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Vertex numbers of simplicial complexes with free abelian fundamental group*

Florian Frick †

Dept. Math. Sciences, Carnegie Mellon University, Pittsburgh, PA 15213, USA and *Inst. Math., Freie Universität Berlin, Arnimallee 2, 14195 Berlin, Germany*

Matt Superdock ‡

Dept. Computer Science, Rhodes College, Memphis, TN 38112, USA

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Abstract

We show that the minimum number of vertices of a simplicial complex with fundamental group $\mathbb{Z}^n$ is at most $O(n)$ and at least $\Omega(n^{3/4})$. For the upper bound, we use a result on orthogonal 1-factorizations of K_{2n}. For the lower bound, we use a fractional Sylvester–Gallai result. This application of extremal results in discrete geometry seems to be new. We also prove that any group presentation $\langle S|R\rangle \cong \mathbb{Z}^n$ whose relations are of the form $g^a h^b i^c$ for $g, h, i \in S$ has at least $\Omega(n^{3/2})$ generators.

Keywords: Simplicial complex, fundamental group, incidence geometry.

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1 Introduction

Given a space X, a *vertex-minimal triangulation* of X is a simplicial complex homeomorphic to X using as few vertices as possible. Such triangulations are known for only a few manifolds [8, 16, 18], and upper and lower bounds differ significantly for many others, despite recent improvements such as [1]. For example, the n-dimensional torus can be

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‡The author was supported by NSF grant DMS 1855591.

E-mail addresses: frick@cmu.edu (Florian Frick), superdockm@rhodes.edu (Matt Superdock)

triangulated on $2^{n+1}-1$ vertices [17], but the best known lower bounds are quadratic in n; see [5].

The number of faces of a simplicial complex X can be bounded in terms of the Betti numbers of X [7] or in terms of the minimal number of generators of $\pi_1(X)$ [22]. The effect of relations of $\pi_1(X)$ on vertex numbers has been studied for cyclic torsion groups [15, 23] and for triangulations of manifolds with non-free fundamental group [25]. In this paper, we consider the minimal number of vertices of a simplicial complex with fundamental group $\mathbb{Z}^n$:

Theorem 1.1. *We have the following asymptotic results for vertex numbers of simplicial complexes with fundamental group $\mathbb{Z}^n$:*

(a) *There is a simplicial complex X_n with $\pi_1(X_n) \cong \mathbb{Z}^n$ on $O(n)$ vertices.*

(b) *Every simplicial complex X_n with $\pi_1(X_n) \cong \mathbb{Z}^n$ has $\Omega(n^{3/4})$ vertices.*

These results appear separately as Theorems 2.8 and 3.21; our precise upper bound depends on parity, but is asymptotically $4n$ in both cases:

Theorem 2.8. *For $n \in \mathbb{N}$ with $n \neq 2, 3$, there exists:*

- *A simplicial complex X_{2n} with $8n-1$ vertices, $\pi_1(X_{2n}) \cong \mathbb{Z}^{2n}$.*
- *A simplicial complex X_{2n-1} with $8n-3$ vertices, $\pi_1(X_{2n-1}) \cong \mathbb{Z}^{2n-1}$.*

To prove the $O(n)$ upper bound, we construct a complex W_n on n^2+n+1 vertices with fundamental group $\pi_1(W_n) \cong \mathbb{Z}^n$, and then perform identifications that preserve $\pi_1(W_n)$. The latter step uses a result on orthogonal 1-factorizations of the complete graph K_{2n}, which is implied by a result on Room squares; see [14, 20, 21].

To prove the $\Omega(n^{3/4})$ lower bound, we relate simplicial complexes to group presentations. Specifically, we define a *3-presentation* as a group presentation $\langle S|R\rangle$ whose relations are of the form $g^a h^b i^c$ for $g, h, i \in S$; this is a generalization of triangular presentations as studied in [2, 3, 4]. Then we show that simplicial complexes give rise to 3-presentations.

For any group presentation $\langle S|R\rangle \cong \mathbb{Z}^n$, it is known that $|R| \geq \binom{n}{2}$; this bound is sharp, by the presentation

$$\langle g_1, \ldots, g_n \,|\, g_i g_j g_i^{-1} g_j^{-1},\ i<j\rangle \cong \mathbb{Z}^n.$$

The bound $|R| \geq \binom{n}{2}$ already gives a $\Omega(n^{2/3})$ lower bound for vertex numbers in Theorem 1.1(b). To strengthen this homological bound, we develop a novel application of extremal results in discrete geometry (specifically, the fractional Sylvester–Gallai results in [6, 10, 11]), to show that any 3-presentation $\langle S|R\rangle \cong \mathbb{Z}^n$ has $|S| = \Omega(n^{3/2})$. This translates to a $\Omega(n^{3/4})$ lower bound in Theorem 1.1(b).

We pose the following questions for further research:

Question 1.2. Can we close the gap in Theorem 1.1 between the upper and lower bounds?

Question 1.3. Can we prove an analogue to Theorem 1.1 for fundamental group $(\mathbb{Z}_k)^n$ (perhaps restricting k to primes or prime powers)?

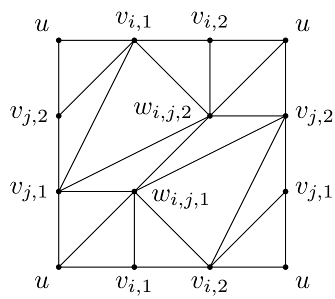
We will generally assume simplicial complexes X are connected, since if X is disconnected, and $\pi_1(X, v) \cong G$, then the component C of X containing v has fewer vertices than X, and $\pi_1(C, v) \cong G$. Under this assumption, $\pi_1(X, v)$ is independent of v, so we will write $\pi_1(X)$ instead.

2 Upper bound

In this section, we show that for all $n \in \mathbb{N}$, there exists a simplicial complex X_n on $O(n)$ vertices with fundamental group $\pi_1(X_n) \cong \mathbb{Z}^n$. We start by constructing a complex W_n on $n^2 + n + 1$ vertices with fundamental group $\pi_1(W_n) \cong \mathbb{Z}^n$, and then obtain X_n by identifying vertices and edges. We write $[n]$ for $\{1, 2, \dots, n\}$.

Definition 2.1. For $n \in \mathbb{N}$, the simplicial complex W_n is defined as follows:

- The vertex set consists of a vertex u, vertices $v_{i,k}$ for $i \in [n]$, $k \in [2]$, and vertices $w_{i,j,k}$ for $i, j \in [n]$, $i < j$, $k \in [2]$.
- The edge set includes edges $\{u, v_{i,1}\}$, $\{v_{i,1}, v_{i,2}\}$, $\{v_{i,2}, u\}$ for all $i \in [n]$.
- For each $i, j \in [n]$ with $i < j$, we include edges and triangles as in the following diagram. (The vertices and edges on the boundary are those defined above, and each planar region corresponds to a triangle.)



(This diagram also gives a vertex-minimal triangulation of the torus.)

Lemma 2.2. *For all $n \in \mathbb{N}$, we have $\pi_1(W_n) \cong \mathbb{Z}^n$.*

Proof. Note that W_n is homeomorphic to a CW complex W_n' consisting of:

- A single 0-cell u.
- A 1-cell e_i from u to itself for each $i \in [n]$, corresponding to the edges $\{u, v_{i,1}\}$, $\{v_{i,1}, v_{i,2}\}$, and $\{v_{i,2}, u\}$.
- A 2-cell $f_{i,j}$ attached along $e_i e_j e_i^{-1} e_j^{-1}$ for each $i, j \in [n]$, $i < j$, corresponding to the triangles in the diagram in Definition 2.1 for i, j. (By e_i^{-1} we denote attaching in the opposite direction along e_i.)

This gives a group presentation $\pi_1(W_n') \cong \langle S|R\rangle$, where

$$S = \{e_i : i \in [n]\}, \qquad R = \{e_i e_j e_i^{-1} e_j^{-1} : i, j \in [n],\, i < j\}.$$

But $\langle S|R\rangle \cong \mathbb{Z}^n$, so $\pi_1(W_n) \cong \pi_1(W_n') \cong \mathbb{Z}^n$, as desired. □

We now establish the tools we need to perform identifications on W_n.

Definition 2.3. Let X be a simplicial complex with vertex $u \in V(X)$. We say that a set $S \subseteq V(X) \setminus \{u\}$ is a *spur* in (X, u) if the following properties hold:

(1) Each $v \in S$ is adjacent to u in X.

(2) No two distinct $v, v' \in S$ are adjacent in X.

(3) No two distinct $v, v' \in S$ have a common neighbor in X other than u.

Definition 2.4. Let X be a simplicial complex with vertex $u \in V(X)$. We say that two spurs S, S' in (X, u) are *compatible* if $S \cap S' = \emptyset$, and there is at most one edge $\{v, v'\}$ in X with $v \in S$, $v' \in S'$.

Lemma 2.5. *Let X be a simplicial complex with vertex $u \in V(X)$. If S is a spur in (X, u), then we can "collapse" S to obtain a simplicial complex, which we denote X/S, as follows:*

- *Identify all vertices $v \in S$ to a single new vertex w.*
- *Identify all edges $\{u, v\}$ for $v \in S$ to a single edge $\{u, w\}$.*

Moreover, $\pi_1(X/S) \cong \pi_1(X)$.

Proof. We may perform the identifications in the category of CW complexes, but we need to prove that the result is a simplicial complex. Since no adjacent vertices are identified, it remains to prove that no two distinct faces f, f' of X have the same vertex set in X/S, other than those explicitly identified.

If f, f' are distinct faces of X with the same vertex set in X/S, then there exist $v, v' \in S$ with $v \in f$, $v' \in f'$. If $f, f' \subseteq \{u\} \cup S$, then f, f' are either $\{v\}, \{v'\}$ or $\{u, v\}, \{u, v'\}$, and are explicitly identified. Hence we may assume that f, f' both contain a vertex $x \notin \{u\} \cup S$. But then x is a common neighbor of v, v', a contradiction. Hence X/S is a simplicial complex.

Now let A be the subcomplex of X with vertices u, S and edges $\{u, v\}$ for all $v \in S$, and let B be the subcomplex of X/S with vertices u, w and edge $\{u, w\}$. Consider the quotients X/A, $(X/S)/B$ in the category of CW complexes, and note that $X/A \cong (X/S)/B$. Since A, B are contractible, we have homotopy equivalences $X/A \simeq X$ and $(X/S)/B \simeq X/S$ (see e.g., [13, Proposition 0.17]). By transitivity, we have $X \simeq X/S$, so $\pi_1(X) \cong \pi_1(X/S)$ as desired. □

Lemma 2.6. *Let X be a simplicial complex with vertex $u \in V(X)$.*

(a) *If S, S' are compatible spurs in (X, u), then S' is a spur in $(X/S, u)$.*

(b) *If S, S', S'' are pairwise compatible spurs in (X, u), then S', S'' are compatible spurs in $(X/S, u)$.*

Proof. For (a), conditions (1), (2) in Definition 2.3 hold since S, S' are disjoint, and condition (3) holds since S, S' have at most one edge between them.

For (b), S', S'' are spurs in $(X/S, u)$ by (a), and compatibility follows from the fact that collapsing S does not affect vertices or edges among $S' \cup S''$. □

Lemma 2.7. *For $n \in \mathbb{N}$ with $n \neq 2, 3$, the vertices $w_{i,j,k}$ of W_{2n} (or W_{2n-1}) can be partitioned into $4n - 2$ pairwise compatible spurs in (W_{2n}, u) (or (W_{2n-1}, u)).*

Proof. A *1-factorization* of the complete graph on vertex set $[2n]$ is a partition of its edges into perfect matchings. An *orthogonal pair* of 1-factorizations is a pair of 1-factorizations, such that no two edges appear in the same matching in both factorizations. By [21] (see also [14, 20]), such a pair (F_1, F_2) exists for all $n \in \mathbb{N}$ with $n \neq 2, 3$.

Then for each matching $M \in F_k$, we construct a spur S_M in (W_{2n}, u):

$$S_M = \{w_{i,j,k} : \{i, j\} \in M\}$$

To see that S_M is a spur, note that the neighbors of $w_{i,j,k}$ in W_{2n} are u, and some vertices of the form $v_{i,k'}, v_{j,k'}, w_{i,j,k'}$ for $k' \in [2]$, so conditions (1) and (2) hold. Since M is a matching, condition (3) holds.

Then the S_M are disjoint, and the only edges between vertices in $S_M, S_{M'}$ for distinct M, M' are the edges $\{w_{i,j,1}, w_{i,j,2}\}$, which arise for $M \in F_1$, $M' \in F_2$ with $\{i, j\} \in M$, $\{i, j\} \in M'$. Then the orthogonality of (F_1, F_2) implies that the S_M are pairwise compatible, and there are $2(2n - 1) = 4n - 2$ such S_M.

Viewing W_{2n-1} as an induced subcomplex of W_{2n}, the S_M remain pairwise compatible spurs in (W_{2n-1}, u), upon deleting the missing vertices. □

Our promised upper bound follows:

Theorem 2.8. *For $n \in \mathbb{N}$ with $n \neq 2, 3$, there exists:*

- *A simplicial complex X_{2n} with $8n - 1$ vertices, $\pi_1(X_{2n}) \cong \mathbb{Z}^{2n}$.*
- *A simplicial complex X_{2n-1} with $8n - 3$ vertices, $\pi_1(X_{2n-1}) \cong \mathbb{Z}^{2n-1}$.*

Proof. Starting with W_{2n} or W_{2n-1}, apply Lemma 2.7 to obtain $4n - 2$ pairwise compatible spurs. Collapse these spurs, one by one, via Lemma 2.5, to obtain X_{2n} or X_{2n-1} with $\pi_1(X_{2n}) \cong \mathbb{Z}^{2n}$, $\pi_1(X_{2n-1}) \cong \mathbb{Z}^{2n-1}$; note that Lemma 2.6 guarantees that the remaining spurs remain compatible. The remaining vertices are u, the $v_{i,k}$, and one vertex for each spur, so:

- The number of vertices in X_{2n} is $1 + 4n + (4n - 2) = 8n - 1$.
- The number of vertices in X_{2n-1} is $1 + 2(2n - 1) + (4n - 2) = 8n - 3$.

This completes the proof. □

3 Lower bound

In this section, we show that a simplicial complex X with fundamental group $\pi_1(X) \cong \mathbb{Z}^n$ has $\Omega(n^{3/4})$ vertices. We begin by relating simplicial complexes to group presentations:

Definition 3.1. Given a group G, a *3-presentation* of G is a group presentation $\langle S|R\rangle \cong G$ where each relation in R is one of the following:

- $\langle\rangle$ (the empty word).
- g^a, where $g \in S$ and $a \in \mathbb{Z}$.
- $g^a h^b$, where $g, h \in S$ and $a, b \in \mathbb{Z}$.
- $g^a h^b i^c$, where $g, h, i \in S$ and $a, b, c \in \mathbb{Z}$.

We say such a word w is in *normal form* if the generators used are all distinct, and $a, b, c \neq 0$. To "write $r \in R$ in normal form" means to find w in normal form such that r and w are conjugates in $\langle S \rangle$; in this case we write $r \rightsquigarrow w$.

For example, we describe a 3-presentation $\langle S|R \rangle \cong \mathbb{Z}^n$, derived from the presentation for $\mathbb{Z}^n$ given in the introduction:

$$S = \{g_i : i \in [n]\} \cup \{h_{i,j} : i, j \in [n],\, i < j\}$$
$$R = \{g_i g_j h_{i,j},\, g_j g_i h_{i,j} : i, j \in [n],\, i < j\}$$

We will use the phrase, "Let $\phi\colon \langle S|R \rangle \cong G$ be a 3-presentation," to mean, "Let $\langle S|R \rangle \cong G$ be a 3-presentation, and fix an isomorphism $\phi\colon \langle S|R \rangle \to G$."

Lemma 3.2. *Let $\langle S|R \rangle \cong G$ be a 3-presentation. Then any relation $r \in R$ can be written uniquely in normal form, up to the following conjugacies:*

- *$g^a h^b, h^b g^a$ are conjugates in $\langle S \rangle$.*
- *$g^a h^b i^c, h^b i^c g^a, i^c g^a h^b$ are conjugates in $\langle S \rangle$.*

Proof. If r is not in normal form, we can apply one of the following steps:

- If r has a zero exponent or identical adjacent generators, rewrite r with fewer generators. (For example, $g^0 hi$ becomes hi; $g^1 g^2 h$ becomes $g^3 h$.)
- If $r = g^a h^b g^c$, then replace r with its conjugate $g^{a+c} h^b$.

Each such step reduces k in $r = \prod_{i=1}^{k} g_i^{a_i}$, so this process terminates. Uniqueness follows from considering the conjugates of reduced words w. □

Lemma 3.3. *If X is a simplicial complex on k vertices with fundamental group $\pi_1(X) \cong G$, then there exists a 3-presentation $\langle S|R \rangle \cong G$ with $|S| \leq \binom{k}{2}$ and $|R| \leq \binom{k}{3}$.*

Proof. Assume X is connected, otherwise reduce to the component of X containing the basepoint. Then the 1-skeleton of X is a connected graph; choose a spanning tree T of this graph. View X as a CW complex and T as a contractible subcomplex of X, and consider the quotient complex X/T, which is homotopy equivalent to X (see e.g., [13, Proposition 0.17]), so $\pi_1(X/T) \cong G$.

Now X/T has a single 0-cell, and hence can be viewed as the presentation complex of some group presentation $\langle S|R \rangle \cong G$ upon choosing a direction for each 1-cell. The 1-cells correspond bijectively to the generators, and arise from distinct edges of X, so $|S| \leq \binom{k}{2}$. The 2-cells correspond bijectively to the relations, and arise from distinct triangles of X, so $|R| \leq \binom{k}{3}$.

Moreover, each $r \in R$ is of one of the following forms, depending on how many edges of the corresponding triangle in X lie in T:

- g^a, where $g \in S$ and $a \in \{\pm 1\}$.
- $g^a h^b$, where $g, h \in S$ are distinct and $a, b \in \{\pm 1\}$.
- $g^a h^b i^c$, where $g, h, i \in S$ are distinct and $a, b, c \in \{\pm 1\}$.

In particular, $\langle S|R \rangle$ is a 3-presentation, which completes the proof. □

Hence we may turn our attention to proving lower bounds on $|S|$ and $|R|$ for 3-presentations $\langle S|R\rangle$ of given groups. We will use the concept of deficiency:

Definition 3.4. The *deficiency* of a group presentation $P = \langle S|R\rangle$ is $\operatorname{def} P = |S| - |R|$. The *deficiency*, $\operatorname{def} G$, of a group G is the maximum of $\operatorname{def} P$ over all finite presentations P of G.

Then we have an inequality in group homology due to Epstein [12]:

$$\operatorname{def} G \leq \operatorname{rank} H_1(G; \mathbb{Z}) - s(H_2(G; \mathbb{Z})),$$

where $s(H_2(G;\mathbb{Z}))$ is the minimum number of generators of $H_2(G;\mathbb{Z})$. (For a related inequality in terms of free resolutions, see [26].) In particular, when G is $\mathbb{Z}^n$, we obtain $\operatorname{def} \mathbb{Z}^n \leq n - \binom{n}{2}$. Thus we have several constraints on the size of a presentation of $\mathbb{Z}^n$; if $\langle S|R\rangle \cong \mathbb{Z}^n$, then

- $|S| \geq n$.
- $|R| - |S| \geq \binom{n}{2} - n$.
- $|R| \geq \binom{n}{2}$ (by adding the previous two inequalities).

For the presentation of $\mathbb{Z}^n$ described in the introduction, we have equality in all three of these bounds. Hence $\operatorname{def} \mathbb{Z}^n = n - \binom{n}{2}$.

Lemma 3.3 allows us to translate these bounds into lower bounds for the number of vertices in a simplicial complex X with fundamental group $\pi_1(X) \cong \mathbb{Z}^n$. The first inequality above gives a bound of $\Omega(n^{1/2})$, and the third gives a stronger bound of $\Omega(n^{2/3})$. We present the latter bound in more detail:

Proposition 3.5. *A simplicial complex X with fundamental group $\pi_1(X) \cong \mathbb{Z}^n$ has at least $\Omega(n^{2/3})$ vertices.*

Proof. Let $f(n)$ be the minimum number of vertices in a simplicial complex X_n with fundamental group $\pi_1(X_n) \cong \mathbb{Z}^n$. By Lemma 3.3, for each n we obtain a 3-presentation $\langle S_n|R_n\rangle \cong \mathbb{Z}^n$ with $|R_n| \leq \binom{f(n)}{3}$. But $|R_n| \geq \binom{n}{2}$, so $\binom{f(n)}{3} \geq \binom{n}{2}$, hence $f(n) = \Omega(n^{2/3})$. □

Up to now, we have considered bounds on the size of arbitrary presentations of $\mathbb{Z}^n$. Now we turn to proving that for 3-presentations $\langle S|R\rangle \cong \mathbb{Z}^n$, we have a stronger bound $|S| = \Omega(n^{3/2})$. First we introduce a notion of dimension:

Definition 3.6. Let $\phi\colon \langle S|R\rangle \cong \mathbb{Z}^n$ be a 3-presentation. Then the *dimension* of a subset $S' \subseteq S$, denoted $\dim S'$, is

$$\dim(\operatorname{span}(\{\phi(g) : g \in S'\})),$$

where we view each $\phi(g)$ as a vector in $\mathbb{R}^n \supseteq \mathbb{Z}^n$.

For $r \in R$, let $r \rightsquigarrow \prod_i g_i^{a_i}$ by Lemma 3.2. The *dimension* of r is the dimension of the subset $\{g_i\} \subseteq S$. (Note that the set $\{g_i\}$ is independent of the choice of normal form, so this definition is valid.)

Note that for a relation r with $r \rightsquigarrow \prod_{i=1}^{k} g_i^{a_i}$, we have $\sum_{i=1}^{k} a_i \phi(g_i) = 0$, a linear dependence among the $\phi(g_i)$. It follows that $\dim r < k$. In particular, all relations of a 3-presentation $\langle S|R \rangle \cong \mathbb{Z}^n$ have dimension at most two.

Our next goal is to show that for 3-presentations $\langle S|R \rangle \cong \mathbb{Z}^n$ with $|S|$ minimal, all nonempty relations have dimension exactly two. To do this, we use Tietze transformations ([27]; see also [19]):

Remark 3.7 ([27, Tietze])**.** Consider a group presentation $\langle S|R \rangle \cong G$. Then:

- Let r be a word over S which is the identity in $\langle S|R \rangle$. Then $\langle S|R \cup \{r\} \rangle \cong G$.
- Let w be a word over S, and let g be fresh. Then $\langle S \cup \{g\}|R \cup \{g^{-1}w\} \rangle \cong G$.

We refer to the passage from one presentation to another in either of these ways, in either direction, as a *Tietze transformation*.

We now establish several transformations of 3-presentations:

Lemma 3.8. *Let $\langle S|R \rangle \cong \mathbb{Z}^n$ be a 3-presentation, and suppose $g = e$ (the identity) in $\langle S|R \rangle$, where $g \in S$. Then we obtain a 3-presentation $\langle S'|R' \rangle \cong \mathbb{Z}^n$ where:*

- *$S' = S \setminus \{g\}$.*
- *R' is obtained from R by removing g wherever it appears in relations $r \in R$. (For example, $ghi \in R$ becomes $hi \in R'$.)*

Proof. We apply Tietze transformations:

- Add the redundant relation g to R to obtain R'.
- Remove g wherever it appears in relations $r \in R'$, except in the relation $g \in R'$. This is valid since $g = e$ in $\langle S|R' \rangle$ by the relation $g \in R'$. (Each such removal is two Tietze transformations, adding and removing a relation.)
- Remove the generator g, along with the relation g.

This gives the desired 3-presentation of $\mathbb{Z}^n$. □

Lemma 3.9. *Let $\langle S|R \rangle \cong \mathbb{Z}^n$ be a 3-presentation, and suppose $g^a h^b = e$ (the identity) in $\langle S|R \rangle$, where $g, h \in S$ are distinct, $a, b \neq 0$, and a, b are relatively prime. Then we obtain a 3-presentation $\langle S'|R' \rangle \cong \mathbb{Z}^n$ where:*

- *$S' = S \cup \{i\} \setminus \{g, h\}$, where i is a fresh generator.*
- *R' is obtained from R by replacing g with i^b and h with i^{-a} wherever they appear in relations $r \in R$.*

Proof. There exist $c, d \in \mathbb{Z}$ with $ac + bd = 1$. We apply Tietze transformations:

- Add the relation $g^a h^b$, which is redundant by assumption.
- Add a generator i, along with the relation $i^{-1} g^d h^{-c}$, to obtain a new 3-presentation

$$\phi' \colon \langle S'|R' \rangle \cong \mathbb{Z}^n.$$

- Add the relation $g^{-1}i^b$, which is redundant since

$$i^b = g^{bd}h^{-bc} = g^{1-ac}h^{-bc} = g(g^a h^b)^{-c} = g$$

  in $\langle S'|R'\rangle$. (We use commutativity of g, h in $\langle S'|R'\rangle$, which follows from commutativity of $\phi'(g), \phi'(h)$ in $\mathbb{Z}^n$.)

- Similarly, add the relation $h^{-1}i^{-a}$, which is redundant since

$$i^{-a} = g^{-ad}h^{ac} = g^{-ad}h^{1-bd} = h(g^a h^b)^{-d} = h$$

  in $\langle S'|R'\rangle$.

- Replace g with i^b and h with i^{-a} wherever they appear in relations $r \in R'$ (i.e. in all relations other than the new relations $g^{-1}i^b, h^{-1}i^{-a}$).

- Remove the generators g, h, along with the relations $g^{-1}i^b, h^{-1}i^{-a}$.

- Remove the relation $i^{-1}g^d h^{-c}$, which is now $i^{-1}i^{bd}i^{ac} = e$.

- Remove the relation $g^a h^b$, which is now $(i^b)^a(i^{-a})^b = e$.

This gives the desired 3-presentation of $\mathbb{Z}^n$. □

Lemma 3.10. *Let $\phi\colon \langle S|R\rangle \cong \mathbb{Z}^n$ be a 3-presentation with $|S|$ minimal. Then for each nonempty $r \in R$, we have $r \rightsquigarrow g^a h^b i^c$, where $g, h, i \in S$ are distinct and $a, b, c \neq 0$, and* $\dim r = 2$.

Proof. Let $r \rightsquigarrow \prod_{i=1}^{k} g_i^{a_i}$ by Lemma 3.2. By Lemma 3.8, no $g_i = e$ in $\langle S|R\rangle$, which implies $k \neq 1$. By Lemma 3.9, no distinct g_i, g_j have $\phi(g_i)$, $\phi(g_j)$ in a common one-dimensional subspace of $\mathbb{R}^n$, which implies $k \neq 2$.

Hence $k = 3$, so $r \rightsquigarrow g^a h^b i^c$ for $g, h, i \in S$ distinct and $a, b, c \neq 0$. Then the considerations above imply $\dim\{g, h\} = 2$, so $\dim r \geq 2$. Since $\phi(g), \phi(h), \phi(i)$ are dependent in $\mathbb{R}^n$, we have $\dim r = 2$. □

Now we turn our attention to subsets of relations of 3-presentations of $\mathbb{Z}^n$.

Definition 3.11. Let $\langle S|R\rangle \cong \mathbb{Z}^n$ be a 3-presentation, and let $A \subseteq S$, $R' \subseteq R$. Then define the set $R'[A] \subseteq R'$ as

$$R'[A] = \{r \in R' : r \rightsquigarrow w, \text{ and } w \text{ uses only generators in } A\}.$$

Note that any group presentation of a two-dimensional lattice using k generators requires $k-1$ relations, e.g., $\langle g, h, i \,|\, ghi, ihg\rangle \cong \mathbb{Z}^2$. This motivates the following definition:

Definition 3.12. Let $\phi\colon \langle S|R\rangle \cong \mathbb{Z}^n$ be a 3-presentation, and let $R' \subseteq R$.

- For $A \subseteq S$ with $\dim A = 2$, R' is *sparse on* A if $|R'[A]| \leq |A| - 1$.

- R' is *sparse* if R' is sparse on all $A \subseteq S$ with $\dim A = 2$.

- A set $A \subseteq S$ is *critical for* R' if $\dim A = 2$ and $|R'[A]| = |A| - 1$.

Lemma 3.13. *Let $\phi\colon \langle S|R\rangle \cong \mathbb{Z}^n$ be a 3-presentation with $|S|$ minimal, so that* Lemma 3.10 *applies. Suppose $R' \subseteq R$ is sparse, and $A, B \subseteq S$ are critical for R'. If $R[A] \cap R[B] \neq \emptyset$, then $A \cup B$ is also critical for R'.*

Proof. Let $r \in R[A] \cap R[B]$, and write $r \rightsquigarrow g^a h^b i^c$ by Lemma 3.10. Then the set $\{\phi(g), \phi(h), \phi(i)\}$ spans a 2-dimensional subspace $U \subseteq \mathbb{R}^n$. Since $\dim A = \dim B = 2$, it follows that $\operatorname{span}(\phi(A)) = \operatorname{span}(\phi(B)) = U$. Then

$$U \subseteq \operatorname{span}(\phi(A \cap B)) \subseteq \operatorname{span}(\phi(A)) = U,$$

so $\operatorname{span}(\phi(A \cap B)) = U$, and $\operatorname{span}(\phi(A \cup B)) = U + U = U$. In particular, we have $\dim(A \cap B) = \dim(A \cup B) = 2$. Therefore, we have

$$\begin{aligned} |R'[A \cup B]| &\geq |R'[A]| + |R'[B]| - |R'[A \cap B]| \\ &\geq (|A| - 1) + (|B| - 1) - (|A \cap B| - 1) \\ &\geq |A \cup B| - 1. \end{aligned}$$

Since R' is sparse, we have equality above, so $A \cup B$ is critical for R'. □

Corollary 3.14. *Let $\phi\colon \langle S|R\rangle \cong \mathbb{Z}^n$ be a 3-presentation with $|S|$ minimal, so that* Lemma 3.10 *applies. Suppose $R' \subseteq R$ is sparse. Then there exists a collection $\mathcal{C}$ of certain critical sets $A \subseteq S$ for R', such that:*

(1) *If $B \subseteq S$ is critical for R', then there exists $A \in \mathcal{C}$ with $B \subseteq A$.*

(2) *If $A, B \in \mathcal{C}$, then $R[A] \cap R[B] = \emptyset$.*

Proof. First take the collection $\mathcal{C} = \{A \subseteq S : A \text{ critical for } R'\}$; then (1) holds. If $A, B \in \mathcal{C}$ with $R[A] \cap R[B] \neq \emptyset$, then by Lemma 3.13, $A \cup B$ is critical for R'. Then consider removing A, B from $\mathcal{C}$, and adding $A \cup B$ if it is not present.

While (2) fails, apply the step above repeatedly. Each step preserves (1) and reduces $|\mathcal{C}|$, so this process terminates with $\mathcal{C}$ such that (1), (2) both hold. □

Lemma 3.15. *Let $\phi\colon \langle S|R\rangle \cong \mathbb{Z}^n$ be a 3-presentation with $|S|$ minimal, so that* Lemma 3.10 *applies. Partition R as $R = R_s \sqcup R_e \sqcup R_o$ (mnemonic: "sparse," "extra," "other"), such that R_s is sparse, and R_e and R_o are determined from R_s as follows:*

- *For each $r \in R_e$ with $r \rightsquigarrow g^a h^b i^c$, we have $\{g, h, i\} \subseteq A$ for some critical $A \subseteq S$ for R_s.*
- *For each $r \in R_o$ with $r \rightsquigarrow g^a h^b i^c$, we have $\{g, h, i\} \not\subseteq A$ for all critical $A \subseteq S$ for R_s.*

Then we obtain a 3-presentation $\langle S'|R'\rangle \cong \mathbb{Z}^n$ where:

- *S' includes all generators in S.*
- *R' includes all relations in R_o.*
- *$|R'| - |S'| = |R_s| + |R_o| - |S|$.*

Proof. Obtain a collection $\mathcal{C}$ of critical sets for R_s via Corollary 3.14. Then for each $A \in \mathcal{C}$, consider the integer span of $\phi(A)$ in $\mathbb{Z}^n$, that is, the set

$$\Lambda(A) = \left\{ \sum_{i=1}^{k} a_i \phi(g_i) : k \in \mathbb{N},\, a_i \in \mathbb{Z},\, g_i \in A \right\}.$$

We have $\Lambda(A) \cong \mathbb{Z}^2$ (see [9, Theorem 1.12.3]), so $\Lambda(A)$ has a basis $\{x_1, x_2\}$. We apply Tietze transformations (for each $A \in \mathcal{C}$) to $\langle S|R \rangle$. (We will introduce some relations with more than three generators, but we remove these later.)

- For each $j \in [2]$, write $x_j = \sum_i a_i \phi(g_i)$ for $a_i \in \mathbb{Z}$, $g_i \in A$. Then add a generator h_j, along with the relation $h_j^{-1} \prod_i g_i^{a_i}$, to obtain $\phi' \colon \langle S'|R' \rangle \cong \mathbb{Z}^n$. Note that

$$\phi'(h_j) = \phi\left(\prod_i g_i^{a_i}\right) = \sum_i a_i \phi(g_i) = x_j.$$

- For each $g \in A$, write $\phi(g) = \sum_j b_j x_j$ for $b_j \in \mathbb{Z}$. Then add the relation $g^{-1} \prod_j h_j^{b_j}$, which is redundant since

$$\phi'\left(g^{-1} \prod_j h_j^{b_j}\right) = -\phi(g) + \sum_j b_j \phi'(h_j) = 0,$$

  where we use $\phi'(h_j) = x_j$ in the last step.

- Add a generator h_*, along with the relation $h_*^{-1} h_1 h_2$.

- Add the relation $h_*^{-1} h_2 h_1$, which is redundant since $\mathbb{Z}^n$ (and hence our current $\langle S'|R' \rangle \cong \mathbb{Z}^n$) is abelian. Note that the relation $h_1 h_2 h_1^{-1} h_2^{-1}$ is now implied by the relations $h_*^{-1} h_1 h_2$ and $h_*^{-1} h_2 h_1$.

- Remove all relations $r \in R[A]$, which are now redundant. To see this, first rewrite r in terms of only the h_j, via the relations $g^{-1} \prod_j h_j^{b_j}$. Then rewrite r as $\prod_j h_j^{b_j}$ for $b_j \in \mathbb{Z}$, via the relations $h_i h_j h_i^{-1} h_j^{-1}$. Applying ϕ', we obtain $\sum_j b_j x_j = 0$, so $b_j = 0$ by the lattice structure of $\Lambda(A) \cong \mathbb{Z}^2$. Hence we have rewritten r as the empty word, so r is redundant.

- Remove the relations $h_j^{-1} \prod_i g_i^{b_i}$ added in the first step, which are now redundant, since we may rewrite any such relation in terms of only the h_j, and then apply the previous argument.

After applying these steps for each $A \in \mathcal{C}$, we call the resulting 3-presentation $\langle S'|R' \rangle$. For each $A \in \mathcal{C}$, we have added three generators and a net of $|A| - |R[A]| + 2$ relations. By definition of $\mathcal{C}$, the sets $R[A]$ are disjoint for distinct $A \in \mathcal{C}$. Also, $R_o[A] = \emptyset$, since A is critical. Therefore,

$$\begin{aligned}|R'| - |S'| &= |R| - |S| + \sum_{A \in \mathcal{C}} (|A| - |R[A]| - 1) \\ &= |R| - |S| + \sum_{A \in \mathcal{C}} (|A| - |R_s[A]| - 1) - \sum_{A \in \mathcal{C}} |R_e[A]| \\ &= |R| - |S| - |R_e| \\ &= |R_s| + |R_o| - |S|.\end{aligned}$$

This completes the proof. □

We need one more transformation of presentations:

Lemma 3.16. *Let $\phi\colon \langle S|R\rangle \cong \mathbb{Z}^n$ be a 3-presentation, let $S' \subseteq S$, and let $d = \dim S'$. Then we obtain a presentation $\langle S''|R''\rangle \cong \mathbb{Z}^{n-d}$ where:*

- *$S'' = S \setminus S'$.*
- *R'' is obtained from R by adding d relations to form R', then removing each $g \in S'$ wherever it appears in relations $r \in R'$.*

Proof. Let $U = \operatorname{span}(\phi(S'))$ in $\mathbb{R}^n$. Then $U \cap \mathbb{Z}^n$ is a lattice of dimension d, so we may take a basis $\{x_1, \dots, x_d\}$ of $U \cap \mathbb{Z}^n$, and extend to a basis $\{x_1, \dots, x_n\}$ of $\mathbb{Z}^n$ (see Chapter 2, Lemma 4 of [24]). Then for each $i \in [d]$, let w_i be a word in $\langle S\rangle$ with $\phi(w_i) = x_i$ in $\mathbb{Z}^n$. Let $R' = R \cup \{w_1, \dots, w_d\}$.

We claim $\langle S|R'\rangle \cong \mathbb{Z}^{n-d}$. To prove this, we will construct an isomorphism $\psi\colon \langle S|R'\rangle \to \mathbb{Z}^{n-d}$. Let $p\colon \mathbb{Z}^n \to \mathbb{Z}^{n-d}$ be the projection to the last $n-d$ coordinates under the basis $\{x_1, \dots, x_n\}$; more precisely,

$$p\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=d+1}^{n} a_i y_i,$$

where $\{y_{d+1}, \dots, y_n\}$ is a basis for $\mathbb{Z}^{n-d}$. Note that p is linear. Now define ψ on S by $\psi(g) = p(\phi(g))$ for all $g \in S$, and extend ψ to $\langle S\rangle$ by the universal property of the free group. Then for any word $w = \prod_i g_i^{a_i}$ in $\langle S\rangle$, we have

$$\psi(w) = \sum_i a_i \psi(g_i) = \sum_i a_i p(\phi(g_i)) = p\left(\sum_i a_i \phi(g_i)\right) = p(\phi(w)).$$

In particular, for $r \in R$, we have $\psi(r) = p(\phi(r)) = p(0) = 0$. For the w_i above, we have $\psi(w_i) = p(\phi(w_i)) = p(x_i) = 0$. Therefore, ψ is well-defined on $\langle S|R'\rangle$.

To show ψ is injective, suppose $\psi(w) = 0$ for $w \in \langle S\rangle$. Then $p(\phi(w)) = 0$, so $\phi(w) = \sum_{i=1}^{d} a_i x_i$ for some $a_i \in \mathbb{Z}$. Then $\phi(w) = \phi(\prod_{i=1}^{d} w_i^{a_i})$ so $w = \prod_{i=1}^{d} w_i^{a_i}$ in $\langle S|R\rangle$ by the injectivity of ϕ. Since $R \subseteq R'$, we have $w = \prod_{i=1}^{d} w_i^{a_i}$ in $\langle S|R'\rangle$ also. But since $w_i \in R'$, this implies $w = e$ in $\langle S|R'\rangle$.

To show ψ is surjective, it suffices to show that for each $d < i \leq n$, there exists $w \in \langle S\rangle$ with $\psi(w) = y_i$. By the surjectivity of ϕ, take w with $\phi(w) = x_i$. Then $\psi(w) = p(\phi(w)) = p(x_i) = y_i$ as desired. Hence $\langle S|R'\rangle \cong \mathbb{Z}^{n-d}$.

Now all generators $g \in S'$ have $g = e$ in $\langle S|R'\rangle$, so repeated application of Lemma 3.8 gives the desired result. □

Next, we need a variant of the Sylvester–Gallai-type results in [6, 10, 11]. We begin by stating the relevant definition and theorem from [11]:

Definition 3.17 ([11, Definition 1.7])**.** Given a set of points $v_1, \dots, v_n \in \mathbb{R}^d$, a *special line* is a line in $\mathbb{R}^d$ containing at least three of the points v_i. We say that $v_1, \dots, v_n$ is a *δ-SG configuration* if for each v_i, $i \in [n]$, at least $\delta(n-1)$ of the remaining points lie on special lines through v_i.

Theorem 3.18 ([11, Theorem 5.1])**.** *If $v_1, \dots, v_n$ is a δ-SG configuration, then the affine dimension of $\{v_1, \dots, v_n\}$ is at most $12/\delta$.*

Now we give our variant; we translate the average case result in [11] from an affine setting to a linear one (as in [10]), with a guarantee on $|E'|$:

Theorem 3.19. *Let $V \subseteq \mathbb{R}^d \setminus \{0\}$ be a finite set of points, such that no two points in V lie in a common 1-dimensional subspace of $\mathbb{R}^d$. Let E be a finite multiset of triples $\{u, v, w\}$, each consisting of distinct points $u, v, w \in V$ lying in a common 2-dimensional subspace of $\mathbb{R}^d$, so that (V, E) forms a 3-uniform hypergraph. Suppose that for each induced subhypergraph (V'', E'') of (V, E) with $\dim(\operatorname{span}(V'')) \leq 2$, we have $|E''| \leq |V''| - 1$. Then for $\lambda > 0$, there exists an induced subhypergraph (V', E') of (V, E) with $|E| - |E'| < \lambda|V|$, and*

$$\dim(\operatorname{span}(V')) - 1 \leq 12|V|/\lambda.$$

Proof. Following the proof of [6, Theorem 13], consider (V, E) as a 3-uniform hypergraph, and repeatedly remove vertices of degree less than λ. This removes less than $\lambda|V|$ edges, so we obtain a sub-hypergraph (V', E') with $|E| - |E'| < \lambda|V|$ and minimum degree at least λ.

Fix $u \in V'$; the neighborhood $N(u)$ in (V', E') forms a graph $G(u)$, where we consider two vertices $v, w \in N(u)$ adjacent if and only if $\{u, v, w\} \in E'$. If $v, w \in N(u)$ are adjacent in $G(u)$, then w lies in span$(\{u, v\}) \subseteq \mathbb{R}^d$. Therefore, if $\{v_1, \dots, v_k\}$ form a component C of $G(u)$, then $U = \{u, v_1, \dots, v_k\}$ has $\dim(\operatorname{span}(U)) \leq 2$, so the number of triples in E' using only points in U is at most k. Hence the number of edges in C is at most k. Summing over components C, the number of neighbors of u in (V', E') is at least $\deg_{(V',E')} u \geq \lambda$.

Now choose a nonzero vector $\vec{n} \in \mathbb{R}^d$ not orthogonal to any $v \in V'$, and define an affine hyperplane $H = \{\vec{x} \in \mathbb{R}^d : \vec{x} \cdot \vec{n} = 1\}$. Then to each $v \in V'$ we associate the unique point $\tilde{v} \in$ span$(\{v\}) \cap H$. Note that the $\tilde{v}$ are distinct, since no two points in V lie in a common 1-dimensional subspace of $\mathbb{R}^d$. Also, note that u, v, w lie in a common two-dimensional subspace of $\mathbb{R}^d$ if and only if $\tilde{u}, \tilde{v}, \tilde{w}$ lie on a common line in H. Then the set $\tilde{V'} = \{\tilde{v} : v \in V'\}$ is a δ-SG configuration with $\delta = \lambda/|V|$. By Theorem 3.18, the affine dimension of $\tilde{V'}$ is at most $12/\delta$, so $\dim(\operatorname{span}(V')) - 1 \leq 12|V|/\lambda$. □

Now we prove our bound on the size of 3-presentations of $\mathbb{Z}^n$:

Theorem 3.20. *If $\langle S|R\rangle \cong \mathbb{Z}^n$ is a 3-presentation, then $|S| = \Omega(n^{3/2})$.*

Proof. Fix an isomorphism $\phi\colon \langle S|R\rangle \to \mathbb{Z}^n$. Assume that $|S|$ is minimal, and consider the images $\phi(g)$ for $g \in S$. By Lemma 3.8, all $\phi(g)$ are nonzero; by Lemma 3.9, no two $\phi(g)$ lie in a common 1-dimensional subspace of $\mathbb{R}^n$. Moreover, by Lemma 3.10, for each $r \in R$ we have $r \rightsquigarrow g^a h^b i^c$ for $g, h, i \in S$ distinct, and $\dim r = 2$. Let R' be an inclusion-wise maximal sparse subset of R.

Now let $k = |S|$, let $c > 0$ be a constant to be determined later, and apply Theorem 3.19 with $V = \phi(S)$, $E = \{\{\phi(g), \phi(h), \phi(i)\} : r \rightsquigarrow g^a h^b i^c,\ r \in R'\}$, and $\lambda = ck/n$, to obtain $S' \subseteq S$ such that:

(1) $|R' \setminus R[S']| \leq ck^2/n$.

(2) $\dim S' - 1 \leq 12|V|/\lambda = 12n/c$.

If there exists $g \in S \setminus S'$ with $\dim(S' \cup \{g\}) = \dim S'$, then we may replace S' with $S' \cup \{g\}$, preserving (1) and (2). Therefore, we may assume that for each $g \in S \setminus S'$, we have $\phi(g) \notin \operatorname{span}(\phi(S'))$.

Now partition R as $R = R_s \sqcup R_e \sqcup R_o$, where:

- $R_s = R' \setminus R[S']$.
- $R_e = (R \setminus R') \setminus R[S']$.
- $R_o = R[S']$.

Note that $|R_s| \leq ck^2/n$ by the above. Now we check the conditions of Lemma 3.15.

- The set R_s is sparse, since sparseness is closed under taking subsets.
- For each $r \in R_e$ with $r \rightsquigarrow g^a h^b i^c$, we have $\{g, h, i\} \subseteq A$ for some critical $A \subseteq S$ for R', since $r \notin R'$ and R' is maximal. It suffices to show that A is also critical for R_s. Since $r \notin R[S']$, we have $\{g, h, i\} \not\subseteq S'$; assume WLOG $g \notin S'$. Then $\phi(g) \notin \operatorname{span}(\phi(S'))$ by the above, so $\operatorname{span}(\phi(A)) \not\subseteq \operatorname{span}(\phi(S'))$. Then $R[A] \cap R[S'] = \emptyset$, since any $r' \in R[A]$ determines the 2-dimensional subspace $\operatorname{span}(\phi(A))$. Therefore, A is also critical for R_s.
- For each $r \in R_o$ with $r \rightsquigarrow g^a h^b i^c$, we have $\{g, h, i\} \subseteq S'$. Suppose for contradiction that $\{g, h, i\} \subseteq A$ for some critical $A \subseteq S$ for R_s. Then $\dim\{g, h, i\} = \dim A = 2$, so we have $\operatorname{span}(\phi(\{g, h, i\})) = \operatorname{span}(\phi(A))$. Since $\{g, h, i\} \subseteq S'$, we have $\operatorname{span}(\phi(A)) \subseteq \operatorname{span}(\phi(S'))$. Therefore, any $x \in A$ has $\phi(x) \in \operatorname{span}(\phi(S'))$, so $x \in S'$ by our assumption above. Hence we have $A \subseteq S'$, so $R_s[A] = \emptyset$, contradicting the assumption that A is critical for R_s.

Therefore, we may apply Lemma 3.15 to $\langle S|R\rangle$, to obtain a 3-presentation $\langle S''|R''\rangle \cong \mathbb{Z}^n$ with $|R''| - |S''| = |R_s| + |R_o| - |S|$.

Finally, let $d = \dim S' - 1$, and apply Lemma 3.16 to $\langle S''|R''\rangle$ using $S' \subseteq S''$, to obtain $\langle S'''|R'''\rangle \cong \mathbb{Z}^{n-d}$. Then remove all relations in R''' arising from relations in $R_o = R[S']$, which are now trivial, to obtain $\langle S'''|R''''\rangle \cong \mathbb{Z}^{n-d}$. Then

$$\begin{aligned}
|R''''| - |S'''| &= (|R'''| - |R_o|) - |S'''| \\
&= (|R''| + d + 1 - |R_o|) - (|S''| - |S'|) \\
&= (|R''| - |S''| - |R_o|) + d + |S'| + 1 \\
&= (|R_s| - |S|) + d + |S'| + 1 \\
&= |R_s| + d - |S \setminus S'| + 1 \\
&\leq ck^2/n + d + 1.
\end{aligned}$$

But by the bound def $\mathbb{Z}^m \leq m - \binom{m}{2}$, we have $|R''''| - |S'''| = \Omega((n-d)^2)$. Take $c = 24$; then $d \leq 12n/c = n/2$, so $n - d \geq n/2$. Hence $|R''''| - |S'''| = \Omega(n^2)$. Since $d \leq n/2$, we have $ck^2/n = \Omega(n^2)$. Therefore, $k = \Omega(n^{3/2})$ as desired. □

Theorem 3.21. *A simplicial complex X with fundamental group $\pi_1(X) \cong \mathbb{Z}^n$ has at least $\Omega(n^{3/4})$ vertices.*

Proof. Let $f(n)$ be the minimum number of vertices in a simplicial complex X_n with fundamental group $\pi_1(X_n) \cong \mathbb{Z}^n$. By Lemma 3.3, for each n we obtain a 3-presentation $\langle S_n | R_n \rangle \cong \mathbb{Z}^n$ with $|S_n| \leq \binom{f(n)}{2}$. But $|S_n| = \Omega(n^{3/2})$, so $\binom{f(n)}{2} = \Omega(n^{3/2})$, hence $f(n) = \Omega(n^{3/4})$. □

ORCID iDs

Florian Frick https://orcid.org/0000-0002-7635-744X
Matt Superdock https://orcid.org/0000-0002-5663-6222

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