduce that

$$||\omega||_{H^2(\mathbb{R})} \leq C. \tag{2.52}$$

By virtue of (2.46) and $\psi(\xi) = -\frac{1}{\xi}\omega'$, we find

$$||\varphi||_{H^1(\mathbb{R})} \leq C,\ ||\psi||_{H^1(\mathbb{R})} \leq C. \tag{2.53}$$

Moreover, the equation $(2.7)_2$ can be rewritten as

$$\omega'''' = -|\xi|^4\omega + 2|\xi|^2\omega'' - \frac{1}{\lambda\mu}\left\{\lambda^2[|\xi|^2(\rho_s\omega) - (\rho_s\omega')'] - \lambda\xi f\varphi' + |\xi|^2 g\rho_s'\omega\right\},$$

which leads to

$$\int_{\mathbb{R}} |\omega''''|^2 dz \leq C. \tag{2.54}$$

Then using integration by parts and Hölder inequality, we obtain

$$\int_{\mathbb{R}} |\omega'''|^2 dz = -\int_{\mathbb{R}} \omega''\omega'''' dz \leq (\int_{\mathbb{R}} |\omega''|^2 dz)^{\frac{1}{2}}(\int_{\mathbb{R}} |\omega''''|^2 dz)^{\frac{1}{2}} \leq C. \tag{2.55}$$

Consequently, from (2.52), (2.54) and (2.55), we derive

$$||\omega||_{H^4(\mathbb{R})} \leq C. \tag{2.56}$$

Similarly, from (2.46) and $\psi(\xi) = -\frac{1}{\xi}\omega'$ as well as (2.47), we obtain

$$||\varphi||_{H^3(\mathbb{R})} \leq C, \ ||\psi||_{H^3(\mathbb{R})} \leq C, \ ||\eta||_{H^1(\mathbb{R})} \leq C. \tag{2.57}$$

As a result, differentiating $(2.7)_2$ and using (2.56)-(2.57), repeating the above process, we obtain the inequality (2.48) and (2.49). □

2.3. Exponential growth rate

In this subsection, we shall apply the Fourier synthesis to construct growing solutions to (1.10) for any fixed spatial frequency $\xi \in \mathbb{R}$ with $|\xi| > \sqrt{\beta}$.

Theorem 2.3. *Assume that the parameter f satisfies $f \geq \bar{u}_0 \sup_{z\in\mathbb{R}} \rho_s$. Let $\sqrt{\beta} < R_1 < R_2 < +\infty$ and $h \in C_0^\infty((R_1, R_2))$ be a real-valued function. For $\xi \in \mathbb{R}$ with $|\xi| \in (\sqrt{\beta}, +\infty)$, define*

$$(\hat{v}_1(\xi, z), \hat{v}_2(\xi, z), \hat{v}_3(\xi, z), \hat{q}(\xi, z)) = (-i\varphi(\xi, z), -i\psi(\xi, z), \omega(|\xi|, z), \eta(|\xi|, z)), \tag{2.58}$$

where $(\varphi(\xi, z), \psi(\xi, z), \omega(\xi, z), \eta(\xi, z))$ with $\lambda(|\xi|) > 0$ is the solution constructed by Theorem 2.2. Set

$$\begin{aligned}
\varrho(t, y, z) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} [-\rho_s' h(|\xi|)\hat{v}_3(\xi, z) e^{\lambda(|\xi|)t} e^{iy\xi}] d\xi, \\
\mathbf{v}(t, y, z) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} [\lambda(|\xi|) h(|\xi|)\hat{\mathbf{v}}(\xi, z) e^{\lambda(|\xi|)t} e^{iy\xi}] d\xi, \\
q(t, y, z) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} [\lambda(|\xi|) h(|\xi|)\hat{q}(\xi, z) e^{\lambda(|\xi|)t} e^{iy\xi}] d\xi.
\end{aligned} \tag{2.59}$$

Then $(\varrho(t, y, z), \mathbf{v}(t, v, z), q(t, y, z))$ is a real-valued solution of the linearized equations (1.10). Moreover, for any fixed ξ with $|\xi| > \sqrt{\beta}$, the following estimate is valid

$$\begin{aligned}
&||\varrho(0)||_{H^k(\mathbb{R}^2)} + ||\mathbf{v}(0)||_{(H^k(\mathbb{R}^2))^3} + ||q(0)||_{H^k(\mathbb{R}^2)} \\
&\leq D_k [\int_{\mathbb{R}} (1 + |\xi|^2)|h(|\xi|)|^2 d\xi]^{\frac{1}{2}},
\end{aligned} \tag{2.60}$$

where D_k is a constant depending on R_1, R_2, ρ_s, μ, g and f. Furthermore, for any fixed $t > 0$, we have $(\varrho(t, y, z), v_1(t, y, z), v_2(t, y, z), v_3(t, y, z), q(t, y, z)) \in H^k(\mathbb{R}^2) \times H^{k-1}(\mathbb{R}^2) \times H^{k-1}(\mathbb{R}^2) \times H^k(\mathbb{R}^2) \times H^{k-3}(\mathbb{R}^2)$, and

$$
\begin{aligned}
&e^{t\lambda_0(|\xi|)}||\varrho(0)||_{H^k(\mathbb{R}^2)} \le ||\varrho(t)||_{H^k(\mathbb{R}^2)} \le e^{t\Lambda}||\varrho(0)||_{H^k(\mathbb{R}^2)},\\
&e^{t\lambda_0(|\xi|)}||\mathbf{v}(0)||_{(H^k(\mathbb{R}^2))^3} \le ||\mathbf{v}(t)||_{(H^k(\mathbb{R}^2))^3} \le e^{t\Lambda}||\mathbf{v}(0)||_{(H^k(\mathbb{R}^2))^3},\\
&e^{t\lambda_0(|\xi|)}||q(0)||_{H^k(\mathbb{R}^2)} \le ||q(t)||_{H^k(\mathbb{R}^2)} \le e^{t\Lambda}||q(0)||_{H^k(\mathbb{R}^2)},
\end{aligned}
\tag{2.61}
$$

where

$$
\lambda_0 = \inf_{|\xi|\in[R_1,R_2]} \lambda(|\xi|) > 0, \tag{2.62}
$$

as well as Λ *is given by* (1.11)*. Particularly, if* $h(|\xi|) \neq 0$*, then* $||v_3(0)||_{H^k(\mathbb{R}^2)} > 0$*. In addition, we can choose proper constants* R_1, R_2 *such that* $\lambda_0 = \frac{\Lambda}{2}$*.*

Proof. According to Theorem 2.2, $(\varphi(\xi,z), \psi(\xi,z), \omega(\xi,z), \eta(\xi,z))$ is the solution of (2.5)-(2.6). Thus, $(\hat{\mathbf{v}}(\xi,z), \hat{q}(\xi,z))$ is the solution of the equations (2.4). As a result, for any fixed $|\xi| > \sqrt{\beta}$,

$$
\begin{aligned}
\tilde{\varrho}(t,y,z) &= -\rho_s' h(|\xi|)\hat{v}_3(\xi,z)e^{\lambda(|\xi|)t}e^{iy\xi},\\
\tilde{\mathbf{v}}(t,y,z) &= \lambda(|\xi|)h(|\xi|)\hat{\mathbf{v}}(\xi,z)e^{\lambda(|\xi|)t}e^{iy\xi},\\
\tilde{q}(t,y,z) &= \lambda(|\xi|)h(|\xi|)\hat{q}(\xi,z)e^{\lambda(|\xi|)t}e^{iy\xi},
\end{aligned}
$$

is the solution of the equations (1.10). Moreover, since $h \in C_0^\infty((R_1,R_2))$, by (2.48)-(2.49), we deduce that

$$
\begin{aligned}
&\sup_{\xi\in supp(h)} ||\partial^k\tilde{\varrho}(\xi,\cdot)||_{L^\infty(\mathbb{R}^2)} < \infty, \quad \sup_{\xi\in supp(h)} ||\partial^k\tilde{\mathbf{v}}(\xi,\cdot)||_{L^\infty(\mathbb{R}^2)} < \infty,\\
&\sup_{\xi\in supp(h)} ||\partial^k\tilde{q}(\xi,\cdot)||_{L^\infty(\mathbb{R}^2)} < \infty, \ \forall k\in\mathbb{N},
\end{aligned}
$$

which implies that the solution given by (2.59) is also a solution of the linearized equations (1.10). Additionally, with the help of Remark 2.1 and (2.58), we obtain $\varrho(t,y,z)$ and $\mathbf{v}(t,y,z)$ as well as $q(t,y,z)$ are real-valued functions.

Next, we shall show the estimate (2.60). In fact, combining the estimate (2.48) and the fact h is compactly supported, we deduce that

$$
\begin{aligned}
||\varrho(0)||^2_{H^k(\mathbb{R}^2)} &= \sum_{j=0}^k \int_{\mathbb{R}^2} (1+|\xi|^2)^{k-j}|h(|\xi|)|^2|\partial_z^j(\rho_s'\hat{v}_3(\xi,z))|^2 d\xi dz\\
&= \sum_{j=0}^k \int_{\mathbb{R}} (1+|\xi|^2)^{k-j}|h(|\xi|)|^2||\partial_z^j(\rho_s'\omega(|\xi|,z))||^2_{L^2(\mathbb{R})} d\xi\\
&\le D_k \int_{\mathbb{R}} [(1+|\xi|^2)^{k+2}|h(|\xi|)|^2] d\xi.
\end{aligned}
$$

Similarly, using the same way, we have

$$||\mathbf{v}(0)||^2_{H^k(\mathbb{R}^2)} \le D_k \int\limits_{\mathbb{R}} [(1+|\xi|^2)^{k+2}|h(|\xi|)|^2]d\xi,$$

$$||q(0)||^2_{H^k(\mathbb{R}^2)} \le D_k \int\limits_{\mathbb{R}} [(1+|\xi|^2)^{k+2}|h(|\xi|)|^2]d\xi.$$

Thus, the estimate (2.60) is valid. From (2.50), we derive $||w(0)||_{H^k(\mathbb{R}^2)} > 0$. (2.61) is clearly valid since that fact $e^{\lambda(|\xi|)t}$ is strict increasing on t. Finally, the assertion $\lambda_0 = \frac{\Lambda}{2}$ follows from Lemma 2.5. □

2.4. Uniqueness of solutions to the linearized equations

In order to show the main result in this paper, we need to discuss the uniqueness of solutions to the linearized equations (1.10). To this end, we introduce the relevant function space

$$\begin{aligned} Q_T = &\{(\varrho, \mathbf{v}, q) | \varrho \in C^0([0,T], L^2(\mathbb{R}^2)), \nabla q \in L^2(0,T; H^1_{Loc}(\mathbb{R}^2)), \\ & v_1 \in C^0([0,T], L^2(\mathbb{R}^2)),\ v_{1t} \in L^2((0,T)\times\mathbb{R}^2), \\ & \tilde{\mathbf{v}} \in C^0([0,T], (L^2(\mathbb{R}^2))^2) \cap L^2(0,T; (H^2(\mathbb{R}^2))^2), \\ & \tilde{\mathbf{v}}_t \in (L^2((0,T)\times\mathbb{R}^2))^2 \text{and } \mathrm{div}\tilde{\mathbf{v}} = 0\}. \end{aligned}$$

Thus, we have the following result.

Theorem 2.4. *The strong solution of the linearized equations* (1.10) *is unique.*

Proof. Let $(\varrho^i, \mathbf{v}^i, q^i) \in Q_T (i=1,2)$ be two strong solutions of (1.10) with $(\varrho^i(0), \mathbf{v}^i(0)) = (\varrho^i_0, \mathbf{v}^i_0)$. Denote $(\varrho, \mathbf{v}, q) = (\varrho^1 - \varrho^2, \mathbf{v}^1 - \mathbf{v}^2, q^1 - q^2) \in Q_T$. Thus, $(\varrho, \mathbf{v}, q)$ is also a strong solution of (1.10) with $(\varrho(0), \mathbf{v}(0)) = (0, \mathbf{0})$.

Multiplying $(1.10)_2$, $(1.10)_3$ by v_1, and $\tilde{\mathbf{v}}$, respectively, and integrating by parts over $(0,t)\times\mathbb{R}^2$, we obtain

$$\int\limits_0^t \int\limits_{\mathbb{R}^2} \rho_s \frac{\partial v_1}{\partial t} v_1 dydzds + \int\limits_0^t \int\limits_{\mathbb{R}^2} [\rho_s \bar{u}_0 - f] v_2 v_1 dydzds = 0, \tag{2.63}$$

and

$$\begin{aligned} \int\limits_0^t \int\limits_{\mathbb{R}^2} \rho_s \frac{\partial \tilde{\mathbf{v}}}{\partial t} \cdot \tilde{\mathbf{v}} dydzds + \int\limits_0^t \int\limits_{\mathbb{R}^2} f v_2 v_1 dydzds = -\mu \int\limits_0^t \int\limits_{\mathbb{R}^2} |\nabla \tilde{\mathbf{v}}|^2 dydzds \\ - g \int\limits_0^t \int\limits_{\mathbb{R}^2} \varrho v_3 dydzds. \end{aligned} \tag{2.64}$$

Then, with the help of Cauchy's inequality, (2.63) leads to

$$\int_{\mathbb{R}^2} \rho_s |v_1|^2 dydz \leq \left\| \frac{\rho_s \bar{u}_0 - f}{\rho_s} \right\|_{L^\infty(\mathbb{R})} \int_0^t \int_{\mathbb{R}^2} \rho_s |v_1|^2 dydzds$$
$$+ \left\| \frac{\rho_s \bar{u}_0 - f}{\rho_s} \right\|_{L^\infty(\mathbb{R})} \int_0^t \int_{\mathbb{R}^2} \rho_s |v_2|^2 dydzds. \tag{2.65}$$

Using the same process as [27], from $(1.10)_1$, we derive

$$\varrho(t, y, z) = \int_0^t v_3 \rho_s' ds, \ \forall t > 0. \tag{2.66}$$

Then, utilizing the Fubini's theorem, we deduce that

$$-g \int_0^t \int_{\mathbb{R}^2} \varrho v_3 dx d\tau \leq g \frac{T}{2} \left\| \frac{\rho_s'}{\rho_s} \right\|_{L^\infty(\mathbb{R})} \int_0^t \rho_s |v_3|^2 dydzd\tau. \tag{2.67}$$

Substituting (2.67) into (2.64), we derive that

$$\int_{\mathbb{R}^2} \rho_s \frac{\partial \tilde{\mathbf{v}}}{\partial t} \cdot \tilde{\mathbf{v}} dydz + 2\mu \int_0^t \int_{\mathbb{R}^2} |\nabla \tilde{\mathbf{v}}|^2 dydzds$$
$$\leq \left\| \frac{\rho_s \bar{u}_0 - f}{\rho_s} \right\|_{L^\infty(\mathbb{R})} \int_0^t \rho_s |v_2|^2 dydzds \tag{2.68}$$
$$+ \frac{Tg}{2} \left\| \frac{\rho_s'}{\rho_s} \right\|_{L^\infty(\mathbb{R})} \int_0^t \rho_s |v_3|^2 dydzds.$$

Therefore, combining (2.65) and (2.68), we conclude that

$$\int_{\mathbb{R}^2} \rho_s |\mathbf{v}|^2 dydz + 2\mu \int_0^t \int_{\mathbb{R}^2} |\nabla \tilde{\mathbf{v}}|^2 dydzds$$
$$\leq \left[\left\| \frac{\rho_s \bar{u}_0 - f}{\rho_s} \right\|_{L^\infty(\mathbb{R})} + \left\| \frac{f}{\rho_s} \right\|_{L^\infty(\mathbb{R})} \right] \int_0^t \int_{\mathbb{R}^2} \rho_s [|v_1|^2 + |v_2|^2] dydzds$$
$$+ Tg \left\| \frac{\rho_s'}{\rho_s} \right\|_{L^\infty(\mathbb{R})} \int_0^t \rho_s |v_3|^2 dydzds,$$

which leads to

$$\int_{\mathbb{R}^2} \rho_s |\mathbf{v}|^2 dydz \le M \int_0^t \int_{\mathbb{R}^2} \rho_s |\mathbf{v}|^2 dydzds,$$

where

$$M = \max\left\{ \left\| \frac{\rho_s \bar{u}_0 - f}{\rho_s} \right\|_{L^\infty(\mathbb{R})} + \left\| \frac{f}{\rho_s} \right\|_{L^\infty(\mathbb{R})}, Tg \left\| \frac{\rho_s'}{\rho_s} \right\|_{L^\infty(\mathbb{R})} \right\}.$$

Consequently, with the help of Grownwall's inequality, we find

$$\int_{\mathbb{R}^2} \rho_s |\mathbf{v}|^2 dydz = 0, \ \forall t \in [0, T],$$

which infers $\mathbf{v} = \mathbf{0}$ since $\rho_s > 0$. Namely, $\mathbf{v}^1 = \mathbf{v}^2$. Finally, from $(1.10)_1$, $(1.10)_3$ and (2.3), we obtain

$$(\varrho^1, \mathbf{v}^1, \nabla q^1) = (\varrho^2, \mathbf{v}^2, \nabla q^2), \ \forall t \in [0, T]. \quad \square$$

3. Nonlinear energy estimates for the perturbed problems

In this section, we shall provide some estimates of the perturbed problems (1.7), which is necessary to show our main results. Let $(\varrho, \mathbf{v}, q)$ be a classical solution of (1.7) in $[0, T] \times \mathbb{R}^2$ for any $0 < T < T^*$.

3.1. Estimates for $||\mathbf{v}||_{(L^2(\mathbb{R}^2))^3}$ and $||\nabla \tilde{\mathbf{v}}||_{(L^2(\mathbb{R}^2))^2}$

Lemma 3.1. *Let $||\varrho_0||^2_{L^2(\mathbb{R}^2)} + ||\mathbf{v}_0||^2_{L^2(\mathbb{R}^2)} \le \sigma^2$. Then, for any $0 < T < \infty$, the solution $(\varrho, \mathbf{v}, q)$ to (1.7), emanating from the initial data $(\varrho_0, \mathbf{v}_0)$, satisfies*

$$\sup_{t \in (0,T)} [||\varrho||^2_{L^2(\mathbb{R}^2)} + ||\mathbf{v}||^2_{(L^2(\mathbb{R}^2))^3}] + \int_0^T ||\nabla \tilde{\mathbf{v}}||^2_{(L^2(\mathbb{R}^2))^2} d\tau \le C(T)\sigma^2.$$

Moreover, if $||\varrho_0||^2_{L^2(\mathbb{R}^2)} + ||v_{10}||^2_{L^2(\mathbb{R}^2)} + ||\tilde{\mathbf{v}}_0||^2_{H^1(\mathbb{R}^2)} \le \sigma^2$, then, for any $0 < T < \infty$, the solution $(\varrho, \mathbf{v}, q)$ to (1.7) satisfies

$$\sup_{t \in (0,T)} ||\nabla \tilde{\mathbf{v}}||^2_{(L^2(\mathbb{R}^2))^2} + \int_0^T [||\tilde{\mathbf{v}}_t||^2_{(L^2(\mathbb{R}^2))^2} + ||\nabla^2 \tilde{\mathbf{v}}||^2_{(L^2(\mathbb{R}^2))^2}] dt \le C(T)\sigma^2.$$

Proof. From (1.7)$_1$ we find that for any $t \in (0, T]$,

$$\kappa_1 := \inf_{R^2} \rho_0 \leq \rho(t) \leq \sup_{R^2} \rho_0 := \kappa_2. \tag{3.1}$$

Multiplying (1.7)$_i$ $(i = 1, 2)$ by ϱ, $\mathbf{v}$ and integrating by parts over $\mathbb{R}^2$, we obtain

$$\begin{aligned} \frac{d}{dt}||\varrho||^2_{L^2(\mathbb{R}^2)} &= -2\int_{\mathbb{R}^2} \varrho\rho_s' v_3 dydz \leq 2\left|\left|\frac{\varrho\rho_s'}{\sqrt{\varrho+\rho_s}}\right|\right|_{L^2(\mathbb{R}^2)} ||\sqrt{\varrho+\rho_s}v_3||_{L^2(\mathbb{R}^2)} \\ &\leq 2(\kappa_1 + \inf_{\mathbb{R}^2} \rho_s)^{-\frac{1}{2}} ||\rho_s'||_{L^\infty(\mathbb{R}^2)} ||\varrho||_{L^2(\mathbb{R}^2)} ||\sqrt{\varrho+\rho_s}v_3||_{L^2(\mathbb{R}^2)}, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} &\int_{\mathbb{R}^2} (\varrho+\rho_s)\frac{\partial v_1}{\partial t} v_1 dydz + \int_{\mathbb{R}^2} (\varrho+\rho_s)[(\tilde{\mathbf{v}}\cdot\nabla)v_1 + \bar{u}_0 v_2] v_1 dydz \\ &- \int_{\mathbb{R}^2} f v_1 v_2 dydz = 0, \end{aligned} \tag{3.3}$$

as well as

$$\begin{aligned} &\int_{\mathbb{R}^2} (\varrho+\rho_s)\frac{\partial \tilde{\mathbf{v}}}{\partial t}\cdot\tilde{\mathbf{v}} dydz + \int_{\mathbb{R}^2} (\varrho+\rho_s)(\tilde{\mathbf{v}}\cdot\nabla)\tilde{\mathbf{v}}\cdot\tilde{\mathbf{v}} dydz + \int_{\mathbb{R}^2} f v_1 v_2 dydz \\ &= -\mu \int_{\mathbb{R}^2} |\nabla\tilde{\mathbf{v}}|^2 dydz - g\int_{\mathbb{R}^2} \varrho v_3 dydz. \end{aligned} \tag{3.4}$$

With the help of (1.7)$_1$ and (1.7)$_5$, we deduce that

$$\int_{\mathbb{R}^2} (\varrho+\rho_s)(\tilde{\mathbf{v}}\cdot\nabla)v_1 v_1 dydz = \frac{1}{2}\int_{\mathbb{R}^2} |v_1|^2 \frac{\partial\varrho}{\partial t} dydz. \tag{3.5}$$

Similarly, we have

$$\int_{\mathbb{R}^2} (\varrho+\rho_s)(\tilde{\mathbf{v}}\cdot\nabla)\tilde{\mathbf{v}} dydz = \frac{1}{2}\int_{\mathbb{R}^2} |\tilde{\mathbf{v}}|^2 \frac{\partial\varrho}{\partial t} dydz. \tag{3.6}$$

Then substituting (3.5), (3.6), into (3.3) and (3.4), respectively, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^2} (\varrho+\rho_s)\frac{\partial \mathbf{v}}{\partial t}\cdot\mathbf{v} dydz + \frac{1}{2}\int_{\mathbb{R}^2} \frac{\partial\varrho}{\partial t}|\mathbf{v}|^2 dydz + \mu\int_{\mathbb{R}^2} |\nabla\tilde{\mathbf{v}}|^2 dydz \\ &= -\bar{u}_0 \int_{\mathbb{R}^2} (\varrho+\rho_s) v_1 v_2 dydz - g\int_{\mathbb{R}^2} \varrho v_3 dydz. \end{aligned}$$

Namely, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||\sqrt{\varrho+\rho_s}\mathbf{v}||^2_{(L^2(\mathbb{R}^2))^3}+\mu||\nabla\tilde{\mathbf{v}}||^2_{(L^2(\mathbb{R}^2))^2}\\
&=-\bar{u}_0\int\limits_{\mathbb{R}^2}(\varrho+\rho_s)v_1v_2dydz-g\int\limits_{\mathbb{R}^2}\varrho v_3dydz \qquad (3.7)\\
&\le\bar{u}_0||\sqrt{\varrho+\rho_s}v_1||_{L^2(\mathbb{R}^2)}||\sqrt{\varrho+\rho_s}v_2||_{L^2(\mathbb{R}^2)}\\
&\quad+g(\kappa_1+\inf_{z\in\mathbb{R}}\rho_s)^{-\frac{1}{2}}||\varrho||_{L^2(\mathbb{R}^2)}||\sqrt{\varrho+\rho_s}v_3||_{L^2(\mathbb{R}^2)}.
\end{aligned}$$

As a result, combining (3.2) and (3.7) and using Cauchy's inequality, we find

$$\begin{aligned}
&\frac{d}{dt}[||\varrho||^2_{L^2(\mathbb{R}^2)}+||\sqrt{\varrho+\rho_s}\mathbf{v}||^2_{(L^2(\mathbb{R}^2))^3}]+\mu||\nabla\tilde{\mathbf{v}}||^2_{(L^2(\mathbb{R}^2))^2}\\
&\le C[||\varrho||^2_{L^2(\mathbb{R}^2)}+||\sqrt{\varrho+\rho_s}\mathbf{v}||^2_{(L^2(\mathbb{R}^2))^3}], \qquad (3.8)
\end{aligned}$$

which leads to

$$||\varrho||^2_{L^2(\mathbb{R}^2)}+||\sqrt{\varrho+\rho_s}\mathbf{v}||^2_{(L^2(\mathbb{R}^2))^3}\le\sigma^2e^{Ct}. \qquad (3.9)$$

Particularly, with the help of (3.8) and (3.9), we obtain

$$||\varrho||^2_{L^2(\mathbb{R}^2)}+||\mathbf{v}||^2_{(L^2(\mathbb{R}^2))^3}+\mu\int\limits_0^t||\nabla\tilde{\mathbf{v}}||^2_{(L^2(\mathbb{R}^2))^2}d\tau\le C\sigma^2e^{Ct}. \qquad (3.10)$$

Multiplying $(1.7)_3$ by $\tilde{\mathbf{v}}_t$ and integrating by parts over $\mathbb{R}^2$, we obtain

$$\begin{aligned}
&||\sqrt{\varrho+\rho_s}\tilde{\mathbf{v}}||^2_{(L^2(\mathbb{R}^2))^2}+\mu\frac{d}{dt}||\nabla\tilde{\mathbf{v}}||^2_{L^2(\mathbb{R}^2)^2}\\
&=-\int\limits_{\mathbb{R}^2}(\varrho+\rho_s)[\tilde{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}}]\cdot\tilde{\mathbf{v}}_tdx-f\int\limits_{\mathbb{R}^2}v_1v_{2t}dx-g\int\limits_{\mathbb{R}^2}\varrho v_{3t}dx. \qquad (3.11)
\end{aligned}$$

Using Cauchy's inequality, we deduce that

$$-\int\limits_{\mathbb{R}^2}(\varrho+\rho_s)[\tilde{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}}]\cdot\tilde{\mathbf{v}}_tdx\le\frac{1}{4}||\sqrt{\varrho+\rho_s}\tilde{\mathbf{v}}_t||^2_{L^2(\mathbb{R}^2)}+C||\tilde{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}}||^2_{L^2(\mathbb{R}^2)}, \qquad (3.12)$$

and

$$-f\int\limits_{\mathbb{R}^2}v_1v_{2t}dx\le\frac{1}{4}||\sqrt{\varrho+\rho_s}v_{2t}||^2_{L^2(\mathbb{R}^2)}+C||v_1||^2_{L^2(\mathbb{R}^2)}, \qquad (3.13)$$

as well as

$$-g\int_{\mathbb{R}^2}\varrho v_{3t}dx \le \frac{1}{4}||\sqrt{\varrho+\rho_s}v_{3t}||^2_{L^2(\mathbb{R}^2)} + C||\varrho||^2_{L^2(\mathbb{R}^2)}. \tag{3.14}$$

Then, substituting (3.12)-(3.14) into (3.11), we have

$$\begin{aligned}&\frac{1}{2}||\sqrt{\varrho+\rho_s}\tilde{\mathbf{v}}||^2_{(L^2(\mathbb{R}^2))^2} + \mu\frac{d}{dt}||\nabla\tilde{\mathbf{v}}||^2_{(L^2(\mathbb{R}^2))^2}\\ &\le C||\tilde{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}}||^2_{L^2(\mathbb{R}^2)} + C||v_1||^2_{L^2(\mathbb{R}^2)} + C||\varrho||^2_{L^2(\mathbb{R}^2)}.\end{aligned} \tag{3.15}$$

In addition, the classical regularity on the Stokes equations implies

$$\begin{aligned}&||\nabla^2\tilde{\mathbf{v}}||^2_{(L^2(\mathbb{R}^2))^2} + ||\nabla q||^2_{L^2(\mathbb{R}^2)}\\ &\le C[||\sqrt{\varrho+\rho_s}\tilde{\mathbf{v}}_t||^2_{(L^2(\mathbb{R}^2))^2} + ||\sqrt{\varrho+\rho_s}\tilde{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}}||^2_{L^2(\mathbb{R}^2)}\\ &\quad + ||v_1||^2_{L^2(\mathbb{R}^2)} + ||\varrho||^2_{L^2(\mathbb{R}^2)}].\end{aligned} \tag{3.16}$$

Then, combining (3.15) and (3.16), we obtain

$$\begin{aligned}&\frac{1}{4}||\sqrt{\varrho+\rho_s}\tilde{\mathbf{v}}_t||^2_{(L^2(\mathbb{R}^2))^2} + \epsilon||\nabla^2\tilde{\mathbf{v}}||^2_{(L^2(\mathbb{R}^2))^2} + \mu\frac{d}{dt}||\nabla\tilde{\mathbf{v}}||^2_{(L^2(\mathbb{R}^2))^2}\\ &\le C||\tilde{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}}||^2_{L^2(\mathbb{R}^2)} + C||v_1||^2_{L^2(\mathbb{R}^2)} + C||\varrho||^2_{L^2(\mathbb{R}^2)}.\end{aligned} \tag{3.17}$$

Using Hölder's inequality and interpolation inequality, we have

$$\begin{aligned}&C||\tilde{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}}||^2_{L^2(\mathbb{R}^2)} \le C||\tilde{\mathbf{v}}||^2_{L^\infty(\mathbb{R}^2)}||\nabla\tilde{\mathbf{v}}||^2_{L^2(\mathbb{R}^2)}\\ &\le C||\nabla^2\tilde{\mathbf{v}}||_{L^2(\mathbb{R}^2)}||\tilde{\mathbf{v}}||_{L^2(\mathbb{R}^2)}||\nabla\tilde{\mathbf{v}}||^2_{L^2(\mathbb{R}^2)}\\ &\le \frac{\epsilon}{2}||\nabla^2\tilde{\mathbf{v}}||^2_{L^2(\mathbb{R}^2)} + C(\epsilon)||\nabla\tilde{\mathbf{v}}||^4_{L^2(\mathbb{R}^2)}||\tilde{\mathbf{v}}||^2_{L^2(\mathbb{R}^2)}.\end{aligned} \tag{3.18}$$

Then, substituting (3.18) into (3.17), we get

$$\begin{aligned}&\frac{1}{4}||\sqrt{\varrho+\rho_s}\tilde{\mathbf{v}}_t||^2_{(L^2(\mathbb{R}^2))^2} + \frac{\epsilon}{2}||\nabla^2\tilde{\mathbf{v}}||^2_{(L^2(\mathbb{R}^2))^2} + \mu\frac{d}{dt}||\nabla\tilde{\mathbf{v}}||^2_{(L^2(\mathbb{R}^2))^2}\\ &\le C||\nabla\tilde{\mathbf{v}}||^4_{L^2(\mathbb{R}^2)}||\tilde{\mathbf{v}}||^2_{L^2(\mathbb{R}^2)} + C||v_1||^2_{L^2(\mathbb{R}^2)} + C||\varrho||^2_{L^2(\mathbb{R}^2)}.\end{aligned} \tag{3.19}$$

As a result, with the help of Gronwall inequality, we obtain

$$||\nabla\tilde{\mathbf{v}}||^2_{(L^2(\mathbb{R}^2))^2} + \int_0^t [||\sqrt{\varrho+\rho_s}\tilde{\mathbf{v}}_t||^2_{(L^2(\mathbb{R}^2))^2} + ||\nabla^2\tilde{\mathbf{v}}||^2_{(L^2(\mathbb{R}^2))^2}]dt \le Ce^{C\sigma^4 e^{CT}}\sigma^2. \tag{3.20}$$

From (3.10) and (3.20), we have

$$\int_0^T ||\tilde{\mathbf{v}}||^2_{(H^2(\mathbb{R}^2))^2} dt \le Ce^{C\sigma^4 e^{CT}}\sigma^2. \quad \square \tag{3.21}$$

3.2. Estimates for $||\varrho||_{H^1(\mathbb{R}^2)}$ and $||v_1||_{H^1(\mathbb{R}^2)}$

Lemma 3.2. *Let $||\varrho_0||^2_{H^1(\mathbb{R}^2)} + ||v_{10}||^2_{H^1(\mathbb{R}^2)} + ||\tilde{\mathbf{v}}_0||^2_{H^2(\mathbb{R}^2)} \le \sigma^2$. Then, for any $0 < T < T^*$, the solution $(\varrho, \mathbf{v}, q)$ to (1.7), emanating from the initial data $(\varrho_0, \mathbf{v}_0)$, satisfies*

$$\sup_{t\in(0,T)} [||\nabla\varrho||^2_{L^2(\mathbb{R}^2)} + ||\nabla v_1||^2_{L^2(\mathbb{R}^2)} + ||\sqrt{\rho}\tilde{\mathbf{v}}_t||^2_{L^2(\mathbb{R}^2)} + ||\tilde{\mathbf{v}}||^2_{H^2(\mathbb{R}^2)} + ||\nabla q||^2_{L^2(\mathbb{R}^2)}]$$
$$+ \int_0^T [||\nabla\tilde{\mathbf{v}}_t||^2_{L^2(\mathbb{R}^2)} + ||v_{1t}||^2_{L^2(\mathbb{R}^2)}]ds \le C(T)\sigma^2,$$

where T^* denotes the maximal time of existence of the solution.

Proof. By taking the partial derivative of $(1.7)_1$ with respect to y and z respectively, we obtain

$$\frac{\partial \varrho_y}{\partial t} + [\tilde{\mathbf{v}} \cdot \nabla\varrho]_y + v_{3y}\rho_s' = 0, \tag{3.22}$$

and

$$\frac{\partial \varrho_z}{\partial t} + [\tilde{\mathbf{v}} \cdot \nabla\varrho]_z + v_{3z}\rho_s' + v_3\rho_s'' = 0. \tag{3.23}$$

Then, multiplying (3.22) by ϱ_y and integrating over $\mathbb{R}^2$, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} |\varrho_y|^2 dydz + \int_{\mathbb{R}^2} \varrho_y(\tilde{\mathbf{v}}_y \cdot \nabla\varrho)dydz + \int_{\mathbb{R}^2} \rho_s' v_{3y}\varrho_y dydz = 0. \tag{3.24}$$

Similarly, from (3.23), we derive

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} |\varrho_z|^2 dydz + \int_{\mathbb{R}^2} \varrho_z(\tilde{\mathbf{v}}_z \cdot \nabla\varrho)dydz + \int_{\mathbb{R}^2} (\rho_s' v_{3z} + v_3\rho_s'')\varrho_z dydz = 0. \tag{3.25}$$

Thus, combining (3.24) and (3.25), we find

$$\frac{d}{dt}\int_{\mathbb{R}^2} |\nabla\varrho|^2 \le C||\nabla\tilde{\mathbf{v}}||_{L^\infty(\mathbb{R}^2)}||\nabla\varrho||^2_{L^2(\mathbb{R}^2)} + ||\tilde{\mathbf{v}}||_{H^1(\mathbb{R}^2)}||\nabla\varrho||_{L^2(\mathbb{R}^2)}. \tag{3.26}$$

To estimate $||\varrho||_{H^1(\mathbb{R}^2)}$, we need estimate $||\nabla\tilde{\mathbf{v}}||_{L^\infty(\mathbb{R}^2)}$. It follows from the classical regularity theory for Stokes equations that

$$\begin{aligned}&||\Delta\tilde{\mathbf{v}}||^2_{L^4(\mathbb{R}^2)}+||\nabla q||^2_{L^4(\mathbb{R}^2)}\\&\le C[||(\varrho+\rho_s)\tilde{\mathbf{v}}_t||^2_{L^4(\mathbb{R}^2)}+||(\varrho+\rho_s)\tilde{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}}||^2_{L^4(\mathbb{R}^2)}\\&+||v_1||^2_{L^4(\mathbb{R}^2)}+||\varrho||^2_{L^4(\mathbb{R}^2)}].\end{aligned}\tag{3.27}$$

In addition, with the help of Hölder inequality and Gagliardo-Nirenberg inequality, we find

$$\begin{aligned}&||\rho\tilde{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}}||^2_{L^4(\mathbb{R}^2)}\\&\le C||\tilde{\mathbf{v}}||^2_{L^8(\mathbb{R}^2)}||\nabla\tilde{\mathbf{v}}||^2_{L^8(\mathbb{R}^2)}\\&\le C||\tilde{\mathbf{v}}||^{\frac{3}{4}}_{L^2(\mathbb{R}^2)}||\nabla\tilde{\mathbf{v}}||^{\frac{3}{2}}_{L^2(\mathbb{R}^2)}||\nabla^2\tilde{\mathbf{v}}||^{\frac{7}{4}}_{L^2(\mathbb{R}^2)}.\end{aligned}\tag{3.28}$$

Thus, combining (3.27) and (3.28), we derive

$$\begin{aligned}&\frac{d}{dt}||\nabla\varrho||^2_{L^2(\mathbb{R}^2)}\\&\le\epsilon||\nabla\tilde{\mathbf{v}}_t||^2_{L^2(\mathbb{R}^2)}+C(\epsilon)||\tilde{\mathbf{v}}_t||^2_{L^2(\mathbb{R}^2)}+C||\tilde{\mathbf{v}}||^2_{H^1(\mathbb{R}^2)}\\&+C||\tilde{\mathbf{v}}||^{\frac{3}{4}}_{L^2(\mathbb{R}^2)}||\nabla\tilde{\mathbf{v}}||^{\frac{3}{2}}_{L^2(\mathbb{R}^2)}||\nabla^2\tilde{\mathbf{v}}||^{\frac{7}{4}}_{L^2(\mathbb{R}^2)}+||v_1||^2_{L^2(\mathbb{R}^2)}\\&+C||\varrho||^2_{L^2(\mathbb{R}^2)}+C||\nabla\varrho||^2_{L^2(\mathbb{R}^2)}(||\nabla\varrho||^2_{L^2(\mathbb{R}^2)}+1)+C||\nabla v_1||^2_{L^2(\mathbb{R}^2)}.\end{aligned}\tag{3.29}$$

Next, we estimate the first term on the right-hand side of inequality (3.29). From (1.6), the equation $(1.7)_3$ are rewritten as

$$\rho\frac{\partial\tilde{\mathbf{v}}}{\partial t}+\varrho\tilde{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}}+fv_1\mathbf{e}_1=\mu\Delta\tilde{\mathbf{v}}-\nabla q-g\rho\mathbf{e}_2.\tag{3.30}$$

By taking the partial derivative of (3.30) with respect to t, we derive

$$\begin{aligned}&\rho_t\tilde{\mathbf{v}}_t+\rho\tilde{\mathbf{v}}_{tt}+\rho_t\tilde{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}}+\rho\tilde{\mathbf{v}}_t\cdot\nabla\tilde{\mathbf{v}}+\rho\tilde{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}}_t+fv_{1t}\mathbf{e}_1\\&=\mu\Delta\tilde{\mathbf{v}}_t-\nabla q_t-g\rho_t\mathbf{e}_2.\end{aligned}\tag{3.31}$$

Then, (3.31) dotting $\tilde{\mathbf{v}}_t$ and integrating over $\mathbb{R}^2$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}\rho|\tilde{\mathbf{v}}_t|^2dydz+\mu\int_{\mathbb{R}^2}|\nabla\tilde{\mathbf{v}}_t|^2dydz\\&=\int_{\mathbb{R}^2}\operatorname{div}(\rho\tilde{\mathbf{v}})[|\tilde{\mathbf{v}}_t|^2+(\tilde{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}})\cdot\tilde{\mathbf{v}}_t+g\mathbf{e}_2\cdot\tilde{\mathbf{v}}_t]dydz\\&\quad-\int_{\mathbb{R}^2}fv_{1t}\mathbf{e}_1\cdot\tilde{\mathbf{v}}_t-\int_{\mathbb{R}^2}(\rho\tilde{\mathbf{v}}_t\cdot\nabla\tilde{\mathbf{v}})\cdot\tilde{\mathbf{v}}_tdydz,\end{aligned}\tag{3.32}$$

which yields to

$$\begin{aligned}
&\frac{d}{dt}\int\limits_{\mathbb{R}^2}\rho|\tilde{\mathbf{v}}_t|^2dydz+\mu\int\limits_{\mathbb{R}^2}|\nabla\tilde{\mathbf{v}}_t|^2dydz\\
&\le\int\limits_{\mathbb{R}^2}[2\rho|\tilde{\mathbf{v}}||\tilde{\mathbf{v}}_t||\nabla\tilde{\mathbf{v}}_t|+\rho|\tilde{\mathbf{v}}||\nabla\tilde{\mathbf{v}}|^2|\tilde{\mathbf{v}}_t|+\rho|\tilde{\mathbf{v}}|^2|\nabla^2\tilde{\mathbf{v}}||\tilde{\mathbf{v}}_t| \qquad (3.33)\\
&\quad+\rho|\tilde{\mathbf{v}}|^2|\nabla\tilde{\mathbf{v}}||\nabla\tilde{\mathbf{v}}_t|+f|v_{1t}||\tilde{\mathbf{v}}_t|+g\rho|\tilde{\mathbf{v}}||\nabla\tilde{\mathbf{v}}_t|+\rho|\tilde{\mathbf{v}}_t|^2|\nabla\tilde{\mathbf{v}}|]dydz:=\sum_{i=1}^{7}I_i.
\end{aligned}$$

By Hölder's inequality, Gagliardo-Nirenberg inequality as well as Young's inequality, we obtain

$$\begin{aligned}
I_1&\le C||\rho||_{L^\infty}^{\frac12}||\tilde{\mathbf{v}}||_{L^4(\mathbb{R}^2)}||\sqrt{\rho}\tilde{\mathbf{v}}_t||_{L^4(\mathbb{R}^2)}||\nabla\tilde{\mathbf{v}}_t||_{L^2(\mathbb{R}^2)}\\
&\le C||\rho||_{L^\infty}^{\frac34}||\tilde{\mathbf{v}}||_{L^2(\mathbb{R}^2)}^{\frac12}||\nabla\tilde{\mathbf{v}}||_{L^2(\mathbb{R}^2)}^{\frac12}||\sqrt{\rho}\tilde{\mathbf{v}}_t||_{L^2(\mathbb{R}^2)}^{\frac12}||\nabla\tilde{\mathbf{v}}_t||_{L^2(\mathbb{R}^2)}^{\frac32}\\
&\le\epsilon||\nabla\tilde{\mathbf{v}}_t||_{L^2(\mathbb{R}^2)}^2+C(\epsilon)||\tilde{\mathbf{v}}||_{L^2(\mathbb{R}^2)}^2||\nabla\tilde{\mathbf{v}}||_{L^2(\mathbb{R}^2)}^2||\sqrt{\rho}\tilde{\mathbf{v}}_t||_{L^2(\mathbb{R}^2)}^2,
\end{aligned}$$

where ϵ is a small constant to be specified later, and $C(\epsilon)$ is a constant dependent on ϵ. Likewise, we derive

$$\begin{aligned}
I_2&\le C||\nabla\tilde{\mathbf{v}}||_{L^2(\mathbb{R}^2)}||\nabla\tilde{\mathbf{v}}||_{L^6(\mathbb{R}^2)}||\tilde{\mathbf{v}}||_{L^6(\mathbb{R}^2)}||\sqrt{\rho}\tilde{\mathbf{v}}_t||_{L^6(\mathbb{R}^2)}\\
&\le C||\tilde{\mathbf{v}}||_{L^2(\mathbb{R}^2)}^{\frac12}||\nabla\tilde{\mathbf{v}}||_{L^2(\mathbb{R}^2)}^{\frac53}||\nabla^2\tilde{\mathbf{v}}||_{L^2(\mathbb{R}^2)}^{\frac56}||\tilde{\mathbf{v}}_t||_{L^2(\mathbb{R}^2)}^{\frac13}||\nabla\tilde{\mathbf{v}}_t||_{L^2(\mathbb{R}^2)}^{\frac23}\\
&\le\epsilon||\nabla\tilde{\mathbf{v}}_t||_{L^2(\mathbb{R}^2)}^2+C(\epsilon)||\sqrt{\rho}\tilde{\mathbf{v}}_t||_{L^2}^2+C(\epsilon)||\tilde{\mathbf{v}}||_{L^2(\mathbb{R}^2)}||\nabla\tilde{\mathbf{v}}||_{L^2(\mathbb{R}^2)}^{\frac{10}{3}}||\nabla^2\tilde{\mathbf{v}}||_{L^2(\mathbb{R}^2)}^{\frac53},\\
I_3&\le C||\rho||_{L^\infty(\mathbb{R}^2)}||\nabla^2\tilde{\mathbf{v}}||_{L^2(\mathbb{R}^2)}||\tilde{\mathbf{v}}_t||_{L^6(\mathbb{R}^2)}||\tilde{\mathbf{v}}||_{L^6(\mathbb{R}^2)}^2\\
&\le C||\nabla^2\tilde{\mathbf{v}}||_{L^2(\mathbb{R}^2)}||\nabla\tilde{\mathbf{v}}||_{L^2(\mathbb{R}^2)}^{\frac43}||\tilde{\mathbf{v}}||_{L^2(\mathbb{R}^2)}^{\frac23}||\nabla\tilde{\mathbf{v}}_t||_{L^2(\mathbb{R}^2)}^{\frac23}||\tilde{\mathbf{v}}_t||_{L^2(\mathbb{R}^2)}^{\frac13}\\
&\le\epsilon||\nabla\tilde{\mathbf{v}}_t||_{L^2(\mathbb{R}^2)}^2+C(\epsilon)||\sqrt{\rho}\tilde{\mathbf{v}}_t||_{L^2(\mathbb{R}^2)}^2+C||\tilde{\mathbf{v}}||_{L^2(\mathbb{R}^2)}^{\frac43}||\nabla\tilde{\mathbf{v}}||_{L^2(\mathbb{R}^2)}^{\frac83}||\nabla^2\tilde{\mathbf{v}}||_{L^2(\mathbb{R}^2)}^2,
\end{aligned}$$

$$\begin{aligned}
I_4&\le C||\rho||_{L^\infty(\mathbb{R}^2)}||\nabla\tilde{\mathbf{v}}_t||_{L^2(\mathbb{R}^2)}||\nabla\tilde{\mathbf{v}}||_{L^6(\mathbb{R}^2)}||\tilde{\mathbf{v}}||_{L^6(\mathbb{R}^2)}^2\\
&\le C||\nabla\tilde{\mathbf{v}}_t||_{L^2(\mathbb{R}^2)}||\nabla^2\tilde{\mathbf{v}}||_{L^2(\mathbb{R}^2)}^{\frac56}||\nabla\tilde{\mathbf{v}}||_{L^2(\mathbb{R}^2)}^{\frac43}||\tilde{\mathbf{v}}||_{L^2(\mathbb{R}^2)}^{\frac56}\\
&\le\epsilon||\nabla\tilde{\mathbf{v}}_t||_{L^2(\mathbb{R}^2)}^2+C(\epsilon)||\tilde{\mathbf{v}}||_{L^2(\mathbb{R}^2)}^{\frac53}||\nabla\tilde{\mathbf{v}}||_{L^2(\mathbb{R}^2)}^{\frac83}||\nabla^2\tilde{\mathbf{v}}||_{L^2(\mathbb{R}^2)}^{\frac53}.
\end{aligned}$$

For I_5 and I_6, we use Hölder's inequality and Young's inequality to obtain

$$I_5\le C||v_{1t}||_{L^2(\mathbb{R}^2)}||\sqrt{\rho}\tilde{\mathbf{v}}_t||_{L^2(\mathbb{R}^2)}\le C||\sqrt{\rho}\tilde{\mathbf{v}}_t||_{L^2(\mathbb{R}^2)}^2+C||v_{1t}||_{L^2(\mathbb{R}^2)}^2,$$

and

$$I_6 \leq \epsilon ||\nabla \tilde{\mathbf{v}}_t||^2_{L^2(\mathbb{R}^2)} + C(\epsilon)||\tilde{\mathbf{v}}||^2_{L^2(\mathbb{R}^2)}.$$

I_7 can be estimated in a similar way

$$\begin{aligned} I_7 &\leq ||\rho||_{L^\infty(\mathbb{R}^2)}||\tilde{\mathbf{v}}_t||^2_{L^4}||\nabla\tilde{\mathbf{v}}||_{L^2(\mathbb{R}^2)} \\ &\leq C||\tilde{\mathbf{v}}_t||_{L^2(\mathbb{R}^2)}||\nabla\tilde{\mathbf{v}}_t||_{L^2(\mathbb{R}^2)}||\nabla\tilde{\mathbf{v}}||_{L^2(\mathbb{R}^2)} \\ &\leq \epsilon||\nabla\tilde{\mathbf{v}}_t||^2_{L^2(\mathbb{R}^2)} + C(\epsilon)||\sqrt{\rho}\tilde{\mathbf{v}}_t||^2_{L^2(\mathbb{R}^2)}||\nabla\tilde{\mathbf{v}}||^2_{L^2(\mathbb{R}^2)}. \end{aligned}$$

Thus, submitting the above inequality into (3.33), we have

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^2} \rho|\tilde{\mathbf{v}}_t|^2 dydz + \frac{\mu}{2}\int_{\mathbb{R}^2} |\nabla\tilde{\mathbf{v}}_t|^2 dydz \\ &\leq C||\sqrt{\rho}\tilde{\mathbf{v}}_t||^2_{L^2(\mathbb{R}^2)}[1 + ||\nabla\tilde{\mathbf{v}}||^2_{L^2(\mathbb{R}^2)} + ||\tilde{\mathbf{v}}||^2_{L^2(\mathbb{R}^2)}||\nabla\tilde{\mathbf{v}}||^2_{L^2(\mathbb{R}^2)}] \\ &\quad + C[\mathcal{K}(t) + ||v_{1t}||^2_{L^2(\mathbb{R}^2)}], \end{aligned} \tag{3.34}$$

where

$$\begin{aligned} \mathcal{K}(t) = {} & ||\tilde{\mathbf{v}}||_{L^2(\mathbb{R}^2)}||\nabla\tilde{\mathbf{v}}||^{\frac{10}{3}}_{L^2(\mathbb{R}^2)}||\nabla^2\tilde{\mathbf{v}}||^{\frac{5}{3}}_{L^2(\mathbb{R}^2)} + ||\tilde{\mathbf{v}}||^{\frac{4}{3}}_{L^2(\mathbb{R}^2)}||\nabla\tilde{\mathbf{v}}||^{\frac{8}{3}}_{L^2(\mathbb{R}^2)}||\nabla^2\tilde{\mathbf{v}}||^2_{L^2(\mathbb{R}^2)} \\ & + ||\tilde{\mathbf{v}}||^{\frac{5}{3}}_{L^2(\mathbb{R}^2)}||\nabla\tilde{\mathbf{v}}||^{\frac{8}{3}}_{L^2(\mathbb{R}^2)}||\nabla^2\tilde{\mathbf{v}}||^{\frac{5}{3}}_{L^2(\mathbb{R}^2)} + ||\tilde{\mathbf{v}}||^2_{L^2(\mathbb{R}^2)}. \end{aligned} \tag{3.35}$$

From (3.10) and (3.20), we find $\mathcal{K}(t) \in L^1(0,T)$ for any $0 < T < \infty$. By $(1.7)_4$, we deduce that

$$\begin{aligned} &||\sqrt{\rho}v_{1t}||^2_{L^2(\mathbb{R}^2)} \\ &\leq \frac{1}{2}||\sqrt{\rho}v_{1t}||^2_{L^2(\mathbb{R}^2)} + ||\tilde{\mathbf{v}}\cdot\nabla v_1||^2_{L^2(\mathbb{R}^2)} + C||v_2||^2_{L^2(\mathbb{R}^2)} \\ &\leq \frac{1}{2}||\sqrt{\rho}v_{1t}||^2_{L^2(\mathbb{R}^2)} + ||\tilde{\mathbf{v}}||^2_{L^\infty(\mathbb{R}^2)}||\nabla v_1||^2_{L^2(\mathbb{R}^2)} + C||v_2||^2_{L^2(\mathbb{R}^2)} \\ &\leq \frac{1}{2}||\sqrt{\rho}v_{1t}||^2_{L^2(\mathbb{R}^2)} + ||\tilde{\mathbf{v}}||^2_{H^2(\mathbb{R}^2)}||\nabla v_1||^2_{L^2(\mathbb{R}^2)} + C||v_2||^2_{L^2(\mathbb{R}^2)}. \end{aligned}$$

Namely,

$$||v_{1t}||^2_{L^2(\mathbb{R}^2)} \leq C||\tilde{\mathbf{v}}||^2_{H^2(\mathbb{R}^2)}||\nabla v_1||^2_{L^2(\mathbb{R}^2)} + C||\tilde{\mathbf{v}}||^2_{L^2(\mathbb{R}^2)}. \tag{3.36}$$

Submitting (3.36) into (3.34), we obtain

$$\begin{aligned} &\frac{d}{dt}||\sqrt{\rho}\tilde{\mathbf{v}}_t||^2_{L^2(\mathbb{R}^2)} + \frac{\mu}{2}||\nabla\tilde{\mathbf{v}}_t||^2_{L^2(\mathbb{R}^2)} \\ &\leq C||\sqrt{\rho}\tilde{\mathbf{v}}_t||^2_{L^2(\mathbb{R}^2)}[1 + ||\nabla\tilde{\mathbf{v}}||^2_{L^2(\mathbb{R}^2)} + ||\tilde{\mathbf{v}}||^2_{L^2(\mathbb{R}^2)}||\nabla\tilde{\mathbf{v}}||^2_{L^2(\mathbb{R}^2)}] \\ &+ C[\mathcal{K}(t) + ||\tilde{\mathbf{v}}||^2_{H^2(\mathbb{R}^2)}||\nabla v_1||^2_{L^2(\mathbb{R}^2)} + ||\tilde{\mathbf{v}}||^2_{L^2(\mathbb{R}^2)}]. \end{aligned} \tag{3.37}$$

By taking the partial derivative of (1.7)$_2$ with respect to y and z, respectively, we have

$$\frac{\partial v_{1y}}{\partial t}+[\tilde{\mathbf{v}}\cdot\nabla v_1]_y=\left[\frac{(f-u_0(\varrho+\rho_s))v_2}{\varrho+\rho_s}\right]_y \tag{3.38}$$

and

$$\frac{\partial v_{1z}}{\partial t}+[\tilde{\mathbf{v}}\cdot\nabla v_1]_z=\left[\frac{(f-u_0(\varrho+\rho_s))v_2}{\varrho+\rho_s}\right]_z. \tag{3.39}$$

Then, multiplying (3.38) by v_{1y} and integrating over $\mathbb{R}^2$, we drive

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\limits_{\mathbb{R}^2}|\frac{\partial v_1}{\partial y}|^2dydz+\int\limits_{\mathbb{R}^2}v_{1y}(\tilde{\mathbf{v}}_y\cdot\nabla v_1)dydz\\&=\int\limits_{\mathbb{R}^2}\left[\frac{(f-u_0(\varrho+\rho_s))v_2}{\varrho+\rho_s}\right]_y v_{1y}dydz.\end{aligned} \tag{3.40}$$

Analogously, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\limits_{\mathbb{R}^2}|\frac{\partial v_1}{\partial z}|^2dydz+\int\limits_{\mathbb{R}^2}v_{1z}(\tilde{\mathbf{v}}_z\cdot\nabla v_1)dydz\\&=\int\limits_{\mathbb{R}^2}\left[\frac{(f-u_0(\varrho+\rho_s))v_2}{\varrho+\rho_s}\right]_z v_{1z}dydz.\end{aligned} \tag{3.41}$$

As a result, by (3.40) and (3.41), we have

$$\begin{aligned}&\frac{d}{dt}\int\limits_{\mathbb{R}^2}|\nabla v_1|^2dydz\\&\le C||\nabla\tilde{\mathbf{v}}||_{L^\infty(\mathbb{R}^2)}||\nabla v_1||^2_{L^2(\mathbb{R}^2)}+C||\nabla\tilde{\mathbf{v}}||_{L^2(\mathbb{R}^2)}||\nabla v_1||_{L^2(R^2)}\\&\quad+C||\tilde{\mathbf{v}}||_{H^2(\mathbb{R}^2)}||\nabla\varrho||_{L^2(\mathbb{R}^2)}||\nabla v_1||_{L^2(\mathbb{R}^2)}+C||\tilde{\mathbf{v}}||_{L^2(\mathbb{R}^2)}||\nabla v_1||_{L^2(\mathbb{R}^2)}.\end{aligned} \tag{3.42}$$

Then, combining (3.28) and (3.42), we obtain

$$\begin{aligned}&\frac{d}{dt}||\nabla v_1||^2_{L^2(\mathbb{R}^2)}\\&\le\epsilon||\nabla\tilde{\mathbf{v}}_t||^2_{L^2(\mathbb{R}^2)}+C(\epsilon)||\tilde{\mathbf{v}}_t||^2_{L^2(\mathbb{R}^2)}+C||\tilde{\mathbf{v}}||^2_{H^1(\mathbb{R}^2)}+||v_1||^2_{L^2(\mathbb{R}^2)}\\&+C||\tilde{\mathbf{v}}||^{\frac{3}{4}}_{L^2(\mathbb{R}^2)}||\nabla\tilde{\mathbf{v}}||^{\frac{3}{2}}_{L^2(\mathbb{R}^2)}||\nabla^2\tilde{\mathbf{v}}||^{\frac{7}{4}}_{L^2(\mathbb{R}^2)}+C||\varrho||^2_{L^2(\mathbb{R}^2)}\\&+C||\nabla v_1||^2_{L^2(\mathbb{R}^2)}(||\nabla v_1||^2_{L^2(\mathbb{R}^2)}+1)+C||\nabla\varrho||^2_{L^2(\mathbb{R}^2)}+||\tilde{\mathbf{v}}||^2_{H^2(\mathbb{R}^2)}||\nabla\varrho||^2_{L^2(\mathbb{R}^2)}.\end{aligned} \tag{3.43}$$

According to (3.29), (3.37), (3.43), and choosing $\epsilon=\frac{\mu}{8}$, we derive

$$\begin{aligned}&\frac{d}{dt}[||\nabla\varrho||^2_{L^2(\mathbb{R}^2)}+||\nabla v_1||^2_{L^2(\mathbb{R}^2)}+||\sqrt{\rho}\tilde{\mathbf{v}}_t||^2_{L^2(\mathbb{R}^2)}]+\frac{\mu}{4}||\nabla\tilde{\mathbf{v}}_t||^2_{L^2(\mathbb{R}^2)}\\&\le C||\nabla\varrho||^2_{L^2(\mathbb{R}^2)}(||\nabla\varrho||^2_{L^2(\mathbb{R}^2)}+1)+C||\nabla v_1||^2_{L^2(\mathbb{R}^2)}(||\nabla v_1||^2_{L^2(\mathbb{R}^2)}+1)\\&\quad+||\tilde{\mathbf{v}}||^2_{H^2(\mathbb{R}^2)}(||\nabla\varrho||^2_{L^2(\mathbb{R}^2)}+||\nabla v_1||^2_{L^2(\mathbb{R}^2)})\\&\quad+C||\sqrt{\rho}\tilde{\mathbf{v}}_t||^2_{L^2(\mathbb{R}^2)}(1+||\nabla\tilde{\mathbf{v}}||^2_{L^2(\mathbb{R}^2)}+||\tilde{\mathbf{v}}||^2_{L^2(\mathbb{R}^2)}||\nabla\tilde{\mathbf{v}}||^2_{L^2(\mathbb{R}^2)})\\&\quad+C\mathcal{K}(t)+C\mathcal{L}(t),\end{aligned}\tag{3.44}$$

where

$$\begin{aligned}\mathcal{L}(t)=&||\tilde{\mathbf{v}}_t||^2_{L^2(\mathbb{R}^2)}+||\tilde{\mathbf{v}}||^{\frac{3}{4}}_{L^2(\mathbb{R}^2)}||\nabla\tilde{\mathbf{v}}||^{\frac{3}{2}}_{L^2(\mathbb{R}^2)}||\nabla^2\tilde{\mathbf{v}}||^{\frac{7}{4}}_{L^2(\mathbb{R}^2)}\\&+||\varrho||^2_{L^2(\mathbb{R}^2)}+||\tilde{\mathbf{v}}||^2_{H^1(\mathbb{R}^2)}+||v_1||^2_{L^2(\mathbb{R}^2)},\end{aligned}\tag{3.45}$$

and $\mathcal{K}(t)\in L^1(0,T)$ is given by (3.35). From (3.10) and (3.20), we also find $\mathcal{L}(t)\in L^1(0,T)$.

Denote

$$\begin{aligned}&\mathcal{E}(t)=||\nabla\varrho||^2_{L^2(\mathbb{R}^2)}+||\nabla v_1||^2_{L^2(\mathbb{R}^2)}+||\sqrt{\rho}\tilde{\mathbf{v}}_t||^2_{L^2(\mathbb{R}^2)}+1,\\&\mathcal{F}(t)=C[1+||\tilde{\mathbf{v}}||^2_{H^2(\mathbb{R}^2)}+||\tilde{\mathbf{v}}||^2_{L^2(\mathbb{R}^2)}||\nabla\tilde{\mathbf{v}}||^2_{L^2(\mathbb{R}^2)}],\\&\mathcal{M}(t)=C\mathcal{K}(t)+C\mathcal{L}(t).\end{aligned}$$

Thus, from (3.10) and (3.20), we derive $\mathcal{F}(t),\mathcal{M}(t)\in L^1(0,T)$. Additionally, we find (3.44) can be rewritten as

$$\frac{d}{dt}\mathcal{E}(t)\le\mathcal{F}(t)\mathcal{E}^2(t)+\mathcal{M}(t).\tag{3.46}$$

Then, integrating (3.46) over (τ,t), we obtain

$$\mathcal{E}(t)\le[\mathcal{E}(\tau)+\int_\tau^T\mathcal{M}(s)ds]+\int_\tau^t\mathcal{F}(s)\mathcal{E}^2(s)ds.\tag{3.47}$$

Utilizing the classical Osgood lemma (See Lemma 2.3 in [7]), we derive

$$-\mathcal{N}(\mathcal{E}(t))+\mathcal{N}(c)\le\int_\tau^T\mathcal{F}(s)ds,\tag{3.48}$$

where

$$\mathcal{N}(x)=\int_x^1\frac{dr}{r^2},\ c(\tau)=\mathcal{E}(\tau)+\int_\tau^T\mathcal{M}(s)ds.\tag{3.49}$$

In addition, multiplying (1.7) by $\tilde{\mathbf{v}}_t$ and utilizing $\nabla\cdot\tilde{\mathbf{v}}_t=0$, we find that

$$||\sqrt{\rho}\tilde{\mathbf{v}}_t||^2_{L^2(R^2)} = ||\sqrt{\varrho+\rho_s}\tilde{\mathbf{v}}_t||^2_{L^2(\mathbb{R}^2)}$$
$$= \int_{\mathbb{R}^2}[-f\mathbf{e}_2 v_1 - (\varrho+\rho_s)\tilde{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}} - g\varrho\mathbf{e}_1 + \mu\Delta\tilde{\mathbf{v}} - \nabla q]\cdot\tilde{\mathbf{v}}_t dx \tag{3.50}$$

and then

$$||\sqrt{\varrho+\rho_s}\tilde{\mathbf{v}}_t||^2_{L^2(\mathbb{R}^2)}$$
$$\le C\int_{\mathbb{R}^2}[|f|^2|v_1|^2 + |(\varrho+\rho_s)\tilde{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}}|^2 + |\frac{\varrho}{\varrho+\rho_s}g\mathbf{e}_1|^2$$
$$+ (\varrho+\rho_s)^{-1}|(\mu\Delta\tilde{\mathbf{v}} - \nabla q)|^2]dx.$$

As a result, from the regularity result on the Stokes equations, we deduce that

$$\limsup_{\tau\to 0}||\sqrt{\rho}\tilde{\mathbf{v}}_t||^2_{L^2(\mathbb{R}^2)} \le C\sigma^2. \tag{3.51}$$

From (3.20)-(3.21), (3.35), (3.45), (3.48) and (3.51), we derive

$$\mathcal{E}(t) \le (\frac{1}{c(0)} - \int_0^T \mathcal{F}(s)ds)^{-1} \tag{3.52}$$
$$\le \left[\frac{1}{C\sigma^2 + Ce^{C\sigma^4 e^{CT}}\sigma^2 + 1} - Ce^{C\sigma^4 e^{CT}}\sigma^2\right]^{-1},\ 0 < T < T_1,$$

where

$$T_1 = \frac{1}{C}\ln\left\{\frac{\ln[\frac{-(C\sigma^2+1)+\sqrt{(C\sigma^2+1)^2+4}}{2C\sigma^2}]}{C\sigma^4}\right\}.$$

According to (3.52), we set

$$\left[\frac{1}{C\sigma^2 + Ce^{C\sigma^4 e^{CT}}\sigma^2 + 1} - Ce^{C\sigma^4 e^{CT}}\sigma^2\right]^{-1} = C_1 e^{C\sigma^4 e^{CT}}\sigma^2. \tag{3.53}$$

Namely,

$$\frac{1}{C\sigma^2 + Ce^{C\sigma^4 e^{CT}}\sigma^2 + 1} - Ce^{C\sigma^4 e^{CT}}\sigma^2 = [C_1 e^{C\sigma^4 e^{CT}}\sigma^2]^{-1}. \tag{3.54}$$

Setting $A := A(T) = e^{C\sigma^4 e^{CT}}\sigma^2$, we derive

$$C^2 C_1 A^3 + CC_1 A^2(C\sigma^2+1) + (C - C_1)A + C(C\sigma^2+1) = 0. \tag{3.55}$$

In order for (3.55) to have a solution in $(0, +\infty)$, we need the following condition

$$C^2C_1e^{3C\sigma^4}\sigma^6 + CC_1e^{2C\sigma^4}\sigma^4(C\sigma^2+1) + (C - C_1)e^{C\sigma^4}\sigma^2 + (C\sigma^2+1) < 0. \quad (3.56)$$

Accordingly we choose the constant C_1 to satisfy

$$C_1 > \frac{Ce^{C\sigma^4}\sigma^2 + C\sigma^2 + 1}{e^{C\sigma^4}\sigma^2 - C^2e^{3C\sigma^4}\sigma^6 - Ce^{2C\sigma^4}\sigma^4(C\sigma^2+1)}.$$

Choosing $T^* = \min\{T_1, T_2\}$, where T_2 is a solution of (3.55), then we have

$$\mathcal{E}(t) \le C_1(T)\sigma^2, \quad 0 < t < T^*, \quad (3.57)$$

where the constant $C(T)$ depends on T.

As a result, from (3.57), we derive

$$\sup_{t\in(0,T)} [||\nabla\varrho||^2_{L^2(\mathbb{R}^2)} + ||\nabla v_1||^2_{L^2(\mathbb{R}^2)} + ||\sqrt{\rho}\tilde{\mathbf{v}}_t||^2_{L^2(\mathbb{R}^2)}] \le C(T)\sigma^2, \quad (3.58)$$

and

$$\int_0^T ||\nabla\tilde{\mathbf{v}}_t||^2_{L^2(\mathbb{R}^2)}ds \le C(T)\sigma^2. \quad (3.59)$$

Combining (3.36) and (3.58), we obtain

$$\int_0^T ||v_{1t}||^2_{L^2(\mathbb{R}^2)} \le C(T)\sigma^2. \quad (3.60)$$

From (3.16) and (3.58), we have

$$\sup_{t\in(0,T)} [||\tilde{\mathbf{v}}||^2_{H^2(\mathbb{R}^2)} + ||\nabla q||^2_{L^2(\mathbb{R}^2)}] \le C(T)\sigma^2. \quad \square \quad (3.61)$$

Combining Lemma 3.1 and Lemma 3.2, we have the following result.

Theorem 3.1. *Let* $||\varrho_0||^2_{H^1(\mathbb{R}^2)} + ||v_{10}||^2_{H^1(\mathbb{R}^2)} + ||\tilde{\mathbf{v}}_0||^2_{H^2(\mathbb{R}^2)} \le \sigma^2$. *Then, for any* $0 < T < T^*$, *any classical solution* $(\varrho, \mathbf{v}, q)$ *to* (1.7)*, emanating from the initial data* $(\varrho_0, \mathbf{v}_0)$*, satisfies*

$$\sup_{0<t\le T} [||\varrho(t)||^2_{H^1(\mathbb{R}^2)} + ||v_1(t)||^2_{H^1(\mathbb{R}^2)} + ||\tilde{\mathbf{v}}||^2_{H^2(\mathbb{R}^2)} + ||\tilde{\mathbf{v}}_t||^2_{L^2(\mathbb{R}^2)} + ||\nabla q(t)||^2_{L^2(\mathbb{R}^2)}]$$
$$+ \int_0^T (||v_{1t}(s)||^2_{L^2(\mathbb{R}^2)} + ||\nabla\tilde{\mathbf{v}}_t(s)||^2_{L^2(\mathbb{R}^2)})ds \le C(T)\sigma^2, \quad (3.62)$$

where the constant $C(T)$ *depends on* T *and* T^* *is a solution of* (3.55).

4. Proof of Theorem 1.3

In this section, we aim to demonstrate the nonlinear instability utilizing the method outlined in Jiang's work [27]. Specifically, by contradiction argument, we can show that the nonlinear equations (1.7)-(1.9) possess a strong solution that satisfies the outcome specified in Theorem 1.3.

In view of Theorem 2.3, we obtain that there exists a classical solution $(\varrho^l, \mathbf{v}^l, q^l)$ to linearized system (1.10) satisfying (2.61) and $||v_3^l(0)||_{H^k(\mathbb{R}^2)} > 0$ as well as

$$||\varrho^l(0)||^2_{H^k(\mathbb{R}^2)} + ||\mathbf{v}^l(0)||^2_{H^k(\mathbb{R}^2)} = \sigma^2. \tag{4.1}$$

Next, we set

$$i_0 := i_0(k) = \frac{||v_3^l(0)||_{L^2(\mathbb{R}^2)}}{\sigma} \le 1. \tag{4.2}$$

Then $i_0 > 0$. As a result, defining $t_K = \frac{2}{\Lambda}\ln\frac{2K}{i_0}$, we have $0 < K < \frac{i_0}{2}e^{\frac{\Lambda T^*}{2}}$. Thus, we obtain $0 < t_K < T^*$. Moreover, from (2.61), we have

$$||v_3^l(t_K)||_{L^2(\mathbb{R}^2)} \ge e^{\frac{t_K\Lambda}{2}} i_0\sigma \ge 2K\sigma. \tag{4.3}$$

Denote $(\varrho_0^\epsilon, \mathbf{v}_0^\epsilon) := \epsilon(\varrho^l(0), \mathbf{v}^l(0)), \epsilon \in (0, 1)$. Then, we obtain

$$||\varrho_0^\epsilon||^2_{H^k(\mathbb{R}^2)} + ||\mathbf{v}_0^\epsilon||^2_{H^k(\mathbb{R}^2)} = \epsilon^2\sigma^2 < \sigma^2. \tag{4.4}$$

Hence, by virtue of Theorem 3.1 and Theorem A.2, the perturbed problem (1.7)-(1.9) admits a family of strong solutions $(\varrho^\epsilon, \mathbf{v}^\epsilon, q^\epsilon)$ such that

$$\begin{aligned}
&\sup_{0<t\le T}[||\varrho^\epsilon(t)||^2_{H^1(\mathbb{R}^2)} + ||v_1^\epsilon(t)||^2_{H^1(\mathbb{R}^2)} + ||\tilde{\mathbf{v}}^\epsilon||^2_{(H^2(\mathbb{R}^2))^2} + ||\tilde{\mathbf{v}}_t^\epsilon||^2_{(L^2(\mathbb{R}^2))^2} \\
&+ ||\nabla q^\epsilon(t)||^2_{L^2(\mathbb{R}^2)}] + \int_0^t (||v_{1t}^\epsilon(s)||^2_{L^2(\mathbb{R}^2)} + ||\nabla\tilde{\mathbf{v}}_t^\epsilon(s)||^2_{(L^2(\mathbb{R}^2))^2})ds \\
&\le C(T)\sigma^2\epsilon^2,
\end{aligned} \tag{4.5}$$

where the constant $C(T)$ does not depend on ϵ. Additionally, from (3.25), we have

$$\sup_{0<t\le T} ||\varrho^\epsilon||_{L^\infty(\mathbb{R}^2)} \le C(T)\sigma^2\epsilon^2, \tag{4.6}$$

which means

$$\sup_{0<t\le T} ||\varrho^\epsilon + \rho_s||_{L^\infty(\mathbb{R}^2)} \le C(\sigma), \tag{4.7}$$

where $C(\sigma)$ depends on σ. From (4.5), we have

$$\sup_{0<t\le T} ||q^\epsilon||^2_{H^1(\mathbb{R}^2)} \le C(T)\sigma^2\epsilon^2. \tag{4.8}$$

Lemma 4.1. *There exists an* $\epsilon \in (0, 1)$ *such that the strong solution* $(\varrho^\epsilon, \mathbf{v}^\epsilon, q^\epsilon)$ *emanating from the initial data* $(\varrho_0^\epsilon, \mathbf{v}_0^\epsilon)$*, satisfies*

$$||v_3^\epsilon(t_K)||_{L^2(\mathbb{R}^2)} > F(||\varrho_0^\epsilon, \mathbf{v}_0^\epsilon)||_{H^k(\mathbb{R}^2)}), \text{ for } t_K \in (0, \frac{2}{\Lambda}\ln\frac{2K}{i_0}) \subset (0, T^*).$$

Proof. Assume that for any $\epsilon \in (0, 1)$, the strong solution $(\varrho^\epsilon, \mathbf{v}^\epsilon, q^\epsilon)$ satisfies

$$||v_3^\epsilon(t_K)||_{L^2(\mathbb{R}^2)} \le F(||\varrho^\epsilon(0), \mathbf{v}^\epsilon(0))||_{H^k(\mathbb{R}^2)}) \le K\sigma\epsilon, \ \forall t \in (0, T^*). \tag{4.9}$$

We denote $(\bar\varrho^\epsilon, \bar{\mathbf{v}}^\epsilon, \bar q^\epsilon) = \frac{1}{\epsilon}(\varrho^\epsilon, \mathbf{v}^\epsilon, q^\epsilon)$, thus they satisfy

$$\begin{aligned}
&\frac{\partial\bar\varrho^\epsilon}{\partial t} + \tilde{\bar{\mathbf{v}}}^\epsilon\cdot\nabla(\epsilon\bar\varrho^\epsilon + \rho_s) = 0,\\
&(\epsilon\bar\varrho^\epsilon + \rho_s)\frac{\partial\bar v_1^\epsilon}{\partial t} + (\epsilon\bar\varrho^\epsilon + \rho_s)[\epsilon\tilde{\bar{\mathbf{v}}}_2^\epsilon\cdot\nabla\bar v_1^\epsilon + \bar u_0\bar v_2^\epsilon] - f\bar v_2^\epsilon = 0,\\
&(\epsilon\bar\varrho^\epsilon + \rho_s)\frac{\partial\tilde{\bar{\mathbf{v}}}^\epsilon}{\partial t} + (\epsilon\bar\varrho^\epsilon + \rho_s)\epsilon\tilde{\bar{\mathbf{v}}}^\epsilon\cdot\nabla\tilde{\bar{\mathbf{v}}}^\epsilon + f\bar v_1^\epsilon\mathbf{e}_1 = \mu\Delta\tilde{\bar{\mathbf{v}}}_2^\epsilon - \nabla\bar q^\epsilon - g\bar\varrho^\epsilon\mathbf{e}_2,\\
&\nabla\cdot\tilde{\bar{\mathbf{v}}}^\epsilon = 0,
\end{aligned} \tag{4.10}$$

with initial data

$$(\bar\varrho^\epsilon(0), \bar{\mathbf{v}}^\epsilon(0)) = (\varrho^l(0), \mathbf{v}^l(0)). \tag{4.11}$$

Then, with the help of (4.5)-(4.8) and (3.53), we obtain the following estimates

$$\begin{aligned}
&\sup_{0<t\le T}[||\bar\varrho^\epsilon(t)||^2_{H^1(\mathbb{R}^2)} + ||\bar v_1^\epsilon(t)||^2_{H^1(\mathbb{R}^2)} + ||\tilde{\bar{\mathbf{v}}}^\epsilon||^2_{(H^2(\mathbb{R}^2))^2} + ||\tilde{\bar{\mathbf{v}}}_t^\epsilon||^2_{(L^2(\mathbb{R}^2))^2}\\
&+ ||\nabla\bar q^\epsilon(t)||^2_{L^2(\mathbb{R}^2)}] + \int_0^t (||\bar v_{1t}^\epsilon(s)||^2_{L^2(\mathbb{R}^2)} + ||\nabla\tilde{\bar{\mathbf{v}}}_t^\epsilon(s)||^2_{(L^2(\mathbb{R}^2))^2})ds\\
&\le C(T)\sigma^2,
\end{aligned} \tag{4.12}$$

and

$$\sup_{0<t\le T}||(\epsilon\bar\varrho^\epsilon + \rho_s)||_{L^\infty(\mathbb{R}^2)} \le C(\sigma), \quad \sup_{0<t\le T}||\bar q^\epsilon||^2_{H^1(\mathbb{R}^2)} \le C(T)\sigma^2. \tag{4.13}$$

From $(4.10)_1$ and $(4.10)_2$, we infer that

$$\sup_{0<t\le T}||\bar\varrho_t^\epsilon||^2_{L^2(\mathbb{R}^2)} \le C(T)\sigma^2(1+\sigma^2), \quad \sup_{0<t\le T}||\bar v_{1t}^\epsilon||^2_{L^2(\mathbb{R}^2)} \le C(T)\sigma^2(1+\sigma^2). \tag{4.14}$$

Thus, we deduce that there exists a subsequence of $\{(\bar{\varrho}^\epsilon, \bar{\mathbf{v}}^\epsilon, \bar{q}^\epsilon)\}$ such that

$$
\begin{aligned}
&(\bar{\varrho}^\epsilon_t, \bar{\mathbf{v}}^\epsilon_t, \bar{q}^\epsilon) \to (\bar{\varrho}_t, \bar{\mathbf{v}}_t, \bar{q}) \text{ weakly-star in } L^\infty(0,T;(L^2(\mathbb{R}^2))^4 \times H^1(\mathbb{R}^2)),\\
&(\bar{\varrho}^\epsilon, \bar{\mathbf{v}}^\epsilon) \to (\bar{\varrho}, \bar{\mathbf{v}}) \text{ weakly-star in } L^\infty(0,T;(H^1(\mathbb{R}^2))^2 \times (H^2(\mathbb{R}^2))^2),\\
&(\bar{\varrho}^\epsilon, \bar{\mathbf{v}}^\epsilon) \to (\bar{\varrho}, \bar{\mathbf{v}}) \text{ strongly in } C^0(0,T;(L^2_{Loc}(\mathbb{R}^2))^4).
\end{aligned}
$$

Taking the limit as $\epsilon \to 0$ in (4.10), we find

$$
\begin{aligned}
&\frac{\partial \bar{\varrho}^\epsilon}{\partial t} + \tilde{\bar{\mathbf{v}}}^\epsilon \cdot \nabla \rho_s = 0,\\
&\rho_s \frac{\partial \bar{v}_1^\epsilon}{\partial t} + \rho_s \bar{u}_0 \bar{v}_2^\epsilon - f \bar{v}_2^\epsilon = 0,\\
&\rho_s \frac{\partial \tilde{\bar{\mathbf{v}}}^\epsilon}{\partial t} + f \bar{v}_1^\epsilon \mathbf{e}_1 = \mu \Delta \tilde{\bar{\mathbf{v}}}_2^\epsilon - \nabla \bar{q}^\epsilon - g \bar{\varrho}^\epsilon \mathbf{e}_2,\\
&\nabla \cdot \tilde{\bar{\mathbf{v}}}^\epsilon = 0.
\end{aligned}
\tag{4.15}
$$

Then we deduce that $(\bar{\varrho}, \bar{\mathbf{v}})$ is a strong solution of the linearized problem (1.10). We recall that $(\varrho^l, \mathbf{v}^l)$ is also a strong solution of the linearized problem (1.10) and the initial data satisfies $(\bar{\varrho}^\epsilon(0), \bar{\mathbf{v}}^\epsilon(0)) = (\varrho^l(0), \mathbf{v}^l(0))$. Consequently, according to Theorem 2.4, we obtain

$$
(\bar{\varrho}, \bar{\mathbf{v}}) = (\varrho^l, \mathbf{v}^l), \text{ on } [0,T] \times \mathbb{R}^2. \tag{4.16}
$$

Consequently, with the help of (4.3), we obtain

$$
2K\sigma \le ||\bar{v}_3(t_K)||_{L^2(\mathbb{R}^2)} \le ||v_3^l(t_K)||_{L^2(\mathbb{R}^2)} \le K\sigma, \tag{4.17}
$$

which is a contraction. □

From Lemma 4.1, we deduce that the conclusion of Theorem 1.3 is valid.

Data availability

No data was used for the research described in the article.

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Appendix A

In this section, we shall investigate the existence of strong solution of (1.7)-(1.9). We first consider the strong solution of (1.7)-(1.9) in a bounded region $B_0(R) = \{(y,z)||y|^2 + |z|^2 < R^2\}$. Namely, (1.7) subject to the boundary condition

$$\mathbf{v}(t,\mathbf{x}) = 0, \text{ for } t > 0, \ \mathbf{x} \in \partial B_0(R), \tag{A.1}$$

and the initial conditions are expressed by

$$\mathbf{v}(0,\mathbf{x}) = \mathbf{v}_0(\mathbf{x}), \ \varrho(0,\mathbf{x}) = \varrho_0(\mathbf{x}), \tag{A.2}$$

where $v_{i0}(i = 1, 2, 3)$ and $\varrho_0(\mathbf{x})$ are given functions.

We introduce the following function space

$$W = \{\phi \in (C_0^\infty(B_0(R)))^2 : \mathrm{div}\phi = 0\}.$$

Then V is defined to be the completion of W with respect to the norm of $(H^1(B_0(R)))^2$. From (1.6), (1.7) is equivalent to

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \tilde{\mathbf{v}} \cdot \nabla \rho = 0,\\
&\rho \frac{\partial v_1}{\partial t} + \rho \tilde{\mathbf{v}} \cdot \nabla v_1 = [f - \rho \bar{u}_0] v_2,\\
&\rho \frac{\partial \tilde{\mathbf{v}}}{\partial t} + (\rho \tilde{\mathbf{v}} \cdot \nabla)\tilde{\mathbf{v}} + f v_1 \mathbf{e}_1 = \mu \Delta \tilde{\mathbf{v}} - \nabla q - g(\rho - \rho_s)\mathbf{e}_2,\\
&\nabla \cdot \tilde{\mathbf{v}} = 0,
\end{aligned} \tag{A.3}$$

where $\mathbf{e}_1 = (1, 0)$, $\mathbf{e}_2 = (0, 1)$.

The system (A.3) subject to the same boundary condition (A.1), but the initial condition (A.2) becomes

$$\rho(0,\mathbf{x}) = \varrho_0(\mathbf{x}) - \rho_s(\mathbf{x}) := \rho_0. \tag{A.4}$$

Now we give the definition of weak solution to (A.1)-(A.4).

Definition A.1. A weak solution of (A.1)-(A.4) is a pair of functions $\mathbf{v}(t,\mathbf{x})$, $\varrho(t,\mathbf{x})$ such that $\mathbf{v}(t,\mathbf{x}) = (v_1(t,\mathbf{x}), v_2(t,\mathbf{x}), v_3(t,\mathbf{x})) \in L^2(0,T; B_0(R)) \times L^2(0,T; V)$, $\varrho(t,\mathbf{x}) \in L^\infty(0,T; B_0(R))$ and

$$-\int_0^T \int_{B_0(R)} \rho \frac{\partial \psi}{\partial t} d\mathbf{x} dt - \int_0^T \int_{B_0(R)} \rho \tilde{v} \cdot \nabla \psi d\mathbf{x} \int_{B_0(R)} \rho_0 \psi(0,\mathbf{x}) d\mathbf{x}, \tag{A.5}$$

and

$$
-\int_0^T\int_{B_0(R)} \rho v_1 \frac{\partial \varphi_1}{\partial t} d\mathbf{x}dt - \int_0^T\int_{B_0(R)} \rho v_1 \tilde{\mathbf{v}}\cdot\nabla\varphi_1 d\mathbf{x}dt
$$

$$
+\int_0^T\int_{B_0(R)} [\bar{u}_0\rho - f]v_2\varphi_1 d\mathbf{x}dt = \int_{B_0(R)} \rho_0 v_1(\mathbf{x})\varphi_1(\mathbf{x},0)d\mathbf{x}, \tag{A.6}
$$

as well as

$$
-\int_0^T\int_{B_0(R)} \rho\tilde{\mathbf{v}}\cdot\frac{\partial \varphi}{\partial t} d\mathbf{x}dt - \sum_{i=2}^{3}\int_0^T\int_{B_0(R)} \rho v_i \tilde{\mathbf{v}}\cdot\nabla\varphi_i d\mathbf{x}dt
$$

$$
+\int_0^T\int_{B_0(R)} f v_1\varphi_2 d\mathbf{x}dt + \mu\int_0^T\int_{B_0(R)} \nabla\tilde{\mathbf{v}}\cdot\nabla\varphi d\mathbf{x}dt \tag{A.7}
$$

$$
+\int_0^T\int_{B_0(R)} g(\rho-\rho_s)\varphi_3 d\mathbf{x}dt = \int_{B_0(R)} \rho_0(\mathbf{x})\tilde{\mathbf{v}}_0(\mathbf{x})\varphi_2(\mathbf{x},0)d\mathbf{x},
$$

hold for all $\psi \in C^1(0,T;H^1(B_R))$, $\varphi_1 \in C^1(0,T;H^1(B_R))$ and $\varphi = (\varphi_2,\varphi_3) \in C^1(0,T;V)$ satisfying $(\varphi_2(T,\mathbf{x}),\varphi_3(T,\mathbf{x})) = (0,0)$, a.e. in $B_0(R)$.

We first consider the following initial value problem

$$
\begin{cases} \frac{\partial\rho}{\partial t} + \tilde{\mathbf{v}}\cdot\nabla\rho = 0, \\ \rho(0,\mathbf{x}) = \rho_0. \end{cases} \tag{A.8}
$$

Then, from Lemma 2.2 and Lemma 2.3 in [29], the following two conclusions hold true.

Lemma A.1. *Assume* $\tilde{\mathbf{v}} \in C^1([0,T];C^1(B_0(R)))^2$, $\nabla\cdot\tilde{\mathbf{v}} = 0$ *for all* $(\mathbf{x},t) \in \overline{B_0(R)}\times[0,T]$, *and* $\tilde{\mathbf{v}} = 0$ *for all* $(\mathbf{x},t) \in \partial\overline{B_0(R)}\times[0,T]$, *and* $\varrho_0 \in C^1(\overline{B_0(R)})$, $\kappa_1 \le \varrho_0 \le \kappa_2$ *for all* $\mathbf{x} \in \overline{B_0(R)}$, *where* $\kappa_1,\kappa_2 > 0$. *Then* (A.8) *has a unique solution* $\rho(t,x) \in C^1([0,T]\times\overline{B_0(R)})$, *and* $\kappa_1 + \inf\rho_s \le \rho(t,\mathbf{x}) \le \kappa_2 + \sup\rho_s$ *holds for all* $\mathbf{x} \in \overline{B_0(R)}$.

Lemma A.2. *For each* $n = 1,2,...,$ *assume* $\tilde{\mathbf{v}}_n(\mathbf{x},t) \in C([0,T];C^1(\overline{B_0(R)}))$, $\nabla\cdot\tilde{\mathbf{v}}_n = 0$ *for all* $(\mathbf{x},t) \in \overline{B_0(R)}\times[0,T]$ *and* $\tilde{\mathbf{v}}_n = 0$ *for all* $(\mathbf{x},t) \in \partial B_0(R)\times[0,T]$. *Suppose that* $\tilde{\mathbf{v}}_n \to \tilde{\mathbf{v}}$ *in* $C([0,T];C^1(\overline{B_0(R)}))$, *and denote by* $\rho_n(\mathbf{x},t)$, $\rho(\mathbf{x},t)$ *the unique solution of*

$$
\begin{cases} \frac{\partial\rho_n}{\partial t} + \tilde{\mathbf{v}}_n\cdot\nabla\rho_n = 0, \\ \rho_n(0,\mathbf{x}) = \rho_0(\mathbf{x}), \end{cases} \tag{A.9}
$$

and the unique solution of (A.8), *respectively. Then* $\rho_n(\mathbf{x},t) \to \rho(\mathbf{x},t)$ *in* $C([0,T]\times(\overline{B_0(R)}))$.

Next, we consider the following initial value problem

$$\frac{\partial v_1}{\partial t} + \tilde{\mathbf{v}} \cdot \nabla v_1 = h,$$
$$v_1(0, \mathbf{x}) = v_{10}(\mathbf{x}), \tag{A.10}$$

where

$$h = \frac{[f - \bar{u}_0 \rho] v_2}{\rho}. \tag{A.11}$$

Similar to the proof process of Lemma 2.2 and Lemma 2.3 in [29], we obtain the following conclusion.

Lemma A.3. *Suppose* $\tilde{\mathbf{v}} \in C([0,T]; C^1(B_0(R)))^2$, $\nabla \cdot \tilde{\mathbf{v}} = 0$ *for all* $(t, \mathbf{x}) \in [0,T] \times B_0(R)$, *and* $\tilde{\mathbf{v}} = 0$ *for all* $(t, \mathbf{x}) \in [0,T] \times \partial B_0(R)$. *Let* $\rho(t, \mathbf{x}) \in C^1([0,T]; \overline{B_0(R)})$ *and* $\kappa_1 + \inf \rho_s \le \rho \le \kappa_2 + \sup \rho_s$. *Then* (A.10)-(A.11) *possesses a solution* $v_1 \in C^1([0,T]; \overline{B_0(R)})$.

Proof. We use the classical method of characteristics to construct a solution. Let E be an open ball in R^2 such that $\overline{B_0(R)} \subset E$. We extend $\tilde{\mathbf{v}}$ to $\tilde{\mathbf{w}} \in C([0,T]; C^1(E)^2)$ so that $\tilde{\mathbf{v}} = \tilde{\mathbf{w}}$ for all $(\mathbf{x}, t) \in \overline{B_0(R)} \times [0,T]$. Consider the system

$$\begin{cases} \frac{d\mathbf{X}}{dt} = \tilde{\mathbf{w}}(\mathbf{X}(t,a), t), \\ \mathbf{X}(0) = a. \end{cases} \tag{A.12}$$

Then, from the proof of Lemma 2.2 in [29], the solution of (A.10) is given by

$$v_1(t, \mathbf{x}) = v_{10}(A(t, \mathbf{x})) + \int_0^t h(\tau, A(t, \mathbf{x})) d\tau, \tag{A.13}$$

where $A = X^{-1}$. □

Using the proof method of Lemma 2.2, we can get the following conclusion.

Lemma A.4. *For each* $n = 1, 2, ...,$ *assume* $\tilde{\mathbf{v}}_n(\mathbf{x}, t) \in C([0,T]; C^1(\overline{B_0(R)}))$, $\nabla \cdot \tilde{\mathbf{v}}_n = 0$ *for all* $(\mathbf{x}, t) \in \overline{B_0(R)} \times [0,T]$ *and* $\tilde{\mathbf{v}}_n = 0$ *for all* $(\mathbf{x}, t) \in \partial B_0(R) \times [0,T]$. *Suppose that* $\tilde{\mathbf{v}}_n \to \tilde{\mathbf{v}}$ *in* $C([0,T]; C^1(\overline{B_0(R)}))$, *and denote by* $v_{1n}(\mathbf{x}, t)$, $v_1(\mathbf{x}, t)$ *the unique solution of*

$$\begin{cases} \frac{\partial v_{1n}}{\partial t} + \tilde{\mathbf{v}}_n \cdot \nabla v_{1n} = \frac{[f - \bar{u}_0 \rho_n] v_{2n}}{\rho_n} := h_n, \\ v_{1n}(0, \mathbf{x}) = v_{10}(\mathbf{x}), \end{cases} \tag{A.14}$$

and the unique solution of (A.10), *respectively. In addition, assume that* ρ_n, ρ *are the solution of* (A.9) *and* (A.8) *with* $\rho_n \to \rho$ *in* $C([0,T]; \overline{B_0(R)})$. *Then* $v_{1n}(\mathbf{x}, t) \to v_1(\mathbf{x}, t)$ *in* $C([0,T] \times (\overline{B_0(R)}))$.

Proof. By the proof of Lemma 2.3 in [29], we obtain that $A_n(t,\mathbf{x}) \to A(t,\mathbf{x})$ uniformly in $[0,T]\times\overline{B_0(R)}$. In addition, we find $\rho_n \to \rho$ uniformly in $[0,T]\times\overline{B_0(R)}$ and $\tilde{\mathbf{v}}_n \to \tilde{\mathbf{v}}$ uniformly in $[0,T]\times\overline{B_0(R)}$, since $\rho_n \to \rho$ in $C([0,T],C(\overline{B_0(R)}))$ and $\tilde{\mathbf{v}}_n \to \tilde{\mathbf{v}}$ in $C([0,T],C^1(\overline{B_0(R)}))$. Thus, we deduce that $h_n \to h := \frac{[f-\bar{u}_0\rho]v_2}{\rho}$ uniformly in $[0,T]\times\overline{B_0(R)}$. Consequently, we find $v_{1n}(t,\mathbf{x}) \to v_1(t,\mathbf{x})$ uniformly in $[0,T]\times\overline{B_0(R)}$. $\square$

Next, we choose sequences of functions $\{\varrho_{0m}(\mathbf{x})\}_{m=1}^{\infty}$ such that $\varrho_{0m}(\mathbf{x}) \in C^1(B_0(R))$, $\kappa_1 + \frac{1}{m} \le \rho_{0m}(\mathbf{x}) \le \kappa_2 + \frac{1}{m}$ for all $\mathbf{x}\in B_0(R)$, $\rho_{0m}(\mathbf{x}) \to \rho_0(\mathbf{x})$ in $L^2(B_0(R))$. We set

$$\tilde{\mathbf{v}}_m(t,\mathbf{x}) = \sum_{k=1}^{m} A_{mk}\phi_k(\mathbf{x}), \tag{A.15}$$

and consider the following equations

$$\begin{aligned}
&\frac{\partial\rho_m}{\partial t} + \tilde{\mathbf{v}}_m\cdot\nabla\rho_m = 0,\\
&\rho_m\frac{\partial v_{1m}}{\partial t} + \rho_m\tilde{\mathbf{v}}_m\cdot\nabla v_{1m} = [f-\bar{u}_0\rho_m]v_{2m},\\
&\sum_{k=1}^{m} b_{jk}^m(t)\frac{dA_{mk}}{dt} + \sum_{k,l=1}^{m} C_{jkl}^m(t)A_{mk}(t)A_{ml}(t) + fc_j^m(t)\\
&= \mu\lambda_j A_{mj}(t) - d_j^m(t),\ \ j=1,2,\dots,m,
\end{aligned} \tag{A.16}$$

where

$$\begin{aligned}
&b_{jk}^m(t) = \int_{B_0(R)} \rho_m\phi_k\cdot\phi_j d\mathbf{x}d\mathbf{x},\ C_{jkl}^m(t) = \int_{B_0(R)} (\rho_m\phi_k\cdot\nabla)\phi_l\cdot\phi_j d\mathbf{x},\\
&c_j^m(t) = \int_{B_0(R)} f v_{1m}e_1\cdot\phi_j d\mathbf{x},\ d_j^m(t) = \int_{B_0(R)} g(\rho_m-\rho_s)e_2\cdot\phi_j d\mathbf{x}.
\end{aligned}$$

(A.16) subjects to the following initial conditions

$$\begin{aligned}
&\rho_m(0,\mathbf{x}) = \rho_{0m},\\
&v_{1m}(0,\mathbf{x}) = v_{10m}(\mathbf{x}),\\
&A_{mk}(0) = \int_{B_0(R)} \tilde{\mathbf{v}}_0(\mathbf{x})\cdot\phi_k(\mathbf{x})d\mathbf{x},\ \ k=1,2,...,m.
\end{aligned} \tag{A.17}$$

Following the proof of Lemma 2.5 in [29], we arrive at the result.

Lemma A.5. *Assume $\rho_m(t,\mathbf{x}) \in C^1([0,T]\times\overline{B_0(R)})$ and $\kappa_1 + \inf_{z\in B_0(R)}\rho_s \le \rho_m \le \kappa_2 + \sup_{z\in B_0(R)}\rho_s$, for any $(t,\mathbf{x})\in[0,T]\times\overline{B_0(R)}$. Then the matrix $\{b_{jk}^m(t)\}$ is nonsingular and each component of its inverse belongs to $C^1[0,T]$.*

By Theorem 3.1, we obtain

Lemma A.6. *Let $||\varrho_0||^2_{H^1(\mathbb{R}^2)} + ||v_{10}||^2_{H^1(\mathbb{R}^2)} + ||\tilde{\mathbf{v}}_0||^2_{H^2(\mathbb{R}^2)} \le \sigma^2$. Then, for any $T > 0$ independent of m, there exist solutions $\rho_m \in C^1([0,T] \times \overline{B_0(R)})$, $A_{mk}(t) \in C^1[0,T]$, $k = 1, 2, ..., m$ of (A.16)-(A.17) such that*

$$
\begin{aligned}
&\left|\left|\sqrt{\rho_m}\frac{\partial \tilde{\mathbf{v}}_m}{\partial t}\right|\right|_{L^2(0,T;H^1(B_0(R)))} \le C(T)\sigma^2, \ ||\tilde{\mathbf{v}}_m||_{L^2(0,T;H^2(B_0(R)))} \le C(T)\sigma^2,\\
&||v_{1m}||_{L^\infty(0,T;H^1(B_0(R)))} \le C(T)\sigma^2, \left|\left|\frac{\partial v_{1m}}{\partial t}\right|\right|_{L^\infty(0,T;L^2(B_0(R)))} \le C(T)\sigma^2,\\
&\left|\left|\frac{\partial \rho_m}{\partial t}\right|\right|_{L^\infty(0,T;L^2(B_0(R)))} \le C(T)\sigma^2, ||\rho_m||_{L^\infty(0,T;H^1(B_0(R)))} \le C(T)\sigma^2, \qquad \text{(A.18)}\\
&\kappa_1 + \inf_{z\in B_0(R)} \rho_s \le \rho_m \le \kappa_2 + \sup_{z\in B_0(R)} \rho_s.
\end{aligned}
$$

Proof. Let $\overline{B_r}$ be a closed ball in $C([0,T])^m$ with large enough radius $r \ge \sigma\sqrt{\frac{C}{\lambda_1}}$. Suppose $(A_{m1}(t), A_{m2}(t), ..., A_{mm}(t)) \in \overline{B_r}$, where λ_1 is the first eigenvalue of $-P\Delta$ and C is a fixed constant. In addition, we set $\tilde{\mathbf{v}}_m(t,\mathbf{x}) = \sum_{k=1}^m A_{km}(t)\phi_k(\mathbf{x})$. According to Lemma A.1, we find a solution ρ_m of $(\text{A.16})_1$ in $C^1(0,T;\overline{B_0(R)})$. Furthermore, by Lemma A.3, $(\text{A.16})_2$ has a solution $v_{1m} \in C^1(0,T;\overline{B_0(R)})$. Thus, with the help of ρ_m and v_{1m}, we obtain a solution $(\tilde{A}_{1m}, \tilde{A}_{2m}, ..., \tilde{A}_{km})$ of $(\text{A.16})_3$ in $C^1(0,T)^m \cap \overline{B_r}$. From Lemma A.2 and Lemma A.4 as well as Lemma A.5, we deduce that the mapping $(A_{m1}(t), A_{m2}(t), ..., A_{mm}(t)) \longrightarrow (\tilde{A}_{m1}(t), \tilde{A}_{m2}(t), ..., \tilde{A}_{mm}(t))$ is completely continuous from $\overline{B_r}$ to itself. Consequently, the mapping has a fixed point, which together with ρ_m and v_{1m} are solutions of $(\text{A.16})_1$ and $(\text{A.16})_2$, respectively. Finally, (A.18) is valid from the proof of Theorem 3.1. □

Theorem A.1. *Assume that $\varrho_0, v_{10} \in H^1(B_0(R))$ and $\tilde{\mathbf{v}}_0 \in H^2(B_0(R))$. Then, there exists a local strong solution $\rho(t,\mathbf{x})$, $v_1(t,\mathbf{x})$, $\tilde{\mathbf{v}}(t,\mathbf{x})$ of (A.3) such that*

$$
\begin{aligned}
&\varrho(t) \in L^\infty(0,T;H^1(B_0(R))),\ v_1(t) \in L^\infty(0,T;H^1(B_0(R))),\ v_{1t} \in L^2((0,T)\times B_0(R)),\\
&\tilde{\mathbf{v}} \in L^\infty(0,T;H^2(B_0(R))),\ \tilde{\mathbf{v}}_t \in L^2(0,T;H^1(B_0(R))) \cap L^\infty(0,T;L^2(B_0(R))),
\end{aligned}
$$

where $0 < T < T^*$ and T^* is the maximal time of existence of the solution.

Proof. From (A.18), we can extract a subsequence $\{\tilde{\mathbf{v}}_m\}$ and $\{\rho_m\}$ as well as $\{v_{1m}\}$ such that $\tilde{\mathbf{v}}_m \to \tilde{\mathbf{v}}$ weak in $L^2(0,T,H^2(B_0(R)))$, $\frac{\partial \tilde{\mathbf{v}}_m}{\partial t} \to \frac{\partial \tilde{\mathbf{v}}}{\partial t}$ in $L^2(0,T;H^1(B_0(R)))$, $\rho_m \to \rho$ weak star in $L^\infty(0,T;H^1(B_0(R)))$, and $\frac{\partial \rho_m}{\partial t} \to \frac{\partial \rho}{\partial t}$ weak star in $L^\infty(0,T;L^2(B_0(R)))$, $v_{1m} \to v_1$ weak star in $L^\infty(0,T;H^1(B_0(R)))$, $\frac{\partial v_{1m}}{\partial t} \to \frac{\partial v_1}{\partial t}$ weakly star in $L^\infty(0,T;L^2(B_0(R)))$. With the help of Aubin Theorem [12], we deduce that $\rho_m \to \rho$ strongly in $L^2(0,T;L^4(B_0(R)))$. Similarly, we also find $v_{1m} \to v_1$ strongly in $L^2(0,T;L^4(B_0(R)))$. Thus, $\rho_m v_{1m} \to \rho v_1$ weakly in $L^2(0,T;L^2(B_0(R)))$ as well as $\rho_m\tilde{\mathbf{v}}_m \to \rho\tilde{\mathbf{v}}$ in $\mathcal{D}'(0,T;B_0(R))^2$ and then, $\rho_m\tilde{\mathbf{v}}_m \to \rho\tilde{\mathbf{v}}$ weak star in $L^\infty(0,T;L^2(B_0(R)))$. Furthermore, we deduce that $\{\tilde{\mathbf{v}}_m\frac{\partial \rho_m}{\partial t}\}$ is bounded in $L^2(0,T;L^2(B_0(R)))$. Additionally, from (A.18), we find that $\{\rho_m\frac{\partial \tilde{\mathbf{v}}_m}{\partial t}\}$ is bounded in

$L^2(0,T;L^2(B_0(R)))$. As a result, $\{\frac{\partial(\rho_m\tilde{\mathbf{v}}_m)}{\partial t}\}$ is bounded in $L^2(0,T;L^2(B_0(R)))$. Namely, $\{\frac{\partial(\rho_m\tilde{\mathbf{v}}_m)}{\partial t}\}$ is bounded in $L^2(0,T;H^{-1}(B_0(R)))$. Then, $\rho_m\tilde{\mathbf{v}}_m \to \rho\tilde{\mathbf{v}}$ strongly in $L^2(0,T;H^{-\frac{1}{2}}(B_0(R)))$. Since $\tilde{\mathbf{v}}_m \to \tilde{\mathbf{v}}$ weakly in $L^2(0,T;H^2(B_0(R)))$, $\rho_m\tilde{\mathbf{v}}_m v_{mk} \to \rho\tilde{\mathbf{v}}v_k$ in $\mathcal{D}'((0,T)\times B_0(R))$, where $k=1,2,3$ and $\tilde{\mathbf{v}}=(v_2,v_3)$. Then, $\rho_m\tilde{\mathbf{v}}_m v_{mk} \to \rho\tilde{\mathbf{v}}v_k$ weakly in $L^2(0,T;L^2(B_0(R)))^2$ for $k=1,2,3$. Thus, we derive that $\tilde{\mathbf{v}}_m\cdot\nabla\rho_m \to \tilde{\mathbf{v}}\cdot\nabla\rho$ weakly in $L^2(0,T;H^{-1}(B_0(R)))$. Choosing arbitrary $\xi_j(t)\in C^1([0,T])$ satisfied $\xi_j(T)=0$, $j=1,2,...m$, with the help of (A.16), it holds that $(\rho,v_1,\tilde{\mathbf{v}})$ is a weak solution of (A.1)-(A.4).

Moreover, using the method in the proof of Theorem 3.1, we obtain the following estimates

$$\sup_{0<t\le T}[||\varrho(t)||^2_{H^1(B_0(R))}+||v_1(t)||^2_{H^1(B_0(R))}+||\tilde{\mathbf{v}}||^2_{H^2(B_0(R))}+||\tilde{\mathbf{v}}_t||^2_{L^2(B_0(R))}+||\nabla q(t)||^2_{L^2(B_0(R))}]$$
$$+\int_0^T(||v_{1t}(s)||^2_{L^2(B_0(R))}+||\nabla\tilde{\mathbf{v}}_t(s)||^2_{L^2(B_0(R))})ds\le C, \tag{A.19}$$

where the constant C independents on R. From (A.19), the assertions stated in Theorem A.1 hold true. □

Finally, we use the expanding domain method to explore existence of the strong solution of (1.7)-(1.9).

Theorem A.2. *Assume that* $||\varrho_0||^2_{H^1(\mathbb{R}^2)}+||v_{10}||^2_{H^1(\mathbb{R}^2)}+||\tilde{\mathbf{v}}_0||^2_{H^2(\mathbb{R}^2)}\le\sigma^2$. *Then there exists a strong solution* $(\rho,v_1,\tilde{\mathbf{v}})$ *to* (A.3) *such that*

$$\rho\in L^\infty(0,T;H^1(\mathbb{R}^2)),\, v_1\in L^\infty(0,T;H^1(\mathbb{R}^2)),\;\; v_{1t}\in L^2(\mathbb{R}^2\times(0,T)), \tag{A.20}$$
$$\tilde{\mathbf{v}}\in L^\infty(0,T;H^2(\mathbb{R}^2)),\;\tilde{\mathbf{v}}_t\in L^\infty(0,T;L^2(\mathbb{R}^2)),\;\nabla\tilde{\mathbf{v}}_t\in L^2(\mathbb{R}^2\times(0,T)), \textit{where } 0<T<T^*.$$

Proof. Since $\rho_0\in H^1(\mathbb{R}^2)$, $v_{10}\in H^1(\mathbb{R}^2)$, we can choose $\rho_0^R, v_{10}^R\in C_0^\infty(B_0(R))$ such that

$$\rho_0^R\to\rho_0,\;\; v_{10}^R\to v_{10} \text{ in } \; H^1(\mathbb{R}^2),\text{ as } R\to\infty. \tag{A.21}$$

In addition, since $\tilde{\mathbf{v}}_0\in H^2(\mathbb{R}^2)$, we select $\tilde{\mathbf{u}}_i^R\in C_0^\infty(B_0(R))(i=1,2)$ such that

$$\lim_{R\to\infty}||\partial_i\tilde{\mathbf{u}}_i^R-\Delta\tilde{\mathbf{v}}_0||_{L^2(\mathbb{R}^2)}=0. \tag{A.22}$$

We consider the following Stokes problem

$$\begin{aligned} &-\Delta\tilde{\mathbf{v}}_0^R+\tilde{\mathbf{v}}_0^R+\nabla p_0^R=\mathbf{h}^R-\partial_i\tilde{\mathbf{u}}_i^R,\\ &\mathrm{div}\tilde{\mathbf{v}}_0^R=0,\\ &\tilde{\mathbf{v}}_0^R=0,\;\text{on }\;\partial B_0(R),\end{aligned} \tag{A.23}$$

where $\mathbf{h}^R=\tilde{\mathbf{v}}_0 * j_{1/R}$ with j_ϵ being the standard mollifying kernel of width ϵ. Obviously, (A.23) posses a unique solution $\tilde{\mathbf{v}}_0^R$. Then we extend $\tilde{\mathbf{v}}_0^R$ to $\mathbb{R}^2$ by defining 0 outside $B_0(R)$, we shall show

$$\tilde{\mathbf{v}}_0^R \to \tilde{\mathbf{v}}_0, \text{ in } H^2(\mathbb{R}^2), \quad \text{as } R \to \infty. \tag{A.24}$$

Multiplying (A.23) by $\tilde{\mathbf{v}}_0^R$ and integrating the result over $\mathbb{R}^2$ lead to

$$\begin{aligned}
&\int_{\mathbb{R}^2} |\nabla \tilde{\mathbf{v}}_0^R|^2 d\mathbf{x} + \int_{\mathbb{R}^2} |\tilde{\mathbf{v}}_0^R|^2 d\mathbf{x} \\
&\le ||\mathbf{h}^R||_{L^2(\mathbb{R}^2)} ||\tilde{\mathbf{v}}_0^R||_{L^2(\mathbb{R}^2)} + ||\tilde{\mathbf{v}}_0^R||_{L^2(\mathbb{R}^2)} ||\partial_i \tilde{\mathbf{u}}_i^R||_{L^2(\mathbb{R}^2)} \\
&\le \frac{1}{2} ||\tilde{\mathbf{v}}_0^R||^2_{L^2(\mathbb{R}^2)} + C[||\mathbf{h}^R||^2_{L^2(\mathbb{R}^2)} + ||\partial_i \tilde{\mathbf{u}}_i^R||^2_{L^2(\mathbb{R}^2)}],
\end{aligned} \tag{A.25}$$

which yields to

$$||\nabla \tilde{\mathbf{v}}_0^R||^2_{L^2(\mathbb{R}^2)} + ||\tilde{\mathbf{v}}_0^R||^2_{L^2(\mathbb{R}^2)} \le C, \tag{A.26}$$

for some C independent of R. Moreover, according to the regular theory of Stokes equations, we find

$$||\Delta \tilde{\mathbf{v}}_0^R||^2_{L^2(\mathbb{R}^2)} \le C[||\tilde{\mathbf{v}}_0^R||^2_{L^2(\mathbb{R}^2)} + ||\mathbf{h}^R||_{L^2(\mathbb{R}^2)} + ||\partial_i \tilde{\mathbf{u}}_i^R||^2_{L^2(\mathbb{R}^2)}]. \tag{A.27}$$

Combining (A.26) and (A.27), we deduce that

$$||\tilde{\mathbf{v}}_0^R||_{H^2(\mathbb{R}^2)} \le C,$$

which yields that there exists a subsequence R_j such that

$$\begin{aligned}
&\tilde{\mathbf{v}}_0^{R_j} \to \bar{\tilde{\mathbf{v}}}_0 \text{ weakly in } L^2(\mathbb{R}^2), \\
&\nabla \tilde{\mathbf{v}}_0^{R_j} \to \nabla \bar{\tilde{\mathbf{v}}}_0 \text{ weakly in } L^2(\mathbb{R}^2), \\
&\Delta \tilde{\mathbf{v}}_0^{R_j} \to \Delta \bar{\tilde{\mathbf{v}}}_0 \text{ weakly in } L^2(\mathbb{R}^2).
\end{aligned} \tag{A.28}$$

Now, we will show $\bar{\tilde{\mathbf{v}}}_0 = \tilde{\mathbf{v}}_0$. Indeed, multiplying (A.23) by $\pi \in C_0^\infty(\mathbb{R}^2)$ with $\nabla \cdot \pi = 0$, we obtain

$$\int_{\mathbb{R}^2} (-\Delta \tilde{\mathbf{v}}_0^{R_j} + \partial_i \tilde{\mathbf{u}}_i^{R_j}) \cdot \pi \, d\mathbf{x} + \int_{\mathbb{R}^2} (\tilde{\mathbf{v}}_0^{R_j} - \mathbf{h}^{R_j}) \cdot \pi \, d\mathbf{x} = 0.$$

Let $R_j \to \infty$ and it follows from (A.22) as well as (A.28) that

$$\int_{\mathbb{R}^2} (-\Delta \bar{\tilde{\mathbf{v}}}_0 + \Delta \tilde{\mathbf{v}}_0) \cdot \pi \, d\mathbf{x} + \int_{\mathbb{R}^2} (\bar{\tilde{\mathbf{v}}}_0 - \tilde{\mathbf{v}}_0) \cdot \pi \, d\mathbf{x} = 0. \tag{A.29}$$

Thus, with the help of (A.29), we obtain $\bar{\tilde{\mathbf{v}}}_0 = \tilde{\mathbf{v}}_0$.

Furthermore, multiplying (A.23) by $\tilde{\mathbf{v}}_0^{R_j}$ and $-\Delta\tilde{\mathbf{v}}_0^{R_j}$ respectively, and integrating the results, we derive that

$$\lim_{R_j\to\infty}\Big[\int_{\mathbb{R}^2}|\nabla\tilde{\mathbf{v}}_0^{R_j}|^2d\mathbf{x}+\int_{\mathbb{R}^2}|\tilde{\mathbf{v}}_0^{R_j}|^2d\mathbf{x}\Big]=\int_{\mathbb{R}^2}|\nabla\tilde{\mathbf{v}}_0|^2d\mathbf{x}+\int_{\mathbb{R}^2}|\tilde{\mathbf{v}}_0|^2d\mathbf{x}, \tag{A.30}$$

and

$$\lim_{R_j\to\infty}\Big[\int_{\mathbb{R}^2}|\Delta\tilde{\mathbf{v}}_0^{R_j}|^2d\mathbf{x}+\int_{\mathbb{R}^2}|\nabla\tilde{\mathbf{v}}_0^{R_j}|^2d\mathbf{x}\Big]=\int_{\mathbb{R}^2}|\Delta\tilde{\mathbf{v}}_0|^2d\mathbf{x}+\int_{\mathbb{R}^2}|\nabla\tilde{\mathbf{v}}_0|^2d\mathbf{x}. \tag{A.31}$$

Thus, utilizing (A.22) and (A.28) we have

$$\begin{aligned}
&\lim_{R_j\to\infty}\int_{\mathbb{R}^2}|\tilde{\mathbf{v}}_0^{R_j}|^2d\mathbf{x}=\int_{\mathbb{R}^2}|\tilde{\mathbf{v}}_0|^2d\mathbf{x},\ \lim_{R_j\to\infty}\int_{\mathbb{R}^2}|\nabla\tilde{\mathbf{v}}_0^{R_j}|^2d\mathbf{x}=\int_{\mathbb{R}^2}|\nabla\tilde{\mathbf{v}}_0|^2d\mathbf{x},\\
&\lim_{R_j\to\infty}\int_{\mathbb{R}^2}|\Delta\tilde{\mathbf{v}}_0^{R_j}|^2d\mathbf{x}=\int_{\mathbb{R}^2}|\Delta\tilde{\mathbf{v}}_0|^2d\mathbf{x}.
\end{aligned} \tag{A.32}$$

This leads to (A.24).

Let $(\rho^R, v_1^R, \tilde{\mathbf{v}}^R)$ be a strong solution of (A.3) with the initial data $(\rho_0^R, v_{10}^R, \tilde{\mathbf{v}}_0^R)$. Extending $(\rho^R, v_1^R, \tilde{\mathbf{v}}^R)$ to $\mathbb{R}^2$ by defining 0 outside $B_0(R)$, thus, from Theorem 3.1, we find that $(\rho^R, v_1^R, \tilde{\mathbf{v}}^R)$ satisfies the estimate (3.62) with T^* and C being independent of R. As a result, there exists a subsequence R_j, $R_j\to\infty$, such that $(\rho^{R_j}, v_1^{R_j}, \tilde{\mathbf{v}}^{R_j})$ converges to a limit $(\rho, v_1, \tilde{\mathbf{v}})$ in weak sense. Namely, as $R_j\to\infty$, we have

$$\begin{aligned}
&\rho^{R_j}\to\rho,\ v_1^{R_j}\to v_1, \text{weakly} * \text{ in } L^\infty(0,T;H^1(\mathbb{R}^2)),\\
&\tilde{\mathbf{v}}^{R_j}\to\tilde{\mathbf{v}} \text{ weakly} * \text{ in } L^\infty(0,T;H^2(\mathbb{R}^2)),\\
&\tilde{\mathbf{v}}_t^{R_j}\to\tilde{\mathbf{v}}_t \text{ weakly} * \text{ in } L^\infty(0,T;L^2(\mathbb{R}^2)),\\
&v_{1t}^{R_j}\to v_{1t},\ \nabla\tilde{\mathbf{v}}_t^{R_j}\to\nabla\tilde{\mathbf{v}}_t,\ \text{weakly in } L^2(0,T;L^2(\mathbb{R}^2)).
\end{aligned}$$

We take $\phi\in C_0^\infty(\mathbb{R}^2\times T)$ as a text function in (A.3) with initial data $(\rho^R, v_1^R, \tilde{\mathbf{v}}^R)$. Then, letting $R_j\to\infty$, we obtain $(\rho, v_1, \tilde{\mathbf{v}})$ is a strong solution. □

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