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Fast Consensus Topology Design via Minimizing Laplacian Energy*

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Abstract— **This paper characterizes the graphical properties of an optimal topology with minimal Laplacian energy under the constraint of fixed numbers of vertices and edges, and devises an algorithm to construct such connected optimal graphs. These constructed graphs possess maximum vertex and edge connectivity, and more importantly, generically exhibit large algebraic connectivity of an optimal order provided they are not sparse. These properties guarantee fast and resilient consensus processes over these graphs.**

I. Introduction

Over the past two decades, consensus has achieved great success and attracted significant attention [2]–[5], being applied to a wide range of distributed control and computation problems [6]–[10].

A continuous-time linear consensus process over a simple connected graph $\mathbb{G}$ can be typically modeled by a linear differential equation of the form $\dot{x}(t) = -Lx(t)$, where $x(t)$ is a vector in $\mathbb{R}^n$ and L is the "Laplacian matrix" of $\mathbb{G}$. For any simple graph $\mathbb{G}$ with n vertices, we use $D(\mathbb{G})$ and $A(\mathbb{G})$ to denote its degree matrix and adjacency matrix, respectively. Specifically, $D(\mathbb{G})$ is an $n\times n$ diagonal matrix whose ith diagonal entry equals the degree of vertex i, and $A(\mathbb{G})$ is an $n \times n$ matrix whose ijth entry equals 1 if (i, j) is an edge in $\mathbb{G}$ and otherwise equals 0. The Laplacian matrix of $\mathbb{G}$ is defined as $L(\mathbb{G}) = D(\mathbb{G}) - A(\mathbb{G})$. It is easy to see that any Laplacian matrix is symmetric and thus has a real spectrum. It is well known that $L(\mathbb{G})$ is positive-semidefinite, its smallest eigenvalue equals 0, and its second smallest eigenvalue, called the algebraic connectivity of $\mathbb{G}$ and denoted as $a(\mathbb{G})$, is positive if and only if $\mathbb{G}$ is connected [11]. It has been shown that the convergence rate of continuous-time linear consensus is determined by the algebraic connectivity, in that the larger the algebraic connectivity is, the faster the consensus can be reached [3].

With the preceding facts in mind, a natural and fundamental research problem is how to design network topology to achieve faster or even the fastest consensus. The problem has been studied for many years [12]–[17], to name a few. Notwithstanding these developments, the following question is still largely unsolved: *Given a fixed number of vertices and edges, what are the optimal graphs that achieve maximal algebraic connectivity?*

The above question presents a challenging combinatorial optimization problem, and thus, it was only partially answered for some special cases in [17]. Even though a powerful computer can execute such a combinatorial search, identifying the graphical properties of optimal graphs with maximal algebraic connectivity remains a mystery, not to mention the associated computational complexity.

In this paper, we propose approximating the maximal algebraic connectivity by minimizing the "Laplacian energy" defined as follows.

Definition 1: The Laplacian energy of a simple graph $\mathbb{G}$ with n vertices is $E(\mathbb{G}) = \sum_{i=1}^{n} \lambda_i^2$, where λ_i, $i \in \{1, \ldots, n\}$ are eigenvalues of the Laplacian matrix of $\mathbb{G}$.

The above concept was first proposed in [18] and finds applications/connections to ordinary energy for π-electron energy in molecules [19] and the first Zagreb index [20]. It is worth mentioning that there have been various mathematical definitions for network energy [21], including the earliest version of Laplacian energy [22].

We are motivated to appeal to the concept of Laplacian energy for designing fast/optimal consensus network topologies due to the following observations: We list all maximal algebraic connectivity graphs under the constraint of fixed numbers of vertices and edges for the cases where the vertex number n ranges from 4 to 7. These are respectively given in Figures 1 through 4.[1] We omit the case of $n = 3$ as well as some complete graphs, as these graphs are unique. For $n \geq 8$, it will be very computationally expensive to go through all possible graphs. For each of these graphs, we list its corresponding Laplacian energy E, and for each pair of vertex number n and edge number m, we list the minimal Laplacian energy $E_{\min}$ among all possible graphs. It is readily apparent that among all simple graphs with a fixed number of vertices and edges, maximal algebraic connectivity and minimal Laplacian energy coincide in most cases. The non-matching cases, highlighted in orange, are always centered in sparse cases and occasionally scattered in medium-density cases. This suggests that we may design fast consensus topologies by minimizing Laplacian energy for most scenarios. It turns out that, given a fixed number of vertices and edges, minimizing Laplacian energy is a much easier task and considerably more computationally efficient.

In this paper, we first characterize the degree distribution properties of minimal Laplacian energy graphs under the constraint of fixed numbers of vertices and edges, and then devise an algorithm to construct such connected optimal

*The proofs of all assertions in this paper are omitted due to space limitations and can be found in the arXiv version of the paper [1].

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[1]The maximal algebraic connectivity graphs depicted in Figures 2 through 4 are sourced from [17].

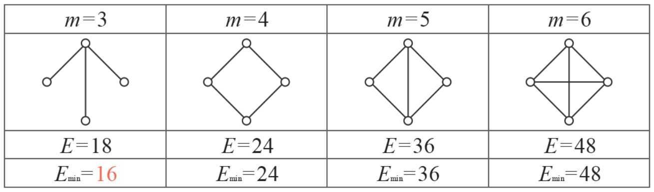

Fig. 1: Maximal algebraic connectivity graphs with 4 vertices

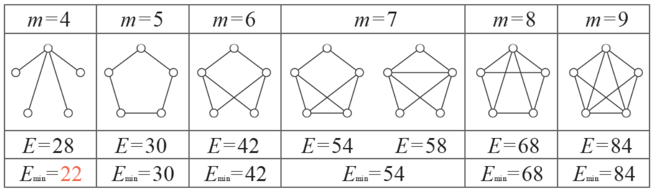

Fig. 2: Maximal algebraic connectivity graphs with 5 vertices

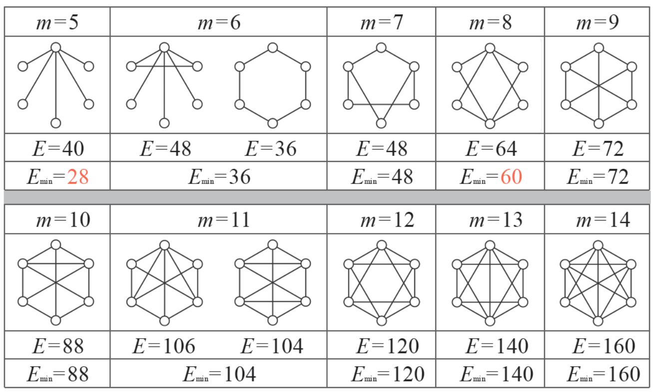

Fig. 3: Maximal algebraic connectivity graphs with 6 vertices

graphs (cf. Section II). Next, we show that the minimal Laplacian energy graphs generated by the proposed algorithm exhibit strong resilience, featuring maximum vertex and edge connectivity (cf. Section III). Finally, we investigate the spectral properties of the Laplacian matrices of these generated minimal Laplacian energy graphs, and show that they generically possess large algebraic connectivity of an optimal order, provided they are not sparse (cf. Section IV). Overall, we propose a computationally efficient approach to designing fast and resilient consensus topologies by minimizing Laplacian energy.

II. Minimal Laplacian Energy

It has been proved in [18] that $E(\mathbb{G}) = \sum_{i=1}^{n}(d_i^2 + d_i)$, where d_i denotes the degree of vertex i. It immediately implies that the Laplacian energy of a simple graph will increase after adding any additional edge. Thus, the Laplacian energy of an n-vertex graph achieves its maximum value, $n^2(n-1)$, when the graph is complete. Various upper and lower bounds on $E(\mathbb{G})$ have been established [23]. There has been an effort in the literature to identify optimal topologies that minimize the Laplacian energy for certain types of graphs. For example, among all n-vertex connected graphs, the Laplacian energy achieves its minimum value on the path [18]. Another example is that among all connected graphs with chromatic number χ, the Laplacian energy achieves its minimum value, $\chi^2(\chi-1)$, on the χ-vertex complete graph [24]. Notwithstanding these results, the following question has never been studied: *Given a fixed number of vertices and edges, what are the optimal graphs that achieve minimal Laplacian energy?*

This section solves the above open problem. To state our main results, we use $\lfloor\cdot\rfloor$ to denote the floor function.

Theorem 1: Among all simple graphs with n vertices and m edges, the minimal Laplacian energy is $(k+1)(4m-nk)$ with $k = \lfloor\frac{2m}{n}\rfloor$, which is achieved if, and only if, $n(k+1)-2m$ vertices are of degree k and the remaining $2m-nk$ vertices are of degree $k+1$.

The theorem states that the sequence of vertex degrees, arranged in descending order, follows the following pattern:

$$(d_1,\dots,d_n) = (\underbrace{k+1,\dots,k+1}_{2m-nk},\ \underbrace{k,\dots,k}_{n(k+1)-2m}) \qquad (1)$$

In the special case when $\frac{2m}{n}$ is an integer, all n vertices are of degree $k = \frac{2m}{n}$. Thus, Theorem 1 implies that minimal Laplacian energy graphs have an (almost) uniform degree distribution, which is intuitive from the fact that $E(\mathbb{G}) = \sum_{i=1}^{n}(d_i^2 + d_i) = \sum_{i=1}^{n} d_i^2 + 2m$. Such a graph, in which the degree difference is at most 1, is called an almost regular graph [25].

It is easy to check that the total degree sum of a minimal Laplacian energy graph specified by Theorem 1 equals $k(n(k+1)-2m) + (k+1)(2m-nk) = 2m$, which is consistent with the assumption of m edges. Moreover, it can be proved that such a degree distribution always admits a graph using the Erdős-Gallai theorem [26], [27].

The proof of Theorem 1 utilizes the following results.

Lemma 1: (Theorem 3 in [18]) For any simple graph $\mathbb{G}$ with n vertices, $E(\mathbb{G}) = \sum_{i=1}^{n}(d_i^2 + d_i)$.

Lemma 2: (Erdős-Gallai Theorem [26], [27]) A nonincreasing sequence of nonnegative integers $d_1,\dots,d_n$ constitutes the degree sequence of an n-vertex simple graph if, and only if, $\sum_{i=1}^{n} d_i$ is even and $\sum_{i=1}^{j} d_i \le j(j-1) + \sum_{i=j+1}^{n}\min\{j, d_i\}$ for all $j \in \{1,\dots,n\}$.

Theorem 1 provides a simple graphical condition, dependent solely on the degree distribution, that characterizes minimal Laplacian energy graphs. The following example shows that such a degree distribution condition does not guarantee connectivity. Consider all simple graphs with 6 vertices and 6 edges (i.e., $n = m = 6$), Theorem 1 identifies two minimal Laplacian energy graphs, as shown in Figure 5.

In many network applications, such as distributed control and optimization [7], [28], connected graphs are often desired. The following theorem states that a connected optimal graph with the same minimal Laplacian energy always exists.

Theorem 2: Among all connected simple graphs with n vertices and m edges, the minimal Laplacian energy is $(k+1)(4m-nk)$ with $k = \lfloor\frac{2m}{n}\rfloor$, which is achieved if, and only

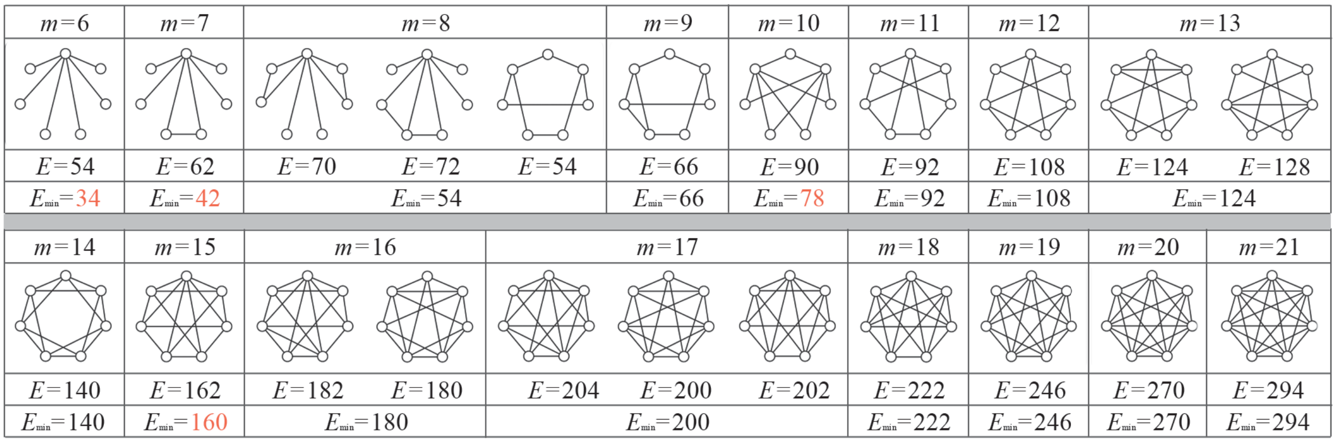

Fig. 4: Maximal algebraic connectivity graphs with 7 vertices

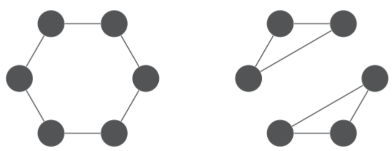

Fig. 5: All minimal Laplacian energy graphs with 6 vertices and 6 edges

if, $n(k+1)-2m$ vertices are of degree k and the remaining $2m-nk$ vertices are of degree $k+1$.

The above theorem implicitly assumes that $m \geq n-1$; otherwise, there is no such connected graph. In the special case when $m=n-1$, all connected graphs are trees. It is not hard to see that the following result is a direct consequence of Theorem 2.

Corollary 1: Among all simple trees with n vertices, the minimal Laplacian energy is $6n-8$, which is achieved on the path.

For general pairs of n and m with $m > n-1$, the optimal connected graph may not be unique. For example, for the case when $n = 6$ and $m = 8$, two connected minimal Laplacian energy graphs are illustrated in Figure 6.

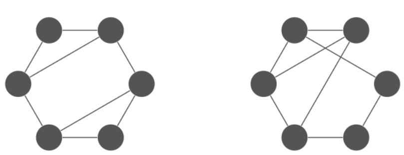

Fig. 6: Two connected minimal Laplacian energy graphs with 6 vertices and 8 edges

We next present the following algorithm to generate a minimal Laplacian energy graph that is connected and satisfies the degree distribution specified by Theorem 2.

Algorithm 1: Given n and m with $m \geq n-1 > 0$, without loss of generality, label n vertices from 1 to n. Set $k = \lfloor \frac{2m}{n} \rfloor$, which implies that $nk \leq 2m < n(k+1)$.

Case 1: The integer k is even.

(1) If $2m = nk$, for each $i \in \{1, \ldots, n\}$, connect vertex i with each of vertices $(i+j) \bmod n$, $j \in \{\pm 1, \ldots, \pm \frac{k}{2}\}$.
(2) If $2m = nk + l$ where $l \in [1, n)$ is an even integer, first construct the graph as done in Case 1 (1), and then for each $i \in \{1, \ldots, \frac{l}{2}\}$, connect vertex i and vertex $i + \lfloor \frac{n}{2} \rfloor$.

Case 2: The integer k is odd.

(1) If n is even and $2m = nk$, for each $i \in \{1, \ldots, n\}$, connect vertex i with each of vertices $(i+j) \bmod n$, $j \in \{\pm 1, \ldots, \pm \frac{k-1}{2}, \frac{n}{2}\}$.
(2) If n is even and $2m = nk + l$ where $l \in [1, n)$ is an even integer, first construct the graph as done in Case 2 (1), and then for each $i \in \{1, \ldots, \frac{l}{2}\}$, connect vertex i and vertex $i + \frac{n-2}{2}$.
(3) If n is odd and $2m = nk + 1$, first for each $i \in \{1, \ldots, n\}$, connect vertex i with each of vertices $(i+j) \bmod n$, $j \in \{\pm 1, \ldots, \pm \frac{k-1}{2}\}$, and then for each $i \in \{1, \ldots, \frac{n+1}{2}\}$, connect vertex i and vertex $i + \frac{n-1}{2}$.
(4) If n is odd and $2m = nk + 1 + l$ where $l \in [1, n-1)$ is an even integer, first construct the graph as done in Case 2 (3), and then for each $i \in \{\frac{n+3}{2}, \ldots, \frac{n+1+l}{2}\}$, connect vertex i and vertex $(i + \frac{n-1}{2}) \bmod n$.

The computational complexity of Algorithm 1 is $O(m)$, as identifying the endpoints for each of the m edges takes $O(1)$ time.

It is worth noting that Algorithm 1 generates a specific subclass of Harary graphs [29].

Theorem 3: Algorithm 1 constructs a connected simple graph with $n(k+1)-2m$ vertices of degree k and $2m-nk$ vertices of degree $k+1$.

It can be straightforwardly checked that in the case when $n = 6$ and $m = 5$, Algorithm 1 will follow Case 2 (2) and construct the 6-vertex path, which is consistent with Corollary 1. We further present six tailored examples, each corresponding to a distinct case outlined in Algorithm 1, as depicted in Figures 7 to 9. These examples collectively validate Theorem 3.

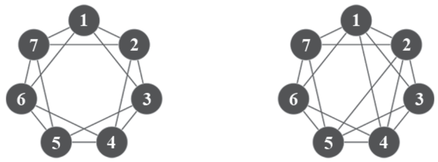

Fig. 7: Left is the graph with 7 vertices and 14 edges, generated by Algorithm 1 following Case 1 (1); right is the graph with 7 vertices and 16 edges, generated by Algorithm 1 following Case 1 (2).

Fig. 8: Left is the graph with 6 vertices and 9 edges, generated by Algorithm 1 following Case 2 (1); right is the graph with 6 vertices and 11 edges, generated by Algorithm 1 following Case 2 (2).

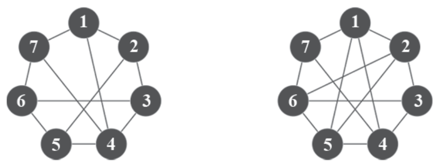

Fig. 9: Left is the graph with 7 vertices and 11 edges, generated by Algorithm 1 following Case 2 (3); right is the graph with 7 vertices and 13 edges, generated by Algorithm 1 following Case 2 (4).

III. Connectivity Resilience

The graphs generated by Algorithm 1 exhibit "optimal" connectivity properties. To see this, we introduce two well-known connectivity concepts in graph theory: vertex connectivity, denoted as $v(\mathbb{G})$, which is defined as the minimum number of vertices whose removal would disconnect graph $\mathbb{G}$, and edge connectivity, denoted as $e(\mathbb{G})$, which is defined as the minimum number of edges whose removal would disconnect graph $\mathbb{G}$. For the complete graph with n vertices, its edge connectivity clearly equals $n-1$. However, there is no subset of vertices whose removal disconnects the complete graph, so it is conventional to set its vertex connectivity as $n-1$ [30, page 149].

Theorem 4: Let $\mathbb{G}$ be the graph generated by Algorithm 1 with n vertices and m edges. Then, $v(\mathbb{G}) = e(\mathbb{G}) = \lfloor \frac{2m}{n} \rfloor$.

The theorem implies that the graphs constructed by Algorithm 1 always have the maximum vertex and edge connectivity. To see this, consider any graph $\mathbb{G}$ with n vertices and m edges. Its minimum degree $\delta(\mathbb{G})$ is at most $\lfloor \frac{2m}{n} \rfloor$. Since $e(\mathbb{G}) \leq \delta(\mathbb{G})$ by definition and $v(\mathbb{G}) \leq e(\mathbb{G})$ [31, Theorem 5], it follows that $v(\mathbb{G}) \leq e(\mathbb{G}) \leq \delta(\mathbb{G}) \leq \lfloor \frac{2m}{n} \rfloor$.

Recall that all graphs generated by Algorithm 1 are almost regular graphs with the degree sequence specified in (1). In general, almost regular graphs do not necessarily have $v(\mathbb{G}) = e(\mathbb{G}) = \lfloor \frac{2m}{n} \rfloor$. To see this, consider two examples in Figure 10. The left almost regular graph has $n = 6$ vertices and $m = 10$ edges, but its vertex connectivity is 2 (by removing vertices 2 and 5), which is less than $\lfloor \frac{2m}{n} \rfloor = 3$. The right almost regular graph has $n = 6$ vertices and $m = 7$ edges, but its vertex connectivity is 1 (by removing vertex 6), so is its edge connectivity (by removing the edge between vertices 5 and 6), which are both less than $\lfloor \frac{2m}{n} \rfloor = 2$.

Fig. 10: Two almost regular graphs

IV. Fast Consensus

In this section, we study the algebraic connectivity of the minimal Laplacian energy graphs generated by Algorithm 1. We will show that the generated "dense" graphs possess large algebraic connectivity, while the generated "sparse" graphs do not. This finding is consistent with the observations from the figures in the introduction.

Among all non-complete graphs with n vertices and m edges, it is known that $a(\mathbb{G}) \leq v(\mathbb{G})$ [11, Theorem 4.1]. From the preceding discussion, it follows that $a(\mathbb{G}) \leq \lfloor \frac{2m}{n} \rfloor$ for all non-complete graphs. Theorem 5 gives a lower bound on algebraic connectivity of graphs generated by Algorithm 1.

Theorem 5: Let $\mathbb{G}$ be the graph generated by Algorithm 1 with n vertices and $m \geq n$ edges. Then,

$$a(\mathbb{G}) \geq \bar{k} - \frac{\sin(\bar{k}\pi/n)}{\sin(\pi/n)}, \qquad (2)$$

where $k = \lfloor \frac{2m}{n} \rfloor$ and $\bar{k} = 2\lfloor \frac{k}{2} \rfloor + 1$, with equality holding if the graph is constructed in Case 1 (1) or Case 2 (1).

Note that $\bar{k} \leq k + 1 \leq \frac{2m}{n} + 1 \leq n$. We will use this fact without special mention in the sequel.

The proof of Theorem 5 makes use of the following concept and results. A circulant matrix is a square matrix in which each row is rotated one entry to the right relative to the preceding row. The spectrum of any circulant matrix can be completely determined by its first row entries, as specified in the following lemma.

Lemma 3: (Theorem 6 in [32]) If C is an $n \times n$ circulant matrix whose first row entries are $c_0, c_1, \ldots, c_{n-1}$, then its n eigenvalues are $\lambda_i = \sum_{p=0}^{n-1} c_p e^{\frac{j2pi\pi}{n}}$, $i \in \{0, 1, \ldots, n-1\}$, where j is the imaginary unit.

Lemma 4: For any integers $n \geq 2$ and $2 \leq k \leq n-2$,

$$\max_{i \in \{1,2,\ldots,n-1\}} 2 \sum_{p=1}^{\lfloor \frac{k}{2} \rfloor} \cos\left(\frac{2pi\pi}{n}\right) = \frac{\sin(\bar{k}\pi/n)}{\sin(\pi/n)} - 1,$$

where $\bar{k} = 2\lfloor \frac{k}{2} \rfloor + 1$, and the maximum is achieved if, and only if, $i = 1$ or $i = n-1$.

From Theorem 5, the graphs constructed by Algorithm 1 in Case 1 (1) and Case 2 (1) have an explicit algebraic connectivity expression, $a(\mathbb{G}) = \bar{k} - \frac{\sin(\bar{k}\pi/n)}{\sin(\pi/n)}$, which can be bounded as follows.

Lemma 5: For any graph $\mathbb{G}$ generated by Algorithm 1 in Case 1 (1) or Case 2 (1),

$$\frac{\pi^2(0.5\bar{k}^3 - \bar{k})}{6n^2 - \pi^2} < a(\mathbb{G}) < \frac{\bar{k}^3\pi^2}{6n^2},$$

where $\bar{k} = 2\lfloor \frac{k}{2} \rfloor + 1$ and $k = \lfloor \frac{2m}{n} \rfloor$.

Let us agree to call a graph with n vertices and m edges sparse if its average degree $\frac{2m}{n}$ is much smaller than $O(n)$, and dense if $\frac{2m}{n} = O(n)$. Consider a generated sparse graph for which $k = \lfloor \frac{2m}{n} \rfloor \leq \sqrt[3]{6n^2/\pi^2} - 1$. Then, $\bar{k} \leq k + 1 \leq \sqrt[3]{6n^2/\pi^2}$. From Lemma 5, $a(\mathbb{G}) < 1$. We have thus proved the following:

Corollary 2: Let $\mathbb{G}$ be any graph generated by Algorithm 1 in Case 1 (1) or Case 2 (1). If $k = \lfloor \frac{2m}{n} \rfloor \leq \sqrt[3]{6n^2/\pi^2} - 1$, then $a(\mathbb{G}) < 1$.

It is known that $a(\mathbb{G}) \geq 2e(\mathbb{G})(1 - \cos(\pi/n))$ [11, Theorem 4.3]. Since $e(\mathbb{G}) \leq n-1$, this lower bound for $a(\mathbb{G})$ is strictly less than 1 if $n \geq 9$. Corollary 2 implies that when the minimal Laplacian energy graph generated by Algorithm 1 in Case 1 (1) or Case 2 (1) is sparse, its algebraic connectivity is small. This suggests that small/minimal Laplacian energy and large/maximal algebraic connectivity do not match for sparse graphs. This observation is not surprising: for instance, in the special case of trees where $m = n-1$, the maximal algebraic connectivity graph is the star, while the minimal Laplacian energy graph is the path, which are opposites. The maximal algebraic connectivity graphs in special sparse cases were theoretically identified in [17, Theorems 1, 2, 4].

In contrast to sparse graphs, the following result shows that generated dense graphs have large algebraic connectivity of an optimal order.

Corollary 3: Let $\mathbb{G}$ be the graph generated by Algorithm 1 with n vertices and m edges. If $k = \lfloor \frac{2m}{n} \rfloor \geq n + 3 - 2\sqrt{n-1}$, then $a(\mathbb{G}) \geq k + 1 - 2\sqrt{k}$.

Among all connected non-complete graphs with n vertices and m edges, it is known that $a(\mathbb{G}) \leq v(\mathbb{G})$ [11, Theorem 4.1]. From the discussion in Section III, $a(\mathbb{G}) \leq v(\mathbb{G}) \leq k$. Note that $n+1-\sqrt{2n-3} = O(n)$. Thus, Corollary 3 implies that when the minimal Laplacian energy graph generated by Algorithm 1 is dense, its algebraic connectivity is large. More can be said. Recall that the graphs generated by Algorithm 1 are almost regular. Theorem 1 and the subsequent paragraph in [33] imply that $a(\mathbb{G}) \leq k + 1 - 2\sqrt{k} + \frac{4\sqrt{k-2}}{\log_k(n) - O(1)}$ for all almost regular graphs with n vertices and m edges [33, Theorem 1]; thus, $\lim_{n\to\infty} a(\mathbb{G}) \leq k+1-2\sqrt{k}$ for all almost regular graphs with $k = \lfloor \frac{2m}{n} \rfloor$. Corollary 3 therefore implies that the dense graphs generated by Algorithm 1 exhibit nearly optimal algebraic connectivity when n is large.

Although both sparse and dense graphs have been analyzed, and the findings are consistent with the observations in the introduction, the scattered non-matching cases for medium-dense graphs (see Figures 3 and 4) have not been addressed. Most of these graphs belong to the class of complete bipartite graphs, which we will study next.

A simple graph is called bipartite if its vertices can be partitioned into two classes so that every edge has endpoints in different classes. The complete bipartite graph $\mathbb{K}_{p,n-p}$ is the bipartite graph with p vertices in one class, $n-p$ vertices in the other class, and all $p(n-p)$ edges between vertices of different classes [34, page 17]. It has been shown in [17, Corollary 1] that $\mathbb{K}_{p,n-p}$ possesses maximal algebraic connectivity if any of the following conditions hold:[2]

A) $3 \leq n \leq 7$ and $1 \leq p \leq n/2$;
B) $n \geq 8$ and $p = 1$;
C) $n \geq 8$ and $(n + \sqrt{n^2 - 8n})/4 \leq p \leq n/2$.

Case A explains the two medium-dense non-matching cases, $(n, m) = (6, 8)$ and $(7, 10)$, as shown in Figures 3 and 4, respectively. The following theorem addresses Case C.

Theorem 6: For complete bipartite graph $\mathbb{K}_{p,n-p}$ with $n \geq 8$ and $(n + \sqrt{n^2 - 8n})/4 \leq p \leq n/2$, there hold $E(\mathbb{K}_{p,n-p}) - E_{\min} \leq n/4$, $E_{\max} - E_{\min} > n^3/10 - n^2 - 4n$, and $\frac{E(\mathbb{K}_{p,n-p}) - E_{\min}}{E_{\max} - E_{\min}} < \frac{5}{2n^2 - 20n - 80}$, where $E_{\min}$ and $E_{\max}$ denote the minimum and maximum Laplacian energy, respectively, among all simple graphs with n vertices and $p(n-p)$ edges.

The theorem implies that while the length of the interval $[E_{\min}, E_{\max}]$, which covers all possible Laplacian energy levels, grows at least on the order of n^3, the Laplacian energy of $\mathbb{K}_{p,n-p}$ is at most $n/4$ larger than $E_{\min}$. Thus, complete bipartite graphs $\mathbb{K}_{p,n-p}$ with large or even maximal algebraic connectivity tend to have relatively low Laplacian energy compared to other graphs with the same number of vertices and edges, under the theorem condition.

It is worth emphasizing that the remaining medium-density non-matching graph, namely $(n, m) = (7, 15)$ shown in Figure 4, is not a complete bipartite graph. Further investigating such scattered medium-density non-matching cases will be a critical step toward comprehensively constructing all optimal graphs with maximal algebraic connectivity.

V. Discussion

The graphs constructed by Algorithm 1 in Case 1 (1) actually belong to the so-called regular lattices. A simple graph with $n \geq 3$ vertices is called a d-regular lattice, with d being an even integer in the interval $[2, n-1]$, if each vertex i is adjacent to each of those vertices whose indices

[2] Since p is an integer, the condition $p \leq n/2$ is equivalent to $p \leq \lfloor n/2 \rfloor$, which was used in [17, Corollary 1].

are $(i+j) \bmod n$, $j \in \{\pm 1, \ldots, \pm \frac{d}{2}\}$ [35]. It is easy to see that any graph with $n \geq 3$ vertices generated by Algorithm 1 in Case 1 (1) is a k-regular lattice. Lemma 5 immediately implies that the algebraic connectivity of a k-regular lattice is of the order $O(k^3/n^2)$.

Regular lattices are closely related to Watts-Strogatz small-world networks, which are generated by randomly rewiring edges in a regular lattice. The rewiring procedure involves iterating through each edge, and with probability p, one endpoint is moved to a new vertex chosen randomly from the lattice. Double edges and self-loops are not allowed in this process, so small-world networks are simple graphs [35].

The work of [36] defines the algebraic connectivity gain, $\lambda_2(p)/\lambda_2(0)$, as the algebraic connectivity of the small-world network formed by rewiring with probability p divided by the algebraic connectivity of the regular lattice [36, Definition 1]. Through simulations with $k \approx \log(n)$, the paper conjectures that the maximum $\lambda_2(p)/\lambda_2(0)$ is on the order of $O(n)$ [36, Observation (ii), page 4], which implies that (sparse) small-world networks can reach consensus significantly faster than regular lattices.

Lemma 5 implies that $\lambda_2(0)$ is on the order of $O(k^3/n^2)$. Thus, if one can show that $\lambda_2(p)$ is on the order of $O(k^3/n)$, then the conjecture in [36] would be mathematically verified. Whether small-world networks can achieve algebraic connectivity of order $O(k^3/n)$ has so far eluded us, but Lemma 5 provides a helpful first step in addressing this question.

In summary, this paper proposes a novel approach to designing fast consensus topologies by minimizing Laplacian energy, marking the first step in this direction.

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