Safe Physics-Informed Machine Learning for Optimal Predefined-Time Stabilization: A Lyapunov-Based Approach

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***Abstract*—In this article, we introduce the notion of safe predefined-time stability and address an optimal safe predefined-time stabilization problem. In particular, safe predefined-time stability characterizes parameter-dependent nonlinear dynamical systems whose trajectories starting in a given set of admissible states remain in the set of admissible states for all time and converge to an equilibrium point in a predefined time. Furthermore, we provide a Lyapunov theorem establishing sufficient conditions for safe predefined-time stability. We address the optimal safe predefined-time stabilization problem by synthesizing feedback controllers that guarantee closed-loop system safe predefined-time stability while optimizing a given performance measure. Specifically, safe predefined-time stability of the closed-loop system is guaranteed via a Lyapunov function satisfying a differential inequality while simultaneously serving as a solution to the steady-state Hamilton-Jacobi–Bellman (HJB) equation ensuring optimality. Given that the HJB equation is generally difficult to solve, we develop a physics-informed machine learning-based algorithm for learning the safely predefined-time stabilizing solution to the steady-state HJB equation. Several simulation results are provided to demonstrate the efficacy of the proposed approach.**

***Index Terms*—Optimal feedback control, physics-informed neural networks (PINNs), predefined-time stability, safety-critical control.**

I. Introduction

IN CONTROL systems engineering, the term *autonomy* concerns controlled systems that can function without involving a supervisor [1]. Systems featuring this property are commonly referred to as intelligent autonomous systems (IASs) including drones, humanoid robots, and self-driving cars, to mention but a few examples. However, there have been many reported cases wherein IASs have accidentally crashed, demonstrating that IASs are *safety-critical systems*. Consequently, ensuring safety becomes a necessity, leading to the emergence of *safe autonomy* [2]. To enable safe autonomy, the control systems community can leverage the benefits of *nonlinear* [3] and *optimal control theory* [4] to endow IASs with control mechanisms guaranteeing safety, stability, and performance. To ensure the efficient and safe operation of IASs in view of an uncertain and dynamic environment, it is necessary for the decision-making mechanism to generate optimal safe policies that allow for adaptability in a predefined time rather than in an infinite or finite time.

Optimal control theory deals with finding a control law for a given dynamical system so that a user-prescribed cost functional is optimized [4]. In the infinite horizon optimal control problem, the notions of *optimality* and *asymptotic stability* are intertwined [5]. Specifically, the optimal control policy is a state feedback control law ensuring closed-loop system asymptotic stability while minimizing a given cost functional. The concept of *asymptotic stability* in dynamical systems enables the convergence of system trajectories to a Lyapunov stable equilibrium point over the infinite horizon [3]. In contrast, the notion of *finite-time stability* allows the convergence of the system solutions to a Lyapunov stable equilibrium state in finite-time [6]. The problem of *optimal finite-time stabilization* [7] merges the notions of *finite-time stability* and *optimality* characterizing feedback controllers that optimize a given cost functional while establishing finite-time stability of the closed-loop system. A key limitation of finite-time stability is that the settling-time function is not necessarily uniformly upper-bounded, whereas the stronger notion of *fixed-time stability* involves a *uniformly* upper-bounded settling-time function, which implies the convergence of the system trajectories to a finite-time stable equilibrium point in a fixed-time [8], [9]. However, the upper bound of the settling-time function may not necessarily be predefined. Alternatively, the concept of *predefined-time stability* [10], [11] involves fixed-time stable parameter-dependent dynamical systems whose upper bound of the settling-time function can be chosen via an appropriate selection of the system param-

Received 12 March 2024; revised 21 November 2024; accepted 13 March 2025. This work was supported in part by NSF under Grant CAREER CPS-1851588, Grant CPS-2227185, Grant S&AS-1849198, and Grant CPS-2038589; and in part by Onassis Foundation Scholarship under Grant F ZR 025-1/2021-2022. The work of Zhen Zhang, Shanqing Liu, Jérôme Darbon, and George Em Karniadakis was supported in part by the Office of the Secretary of Defense (OSD)/Air Force Office of Scientific Research (AFOSR) Multidisciplinary University Research Initiatives (MURI) under Grant FA9550-20-1-0358. *(Corresponding author: Nick-Marios T. Kokolakis.)*

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Digital Object Identifier 10.1109/TNNLS.2025.3554713

eters. The *optimal predefined-time stabilization* problem [12] brings together the concepts of *predefined-time stability* and *optimality* to synthesize feedback control laws guaranteeing closed-loop system predefined-time stability while optimizing a given cost functional. In light of the above, while the aforementioned control architectures ensure optimality and stability, safety is not a design consideration.

Safety-critical control theory involves the analysis and synthesis of a controller for a safety-critical dynamical system to ensure the satisfaction of safety specifications [13]. Safety specifications can be expressed as forward invariance [3] of a set of safe system states. *Control barrier functions* (CBF) have been widely employed for ensuring the safety of a control system by rendering a safe set forward invariant [14], [15], [16]. Romdlony and Jayawardhana [17] proposed a control method predicated on a control Lyapunov–Barrier function (CLBF) that addresses the problem of *asymptotic* stabilization with guaranteed safety by merging a control Lyapunov function (CLF) [18] and a CBF [14]. However, it is shown in [19] that a CLBF cannot exist. Alternatively, quadratic programming (QP) has been used to combine a CLF and a CBF to construct controllers for safe *asymptotic* stabilization of nonlinear systems [20], [21], [22]. However, CBF-based QPs introduce undesirable *asymptotically* stable equilibria [23], [24], do not optimize the closed-loop system performance, and do not incorporate time constraints. To the best of our knowledge, a control architecture that *simultaneously* ensures *safety*, *predefined-time stability*, and *optimality* is absent from the literature.

To derive the optimal control policy for an *optimal stabilization problem*, one needs first to determine the optimal cost function (value function), which is a *stabilizing* solution to a nonlinear partial differential equation, the steady-state Hamilton–Jacobi–Bellman (HJB) equation, which is generally intractable aside from special cases [25]. *Adaptive dynamic programming* (ADP) [26], [27], [28], [29], [30], [31], [32], [33], [34], [35] unifies *optimal* and *adaptive* [36] control by designing adaptive learning algorithms to learn a solution to the steady-state HJB online via data measured along the system trajectories. Most adaptive learning algorithms [37], [38] converge to a near-optimal control if a *persistence of excitation* (PE) [36] condition is satisfied. Alternatively, *concurrent learning/experience replay*-based ADP algorithms [39], [40] enable the learning of a solution to an optimal stabilization problem by requiring a weaker form of a PE condition to be satisfied to compose a sufficiently rich dataset [41], [42], [43]. Building on these results, *safe experience replay-based reinforcement learning* architectures have been developed [44], [45], [46], [47], [48], [49] assuming a sufficiently rich dataset is given, for which, however, there are no sufficient conditions guaranteeing its existence for a nonlinear dynamical system [36].

Physics-informed neural networks (PINNs), initially introduced in [50], showcase the capability of learning a solution to the steady-state HJB equation associated with the *optimal asymptotic stabilization* problem [51]. One challenge in applying PINNs for approximately solving the steady-state HJB equation lies in the presence of multiple solutions. Fotiadis and Vamvoudakis [52] overcome this issue by proposing a learning framework applying PINNs to a finite-horizon variant of the steady-state HJB equation with a unique solution, which uniformly approximates the infinite-horizon optimal cost function as the horizon increases. Nevertheless, the fact that the value function is a Lyapunov function and is the *unique* asymptotically stabilizing solution to the steady-state HJB equation is not considered in the aforementioned learning frameworks. To the best of our knowledge, a physics-informed learning architecture approximating the unique stabilizing solution to the steady-state HJB is absent from the literature.

Contributions: The contributions of the present article are fourfold. First, the notion of *safe predefined-time stability* for general parameter-dependent nonlinear dynamical systems is introduced. Then, sufficient conditions for safe predefined-time stability are given in terms of a Lyapunov function. Subsequently, an *optimal safe predefined-time stabilization* problem is addressed, and sufficient conditions for characterizing an optimal nonlinear feedback controller ensuring safe predefined-time stability of the closed-loop system are provided. Finally, a *physics-informed machine learning*-based algorithm is developed for learning the solution to the optimal safe predefined-time stabilization problem.

Structure: The remainder of the article is structured as follows. Section II defines the notion of safe predefined-time stability for general parameter-dependent nonlinear dynamical systems, whereas the optimal safe predefined-time stabilization problem is stated in Section III. In Section IV, the optimal and inverse optimal safe predefined-time stabilization problem tailored to parameter-dependent nonlinear affine dynamical systems is introduced. Section V develops a physics-informed machine learning-based algorithm for learning the solution to the optimal safe predefined-time stabilization problem. Section VI presents two illustrative numerical examples. Finally, Section VII provides conclusions and outlines future research directions.

Notation: The notation used in this article is standard. Specifically, $\|\cdot\|_p \triangleq \left[\sum_{i=1}^{n} |x_i|^p\right]^{1/p}$, $1 \leq p < \infty$, denotes the ℓ^p-norm of a vector. We interchangeably use the notation $V'(x)$ and $V_x(x)$ to denote the gradient of a scalar-valued function V with respect to a vector-valued variable x, which is defined as a row vector. The signum function $\mathrm{sgn}\colon \mathbb{R} \to \{-1, 0, 1\}$ is defined as $\mathrm{sgn(x)} \triangleq \mathrm{x}/|\mathrm{x}|, \mathrm{x} \neq 0$, and $\mathrm{sgn}(0) \triangleq 0$. We define the gamma function $\Gamma(\cdot)$ as $\Gamma(x) \triangleq \int_0^\infty e^{-t} t^{x-1} \mathrm{d}t$, $x > 0$. The indicator function of a set $A \subseteq \mathbb{R}^n$ is the function $\mathbb{1}_A\colon \mathbb{R}^n \to \{0, 1\}$ defined by $\mathbb{1}_A(x) \triangleq 1, x \in A$, and $\mathbb{1}_A(x) \triangleq 0, x \notin A$. Let $\lceil\cdot\rfloor^\eta \triangleq |\cdot|^\eta \mathrm{sgn}(\cdot)$, where $|\cdot|$ and $\mathrm{sgn}(\cdot)$ operate componentwise and $\eta > 0$. The distance of a point $x_0 \in \mathbb{R}^n$ to a closed set $C \subseteq \mathbb{R}^n$ in the norm $\|\cdot\|$ is defined as $\mathrm{dist(x_0, C)} \triangleq \inf_{\mathrm{x \in C}}\{\|\mathrm{x_0 - x}\|\}$. Finally, the notation $\partial\mathcal{S}$ denotes the boundary of the set $\mathcal{S}$.

II. Safe Predefined-Time Stability

In this section, we define the notion of *safe predefined-time stability* to characterize a class of nonlinear dynamical systems with the property that every trajectory starting in a given set of admissible states containing an equilibrium point remains in the set of admissible states for all time and converges to the equilibrium point in a predefined time. Moreover, we provide sufficient conditions for safe predefined-time stability in terms of a Lyapunov function.

To define the notions of *finite-* and *fixed-time stability*, consider the *parameter-independent* nonlinear dynamical system given by

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \; t \geq 0 \tag{1}$$

where $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n, t \geq 0$, is a system state vector, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $f\colon \mathcal{D} \to \mathbb{R}^n$ is continuous on $\mathcal{D}$ and satisfies $f(0) = 0$.

Definition 1 [6]: The equilibrium point $x_e = 0$ of (1) is *finite-time stable* if it is Lyapunov stable and finite-time convergent, i.e., for all $x(0) \in \mathcal{N}\backslash\{0\}$, where $\mathcal{N} \subseteq \mathcal{D}$ is an open neighborhood of the origin, $\lim_{t\to T(x(0))} x(t) = 0$, where $T(\cdot)$ is the settling-time function such that $T(x(0)) < \infty, x(0) \in \mathcal{N}$. The equilibrium point $x_e = 0$ of (1) is *globally finite-time stable* if it is finite-time stable with $\mathcal{N} = \mathcal{D} = \mathbb{R}^n$. □

Definition 2 [9]: The equilibrium point $x_e = 0$ of (1) is *fixed-time stable* if it is finite-time stable and the settling-time function $T(\cdot)$ is uniformly bounded, i.e., there exists $T_{\max} > 0$ such that $T(x(0)) \leq T_{\max}, x(0) \in \mathcal{N}$. The equilibrium point $x_e = 0$ of (1) is *globally fixed-time stable* if it is fixed-time stable with $\mathcal{N} = \mathcal{D} = \mathbb{R}^n$. □

The key difference between the notions of finite- and fixed-time stability arises from the fact that in finite-time stability the settling-time function is not necessarily uniformly upper-bounded unlike fixed-time stability involving a uniformly upper-bounded settling-time function.

Alternatively, to define the notion of *predefined-time stability*, consider the *parameter-dependent* nonlinear dynamical system given by

$$\dot{x}(t) = f(x(t), \theta), \quad x(0) = x_0, \; t \geq 0 \tag{2}$$

where for every $t \geq 0$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$ is the system state vector, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $\theta \in \mathbb{R}^N$ is a system parameter vector, $f\colon \mathcal{D}\times\mathbb{R}^N \to \mathbb{R}^n$ is such that $f(\cdot, \theta)$ is continuous on $\mathcal{D}$ for all $\theta \in \mathbb{R}^N$ and $f(0, \cdot) = 0$. We write $s(t, x_0, \theta), t \geq 0$, to denote the solution to (2) with initial condition x_0 and system parameter θ.

Definition 3 [11]: The equilibrium point $x_e = 0$ of (2) is *predefined-time stable* with a predefined time $T_p > 0$ if there exists a system parameter vector $\theta \in \mathbb{R}^N$ such that the equilibrium point $x_e = 0$ of (2) is fixed-time stable with the settling-time function $T(\cdot, \theta)$ being uniformly bounded by T_p, i.e., $T(x(0), \theta) \leq T_p, x(0) \in \mathcal{N}$. The equilibrium point $x_e = 0$ of (2) is *globally predefined-time stable* with a predefined time $T_p > 0$ if it is predefined stable with a predefined time $T_p > 0$ with $\mathcal{N} = \mathcal{D} = \mathbb{R}^n$. □

The key difference between the notions of fixed- and predefined-time stability lies in the fact that in predefined-time stability, the upper bound of the settling-time function is defined a priori as an explicit function of the system parameters. In contrast, the upper bound of the settling-time function associated with fixed-time stability is not necessarily predefined. In essence, fixed-time stability concerns the stability of a parameter-independent nonlinear system, whereas predefined-time stability involves parameter-dependent nonlinear systems.

Lyapunov theorems for finite-, fixed-, and predefined-time stability are given in [6], [9], and [11], respectively. However, for completeness of exposition, the next theorem gives sufficient conditions for predefined-time stability of the nonlinear dynamical system given by (2).

Theorem 1 [11]: Consider the nonlinear dynamical system (2). Let $T_p > 0$ be a predefined time. Assume that there exist a continuously differentiable function $V\colon \mathcal{D} \to \mathbb{R}$, a system parameter vector $\theta \in \mathbb{R}^N$, a neighborhood $\mathcal{N} \subseteq \mathcal{D}$ of the origin, and real numbers $\alpha, \beta, p, q, r > 0$ such that $pr < 1, qr > 1$, and

$$V(0) = 0$$
$$V(x) > 0, \quad x \in \mathcal{N}\backslash\{0\}$$
$$V'(x)f(x,\theta) \leq -\frac{\gamma}{T_p}\left(\alpha V^p(x) + \beta V^q(x)\right)^r, \quad x \in \mathcal{N}$$

where

$$\gamma \triangleq \frac{\Gamma\left(\frac{1-rp}{q-p}\right)\Gamma\left(\frac{rq-1}{q-p}\right)}{\alpha^r\Gamma(r)(q-p)}\left(\frac{\alpha}{\beta}\right)^{\frac{1-rp}{q-p}}.$$

Then, the equilibrium point $x_e = 0$ of (2) is predefined-time stable with predefined time T_p. If, in addition, $\mathcal{D} = \mathcal{N} = \mathbb{R}^n$ and $V(\cdot)$ is radially unbounded, then the equilibrium point $x_e = 0$ of (2) is globally predefined-time stable with predefined time T_p.

Remark 1: Note that the system parameter vector $\theta \in \mathbb{R}^N$ is constant and implicitly depends on the predefined time T_p. □

A set of *admissible states* $\mathcal{S} \subset \mathcal{D}$ is called a *safe set* with respect to the parameter-dependent nonlinear dynamical system (2) if there exists a system parameter $\theta \in \mathbb{R}^N$ such that, for every $x_0 \in \mathcal{S}$, the solution $s(t, x_0, \theta), t \geq 0$, to (2) satisfies $\mathrm{dist}(\mathrm{s}(\mathrm{t}, \mathrm{x}_0, \theta), \mathcal{S}) \equiv 0$. In other words, the set $\mathcal{S}$ is safe if there exists a system parameter vector such that $\mathcal{S}$ is positively invariant with respect to (2).

The following definition introduces the notion of *safe predefined-time stability*.

Definition 4: Let $\mathcal{S} \subset \mathcal{D}$ be a set of admissible states with $0 \in \mathcal{S}$ and let $T_p > 0$ be a predefined time. The equilibrium point $x_e = 0$ of (2) is *safely predefined-time stable* with predefined time T_p with respect to the set of admissible states $\mathcal{S}$ if there exists a system parameter $\theta \in \mathbb{R}^N$ such that, for every $x_0 \in \mathcal{S}$, the solution $s(t, x_0, \theta), t \geq 0$, to (2) satisfies $\mathrm{dist}(\mathrm{s}(\mathrm{t}, \mathrm{x}_0, \theta), \mathcal{S}) \equiv 0$ and $s(t, x_0, \theta) = 0$ for all $t \geq T_p$. □

Remark 2: Note that safe predefined-time stability unifies safety and predefined-time stability since positive invariance of the set of admissible states and predefined-time stability of the origin are simultaneously established. Furthermore, note that the system parameter vector θ implicitly depends on the predefined time T_p and the set of admissible states $\mathcal{S}$. □

The following key lemma is needed for the main result of this section. First, however, we give the definition of a *coercive* function.

Definition 5 [53]: Let $\mathcal{S} \subseteq \mathbb{R}^n$ be an unbounded set. A function $V : \mathcal{S} \to \mathbb{R}$ is called *coercive* if, for every sequence $\{x_k\}_{k=1}^{\infty}$ in $\mathcal{S}$ such that $\lim_{k\to\infty} \|x_k\| = \infty$, we have $\lim_{k\to\infty} V(x_k) = \infty$. □

Lemma 1: Let $\mathcal{S} \subset \mathcal{D}$ be a set of admissible states. Suppose that there exists a continuous function $V : \mathcal{S} \to \mathbb{R}$ such that

$$V(0) = 0 \tag{3}$$

$$V(x) > 0, \quad x \in \mathcal{S}\backslash\{0\} \tag{4}$$

$$V(x) \to \infty \text{ as } x \to \partial\mathcal{S}. \tag{5}$$

If either $\mathcal{S}$ is bounded or both $\mathcal{S}$ is unbounded and $V(\cdot)$ is coercive, then the set $\mathcal{S}_c \triangleq \{x \in \mathcal{S} : V(x) \leq c\}$ is compact and is contained in $\mathcal{S}$ for all $c > 0$.

Proof: First, suppose that $\mathcal{S}$ is bounded. Note that, for all $c > 0$, $\mathcal{S}_c$ is compact if it is closed and bounded. By continuity of $V(\cdot)$, $\mathcal{S}_c$ is closed for all $c > 0$. Furthermore, $\mathcal{S}_c$ is bounded for all $c > 0$ since $\mathcal{S}$ is bounded and $\mathcal{S}_c$ is a subset of $\mathcal{S}$. Hence, $\mathcal{S}_c$ is compact for all $c > 0$.

Next, suppose that $\mathcal{S}$ is unbounded and $V(\cdot)$ is coercive. Using an identical argument as above, it can be shown that $\mathcal{S}_c$ is closed for all $c > 0$. Now, to show boundedness, suppose, *ad absurdum*, that there exists $c^\star > 0$ such that $\mathcal{S}_{c^\star}$ is unbounded, which implies that there exists $x^\star \in \mathcal{S}_{c^\star}$ such that $\|x^\star\| \to \infty$ and $V(x^\star) \leq c^\star$, which leads to a contradiction. Hence, $\mathcal{S}_c$ is bounded for all $c > 0$, which implies that $\mathcal{S}_c$ is compact. ■

The next theorem gives sufficient conditions for safe predefined-time stability of a parameter-dependent nonlinear dynamical system.

Theorem 2: Consider the parameter-dependent nonlinear dynamical system (2). Let $\mathcal{S} \subset \mathcal{D}$ be a set of admissible states with $0 \in \mathcal{S}$ and let $T_p > 0$ be a predefined time. Assume that there exist a continuously differentiable function $V : \mathcal{S} \to \mathbb{R}$, a system parameter vector $\theta \in \mathbb{R}^N$, and real numbers $\alpha, \beta, p, q, r > 0$ such that $pr < 1, qr > 1$, and

$$V(0) = 0 \tag{6}$$

$$V(x) > 0, \quad x \in \mathcal{S}\backslash\{0\} \tag{7}$$

$$V(x) \to \infty \text{ as } x \to \partial\mathcal{S} \tag{8}$$

$$V'(x)f(x,\theta) \leq -\frac{\gamma}{T_p}\left(\alpha V^p(x) + \beta V^q(x)\right)^r, \quad x \in \mathcal{S} \tag{9}$$

where

$$\gamma \triangleq \frac{\Gamma\left(\frac{1-rp}{q-p}\right)\Gamma\left(\frac{rq-1}{q-p}\right)}{\alpha^r\Gamma(r)(q-p)}\left(\frac{\alpha}{\beta}\right)^{\frac{1-rp}{q-p}}. \tag{10}$$

If either $\mathcal{S}$ is bounded or both $\mathcal{S}$ is unbounded and $V(\cdot)$ is coercive, then the equilibrium point $x_e = 0$ of (2) is safely predefined-time stable with predefined time T_p with respect to the set of admissible states $\mathcal{S}$.

Proof: Let $x_0 \in \mathcal{S}$ and let $x(t), t \geq 0$, be the solution to (2). Let $c_{x_0} > 0$ be such that $V(x_0) \leq c_{x_0}$ and define

$$\mathcal{S}_{c_{x_0}} \triangleq \left\{x \in \mathcal{S} : V(x) \leq c_{x_0}\right\}. \tag{11}$$

Note that $\mathcal{S}_{c_{x_0}}$ contains 0 and x_0 by definition and is a compact subset of $\mathcal{S}$ by Lemma 1. Furthermore, it follows from (9) that $\dot{V}(x) \leq 0$ for all $x \in \mathcal{S}_{c_{x_0}}$, which implies that $V(x(t)) \leq V(x_0) \leq c_{x_0}$ for all $t \geq 0$, and hence, $\mathrm{dist}(x(t), \mathcal{S}_{c_{x_0}}) \equiv 0$, which shows that $\mathcal{S}_{c_{x_0}}$ is a compact positively invariant set with respect to (2). Thus, it follows that for every $x_0 \in \mathcal{S}$, there exists a compact positively invariant set $\mathcal{S}_{c_{x_0}} \subset \mathcal{S}$ given by (11), which implies that $\mathcal{S}$ is a positively invariant set with respect to (2).

Finally, Theorem 1, (6), (7), and (9) imply predefined-time stability of the equilibrium point $x_e = 0$ of (2), which along with the positive invariance of $\mathcal{S}$, implies safe predefined-time stability of the equilibrium point $x_e = 0$ of (2). ■

A continuously differentiable function $V(\cdot)$ satisfying (6)–(8) is called a *safely predefined-time stabilizing Lyapunov function candidate* for the nonlinear dynamical system (2). If, additionally, $V(\cdot)$ satisfies (9), $V(\cdot)$ is called a *safely predefined-time stabilizing Lyapunov function* for the nonlinear dynamical system (2).

Remark 3: Theorem 2 examines safe predefined-time stability for the nonlinear dynamical system (2) without knowledge of the system trajectories. □

III. Optimal Safe Predefined-Time Stabilization

In this section, we address the problem of characterizing optimal feedback controllers that render the equilibrium point of the closed-loop system safely predefined-time stable while optimizing the closed-loop system performance. Specifically, we provide sufficient conditions for nonlinear system optimal safe predefined-time stabilization.

Consider the controlled parameter-dependent nonlinear dynamical system given by

$$\dot{x}(t) = F(x(t), \theta_s, u(t)), \quad x(0) = x_0, \ t \geq 0 \tag{12}$$

where for every $t \geq 0$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$ is the state vector, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $\theta_s \in \mathbb{R}^{N_s}$ is the system parameter vector, $u(t) \in U \subseteq \mathbb{R}^m$ is the control input with $0 \in U$, $F\colon \mathcal{D}\times\mathbb{R}^{N_s}\times U \to \mathbb{R}^n$ is such that $F(\cdot, \theta_s, \cdot)$ is jointly continuous on $\mathcal{D}\times U$ for all $\theta_s \in \mathbb{R}^{N_s}$ and $F(0, \cdot, 0) = 0$. The control $u(\cdot)$ in (12) belongs to the class of *admissible* controls $\mathcal{U} \triangleq \{u : [0, \infty) \to U : u(\cdot)$ is Lebesgue measurable$\}$. We assume that the required properties for the existence and uniqueness of solutions to (12) are satisfied, and we write $s(t, x_0, \theta_s, u(\cdot)), t \geq 0$, to denote the solution to (12) with initial condition x_0, system parameter θ_s, and admissible control $u(\cdot)$.

A mapping $u^\star\colon \mathcal{S}\times\mathbb{R}^{N_c} \to U$ such that $u^\star(\cdot, \theta_c)$ is a Lebesgue measurable function on $\mathcal{S}$ for every control parameter vector $\theta_c \in \mathbb{R}^{N_c}$ and $u^\star(0, \cdot) = 0$ is called a *control law*. Moreover, If $u(t) = u^\star(x(t), \theta_c), t \geq 0$, where $u^\star(\cdot, \cdot)$ is a control law and $x(t)$ is a solution to (12), then we call $u(\cdot)$ a *feedback control law*. Note that a feedback control law is an admissible control since $u^\star(\cdot, \cdot)$ takes values in U and is Lebesgue measurable. Given a control law $u^\star(\cdot, \cdot)$ and a feedback control law $u(t) = u^\star(x(t), \theta_c), t \geq 0$, the closed-loop system is given by

$$\dot{x}(t) = F\left(x(t), \theta_s, u^\star\left(x(t), \theta_c\right)\right), \quad x(0) = x_0, \ t \geq 0 \tag{13}$$

which can be cast in the form of (2) with $N = N_s + N_c$, $\theta = [\theta_s^{\mathrm{T}}, \theta_c^{\mathrm{T}}]^{\mathrm{T}}$, and $f(x, \theta) = F(x, \theta_s, u^\star(x, \theta_c))$.

We now define the notion of a *safely predefined-time stabilizing feedback control law*.

Definition 6: Consider the controlled parameter-dependent nonlinear dynamical system given by (12). Let $\mathcal{S} \subset \mathcal{D}$ be a set of admissible states with $0 \in \mathcal{S}$ and let $T_p > 0$ be a predefined time. The feedback control law $u(\cdot) = u^\star(x(\cdot), \theta_c)$ is *safely predefined-time stabilizing* if there exists a system parameter $\theta_s \in \mathbb{R}^{N_s}$ such that the equilibrium point $x_e = 0$ of the closed-loop system (13) is safely predefined-time stable with predefined time T_p with respect to the set of admissible states $\mathcal{S}$. □

Given a set of admissible states $\mathcal{S} \subset \mathcal{D}$ with $0 \in \mathcal{S}$ and a predefined time $T_p > 0$, we define for every $x_0 \in \mathcal{S}$ the set of safely predefined-time stabilizing feedback controllers by $\mathcal{F}(x_0, \mathcal{S}, T_p) \triangleq \{u : [0, \infty) \to U : u(\cdot)$ is a feedback control law and $s(t, x_0, \theta_s, u(\cdot)), t \geq 0$, is a solution to (12) satisfying $\mathrm{dist}(s(t, x_0, \theta_s, u(\cdot)), \mathcal{S}) \equiv 0$ and $s(t, x_0, \theta_s, u(\cdot)) = 0$ for all $t \geq T_p\} \subset \mathcal{U}$.

To assess the performance of the controlled parameter-dependent nonlinear dynamical system (12) over the time interval $[0, T_p]$ for a given predefined time $T_p > 0$ and a set of admissible states $\mathcal{S}$ with $0 \in \mathcal{S}$, we define for every $x_0 \in \mathcal{S}, \theta_s \in \mathbb{R}^{N_s}$, and $u(\cdot) \in \mathcal{U}$ the cost functional

$$J(x_0, \theta_s, u(\cdot)) \triangleq \int_0^{T_p} L(x(t), u(t))\mathrm{d}t \tag{14}$$

where $L\colon \mathcal{S} \times U \to \mathbb{R}$ is jointly continuous in x and u.

In light of the above, we now state the optimal safe predefined-time stabilization problem.

Problem 1: Consider the controlled parameter-dependent nonlinear dynamical system given by (12) with the cost functional (14). Let $\mathcal{S} \subset \mathcal{D}$ be a set of admissible states with $0 \in \mathcal{S}$ and let $T_p > 0$ be a predefined time. For every $x_0 \in \mathcal{S}$, let $\mathcal{F}(x_0, \mathcal{S}, T_p) \subset \mathcal{U}$ be the set of safely predefined-time stabilizing feedback controllers and suppose that $\mathcal{F}(x_0, \mathcal{S}, T_p)$ is nonempty. Then, for every initial condition $x_0 \in \mathcal{S}$, determine $u^\star(\cdot) \in \mathcal{F}(x_0, \mathcal{S}, T_p)$ such that the equilibrium point $x_e = 0$ of the closed-loop system (13) is safely predefined-time stable with predefined time T_p with respect to the set of admissible states $\mathcal{S}$ and the cost functional (14) is minimized. □

The optimal safe predefined-time stabilization problem involves the minimization

$$\min_{u(\cdot) \in \mathcal{F}(x_0, \mathcal{S}, T_p)} J(x_0, \theta_s, u(\cdot)), \quad x_0 \in \mathcal{S}$$

subject to (12).

In the sequel, we present a theorem providing sufficient conditions for the existence of a safely predefined-time stabilizing feedback control law solving the optimal safe predefined-time stabilization problem. For the statement of this result, we need to define the Hamiltonian function

$$\begin{aligned} &H(x, \theta_s, u, \lambda) \triangleq L(x, u) + \lambda^{\mathrm{T}} F(x, \theta_s, u) \\ &(x, \theta_s, u, \lambda) \in \mathcal{S} \times \mathbb{R}^{N_s} \times U \times \mathbb{R}^n. \end{aligned} \tag{15}$$

Theorem 3: Consider the controlled parameter-dependent nonlinear dynamical system given by (12) with the cost functional (14). Let $\mathcal{S} \subset \mathcal{D}$ be a set of admissible states with $0 \in \mathcal{S}$ and let $T_p > 0$ be a predefined time. For every $x_0 \in \mathcal{S}$, let $\mathcal{F}(x_0, \mathcal{S}, T_p) \subset \mathcal{U}$ be the set of safely predefined-time stabilizing feedback controllers and suppose that $\mathcal{F}(x_0, \mathcal{S}, T_p)$ is nonempty. Assume that there exist a continuously differentiable function $V\colon \mathcal{S} \to \mathbb{R}$, a system parameter vector $\theta_s \in \mathbb{R}^{N_s}$, real numbers $\alpha, \beta, p, q, r > 0$ such that $pr < 1$ and $qr > 1$, and a control law $u^\star\colon \mathcal{S} \times \mathbb{R}^{N_c} \to U$ such that

$$H\left(x, \theta_s, u^\star(x, \theta_c), V'^{\mathrm{T}}(x)\right) = 0, \quad x \in \mathcal{S} \tag{16}$$

$$H\left(x, \theta_s, u, V'^{\mathrm{T}}(x)\right) \geq 0, \quad (x, u) \in \mathcal{S} \times U \tag{17}$$

$$u^\star(0, \theta_c) = 0 \tag{18}$$

$$V(0) = 0 \tag{19}$$

$$V(x) > 0, \quad x \in \mathcal{S}\backslash\{0\} \tag{20}$$

$$V(x) \to \infty, \quad \text{as } x \to \partial\mathcal{S} \tag{21}$$

$$V'(x)F\left(x, \theta_s, u^\star(x, \theta_c)\right) \leq -\frac{\gamma}{T_p}\left(\alpha V^p(x) + \beta V^q(x)\right)^r, \quad x \in \mathcal{S} \tag{22}$$

where

$$\gamma \triangleq \frac{\Gamma\left(\frac{1-rp}{q-p}\right)\Gamma\left(\frac{rq-1}{q-p}\right)}{\alpha^r \Gamma(r)(q-p)}\left(\frac{\alpha}{\beta}\right)^{\frac{1-rp}{q-p}}.$$

If either $\mathcal{S}$ is bounded or both $\mathcal{S}$ is unbounded and $V(\cdot)$ is coercive, then with the feedback control $u(\cdot) = u^\star(x(\cdot), \theta_c)$, the equilibrium point $x_e = 0$ of (12) is safely predefined-time stable with predefined time T_p with respect to the set of admissible states $\mathcal{S}$, and

$$J\left(x_0, \theta_s, u^\star(x(\cdot), \theta_c)\right) = V(x_0), \quad x_0 \in \mathcal{S}. \tag{23}$$

Furthermore, if $x_0 \in \mathcal{S}$, then the feedback control $u(\cdot) = u^\star(x(\cdot), \theta_c)$ minimizes $J(x_0, \theta_s, u(\cdot))$ in the sense that

$$J\left(x_0, \theta_s, u^\star(x(\cdot), \theta_c)\right) = \min_{u(\cdot) \in \mathcal{F}(x_0, \mathcal{S}, T_p)} J(x_0, \theta_s, u(\cdot)). \tag{24}$$

Proof: Safe predefined-time-time stability follows from (19)–(22) using Theorem 2.

Next, to show optimality, let $x_0 \in \mathcal{S}$, let $u(\cdot) = u^\star(x(\cdot), \theta_c)$, and let $x(t) = s(t, x_0, \theta_s, u^\star(x(\cdot), \theta_c)),\ t \geq 0$, be the solution to (12). Then, since safe predefined-time stability of (13) implies $x(t) \in \mathcal{S}, t \geq 0$, and the time derivative of $V(\cdot)$ along the trajectories of the closed-loop system (13) is defined by

$$\dot{V}(x(t)) \triangleq V'(x(t))F\left(x(t), \theta_s, u^\star(x(t), \theta_c)\right), \quad t \geq 0$$

it follows from (16) that

$$\begin{aligned} &H(x(t), \theta_s, u^\star(x(t), \theta_c), V'^{\mathrm{T}}(x(t))) \\ &\quad = L\left(x, u^\star(x(t), \theta_c)\right) + \dot{V}(x(t)) \\ &\quad = 0, \quad x(t) \in \mathcal{S},\ t \geq 0. \end{aligned} \tag{25}$$

Now, integrating (25) over the time interval $[0, T_p]$ and using the definition of the cost functional (14) yields

$$J\left(x_0, \theta_s, u^\star(x(\cdot), \theta_c)\right) + V(x(T_p)) = V(x_0)$$

which, since $V(x(T_p)) = 0$ using (19) along with safe predefined-time stability of (13), implies (23).

Next, let $x_0 \in \mathcal{S}$, let $u(\cdot) \in \mathcal{F}(x_0, \mathcal{S}, T_p)$, and let $x(t) = s(t, x_0, \theta_s, u(\cdot)), t \geq 0$, be the solution to (12). Then, since $u(\cdot) \in \mathcal{F}(x_0, \mathcal{S}, T_p)$ implies $x(t) \in \mathcal{S}, t \geq 0$, and

$$\dot{V}(x(t)) \triangleq V'(x(t))F(x(t), \theta_s, u(x(t))), \quad t \geq 0$$

it follows from (17) that

$$\begin{aligned} H(x(t), \theta_s, u(x(t)), V'^{\mathrm{T}}(x(t))) &= L(x(t), u(x(t))) + \dot{V}(x(t)) \\ &\geq 0, \quad x(t) \in \mathcal{S},\ t \geq 0. \end{aligned} \tag{26}$$

Now, integrating (26) over $[0, T_p]$ and using (14), (19), and (23), along with the fact that $u(\cdot) \in \mathcal{F}(x_0, \mathcal{S}, T_p)$, we obtain

$$\begin{aligned} J(x_0, \theta_s, u(\cdot)) &\geq V(x_0) \\ &= J\left(x_0, \theta_s, u^\star(x(\cdot), \theta_c)\right) \end{aligned}$$

which yields (24). ■

It is important to note that Theorem 3 examines optimal safe predefined-time stabilization for the controlled parameter-dependent nonlinear system (12) with the cost functional (14) without knowledge of the system trajectories, unlike QP-based methods being system trajectory dependent.

Remark 4: The following observations provide insights into the optimal safe predefined-time stabilization problem.

1) Equation (16) is the steady-state HJB equation for the controlled parameter-dependent nonlinear dynamical system (12) with the cost functional (14). Furthermore, note that (16) and (17) imply

$$\min_{u \in U} \left[H(x, \theta_s, u, V'^{\mathrm{T}}(x))\right] = 0, \quad x \in \mathcal{S}$$

which is an alternative expression for the steady-state HJB equation (16).

2) Equation (24) asserts that the safely predefined-time stabilizing feedback control law $u^\star(x(\cdot), \theta_c)$ is optimal with respect to the set of safely predefined-time stabilizing feedback controllers $\mathcal{F}(x_0, \mathcal{S}, T_p)$ for all $x_0 \in \mathcal{S}$. However, an explicit description of $\mathcal{F}(x_0, \mathcal{S}, T_p)$ is not required.

3) Conditions (19)–(22) ensure safe predefined-time stability, whereas conditions (16) and (17) establish optimality over the set of admissible control values U. The optimal safely predefined-time stabilizing control law $u^\star(x, \theta_c)$ minimizes the Hamiltonian function (15) for every $x \in \mathcal{S}$ with respect to U, that is, $u^\star(x, \theta_c) = \arg\min_{u \in U} H(x, \theta_s, u, V'^{\mathrm{T}}(x))$, and hence, it follows that $u^\star(x, \theta_c)$ is optimal regardless of the initial condition x_0.

4) Equation (23) shows that the safely predefined-time stabilizing Lyapunov function $V(\cdot)$ for the closed-loop system (13) serves as a value function.

5) As shown by [6], the zero solution $x(t) \equiv 0$ to (12) can be extended over the entire infinite time interval, allowing us to assess the system performance over the infinite horizon even though our optimal feedback control problem exhibits predefined-time stability properties. Hence, it follows that $L(x(t), u^\star(x(t), \theta_c)) = 0, t \geq T_p$, which in turn implies

$$\begin{aligned} &\int_0^{T_p} L(x(t), u^\star(x(t), \theta_c))\mathrm{d}t \\ &\quad = \int_0^{\infty} L(x(t), u^\star(x(t), \theta_c))\mathrm{d}t. \end{aligned}$$

6) Although an explicit expression for the settling-time function $T(\cdot, \cdot)$ cannot be provided, safe predefined-time stability ensures that there exists a parameter vector $\theta \in \mathbb{R}^N$ such that the settling-time function $T(\cdot, \theta)$ is uniformly bounded by T_p, that is, $\sup_{x_0 \in \mathcal{S}} T(x_0, \theta) \leq T_p$.

7) The system parameter vector $\theta_s \in \mathbb{R}^{N_s}$ and the control parameter vector $\theta_c \in \mathbb{R}^{N_c}$ implicitly depend on the predefined time T_p and the set of admissible states $\mathcal{S}$. □

IV. Optimal and Inverse Optimal Safe Predefined-Time Stabilization for Nonlinear Affine Systems

In this section, we specialize the results of Section III to parameter-dependent nonlinear affine dynamical systems of the form

$$\dot{x}(t) = f(x(t), \theta_f) + G(x(t), \theta_G)u(t), \quad x(0) = x_0, \ t \geq 0 \tag{27}$$

where for every $t \geq 0$, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $f\colon \mathbb{R}^n \times \mathbb{R}^{N_f} \to \mathbb{R}^n$ is such that $f(\cdot, \theta_f)$ is continuous on $\mathbb{R}^n$ for all $\theta_f \in \mathbb{R}^{N_f}$ and $f(0, \cdot) = 0$, and $G\colon \mathbb{R}^n \times \mathbb{R}^{N_G} \to \mathbb{R}^{n \times m}$ is such that $G(\cdot, \theta_G)$ is continuous on $\mathbb{R}^n$ for all $\theta_G \in \mathbb{R}^{N_G}$. Furthermore, we consider running costs $L(x, u)$ of the form

$$L(x, u) \triangleq L_1(x) + L_2(x)u + u^{\mathrm{T}}R(x)u, \quad (x, u) \in \mathcal{S} \times \mathbb{R}^m \tag{28}$$

where $L_1\colon \mathcal{S} \to \mathbb{R}, L_2\colon \mathcal{S} \to \mathbb{R}^{1 \times m}$, and $R\colon \mathcal{S} \to \mathbb{R}^{m \times m}$ are continuous on $\mathcal{S}$, and $R(x) > 0, x \in \mathcal{S}$. In this case, the cost functional (14) with $\theta_s = [\theta_f^{\mathrm{T}}, \theta_G^{\mathrm{T}}]^{\mathrm{T}}$ becomes

$$\begin{aligned} J\left(x_0, \theta_s, u(\cdot)\right) = \int_0^{T_p} &(L_1(x(t)) + L_2(x(t))u(t) \\ &+ u^{\mathrm{T}}(t)R(x(t))u(t)) \,\mathrm{d}t. \end{aligned} \tag{29}$$

Next, we specialize Theorem 3 to parameter-dependent nonlinear affine dynamical systems (27) with the cost functional (29).

Corollary 1: Consider the parameter-dependent nonlinear affine dynamical system (27) with cost functional (29). Let $\mathcal{S} \subset \mathbb{R}^n$ be a set of admissible states with $0 \in \mathcal{S}$ and let $T_p > 0$ be a predefined time. For every $x_0 \in \mathcal{S}$, let $\mathcal{F}(x_0, \mathcal{S}, T_p) \subset \mathcal{U}$ be the set of safely predefined-time stabilizing feedback controllers and suppose that $\mathcal{F}(x_0, \mathcal{S}, T_p)$ is nonempty. Assume that there exist a continuously differentiable function $V\colon \mathcal{S} \to \mathbb{R}$, system parameter vectors $\theta_f \in \mathbb{R}^{N_f}$ and $\theta_G \in \mathbb{R}^{N_G}$, and real numbers $\alpha, \beta, p, q, r > 0$ such that $pr < 1, qr > 1$, and

$$\begin{aligned} &L_1(x) + V'(x)f(x, \theta_f) - \frac{1}{4}\left(V'(x)G(x, \theta_G) + L_2(x)\right) \\ &\cdot R^{-1}(x)\left(V'(x)G(x, \theta_G) + L_2(x)\right)^{\mathrm{T}} = 0, \quad x \in \mathcal{S} \end{aligned} \tag{30}$$

$$L_2(0) = 0 \tag{31}$$

$$V(0) = 0 \tag{32}$$

$$V(x) > 0, \quad x \in \mathcal{S}\backslash\{0\} \tag{33}$$

$$V(x) \to \infty \text{ as } x \to \partial\mathcal{S} \tag{34}$$

$$\begin{aligned} &V'(x)\bigg(f(x, \theta_f) - \frac{1}{2}G(x, \theta_G)R^{-1}(x)L_2^{\mathrm{T}}(x) \\ &\qquad - \frac{1}{2}G(x, \theta_G)R^{-1}(x)G^{\mathrm{T}}(x, \theta_G)V'^{\mathrm{T}}(x)\bigg) \\ &\leq -\frac{\gamma}{T_p}\left(\alpha V^p(x) + \beta V^q(x)\right)^r, \quad x \in \mathcal{S} \end{aligned} \tag{35}$$

where

$$\gamma \triangleq \frac{\Gamma\left(\frac{1-rp}{q-p}\right)\Gamma\left(\frac{rq-1}{q-p}\right)}{\alpha^r\Gamma(r)(q-p)}\left(\frac{\alpha}{\beta}\right)^{\frac{1-rp}{q-p}}.$$

If either $\mathcal{S}$ is bounded or both $\mathcal{S}$ is unbounded and $V(\cdot)$ is coercive, then there exists a control parameter vector $\theta_c \in \mathbb{R}^{N_c}$ such that with the feedback control

$$\begin{aligned} u(t) &= u^\star(x(t), \theta_c) \\ &= -\frac{1}{2}R^{-1}(x(t))\left(L_2(x(t)) + V'(x(t))G(x(t), \theta_G)\right)^{\mathrm{T}} \\ &\qquad x(t) \in \mathcal{S},\ t \geq 0 \end{aligned} \tag{36}$$

the equilibrium point $x_e = 0$ of

$$\begin{aligned} \dot{x}(t) &= f(x(t), \theta_f) + G(x(t), \theta_G)u^\star(x(t), \theta_c) \\ &\qquad x(0) = x_0,\ t \geq 0 \end{aligned} \tag{37}$$

is safely predefined-time stable with predefined time T_p with respect to the set of admissible states $\mathcal{S}$, (23) holds, and the cost functional (29) is minimized in the sense of (24).

Proof: The proof follows directly from Theorem 3 with $\mathcal{D} = \mathbb{R}^n$, $U = \mathbb{R}^m$, $\theta_s = [\theta_f^{\mathrm{T}}, \theta_G^{\mathrm{T}}]^{\mathrm{T}}$, $F(x, \theta_s, u) = f(x, \theta_f) + G(x, \theta_G)u$, and $L(x, u) = L_1(x) + L_2(x)u + u^{\mathrm{T}}R(x)u$. In particular, the Hamiltonian function becomes

$$\begin{aligned} H\left(x, \theta_s, u, V'^{\mathrm{T}}(x)\right) &= L_1(x) + L_2(x)u + u^{\mathrm{T}}R(x)u \\ &\quad + V'(x)\left(f(x, \theta_f) + G(x, \theta_G)u\right) \\ &\quad (x, \theta_s, u) \in \mathcal{S} \times \mathbb{R}^{N_s} \times \mathbb{R}^m \end{aligned} \tag{38}$$

which, using (16), yields

$$\begin{aligned} &H\left(x, \theta_s, u, V'^{\mathrm{T}}(x)\right) \\ &= H\left(x, \theta_s, u, V'^{\mathrm{T}}(x)\right) - H\left(x, \theta_s, u^\star(x, \theta_c), V'^{\mathrm{T}}(x)\right) \\ &= L(x, u) + V'(x)\left(f(x, \theta_f) + G(x, \theta_G)u\right) \\ &\quad - L\left(x, u^\star(x, \theta_c)\right) \\ &\quad - V'(x)\left(f(x, \theta_f) + G(x, \theta_G)u^\star(x, \theta_c)\right) \\ &= \left(L_2(x) + V'(x)G(x, \theta_G)\right)\left(u - u^\star(x, \theta_c)\right) \\ &\quad + u^{\mathrm{T}}R(x)u - u^{\star\mathrm{T}}(x, \theta_c)R(x)u^\star(x, \theta_c) \end{aligned}$$

and hence, using (36), we obtain

$$\begin{aligned} &H\left(x, \theta_s, u, V'^{\mathrm{T}}(x)\right) \\ &= -2u^{\star\mathrm{T}}(x, \theta_c)R(x)\left(u - u^\star(x, \theta_c)\right) \\ &\quad + u^{\mathrm{T}}R(x)u - u^{\star\mathrm{T}}(x, \theta_c)R(x)u^\star(x, \theta_c) \\ &= \left(u - u^\star(x, \theta_c)\right)^{\mathrm{T}}R(x)\left(u - u^\star(x, \theta_c)\right). \end{aligned}$$

Now, since $R(x) > 0, x \in \mathcal{S}$, it follows that

$$\begin{aligned} H(x, \theta_s, u, V'^{\mathrm{T}}(x)) &= \left(u - u^\star(x, \theta_c)\right)^{\mathrm{T}}R(x)\left(u - u^\star(x, \theta_c)\right) \\ &\geq 0, \quad (x, u) \in \mathcal{S} \times \mathbb{R}^m \end{aligned}$$

which implies (17).

Next, applying the stationarity condition to the Hamiltonian function (38), we obtain the feedback control law (36) as a global minimizer of the Hamiltonian function since $H(x, \theta_s, u, V'^{\mathrm{T}}(x))$ is convex in u for every $x \in \mathcal{S}$. Furthermore, since $V(\cdot)$ is continuously differentiable and $x = 0$ is a local minimum of $V(\cdot)$, we have $V'(0) = 0$, and hence, it follows from (31) and (36) that $u^\star(0, \theta_c) = 0$, which verifies (18). Finally, (32)–(35) are equivalent to (19)–(22), respectively, with $u^\star(x, \theta_c)$ given by (36). The result now follows as a direct consequence of Theorem 3. ■

Remark 5: Using (16), note that $L(x, u)$ can be written as

$$\begin{aligned} L(x, u) &= L(x, u) - H(x, \theta_s, u^\star(x, \theta_c), V'^{\mathrm{T}}(x)) \\ &= L_2(x)\left(u - u^\star(x, \theta_c)\right) \\ &\quad + u^{\mathrm{T}}R(x)u - u^{\star\mathrm{T}}(x, \theta_c)R(x)u^\star(x, \theta_c) \\ &\quad - V'(x)\left(f(x, \theta_f) + G(x, \theta_G)u^\star(x, \theta_c)\right) \\ &\quad (x, u) \in \mathcal{S} \times \mathbb{R}^m \end{aligned}$$

which, with $u^\star(x, \theta_c)$ given by (36), can be rewritten as

$$\begin{aligned} L(x, u) &= \left(u + \frac{1}{2}R^{-1}(x)L_2^{\mathrm{T}}(x)\right)^{\mathrm{T}}R(x) \\ &\quad \cdot \left(u + \frac{1}{2}R^{-1}(x)L_2^{\mathrm{T}}(x)\right) \\ &\quad - \frac{1}{4}V'(x)G(x, \theta_G)R^{-1}(x)G^{\mathrm{T}}(x, \theta_G)V'^{\mathrm{T}}(x) \\ &\quad - V'(x)\left(f(x, \theta_f) + G(x, \theta_G)u^\star(x, \theta_c)\right) \\ &\quad (x, u) \in \mathcal{S} \times \mathbb{R}^m. \end{aligned}$$

Since $R(x) > 0, x \in \mathcal{S}$, and $\dot{V}(x) \triangleq V'(x)\left(f(x, \theta_f) + G(x, \theta_G)u^\star(x, \theta_c)\right) \leq 0, x \in \mathcal{S}$, by (35), we have

$$\begin{aligned} &\left(u + \frac{1}{2}R^{-1}(x)L_2^{\mathrm{T}}(x)\right)^{\mathrm{T}}R(x)\left(u + \frac{1}{2}R^{-1}(x)L_2^{\mathrm{T}}(x)\right) \\ &\quad - V'(x)\left(f(x, \theta_f) + G(x, \theta_G)u^\star(x, \theta_c)\right) \geq 0 \\ &\quad (x, u) \in \mathcal{S} \times \mathbb{R}^m \end{aligned}$$

and hence, it follows that

$$\begin{aligned} L(x, u) &\geqslant -\frac{1}{4}V'(x)G(x, \theta_G)R^{-1}(x)G^{\mathrm{T}}(x, \theta_G)V'^{\mathrm{T}}(x) \\ &\quad (x, u) \in \mathcal{S} \times \mathbb{R}^m \end{aligned}$$

which shows that $L(x, u)$ may be negative since

$$\begin{aligned} &-\frac{1}{4}V'(x)G(x, \theta_G)R^{-1}(x)G^{\mathrm{T}}(x, \theta_G)V'^{\mathrm{T}}(x) \leq 0 \\ &\quad (x, u) \in \mathcal{S} \times \mathbb{R}^m. \end{aligned}$$

Consequently, there may exist an admissible control input $u(\cdot) \in \mathcal{U}$ for which the cost functional $J(x_0, \theta_s, u(\cdot))$ is negative. However, if the control law $u(\cdot)$ is a safely predefined-time stabilizing feedback controller, i.e., $u(\cdot) \in \mathcal{F}(x_0, \mathcal{S}, T_p)$, then it follows from (23) and (24) that

$$\begin{aligned} J\left(x_0, \theta_s, u(\cdot)\right) \geqslant V\left(x_0\right) \geqslant 0, \quad x_0 \in \mathcal{S} \\ u(\cdot) \in \mathcal{F}\left(x_0, \mathcal{S}, T_p\right). \end{aligned}$$

Moreover, if $u(\cdot) = u^\star(x(\cdot), \theta_c)$, it follows from (16) that

$$\begin{aligned} L(x, u^\star(x, \theta_c)) &= -V'(x)\left(f(x, \theta_f) + G(x, \theta_G)u^\star(x, \theta_c)\right) \\ &\geq 0, \quad x \in \mathcal{S}. \end{aligned}$$

□

Next, to circumvent the complexity in solving the steady-state HJB equation, we consider an *inverse optimal feedback control problem* [54], [55], wherein we do not aim to minimize a *given* cost functional, but instead, we parameterize a class of safely predefined-time stabilizing feedback control laws that minimize a *derived* cost functional, thereby providing flexibility in specifying the controller.

Corollary 2: Consider the parameter-dependent nonlinear affine dynamical system (27) with cost functional (29). Let $\mathcal{S} \subset \mathbb{R}^n$ be a set of admissible states with $0 \in \mathcal{S}$ and let $T_p > 0$ be a predefined time. For every $x_0 \in \mathcal{S}$, let $\mathcal{F}(x_0, \mathcal{S}, T_p) \subset \mathcal{U}$ be the set of safely predefined-time stabilizing feedback controllers and suppose that $\mathcal{F}(x_0, \mathcal{S}, T_p)$ is nonempty. Assume that there exist a continuously differentiable function $V : \mathcal{S} \to \mathbb{R}$, a continuous function $L_2 : \mathcal{S} \to \mathbb{R}^{1\times m}$, a continuous positive-definite matrix function $R : \mathcal{S} \to \mathbb{R}^{m\times m}$, system parameter vectors $\theta_f \in \mathbb{R}^{N_f}$ and $\theta_G \in \mathbb{R}^{N_G}$, and real numbers $\alpha, \beta, p, q, r > 0$ such that $pr < 1, qr > 1$, and (31)–(35) hold. If either $\mathcal{S}$ is bounded or both $\mathcal{S}$ is unbounded and $V(\cdot)$ is coercive, then with the feedback control (36), the equilibrium point $x_e = 0$ of (37) is safely predefined-time stable with predefined time T_p with respect to the set of admissible states $\mathcal{S}$, (23) holds, and the cost functional (29), with

$$L_1(x) = u^{\star\mathrm{T}}(x, \theta_c)R(x)u^\star(x, \theta_c) - V'(x)f(x, \theta_f), \quad x \in \mathcal{S} \quad (39)$$

is minimized in the sense of (24).

Proof: The proof is similar to the proof of Corollary 1 and thus is omitted. ■

Remark 6: The following observations are important.

1) The function $L_1(x)$ in the running cost (28) given by (39) explicitly depends on the nonlinear system dynamics, the Lyapunov function of the closed-loop system, and the safely predefined-time stabilizing feedback control law, wherein the coupling arises from the steady-state HJB equation (16).
2) The function $L_2(x)$ in the running cost (28) provides flexibility in the design of the controller since $L_2(x)$ is an arbitrary function of $x \in \mathcal{S}$ subject to conditions (31) and (35). □

In light of the above observations, note that by varying the parameters in the Lyapunov function and the running cost, we can characterize a family of safely predefined-time stabilizing feedback control laws that can satisfy closed-loop system response requirements.

V. PINNs for Optimal Safety-Critical Control

Unlike the *inverse optimal control problem*, whose value function is known, the problem of the *optimal safe predefined-time stabilization* amounts to solving the steady-state HJB equation (30) subject to the constraints (32)–(35), which is, in general, intractable aside from special cases. In this section, we build on the results of [56] to develop a novel *physics-informed machine learning* architecture for approximating the safely predefined-time stabilizing solution of the steady-state HJB equation.

To find the safely predefined-time stabilizing solution $V(\cdot)$ to (30), a surrogate model $\hat{V}(\cdot, w)$ is introduced as an approximator using the universal approximation property of deep neural networks for any continuously differentiable function [57], where $w \in \mathbb{R}^N$ denotes the trainable model parameters. To enforce the constraints (32)–(34), we set

$$\hat{V}(x, w) = h(V_{\mathrm{NN}}(x, w))B(x), \quad x \in \mathcal{S} \quad (40)$$

where $h\colon \mathbb{R} \to (0, \infty)$ is a user-defined continuously differentiable function, $V_{\mathrm{NN}}\colon \mathcal{S}\times\mathbb{R}^N \to \mathbb{R}$ is a standard fully connected neural network, and $B\colon \mathcal{S} \to \mathbb{R}$ is a user-defined continuously differentiable function satisfying $B(0) = 0$, $B(x) > 0, x \in \mathcal{S}\backslash\{0\}$, and $B(x) \to \infty$ as $x \to \partial\mathcal{S}$. If $\mathcal{S}$ is unbounded, then $B(\cdot)$ is additionally coercive.

Suppose M points $\{x_i\}_{i=1}^M$ are randomly sampled in $\mathcal{S}$. The problem of learning the safely predefined-time stabilizing solution $V(\cdot)$ to (30) can be cast as a constrained optimization problem, which involves the minimization

$$\min_{w\in\mathbb{R}^N} \mathcal{E}(w) \quad (41)$$

subject to

$$l_i(w) \le 0, \quad i = 1, \ldots, M \quad (42)$$

where

$$\begin{aligned}\mathcal{E}(w) \triangleq \sum_{i=1}^{M} \Big| &L_1(x_i) + \hat{V}_x(x_i, w)f(x_i, \theta_f) \\ &- \frac{1}{4}\left(\hat{V}_x(x_i, w)G(x_i, \theta_G) + L_2(x_i)\right) \\ &\cdot R^{-1}(x_i)\left(\hat{V}_x(x_i, w)G(x_i, \theta_G) + L_2(x_i)\right)^{\mathrm{T}}\Big|^2 \end{aligned} \quad (43)$$

and

$$\begin{aligned} l_i(w) \triangleq{}& \hat{V}_x(x_i, w) \\ &\cdot \bigg(f(x_i, \theta_f) - \frac{1}{2}G(x_i, \theta_G)R^{-1}(x_i)L_2^{\mathrm{T}}(x_i) \\ &\quad - \frac{1}{2}G(x_i, \theta_G)R^{-1}(x_i)G^{\mathrm{T}}(x_i, \theta_G)\hat{V}_x^{\mathrm{T}}(x_i, w)\bigg) \\ &+ \frac{\gamma}{T_p}\left(\alpha\hat{V}^p(x_i, w) + \beta\hat{V}^q(x_i, w)\right)^r. \end{aligned} \quad (44)$$

We now convert the constrained optimization problem described by (41) and (42) into an unconstrained optimization problem via the augmented Lagrangian method. Thus, the optimization process is formulated as a K-step iterative scheme, where the weights for constraints vary at each iteration. Specifically, the optimization problem at the kth iteration involves the minimization

$$\min_{w\in\mathbb{R}^N} \ \mathcal{E}_k(w) \quad (45)$$

where

$$\mathcal{E}_k(w) \triangleq \mathcal{E}(w) + \sum_{i=1}^{M}\left(\mu_{k-1}\mathbb{1}_{\{l_i(w)\ge 0 \vee \lambda_{k-1}^i > 0\}} l_i^2(w) + \lambda_{k-1}^i l_i(w)\right)$$

with $\vee$ being the or operator. The Lagrange multipliers μ_k and λ_k^i are updated based on the rules

$$\mu_k = \delta\mu_{k-1}$$

and

$$\lambda_k^i = \max\left\{0, \lambda_{k-1}^i + 2\mu_{k-1}l_i(w)\right\}$$

where δ and μ_0 are hyperparameters that can be tuned, and $\{\lambda_0^i\}_{i\in\{1,\ldots,M\}}$ are initialized to be 0. Note that the unconstrained

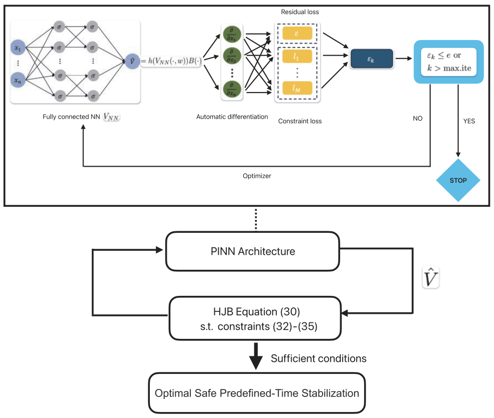


Fig. 1. PINN architecture for solving the optimal safe predefined-time stabilization problem.

optimization problem (45) can be solved efficiently using modern numerical optimization solvers, such as Adam [58] and L-BFGS [59].

Assume that the minimizer w_K is obtained after K iterations, then, $\hat{V}(\cdot, w_K)$ represents the approximate value function. The corresponding optimal controller $\hat{u}(\cdot, w_K)$ can be found by substituting the approximate value function $\hat{V}(\cdot, w_K)$ into (36). A pseudocode describing the proposed physics-informed machine learning optimal safety-critical control framework is given in Algorithm 1, whose architecture is illustrated in Fig. 1.

Algorithm 1 Training Procedure of Safe PINN

Hyperparameters: $\alpha, \beta, \gamma, T_p, p, q, r, \delta, \mu_0$
Input: Training data points $\{x_i\}_{i=1}^{M} \in \mathcal{S}$
Output: Value function $\hat{V}(\cdot, w_K)$

procedure
  $\lambda_0^i \leftarrow 0$ for $1 \le i \le M$
  Initialize network parameters $w_0 \in \mathbb{R}^N$
  for $k = 1 \cdots K$ **do**
    $\tilde{\mathcal{E}} \leftarrow \mathcal{E}(w_{k-1})$ ▷ (43)
    $\tilde{l}_i \leftarrow l_i(w_{k-1})$ ▷ (44)
    $\mathcal{E}_k \leftarrow \tilde{\mathcal{E}} + \sum_{i=1}^{M} (\mu_{k-1} \mathbb{1}_{\{\tilde{l}_i \ge 0 \vee \lambda_{k-1}^i > 0\}} \tilde{l}_i^2 + \lambda_{k-1}^i \tilde{l}_i)$
    $w_k \leftarrow \arg\min_w \mathcal{E}_k$
    $\mu_k \leftarrow \delta \mu_{k-1}$
    $\lambda_k^i \leftarrow \max\{0, \lambda_{k-1}^i + 2\mu_{k-1} \tilde{l}_i\}$
  end for
end procedure

The next theorem shows the asymptotic convergence of Algorithm 1.

Theorem 4: Let $V \colon \mathcal{S} \to \mathbb{R}$ be the unique stabilizing solution of the steady-state HJB equation (30), which satisfies the constraints (32)–(35). Furthermore, let $\hat{V}(\cdot, w_K)$ be the approximate value function generated by Algorithm 1 after K iterations. Then, as $K \to \infty$, $\hat{V}(\cdot, w_K)$ converges to $\hat{V}^\star(\cdot)$, an approximation of $V(\cdot)$, and satisfies the constraints (32)–(35).

Proof: As $K \to \infty$, the sequence of solutions to the unconstrained optimization problem (45) converges to the solution $\hat{V}^\star(\cdot)$ of the constrained optimization problem (41) and (42), which is an approximation of $V(\cdot)$. However, the convergence analysis of Algorithm 1 is similar to that of the standard augmented Lagrangian method (see [60] and [61]), and is hence omitted. ■

Remark 7: The criticality of enforcing stability conditions (32)–(35) to avoid multiple solutions when solving the optimal stabilization control problem using PINNs has not been considered. Our article fills this gap using the augmented Lagrangian method that integrates the constraints effectively. □

VI. Illustrative Numerical Examples

In this section, we provide two illustrative numerical examples to demonstrate the proposed optimal safe predefined-time control and physics-informed learning framework.

A. Abstract Dynamical System

Consider the nonlinear affine dynamical system given by

$$\dot{x}_1(t) = -\tanh(x_1(t)x_2(t))\, x_2(t) + u_1(t)$$
$$x_1(0) = x_{10}, \quad t \ge 0 \tag{46}$$
$$\dot{x}_2(t) = \tanh(x_1(t)x_2(t))\, x_1(t) + u_2(t), \quad x_2(0) = x_{20}. \tag{47}$$

Note that (46) and (47) can be cast in the form of (27) with $n = 2, m = 2, x = [x_1, x_2]^{\mathrm{T}}, \; u = [u_1, u_2]^{\mathrm{T}}$

$$f(x, \theta_f) = \begin{bmatrix} -\tanh(x_1 x_2)\, x_2 \\ \tanh(x_1 x_2)\, x_1 \end{bmatrix}$$

and $G(x, \theta_G) = I_2$.

To characterize the set of admissible states $\mathcal{S}$, let $s \colon \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function such that $s(0) > 0$ and define

$$\mathcal{S} \triangleq \left\{x \in \mathbb{R}^2 : s(x) > 0\right\} \tag{48}$$

whose boundary is given by

$$\partial\mathcal{S} \triangleq \left\{x \in \mathbb{R}^2 : s(x) = 0\right\}.$$

Note that $\mathcal{S}$ is an open set with $0 \in \mathcal{S}$ defined as a zero-strict superlevel set of $s(\cdot)$.

Next, given the set of admissible states $\mathcal{S}$ and a predefined time T_p, we use Corollary 2 to design an inverse optimal safely predefined-time stabilizing control law $u^\star(x, \theta_c)$. Specifically, we consider both the case of a bounded and the case of an unbounded set of admissible states.

1) Bounded Safe Set: Let $s(x) = 1 - \|x\|_2^2, x \in \mathbb{R}^2$, and let $V(x) = \|x\|_2^2/(2s(x)),\; x \in \mathcal{S}$, be the value function, whereas the terms composing the running cost (28) are given by

$$L_1(x) = \frac{1}{2}\left\| \frac{\lceil x \rfloor^{\gamma_1}}{s^{\frac{\gamma_1 - 1}{2}}(x)} + \frac{\lceil x \rfloor^{\gamma_2}}{s^{\frac{\gamma_2 - 1}{2}}(x)} \right\|_2^2, \quad x \in \mathcal{S}$$

$$L_2(x) = \left(\frac{\lceil x \rfloor^{\gamma_1}}{s^{\frac{\gamma_1 - 1}{2}}(x)} + \frac{\lceil x \rfloor^{\gamma_2}}{s^{\frac{\gamma_2 - 1}{2}}(x)} \right)^{\mathrm{T}} - \left(1 + \frac{\|x\|_2^2}{s(x)}\right) \frac{x^{\mathrm{T}}}{s(x)}, \quad x \in \mathcal{S}$$

and $R(x) = (1/2)I_2, x \in \mathcal{S}$, where $\gamma_1 \in (0, 1)$ and $\gamma_2 > 1$. Hence, with $\theta_c = [\gamma_1, \gamma_2]^{\mathrm{T}}$, the inverse optimal controller is given by

$$u^\star(x, \theta_c) = -\left(\frac{\lceil x \rfloor^{\gamma_1}}{s^{\frac{\gamma_1 - 1}{2}}(x)} + \frac{\lceil x \rfloor^{\gamma_2}}{s^{\frac{\gamma_2 - 1}{2}}(x)} \right), \quad x \in \mathcal{S}.$$

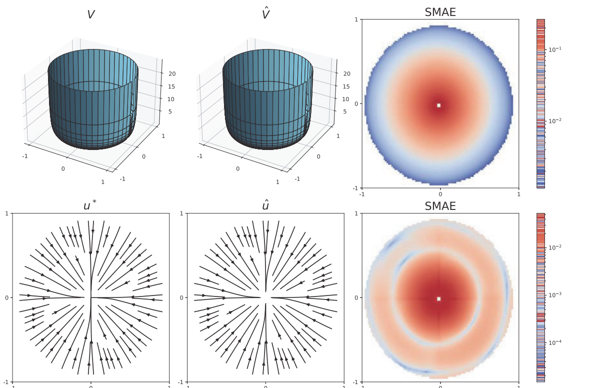


Fig. 2. Value function and optimal control in the case of the bounded safe set. Exact value function V and exact optimal controller $u^\star$ (left). Approximate value function $\hat{V}$ and approximate optimal controller $\hat{u}$ (middle). The symmetric mean absolute error of learning the value function and optimal control policy (right).

Furthermore, note that

$$\dot{V}(x) = -\left(\frac{\|x\|_{\gamma_1+1}^{\gamma_1+1}}{s^{\frac{\gamma_1+1}{2}}(x)} + \frac{\|x\|_{\gamma_2+1}^{\gamma_2+1}}{s^{\frac{\gamma_2+1}{2}}(x)}\right) - \frac{\|x\|_2^2}{s^2(x)}\left(\frac{\|x\|_{\gamma_1+1}^{\gamma_1+1}}{s^{\frac{\gamma_1-1}{2}}(x)} + \frac{\|x\|_{\gamma_2+1}^{\gamma_2+1}}{s^{\frac{\gamma_2-1}{2}}(x)}\right) \leq -\left(\frac{\|x\|_{\gamma_1+1}^{\gamma_1+1}}{s^{\frac{\gamma_1+1}{2}}(x)} + \frac{\|x\|_{\gamma_2+1}^{\gamma_2+1}}{s^{\frac{\gamma_2+1}{2}}(x)}\right), \quad x \in \mathcal{S}. \tag{49}$$

Now, by the monotonicity property of ℓ^p-norms [62], we have $\|x\|_2 \leq \|x\|_{\gamma_1+1}$ since $2 > \gamma_1 + 1 > 0$. Also, by the equivalence of vector norms on $\mathbb{R}^n$ [62], it follows that $\|x\|_2 \leq 2^{(\gamma_2-1)/(2(\gamma_2+1))}\|x\|_{\gamma_2+1}$ since $2 < \gamma_2 + 1$, and hence, (49) can be further bounded as

$$\dot{V}(x) \leq -2^{\frac{\gamma_1+1}{2}} V^{\frac{\gamma_1+1}{2}}(x) - 2V^{\frac{\gamma_2+1}{2}}(x), \quad x \in \mathcal{S}$$

which implies that (35) is satisfied with $\gamma = T_p$, $\alpha = 2^{(\gamma_1+1)/2}, \beta = 2, p = (\gamma_1 + 1)/2, q = (\gamma_2 + 1)/2$, and $r = 1$, and hence, by Corollary 2, the equilibrium point $x_e = 0$ of the closed-loop system is safely predefined-time stable.

Let $T_p = 3.4259$ so that $\gamma_1 = 0.5$ and $\gamma_2 = 1.5$. For our PINN (40), we set $h(x) = e^x, x \in \mathbb{R}$, and $B(x) = \|x\|_2^2/(1 - \|x\|_2), x \in \mathcal{S}$. Furthermore, our PINN architecture consists of a 6-layer fully connected neural network with 100 neurons in each hidden layer, using the hyperbolic tangent activation function $\tanh(\cdot)$. Note that we use the L-BFGS-B optimizer to simultaneously minimize the loss function with respect to all trainable parameters (see [59]).

Fig. 2 illustrates the approximate value function $\hat{V}$ and the approximate optimal control $\hat{u}$ generated by Algorithm 1, alongside the exact value function V and optimal controller $u^\star$ for comparison. To assess the performance of our learning algorithm, we define the symmetric mean absolute error (SMAE) of learning the value function V and the optimal controller $u^\star$ as

$$\mathrm{SMAE}(V, \hat{V}) \triangleq \frac{1}{M}\sum_{i=1}^{M} \frac{|V(x_i) - \hat{V}(x_i)|}{|V(x_i)| + |\hat{V}(x_i)|} \tag{50}$$

and

$$\mathrm{SMAE}(u^\star, \hat{u}) \triangleq \frac{1}{M}\sum_{i=1}^{M} \frac{\|u^\star(x_i) - \hat{u}(x_i)\|_1}{\|u^\star(x_i)\|_1 + \|\hat{u}(x_i)\|_1}. \tag{51}$$

Note that this metric is also referred to as symmetric mean absolute percentage error (SMAPE) in the field of time series modeling [63]. It can be seen that our learning architecture achieves a low symmetric mean absolute error in $\mathcal{S}$. The left plot of Fig. 3 shows the controlled state trajectories, which evolve within the safe set $\mathcal{S}$ and converge to the origin. The right plot shows the predefined-time convergence of the Euclidean norm of the controlled state trajectories, which confirms the result $T(x(0), \theta_c) \leq 3.4259, x(0) \in \mathcal{S}$.

2) Unbounded Safe Set: Let $s(x) = 1 - x_1^2, x \in \mathbb{R}^2$, and let $V(x) = \|x\|_2^2/(2s(x)), x \in \mathcal{S}$, be the value function, while the derived terms in the running cost (28) are given by

$$L_1(x) = \frac{1}{2}\left\| \frac{\lceil x \rfloor^{\gamma_1}}{s^{\frac{\gamma_1-1}{2}}(x)} + \frac{\lceil x \rfloor^{\gamma_2}}{s^{\frac{\gamma_2-1}{2}}(x)} \right\|_2^2 + \frac{\|x\|_2^2}{s^2(x)} \tanh(x_1 x_2)\, x_1 x_2, \quad x \in \mathcal{S}$$

$$L_2(x) = \left(\frac{\lceil x \rfloor^{\gamma_1}}{s^{\frac{\gamma_1-1}{2}}(x)} + \frac{\lceil x \rfloor^{\gamma_2}}{s^{\frac{\gamma_2-1}{2}}(x)}\right)^{\mathrm{T}} - \left[\frac{x_1}{s(x)}\left(1 + \frac{\|x\|_2^2}{s(x)}\right), \frac{x_2}{s(x)}\right], \quad x \in \mathcal{S}$$

and $R(x) = (1/2)I_2, x \in \mathcal{S}$, where $\gamma_1 \in (0, 1)$ and $\gamma_2 > 1$. Hence, with $\theta_c = [\gamma_1, \gamma_2]^{\mathrm{T}}$, the inverse optimal controller is given by

$$u^\star(x, \theta_c) = -\left(\frac{\lceil x \rfloor^{\gamma_1}}{s^{\frac{\gamma_1-1}{2}}(x)} + \frac{\lceil x \rfloor^{\gamma_2}}{s^{\frac{\gamma_2-1}{2}}(x)}\right), \quad x \in \mathcal{S}.$$

Moreover, we have

$$\dot{V}(x) = -\left(\frac{\|x\|_{\gamma_1+1}^{\gamma_1+1}}{s^{\frac{\gamma_1+1}{2}}(x)} + \frac{\|x\|_{\gamma_2+1}^{\gamma_2+1}}{s^{\frac{\gamma_2+1}{2}}(x)}\right) - \frac{\|x\|_2^2}{s^2(x)}\left(\tanh(x_1 x_2)\, x_1 x_2 + \frac{|x_1|^{\gamma_1+1}}{s^{\frac{\gamma_1-1}{2}}(x)} + \frac{|x_1|^{\gamma_2+1}}{s^{\frac{\gamma_2-1}{2}}(x)}\right) \leq -\left(\frac{\|x\|_{\gamma_1+1}^{\gamma_1+1}}{s^{\frac{\gamma_1+1}{2}}(x)} + \frac{\|x\|_{\gamma_2+1}^{\gamma_2+1}}{s^{\frac{\gamma_2+1}{2}}(x)}\right), \quad x \in \mathcal{S}. \tag{52}$$

Now, as in the previous case, by the monotonicity property of ℓ^p-norms and by the equivalence of vector norms on $\mathbb{R}^n$, it follows that (52) can be further bounded as

$$\dot{V}(x) \leq -2^{\frac{\gamma_1+1}{2}} V^{\frac{\gamma_1+1}{2}}(x) - 2V^{\frac{\gamma_2+1}{2}}(x), \quad x \in \mathcal{S}$$

and hence, (35) is satisfied with $\gamma = T_p$, $\alpha = 2^{(\gamma_1+1)/2}, \beta = 2, p = (\gamma_1 + 1)/2, q = (\gamma_2 + 1)/2$, and $r = 1$, which, by Corollary 2, implies that the equilibrium point $x_e = 0$ of the closed-loop system is safely predefined-time stable.

We utilize identical values as in the previous case for the parameters T_p, γ_1, and γ_2. Regarding our PINN (40), we set $h(x) = e^x, x \in \mathbb{R}$, and $B(x) = \|x\|_2^2/(\sqrt{2} - (x_1^2 + 1)^{1/2}), x \in \mathcal{S}$. Moreover, our PINN has the same architecture as in the previous case. Fig. 4 shows the estimated value function $\hat{V}$ and the estimated optimal control $\hat{u}$ generated by Algorithm 1 versus the exact value function V and the exact optimal

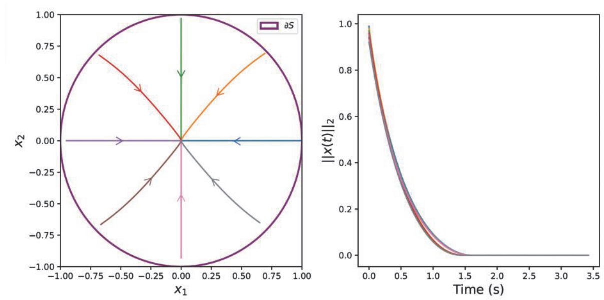


Fig. 3. Optimal safe predefined-time stabilization in the case of the bounded safe set. Controlled state trajectories $x(t), t \geq 0$, starting from different initial conditions in the safe set $\mathcal{S}$, each marked with a unique color (left). The arrows indicate the direction of time evolution for each trajectory. Time evolution of the Euclidean norm of the controlled state trajectories $\|x(t)\|_2, t \geq 0$, starting from the same initial conditions in the safe set $\mathcal{S}$ as the trajectories shown in the left plot (right). Note that trajectories starting from identical initial conditions are marked with the same color in both plots to indicate their correspondence.

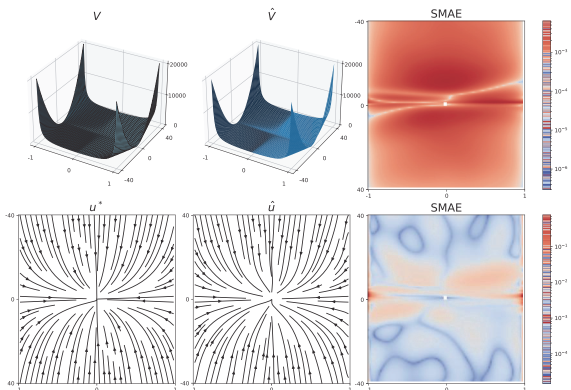


Fig. 4. Value function and optimal control in the case of the unbounded safe set. Exact value function V and exact optimal controller $u^\star$ (left). Approximate value function $\hat{V}$ and approximate optimal controller $\hat{u}$ (middle). The symmetric mean absolute error of learning the value function and optimal control policy (right).

control $u^\star$. Note that our learning framework yields a low symmetric mean absolute error within the training domain $(-1, 1)\times[-40, 40]$. The left plot of Fig. 5 shows the evolution of the controlled state trajectories within the safe set $\mathcal{S}$ and the convergence to the origin, while the right plot of Fig. 5 shows the predefined-time convergence of the Euclidean norm of the controlled state trajectories, which verifies that $T(x(0), \theta_c) \leq 3.4259, x(0) \in \mathcal{S}$.

B. Spacecraft Dynamical System

Unlike our first example involving an abstract dynamical system, we now verify the efficacy of our framework on a real-world dynamical system. Consider a spacecraft with one axis of symmetry given by [64]

$$\dot{\omega}_1(t) = I_{23}\omega_3\omega_2(t) + u_1(t), \quad \omega_1(0) = \omega_{10}, \ t \geq 0 \tag{53}$$

$$\dot{\omega}_2(t) = -I_{23}\omega_3\omega_1(t) + u_2(t), \quad \omega_2(0) = \omega_{20} \tag{54}$$

where $I_{23} \triangleq (I_2 - I_3)/I_1, I_1, I_2$, and I_3 represent the spacecraft principal moments of inertia such that $0 < I_1 = I_2 < I_3, \omega_1 : [0, \infty) \to \mathbb{R}, \omega_2 : [0, \infty) \to \mathbb{R}$, and $\omega_3 \in \mathbb{R}$ are the components of the angular velocity vector with respect to a given inertial reference frame expressed in a central body reference frame,

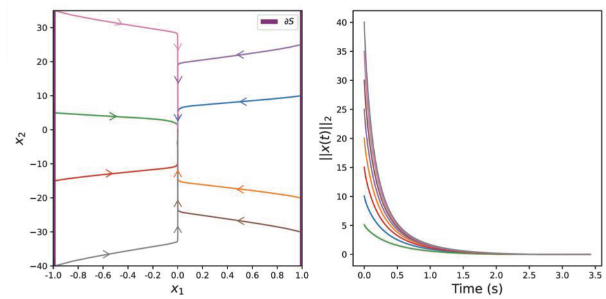


Fig. 5. Optimal safe predefined-time stabilization in the case of the unbounded safe set. Controlled state trajectories $x(t), t \geq 0$, starting from various initial states within the safe set $\mathcal{S}$, each marked with a unique color (left). The arrows indicate the direction of time evolution for each trajectory. Time evolution of the Euclidean norm of the controlled state trajectories $\|x(t)\|_2, t \geq 0$, starting from the same initial conditions in the safe set $\mathcal{S}$ as the trajectories shown in the left plot (right). Note that trajectories starting from identical initial conditions are marked with the same color in both plots to indicate their correspondence.

and u_1 and u_2 denote the spacecraft control moments. The dynamical system (53) and (54) can be cast in the form of (27) with $n = 2, m = 2, x = [\omega_1, \omega_2]^{\mathrm{T}}, u = [u_1, u_2]^{\mathrm{T}}$

$$f(x, \theta_f) = \begin{bmatrix} I_{23}\omega_3\omega_2 \\ -I_{23}\omega_3\omega_1 \end{bmatrix}$$

and $G(x, \theta_G) = I_2$.

Next, we use Corollary 2 to synthesize an inverse optimal safely predefined-time stabilizing control $u^\star(x, \theta_c)$ for a given set of admissible states $\mathcal{S}$ (48) and a predefined time T_p. Let $s(x) = 2-\|x\|_2^2-e^{\|x\|_2^2}, x \in \mathbb{R}^2$, and let $V(x) = \|x\|_2^2/(2s(x)), x \in \mathcal{S}$, be the value function. The terms of the running cost (28) are given by

$$L_1(x) = \frac{1}{2}\left\| \frac{\lceil x \rfloor^{\gamma_1}}{s^{\frac{\gamma_1-1}{2}}(x)} + \frac{\lceil x \rfloor^{\gamma_2}}{s^{\frac{\gamma_2-1}{2}}(x)} \right\|_2^2, \quad x \in \mathcal{S}$$

$$L_2(x) = \left(\frac{\lceil x \rfloor^{\gamma_1}}{s^{\frac{\gamma_1-1}{2}}(x)} + \frac{\lceil x \rfloor^{\gamma_2}}{s^{\frac{\gamma_2-1}{2}}(x)} \right)^{\mathrm{T}} - \left[1 + \frac{\|x\|_2^2 \left(1 + e^{\|x\|_2^2}\right)}{s(x)} \right] \frac{x^{\mathrm{T}}}{s(x)}, \quad x \in \mathcal{S}$$

and $R(x) = (1/2)I_2, x \in \mathcal{S}$, where $\gamma_1 \in (0, 1)$ and $\gamma_2 > 1$. Hence, with $\theta_c = [\gamma_1, \gamma_2]^{\mathrm{T}}$, the inverse optimal controller is given by

$$u^\star(x, \theta_c) = -\left(\frac{\lceil x \rfloor^{\gamma_1}}{s^{\frac{\gamma_1-1}{2}}(x)} + \frac{\lceil x \rfloor^{\gamma_2}}{s^{\frac{\gamma_2-1}{2}}(x)} \right), \quad x \in \mathcal{S}.$$

Now, computing the time derivative of $V(x)$ along the trajectories of (53) and (54) yields

$$\begin{aligned} \dot{V}(x) = &-\left(\frac{\|x\|_{\gamma_1+1}^{\gamma_1+1}}{s^{\frac{\gamma_1+1}{2}}(x)} + \frac{\|x\|_{\gamma_2+1}^{\gamma_2+1}}{s^{\frac{\gamma_2+1}{2}}(x)} \right) \\ &- \frac{\|x\|_2^2 \left(1 + e^{\|x\|_2^2}\right)}{s^2(x)} \left(\frac{\|x\|_{\gamma_1+1}^{\gamma_1+1}}{s^{\frac{\gamma_1-1}{2}}(x)} + \frac{\|x\|_{\gamma_2+1}^{\gamma_2+1}}{s^{\frac{\gamma_2-1}{2}}(x)} \right) \\ \leq &-\left(\frac{\|x\|_{\gamma_1+1}^{\gamma_1+1}}{s^{\frac{\gamma_1+1}{2}}(x)} + \frac{\|x\|_{\gamma_2+1}^{\gamma_2+1}}{s^{\frac{\gamma_2+1}{2}}(x)} \right), \quad x \in \mathcal{S} \end{aligned}$$

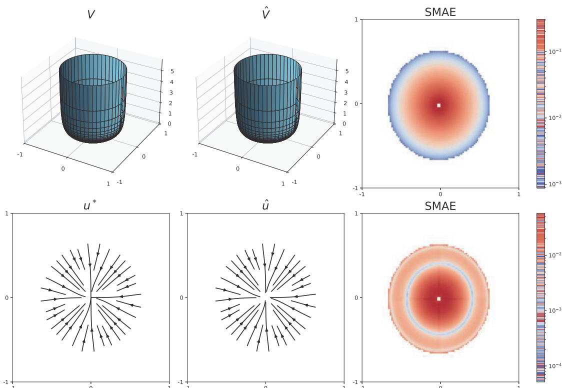


Fig. 6. Value function and optimal control for the optimal safe predefined-time stabilization of spacecraft. Exact value function V and exact optimal controller $u^\star$ (left). Approximate value function $\hat{V}$ and approximate optimal controller $\hat{u}$ (middle). The symmetric mean absolute error of learning the value function and optimal control policy (right).

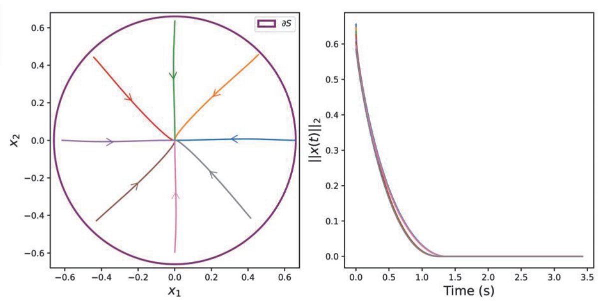


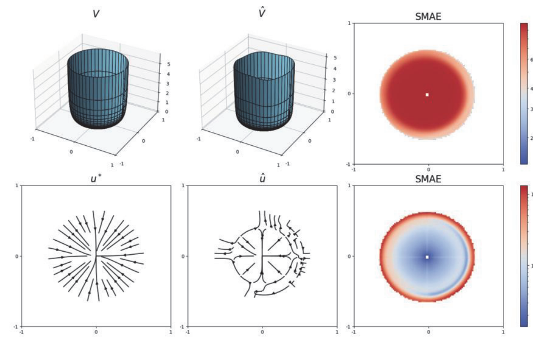
Fig. 7. Optimal safe predefined-time stabilization of spacecraft. Controlled state trajectories $x(t), t \geq 0$, starting from different initial conditions in the safe set $\mathcal{S}$, each marked with a unique color (left). The arrows indicate the direction of time evolution for each trajectory. Time evolution of the Euclidean norm of the controlled state trajectories $\|x(t)\|_2, t \geq 0$, starting from the same initial conditions in the safe set $\mathcal{S}$ as the trajectories shown in the left plot (right). Note that trajectories starting from identical initial conditions are marked with the same color in both plots to indicate their correspondence.

which, by the monotonicity property of ℓ^p-norms and by the equivalence of vector norms on $\mathbb{R}^n$, can be further bounded as

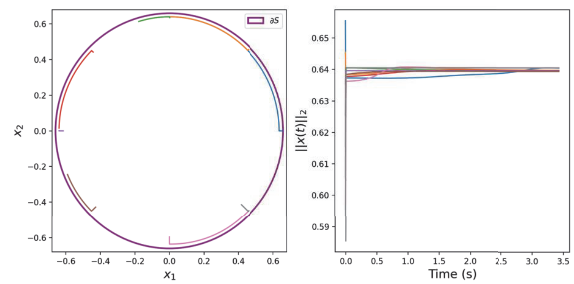
$$\dot{V}(x) \leq -2^{\frac{\gamma_1+1}{2}} V^{\frac{\gamma_1+1}{2}}(x) - 2V^{\frac{\gamma_2+1}{2}}(x), \quad x \in \mathcal{S}$$

which, by Corollary 2, implies that the equilibrium point $x_e = 0$ of the closed-loop system is safely predefined-time stable since (35) is satisfied with $\gamma = T_p$, $\alpha = 2^{(\gamma_1+1)/2}, \beta = 2, p = (\gamma_1+1)/2, q = (\gamma_2+1)/2$, and $r = 1$.

We use identical values as in our first example for the parameters T_p, γ_1, and γ_2. For our PINN (40), we set $h(x) = e^x, x \in \mathbb{R}$, and $B(x) = \|x\|_2^2/(\kappa - \|x\|_2), x \in \mathcal{S}$, where κ is the positive root of $2 - \kappa^2 - e^{\kappa^2} = 0$. Furthermore, our PINN has the same architecture as in our first example. Fig. 6 shows the approximate value function $\hat{V}$ and the approximate optimal control $\hat{u}$ generated by Algorithm 1, along with the exact value function V and the exact optimal controller $u^\star$. Note that our learning architecture achieves a low symmetric mean absolute error in $\mathcal{S}$. The left plot of Fig. 7 shows the evolution of the controlled state trajectories within the safe set $\mathcal{S}$, converging to the origin. The right plot shows the predefined-time convergence of the Euclidean norm of the controlled state trajectories, confirming the result $T(x(0), \theta_c) \leq 3.4259, x(0) \in \mathcal{S}$.



(a)



(b)

Fig. 8. Benchmark comparison for optimal safe predefined-time stabilization of a spacecraft using the physics-informed learning framework developed in [50]. Note that the controlled state trajectories $x(t), t \geq 0$, starting from several initial conditions in a neighborhood of the boundary of the safe set ∂S *fail* to converge to the equilibrium point $x_e = 0$. (a) Value function and optimal control. (b) Controlled state trajectories.

Finally, we further demonstrate the efficacy of the proposed approach by benchmarking our physics-informed learning framework against [50], which does *not* incorporate the stability conditions (33)–(35). The results are shown in Fig. 8. Stress that the controlled state trajectories starting from several initial conditions in a neighborhood of the boundary of the safe set ∂S fail to converge to the equilibrium point since the value function and the optimal control are not effectively approximated in the neighborhood of ∂S as shown in Fig. 8(a).

VII. Conclusion

In this article, the notion of safe predefined-time stability is introduced characterizing parameter-dependent nonlinear dynamical systems whose trajectories starting in a given set of admissible states remain in the set of admissible states for all time and converge to an equilibrium point in a predefined time. Sufficient conditions for safe predefined-time stability are presented in a Lyapunov theorem. An optimal safe predefined-time stabilization problem is stated, and sufficient conditions are provided characterizing an optimal feedback controller that guarantees closed-loop system safe predefined-time stability. Specifically, safe predefined-time stability of the closed-loop system is guaranteed via a Lyapunov function that simultaneously satisfies a certain differential inequality and the steady-state HJB equation. In light of the intractability of the latter, we developed a physics-informed machine learning-based algorithm for learning the safely predefined-time stabilizing solution to the steady-state HJB equation. Future research will focus on developing reinforcement learning-based techniques to solve the optimal safe predefined-time stabilization problem.

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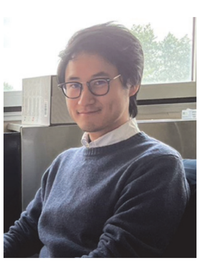

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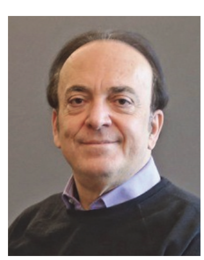

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