



RESEARCH ARTICLE

Polynomial progressions in topological fields

Ben Krause¹, Mariusz Mirek², Sarah Peluse³ and James Wright⁴

¹School of Mathematics, University of Bristol, Bristol, BS8 1UG, England; E-mail: ben.krause@bristol.ac.uk.

²Department of Mathematics, Rutgers University, Piscataway, NJ 08854-8019, USA & Instytut Matematyczny, Uniwersytet Wrocławski, Plac Grunwaldzki 2/4, 50-384 Wrocław, Poland; E-mail: mariusz.mirek@rutgers.edu (corresponding author).

³Department of Mathematics, Stanford University 450, Serra Mall, Building 380, Stanford, CA 94305, USA; E-mail: speluse@stanford.edu.

⁴Maxwell Institute of Mathematical Sciences and the School of Mathematics, University of Edinburgh, JCMB, The King's Buildings, Peter Guthrie Tait Road, Edinburgh, EH9 3FD, Scotland; E-mail: j.r.wright@ed.ac.uk.

Received: 12 February 2024; **Revised:** 25 August 2024; **Accepted:** 10 September 2024

2020 Mathematics Subject Classification: 11B25, 11B30

Abstract

Let $P_1, \dots, P_m \in \mathbb{K}[y]$ be polynomials with distinct degrees, no constant terms and coefficients in a general local field $\mathbb{K}$. We give a quantitative count of the number of polynomial progressions $x, x + P_1(y), \dots, x + P_m(y)$ lying in a set $S \subseteq \mathbb{K}$ of positive density. The proof relies on a general L^∞ inverse theorem which is of independent interest. This inverse theorem implies a Sobolev improving estimate for multilinear polynomial averaging operators which in turn implies our quantitative estimate for polynomial progressions. This general Sobolev inequality has the potential to be applied in a number of problems in real, complex and p -adic analysis.

1. Introduction

Szemerédi's famous theorem [43] states that any set S of integers with positive (upper) density must necessarily contain arbitrarily long arithmetic progressions. Quantitative versions have been obtained by several authors, first by Roth [40] for three-term arithmetic progressions and by Gowers [18] in general, with the current best bounds due to Bloom and Sisask [7], Kelley and Meke [21] in the three-term case and Leng, Sah and Sawhney [26] for longer progressions (see also Green and Tao [17] and Gowers [18]). More generally, one can consider polynomial progressions $x, x + P_1(y), \dots, x + P_m(y)$ for $x, y \in \mathbb{Z}$ with $y \neq 0$, where $P_j \in \mathbb{Z}[y]$ is a sequence of polynomials with integer coefficients and no constant terms (the case of arithmetic progressions corresponding to linear polynomials). Bergelson and Leibman [6], extending earlier work of Bergelson, Furstenberg and Weiss [5], generalised Szemerédi's theorem to polynomial progressions. Obtaining quantitative versions of Bergelson and Leibman's result has been a challenging problem and no progress (outside a few results on two-term progressions) has been made until very recently.

Inspired by the earlier work of Bergelson, Furstenberg and Weiss, Bourgain obtained a quantitative lower bound on the count of three-term polynomial progressions in the setting of the real field $\mathbb{R}$. He accomplished this by coupling a technique he developed in his work on arithmetic progressions [2], together with Fourier-analytic methods.

Theorem 1.1 (Bourgain [3]). *Given $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that for any $N \geq 1$ and measurable set $S \subseteq [0, N]$ satisfying $|S \cap [0, N]| \geq \varepsilon N$, we have*

$$|\{(x, y) \in [0, N] \times [0, N^{1/d}] : x, x + y, x + y^d \in S\}| \geq \delta N^{1+1/d}. \quad (1.2)$$

In particular, we have the existence of a triple $x, x + y$ and $x + y^d$ belonging to S with y satisfying the gap condition $y \geq \delta N^{1/d}$.

The bound (1.2) implies a quantitative multiple recurrence result. Only recently have there been extensions to more general three-term progressions $x, x + P_1(y), x + P_2(y)$; see the work of Durcik, Guo and Roos [11] when $P_1(y) = y$ and general P_2 and of Chen, Guo and Li [8] for general $P_1, P_2 \in \mathbb{R}[y]$ with distinct degrees. The methods in these papers, using delicate oscillatory integral operator bounds, seem limited to three-term progressions.

In another direction, Bourgain and Chang [4] gave quantitative bounds for three-term progressions of the form $x, x + y, x + y^2$ in the setting of finite fields $\mathbb{F}_q$. This result was extended to more general three-term polynomial progressions by Peluse [36] and Dong, Li and Sawin [10]. The techniques in these papers, using a Fourier-analytic approach which relies on sophisticated exponential sum bounds over finite fields, also seem limited to three-term progressions.

By using new ideas in additive combinatorics, by-passing the need of inverse theorems for Gowers' uniformity norms of degree greater than 2, Peluse [37] recently made a significant advance, giving quantitative bounds for general polynomial progressions $x, x + P_1(y), \dots, x + P_m(y)$ in $\mathbb{F}_q$, where $\{P_1, \dots, P_m\} \subseteq \mathbb{Z}[y]$ are linearly independent over $\mathbb{Q}$.

Inspired by this work, Peluse and Prendiville [39] obtained the first quantitative bounds for three-term polynomial progressions in the setting of the integers $\mathbb{Z}$. This has been extended recently to general polynomial progressions $x, x + P_1(y), \dots, x + P_m(y)$ with $P_j \in \mathbb{Z}[y]$ having distinct degrees by Peluse [38]. So although the first quantitative bounds for polynomial progressions were made in the setting of the real field $\mathbb{R}$, we have seen major advances in both the finite field $\mathbb{F}_q$ and integer $\mathbb{Z}$ settings by employing new ideas in additive combinatorics.

One purpose of this paper is to rectify this situation for the continuous setting by establishing quantitative bounds for general polynomial progressions in the real field $\mathbb{R}$, bringing it in line with the recent advances in the finite field and integer settings. Another purpose is to illustrate how one can marry these new ideas in additive combinatorics with other ideas, notably from the work of Krause, Mirek and Tao [23] to obtain compactness results for general multilinear polynomial averaging operators which have implications for problems in euclidean harmonic analysis. These ideas and arguments are robust enough to allow us to obtain quantitative bounds for polynomial progressions in a general local field.

Theorem 1.3. *Let $\mathbb{K}$ be a local field with Haar measure μ . Let $\mathcal{P} = \{P_1, \dots, P_m\}$ be a sequence of polynomials in $\mathbb{K}[y]$ with distinct degrees and no constant terms, and let d denote the largest degree among the polynomials in $\mathcal{P}$. When $\mathbb{K}$ has positive characteristic, we assume the characteristic is larger than d .*

For any $\varepsilon > 0$, there exists a $\delta(\varepsilon, \mathcal{P}) > 0$ and $N(\varepsilon, \mathcal{P}) \geq 1$ such that for any $N \geq N(\varepsilon, \mathcal{P})$ and measurable set $S \subseteq \mathbb{K}$ satisfying $\mu(S \cap B_N) \geq \varepsilon N$, we have

$$\mu(\{(x, y) \in B_N \times B_{N^{1/d}} : x, x + P_1(y), \dots, x + P_m(y) \in S\}) \geq \delta N^{1+1/d}. \quad (1.4)$$

In particular, we have the existence of a progression $x, x + P_1(y), \dots, x + P_m(y)$ belonging to S with y satisfying the gap condition $|y| \geq \delta N^{1/d}$. The proof will show that we can take $\delta = \varepsilon^C \varepsilon^{-2m-2}$ for some $C = C_{\mathcal{P}} > 0$ and $N(\varepsilon, \mathcal{P}) = \varepsilon^{-C'} \varepsilon^{-2m-2}$ for a slightly larger $C' > C_{\mathcal{P}}$.

When $\mathbb{K} = \mathbb{R}$ is the real field, Theorem 1.3 extends the work in [3], [11] and [8] from three-term polynomial progressions to general polynomial progressions albeit for large N , depending on ε .

When $\mathbb{K} = \mathbb{C}$, Theorem 1.3 represents the first known results for complex polynomial progressions. The absolute value $|\cdot|$ used in the statement of Theorem 1.3 is normalised so that we can express the result in this generality (see Section 3). For any sequence of complex polynomials $\{P_1, \dots, P_m\} \subseteq \mathbb{C}[z]$ with distinct degrees and $P_j(0) = 0$, Theorem 1.3 has the following consequence: Given $\varepsilon > 0$, there is a $\delta > 0$ such that for sufficiently large N and any set S in the complex plane satisfying $|S \cap \mathbb{D}_N| \geq \varepsilon N^2$, we can find a progression of the form $w, w + P_1(z), \dots, w + P_m(z)$ lying in S such that $|z| \geq \delta N^{2/d}$.

Important in our analysis are certain properties for $m + 1$ linear forms formed from our collection $\mathcal{P} = \{P_1, \dots, P_m\} \subseteq \mathbb{K}[y]$ of m polynomials with distinct degrees, say $1 \leq \deg(P_1) < \dots < \deg(P_m) =: d$. Let $N \geq 1$ and consider the form

$$\Lambda_{\mathcal{P};N}(f_0, \dots, f_m) := \frac{1}{N^d} \int_{\mathbb{K}^2} f_0(x) \prod_{i=1}^m f_i(x - P_i(y)) d\mu_{[N]}(y) d\mu(x).$$

Here, $d\mu_{[N]}(y) = N^{-1} \mathbb{1}_{B_N(0)}(y) d\mu(y)$ is normalised measure on the ball $B_N(0)$ (we will describe notation used in the paper in Section 4). The key result in the proof of Theorem 1.3 is the following L^∞ inverse theorem for $\Lambda_{\mathcal{P};N}$ which is of independent interest.

Theorem 1.5 (Inverse theorem for $(m + 1)$ -linear forms). *With the setup above, let $f_0, f_1, \dots, f_m$ be 1-bounded functions supported on a ball $B \subset \mathbb{K}$ of measure N^d . Suppose that*

$$|\Lambda_{\mathcal{P};N}(f_0, \dots, f_m)| \geq \delta.$$

Then there exists $N_1 \simeq \delta^{O_{\mathcal{P}}(1)} N^{\deg(P_1)}$ such that

$$N^{-d} \|\mu_{[N_1]} * f_1\|_{L^1(\mathbb{K})} \gtrsim_{\mathcal{P}} \delta^{O_{\mathcal{P}}(1)}.$$

The main application of Theorem 1.5 for us will be to prove a precise structural result for multilinear polynomial operators of the form

$$A_N^{\mathcal{P}}(f_1, \dots, f_m)(x) = \int_{\mathbb{K}} f_1(x + P_1(y)) \cdots f_m(x + P_m(y)) d\mu_{[N]}(y).$$

We will use ideas in the recent work of Krause, Mirek and Tao [23] to accomplish this, and consequently, we will be able to establish the following important Sobolev estimate.

Theorem 1.6 (A Sobolev inequality for $A_N^{\mathcal{P}}$). *Let $1 < p_1, \dots, p_m < \infty$ satisfying $\frac{1}{p_1} + \dots + \frac{1}{p_m} = 1$ be given. Then for $N_j \simeq \delta^{O_{\mathcal{P}}(1)} N^{\deg(P_j)}$, we have*

$$\|A_N^{\mathcal{P}}(f_1, \dots, f_{j-1}, (\delta_0 - \varphi_{N_j}) * f_j, f_{j+1}, \dots, f_m)\|_{L^1(\mathbb{K})} \lesssim \delta^{1/8} \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{K})},$$

provided $N \gtrsim \delta^{-O_{\mathcal{P}}(1)}$. Here, φ_{N_j} is a smooth cut-off function such that $\widehat{\varphi_{N_j}}(\xi) \equiv 1$ for $\xi \in B_{N_j^{-1}}(0)$.

Following an argument of Bourgain in [3], we will show how Theorem 1.6 implies Theorem 1.3. Versions of Theorem 1.6 for two real polynomials $\{P_1, P_2\} \subseteq \mathbb{R}[y]$ were established in [3], [11] and [8] using delicate oscillatory integral operator bounds. Our arguments are much more elementary in nature and do not require deep oscillatory integral/exponential sum/character sum bounds outside a standard application of van der Corput bounds (see [41]) when $\mathbb{K} = \mathbb{R}$ or Hua's exponential sum bound [13] when $\mathbb{K} = \mathbb{Q}_p$ (which extends Mordell's classical bound from the finite field setting to complete exponential sums over $\mathbb{Z}/p^m\mathbb{Z}$) – these bounds extend readily to any local field $\mathbb{K}$; see Section 3. Furthermore, the Sobolev inequalities in [11] and [8] were only established for certain sparse sequences of scales N . The bound in Theorem 1.6 holds for all sufficiently large scales N .

The Sobolev bound in Theorem 1.6 potentially has many other applications. See [3] for a discussion on the implications of Theorem 1.6 to compactness properties of the multilinear operator $A_N^{\mathcal{P}}$. Pointwise convergence results for multilinear polynomial averages are common applications of such Sobolev bounds. See [8] where the Sobolev inequality is used to prove the existence of polynomial progressions in sets of sufficiently large Hausdorff dimension. See also [22], [24], [19], [20] and [9].

Our results require the scales N to be large. It would be interesting, for various applications, to establish these results for small scales as well.

2. Structure of the paper

After a review of analysis in the setting of local fields, including some essential but basic oscillatory integral bounds, we set up some notation and detail some tools involving the Gowers uniformity norms. In Section 5, we give some preliminary results necessary to carry out the core arguments. In Section 6, we give the proof of Theorem 1.5 which is based on a polynomial ergodic theorem (PET) induction scheme and a degree lowering argument developed by the third author in earlier work. In Section 7, we will prove Theorem 1.6. Finally, in Section 8, we show how Theorem 1.3 follows as a consequence of Theorem 1.6.

3. Review of basic analysis on local fields

A basic reference for the material reviewed in this section is [35].

Let $\mathbb{K}$ be a locally compact topological field with a nondiscrete topology. Such fields are called *local fields* and have a unique (up to a positive multiple) Haar measure μ . They also carry a nontrivial absolute value $|\cdot|$ such that the corresponding balls $B_r(x) = \{y \in \mathbb{K} : |y - x| \leq r\}$ generate the topology.

Recall that an absolute value on a field $\mathbb{K}$ is a map $|\cdot| : \mathbb{K} \rightarrow \mathbb{R}^+$ satisfying

$$(a) |x| = 0 \Leftrightarrow x = 0, \quad (b) |xy| = |x||y| \quad \text{and} \quad (c) |x + y| \leq C(|x| + |y|)$$

for some $C \geq 1$. It is *nontrivial* if there is an $x \neq 0$ such that $|x| \neq 1$. Two absolute values $|\cdot|_1$ and $|\cdot|_2$ are said to be *equivalent* if there is a $\theta > 0$ such that $|x|_2 = |x|_1^\theta$ for all $x \in \mathbb{K}$. Equivalent absolute values give the same topology. There is always an equivalent absolute value such that the triangle inequality (c) holds with $C = 1$. If $|\cdot|$ satisfies the stronger triangle inequality (c') $|x + y| \leq \max(|x|, |y|)$, we say that $|\cdot|$ is non-Archimedean. Note that if $|\cdot|$ is non-Archimedean, then all equivalent absolute values are non-Archimedean. The field $\mathbb{K}$ is said to be *non-Archimedean* if the underlying absolute value (and hence all equivalent ones) is non-Archimedean. Otherwise, we say $\mathbb{K}$ is Archimedean.

When $\mathbb{K}$ is Archimedean, then it is isomorphic to the real $\mathbb{R}$ or complex $\mathbb{C}$ field with the usual topology. In this case, Haar measure is a multiple of Lebesgue measure. When $\mathbb{K}$ is non-Archimedean, then it is a finite extension of a p -adic field $\mathbb{Q}_p$ in the characteristic zero case and a function field of Laurent series over a finite field in the positive characteristic case. Furthermore, the *ring of integers* $\mathcal{o}_{\mathbb{K}} := \{x \in \mathbb{K} : |x| \leq 1\}$ and the unique maximal ideal $\mathfrak{m}_{\mathbb{K}} := \{x \in \mathbb{K} : |x| < 1\}$ do not depend on the choice of absolute value (it is invariant when we pass to an equivalent absolute value). For any $\mathbb{K}$, we normalise Haar measure so that $\mu(B_1(0)) = 1$.

When $\mathbb{K}$ is non-Archimedean, the unique maximal ideal $\mathfrak{m}_{\mathbb{K}} = (\pi)$ is principal and we call any generating element π a *uniformizer*. Furthermore, the residue field $k := \mathcal{o}_{\mathbb{K}}/\mathfrak{m}_{\mathbb{K}}$ is finite, say with q elements. For $x \in \mathbb{K}$, there is a unique $n \in \mathbb{Z}$ such that $x = \pi^n u$ where u is a unit. We can go further and expand any $x \in \mathbb{K}$ as a Laurent series in π ; $x = \sum_{j \geq -L} x_j \pi^j$, where each x_j belongs to the residue field k . If $x_{-L} \neq 0$, then $x = \pi^{-L} u$, where $u = \sum_{j \geq -L} x_j \pi^{j+L}$ is a unit.

There is a choice of (equivalent) absolute value $|\cdot|$ such that $\mu(B_r(x)) \simeq r$ for all $r > 0$ and $x \in \mathbb{K}$. When $\mathbb{K} = \mathbb{R}$, we have $|x| = x \operatorname{sgn}(x)$ and when $\mathbb{K} = \mathbb{C}$, we have $|z| = z\bar{z}$. When $\mathbb{K}$ is non-Archimedean, then the absolute value $|x| := q^{-m}$, where $x = \pi^m u$ and u a unit has the property that its balls satisfy $\mu(B_r(x)) = q^n$, where $q^n \leq r < q^{n+1}$ and so $\mu(B_r(x)) \simeq r$. We choose the absolute value with this normalisation.

We will need a couple simple change of variable formulae which we will use again and again:

$$\int_{\mathbb{K}} f(x + y) d\mu(x) = \int_{\mathbb{K}} f(x) d\mu(x) \quad \text{and} \quad \int_{\mathbb{K}} f(y^{-1}x) d\mu(x) = |y| \int_{\mathbb{K}} f(x) d\mu(x).$$

The first follows from the translation invariance of the Haar measure μ . For the second formula, the measure $E \rightarrow \mu(yE)$ defined by an element $y \in \mathbb{K}$ is translation-invariant and so by the uniqueness of Haar measure, we have $\mu(yE) = \operatorname{mod}_{\mu}(y)\mu(E)$ for some nonnegative number $\operatorname{mod}_{\mu}(y)$, the so-called *modulus* of the measure μ . In fact $|y| := \operatorname{mod}_{\mu}(y)$ defines the absolute value with the desired

normalisation whose balls $B_r(x)$ satisfy $\mu(B_r(x)) \simeq r$. This proves the second change of variables formula. There is one additional, more sophisticated, nonlinear change of variable formula which we will need at one point, but we will justify this change of variables at the time.

The (additive) character group of $\mathbb{K}$ is isomorphic to itself. Starting with any nonprincipal character e on $\mathbb{K}$, all other characters χ can be identified with an element $y \in \mathbb{K}$ via $\chi(x) = e(yx)$. We fix a convenient choice for e ; when $\mathbb{K} = \mathbb{R}$, we take $e(x) = e^{2\pi i x}$. When $\mathbb{K}$ is non-Archimedean, we choose e so that $e \equiv 1$ on $\mathcal{O}_{\mathbb{K}}$ and nontrivial on $B_q(0)$; that is, there is a x_0 with $|x_0| = q$ such that $e(x_0) \neq 1$. The choice of e on $\mathbb{C}$ does not really matter but a convenient choice is $e(z) = e^{2\pi i \operatorname{Re} z}$. We define the Fourier transform

$$\widehat{f}(\xi) = \int_{\mathbb{K}} f(x) e(-\xi x) d\mu(x).$$

Plancherel's theorem and the Fourier inversion formula hold as in the real setting.

3.1. An oscillatory integral estimate

For $P(x) = a_d x^d + \cdots + a_1 x \in \mathbb{K}[x]$, we will use the following oscillatory integral bound:

$$|I(P)| \leq C_d [\max_j |a_j|]^{-1/d} \quad \text{where} \quad I(P) = \int_{B_1(0)} e(P(x)) d\mu(x). \quad (3.1)$$

When $\mathbb{K} = \mathbb{R}$, it is a simple matter to deduce the bound (3.1) from general oscillatory bounds due to van der Corput (see [41]). When $\mathbb{K} = \mathbb{Q}_p$ is the p -adic field, then

$$I(P) = p^{-s} \sum_{x=0}^{p^s-1} e^{2\pi i Q(x)/p^s} \quad \text{where} \quad p^s = \max_j |a_j| \quad \text{and} \quad Q(x) = b_d x^d + \cdots + b_1 x \in \mathbb{Z}[x]$$

satisfies $\gcd(b_d, \dots, b_1, p) = 1$; hence, a classical result of Hua [13] implies $|I(P)| \leq C_d p^{-s/d}$ which is Equation (3.1) in this case. It is natural to extend Hua's bound to other non-Archimedean fields; see, for example, [45] where character sums are treated over general Dedekind domains which in particular establishes Equation (3.1) for any non-Archimedean field $\mathbb{K}$ when the characteristic of $\mathbb{K}$ (if positive) is larger than d , a basic assumption appearing in our main result Theorem 1.3.

It is not straightforward to apply van der Corput bounds when $\mathbb{K} = \mathbb{C}$. However, we can see the bound (3.1) for both $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$ as a consequence of the following general bound due to Arkhipov, Chubarikov and Karatsuba [1]: Let $P \in \mathbb{R}[X_1, \dots, X_n]$ be a real polynomial of degree d in n variables. If $\mathbb{B}^n$ denotes the unit ball in $\mathbb{R}^n$, then

$$\left| \int_{\mathbb{B}^n} e^{2\pi i P(\underline{x})} d\underline{x} \right| \leq C_{d,n} H(P)^{-1} \quad \text{where} \quad H(P) = \min_{\underline{x} \in \mathbb{B}^n} \max_{\alpha} |\partial^{\alpha} P(\underline{x})|^{1/|\alpha|}. \quad (3.2)$$

A simple equivalence of norms argument shows that $H(P) \geq c_d [\max_{\alpha} |a_{\alpha}|]^{1/d}$, where $P(\underline{x}) = \sum_{\alpha} a_{\alpha} \underline{x}^{\alpha}$ and d is the degree of P . Hence, Equation (3.2) implies Equation (3.1) when $\mathbb{K} = \mathbb{R}$. When $\mathbb{K} = \mathbb{C}$ and $f(z) = a_d z^d + \cdots + a_1 z \in \mathbb{C}[z]$, write $f(x + iy) = P(x, y) + iQ(x, y)$ and note that

$$\int_{B_1(0)} e(f(z)) dz = \int_{\mathbb{B}^2} e^{2\pi i P(x,y)} dx dy$$

for the choice of character $e(z) = e^{2\pi i \operatorname{Re} z}$. From the Cauchy–Riemann equations, we have $H(P) \simeq d \min_{|z| \leq 1} \max_k |f^{(k)}(z)|^{1/2k} \geq c_d [\max_j |a_j|]^{1/2d}$ (recall we are using the absolute value $|z| = z\bar{z}$ on $\mathbb{C}$), and so Equation (3.2) implies Equation (3.1) with exponent $1/2d$ in this case. There is an alternative argument which establishes Equation (3.1) with the exponent $1/d$ when $\mathbb{K} = \mathbb{C}$ but this is unimportant for our purposes.

4. Some notation and basic tools

By a *scale* N , we mean a positive number when $\mathbb{K}$ is Archimedean and when $\mathbb{K}$ is non-Archimedean, it denotes a discrete value $N = q^k$, $k \in \mathbb{Z}$, a power of the cardinality of the residue field k . When N is a scale, we denote by $[N] := B_N(0)$ the ball with centre 0 and radius N . In this case, we have $\mu([N]) \simeq N$ (equality in the non-Archimedean case) by our normalisations of the absolute value $|\cdot|$ and Haar measure μ . An *interval* I is a ball $I = B_{r_I}(x_I)$ with some centre $x_I \in \mathbb{K}$ and radius $r_I > 0$. For an interval I , we associate the measure

$$d\mu_I(x) = \frac{1}{\mu(I)} \mathbb{1}_I(x) d\mu(x).$$

For an interval I , we define the Fejér kernel $\kappa_I(x) = \mu(I)^{-2} \mathbb{1}_I * \mathbb{1}_{-I}(x)$ and the corresponding measure $d\nu_I(x) = \kappa_I(x) d\mu(x)$. When $I = [N]$ for some scale N , we have $-I = I$ and so $\kappa_{[N]}(x) = N^{-2} \mathbb{1}_{[N]} * \mathbb{1}_{[N]}(x)$. Furthermore, when $\mathbb{K}$ is non-Archimedean, we have $\kappa_{[N]}(x) = N^{-1} \mathbb{1}_{[N]}(x)$ and so $d\nu_I = d\mu_I$ in this case. When $\mathbb{K} = \mathbb{R}$ and $I = [0, N]$, we have $\kappa_I(x) = N^{-1}(1 - |x|/N)$ when $|x| \leq N$ and zero otherwise.

We now give precise notation which we will use throughout the paper.

4.1. Basic notation

As usual, $\mathbb{Z}$ will denote the ring of rational integers.

1. We use $\mathbb{Z}_+ := \{1, 2, \dots\}$ and $\mathbb{N} := \mathbb{Z}_+ \cup \{0\}$ to denote the sets of positive integers and nonnegative integers, respectively.
2. For any $L \in \mathbb{R}_+$, we will use the notation

$$\llbracket L \rrbracket_0 := \{\ell \in \mathbb{N} : \ell \leq L\} \quad \text{and} \quad \llbracket L \rrbracket := \{\ell \in \mathbb{Z}_+ : \ell \leq L\}.$$

3. We use $\mathbb{1}_A$ to denote the indicator function of a set A . If S is a statement, we write $\mathbb{1}_S$ to denote its indicator, equal to 1 if S is true and 0 if S is false. For instance, $\mathbb{1}_A(x) = \mathbb{1}_{x \in A}$.

4.2. Asymptotic notation and magnitudes

The letters $C, c, C_0, C_1, \dots > 0$ will always denote absolute constants; however, their values may vary from occurrence to occurrence.

1. For two nonnegative quantities A, B , we write $A \lesssim_\delta B$ ($A \gtrsim_\delta B$) if there is an absolute constant $C_\delta > 0$ (which possibly depends on $\delta > 0$) such that $A \leq C_\delta B$ ($A \geq C_\delta B$). We will write $A \simeq_\delta B$ when $A \lesssim_\delta B$ and $A \gtrsim_\delta B$ hold simultaneously. We will omit the subscript δ if irrelevant.
2. For a function $f : X \rightarrow \mathbb{C}$ and positive-valued function $g : X \rightarrow (0, \infty)$, write $f = O(g)$ if there exists a constant $C > 0$ such that $|f(x)| \leq Cg(x)$ for all $x \in X$. We will also write $f = O_\delta(g)$ if the implicit constant depends on δ . For two functions $f, g : X \rightarrow \mathbb{C}$ such that $g(x) \neq 0$ for all $x \in X$ we write $f = o(g)$ if $\lim_{x \rightarrow \infty} f(x)/g(x) = 0$.

4.3. Polynomials

Let $\mathbb{K}[t]$ denote the space of all polynomials in one indeterminate t with coefficients in $\mathbb{K}$. Every polynomial $P \in \mathbb{K}[t]$ can be written as a formal power series

$$P(t) = \sum_{j=0}^{\infty} c_j t^j, \tag{4.1}$$

where all but finitely many coefficients $c_j \in \mathbb{K}$ vanish.

1. We define the degree of $P \in \mathbb{K}[t]$ by

$$\deg(P) := \max\{j \in \mathbb{Z}_+ : c_j \neq 0\}.$$

2. A finite collection $\mathcal{P} \subset \mathbb{K}[t]$ has degree $d \in \mathbb{N}$, if $d = \max\{\deg(P) : P \in \mathcal{P}\}$.

3. For a polynomial $P \in \mathbb{K}[t]$ and $j \in \mathbb{N}$, let $c_j(P)$ denote j -th coefficient of P . We also let $\ell(P)$ denote the leading coefficient of P ; that is, for P as in Equation (4.1) we have $c_j(P) = c_j$ for $j \in \mathbb{N}$ and $\ell(P) = c_d$ where $d = \deg P$.

4.4. L^p spaces

$(X, \mathcal{B}(X), \lambda)$ denotes a measure space X with σ -algebra $\mathcal{B}(X)$ and σ -finite measure λ .

1. The set of λ -measurable complex-valued functions defined on X will be denoted by $L^0(X)$.

2. The set of functions in $L^0(X)$ whose modulus is integrable with p -th power is denoted by $L^p(X)$ for $p \in (0, \infty)$, whereas $L^\infty(X)$ denotes the space of all essentially bounded functions in $L^0(X)$.

3. We will say that a function $f \in L^0(X)$ is 1-bounded if $f \in L^\infty(X)$ and $\|f\|_{L^\infty(X)} \leq 1$.

4. For any $n \in \mathbb{Z}_+$ the measure $\lambda^{\otimes n}$ will denote the product measure $\lambda \otimes \dots \otimes \lambda$ on the product space X^n with the product σ -algebra $\mathcal{B}(X) \otimes \dots \otimes \mathcal{B}(X)$.

4.5. Gowers box and uniformity norms

We will use the Gowers norm and Gowers box norm of a function f which is defined in terms of the multiplicative discrete derivatives $\Delta_{h_1, \dots, h_s} f(x)$: for $x, h \in \mathbb{K}$, we set $\Delta_h f(x) = f(x)f(x+h)$, and iteratively, we define

$$\Delta_{h_1, \dots, h_s} f(x) = \Delta_{h_1}(\Delta_{h_2}(\dots(\Delta_{h_s} f(x)) \dots)) \quad \text{where } x, h_1, \dots, h_s \in \mathbb{K}.$$

When $h = (h_1, \dots, h_s) \in \mathbb{K}^s$, we often write $\Delta_{h_1, \dots, h_s} f(x)$ as $\Delta_h f(x)$ or $\Delta_h^s f(x)$. For $\omega = (\omega_1, \dots, \omega_s) \in \{0, 1\}^s$, we write $\omega \cdot h := \sum_{i=1}^s \omega_i h_i$ and $|\omega| := \omega_1 + \dots + \omega_s$. If $\overline{\cdot}$ denotes the conjugation operator, we observe that

$$\Delta_h f(x) = \prod_{\omega \in \{0, 1\}^s} \mathcal{C}^{|\omega|} f(x + \omega \cdot h). \quad (4.2)$$

For any integer $s \geq 1$, we define the Gowers U^s norm of f by

$$\|f\|_{U^s}^{2^s} = \int_{\mathbb{K}^{s+1}} \Delta_{h_1, \dots, h_s} f(x) d\mu(h_1) \dots d\mu(h_s) d\mu(x).$$

We note that $\|f\|_{U^2} = \|\widehat{f}\|_{L^4}$.

For intervals $I, I_1, \dots, I_s$, we define the Gowers box norm as

$$\|f\|_{\square_{I_1, \dots, I_s}^s(I)}^{2^s} = \frac{1}{\mu(I)} \int_{\mathbb{K}^{s+1}} \Delta_{h_1, \dots, h_s} f(x) d\nu_{I_1}(h_1) \dots d\nu_{I_s}(h_s) d\mu(x).$$

From Equation (4.2), we see that

$$\|f\|_{\square_{I_1, \dots, I_{s+1}}^{s+1}(I)}^{2^{s+1}} = \int_{\mathbb{K}} \|\Delta_h f\|_{\square_{I_1, \dots, I_s}^s(I)}^{2^s} d\nu_{I_{s+1}}(h). \quad (4.3)$$

A similar formula relates the Gowers U^{s+1} norm to the Gowers U^s norm.

4.6. The Gowers–Cauchy–Schwarz inequality

When $s \geq 2$, both the Gowers uniformity norm and the Gowers box norm are in fact norms. In particular, the triangle inequality holds. The triangle inequality also holds when $s = 1$ and so we have that

$$\|f + g\|_{U^s} \leq \|f\|_{U^s} + \|g\|_{U^s} \quad \text{and} \quad \|f + g\|_{\square_{I_1, \dots, I_s}^s(I)} \leq \|f\|_{\square_{I_1, \dots, I_s}^s(I)} + \|g\|_{\square_{I_1, \dots, I_s}^s(I)} \quad (4.4)$$

holds for every $s \geq 1$. These inequalities follow from a more general inequality which we will find useful.

Let A be a finite set and for each $\alpha \in A$; let (X_α, du_α) be a probability space. Set $X = \prod_{\alpha \in A} X_\alpha$, and let $f : X \rightarrow \mathbb{C}$ be a complex-valued function. For any $x^{(0)} = (x_\alpha^{(0)})_{\alpha \in A}$ and $x^{(1)} = (x_\alpha^{(1)})_{\alpha \in A}$ in X and $\omega = (\omega_\alpha)_{\alpha \in A} \in \{0, 1\}^A$, we write $x^{(\omega)} = (x_\alpha^{(\omega_\alpha)})_{\alpha \in A}$. We define the *generalised Gowers box norm* of f on X as

$$\|f\|_{\square(X)}^{2|A|} = \iint_{X^2} \prod_{\omega \in \{0, 1\}^A} \mathcal{C}^{|\omega|} f(x^{(\omega)}) du(x^{(0)}) du(x^{(1)}),$$

where du denotes the product measure $\otimes_{\alpha \in A} du_\alpha$. The following lemma is established in [16].

Lemma 4.5 (Gowers–Cauchy–Schwarz inequality). *With the setup above, let $f_\omega : X \rightarrow \mathbb{C}$ for every $\omega \in \{0, 1\}^A$. We have*

$$\left| \iint_{X^2} \prod_{\omega \in \{0, 1\}^A} \mathcal{C}^{|\omega|} f_\omega(x^{(\omega)}) du(x^{(0)}) du(x^{(1)}) \right| \leq \prod_{\omega \in \{0, 1\}^A} \|f_\omega\|_{\square(X)}. \quad (4.6)$$

We will need the following consequence.

Corollary 4.7. *Let $f : X \rightarrow \mathbb{C}$ and for each $\alpha \in A$, suppose $g_\alpha : X \rightarrow \mathbb{C}$ is a 1-bounded function that is independent of the x_α variable. Then*

$$\left| \int_X f(x) \prod_{\alpha \in A} g_\alpha(x) du(x) \right|^{2|A|} \leq \int_{X^2} \prod_{\omega \in \{0, 1\}^A} \mathcal{C}^{|\omega|} f(x^{(\omega)}) du(x^{(0)}) du(x^{(1)}). \quad (4.8)$$

Proof. For $\omega^0 = (0, \dots, 0)$, set $f_{\omega^0} = f$ and for $\omega^\beta = (\omega_\alpha)_{\alpha \in A}$ with $\omega_\alpha = 0$ when $\alpha \neq \beta$ and $\omega_\beta = 1$, set $f_{\omega^\beta} = \overline{g_\beta}$. For all other choices of $\omega \in \{0, 1\}^A$, set $f_\omega = 1$. Hence,

$$\prod_{\omega \in \{0, 1\}^A} \mathcal{C}^{|\omega|} f_\omega(x^{(\omega)}) = f(x^{(0)}) \prod_{\alpha \in A} g_\alpha(x^{(0)})$$

since g_α is independent of the α variable. Therefore, the inequality (4.6) implies

$$\left| \int_X f(x) \prod_{\alpha \in A} g_\alpha(x) du(x) \right| \leq \prod_{\omega \in \{0, 1\}^A} \|f_\omega\|_{\square(X)} \leq \|f\|_{\square(X)}$$

by the 1-boundedness of each g_α . This proves Equation (4.8). $\square$

5. Some preliminaries

In this section, we establish a few useful results which we will need in our arguments.

5.1. U^2 -inverse theorem

We will use the following inverse theorem for the Gowers box norms.

Lemma 5.1 (U^2 -inverse theorem). *Let H_1 and H_2 be two scales, and let f be a 1-bounded function supported in an interval I . Then*

$$\|f\|_{\square_{[H_1], [H_2]}^2(I)}^4 \leq (H_1 H_2)^{-1} \|\widehat{f}\|_{L^\infty(\mathbb{K})}^2. \quad (5.2)$$

Proof. We have

$$\begin{aligned} \|f\|_{\square_{[H_1], [H_2]}^2(I)}^4 &= \frac{1}{\mu(I)} \iiint_{\mathbb{K}^3} \Delta_{h_1, h_2} f(x) d\nu_{[H_1]}(h_1) d\nu_{[H_2]}(h_2) d\mu(x) \\ &= \iint_{\mathbb{K}^2} g(h_1, h_2) d\nu_{[H_1]}(h_1) d\nu_{[H_2]}(h_2) = \iint_{\mathbb{K}^2} \widehat{g}(\xi_1, \xi_2) \overline{\widehat{\nu}_{[H_1]}(\xi_1)} \overline{\widehat{\nu}_{[H_2]}(\xi_2)} d\mu(\xi_1) d\mu(\xi_2), \end{aligned}$$

where

$$g(h_1, h_2) = \frac{1}{\mu(I)} \int_{\mathbb{K}} \Delta_{h_1, h_2} f(x) d\mu(x).$$

Hence,

$$\begin{aligned} \|f\|_{\square_{[H_1], [H_2]}^2(I)}^4 &\leq \|\widehat{\nu}_{[H_1]}\|_{L^1} \|\widehat{\nu}_{[H_2]}\|_{L^1} \sup_{\underline{\xi} \in \mathbb{K}^2} |\widehat{g}(\xi_1, \xi_2)| \\ &= \frac{H_1^{-1} H_2^{-1}}{\mu(I)} \sup_{\underline{\xi} \in \mathbb{K}^2} \left| \iiint_{\mathbb{K}^3} f_{00}(x) \overline{f_{10}(x + h_1)} \overline{f_{01}(x + h_2)} \overline{f_{11}(x + h_1 + h_2)} d\mu(x) d\mu(h_1) d\mu(h_2) \right| \end{aligned}$$

where $f_{00}(x) = f(x)e(-\xi_1 x - \xi_2 x)$,

$$f_{10}(x) = f(x)e(-\xi_1 x), \quad f_{01}(x) = f(x)e(-\xi_2 x) \quad \text{and} \quad f_{11}(x) = f(x).$$

The final equality follows since $|\widehat{\nu}_{[H_j]}(\xi)| = |\widehat{\mu}_{[H_j]}(\xi)|^2$, and so

$$\|\widehat{\nu}_{[H_j]}\|_{L^1(\mathbb{K})} = \|\widehat{\mu}_{[H_j]}\|_{L^2(\mathbb{K})}^2 = \|H_j^{-1} \mathbb{1}_{[H_j]}\|_2^2 = H_j^{-1} \quad \text{for } j \in \{1, 2\}$$

by Plancherel's theorem. Furthermore,

$$\widehat{g}(\xi_1, \xi_2) = \frac{1}{\mu(I)} \iiint_{\mathbb{K}^3} \Delta_{h_1, h_2} f(x) e(\xi_1 h_1 + \xi_2 h_2) d\mu(h_1) d\mu(h_2) d\mu(x).$$

Appealing to the Gowers–Cauchy–Schwarz inequality (4.6), we see that

$$\|f\|_{\square_{[H_1], [H_2]}^2(I)}^4 \leq (\mu(I) H_1 H_2)^{-1} \|f\|_{U^2}^4 = (\mu(I) H_1 H_2)^{-1} \|\widehat{f}\|_{L^4}^4 \leq (H_1 H_2)^{-1} \|\widehat{f}\|_{L^\infty}^2$$

as desired. The last inequality follows from Plancherel's theorem, the 1-boundedness of f and $\text{supp}(f) \subset I$ which implies

$$\|\widehat{f}\|_{L^4}^4 \leq \|\widehat{f}\|_{L^\infty}^2 \|\widehat{f}\|_{L^2}^2 = \|\widehat{f}\|_{L^\infty}^2 \|f\|_{L^2}^2 \leq \mu(I) \|\widehat{f}\|_{L^\infty}^2.$$

□

5.2. van der Corput's inequality

We will need the following useful inequality.

Lemma 5.3 (van der Corput's inequality). *Let $\mathbf{g} \in L^1(\mathbb{K})$, and let $J = B_{r_J}(x_J)$ be an interval. Then for any scale H , $0 < H \leq \mu(J)$, we have*

$$\left| \int_{\mathbb{K}} \mathbf{g}(y) d\mu_J(y) \right|^2 \leq \frac{C}{\mu(J)} \int_{\mathbb{K}} \int_{J \cap (J-h)} \Delta_h \mathbf{g}(y) d\mu(y) dv_{[H]}(h). \quad (5.4)$$

We can take $C = 4$ when $\mathbb{K}$ is Archimedean. When $\mathbb{K}$ is non-Archimedean, we can take $C = 1$ and furthermore, $\mathbb{1}_J(y)\mathbb{1}_J(y+h) = \mathbb{1}_{J \cap (J-h)}(y) = \mathbb{1}_J(y)$ for any $h \in [H]$ so that the above inequality can be expressed as

$$\left| \int_{\mathbb{K}} \mathbf{g}(y) d\mu_J(y) \right|^2 \leq \iint_{\mathbb{K}^2} \Delta_h \mathbf{g}(y) d\mu_{[H]}(h) d\mu_J(y) \quad (5.5)$$

since $dv_{[H]} = d\mu_{[H]}$ in this case.

Proof. We define $\mathbf{g}_J(y) := \mathbf{g}(y)\mathbb{1}_J(y)$. By a change of variables and Fubini's theorem, we note

$$\int_{\mathbb{K}} \mathbf{g}(y) d\mu_J(y) = \frac{1}{\mu(J)} \iint_{\mathbb{K}^2} \mathbf{g}_J(y+h) d\mu_{[H]}(h) d\mu(y).$$

The function $y \mapsto \int_{\mathbb{K}} \mathbf{g}_J(y+h) d\mu_{[H]}(h)$ is supported on the set $J - [H]$ which in turn lies in $B_{2(r_J+H)}(x_J)$ (in the non-Archimedean case, $J - [H] = J$). Hence, by the Cauchy–Schwarz inequality and a change of variables, we conclude that

$$\begin{aligned} \left| \int_{\mathbb{K}} \mathbf{g}(y) d\mu_J(y) \right|^2 &= \frac{1}{\mu(J)^2} \left| \iint_{\mathbb{K}^2} \mathbf{g}_J(y+h) d\mu_{[H]}(h) d\mu(y) \right|^2 \\ &\leq 2 \frac{\mu(J) + H}{\mu(J)^2} \iiint_{\mathbb{K}^3} \mathbf{g}_J(y+h_1) \overline{\mathbf{g}_J(y+h_2)} d\mu_{[H]}(h_1) d\mu_{[H]}(h_2) d\mu(y) \\ &= 2 \frac{\mu(J) + H}{\mu(J)^2} \iint_{\mathbb{K}^2} \kappa_{[H]}(h) \mathbf{g}_J(y) \overline{\mathbf{g}_J(y+h)} d\mu(h) d\mu(y) \\ &\leq 4\mu(J)^{-1} \int_{\mathbb{K}} \int_{J \cap (J-h)} \mathbf{g}(y) \overline{\mathbf{g}(y+h)} d\mu(y) dv_{[H]}(h), \end{aligned}$$

since $\kappa_{[H]}(h) = H^{-2} \int_{\mathbb{K}} \mathbb{1}_{[H]}(h_1) \mathbb{1}_{[H]}(h+h_1) d\mu(h_1)$. This gives the desired conclusion. $\square$

5.3. Preparation for the PET induction scheme

We now give a simple application of van der Corput's inequality which will be repeatedly applied in the PET induction scheme.

Lemma 5.6. *Let $c \geq 1$, and let $I, J \subset \mathbb{K}$ be two intervals with $\mu(I) = N_0$. Assume that $\mathbf{g}_1 \in L^\infty(\mathbb{K})$ and $\mathbf{g}_2 \in L^\infty(\mathbb{K}^2)$ are 1-bounded functions such that*

$$\|\mathbf{g}_1\|_{L^1(\mathbb{K})} \leq N_0, \quad \text{and} \quad \sup_{y \in \mathbb{K}} \|\mathbf{g}_2(\cdot, y)\|_{L^1(\mathbb{K})} \leq cN_0. \quad (5.7)$$

Suppose H is a scale such that $0 < H \leq \mu(J)$. When $\mathbb{K}$ is Archimedean, we have

$$\begin{aligned} &\left| \frac{1}{N_0} \iint_{\mathbb{K}^2} \mathbf{g}_1(x) \mathbf{g}_2(x, y) d\mu_J(y) d\mu(x) \right|^2 \\ &\leq 4 \left| \frac{1}{N_0} \iiint_{\mathbb{K}^3} \mathbf{g}_2(x, y) \overline{\mathbf{g}_2(x, y+h)} d\mu_J(y) dv_{[H]}(h) d\mu(x) \right| + 8c \left[\frac{\mu([H])}{\mu(J)} \right]^\theta, \end{aligned}$$

where $\theta = 1$ when $\mathbb{K} = \mathbb{R}$ and $\theta = 1/2$ when $\mathbb{K} = \mathbb{C}$. When $\mathbb{K}$ is non-Archimedean, this improves to

$$\begin{aligned} & \left| \frac{1}{N_0} \iint_{\mathbb{K}^2} g_1(x) g_2(x, y) d\mu_J(y) d\mu(x) \right|^2 \\ & \leq \frac{1}{N_0} \iiint_{\mathbb{K}^3} g_2(x, y) \overline{g_2(x, y + h)} d\mu_J(y) d\mu_{[H]}(h) d\mu(x). \end{aligned}$$

Proof. Applying the Cauchy–Schwarz inequality in the x variable, it follows that

$$\left| \frac{1}{N_0} \iint_{\mathbb{K}^2} g_1(x) g_2(x, y) d\mu_J(y) d\mu(x) \right|^2 \leq \frac{1}{N_0} \int_{\mathbb{K}} \left| \int_{\mathbb{K}} g_2(x, y) d\mu_J(y) \right|^2 d\mu(x)$$

since by Equation (5.7) and the 1-boundedness of g_1 , we have $\|g_1\|_{L^2(\mathbb{K})}^2 \leq N_0$. By van der Corput’s inequality in Lemma 5.3, we obtain

$$\begin{aligned} & \int_{\mathbb{K}} \left| \int_{\mathbb{K}} g_2(x, y) d\mu_J(y) \right|^2 d\mu(x) \\ & \leq 4 \int_{\mathbb{K}} \int_{\mathbb{K}} \kappa_{[H]}(h) \frac{1}{\mu(J)} \int_{J \cap (J-h)} g_2(x, y) \overline{g_2(x, y + h)} d\mu(y) d\mu(h) d\mu(x) \end{aligned}$$

when $\mathbb{K}$ is Archimedean. In this case, we have $\mu(J \setminus [J \cap (J - h)]) \leq 2\mu([H])$ when $\mathbb{K} = \mathbb{R}$ and $\mu(J \setminus [J \cap (J - h)]) \leq 2\sqrt{\mu([H])}\mu(J)$ when $\mathbb{K} = \mathbb{C}$. Hence,

$$\frac{4}{N_0} \int_{\mathbb{K}} \kappa_{[H]}(h) \frac{1}{\mu(J)} \int_{J \setminus (J \cap (J-h))} \int_{\mathbb{K}} |g_2(x, y)| d\mu(x) d\mu(y) d\mu(h) \leq 8c \left[\frac{\mu([H])}{\mu(J)} \right]^\theta.$$

In the last line, we used Fubini’s theorem and Equation (5.7) for g_2 . This gives the desired bound when $\mathbb{K}$ is Archimedean.

When $\mathbb{K}$ is non-Archimedean, the bound (5.5) in Lemma 5.3 gives

$$\begin{aligned} & \frac{1}{N_0} \int_{\mathbb{K}} \left| \int_{\mathbb{K}} g_2(x, y) d\mu_J(y) \right|^2 d\mu(x) \\ & \leq \frac{1}{N_0} \iiint_{\mathbb{K}^3} g_2(x, y) \overline{g_2(x, y + h)} d\mu_J(y) d\mu_{[H]}(h) d\mu(x) \end{aligned}$$

which is the desired bound in this case. $\square$

The next result is an essential building block of the PET induction scheme, which will be employed in Section 6.

Proposition 5.8. *Let $N, N_0 > 0$ be two scales, I an interval such that $\mu(I) = N_0$, $m \in \mathbb{N}$, $i_0 \in \llbracket m \rrbracket$, and let $\mathcal{P} := \{P_1, \dots, P_m\}$ be a collection of polynomials. Suppose that $\mathfrak{f}_0, \mathfrak{f}_1, \dots, \mathfrak{f}_m \in L^0(\mathbb{K})$ are 1-bounded functions such that $\|\mathfrak{f}_i\|_{L^1(\mathbb{K})} \leq N_0$ for every $i \in \llbracket m \rrbracket_0$.*

Let $0 < \delta \leq 1$, and suppose that

$$\left| \frac{1}{N_0} \iint_{\mathbb{K}^2} \mathfrak{f}_0(x) \prod_{i=1}^m \mathfrak{f}_i(x - P_i(y)) d\mu_{[N]}(y) d\mu(x) \right| \geq \delta. \quad (5.9)$$

Then there exists an absolute constant $C \gtrsim_P 1$ such that for all $\delta' \leq \delta^4/C$ we have

$$\left| \frac{1}{N_0} \iint_{\mathbb{K}^2} \mathfrak{f}'_0(x) \prod_{i=1}^{m'} \mathfrak{f}'_i(x - P'_i(y)) d\mu_{[N]}(y) d\mu(x) \right| \gtrsim_C \delta^2, \quad (5.10)$$

where $m' < 2m$ and $\mathcal{P}' := \{P'_1, \dots, P'_{m'}\}$ is a new collection of polynomials such that

$$\mathcal{P}' = \{P_1(y) - P_{i_0}(y), P_1(y + h) - P_{i_0}(y), \dots, P_m(y) - P_{i_0}(y), P_m(y + h) - P_{i_0}(y)\},$$

for some $\delta' \delta^2 N / C^2 \leq |h| \leq \delta' N \leq \delta^4 N / C$, where $P'_{m'}(y) := P_m(y) - P_{i_0}(y)$, and $\{\mathbf{f}'_0, \dots, \mathbf{f}'_{m'}\} := \{\mathbf{f}_1, \overline{\mathbf{f}_1}, \dots, \mathbf{f}_m, \overline{\mathbf{f}_m}\}$ with $\mathbf{f}'_{m'} := \mathbf{f}_m$.

Proof. Let $\mathbb{I} := \llbracket m \rrbracket$ and $C \geq 1$ be a large constant to be determined later. We shall apply Lemma 5.6 with $J = \llbracket N \rrbracket$, the functions $\mathbf{g}_1(x) = \mathbf{f}_0(x)$ and $\mathbf{g}_2(x, y) = \prod_{i \in \mathbb{I}} \mathbf{f}_i(x - P_i(y))$, and the parameter $H = \delta' N$. Note that $\|\mathbf{g}_1\|_{L^\infty(\mathbb{K})} \leq 1$ and $\|\mathbf{g}_2\|_{L^\infty(\mathbb{K}^2)} \leq 1$ since $\|\mathbf{f}_i\|_{L^\infty(\mathbb{K})} \leq 1$ for all $i \in \mathbb{I}$. Moreover, $\mathbf{g}_1$ and $\mathbf{g}_2$ satisfy Equation (5.7). If $\delta' \leq \delta^4 / C$ and $C \geq 1$ is sufficiently large, using Lemma 5.6, we conclude

$$\left| \frac{1}{N_0} \iiint_{\mathbb{K}^3} \mathbf{g}_2(x, y) \overline{\mathbf{g}_2(x, y + h)} d\mu_{\llbracket N \rrbracket}(y) d\nu_{\llbracket H \rrbracket}(h) d\mu(x) \right| \gtrsim \delta^2.$$

By the pigeonhole principle, there exists $|h| \geq \delta^2 H / C^2$ so that

$$\left| \frac{1}{N_0} \iint_{\mathbb{K}^2} \mathbf{g}_2(x, y) \overline{\mathbf{g}_2(x, y + h)} d\mu_{\llbracket N \rrbracket}(y) d\mu(x) \right| \gtrsim \delta^2.$$

We make the change of variables $x \mapsto x + P_{i_0}(y)$ to conclude

$$\left| \frac{1}{N_0} \iint_{\mathbb{K}^2} \prod_{i \in \mathbb{I}} \mathbf{f}_i(x - P_i(y) + P_{i_0}(y)) \overline{\mathbf{f}_i(x - P_i(y + h) + P_{i_0}(y))} d\mu_{\llbracket N \rrbracket}(y) d\mu(x) \right| \gtrsim \delta^2.$$

This completes the proof. $\square$

6. The L^∞ -inverse theorem

The goal of this section is to present the proof of Theorem 1.5, the key L^∞ -inverse theorem for general polynomials with distinct degrees, which we now restate in a more formal, precise way.

Theorem 6.1 (Inverse theorem for $(m + 1)$ -linear forms). *Let $N \geq 1$ be a scale, $m \in \mathbb{Z}_+$ and $0 < \delta \leq 1$ be given. Let $\mathcal{P} := \{P_1, \dots, P_m\}$ be a collection of polynomials such that $1 \leq \deg P_1 < \dots < \deg P_m$. Set $N_0 = N^{\deg(P_m)}$, and let $f_0, f_1, \dots, f_m \in L^0(\mathbb{K})$ be 1-bounded functions supported on an interval $I \subset \mathbb{K}$ of measure N_0 . Define an $(m + 1)$ -linear form corresponding to the pair $(\mathcal{P}; N)$ by*

$$\Lambda_{\mathcal{P}; N}(f_0, \dots, f_m) := \frac{1}{N_0} \int_{\mathbb{K}^2} f_0(x) \prod_{i=1}^m f_i(x - P_i(y)) d\mu_{\llbracket N \rrbracket}(y) d\mu(x). \quad (6.2)$$

Suppose that

$$|\Lambda_{\mathcal{P}; N}(f_0, \dots, f_m)| \geq \delta. \quad (6.3)$$

Then there exists $N_1 \simeq \delta^{O_{\mathcal{P}}(1)} N^{\deg(P_1)}$ so that

$$N_0^{-1} \|\mu_{\llbracket N_1 \rrbracket} * f_1\|_{L^1(\mathbb{K})} \gtrsim_{\mathcal{P}} \delta^{O_{\mathcal{P}}(1)}. \quad (6.4)$$

If necessary, we will also write $\Lambda_{\mathcal{P}; N}(f_0, \dots, f_m) = \Lambda_{\mathcal{P}; N, I}(f_0, \dots, f_m)$ in order to emphasize that the functions $f_0, f_1, \dots, f_m$ are supported on I .

Remark

When $\mathbb{K} = \mathbb{C}$ is the complex field, the proof of Theorem 6.1 will also hold if the form $\Lambda_{\mathcal{P};N}$ is defined with the disc $[N] = \mathbb{D}_{\sqrt{N}}$ replaced by the square

$$[N]_{sq} := \{x + iy \in \mathbb{C} : |x| \leq \sqrt{N}, |y| \leq \sqrt{N}\}.$$

In this case, the conclusion is $N_0^{-1} \|\mu_{[N]_{sq}} * f_1\|_{L^1(\mathbb{C})} \gtrsim \delta^{O_{\mathcal{P}}(1)}$. This observation will be needed at one point in the proof of Theorem 1.6.

The proof of Theorem 6.1 breaks into two main steps: First, an application of PET induction to show that whenever

$$|\Lambda_{\mathcal{P};N}(f_0, f_1, \dots, f_m)| \geq \delta$$

is large, then necessarily f_m has a fairly large U^s norm for an appropriately large $s = s_{\mathcal{P}}$. Second, an inductive ‘degree-lowering’ step to reduce U^s control to U^2 control. We accordingly subdivide the argument into two subsections.

6.1. PET induction

Our first goal is to show that whenever the multilinear form $\Lambda_{\mathcal{P};I}$ is large, necessarily f_m has some fairly large (sufficiently high degree) Gowers box norm. We begin with the definition of (d, j) -admissible polynomials. Recall that for a polynomial $P \in \mathbb{K}[y]$, the leading coefficient is denoted by $\ell(P)$.

Definition 6.5 (The class of (d, j) -admissible polynomials). Let $N \geq 1$ be a scale, $0 < \delta \leq 1$, $d \in \mathbb{Z}_+$, $j \in \llbracket d \rrbracket$ and parameters $A_0 \geq 1$ and $A \geq 0$ be given. Assume that a finite collection of polynomials $\mathcal{P}$ has degree j , and define $\mathcal{P}_j := \{P \in \mathcal{P} : \deg(P) = j\}$. We will say that $\mathcal{P}$ is (d, j) -admissible with tolerance (A_0, A) if the following properties are satisfied:

1. For every $P \in \mathcal{P}_j$, we have

$$A_0^{-1} \delta^A N^{d-j} \leq |\ell(P)| \leq A_0 \delta^{-A} N^{d-j}. \quad (6.6)$$

2. Whenever $P, Q \in \mathcal{P}_j$ and $\ell(P) \neq \ell(Q)$, we have

$$A_0^{-1} \delta^A N^{d-j} \leq |\ell(P) - \ell(Q)| \leq A_0 \delta^{-A} N^{d-j}. \quad (6.7)$$

3. Whenever $P, Q \in \mathcal{P}_j$ and $P \neq Q$ and $\ell(P) = \ell(Q)$, we have

$$A_0^{-1} \delta^A N^{d-j+1} \leq |\ell(P - Q)| \leq A_0 \delta^{-A} N^{d-j+1}, \quad (6.8)$$

and $\deg(P - Q) = j - 1$.

In the special case where the polynomials in $\mathcal{P}$ are linear, we require that $\ell(P) \neq \ell(Q)$ for each $P, Q \in \mathcal{P}$. The constants A_0, A will be always independent of δ and N but may depend on $\mathcal{P}$. In our applications, the exact values of A_0, A will be unimportant, and then we will simply say that the collection $\mathcal{P}$ is (d, j) -admissible.

Remark 6.9. Under the hypotheses of Theorem 6.1, it is not difficult to see that the collection of polynomials $\mathcal{P} = \{P_1, \dots, P_m\}$ such that $1 \leq \deg P_1 < \dots < \deg P_m = d$ is (d, d) -admissible with the tolerance $(\max\{|\ell(P_m)|^{-1}, |\ell(P_m)|\}, 0)$. Indeed, condition (6.6) can be easily verified and conditions (6.7) and (6.8) are vacuous as $\mathcal{P}_d = \{P_m\}$.

The main result of this subsection is the following theorem.

Theorem 6.10 (Gowers box norms control $(m+1)$ -linear forms). *Let $\mathcal{P} := \{P_1, \dots, P_m\}$ be a collection of (d, d) -admissible polynomials such that $1 \leq \deg P_1 \leq \dots \leq \deg P_m = d$. Let $N, N_0 \geq 1$ be two scales, I an interval with measure N_0 and $0 < \delta \leq 1$ be given, and let $f_0, f_1, \dots, f_m \in L^0(\mathbb{K})$ be 1-bounded functions such that $\|f_i\|_{L^1(\mathbb{K})} \leq N_0$ for all $i \in \llbracket m \rrbracket_0$. If Equation (6.3) is satisfied, then there exists $s := s_{\mathcal{P}} \in \mathbb{Z}_+$ such that*

$$\|f_m\|_{\square_{[H_1], \dots, [H_s]}^s(I)} \gtrsim_{\mathcal{P}} \delta^{O_{\mathcal{P}}(1)}, \quad (6.11)$$

where $H_i \simeq \delta^{O_{\mathcal{P}}(1)} N^{\deg(P_m)}$ for $i \in \llbracket s \rrbracket$.

The proof of Theorem 6.10 requires a subtle downwards induction based on a repetitive application of Proposition 5.8 on the class of (d, j) -admissible polynomials. To make our induction rigorous, we will assign a weight vector to each collection $\mathcal{P} \subset \mathbb{K}[t]$ of polynomials.

Definition 6.12 (Weight vector). For any finite $\mathcal{P} \subset \mathbb{K}[t]$, define the weight vector

$$v(\mathcal{P}) := (v_1, v_2, \dots) \in \mathbb{N}^{\mathbb{Z}_+},$$

where

$$v_j := v_j(\mathcal{P}) := \#\{\ell(P) : P \in \mathcal{P} \text{ and } \deg(P) = j\},$$

is the number of distinct leading coefficients of $\mathcal{P}$ of degree $j \in \mathbb{Z}_+$.

For example, the weight vector for the family $\mathcal{P} = \{x, 5x, x^2, x^2+x, x^4\}$ is $v(\mathcal{P}) = (2, 1, 0, 1, 0, 0, \dots)$. There is a natural ordering on the set of weight vectors.

Definition 6.13 (Well-ordering on the set of weight vectors). For any two weight vectors $v(\mathcal{P}) = (v_1(\mathcal{P}), v_2(\mathcal{P}), \dots)$ and $v(\mathcal{Q}) = (v_1(\mathcal{Q}), v_2(\mathcal{Q}), \dots)$ corresponding to finite collections $\mathcal{P}, \mathcal{Q} \subset \mathbb{K}[t]$ we define an ordering $<$ on the set of weight vectors by declaring that

$$v(\mathcal{P}) < v(\mathcal{Q})$$

if there is a degree $j \in \mathbb{Z}_+$ such that $v_j(\mathcal{P}) < v_j(\mathcal{Q})$ and $v_k(\mathcal{P}) = v_k(\mathcal{Q})$ for all $k > j$.

It is a standard fact that $<$ is a well ordering, we omit the details.

Proof of Theorem 6.10. We begin by stating the following claim:

Claim 6.14. *Let $N, N_0 \geq 1$ be two scales, $0 < \delta \leq 1$, $d, m \in \mathbb{Z}_+$ and $j \in \llbracket d \rrbracket$ be given, and let $\mathcal{P} := \{P_1, \dots, P_m\}$ be a collection of (d, j) -admissible polynomials with tolerance (A_0, A) such that $\deg P_1 \leq \dots \leq \deg P_m = j$. Let I be an interval with $\mu(I) = N_0$, and let $f_0, f_1, \dots, f_m \in L^0(\mathbb{K})$ be 1-bounded functions such that $\|f_i\|_{L^1(\mathbb{K})} \leq N_0$ for all $i \in \llbracket m \rrbracket_0$. Suppose that*

$$|\Lambda_{\mathcal{P}; N}(f_0, \dots, f_m)| \geq \delta. \quad (6.15)$$

Then there exists a collection $\mathcal{P}' := \{P'_1, \dots, P'_{m'}\}$ of $(d, j-1)$ -admissible polynomials with tolerance (A'_0, A') and $m' := \#\mathcal{P}'$ so that $\deg(P'_1) \leq \dots \leq \deg(P'_{m'}) = j-1$, and 1-bounded functions $f'_0, f'_1, \dots, f'_{m'} \in L^0(\mathbb{K})$ such that $\|f'_i\|_{L^1(\mathbb{K})} \leq N_0$ for all $i \in \llbracket m' \rrbracket_0$ with $f'_{m'} := f_m$ and satisfying

$$|\Lambda_{\mathcal{P}'; N}(f'_0, \dots, f'_{m'})| \gtrsim_{\mathcal{P}} \delta^{O_{\mathcal{P}}(1)}. \quad (6.16)$$

The proof of Claim 6.14 will use the polynomial exhaustion technique based on an iterative application of the PET induction scheme from Proposition 5.8. The key steps of this method are gathered in Proposition 6.20. Assuming momentarily that Claim 6.14 is true, we can easily close the argument to

prove Theorem 6.10. We begin with a collection of (d, d) -admissible polynomials such that $\deg P_1 \leq \dots \leq \deg P_m = d$ and apply our claim $d - 1$ times until we reach a collection of $(d, 1)$ -admissible linear polynomials $\mathcal{L}$ with distinct leading terms, which satisfies Equation (6.16) with $\mathcal{P}' = \mathcal{L}$. In the special case where all polynomials are linear matters simplify and can be handled using the next result, Proposition 6.17, which in turn implies Equation (6.11) from Theorem 6.10 as desired. $\square$

Proposition 6.17. *Let $N, N_0 \geq 1$ be two scales, I an interval with $\mu(I) = N_0$, $0 < \delta \leq 1$, $d, m \in \mathbb{Z}_+$ be given and let $\mathcal{L} := \{L_1, \dots, L_m\}$ be a collection of $(d, 1)$ -admissible linear polynomials. Let $f_0, f_1, \dots, f_m \in L^0(\mathbb{K})$ be 1-bounded functions such that $\|f_i\|_{L^1(\mathbb{K})} \leq N_0$ for all $i \in \llbracket m \rrbracket_0$. Suppose that*

$$|\Lambda_{\mathcal{L};N}(f_0, \dots, f_m)| \geq \delta. \quad (6.18)$$

Then we have

$$\|f_m\|_{\square_{[H_1], \dots, [H_m]}^m(I)} \gtrsim_{\mathcal{L}} \delta^{2^{m-1}}, \quad (6.19)$$

where $H_i \simeq \delta^{O_{\mathcal{L}}(1)} N^d$ for $i \in \llbracket s \rrbracket$.

In fact Proposition 6.17 is a special case of Theorem 6.10 with the collection of linear polynomials $\mathcal{L}$ in place of $\mathcal{P}$.

Proof of Proposition 6.17. Defining $\mathcal{L}' = \{L'_i := L_i - L_i(0) : i \in \llbracket m \rrbracket\}$ we see that each $L' \in \mathcal{L}'$ is linear with vanishing constant term and

$$\Lambda_{\mathcal{L};N}(f_0, \dots, f_m) = \Lambda_{\mathcal{L}';N}(g_0, \dots, g_m),$$

where $g_i(x) = T_{-L_i(0)} f_i(x) = f_i(x + L_i(0))$ for each $i \in \llbracket m \rrbracket$. We now apply Lemma 5.6 with functions $g_1(x) = g_0(x)$ and $g_2(x, y) = \prod_{i=1}^m g_i(x - L'_i(y))$ and intervals $J = [N]$, and a parameter $H = \delta^M N/M$ for some large absolute constant $M \geq 1$, which will be specified later. Using Lemma 5.6 and changing the variables $x \mapsto x - L_1(y)$ we obtain

$$\left| \frac{1}{N_0} \iiint_{\mathbb{K}^3} \Delta_{\ell(L_1)h} g_1(x) \prod_{i=2}^m \Delta_{\ell(L_i)h} g_i(x - (L_i - L_1)(y)) d\mu_{[N]}(y) d\mu(x) dv_{[H]}(h) \right| \gtrsim_M \delta^2.$$

Applying Lemma 5.6 $m - 2$ more times and changing the variables $x \mapsto x - L_m(0)$, we obtain

$$\left| \frac{1}{N_0} \int_{\mathbb{K}^{m+1}} \Delta_{u_1 h_1} \cdots \Delta_{u_{m-1} h_{m-1}} \Delta_{\ell(L_m)h_m} f_m(x) dv_{[H]}^{\otimes m}(h_1, \dots, h_m) d\mu(x) \right| \gtrsim_M \delta^{2^{m-1}},$$

where $u_i := \ell(L_m) - \ell(L_i)$ for $i \in \llbracket m - 1 \rrbracket$. By another change of variables we obtain Equation (6.19) with

$$H_m = |\ell(L_m)| \delta^M N/M, \quad \text{and} \quad H_i = |\ell(L_m) - \ell(L_i)| \delta^M N/M$$

for $i \in \llbracket m - 1 \rrbracket$. Using Equation (6.6) with $P = L_m$, and Equation (6.7) with $P = L_m$ and $Q = L_i$ we obtain that $H_i \simeq \delta^{O_{\mathcal{L}}(1)} N^d$ for $i \in \llbracket s \rrbracket$ provided that $M \geq 1$ is sufficiently large. This completes the proof of Proposition 6.17. $\square$

Proposition 6.20. *Let $N, N_0 > 0$ be two scales, $0 < \delta \leq 1$, $d, m \in \mathbb{Z}_+$ and $i, j \in \llbracket d \rrbracket$ be given, and let $\mathcal{P} := \{P_1, \dots, P_m\}$ be a collection of (d, j) -admissible polynomials with tolerance (A_0, A) such that $i = \deg P_1 \leq \dots \leq \deg P_m = j$. Let I be an interval with $\mu(I) = N_0$, and let $f_0, f_1, \dots, f_m \in L^0(\mathbb{K})$ be 1-bounded functions such that $\|f_i\|_{L^1(\mathbb{K})} \leq N_0$ for all $i \in \llbracket m \rrbracket_0$. Suppose that*

$$|\Lambda_{\mathcal{P};N}(f_0, \dots, f_m)| \geq \delta. \quad (6.21)$$

Then there exists a collection of polynomials $\mathcal{P}' := \{P'_1, \dots, P'_{m'}\}$ with $m' := \#\mathcal{P}' < 2\#\mathcal{P}$ satisfying $P'_{m'} := P_m - P_1$ and $\deg(P'_1) \leq \dots \leq \deg(P'_{m'})$, and 1-bounded functions $f'_0, f'_1, \dots, f'_{m'} \in L^0(\mathbb{K})$ such that $\|f'_i\|_{L^1(\mathbb{K})} \leq N_0$ for all $i \in \llbracket m' \rrbracket_0$ and satisfying

$$|\Lambda_{\mathcal{P}'; N}(f'_0, \dots, f'_{m'})| \gtrsim_{\mathcal{P}} \delta^2. \quad (6.22)$$

We also know that $\{f'_0, f'_1, \dots, f'_{m'}\} = \{f_1, \overline{f_1}, \dots, f_m, \overline{f_m}\}$ with $f'_{m'} = f_m$.

Moreover, $v(\mathcal{P}') < v(\mathcal{P})$, and one of the following three scenarios occurs.

- (i) The collection $\mathcal{P}$ is of type I; that is, $\mathcal{P} \neq \mathcal{P}_j$. In this case, $\mathcal{P}'$ is a (d, j) -admissible collection of polynomials with tolerance (A'_0, A') and for some $1 \leq i \leq j - 1$,

$$v(\mathcal{P}') = (v_1(\mathcal{P}'), \dots, v_{i-1}(\mathcal{P}'), v_i(\mathcal{P}) - 1, v_{i+1}(\mathcal{P}), \dots, v_j(\mathcal{P}), 0, 0, \dots). \quad (6.23)$$

- (ii) The collection $\mathcal{P}$ is of type II; that is, $\mathcal{P} = \mathcal{P}_j$ and $v_j(\mathcal{P}) > 1$. In this case, $\mathcal{P}'$ is a (d, j) -admissible collection of polynomials with tolerance (A'_0, A') and

$$v(\mathcal{P}') = (v_1(\mathcal{P}'), \dots, v_{j-1}(\mathcal{P}'), v_j(\mathcal{P}) - 1, 0, 0, \dots). \quad (6.24)$$

- (iii) The collection $\mathcal{P}$ is of type III; that is, $\mathcal{P} = \mathcal{P}_j$ and $v_j(\mathcal{P}) = 1$. In this case, $\mathcal{P}'$ is a $(d, j - 1)$ -admissible collection of polynomials with tolerance (A'_0, A') and

$$v(\mathcal{P}') = (0, \dots, 0, v_{j-1}(\mathcal{P}'), 0, 0, \dots). \quad (6.25)$$

Moreover, the leading coefficients of the polynomials in $\mathcal{P}'$ are pairwise distinct.

The tolerance (A'_0, A') of the collection $\mathcal{P}'$ only depends on the tolerance (A_0, A) of the collection $\mathcal{P}$ and is independent of δ and N .

Using Proposition 6.20, we now prove Claim 6.14.

Proof of Claim 6.14. We may assume, without loss of generality, that the collection $\mathcal{P}$ from Claim 6.14 is of type I or type II. Then we apply Proposition 6.20 until we reach a collection of polynomials of type III with weight vector $v(\mathcal{P}) = (0, \dots, 0, v_j(\mathcal{P}), 0, 0, \dots)$, where $v_j(\mathcal{P}) = 1$ and such that Equation (6.16) holds. We apply Proposition 6.20 once more to reach a collection of $(d, j - 1)$ -admissible polynomials satisfying Equation (6.16). This completes the proof of the claim. $\square$

Proof of Proposition 6.20. Appealing to Proposition 5.8 with $i_0 = 1$, we may conclude that there exists a collection of polynomials $\mathcal{P}' := \{P'_1, \dots, P'_{m'}\}$ with $m' = \#\mathcal{P}' < 2\#\mathcal{P}$ and $P'_{m'} = P_m - P_1$ such that

$$\mathcal{P}' = \{P_1(y) - P_1(y), P_1(y + h) - P_1(y), \dots, P_m(y) - P_1(y), P_m(y + h) - P_1(y)\},$$

for some $\delta' \delta^2 N / C^2 \leq |h| \leq \delta' N \leq \delta^4 N / C$. Proposition 5.8 also ensures that bound (6.22) holds for certain 1-bounded functions $f'_0, f'_1, \dots, f'_{m'} \in L^0(\mathbb{K})$ such that $\|f'_i\|_{L^1(\mathbb{K})} \leq N_0$ for all $i \in \llbracket m' \rrbracket_0$ and satisfying $\{f'_0, f'_1, \dots, f'_{m'}\} = \{f_1, \overline{f_1}, \dots, f_m, \overline{f_m}\}$ with $f'_{m'} = f_m$. Now, it remains to verify conclusions from (i), (ii) and (iii). For this purpose, we will have to adjust $\delta' \leq \delta^4 / C$, which can be made as small as necessary. $\square$

Proof of the conclusion from (i)

Suppose that the collection $\mathcal{P}$ is of type I. Then $i = \deg(P_1) < \deg(P_m) = j$ and $v(\mathcal{P}) = (0, \dots, 0, v_i(\mathcal{P}), \dots, v_j(\mathcal{P}), 0, 0, \dots)$. To establish Equation (6.23), we consider three cases. Let $P \in \mathcal{P}$.

If $\deg(P) > i$, then

$$\begin{aligned}\deg(P - P_1) &= \deg(P(\cdot + h) - P_1) = \deg(P), \\ \ell(P - P_1) &= \ell(P(\cdot + h) - P_1) = \ell(P),\end{aligned}\tag{6.26}$$

which yields that $v_k(\mathcal{P}') = v_k(\mathcal{P})$ for all $k > i$. If $\deg(P) = i$ and $\ell(P) \neq \ell(P_1)$, then

$$\begin{aligned}\deg(P - P_1) &= \deg(P(\cdot + h) - P_1) = i, \\ \ell(P - P_1) &= \ell(P(\cdot + h) - P_1) = \ell(P) - \ell(P_1).\end{aligned}\tag{6.27}$$

If $\deg(P) = i$ and $\ell(P) = \ell(P_1)$, then

$$\deg(P - P_1) < i, \quad \text{and} \quad \deg(P(\cdot + h) - P_1) < i.$$

The latter two cases show that $v_k(\mathcal{P}') \geq 0$ for all $k \in \llbracket i - 1 \rrbracket$ and $v_i(\mathcal{P}') = v_i(\mathcal{P}) - 1$. Hence, Equation (6.23) holds. We now show that $\mathcal{P}'$ is (d, j) -admissible.

We begin with verifying Equation (6.6) for $P' \in \mathcal{P}'_j$. We may write $P' = P(\cdot + \varepsilon h) - P_1$ for some $P \in \mathcal{P}_j$ and $\varepsilon \in \{0, 1\}$. By Equations (6.26) and (6.6) for $P \in \mathcal{P}_j$, we obtain

$$A_0^{-1} \delta^A N^{d-j} \leq |\ell(P')| \leq A_0 \delta^{-A} N^{d-j}.\tag{6.28}$$

We now verify Equation (6.7) for $Q'_1, Q'_2 \in \mathcal{P}'_j$ with $\ell(Q'_1) \neq \ell(Q'_2)$. We may write

$$Q'_1 = Q_1(\cdot + \varepsilon_1 h) - P_1, \quad \text{and} \quad Q'_2 = Q_2(\cdot + \varepsilon_2 h) - P_1\tag{6.29}$$

for some $Q_1, Q_2 \in \mathcal{P}_j$ and $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$. By Equation (6.26), we have $\ell(Q'_1) = \ell(Q_1)$ and $\ell(Q'_2) = \ell(Q_2)$. Then $\ell(Q_1) \neq \ell(Q_2)$ and by Equation (6.7) for $Q_1, Q_2 \in \mathcal{P}_j$, we deduce

$$A_0^{-1} \delta^A N^{d-j} \leq |\ell(Q'_1) - \ell(Q'_2)| \leq A_0 \delta^{-A} N^{d-j}.\tag{6.30}$$

We finally verify Equation (6.8) for $Q'_1, Q'_2 \in \mathcal{P}'_j$ as in Equation (6.29) such that $Q'_1 \neq Q'_2$ and $\ell(Q'_1) = \ell(Q'_2) = \ell$. By Equation (6.26), we see that $\ell(Q_1) = \ell(Q_2) = \ell$. Since $\mathcal{P}$ is (d, j) -admissible, using Equation (6.6), we also have

$$A_0^{-1} \delta^A N^{d-j} \leq |\ell| \leq A_0 \delta^{-A} N^{d-j}.\tag{6.31}$$

Recall that $\delta' \delta^2 N / C^2 \leq |h| \leq \delta' N$, where $\delta' > 0$ is an arbitrarily small number such that $\delta' \leq \delta^4 / C$. Set $\delta' := \delta^M (CM)^{-1}$ for a large number $M \geq 1$, which will be chosen later.

First, suppose $Q_1 = Q_2$. Then $\varepsilon_1 \neq \varepsilon_2$ and $\deg(Q'_1 - Q'_2) = j - 1$. Furthermore, $\ell(Q'_1 - Q'_2) = j\ell h(\varepsilon_1 - \varepsilon_2)$ implying $|\ell(Q'_1 - Q'_2)| = |j\ell h|$, and so by Equation (6.31),

$$|j|(A_0 C^3 M)^{-1} \delta^{A+M+2} N^{d-j+1} \leq |j\ell h| \leq |j| A_0 (CM)^{-1} \delta^{M-A} N^{d-j+1},\tag{6.32}$$

and this verifies Equations (6.8) in the case $Q_1 = Q_2$.

Now, suppose $Q_1 \neq Q_2$ so that $\deg(Q_1 - Q_2) = j - 1$ and Equation (6.8) holds for $\ell(Q_1 - Q_2)$; that is,

$$A_0^{-1} \delta^A N^{d-j+1} \leq |\ell(Q_1 - Q_2)| \leq A_0 \delta^{-A} N^{d-j+1}.\tag{6.33}$$

Taking $M := \max\{2A, 2|j|A_0^2\}$ in Equation (6.32), we see that $|j\ell h| \leq \frac{1}{2} A_0^{-1} \delta^A N^{d-j+1}$ if $C > 1$ is large enough.

In this case, $\ell(Q'_1 - Q'_2) = \ell(Q_1 - Q_2) + jh\ell(\varepsilon_1 - \varepsilon_2)$ and so

$$|\ell(Q_1 - Q_2)| - |j\ell h| \leq |\ell(Q'_1 - Q'_2)| \leq |\ell(Q_1 - Q_2)| + |j\ell h|.$$

From Equation (6.33) and $|j\ell h| \leq \frac{1}{2}A_0^{-1}\delta^A N^{d-j+1}$, we conclude

$$\frac{1}{2}A_0^{-1}\delta^A N^{d-j+1} \leq |\ell(Q'_1 - Q'_2)| \leq \frac{3}{2}A_0\delta^{-A}N^{d-j+1}. \quad (6.34)$$

This verifies Equation (6.8) in the case $Q_1 \neq Q_2$.

In either case, we see that $\deg(Q'_1 - Q'_2) = j - 1$ and (see Equations (6.32) and (6.34)) we can find a tolerance pair (A'_0, A') for $\mathcal{P}'$ depending on the tolerance (A_0, A) of $\mathcal{P}$ and the constants C and M such that

$$(A'_0)^{-1}\delta^{A'}N^{d-j+1} \leq |\ell(Q'_1 - Q'_2)| \leq A'_0\delta^{-A'}N^{d-j+1} \quad (6.35)$$

holds, establishing Equation (6.8).

Proof of the conclusion from (ii)

Suppose that the collection $\mathcal{P}$ is of type II. Then $\deg(P_1) = \dots = \deg(P_m) = j$ and $v(\mathcal{P}) = (0, \dots, 0, v_j(\mathcal{P}), 0, 0, \dots)$ with $v_j(\mathcal{P}) > 1$. To establish Equation (6.24), we will proceed in a similar way as in (i). If $P \in \mathcal{P} = \mathcal{P}_j$ and $\ell(P) \neq \ell(P_1)$, then

$$\begin{aligned} \deg(P - P_1) &= \deg(P(\cdot + h) - P_1) = j, \\ \ell(P - P_1) &= \ell(P(\cdot + h) - P_1) = \ell(P) - \ell(P_1). \end{aligned} \quad (6.36)$$

If $P \in \mathcal{P} = \mathcal{P}_j$ and $\ell(P) = \ell(P_1)$, then by the fact that $\mathcal{P}$ is (d, j) -admissible, and by Equation (6.8), we see that

$$\deg(P - P_1) < j, \quad \text{and} \quad \deg(P(\cdot + h) - P_1) < j. \quad (6.37)$$

This shows that $v_k(\mathcal{P}') \geq 0$ for all $k \in \llbracket j-1 \rrbracket$ and $v_j(\mathcal{P}') = v_j(\mathcal{P}) - 1$. Hence, Equation (6.24) holds. We now show that $\mathcal{P}'$ is (d, j) -admissible.

We begin with verifying Equation (6.6) for $P' \in \mathcal{P}'_j$. We may write $P' = P(\cdot + \varepsilon h) - P_1$ for some $P \in \mathcal{P}_j$ such that $\ell(P) \neq \ell(P_1)$ and $\varepsilon \in \{0, 1\}$. Since $\mathcal{P}$ is (d, j) -admissible, using Equations (6.36) and (6.7) (with $\ell(P) - \ell(P_1)$ in place of $\ell(P) - \ell(Q)$), we obtain Equation (6.28) which is Equation (6.6) for $P' \in \mathcal{P}'_j$.

We now verify Equation (6.7) for $Q'_1, Q'_2 \in \mathcal{P}'_j$ with $\ell(Q'_1) \neq \ell(Q'_2)$. As in Equation (6.29), we may write $Q'_1 = Q_1(\cdot + \varepsilon_1 h) - P_1$, and $Q'_2 = Q_2(\cdot + \varepsilon_2 h) - P_1$ for some $Q_1, Q_2 \in \mathcal{P}_j$ and $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$ such that $\ell(Q_1) \neq \ell(P_1)$ and $\ell(Q_2) \neq \ell(P_1)$. By Equation (6.36), we have $\ell(Q'_1) = \ell(Q_1) - \ell(P_1)$ and $\ell(Q'_2) = \ell(Q_2) - \ell(P_1)$. Then $\ell(Q'_1) \neq \ell(Q'_2)$ and Equation (6.30) is verified by appealing to Equation (6.7) (with $\ell(Q_1) - \ell(Q_2)$ in place of $\ell(P) - \ell(Q)$).

We finally verify Equation (6.8) for $Q'_1, Q'_2 \in \mathcal{P}'_j$ as in Equation (6.29) such that $Q'_1 \neq Q'_2$ and $\ell(Q'_1) = \ell(Q'_2) = \ell$. By Equation (6.36), $\ell(Q'_1) - \ell(P_1) = \ell(Q'_2) - \ell(P_1) = \ell$ and since $\mathcal{P}$ is (d, j) -admissible, we see that ℓ satisfies Equation (6.31). Now, by following the last part of the proof from (i), we conclude that Equation (6.35) holds.

Proof of the conclusion from (iii)

Suppose that the collection $\mathcal{P}$ is of type III. Then $\deg(P_1) = \dots = \deg(P_m) = j$ and $v(\mathcal{P}) = (0, \dots, 0, v_j(\mathcal{P}), 0, 0, \dots)$ with $v_j(\mathcal{P}) = 1$, thus $\ell(P_1) = \dots = \ell(P_m) := \ell$. To establish Equation (6.25), we will proceed in a similar way as in (i) and (ii). If $P \in \mathcal{P}_j$ and $\ell(P) = \ell$, then Equation (6.31) holds for ℓ and once again Equation (6.37) holds. This in turn implies that $v_{j-1}(\mathcal{P}') > 0$ and $v_k(\mathcal{P}') = 0$ for all $k \neq j-1$. Hence, Equation (6.25) holds. We now show that $\mathcal{P}'$ is $(d, j-1)$ -admissible.

We begin with verifying Equation (6.6) (or equivalently Equation (6.28) with j replaced by $j-1$) for $P' \in \mathcal{P}'_{j-1}$. We may write $P' = P(\cdot + \varepsilon h) - P_1$ for some $P \in \mathcal{P}_j$ such that $\ell(P) = \ell(P_1)$ and $\varepsilon \in \{0, 1\}$.

Then

$$\ell(P') = \ell(P(\cdot + \varepsilon h) - P_1) = \ell(P - P_1) + jh\ell\varepsilon. \quad (6.38)$$

As in (i) we have $\delta'\delta^2N/C^2 \leq |h| \leq \delta'N$, where $\delta' := \delta^M(CM)^{-1}$ for a large number $M \geq 1$, which will be chosen later. Furthermore if $P \neq P_1$, then $A_0^{-1}\delta^AN^{d-j+1} \leq |\ell(P - P_1)| \leq A_0\delta^{-A}N^{d-j+1}$ since $\mathcal{P}$ is (d, j) -admissible and so Equation (6.8) holds with $Q = P_1$. This takes care of the case $\varepsilon = 0$.

If $\varepsilon = 1$ and $P = P_1$, then Equation (6.32) gives the desired bound for $|\ell(P')|$. When $P \neq P_1$, we use the upper bound from Equation (6.32)

$$|jh\ell| \leq |j|A_0(CM)^{-1}\delta^{M-A}N^{d-j+1} \leq \frac{1}{2}A_0^{-1}\delta^{-A}N^{d-j+1} \quad (6.39)$$

when $M = \max(2A, 2|j|A_0^2)$ and $C > 1$ chosen large enough. Thus, as before, condition (6.6) holds for P' with some tolerance pair (A'_0, A') as desired.

For $Q'_1 \neq Q'_2 \in \mathcal{P}'_{j-1}$, we may write $Q'_1 = Q_1(\cdot + \varepsilon_1 h) - P_1$, and $Q'_2 = Q_2(\cdot + \varepsilon_2 h) - P_1$ for some $Q_1, Q_2 \in \mathcal{P}_j$ and $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$ such that $\ell(Q_1) = \ell(Q_2) = \ell(P_1) = \ell$. We have $\ell(Q_1 - P_1) - \ell(Q_2 - P_1) = \ell(Q_1 - Q_2)$ and so by Equation (6.38),

$$\ell(Q'_1) - \ell(Q'_2) = \ell(Q_1 - Q_2) + jh\ell(\varepsilon_1 - \varepsilon_2). \quad (6.40)$$

We consider two cases.

If $Q_1 = Q_2$, then necessarily $|\varepsilon_1 - \varepsilon_2| = 1$ and so $\ell(Q'_1) \neq \ell(Q'_2)$, $\deg(Q'_1 - Q'_2) = j - 1$, and Equation (6.32) shows that Equation (6.7) holds for $Q'_1, Q'_2 \in \mathcal{P}'_{j-1}$.

If $Q_1 \neq Q_2$, then $A_0^{-1}\delta^AN^{d-j+1} \leq |\ell(Q_1 - Q_2)| \leq A_0\delta^{-A}N^{d-j+1}$ since $\mathcal{P}$ is (d, j) -admissible, and so Equation (6.8) holds with $P = Q_1$ and $Q = Q_2$. From Equation (6.39), we see that $\ell(Q'_1) \neq \ell(Q'_2)$ and Equation (6.40) implies that Equation (6.7) holds for $Q'_1, Q'_2 \in \mathcal{P}'_{j-1}$.

In either case, we see that Equation (6.8) is vacuously satisfied by P' and Equation (6.7) holds for $Q'_1, Q'_2 \in \mathcal{P}'_{j-1}$ with (necessarily) $\ell(Q'_1) \neq \ell(Q'_2)$.

Concluding, we are able to find a tolerance pair (A'_0, A') for $\mathcal{P}'$ depending on the tolerance (A_0, A) of $\mathcal{P}$ and the constants C and M such that the required estimates for Equations (6.38) and (6.40) hold. This completes the proof of Proposition 6.20.

6.2. Degree-lowering

Here, we establish a modulated version of the inverse theorem, which will imply Theorem 6.1.

Theorem 6.41 (Inverse theorem for modulated $(m + 1)$ -linear forms). *Let $N \geq 1$ be a scale, and let $0 < \delta \leq 1$, $m \in \mathbb{Z}_+$ and $n \in \mathbb{N}$ be given. Let $\mathcal{P} := \{P_1, \dots, P_m\}$ and $\mathcal{Q} := \{Q_1, \dots, Q_n\}$ be collections of polynomials such that*

$$1 \leq \deg P_1 < \dots < \deg P_m < \deg Q_1 < \dots < \deg Q_n.$$

Let $f_0, f_1, \dots, f_m \in L^0(\mathbb{K})$ be 1-bounded functions supported on an interval $I \subset \mathbb{K}$ of measure $N_0 := N^{\deg P_m}$. For $n \in \mathbb{Z}_+$, we define an $(m + 1)$ -linear form corresponding to the triple $(\mathcal{P}, \mathcal{Q}; N)$ and a frequency vector $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{K}^n$ by

$$\Lambda_{\mathcal{P}; N}^{\mathcal{Q}; \xi}(f_0, \dots, f_m) := \frac{1}{N_0} \int_{\mathbb{K}^2} f_0(x) \prod_{i=1}^m f_i(x - P_i(y)) e\left(\sum_{j=1}^n \xi_j Q_j(y)\right) d\mu_{[N]}(y) d\mu(x). \quad (6.42)$$

For $n = 0$, we set $\mathcal{Q} = \emptyset$ and we simply write $\Lambda_{\mathcal{P}; N}^{\mathcal{Q}; \xi}(f_0, \dots, f_m) := \Lambda_{\mathcal{P}; N}(f_0, \dots, f_m)$ as in Equation (6.2). Suppose that

$$|\Lambda_{\mathcal{P};N}^{\mathcal{Q};\xi}(f_0, \dots, f_m)| \geq \delta. \quad (6.43)$$

Then there exists a $C_1 = C_1(\mathcal{P}) \gg 1$ such that

$$N_0^{-1} \|\mu_{[N_1]} * f_1\|_{L^1(\mathbb{K})} \gtrsim_{\mathcal{P}} \delta^{O_{\mathcal{P}}(1)}, \quad (6.44)$$

for any $N_1 = \delta^C N^{\deg P_1}$ with $C \geq C_1$.

If necessary we will also write $\Lambda_{\mathcal{P};N}^{\mathcal{Q};\xi}(f_0, \dots, f_m) = \Lambda_{\mathcal{P};N,I}^{\mathcal{Q};\xi}(f_0, \dots, f_m)$ in order to emphasise that the functions $f_0, f_1, \dots, f_m$ are supported on I .

We first show how the Gowers box norms control the dual functions. The dual function, or more precisely the m -th dual function, corresponding to Equation (6.42) is defined as

$$F_m^{\xi}(x) := \int_{\mathbb{K}} F_{m;y}^{\xi}(x) d\mu_{[N]}(y), \quad x \in \mathbb{K}, \quad (6.45)$$

where

$$F_{m;y}^{\xi}(x) := f_0(x + P_m(y)) \prod_{i=1}^{m-1} f_i(x - P_i(y) + P_m(y)) e\left(\sum_{j=1}^n \xi_j Q_j(y)\right). \quad (6.46)$$

Proposition 6.47 (Gowers box norms control the dual functions). *Let $N \geq 1$ be a scale, and let $0 < \delta \leq 1$, $d, m \in \mathbb{Z}_+$ with $m \geq 2$ and $n \in \mathbb{N}$ be given. Let $\mathcal{P} := \{P_1, \dots, P_m\}$ and $\mathcal{Q} := \{Q_1, \dots, Q_n\}$ be collections of polynomials such that $\mathcal{P}$ is (d, d) -admissible and*

$$1 \leq \deg P_1 \leq \dots \leq \deg P_m \leq \deg Q_1 \leq \dots \leq \deg Q_n.$$

Let $f_0, f_1, \dots, f_m \in L^0(\mathbb{K})$ be 1-bounded functions supported on an interval $I \subset \mathbb{K}$ of measure $N_0 := N^{\deg P_m}$. For $\xi \in \mathbb{K}^n$, let F_m^{ξ} be the dual function defined in Equation (6.45). Suppose that Equation (6.43) is satisfied. Then for the exponent $s \in \mathbb{Z}_+$ which appears in the conclusion of Theorem 6.10, we have

$$\|F_m^{\xi}\|_{\square_{[H_1], \dots, [H_{s+1}]}^{s+1}(I)} \gtrsim_{\mathcal{P}} \delta^{O_{\mathcal{P}}(1)}, \quad (6.48)$$

where $H_i \simeq \delta^{O_{\mathcal{P}}(1)} N^{\deg(P_m)}$ for $i \in \llbracket s+1 \rrbracket$.

Proof. By changing the variables $x \mapsto x + P_m(y)$ in Equation (6.42), we may write

$$\Lambda_{\mathcal{P};N}^{\mathcal{Q};\xi}(f_0, \dots, f_m) = \frac{1}{N_0} \int_{\mathbb{K}} \left(\int_{\mathbb{K}} F_{m;y}^{\xi}(x) d\mu_{[N]}(y) \right) f_m(x) d\mu(x).$$

By the Cauchy–Schwarz inequality (observing once again that $\|f_m\|_{L^2(\mathbb{K})}^2 \leq N_0$), we have

$$\begin{aligned} \delta^2 &\leq \frac{1}{N_0} \int_{\mathbb{K}} \left| \int_{\mathbb{K}} F_{m;y}^{\xi}(x) d\mu_{[N]}(y) \right|^2 d\mu(x) \\ &= \frac{1}{N_0} \left| \int_{\mathbb{K}^3} F_{m;y_1}^{\xi}(x) \overline{F_{m;y_2}^{\xi}(x)} d\mu_{[N]}^{\otimes 2}(y_1, y_2) d\mu(x) \right| \\ &= |\Lambda_{\mathcal{P};N}^{\mathcal{Q};\xi}(f_0, f_1, \dots, f_{m-1}, \overline{F_m^{\xi}})|, \end{aligned}$$

where in the last step we changed variables $x \mapsto x - P_m(y_1)$. Denote $g_m := \overline{F_m^{\xi}}$, and $g_j := f_j$ for $j \in \llbracket m-1 \rrbracket_0$. Our strategy will be to reduce the matter to Theorem 6.10 with the family $\mathcal{P}$. Observe

that g_j is a 1-bounded function and $\|g_j\|_{L^1(\mathbb{K})} \lesssim N_0$ for all $j \in \llbracket m \rrbracket_0$. Changing the variables $x \mapsto x + h$ in the definition of $\Lambda_{\mathcal{P};N}^{\mathcal{Q};\xi}$ and averaging over $h \in [H_{s+1}]$ where $H_{s+1} = \delta^{O(1)} N^{\deg P_m}$, we have

$$\begin{aligned} \delta^4 &\leq |\Lambda_{\mathcal{P};N}^{\mathcal{Q};\xi}(g_0, \dots, g_m)|^2 \\ &\lesssim \frac{1}{N_0} \int_{\mathbb{K}^2} \left| \int_{\mathbb{K}} g_0(x+h) \prod_{i=1}^m g_i(x+h-P_i(y)) d\mu_{[H_{s+1}]}(h) \right|^2 d\mu_{[N]}(y) d\mu(x), \end{aligned}$$

where in the last line we have used the Cauchy–Schwarz inequality in the x and y variables, noting that $x \rightarrow g_0(x+h)$ is supported a fixed dilate of I for every $h \in [H_{s+1}]$. By another change of variables, we obtain

$$\int_{\mathbb{K}} \Lambda_{\mathcal{P};N}(\Delta_h g_0, \dots, \Delta_h g_m) d\nu_{[H_{s+1}]}(h) \gtrsim \delta^4.$$

Now, we may find a measurable set $X \subseteq [H_{s+1}]$ such that

$$|\Lambda_{\mathcal{P};N}(\Delta_h g_0, \dots, \Delta_h g_m)| \gtrsim \delta^4$$

for all $h \in X$ and $\nu_{[H_{s+1}]}(X) \gtrsim \delta^4$. Since $\Delta_h g_j$ is a 1-bounded function and $\|\Delta_h g_j\|_{L^1(\mathbb{K})} \lesssim N_0$ for all $j \in \llbracket m \rrbracket_0$, we may invoke Theorem 6.10 and conclude that

$$\|\Delta_h F_m^\xi\|_{\square_{[H_1], \dots, [H_s]}^s}(I) = \|\Delta_h g_m\|_{\square_{[H_1], \dots, [H_s]}^s}(I) \gtrsim_{\mathcal{P}} \delta^{O_{\mathcal{P}}(1)}$$

for all $h \in X$, where $H_i \simeq \delta^{O_{\mathcal{P}}(1)} N^{\deg(P_m)}$ for $i \in \llbracket s \rrbracket$. Averaging over $h \in X$ and using $\nu_{[H_{s+1}]}(X) \gtrsim \delta^4$, we obtain

$$\|F_m^\xi\|_{\square_{[H_1], \dots, [H_{s+1}]}^{s+1}}^{2^{s+1}}(I) = \int_{\mathbb{K}} \|\Delta_h F_m^\xi\|_{\square_{[H_1], \dots, [H_s]}^s}(I)^{2^s} d\nu_{[H_{s+1}]}(h) \gtrsim_{\mathcal{P}} \delta^{O_{\mathcal{P}}(1)},$$

which is Equation (6.48) as desired. $\square$

We first establish a simple consequence of the oscillatory integral bound (3.1) which will be important later.

Lemma 6.49. *Let $N > 1$ be a scale, $m \in \mathbb{Z}_+$ and $n \in \mathbb{N}$ be given. Let $\mathcal{P} := \{P_1, \dots, P_m\}$ and $\mathcal{Q} := \{Q_1, \dots, Q_n\}$ be collections of polynomials such that*

$$1 \leq \deg P_1 < \dots < \deg P_m < \deg Q_1 < \dots < \deg Q_n. \quad (6.50)$$

Define the multiplier corresponding to the families $\mathcal{P}$ and $\mathcal{Q}$ as follows:

$$m_N^{\mathcal{P}, \mathcal{Q}}(\zeta, \xi) := \int_{\mathbb{K}} e\left(\sum_{i=1}^m \zeta_i P_i(y) + \sum_{j=1}^n \xi_j Q_j(y)\right) d\mu_{[N]}(y),$$

where $\zeta = (\zeta_1, \dots, \zeta_m) \in \mathbb{K}^m$ and $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{K}^n$. Let $0 < \delta \leq 1$ and suppose that

$$|m_N^{\mathcal{P}, \mathcal{Q}}(\zeta, \xi)| \geq \delta. \quad (6.51)$$

Then there exists a large constant $A \gtrsim_{\mathcal{P}, \mathcal{Q}} 1$ such that

$$\begin{aligned} N^{\deg(Q_j)} |\xi_j| &\lesssim \delta^{-A}, \quad \text{for } j \in \llbracket n \rrbracket, \\ N^{\deg(P_j)} |\zeta_j| &\lesssim \delta^{-A}, \quad \text{for } j \in \llbracket m \rrbracket. \end{aligned} \quad (6.52)$$

Proof. Fix an element $\alpha \in \mathbb{K}$ such that $|\alpha| = N$, and make the change of variables $y \rightarrow \alpha y$ to write

$$m_N^{\mathcal{P}, \mathcal{Q}}(\zeta, \xi) = \int_{B_1(0)} e\left(\sum_{i=1}^m \zeta_i P_i(\alpha y) + \sum_{j=1}^n \xi_j Q_j(\alpha y)\right) d\mu(y).$$

Define $R(y) := \sum_{i=1}^m \zeta_i P_i(y) + \sum_{j=1}^n \xi_j Q_j(y)$. Then $R(y)$ may be rewritten as

$$R(y) = \sum_{l=1}^{\deg Q_n} c_l(R) y^l$$

The oscillatory integral bound (3.1) implies

$$|m_N^{\mathcal{P}, \mathcal{Q}}(\zeta, \xi)| \lesssim \left(1 + \sum_{l=1}^{\deg Q_n} |c_l(R)| N^l\right)^{-1/\deg Q_n}. \quad (6.53)$$

Hence, Equation (6.51) implies $\max_l |c_l(R)| N^l \lesssim \delta^{-d_*}$, where $d_* = \deg Q_n$ and the maximum is taken over all $l \in \llbracket \deg(Q_n) \rrbracket$. From this, we see that for any sufficiently large $A \geq d_*$,

$$|c_l(R)| N^l \leq \delta^{-A}/A \quad (6.54)$$

for all $l \in \llbracket \deg(Q_n) \rrbracket$.

Using Equation (6.50), we observe that

$$c_{\deg Q_j}(R) = \sum_{k=j}^n c_{\deg Q_j}(Q_k) \xi_k, \quad \text{for } j \in \llbracket n \rrbracket, \quad (6.55)$$

$$c_{\deg P_j}(R) = \sum_{k=1}^n c_{\deg P_j}(Q_k) \xi_k + \sum_{k=j}^m c_{\deg P_j}(P_k) \zeta_k, \quad \text{for } j \in \llbracket m \rrbracket. \quad (6.56)$$

Using Equation (6.55) for $j = n$, we see that Equation (6.54) implies Equation (6.52) for $N^{\deg Q_n} |\xi_n|$. Inductively, we now deduce, using Equation (6.55), that Equation (6.54) implies that Equation (6.52) holds for all $N^{\deg Q_j} |\xi_j|$, $j \in \llbracket n \rrbracket$. Similarly, using Equations (6.56) and (6.54), we see that the second displayed equation in Equation (6.52) holds. $\square$

The key ingredient in the proof of Theorem 6.41 will be a degree-lowering argument, which reads as follows.

Theorem 6.57 (Degree-lowering argument). *Let $N \geq 1$ be a scale, and let $0 < \delta \leq 1$, $m \in \mathbb{Z}_+$ and $n \in \mathbb{N}$ be given. Let $\mathcal{P} := \{P_1, \dots, P_m\}$ and $\mathcal{Q} := \{Q_1, \dots, Q_n\}$ be collections of polynomials such that*

$$1 \leq \deg P_1 < \dots < \deg P_m < \deg Q_1 < \dots < \deg Q_n.$$

For $\xi \in \mathbb{K}^n$, let F_m^ξ be the dual function from Equation (6.45) corresponding to the form (6.42) and 1-bounded functions $f_0, f_1, \dots, f_{m-1} \in L^0(\mathbb{K})$ supported on an interval $I \subset \mathbb{K}$ of measure $N_0 := N^{\deg P_m}$. Suppose that for some integer $s \in \mathbb{Z}_+$ one has

$$\|F_m^\xi\|_{\square_{[H_1], \dots, [H_s]}^s(I)} \geq \delta, \quad (6.58)$$

where $H_i \simeq \delta^{O_{\mathcal{P}}(1)} N^{\deg(P_m)}$ for $i \in \llbracket s \rrbracket$. Then

$$\|F_m^\xi\|_{\square_{[H_1], \dots, [H_{s-1}]}^{s-1}(I)} \gtrsim_{\mathcal{P}} \delta^{O_{\mathcal{P}}(1)}. \quad (6.59)$$

Assuming momentarily Theorem 6.57, we prove Theorem 6.41.

Proof of Theorem 6.41. Our goal is to prove Equation (6.44) when

$$\delta \leq |\Lambda_{\mathcal{P};N}^{\mathcal{Q};\xi}(f_0, \dots, f_m)|.$$

The proof is by induction on $m \in \mathbb{Z}_+$. We divide the proof into two steps. In the first step, we establish the base case for $m = 1$. In the second step, we will use Theorem 6.57 to establish the inductive step. $\square$

Step 1.

Assume that $m = 1$ so that $N_0 = N^{\deg P_1}$. For $\zeta \in \mathbb{K}$ and $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{K}^n$, we define the multiplier

$$m_N(\zeta, \xi) := \int_{\mathbb{K}} e\left(-\zeta P_1(y) + \sum_{j=1}^n \xi_j Q_j(y)\right) d\mu_{[N]}(y).$$

We now express

$$\Lambda_{\mathcal{P};N}^{\mathcal{Q};\xi}(g_0, g_1) = N_0^{-1} \int_{\mathbb{K}} \widehat{g_0}(-\zeta) \widehat{g_1}(\zeta) m_N(\zeta, \xi) d\mu(\zeta).$$

Using the Cauchy–Schwarz inequality and Plancherel’s theorem, we see

$$|\Lambda_{\mathcal{P};N}^{\mathcal{Q};\xi}(g_0, g_1)| \leq N_0^{-1} \|g_0\|_{L^2(\mathbb{K})} \|g_1\|_{L^2(\mathbb{K})} \sup_{\zeta \in \text{supp}(\widehat{g_0 g_1})} |m_N(\zeta, \xi)|. \quad (6.60)$$

When $\mathbb{K}$ is non-Archimedean, let $\varphi(x) = \mathbb{1}_{[1]}(x) = \mathbb{1}_{B_1(0)}(x)$ so that $\widehat{\varphi}(\zeta) = \mathbb{1}_{[1]}(\xi)$. When $\mathbb{K}$ is Archimedean, choose a Schwartz function $\varphi : \mathbb{K} \rightarrow \mathbb{K}$ such that

$$\mathbb{1}_{[1]}(\zeta) \leq \widehat{\varphi}(\zeta) \leq \mathbb{1}_{[2]}(\zeta), \quad \zeta \in \mathbb{K}.$$

For a scale M , we set $\varphi_M(x) = M^{-1}\varphi(M^{-1}x)$ when $\mathbb{K} = \mathbb{R}$ and when $\mathbb{K} = \mathbb{C}$, we set $\varphi_M(z) = M^{-1}\varphi(M^{-1/2}z)$. When $\mathbb{K}$ is non-Archimedean, we set $\varphi_M(x) = M^{-1}\mathbb{1}_{[M]}(x)$.

Consider two scales $M_1 \simeq \delta^C N^{\deg P_1}$ and $N_1 \simeq \delta^{2C} N^{\deg P_1} / C$. Then we obtain

$$\delta \leq |\Lambda_{\mathcal{P};N}^{\mathcal{Q};\xi}(f_0, f_1)| \leq |\Lambda_{\mathcal{P};N}^{\mathcal{Q};\xi}(f_0, \varphi_{M_1} * f_1)| + |\Lambda_{\mathcal{P};N}^{\mathcal{Q};\xi}(f_0, f_1 - \varphi_{M_1} * f_1)|.$$

Note that

$$|\Lambda_{\mathcal{P};N}^{\mathcal{Q};\xi}(f_0, \varphi_{M_1} * f_1)| \leq N_0^{-1} \|f_0\|_{L^\infty(\mathbb{K})} \|\varphi_{M_1} * f_1\|_{L^1(\mathbb{K})} \leq N_0^{-1} \|\varphi_{M_1} * f_1\|_{L^1(\mathbb{K})},$$

and

$$\begin{aligned} \|\varphi_{M_1} * f_1\|_{L^1(\mathbb{K})} &\leq \|\varphi_{M_1} * \mu_{[N_1]} * f_1\|_{L^1(\mathbb{K})} + \|(\varphi_{M_1} - \varphi_{M_1} * \mu_{[N_1]}) * f_1\|_{L^1(\mathbb{K})} \\ &\lesssim \|\mu_{[N_1]} * f_1\|_{L^1(\mathbb{K})} + C^{-1} \delta^C N_0 \end{aligned}$$

since $\varphi_{M_1} - \varphi_{M_1} * \mu_{[N_1]} = 0$ when $\mathbb{K}$ is non-Archimedean and when $\mathbb{K}$ is Archimedean, we have the pointwise bound

$$|\varphi_{M_1}(x) - \varphi_{M_1} * \mu_{[N_1]}(x)| \lesssim C^{-1} \delta^C M_1^{-1} (1 + M_1^{-1}|x|)^{-10}.$$

If $C \geq 1$ is sufficiently large, then we may write

$$\delta \lesssim |\Lambda_{\mathcal{P};N}^{\mathcal{Q};\xi}(f_0, f_1)| \leq N_0^{-1} \|\mu_{[N_1]} * f_1\|_{L^1(\mathbb{K})} + |\Lambda_{\mathcal{P};N}^{\mathcal{Q};\xi}(f_0, f_1 - \varphi_{M_1} * f_1)|. \quad (6.61)$$

By Equation (6.60), we have that

$$|\Lambda_{\mathcal{P};N}^{\mathcal{Q};\xi}(f_0, f_1 - \varphi_{M_1} * f_1)| \lesssim \sup_{\zeta \in \mathbb{K}: |\zeta| \geq M_1^{-1}} |m_N(\zeta, \xi)| \quad (6.62)$$

since $\|f_0\|_{L^2(\mathbb{K})} \leq N_0^{1/2}$ and $\|f_1\|_{L^2(\mathbb{K})} \leq N_0^{1/2}$. We now prove that

$$\sup_{\zeta \in \mathbb{K}: |\zeta| \geq M_1^{-1}} |m_N(\zeta, \xi)| \lesssim \delta^2. \quad (6.63)$$

Suppose that inequality (6.63) does not hold, then one has

$$|m_N(\zeta, \xi)| \gtrsim \delta^2$$

for some $\zeta \in \mathbb{K}$ so that $|\zeta| \geq M_1^{-1}$. Then Lemma 6.49 implies $N^{\deg P_1} |\zeta| \lesssim \delta^{-A}$ for some large, fixed $A \gtrsim 1$ by Equation (6.52). Since $M_1 = \delta^C N^{\deg P_1}$, we have $\delta^{-C} \lesssim \delta^{-A}$ which is a contradiction if $C \gg A$. Thus, Equation (6.63) holds.

Hence, by Equations (6.63), (6.62) and (6.61), we see that

$$\delta \lesssim N_0^{-1} \|\mu_{[N_1]} * f_1\|_{L^1(\mathbb{K})}$$

which establishes Theorem 6.41 when $m = 1$.

Step 2.

We now assume that Theorem 6.41 is true for $m-1$ in place of m for some integer $m \geq 2$. Using Theorem 6.57, we show that this implies Theorem 6.41 for $m \geq 2$. Note that bound (6.43) implies inequality (6.48) from Proposition 6.47. Now, by Theorem 6.57 applied $s-2$ times we may conclude that

$$\|F_m^\xi\|_{\square_{[H_1], [H_2]}^2(I)} \gtrsim_{\mathcal{P}} \delta^{O_{\mathcal{P}}(1)},$$

where $H_1, H_2 \simeq \delta^{O_{\mathcal{P}}(1)} N^{\deg P_m}$. By Lemma 5.1, we can find a $\xi_0 \in \mathbb{K}$ such that

$$N_0^{-1} |\widehat{F_m^\xi}(\xi_0)| \gtrsim_{\mathcal{P}} \delta^{O_{\mathcal{P}}(1)} \quad (6.64)$$

since $N_0 = N^{\deg P_m}$. By definitions (6.45) and (6.46) and making the change of variables $x \mapsto x - P_m(y)$, we may write

$$\begin{aligned} N_0^{-1} \widehat{F_m^\xi}(\xi_0) &= N_0^{-1} \iint_{\mathbb{K}^2} F_{m;y}^\xi(x) e(-\xi_0 x) d\mu_{[N]}(y) d\mu(x) \\ &= N_0^{-1} \iint_{\mathbb{K}^2} M_{\xi_0} f_0(x) \prod_{i=1}^{m-1} f_i(x - P_i(y)) e\left(\xi_0 P_m(y) + \sum_{j=1}^n \xi_j Q_j(y)\right) d\mu_{[N]}(y) d\mu(x) \\ &= M^{-1} \Lambda_{\mathcal{P}';N}^{\mathcal{Q}';\xi'}(M_{\xi_0} f_0, f_1, \dots, f_{m-1}), \end{aligned}$$

where $M_{\xi_0} f_0(x) := e(-\xi_0 x) f_0(x)$, $\mathcal{P}' := \mathcal{P} \setminus \{P_m\}$, $\mathcal{Q}' := \mathcal{Q} \cup \{P_m\}$, $\xi' := (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{K}^{n+1}$ and $M := N^{\deg(P_m) - \deg(P_{m-1})}$. The parameter $N'_0 := N_0 M^{-1}$ is what appears in the m -linear form $\Lambda_{\mathcal{P}';N}^{\mathcal{Q}';\xi'}$.

We note that $N'_0 = N^{\deg P_{m-1}}$.

Thus, Equation (6.64) implies

$$M^{-1} |\Lambda_{\mathcal{P}';N,I}^{\mathcal{Q}';\xi'}(M_{\xi_0} f_0, f_1, \dots, f_{m-1})| \gtrsim \delta^{O(1)}.$$

By translation invariance, we may assume that all functions $f_0, f_1, \dots, f_{m-1}$ are supported in $[N_0]$. We can partition $[N_0] = \bigcup_{k \in \llbracket L \rrbracket} E_k$ into $L \simeq M$ sets, each with measure $\simeq N'_0$ contained in an interval I_k lying in an $O(N'_0)$ neighbourhood of E_k . Furthermore, E_k is an $O(N_1)$ neighbourhood of a set F_k such that $\mu(E_k \setminus F_k) \lesssim N_1$ and $\text{supp}(\mathbb{1}_{F_k} * \mu_{[N_1]}) \subseteq E_k$. Here, $N_1 \simeq \delta^{O_{\mathcal{P}}(1)} N^{\deg(P_1)}$. In the non-Archimedean setting, this decomposition is straightforward; in this case, we can take $F_k = E_k = I_k$. If fact if $N_0 = q^{n_0}$ and $N'_0 = q^{n_0 - \ell}$ so that $M = q^\ell$, then

$$[N_0] = B_{q^{n_0}}(0) = \bigcup_{x \in \mathcal{F}} B_{q^{n_0 - \ell}}(x)$$

gives our partition of $[N_0]$, where $\mathcal{F} = \{x = \sum_{j=0}^{\ell-1} x_j \pi^{-n_0+j} : x_j \in o_{\mathbb{K}}/m_{\mathbb{K}}\}$. Note $\#\mathcal{F} = q^\ell = M$. When $\mathbb{K} = \mathbb{R}$, one simply decomposes the interval $[N_0] = [-N_0, N_0]$ into M subintervals $(E_k)_{k \in \llbracket L \rrbracket}$ of equal length and then extend and shrink to obtain intervals I_k and F_k with the desired properties.

When $\mathbb{K} = \mathbb{C}$, the set $[N_0]$ is a disc and the decomposition is not as straightforward but not difficult to construct by starting with a mesh of squares of side length $\sqrt{N'_0}$ which cover $[N_0]$. It is important that for this case (when $\mathbb{K} = \mathbb{C}$) that we allow the sets E_k and F_k to be general sets (not necessarily intervals) with the above properties. The picture should be clear.

Hence, by changing variables $x \rightarrow x + P_1(y)$ and then back again,

$$\begin{aligned} & M^{-1} \Lambda_{\mathcal{P}'; N, I}^{\mathcal{Q}', \xi'}(M_{\xi_0} f_0, f_1, \dots, f_{m-1}) \\ &= N_0^{-1} \sum_{k \in \llbracket L \rrbracket} \iint_{E_k \times \mathbb{K}} M_{\xi_0} f_0(x) \prod_{i=1}^{m-1} f_i(x - P_i(y)) e\left(\xi_0 P_m(y) + \sum_{j=1}^n \xi_j Q_j(y)\right) d\mu_{[N]}(y) d\mu(x) \\ &= N_0^{-1} \sum_{k \in \llbracket L \rrbracket} \iint_{\mathbb{K}^2} f_0^k(x) g^k(x - P_1(y)) \prod_{i=2}^{m-1} f_i^k(x - P_i(y)) e\left(\xi_0 P_m(y) + \sum_{j=1}^n \xi_j Q_j(y)\right) d\mu_{[N]}(y) d\mu(x) \\ &= M^{-1} \sum_{k \in \llbracket L \rrbracket} \Lambda_{\mathcal{P}'; N, I_k}^{\mathcal{Q}', \xi'}(f_0^k, g^k, f_2^k, \dots, f_{m-1}^k), \end{aligned}$$

where $f_0^k := M_{\xi_0} f_0 \mathbb{1}_{I_k}$, $f_2^k := f_2 \mathbb{1}_{I_k}$, $\dots$, $f_{m-1}^k := f_{m-1} \mathbb{1}_{I_k}$ and $g^k = f_1 \mathbb{1}_{E_k}$.

By the pigeonhole principle, there exists $L_0 \subseteq \llbracket L \rrbracket$ such that $\#L_0 \gtrsim \delta^{O_{\mathcal{P}'}(1)} M$, and for every $k \in L_0$, we have

$$|\Lambda_{\mathcal{P}'; N, I_k}^{\mathcal{Q}', \xi'}(f_0^k, g^k, f_2^k, \dots, f_{m-1}^k)| \gtrsim \delta^{O_{\mathcal{P}'}(1)}.$$

By the inductive hypothesis, we have

$$(N'_0)^{-1} \|\mu_{[N_1]} * (f_1 \mathbb{1}_{E_k})\|_{L^1(\mathbb{K})} \gtrsim \delta^{O_{\mathcal{P}'}(1)}$$

for every $k \in L_0$ and for every $N_1 = \delta^C N^{\deg P_1}$ with $C \geq C_1(\mathcal{P}')$. Note that

$$(N'_0)^{-1} \|\mu_{[N_1]} * (f_1(\mathbb{1}_{E_k} - \mathbb{1}_{F_k}))\|_{L^1(\mathbb{K})} \lesssim N_1 (N'_0)^{-1} \lesssim \delta^C \quad (6.65)$$

and hence for $C \gg 1$ large enough,

$$(N'_0)^{-1} \|\mu_{[N_1]} * (f_1 \mathbb{1}_{F_k})\|_{L^1(\mathbb{K})} \gtrsim \delta^{O_{\mathcal{P}}(1)} \quad \text{for every } k \in L_0.$$

Now, we can sum over $k \in L_0$, using the bound $\#L_0 \gtrsim \delta^{O(1)} M$ and the pairwise disjoint supports of $(\mu_{[N_1]} * (f_1 \mathbb{1}_{F_k}))_{k \in \llbracket L \rrbracket}$, we obtain

$$\begin{aligned}
N_0^{-1} \left\| \mu_{[N_1]} * \left(\sum_{k \in \llbracket L \rrbracket} f_1 \mathbb{1}_{F_k} \right) \right\|_{L^1(\mathbb{K})} &\geq N_0^{-1} \sum_{k \in \llbracket L \rrbracket} \left\| \mu_{[N_1]} * (f_1 \mathbb{1}_{F_k}) \right\|_{L^1(\mathbb{K})} \\
&\geq M^{-1} \sum_{k \in L_0} (N'_0)^{-1} \left\| \mu_{[N_1]} * (f_1 \mathbb{1}_{F_k}) \right\|_{L^1(\mathbb{K})} \gtrsim \delta^{O_{\mathcal{P}}(1)},
\end{aligned}$$

which by (6.65) yields

$$N_0^{-1} \left\| \mu_{[N_1]} * f_1 \right\|_{L^1(\mathbb{K})} \gtrsim \delta^{O_{\mathcal{P}}(1)},$$

as desired.

We now state two auxiliary technical lemmas which will be needed in the proof of Theorem 6.57. For $\omega = (\omega_1, \dots, \omega_n) \in \{0, 1\}^n$ and $h = (h_1, \dots, h_n) \in \mathbb{K}^n$, we write $\omega \cdot h = \sum_{i=1}^n \omega_i h_i$ and $1 - \omega = (1 - \omega_1, \dots, 1 - \omega_n)$.

Lemma 6.66. *Let $N \geq 1$ be a scale, and let $0 < \delta \leq 1$, $m \in \mathbb{Z}_+$ with $m \geq 2$, $n \in \mathbb{N}$ and scales $H_1, \dots, H_n$ with each $H_i \leq N$ be given. Assume that $\phi : X \rightarrow \mathbb{R}$ is a measurable function defined on a measurable set $X \subseteq H := \prod_{i=1}^n [H_i]$. Let $\mathcal{P} := \{P_1, \dots, P_m\}$ and $\mathcal{Q} := \{Q_1, \dots, Q_n\}$ be collections of polynomials. For $\xi \in \mathbb{K}^n$, let F_m^ξ be the dual function defined in Equation (6.45) that corresponds to the form (6.42) and 1-bounded functions $f_0, f_1, \dots, f_{m-1} \in L^0(\mathbb{K})$ supported on an interval $I \subset \mathbb{K}$ of measure $N_0 := N^{\deg P_m}$. Suppose that*

$$\int_X \left| N_0^{-1} \widehat{\Delta_h^n F_m^\xi}(\phi(h)) \right|^2 d \left(\bigotimes_{i=1}^n \nu_{[H_i]} \right)(h) \geq \delta. \quad (6.67)$$

Then

$$\int_{\square_n(X)} \left| N_0^{-1} \int_{\mathbb{K}} F_m(x; h, h') e(-\psi(h, h')x) d\mu(x) \right|^2 d \left(\bigotimes_{i=1}^n \nu_{[H_i]} \right)^{\otimes 2}(h, h') \gtrsim \delta^{O(1)}, \quad (6.68)$$

where

$$\square_n(X) := \left\{ (h, h') \in H^2 : \omega \cdot h + (1 - \omega) \cdot h' \in X \text{ for every } \omega \in \{0, 1\}^n \right\},$$

and

$$\begin{aligned}
F_m(x; h, h') &:= \int_{\mathbb{K}} \Delta_{h'-h}^n f_0(x + P_m(y)) \prod_{i=1}^{m-1} \Delta_{h'-h}^n f_i(x - P_i(y) + P_m(y)) d\mu_{[N]}(y), \\
\psi(h, h') &:= \sum_{\omega \in \{0, 1\}^n} (-1)^{|\omega|} \phi(\omega \cdot h + (1 - \omega) \cdot h').
\end{aligned}$$

Proof. We shall write $\nu_n := \bigotimes_{i=1}^n \nu_{[H_i]}$. Using Equations (4.2), (6.45) and (6.46), we see that the left-hand side of Equation (6.67) can be written as

$$\frac{1}{N_0^2} \int_{\mathbb{K}^{2n+1}} \int_{\mathbb{K}^2} \int_{\mathbb{K}^n} G_0(x, z, h; \mathbf{y}) d\nu_n(h) d\mu(x) d\mu(z) d\mu_{[N]}^{\otimes 2n+1}(\mathbf{y}),$$

where for $\mathbf{y} = (y_{(\omega, 0)}, y_{(\omega, 1)})_{\omega \in \{0, 1\}^n} \in \mathbb{K}^{2n+1}$, $x, z \in \mathbb{K}$ and $h \in \mathbb{K}^n$. We have set

$$G_0(x, z, h; \mathbf{y}) := \mathbb{1}_X(h) e(-\phi(h)(x - z)) \prod_{\omega \in \{0, 1\}^n} \mathcal{C}^{|\omega|} F_{m; y_{(\omega, 0)}}^\xi(x + h \cdot \omega) \overline{F_{m; y_{(\omega, 1)}}^\xi(z + h \cdot \omega)}.$$

Write elements in X as (h_1, h) with $h_1 \in \mathbb{K}$, and apply the Cauchy–Schwarz inequality in all but the h_1 variable (noting that $(x, z) \rightarrow G_0(x, z, h; \mathbf{y})$ is supported in a product of intervals of measure $\simeq N_0^2$) to conclude

$$\frac{1}{N_0^2} \int_{\mathbb{K}^{2n}} \int_{\mathbb{K}^2} \int_{\mathbb{K}^{n-1}} H_0(x, z, h; \mathbf{y}) d\nu_{n-1}(h) d\mu(x) d\mu(z) d\mu_{[N]}^{\otimes 2n}(\mathbf{y}) \gtrsim \delta^{O(1)}, \quad (6.69)$$

where

$$H_0(x, z, h; \mathbf{y}) := \left| \int_{\mathbb{K}} G_0^1(x, z, (h_1, h); \mathbf{y}) d\nu_{[H_1]}(h_1) \right|^2,$$

and

$$G_0^1(x, z, (h_1, h); \mathbf{y}) := \mathbb{1}_X(h_1, h) e(-\phi(h_1, h)(x - z)) \\ \times \prod_{\omega \in \{0, 1\}^{n-1}} C^{|\omega|} F_{m; y(1, \omega, 0)}^\xi(x + (h_1, h) \cdot (1, \omega)) \overline{F_{m; y(1, \omega, 1)}^\xi(z + (h_1, h) \cdot (1, \omega))}$$

for $\mathbf{y} = (y(1, \omega, 0), y(1, \omega, 1))_{(j, \omega) \in \{0, 1\}^n} \in \mathbb{K}^{2n}$ and $x, z, h_1 \in \mathbb{K}$, $h \in \mathbb{K}^{n-1}$. Expanding the square and changing variables $x \mapsto x - h_1$ and $z \mapsto z - h_1$, we may rewrite Equation (6.69) as

$$\frac{1}{N_0^2} \int_{\mathbb{K}^{2n}} \int_{\mathbb{K}^2} \int_{\mathbb{K}^{n+1}} G_1(x, z, h_1, h'_1, h; \mathbf{y}) d\nu_{[H_1]}^{\otimes 2}(h_1, h'_1) d\nu_{n-1}(h) d\mu(x) d\mu(z) d\mu_{[N]}^{\otimes 2n}(\mathbf{y}) \\ \gtrsim \delta^{O(1)}, \quad (6.70)$$

where

$$G_1(x, z, h_1, h'_1, h; \mathbf{y}) := \mathbb{1}_X(h_1, h) \mathbb{1}_X(h'_1, h) e(-(\phi(h_1, h) - \phi(h'_1, h))(x - z)) \\ \times \prod_{\omega \in \{0, 1\}^{n-1}} C^{|\omega|} \Delta_{h'_1 - h_1} F_{m; y(1, \omega, 0)}^\xi(x + h \cdot \omega) \overline{\Delta_{h'_1 - h_1} F_{m; y(1, \omega, 1)}^\xi(z + h \cdot \omega)}.$$

Iteratively, for each $i \in \{2, \dots, n\}$, we apply the Cauchy–Schwarz inequality in all but the h_i variable to conclude that

$$\frac{1}{N_0^2} \int_{\mathbb{K}^2} \int_{\mathbb{K}^2} \int_{\square_n(X)} G_n(x, z, h, h'; y, y') d\nu_n^{\otimes 2}(h, h') d\mu(x) d\mu(z) d\mu_{[N]}^{\otimes 2}(y, y') \gtrsim \delta^{O(1)},$$

where

$$G_n(x, z, h, h'; y, y') := \Delta_{h' - h}^n F_{m; y}^\xi(x) \overline{\Delta_{h' - h}^n F_{m; y'}^\xi(z)} e(-\psi((h, h'))(x - z)).$$

We have arrived at Equation (6.68), completing the proof of the lemma □

The following lemma is a slight variant of a result found in [38].

Lemma 6.71. *Given a scale $N \geq 1$, $0 < \delta \leq 1$, $m \in \mathbb{Z}_+$ with $m \geq 2$, $n \in \mathbb{N}$ and scales $H_1, \dots, H_{n+1}$ with each $H_i \leq N$. We assume for every $i \in \llbracket n \rrbracket$ that $\varphi_i : \mathbb{K}^n \rightarrow \mathbb{K}$ is a measurable function independent of the variable h_i in a vector $h = (h_1, \dots, h_n) \in \mathbb{K}^n$. Let $\mathcal{P} := \{P_1, \dots, P_m\}$ and $\mathcal{Q} := \{Q_1, \dots, Q_n\}$ be collections of polynomials. For $\xi \in \mathbb{K}^n$, let F_m^ξ be the dual function defined in Equation (6.45) that corresponds to the form (6.42) and 1-bounded functions $f_0, f_1, \dots, f_{m-1} \in L^0(\mathbb{K})$ supported on an*

interval $I \subset \mathbb{K}$ of measure $N_0 = N^{\deg P_m}$. Suppose that

$$\int_{\mathbb{K}^n} \left| N_0^{-1} \widehat{\Delta_h^n F_m^\xi} \left(\sum_{i=1}^n \varphi_i(h) \right) \right|^2 d \left(\bigotimes_{i=1}^n \nu_{[H_i]} \right) (h) \geq \delta. \quad (6.72)$$

Then

$$\|F_m^\xi\|_{\square_{[H_1], \dots, [H_{n+1}]}^{n+1}(I)} \gtrsim_{\mathcal{P}} \delta^{O_{\mathcal{P}}(1)}.$$

Proof. We shall write as before $\nu_n := \bigotimes_{i=1}^n \nu_{[H_i]}$ and also let $\mu_n := \bigotimes_{i=1}^n \mu_{[H_i]}$. Expanding the Fejér kernel, we may write the left-hand side of Equation (6.72) as

$$\begin{aligned} \mathcal{I} &:= \int_{\mathbb{K}^n} \left| N_0^{-1} \widehat{\Delta_h^n F_m^\xi} \left(\sum_{i=1}^n \varphi_i(h) \right) \right|^2 d\nu_n(h) \\ &= \int_{\mathbb{K}^{2n}} \left| N_0^{-1} \widehat{\Delta_{h-h'}^n F_m^\xi} \left(\sum_{i=1}^n \varphi_i(h-h') \right) \right|^2 d\mu_n^{\otimes 2}(h, h') \\ &= \frac{1}{N_0^2} \int_{\mathbb{K}^{2n+2}} \Delta_{h-h'}^n F_m^\xi(x) \overline{\Delta_{h-h'}^n F_m^\xi(z)} e \left(- \sum_{i=1}^n \varphi_i(h-h')(x-z) \right) d\mu_n^{\otimes 2}(h, h') d\mu(x) d\mu(z). \end{aligned}$$

We apply the Cauchy–Schwarz inequality in the x, z and h' variables and Corollary 4.7 to deduce that

$$\begin{aligned} \mathcal{I}^{2n} &\leq \frac{1}{N_0^2} \int_{\mathbb{K}^{3n+2}} \prod_{\omega \in \{0,1\}^n} \mathcal{C}^{|\omega|} \left(\Delta_{h^{(\omega)}-h}^n F_m^\xi(x) \overline{\Delta_{h^{(\omega)}-h}^n F_m^\xi(z)} \right) d\mu_n^{\otimes 3}(h^{(0)}, h^{(1)}, h) d\mu(x) d\mu(z) \\ &= \frac{1}{N_0^2} \int_{\mathbb{K}^{3n}} \mathcal{A}(x, z, h'_n, h^{(0)}, h^{(1)}, h') \mathcal{B}(x, z, h^{(0)}, h^{(1)}, h') d\mu(x) d\mu(z) d\mu_{n-1}^{\otimes 3}(h^{(0)}, h^{(1)}, h') d\mu_{[H_n]}(h'_n), \end{aligned}$$

where

$$\begin{aligned} \mathcal{A}(x, z, h'_n, h^{(0)}, h^{(1)}, h') &:= \int_{\mathbb{K}^2} \prod_{\omega' \in \{0,1\}^{n-1}} \mathcal{C}^{|\omega'|} \left[\Delta_{h^{(\omega')}-h'}^{n-1} \left(F_m^\xi(x + h_n^0 - h'_n) \right. \right. \\ &\quad \left. \left. \times \overline{F_m^\xi(x + h_n^1 - h'_n) F_m^\xi(z + h_n^0 - h'_n) F_m^\xi(z + h_n^1 - h'_n)} \right) \right] d\mu_{[H_n]}(h_n^0, h_n^1), \\ \mathcal{B}(x, z, h^{(0)}, h^{(1)}, h') &:= \prod_{\omega' \in \{0,1\}^{n-1}} \mathcal{C}^{|\omega'|} \left[\Delta_{h^{(\omega')}-h'}^{n-1} \left(|F_m^\xi(x)|^2 |F_m^\xi(z)|^2 \right) \right]. \end{aligned}$$

Since $\mathcal{A} \geq 0$, we see that

$$\begin{aligned} \mathcal{I}^{2n} &\leq N_0^{-2} \int_{\mathbb{K}^{3n}} \mathcal{A}(x, z, h'_n, h^{(0)}, h^{(1)}, h') d\mu(x) d\mu(z) d\mu_{n-1}^{\otimes 3}(h^{(0)}, h^{(1)}, h') d\mu_{[H_n]}(h'_n) \\ &= \frac{1}{N_0^2} \int_{\mathbb{K}^{3n+1}} \prod_{\omega' \in \{0,1\}^{n-1}} \mathcal{C}^{|\omega'|} \left[\Delta_{h^{(\omega')}-h'}^{n-1} \left(F_m^\xi(x + h_n^0) \overline{F_m^\xi(x + h_n^1)} \right. \right. \\ &\quad \left. \left. \times \overline{F_m^\xi(z + h_n^0) F_m^\xi(z + h_n^1)} \right) \right] d\mu(x) d\mu(z) d\mu_n^{\otimes 2}(h^{(0)}, h^{(1)}) d\mu_{n-1}(h') \\ &= \frac{1}{N_0^2} \int_{\mathbb{K}^{3n+1}} \mathcal{C}(x, z, h^{(0)}, h^{(1)}, h') d\mu(x) d\mu(z) d\mu_n^{\otimes 2}(h^{(0)}, h^{(1)}) d\mu_{n-1}(h'), \end{aligned}$$

where

$$\mathcal{C}(x, z, h^{(0)}, h^{(1)}, h') := \prod_{\omega' \in \{0,1\}^{n-1}} \mathcal{C}^{|\omega'|} \left[\Delta_{h^{(0)} - h'}^{n-1} \Delta_{h_n^1 - h_n^0} \left(F_m^\xi(x) \overline{F_m^\xi(z)} \right) \right].$$

In the penultimate equality, we made the change of variables $x \rightarrow x - h_n^0 + h_n'$ and $z \rightarrow z - h_n^0 + h_n'$. Now, proceeding inductively we see that

$$\mathcal{I}^{2^n} \leq \frac{1}{N_0^2} \int_{\mathbb{K}^{2n+2}} \Delta_{h-h'}^n F_m^\xi(x) \overline{\Delta_{h-h'}^n F_m^\xi(z)} d\mu(x) d\mu(z) d\mu_n^{\otimes 2}(h, h').$$

Inserting an extra average in the x variable and using the pigeonhole principle to fix z , it follows that

$$\mathcal{I}^{2^n} \leq \frac{1}{N_0} \int_{\mathbb{K}^{2n+1}} \overline{\Delta_{h-h'}^n F_m^\xi(z)} \int_{\mathbb{K}} \Delta_{h-h'}^n F_m^\xi(x+w) d\mu_{[H_{n+1}]}(w) d\mu_n^{\otimes 2}(h, h') d\mu(x).$$

To conclude, we apply the Cauchy–Schwarz inequality to double the w variable and so

$$\delta^{2^{n+1}} \leq \mathcal{I}^{2^{n+1}} \leq \frac{1}{N_0} \int_{\mathbb{K}^{2n+3}} \Delta_{h-h'}^{n+1} F_m^\xi(x) d\mu_{n+1}^{\otimes 2}(h, h') d\mu(x) = \|F_m^\xi\|_{\square_{[H_1], \dots, [H_{n+1}]}^{n+1}(I)}^2.$$

This completes the proof of the lemma. $\square$

Proof of Theorem 6.57. The proof is by induction on $m \in \mathbb{Z}_+$. The proof will consist of several steps. We begin by establishing the following claim.

Claim 6.73. *Let $N \geq 1$ be a scale, $0 < \delta \leq 1$, $m \in \mathbb{Z}_+$ with $m \geq 2$ and $n \in \mathbb{N}$ be given. Let $\mathcal{P} := \{P_1, \dots, P_m\}$ and $\mathcal{Q} := \{Q_1, \dots, Q_n\}$ be collections of polynomials such that*

$$1 \leq \deg P_1 < \dots < \deg P_m < \deg Q_1 < \dots < \deg Q_n.$$

For $\xi \in \mathbb{K}^n$, let F_m^ξ be the dual function defined in Equation (6.45) that corresponds to the form (6.42) and 1-bounded functions $f_0, f_1, \dots, f_{m-1} \in L^0(\mathbb{K})$ supported on an interval $I \subset \mathbb{K}$ of measure $N_0 := N^{\deg P_m}$. Suppose that

$$N_0^{-1} |\widehat{F_m^\xi}(\zeta)| \geq \delta. \quad (6.74)$$

Then for any sufficiently large constant $C \gtrsim_{\mathcal{P}, \mathcal{Q}} 1$ one has

$$|\zeta| \lesssim \delta^{-C} N^{-\deg(P_m)}, \quad \text{and} \quad |\xi_j| \lesssim \delta^{-C} N^{-\deg(Q_j)} \quad \text{for all } j \in [n]. \quad (6.75)$$

The proof of Claim 6.73 for each integer $m \geq 2$ is itself part of the inductive proof of Theorem 6.57. In the first step, we prove Claim 6.73 for $m = 2$. In the second step, we show that Claim 6.73 for all integers $m \geq 2$ implies Theorem 6.57, this in particular will establish Theorem 6.57 for $m = 2$. In the third step, we finally show that Claim 6.73 for all integers $m \geq 3$ follows from Claim 6.73 and Theorem 6.57 for $m - 1$. Taken together, this shows that Claim 6.73 and Theorem 6.57 hold for each integer $m \geq 2$, completing the proof of Theorem 6.57. $\square$

Step 1.

We now prove Claim 6.73 for $m = 2$. Here, $N_0 = N^{\deg P_2}$. For $\zeta_1, \zeta_2 \in \mathbb{K}$ and $\xi \in \mathbb{K}^n$, we define the multiplier

$$m_N(\zeta_1, \zeta_2, \xi) := \int_{B_1(0)} e\left(-\zeta_1 P_1(\alpha y) + \zeta_2 P_2(\alpha y) + \sum_{j=1}^n \xi_j Q_j(\alpha y)\right) d\mu(y),$$

where $\alpha \in \mathbb{K}$ satisfies $|\alpha| = N$. By definitions (6.45) and (6.46) and making the change of variables $x \mapsto x - P_2(y)$, we may write

$$\begin{aligned} N_0^{-1} \widehat{F_2^\xi}(\zeta_2) &= N_0^{-1} \int_{\mathbb{K}} \int_{\mathbb{K}} F_{2;y}^\xi(x) e(-\zeta_2 x) d\mu_{[N]}(y) d\mu(x) \\ &= N_0^{-1} \int_{\mathbb{K}} \widehat{f_0}(\zeta_2 - \zeta_1) \widehat{f_1}(\zeta_1) m_N(\zeta_1, \zeta_2, \xi) d\zeta_1. \end{aligned}$$

By the Cauchy–Schwarz inequality and Plancherel’s theorem, we obtain

$$\delta \leq N_0^{-1} |\widehat{F_2^\xi}(\zeta_2)| \lesssim N_0^{-1} \|f_0\|_{L^2(\mathbb{K})} \|f_1\|_{L^2(\mathbb{K})} \sup_{\zeta_1 \in \mathbb{K}} |m_N(\zeta_1, \zeta_2, \xi)|,$$

which gives for some $\zeta_1 \in \mathbb{K}$ that

$$\delta \lesssim |m_N(\zeta_1, \zeta_2, \xi)|$$

since $\|f_0\|_{L^2(\mathbb{K})}, \|f_1\|_{L^2(\mathbb{K})} \lesssim N_0^{1/2}$. Applying Lemma 6.49 with $\mathcal{P} = \{-P_1, P_2\}$ and $\mathcal{Q} = \{Q_1, \dots, Q_n\}$, we deduce that for every sufficiently large $C \gtrsim 1$ one has

$$|\zeta_j| \lesssim \delta^{-C} N^{-\deg(P_j)} \quad \text{for all } j \in \llbracket 2 \rrbracket, \quad \text{and} \quad |\xi_j| \lesssim \delta^{-C} N^{-\deg(Q_j)} \quad \text{for all } j \in \llbracket n \rrbracket.$$

This completes the proof of Claim 6.73 for $m = 2$.

Step 2.

In this step, we show that Claim 6.73 for all integers $m \geq 2$ implies Theorem 6.57. In view of Step 1, this will in particular establish Theorem 6.57 for $m = 2$, which is the base case of our double induction. As before, we shall write $\nu_j := \bigotimes_{i=1}^j \nu_{[H_i]}$ for any $j \in \mathbb{Z}_+$. Recall that $N_0 = N^{\deg(P_m)}$ and note

$$\|F_m^\xi\|_{\square_{[H_1], \dots, [H_s]}^s}^{2^s} (I) = \int_{\mathbb{K}^{s-2}} \|\Delta_h^{s-2} F_m^\xi\|_{\square_{[H_{s-1}], [H_s]}^2}^4 (I) d\nu_{s-2}(h).$$

By Equation (6.58) and the pigeonhole principle, there exists a measurable set $X \subseteq \prod_{i=1}^{s-2} [H_i]$ so that $\nu_{s-2}(X) \gtrsim \delta^{O(1)}$, and for all $h \in X$ one has

$$\|\Delta_h^{s-2} F_m^\xi\|_{\square_{[H_{s-1}], [H_s]}^2}^4 (I) \gtrsim \delta^{O(1)}.$$

Here, we used that $\text{supp } F_m^\xi$ is a subset of an interval whose measure is at most $O(N_0)$. By Lemma 5.1, we have

$$N_0^{-1} \|\widehat{\Delta_h^{s-2} F_m^\xi}\|_{L^\infty(\mathbb{K})} \gtrsim \delta^{O(1)}.$$

Next, we claim that there is a countable set $\mathcal{F} \subset \mathbb{K}$, depending on N and δ such that

$$\sup_{\phi \in \mathcal{F}} N_0^{-1} |\widehat{\Delta_h^{s-2} F_m^\xi}(\phi)| \gtrsim \delta^{C_0} \tag{6.76}$$

for some absolute constant $C_0 \in \mathbb{Z}_+$ and for all $h \in X$. When $\mathbb{K}$ is non-Archimedean, we take

$$\mathcal{F} = \bigcup_{M \geq 1} \left\{ z = \sum_{j=-M}^{L-1} z_j \pi^j \in \mathbb{K} : z_j \in o_{\mathbb{K}}/m_{\mathbb{K}} \right\},$$

where $N_0 = q^L$. Let $x \in I = B_{N_0}(x_0)$. For any $\zeta \in \mathbb{K}$, we have $\zeta \in B_{N_0^{-1}}(\zeta_0)$ for some $\zeta_0 \in \mathcal{F}$. Note that

$$e(-\zeta x) = e(-x\zeta_0) e(-(x - x_0)(\zeta - \zeta_0)) e(-x_0(\zeta - \zeta_0)) = e(-x\zeta_0) e(-x_0(\zeta - \zeta_0))$$

since $|(x - x_0)(\zeta - \zeta_0)| \leq N_0 N_0^{-1} = 1$ and $e = 1$ on $\mathcal{O}_{\mathbb{K}}$. Therefore, $|\widehat{\Delta_h^{s-2} F_m^\xi}(\zeta)| = |\widehat{\Delta_h^{s-2} F_m^\xi}(\zeta_0)|$ since $\Delta_h^{s-2} F_m$ is supported in I whenever $h \in X$. This shows that Equation (6.76) holds for non-Archimedean fields.

When $\mathbb{K} = \mathbb{R}$, we take $\mathcal{F} := T_0 \mathbb{Z}$, where

$$T_0 := \delta^{C_0} (CN_0)^{-1}$$

for a sufficiently large constant $C \gtrsim 1$. When $\mathbb{K} = \mathbb{C}$, we take $\mathcal{F} := T_1 \mathbb{Z}^2$, where $T_1 := \delta^{C_0} (C\sqrt{N_0})^{-1}$. By the Lipschitz nature of characters on $\mathbb{R}$ or $\mathbb{C}$, we again see that Equation (6.76) holds in the Archimedean cases. In particular, there exists a measurable function $\phi : X \rightarrow \mathcal{F}$ so that

$$N_0^{-1} |\widehat{\Delta_h^{s-2} F_m^\xi}(\phi(h))| \gtrsim \delta^{C_0} \quad (6.77)$$

for all $h \in X$. If necessary, we may additionally assume that the range of ϕ is finite.

By Lemma 6.66, it follows that

$$\int_{\square_{s-2}(X)} \left| N_0^{-1} \int_{\mathbb{K}} F_m(x; h, h') e(-\psi((h, h')x)) d\mu(x) \right|^2 d\nu_{s-2}^{\otimes 2}(h, h') \gtrsim \delta^{O(1)},$$

where

$$\begin{aligned} F_m(x; h, h') &:= \int_{\mathbb{K}} \Delta_{h'-h}^{s-2} f_0(x + P_m(y)) \prod_{i=1}^{m-1} \Delta_{h'-h}^{s-2} f_i(x - P_i(y) + P_m(y)) d\mu_{[N]}(y), \\ \psi(h, h') &:= \sum_{\omega \in \{0,1\}^{s-2}} (-1)^{|\omega|} \phi(\omega \cdot h + (1 - \omega) \cdot h'). \end{aligned}$$

Thus, by the pigeonhole principle, there exists a measurable set $X_0 \subseteq \square_{s-2}(X)$ with $\nu_{s-2}^{\otimes 2}(X_0) \gtrsim \delta^{O(1)}$ such that for every $(h, h') \in X_0$ one has

$$\left| N^{-\deg(P_m)} \int_{\mathbb{K}} F_m(x; h, h') e(-\psi((h, h')x)) d\mu(x) \right| \gtrsim \delta^{O(1)}.$$

By Claim 6.73, there is a $c := c_{m,s} \geq 1$ such that for each $(h, h') \in X_0$, one has

$$|\psi((h, h'))| \lesssim_{m,s} \delta^{-c} N^{-\deg(P_m)}.$$

By the pigeonhole principle, there exists $h' \in \prod_{i=1}^{s-2} [H_i]$ and a measurable set

$$X_0(h') := \{h \in X : (h, h') \in X_0 \text{ and } |\psi((h, h'))| \lesssim \delta^{-c} N^{-\deg(P_m)}\}$$

satisfying $\nu_{s-2}(X_0(h')) \gtrsim \delta^{O(1)}$. Since $\psi((h, h')) \in \mathcal{F}$, we see that

$$X_0(h') \subseteq \bigcup_{k \in \mathbb{K}} X_0^k(h'),$$

where $\mathbb{K} = [O(\delta^{-O(1)})] \cap \mathbb{Z}$ when $\mathbb{K} = \mathbb{R}$. In this case, $X_0^k(h') := \{h \in X : \psi((h, h')) = T_0 k\}$. When $\mathbb{K} = \mathbb{C}$, we have $\mathbb{K} = [O(\delta^{-O(1)})] \cap \mathbb{Z}^2$ and $X_0^k(h') := \{h \in X : \psi((h, h')) = T_1 k\}$. Finally when $\mathbb{K}$ is

non-Archimedean,

$$\mathbb{K} = [O(\delta^{-O(1)})] \cap \left\{ k = \sum_{j=-M}^{-1} k_j \pi^j \in \mathbb{K} : k_j \in o_{\mathbb{K}}/m_{\mathbb{K}} \right\}$$

and $X_0^k(h') := \{h \in X : \psi((h, h')) = \pi^L k\}$.

Thus, by the pigeonhole principle there is $k_0 \in \mathbb{K}$ such that $\nu_{s-2}(X_0^{k_0}(h')) \gtrsim \delta^{O(1)}$. When $\mathbb{K} = \mathbb{R}$, this shows that $\psi(h, h') = T_0 k_0 =: \phi_m$ for all $h \in X_0^{k_0}(h')$. When $\mathbb{K} = \mathbb{C}$, we have $\phi(h, h') = T_1 k_0$ for all $h \in X_0^{k_0}(h')$ and when $\mathbb{K}$ is non-Archimedean, $\phi(h, h') = \pi^L k_0$ for all $h \in X_0^{k_0}(h')$. We will denote these values by ϕ_m in all cases.

Set

$$\psi_1(h, h') := (-1)^{s+1} \sum_{\substack{\omega \in \{0,1\}^{s-2} \\ \omega_1=0}} (-1)^{|\omega|} \phi((\omega \cdot h + (1-\omega) \cdot h')) + (-1)^s \phi_m$$

and, for $i = \llbracket s-2 \rrbracket \setminus \{1\}$, set

$$\psi_i(h, h') := (-1)^{s+1} \sum_{\substack{\omega \in \{0,1\}^{s-2} \setminus \{0\} \\ \omega_1=\dots=\omega_{i-1}=1 \\ \omega_i=0}} (-1)^{|\omega|} \phi((\omega \cdot h + (1-\omega) \cdot h')).$$

Note that ψ_i does not depend on h_i and we can write

$$\phi(h) = \sum_{i=1}^{s-2} \psi_i(h, h').$$

Averaging Equation (6.77) over $\mathbb{X} := X_0^{k_0}(h')$ and using positivity, we obtain

$$\begin{aligned} & \int_{\mathbb{K}^{s-2}} \left| N_0^{-1} \widehat{\Delta_h^{s-2} F_m^\xi} \left(\sum_{i=1}^{s-2} \psi_i(h, h') \right) \right|^2 d\nu_{s-2}(h) \\ & \geq \int_{\mathbb{X}} \left| N_0^{-1} \widehat{\Delta_h^{s-2} F_m^\xi}(\phi(h)) \right|^2 d\nu_{s-2}(h) \gtrsim \delta^{O(1)}. \end{aligned}$$

Invoking Lemma 6.71, we conclude that

$$\|F_m^\xi\|_{\square_{[H_1], \dots, [H_{s-1}]}^{s-1}}(I) \gtrsim \delta^{O(1)}.$$

Step 3.

Gathering together the conclusions of Step 1 and Step 2 (for $m = 2$), we see that the base step of a double induction has been established. In this step, we shall illustrate how to establish the inductive step. We assume that Claim 6.73 and Theorem 6.57 hold for $m-1$ in place m for some integer $m \geq 3$. Then we will prove that Claim 6.73 holds for $m \geq 3$, which in view of Step 2 will allow us to deduce that Theorem 6.57 also holds for $m \geq 3$. This will complete the proof of Theorem 6.57.

Recall that $N_0 = N^{\deg(P_m)}$. By definitions (6.45) and (6.46) and making the change of variables $x \mapsto x - P_m(y)$, we may write

$$\begin{aligned} N_0^{-1} \widehat{F_m^\xi}(\zeta_m) &= N_0^{-1} \int_{\mathbb{K}^2} F_{m;y}^\xi(x) e(-\zeta_m x) d\mu(x) d\mu_{[N]}(y) \\ &= N_0^{-1} \int_{\mathbb{K}^2} M_{\zeta_m} f_0(x) \prod_{i=1}^{m-1} f_i(x - P_i(y)) e\left(\zeta_m P_m(y) + \sum_{j=1}^n \xi_j Q_j(y)\right) d\mu_{[N]}(y) d\mu(x) \\ &=: M^{-1} \Lambda_{\mathcal{P}'; N}^{\mathcal{Q}', \xi'}(M_{\zeta_m} f_0, f_1, \dots, f_{m-1}), \end{aligned}$$

where $M_{\zeta_m} f_0(x) := e(-\zeta_m x) f_0(x)$, $\mathcal{P}' := \mathcal{P} \setminus \{P_m\}$, $\mathcal{Q}' := \mathcal{Q} \cup \{P_m\}$, $\xi' := (\zeta_m, \xi_1, \dots, \xi_n) \in \mathbb{K}^{n+1}$ and $M = N_0 N_0'^{-1}$, where N_0' is the scale $N^{\deg(P_{m-1})}$.

Thus, Equation (6.74) implies

$$M^{-1} |\Lambda_{\mathcal{P}'; N}^{\mathcal{Q}', \xi'}(M_{\zeta_m} f_0, f_1, \dots, f_{m-1})| \gtrsim \delta^{O(1)}.$$

As in the proof of Theorem 6.1, by the pigeonhole principle, we can find an interval $I' \subset \mathbb{K}$ of measure about N_0' such that

$$|\Lambda_{\mathcal{P}'; N, I'}^{\mathcal{Q}', \xi'}(f'_0, f'_1, \dots, f'_{m-1})| \gtrsim \delta^{O(1)},$$

where $f'_0 := M_{\zeta_m} f_0 \mathbb{1}_{I'}$, $f'_1 := f_1 \mathbb{1}_{I'}$, $\dots$, $f'_{m-1} := f_{m-1} \mathbb{1}_{I'}$.

Consequently, by Proposition 6.47, there exists an $s \in \mathbb{Z}_+$ such that

$$\|F_{m-1}^{\xi'}\|_{\square_{[H_1], \dots, [H_s]}^s(N_0')} \gtrsim \delta^{O(1)},$$

where $F_{m-1}^{\xi'}$ is the dual function respect the form $\Lambda_{\mathcal{P}'; N, I'}^{\mathcal{Q}', \xi'}(f'_0, f'_1, \dots, f'_{m-1})$ and $H_i \simeq \delta^{O_{\mathcal{P}'}(1)} N^{\deg(P_{m-1})}$ for $i \in [s]$. By the induction hypothesis (for Theorem 6.57), we deduce that

$$\|F_{m-1}^{\xi'}\|_{\square_{[H_1], [H_2]}^2(N_0')} \gtrsim \delta^{O(1)},$$

which in turn by Lemma 5.1 implies

$$(N_0')^{-1} |\widehat{F_{m-1}^{\xi'}}(\zeta_{m-1})| \gtrsim \delta^{O(1)}$$

for some $\zeta_{m-1} \in \mathbb{K}$. By the induction hypothesis (for Claim 6.73), we deduce that

$$\begin{aligned} |\zeta_j| &\lesssim \delta^{-C} N^{-\deg(P_j)} & \text{for all } j \in [m] \setminus [m-2], & \text{and} \\ |\xi_j| &\lesssim \delta^{-C} N^{-\deg(Q_j)} & \text{for all } j \in [n], \end{aligned}$$

which in particular implies Equation (6.75), and we are done.

7. Sobolev estimates

As a consequence of the L^∞ -inverse theorem from the previous section, we establish some Sobolev estimates, which will be critical in the proof of Theorem 1.3.

We begin with a smooth variant of Theorem 6.1. When $\mathbb{K}$ is Archimedean, we fix a Schwartz function φ on $\mathbb{K}$ so that

$$\mathbb{1}_{[1]}(\xi) \leq \widehat{\varphi}(\xi) \leq \mathbb{1}_{[2]}(\xi), \quad \xi \in \mathbb{K}.$$

When $\mathbb{K} = \mathbb{R}$, we set $\varphi_N(x) = N^{-1} \varphi(N^{-1}x)$ for any $N > 0$ and when $\mathbb{K} = \mathbb{C}$, we set $\varphi_N(z) = N^{-1} \varphi(N^{-1/2}z)$ for any $N > 0$. When $\mathbb{K}$ is non-Archimedean, we set $\varphi(x) = \mathbb{1}_{B_1(0)}(x)$ so that $\widehat{\varphi}(\xi) = \mathbb{1}_{B_1(0)}(\xi)$ and we set $\varphi_N(x) = N^{-1} \mathbb{1}_{[N]}(x)$ for any scale N .

Theorem 7.1 (A smooth variant of the inverse theorem). *Let $N \geq 1$ be a scale, $0 < \delta \leq 1$, $m \in \mathbb{Z}_+$ be given. Let $\mathcal{P} := \{P_1, \dots, P_m\}$ be a collection of polynomials such that $1 \leq \deg P_1 < \dots < \deg P_m$. Let $f_0, f_1, \dots, f_m \in L^0(\mathbb{K})$ be 1-bounded functions supported on an interval $I \subset \mathbb{K}$ of measure $N_0 = N^{\deg P_m}$. Suppose that the $(m+1)$ -linear form defined in Equation (6.2) satisfies*

$$|\Lambda_{\mathcal{P};N}(f_0, \dots, f_m)| \geq \delta. \quad (7.2)$$

Then for any $j \in \llbracket m \rrbracket$ there exists an absolute constant $C_j \gtrsim_{\mathcal{P}} 1$ so that

$$N_0^{-1} \|\varphi_{N_j} * f_j\|_{L^1(\mathbb{K})} \gtrsim_{\mathcal{P}} \delta^{O_{\mathcal{P}}(1)}, \quad (7.3)$$

where $N_j \simeq \delta^{C_j} N^{\deg(P_j)}$, provided $N \gtrsim \delta^{-O_{\mathcal{P}}(1)}$.

Proof. By translation invariance, we can assume that f_j is supported on $[N_0]$ for every $j \in \llbracket m \rrbracket$. The proof will consist of two steps. In the first step, we will invoke Theorem 6.1 to prove Equation (7.3) for $j = 1$. In the second step, we will use Equation (7.3) for $j = 1$ to establish Equation (7.3) for $j = 2$, and continuing inductively we will obtain Equation (7.3) for all $j \in \llbracket m \rrbracket$. $\square$

Step 1.

We first establish Equation (7.3) for $j = 1$. When $\mathbb{K}$ is non-Archimedean, this is an immediate consequence of Theorem 6.1 since $\varphi_{N_1} = \mu_{[N_1]}$ in this case. Nevertheless, we make the observation that

$$|\Lambda_{\mathcal{P};N}(f_0, \varphi_{N_1} * f_1, \dots, f_m)| \gtrsim \delta \quad (7.4)$$

holds. In fact, we will see that Equation (7.4) holds for any $\mathbb{K}$, non-Archimedean or Archimedean. First, let us see Equation (7.4) when $\mathbb{K}$ is non-Archimedean. Suppose that $|\Lambda_{\mathcal{P};N}(f_0, \varphi_{N_1} * f_1, \dots, f_m)| \leq c\delta$ for some small $c > 0$. Then, since

$$\delta \leq |\Lambda_{\mathcal{P};N}(f_0, f_1, \dots, f_m)| \leq |\Lambda_{\mathcal{P};N}(f_0, \varphi_{N_1} * f_1, \dots, f_m)| + |\Lambda_{\mathcal{P};N}(f_0, f_1 - \varphi_{N_1} * f_1, \dots, f_m)|,$$

we conclude that $|\Lambda_{\mathcal{P};N}(f_0, f_1 - \varphi_{N_1} * f_1, \dots, f_m)| \gtrsim \delta$. Therefore, Theorem 6.1 implies that $N_0^{-1} \|\varphi_{N_1} * (f_1 - \varphi_{N_1} * f_1)\|_{L^1(\mathbb{K})} \gtrsim \delta^{O(1)}$, but this is a contradiction since $\varphi_{N_1} * \varphi_{N_1} = \varphi_{N_1}$ when $\mathbb{K}$ is non-Archimedean (in which case $\varphi_{N_1} = N_1^{-1} \mathbb{1}_{[N_1]}$) and so $\varphi_{N_1} * (f_1 - \varphi_{N_1} * f_1) \equiv 0$.

We now turn to establish (7.3) for $j = 1$ when $\mathbb{K}$ is Archimedean (when $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). Let $\eta : \mathbb{K} \rightarrow [0, \infty)$ be a Schwartz function so that $\int_{\mathbb{K}} \eta = 1$, $\widehat{\eta} \equiv 1$ near 0 and $\text{supp } \widehat{\eta} \subseteq [2]$. For $t > 0$, we write $\eta_t(x) := t^{-1} \eta(t^{-1}x)$ when $\mathbb{K} = \mathbb{R}$ and $\eta_t(x) := t^{-2} \eta(t^{-1}x)$ when $\mathbb{K} = \mathbb{C}$. We will also need a Schwartz function $\rho : \mathbb{K} \rightarrow [0, \infty)$ such that

$$\mathbb{1}_{[1] \setminus [1-\delta^M]}(x) \leq \rho(x) \leq \mathbb{1}_{[1]}(x), \quad x \in \mathbb{K}$$

for some large absolute constant $M \geq 1$, which will be specified later. We shall also write $\rho_{(t)}(x) := \rho(t^{-1}x)$ for $t > 0$ and $x \in \mathbb{K}$.

Let $N'_0 \simeq N_0$ when $\mathbb{K} = \mathbb{R}$ and $N'_0 \simeq \sqrt{N_0}$ when $\mathbb{K} = \mathbb{C}$. Observe that Equation (7.2) implies that at least one of the following lower bounds holds:

$$|\Lambda_{\mathcal{P};N}(f_0, \varphi_{N_1} * f_1, \dots, f_m)| \gtrsim \delta, \quad (7.5)$$

$$|\Lambda_{\mathcal{P};N}(f_0, \rho_{(N'_0)}(f_1 - \varphi_{N_1} * f_1), \dots, f_m)| \gtrsim \delta, \quad (7.6)$$

$$|\Lambda_{\mathcal{P};N}(f_0, (1 - \rho_{(N'_0)})(f_1 - \varphi_{N_1} * f_1), \dots, f_m)| \gtrsim \delta. \quad (7.7)$$

By Theorem 6.1, it is easy to see that Equation (7.5) yields that

$$N_0^{-1} \|\varphi_{N_1} * f_1\|_{L^1(\mathbb{R})} \gtrsim \delta,$$

which in turn will imply Equation (7.3) for $j = 1$ provided that the remaining two alternatives (7.6) and (7.7) do not hold. If this is the case, then Equation (7.4) also holds when $\mathbb{K} = \mathbb{R}, \mathbb{C}$ is Archimedean.

If the second alternative holds we, let $f'_1 := \rho_{(N'_0)}(f_1 - \varphi_{N_1} * f_1)$ and then Theorem 6.1 implies that

$$N_0^{-1} \|\mu_{[N'_1]} * f'_1\|_{L^1(\mathbb{K})} \gtrsim_{\mathcal{P}} \delta^{C'_0},$$

with $N'_1 \simeq \delta^{C'_1} N^{\deg(P_1)}$. By the Cauchy–Schwarz inequality (the support of $\mu_{[N'_1]} * f'_1$ is contained in a fixed dilate of $[N_0]$), we have

$$N_0^{-1} \|\mu_{[N'_1]} * f'_1\|_{L^2(\mathbb{K})}^2 \gtrsim_{\mathcal{P}} \delta^{2C'_0}.$$

Let $N''_1 := \delta^{A+C'_1} N^{\deg(P_1)} / A$ for some $A \geq 1$ to be determined later. We now show that

$$\|\mu_{[N'_1]} - \mu_{[N'_1]} * \eta_{N'_1}\|_{L^1(\mathbb{K})}^2 \lesssim \sqrt{N''_1 / N'_1} \lesssim \sqrt{\delta^A / A}. \quad (7.8)$$

We note that for $|x| \geq CN'_1$,

$$|\mathbb{1}_{[N'_1]}(x) - \mathbb{1}_{[N'_1]} * \eta_{N'_1}(x)| = \left| \int_{\mathbb{K}} \mathbb{1}_{[N'_1]}(x - y) \eta_{N'_1}(y) d\mu(y) \right|$$

and so

$$\int_{|x| \geq CN'_1} |\mathbb{1}_{[N'_1]}(x) - \mathbb{1}_{[N'_1]} * \eta_{N'_1}(x)| d\mu(x) \lesssim N''_1.$$

When $|x| \leq CN'_1$ is small, we use the Cauchy–Schwarz inequality

$$\int_{|x| \leq CN'_1} |\mathbb{1}_{[N'_1]}(x) - \mathbb{1}_{[N'_1]} * \eta_{N'_1}(x)| d\mu(x) \lesssim \sqrt{N'_1} \|\mathbb{1}_{[N'_1]} * (\delta_0 - \eta_{N'_1})\|_{L^2(\mathbb{K})}$$

and then Plancherel’s theorem,

$$\|\mathbb{1}_{[N'_1]} * (\delta_0 - \eta_{N'_1})\|_{L^2(\mathbb{K})}^2 = \int_{\mathbb{K}} |1 - \widehat{\eta_{N'_1}}(\xi)|^2 |\widehat{\mathbb{1}_{[N'_1]}}(\xi)|^2 d\mu(\xi) \lesssim \sqrt{N'_1 N''_1}.$$

Here, we use the facts that $\widehat{\eta} \equiv 1$ near 0 and the Fourier decay bound for Euclidean balls,

$$|\widehat{\mathbb{1}_{[N'_1]}}(\xi)|^2 \lesssim |\xi|^{-2} \text{ when } \mathbb{K} = \mathbb{R} \text{ and } |\widehat{\mathbb{1}_{[N'_1]}}(\xi)|^2 \lesssim \sqrt{N'_1} |\xi|^{-3} \text{ when } \mathbb{K} = \mathbb{C}.$$

This establishes Equation (7.8) and so

$$\begin{aligned} N_0^{-1} \|(\mu_{[N'_1]} - \mu_{[N'_1]} * \eta_{N'_1}) * f'_1\|_{L^2(\mathbb{K})}^2 &\lesssim \|\mu_{[N'_1]} - \mu_{[N'_1]} * \eta_{N'_1}\|_{L^1(\mathbb{K})}^2 \\ &\lesssim \sqrt{N''_1 / N'_1} \lesssim \sqrt{\delta^A / A}. \end{aligned}$$

Consequently,

$$\delta^{2C'_0} \lesssim_{\mathcal{P}} N_0^{-1} \|\mu_{[N'_1]} * f'_1\|_{L^2(\mathbb{K})}^2 \lesssim N_0^{-1} \|\mu_{[N'_1]} * \eta_{N'_1} * f'_1\|_{L^2(\mathbb{K})}^2 + \sqrt{\delta^A / A},$$

which for sufficiently large $A \geq C'_0$ yields

$$N_0^{-1} \|\eta_{N_1''} * f_1'\|_{L^2(\mathbb{K})}^2 \gtrsim_{\mathcal{P}} \delta^{2C'_0}.$$

Taking $N_1 := \frac{1}{2}N_1''$ and using support properties of $\widehat{\varphi}$ and $\widehat{\eta}$, by the Plancherel theorem we may write (when $\mathbb{K} = \mathbb{R}$)

$$\begin{aligned} N_0^{-1} \|\eta_{N_1''} * f_1'\|_{L^2(\mathbb{R})}^2 &= N_0^{-1} \|\widehat{\eta}_{N_1''}(\widehat{\rho_{(N_1'')}} * ((1 - \widehat{\varphi}_{N_1})\widehat{f}_1))\|_{L^2(\mathbb{R})}^2 \\ &\lesssim N_0^{-1} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{N'_0}{(1 + N'_0|\xi - \zeta|)^{200}} |\widehat{f}_1(\zeta)(1 - \widehat{\varphi}(N_1\zeta))| |\widehat{\eta}(N_1''\xi)| \right)^2 d\mu(\xi) \\ &\lesssim N_0^{-1} \delta^{100(A+C'_1)} \|f_1\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

A similar bound holds when $\mathbb{K} = \mathbb{C}$. Therefore,

$$\delta^{2C'_0} \lesssim_{\mathcal{P}} N_0^{-1} \|\eta_{N_1''} * f_1'\|_{L^2(\mathbb{K})}^2 \lesssim \delta^{100(A+C'_1)},$$

which is impossible if $A \geq 1$ is large enough. Thus, the second alternative (7.6) is impossible. To see that the third alternative (7.7) is also impossible observe that

$$\delta \lesssim |\Lambda_{\mathcal{P};N}(f_0, (1 - \rho_{(N'_0)})(f_1 - \varphi_{N_1} * f_1), \dots, f_m)| \lesssim N_0^{-1} \int_{[N'_0]} (1 - \rho_{(N'_0)})(x) d\mu(x) \lesssim \delta^M,$$

which is also impossible if $M \geq 1$ is sufficiently large. Hence, Equation (7.5) must necessarily hold, and we are done.

Step 2.

Let $M \geq 1$ be a large constant to be determined later, and define $N' \simeq \delta^M N$ and $N'_0 \simeq \delta^M N_0$. The main idea is to partition the intervals $[N]$ and $[N_0]$ into $\mathbb{K} \simeq \delta^{-M}$ disjoint intervals of measure $\simeq N'$ and $\simeq N'_0$, respectively. Such partitions are straightforward when $\mathbb{K} = \mathbb{R}$. When $\mathbb{K}$ is non-Archimedean, we only need to partition $[N]$ and not $[N_0]$. Finally, when $\mathbb{K} = \mathbb{C}$, intervals are discs and it is not possible to partition a disc into subdiscs and so we will need to be careful with this technical issue.

We first concentrate on the case when $\mathbb{K}$ is non-Archimedean. In this case, we only need to partition $[N]$ and not $[N_0]$. Such a partition was given in the proof of Theorem 6.41. In fact, choosing $\ell \gg 1$ such that $q^{-\ell} \simeq \delta^M$ and setting $N = q^n$ so that $N' = q^{n-\ell}$, we have

$$[N] = B_{q^n}(0) = \bigcup_{y \in \mathcal{F}} B_{q^{n-\ell}}(y),$$

which gives a partition of $[N]$ where $\mathcal{F} = \{y = \sum_{j=0}^{\ell-1} y_j \pi^{-n+j} : y_j \in o_{\mathbb{K}}/m_{\mathbb{K}}\}$. Note $\#\mathcal{F} = q^\ell$ so that $\#\mathcal{F} \simeq \delta^{-M}$. Hence, $\Lambda_{\mathcal{P};N}(f_0, \varphi_{N_1} * f_1, \dots, f_m) =$

$$\frac{1}{N_0 N} \sum_{y_0 \in \mathcal{F}} \int_{\mathbb{K}} \int_{B_{q^{n-\ell}}(y_0)} f_0(x) \varphi_{N_1} * f_1(x - P_1(y)) \prod_{i=2}^m f_i(x - P_i(y)) d\mu(y) d\mu(x).$$

We observe that $\varphi_{N_1} * f_1(x - P_1(y)) = \varphi_{N_1} * f_1(x - P_1(y_0))$ for any $y \in B_{q^{n-\ell}}(y_0)$ by the non-Archimedean nature of $\mathbb{K}$, if M is chosen large enough depending on P_1 . Hence, by the pigeonhole principle, we can find a $y_0 \in \mathcal{F}$ such that

$$\left| \frac{1}{N_0 N'} \int_{\mathbb{K}} \int_{B_{q^{n-\ell}}(y_0)} f_0(x) \varphi_{N_1} * f_1(x - P_1(y_0)) \prod_{i=2}^m f_i(x - P_i(y)) d\mu(y) d\mu(x) \right| \gtrsim \delta.$$

Changing variables $y \rightarrow y_0 + y$ allows us to write the above as

$$|\Lambda_{\mathcal{P}', N'}(f'_0, f'_2, \dots, f'_m)| \gtrsim \delta \quad \text{where}$$

$$\Lambda_{\mathcal{P}', N'}(f'_0, f'_2, \dots, f'_m) = \frac{1}{N_0} \int_{\mathbb{K}^2} f'_0(x) \prod_{j=2}^m f'_j(x - P'_j(y)) d\mu_{[N']}(y) d\mu(x),$$

with $P'_j(y) = P_j(y_0 + y) - P_j(y_0)$, $f'_0(x) = f_0(x) \varphi_{N_1} * f_1(x - P_1(y_0))$ and $f'_j(x) = f_j(x + P_j(y_0))$. Note that each f'_j is supported in a fix dilate of I . In order to apply Theorem 6.1, we require $N' \simeq \delta^M N \geq 1$ and here is where the condition $N \gtrsim \delta^{-O_{\mathcal{P}}(1)}$ is needed. Therefore, Theorem 6.1 implies that

$$N_0^{-1} \|\mu_{[N_2]} * f_2\|_{L^1(\mathbb{K})} = N_0^{-1} \|\mu_{[N_2]} * f'_2\|_{L^1(\mathbb{K})} \gtrsim \delta^{O(1)}.$$

The equality of L^1 norms follows from the change of variables $x \rightarrow x + P_2(y_0)$. This completes the proof of Equation (6.4) for $j = 2$ when $\mathbb{K}$ is non-Archimedean since $\mu_{[N_2]} = \varphi_{N_2}$.

We now turn to the Archimedean case, when $\mathbb{K} = \mathbb{R}$ or when $\mathbb{K} = \mathbb{C}$. Here, we argue as in Step 1 and establish the version of Equation (7.4) for the function f_2 . More precisely, writing

$$\Lambda_{\mathcal{P}; N}(f_0, \dots, f_m) = \Lambda_{\mathcal{P}; N}(f_0, f_1, \varphi_{N_2} * f_2, \dots, f_m) + \Lambda_{\mathcal{P}; N}(f_0, f_1, f_2 - \varphi_{N_2} * f_2, \dots, f_m),$$

the argument in Step 1 shows that Equation (7.2) implies

$$|\Lambda_{\mathcal{P}; N}(f_0, f_1, \varphi_{N_2} * f_2, \dots, f_m)| \gtrsim \delta. \quad (7.9)$$

This inequality allows us to reduce matters to showing that Equation (7.2) implies $N_0^{-1} \|\mu_{[N_2]} * f_2\|_{L^1(\mathbb{K})} \gtrsim \delta^{O(1)}$ since then (7.9) would imply

$$\delta^{O(1)} \lesssim N_0^{-1} \|\mu_{[N_2]} * \varphi_{N_2} * f_2\|_{L^1(\mathbb{K})} \leq N_0^{-1} \|\varphi_{N_2} * f_2\|_{L^1(\mathbb{K})},$$

establishing (7.3) for $j = 2$.

We give the details when $\mathbb{K} = \mathbb{C}$ since there are additional technical difficulties alluded to above. The case $\mathbb{R}$ is easier. Given a large, general interval $\mathcal{I}$ in $\mathbb{C}$ (that is, $\mathcal{I}$ is a disc with large radius R), we can clearly find a mesh of $K \simeq \delta^{-M}$ disjoint squares $(S_k)_{k \in [K]}$ of side length $\delta^{M/2} R$ which sit inside $\mathcal{I}$ such that $\mu(\mathcal{I} \setminus \bigcup_{k \in [K]} S_k) \lesssim \delta^2 R^2$. We fix such a mesh of squares $(S_k)_{k \in [K]}$ for $[N]$ and a mesh of squares $(T_j)_{j \in [J]}$ for $[N_0]$ so that

$$\begin{aligned} & \Lambda_{\mathcal{P}; N}(f_0, \varphi_{N_1} * f_1, \dots, f_m) = \\ & \frac{1}{N_0 N} \sum_{j \in [J]} \sum_{k \in [K]} \int_{T_j} \int_{S_k} f_0(x) f_0(x) \varphi_{N_1} * f_1(x - P_1(y)) \prod_{i=2}^m f_i(x - P_i(y)) d\mu(x) d\mu(y) + O(\delta^2). \end{aligned}$$

Since $|\Lambda_{\mathcal{P}; N}(f_0, \varphi_{N_1} * f_1, \dots, f_m)| \gtrsim \delta$ by Equation (7.4) and since the number of terms in each sum above is about δ^{-M} , the pigeonhole principle gives us a square T_0 in $[N_0]$ and a square S_0 in $[N]$ such that

$$\left| \frac{1}{N'_0 N'} \int_{T_0} \int_{S_0} f_0(x) f_0(x) \varphi_{N_1} * f_1(x - P_1(y)) \prod_{i=2}^m f_i(x - P_i(y)) d\mu(x) d\mu(y) \right| \gtrsim \delta.$$

Write $[N']_{sq} = \{z \in \mathbb{C} : |z|_{\infty} \leq \sqrt{N'}\}$, where $|z|_{\infty} = \max(|x|, |y|)$ for $z = x + iy$. Hence, $S_0 = y_0 + [N']_{sq}$ for some $y_0 \in [N]$. For $z \in S_0$, we have $z = y_0 + y$ for some $y \in [N']_{sq}$ and so by the mean value theorem and the 1-boundedness of f_1 ,

$$\begin{aligned} & |\varphi_{N_1} * f_1(x - P_1(y_0 + y)) - \varphi_{N_1} * f_1(x - P_1(y_0))| \\ & \leq \sqrt{\frac{(N')^{\deg P_1}}{N_1}} \int_{\mathbb{C}} \|(\nabla \varphi)_{N_1}(z)\| d\mu(z) \lesssim_{\varphi} \delta^{(M \deg P_1 - C_1)/2}, \end{aligned}$$

where $N_1 = \delta^{C_1} N^{\deg P_1}$. Ensuring that $M \deg P_1 - C_1 \geq 4$, we see that

$$\left| \frac{1}{N'_0 N'} \int_{T_0} \int_{[N']_{sq}} f_0(x) \varphi_{N_1} * f_1(x - P_1(y_0)) \prod_{i=2}^m f'_i(x - P'_i(y)) d\mu(x) d\mu(y) \right| \gtrsim \delta,$$

where $P'_i(y) = P_i(y_0 + y) - P_i(y_0)$ and $f'_i(x) = f_i(x + P_i(y_0))$. For an appropriate interval I' containing T_0 with measure $\simeq N'_0$, we can write the above inequality as $|\Lambda_{\mathcal{P}'; N'}(f'_0, f'_2, \dots, f'_m)| \gtrsim \delta$, where $\mathcal{P}' = \{P'_2, \dots, P'_m\}$, $f'_0(x) = f_0(x) \varphi_{N_1} * f_1(x - P_1(y_0)) \mathbb{1}_{T_0}(x)$ and $f'_i(x) = f'_i(x) \mathbb{1}_{I'}(x)$ for $i \in \llbracket m \rrbracket \setminus \llbracket 1 \rrbracket$. Here,

$$\Lambda_{\mathcal{P}'; N'}(f'_0, \dots, f'_m) = \frac{1}{N'_0} \iint_{\mathbb{C}^2} f'_0(x) \prod_{i=2}^m f'_i(x - P'_i(y)) d\mu_{[N']_{sq}}(y) d\mu(x).$$

Again, in order to apply Theorem 6.1, we need $N' = \delta^M N \geq 1$ which holds provided $N \gtrsim \delta^{-O_{\mathcal{P}}(1)}$. Therefore, by Theorem 6.1 (see the remark following the statement of Theorem 6.1), we conclude that

$$(N'_0)^{-1} \|\mu_{[N_2]_{sq}} * f'_2\|_{L^1(\mathbb{C})} \gtrsim_{\mathcal{P}} \delta^{O(1)}$$

for some $N_2 \simeq \delta^{C_2 + M \deg(P_2)} N^{\deg(P_2)}$. The function $\mu_{[N_2]} * f'_2$ is supported on an interval $I'' \supseteq I'$ such that $\mu(I'' \setminus I') \lesssim N_2$. Furthermore, we can find an interval $I''' \subseteq I'$ so that $\mu(I' \setminus I''') \lesssim N_2$ and for $x \in I'''$, we have $\mathbb{1}_{I'}(x - u) = 1$ for all $u \in [N_2]_{sq}$. Hence,

$$\delta^{O(1)} \lesssim \frac{1}{N'_0} \int_{I'''} \left| \int_{\mathbb{C}} f_2(x + P_2(y_0) - u) d\mu_{[N_2]_{sq}}(u) \right| d\mu(x) + O(N_2 (N'_0)^{-1}),$$

where $N_2/N'_0 \lesssim \delta^{M(\deg P_2 - 1)}$ and $\deg P_2 - 1 \geq 1$. Hence, for $M \gg 1$ sufficiently large, we conclude that

$$\delta^{O(1)} \lesssim \frac{1}{N'_0} \int_{I'''} \left| \int_{\mathbb{C}} f_2(x + P_2(y_0) - u) d\mu_{[N_2]_{sq}}(u) \right| d\mu(x) \lesssim N_0^{-1} \|\mu_{[N_2]_{sq}} * f_2\|_{L^1(\mathbb{C})}. \quad (7.10)$$

In the final inequality, we promoted the integration in x to all of $\mathbb{C}$ and changed variables $x \rightarrow x + P_2(y_0)$. Hence, we have shown that Equation (7.2) implies $N_0^{-1} \|\mu_{[N_2]_{sq}} * f_2\|_{L^1(\mathbb{C})} \gtrsim \delta^{O(1)}$. Since Equation (7.2) holds with f_2 replaced by $\varphi_{N_2} * f_2$ (this is Equation (7.9)), we see that

$$\delta^{O(1)} \lesssim N_0^{-1} \|\mu_{[N_2]_{sq}} * \varphi_{N_2} * f_2\|_{L^1(\mathbb{C})} \leq N_0^{-1} \|\varphi_{N_2} * f_2\|_{L^1(\mathbb{C})},$$

establishing Equation (7.3) for $j = 2$. Now, we can proceed inductively and obtain Equation (7.3) for all $j \in \llbracket m \rrbracket$.

7.1. Multilinear functions and their duals

Recall the multilinear form

$$\Lambda_{\mathcal{P}; N}(f_0, f_1, \dots, f_m) = \frac{1}{N_0} \iint_{\mathbb{K}^2} f_0(x) \prod_{i=1}^m f_i(x - P_i(y)) d\mu_{[N]}(y) d\mu(x).$$

We define the multilinear function

$$A_N^{\mathcal{P}}(f_1, \dots, f_m)(x) := \int_{\mathbb{K}} \prod_{i=1}^m f_i(x - P_i(y)) d\mu_{[N]}(y)$$

so that $\Lambda_{\mathcal{P};N}$ can be written as a pairing of $A_N^{\mathcal{P}}$ with f_0 ,

$$\langle A_N^{\mathcal{P}}(f_1, \dots, f_m), f_0 \rangle = N_0 \Lambda_{\mathcal{P};N, [N_0]}(f_0, f_1, \dots, f_m),$$

where $\langle f, g \rangle = \int_{\mathbb{K}} f(x)g(x)d\mu(x)$. By duality, we have

$$\langle A_N^{\mathcal{P}}(f_1, \dots, f_m), f_0 \rangle = \langle (A_N^{\mathcal{P}})^{*j}(f_1, \dots, f_{j-1}, f_0, f_{j+1}, \dots, f_m), f_j \rangle,$$

where

$$(A_N^{\mathcal{P}})^{*j}(f_1, \dots, f_0, \dots, f_m)(x) := \int_{\mathbb{K}} \prod_{\substack{i=1 \\ i \neq j}}^m f_i(x - P_i(y) + P_j(y)) f_0(x + P_j(y)) d\mu_{[N]}(y).$$

Lemma 7.11 (Application of Hahn–Banach). *Let $A, B > 0$, let $I \subset \mathbb{K}$ be an interval, and let G be an element of $L^2(I)$. Let Φ be a family of vectors in $L^2(I)$, and assume the following inverse theorem: Whenever $f \in L^2(I)$ is such that $\|f\|_{L^\infty(I)} \leq 1$ and $|\langle f, G \rangle| > A$, then $|\langle f, \phi \rangle| > B$ for some $\phi \in \Phi$. Then G lies in the closed convex hull of*

$$V = \{\lambda\phi \in L^2(I) : \phi \in \Phi, |\lambda| \leq A/B\} \cup \{h \in L^2(I) : \|h\|_{L^1(I)} \leq A\}. \quad (7.12)$$

Proof. By way of contradiction, suppose that G does not lie in $W = \overline{\text{conv}V}^{\|\cdot\|_{L^2(I)}}$. From the Hahn–Banach theorem, we can find a continuous linear functional Λ of $L^2(I)$ which separates G from W ; that is, there is a $C \in \mathbb{R}$ such that $\text{Re } \Lambda(h) \leq C < \text{Re } \Lambda(G)$ for all $h \in W$. Scaling Λ allows us to change the constant C , so we can choose Λ such that $C = A$ is in the statement of the lemma. Since W is balanced, we see that $|\Lambda(h)| \leq A < \text{Re } \Lambda(G)$ for all $h \in W$. By the Riesz representation theorem, there is an $f \in L^2(I)$ which represents Λ so that $|\langle f, h \rangle| \leq A < \text{Re } \langle f, G \rangle$ for all $h \in V$. This implies that

$$|\langle f, \phi \rangle| \leq B$$

for all $\phi \in \Phi$ and that

$$\|f\|_{L^\infty(I)} = \sup_{\|h\|_{L^1(I)} \leq 1} |\langle f, h \rangle| \leq 1,$$

contradicting the hypothesis of the lemma. This completes the proof of the lemma. $\square$

Corollary 7.13 (Structure of dual functions). *Let $N \geq 1$ be a scale, $m \in \mathbb{Z}_+$ and $0 < \delta \leq 1$ be given. Let $\mathcal{P} := \{P_1, \dots, P_m\}$ be a collection of polynomials such that $1 \leq \deg P_1 < \dots < \deg P_m$. Let $f_0, f_1, \dots, f_m \in L^0(\mathbb{K})$ be 1-bounded functions supported on an interval of measure $N_0 = N^{\deg(P_m)}$. Then for every $j \in \llbracket m \rrbracket$, provided $N \gtrsim \delta^{-O_{\mathcal{P}}(1)}$, there exist a decomposition*

$$(A_N^{\mathcal{P}})^{*j}(f_1, \dots, f_0, \dots, f_m)(x) = H_j(x) + E_j(x), \quad (7.14)$$

where $H_j \in L^2(\mathbb{K})$ has Fourier transform supported in $[(N_j)^{-1}]$, where $N_j \simeq \delta^{C_j} N^{\deg P_j}$ and C_j is as in Theorem 7.1, and obeys the bounds

$$\|H_j\|_{L^\infty(\mathbb{K})} \lesssim_m 1, \quad \text{and} \quad \|H_j\|_{L^1(\mathbb{K})} \lesssim_m N_0. \quad (7.15)$$

The error term $E_j \in L^1(\mathbb{K})$ obeys the bound

$$\|E_j\|_{L^1(\mathbb{K})} \leq \delta N_0. \quad (7.16)$$

Proof. Fix $j \in \llbracket m \rrbracket$, let $I_0 := \text{supp}(A_N^{\mathcal{P}})^{*j}(f_1, \dots, f_0, \dots, f_m)$ and recall that $N_0 = N^{\deg(P_m)}$. By translation invariance we may assume $\text{supp } f_j \subseteq [N_0]$ for all $j \in \llbracket m \rrbracket$, and that $I_0 := [O(N_0)]$. If there exists $f \in L^\infty(I_0)$ with $\|f\|_{L^\infty(I_0)} \leq 1$ such that

$$|\langle f, (A_N^{\mathcal{P}})^{*j}(f_1, \dots, f_0, \dots, f_m) \rangle| > \delta N_0, \quad (7.17)$$

then proceeding as in the proof of Theorem 7.1 we may conclude that

$$|\langle \varphi_{N_j} * f, (A_N^{\mathcal{P}})^{*j}(\varphi_{N_1} * f_1, \dots, \varphi_{N_{j-1}} * f_{j-1}, f_0, f_{j+1}, \dots, f_m) \rangle| \geq c_m \delta N_0,$$

where $N_i \simeq \delta^{C_i} N^{\deg(P_i)}$ for $i \in \llbracket j \rrbracket$. This implies that there exists a 1-bounded $F \in L^2(\mathbb{K})$ with $\|F\|_{L^1(\mathbb{K})} \leq N_0$ such that $\text{supp } \widehat{F} \subseteq [N_j^{-1}]$ and

$$|\langle f, F \rangle| \geq c_m \delta N_0. \quad (7.18)$$

If fact, we can take

$$F(x) = \tilde{\varphi}_{N_j} * (A_N^{\mathcal{P}})^{*j}(\varphi_{N_1} * f_1, \dots, \varphi_{N_{j-1}} * f_{j-1}, f_0, f_{j+1}, \dots, f_m)(x),$$

where $\tilde{\varphi}(x) = \varphi(-x)$. Let Ψ denote the collection of all 1-bounded $F \in L^2(\mathbb{K})$ with $\text{supp } \widehat{F} \subseteq [N_j^{-1}]$ and $\|F\|_{L^1(\mathbb{K})} \leq N_0$. Invoking Lemma 7.11 with $A = \delta N_0/4$ and $B = c_m \delta N_0$ and the set $\Phi = \{F \mathbb{1}_{I_0} : F \in \Psi\}$, we obtain a decomposition

$$(A_N^{\mathcal{P}})^{*j}(f_1, \dots, f_{j-1}, f_0, f_{j+1}, \dots, f_m) = \sum_{l=1}^{\infty} c_l \phi_l + E(1) + E(2), \quad (7.19)$$

with the following properties:

- (i) for each $l \in \mathbb{Z}_+$, we have that $\phi_l = \lambda_l F_l \mathbb{1}_{I_0}$, $F_l \in \Psi$ and $\lambda_l \in \mathbb{C}$ such that $|\lambda_l| \lesssim_m 1$;
- (ii) the coefficients c_l are nonnegative with $\sum_{l=1}^{\infty} c_l \leq 1$, and all but finitely c_l vanish;
- (iii) the error term $E(1) \in L^1(I_0)$ satisfies $\|E(1)\|_{L^1(I_0)} \leq \delta N_0/2$;
- (iv) the error term $E(2) \in L^2(I_0)$ satisfies $\|E(2)\|_{L^2(I_0)} \leq \delta$.

The latter error term arises as a consequence of the fact that one is working with the closed convex hull instead of the convex hull. In fact, its $L^2(I_0)$ norm can be made arbitrarily small, but δ will suffice for our purposes.

Grouping together terms in the decomposition (7.19), we have

$$(A_N^{\mathcal{P}})^{*j}(f_1, \dots, f_{j-1}, f_0, f_{j+1}, \dots, f_m) = H'_j + E'_j,$$

where

$$H'_j = \left[\sum_{l=1}^{\infty} c_l \lambda_l F_l \right] \mathbb{1}_{I_0} \quad \text{satisfies} \quad \|H'_j\|_{L^1(\mathbb{K})} \leq \sum_{l=1}^{\infty} c_l |\lambda_l| \|F_l\|_{L^1(\mathbb{K})} \lesssim_m N_0 \quad \text{and}$$

$$\|H'_j\|_{L^\infty(\mathbb{K})} \leq \sup_{l \in \mathbb{N}} \|F_l\|_{L^\infty(\mathbb{K})} \sum_{l=1}^{\infty} c_l |\lambda_l| \lesssim_m 1.$$

Also, $E'_j = E(1) + E(2)$ satisfies $\|E'_j\|_{L^1(I_0)} \leq \delta N_0$ by (iii) and (iv) above since by the Cauchy–Schwarz inequality, we have $\|E(2)\|_{L^1(I_0)} \leq \delta N_0^{1/2}$.

We note that the function $F(x) = \sum_{i=1}^{\infty} c_i \lambda_i F_i(x)$ is Fourier supported in the interval $[N_j^{-1}]$.

When $\mathbb{K}$ is non-Archimedean, $\text{supp}(\widehat{\mathbb{1}_{I_0}}) \subseteq [N_0^{-1}]$ and so the Fourier transform of H'_j is supported in $[N_j^{-1}]$. This verifies Equation (7.15) in this case and completes the proof when $\mathbb{K}$ is non-Archimedean since the decomposition $H'_j + E'_j$ of $(A_N^{\mathcal{P}})^{*j}$ satisfies Equations (7.15) and (7.16).

Now, suppose $\mathbb{K}$ is Archimedean. Let ψ be a Schwartz function such that $\int_{\mathbb{K}} \psi(x) d\mu(x) = 1$ and $\text{supp } \widehat{\psi} \subseteq [2]$. Let $M \simeq \delta^{O(1)} N_0$ and as usual, set $\psi_M(x) = M^{-1} \psi(M^{-1}x)$ when $\mathbb{K} = \mathbb{R}$ and $\psi_M(x) = M^{-1} \psi(M^{-1/2}x)$ when $\mathbb{K} = \mathbb{C}$. From the proof of Equation (7.8), we have

$$\|\mathbb{1}_{I_0} - \mathbb{1}_{I_0} * \psi_M\|_{L^1(\mathbb{K})} \lesssim M^{1/4} N_0^{3/4}. \quad (7.20)$$

We set $H_j(x) = F(x) \mathbb{1}_{I_0} * \psi_M(x)$ and $E_j = E(1) + E(2) + (\mathbb{1}_{I_0} - \mathbb{1}_{I_0} * \psi_M)F$ so that

$$(A_N^{\mathcal{P}})^{*j}(f_1, \dots, f_{j-1}, f_0, f_{j+1}, \dots, f_m)(x) = H_j(x) + E_j(x).$$

From Equation (7.20), we see that E_j satisfies Equation (7.16). The properties $\|H_j\|_{L^\infty(\mathbb{K})} \lesssim_m 1$ and $\|H_j\|_{L^1(\mathbb{K})} \lesssim_m N_0$ are still preserved. Moreover, $\text{supp } \widehat{H}_j \subseteq [O(N_j^{-1})]$ since

$$\widehat{H}_j = (\widehat{\mathbb{1}_{I_0}} \widehat{\varphi}_M) * \widehat{F}.$$

This shows that Equation (7.15) holds for H_j and this completes the proof of the corollary. $\square$

We will combine Corollary 7.13 and the following L^p improving bound for polynomial averages to establish the key Sobolev inequality.

Lemma 7.21 (L^p -improving for polynomial averages). *Let $Q \in \mathbb{K}[y]$ with $\deg(Q) = d$, and let $N \gg_Q 1$ be a large scale. Consider the averaging operator*

$$M_N^Q g(x) := \int_{\mathbb{K}} g(x - Q(y)) d\mu_{[N]}(y).$$

For any parameters $1 < r < s < \infty$ satisfying $1/s = 1/r - 1/d$, the following inequality holds:

$$\|M_N^Q g\|_{L^s(\mathbb{K})} \lesssim_Q N^{d(\frac{1}{s} - \frac{1}{r})} \|g\|_{L^r(\mathbb{K})} \quad \text{for } g \in L^r(\mathbb{K}). \quad (7.22)$$

Proof. As our bounds are allowed to depend on Q , we may assume that Q is monic. Let $\alpha \in \mathbb{K}$ be such that $|\alpha| = N$, and change variables $y \rightarrow \alpha y$ to write

$$M_N^Q g(x) = \int_{B_1(0)} g(x - Q(\alpha y)) d\mu(y) = \int_{B_1(0)} g_\alpha(\alpha^{-d}x - Q_\alpha(y)) d\mu(y)$$

where $g_\alpha(x) = g(\alpha^d x)$ and $Q_\alpha(y) = \alpha^{-d}Q(\alpha y) = y^d + \alpha^{-1}a_{d-1}y^{d-1} + \dots + \alpha^{-d}a_0$. Hence, the right-hand side above can be written as $M_{Q_\alpha}^1 g_\alpha(\alpha^{-d}x)$. Since $\|g_\alpha\|_{L^r(\mathbb{K})} = N^{-d/r} \|g\|_{L^r(\mathbb{K})}$, we see that matters are reduced to proving Equation (7.22) for $N = 1$ and $Q = Q_\alpha$ with uniform bounds in α .

The mapping $y \rightarrow Q_\alpha(y)$ is d -to-1, and we can use a generalised change of variables formula to see that

$$|M_{Q_\alpha}^1 g(x)| \lesssim \int_{|s| \leq 2} |g(x - s)| |s|^{-(d-1)/d} d\mu(s)$$

when $N \gg_Q 1$. Hence M_α^1 is controlled by fractional integration, uniformly in α . When $\mathbb{K}$ is Archimedean, such a change of variables formula is well known. Recall that when $\mathbb{K} = \mathbb{C}$, $|s| = s\bar{s}$ is the square of the usual absolute value.

When $\mathbb{K} = \mathbb{Q}_p$ is the p -adic field, such a formula is given in [12]. The argument in [12] generalises to general non-Archimedean fields (when the characteristic, if positive, is larger than d). Alternatively,

one can use a construction in [46], valid in any local field and valid for any polynomial Q where $Q'(x)$ does not equal to zero mod $m_{\mathbb{K}}$ for any nonzero x (we need the condition on the characteristic of the field for this), in which the unit group $U = \bigcup_{j \in \llbracket J \rrbracket} U_j$ is partitioned into $J = \gcd(d, q - 1)$ open sets and analytic isomorphisms $\phi_j : D_j \rightarrow \phi_j(D_j)$ are constructed such that $y = \phi_j(x)$ precisely when $Q(y) = x$. For us, $Q'_\alpha(x) \neq 0 \bmod m_{\mathbb{K}}$ for any nonzero x if $|\alpha| = N \gg_Q 1$ is sufficiently large.

By the Hardy–Littlewood–Sobolev inequality (easily seen to be valid over general locally compact topological fields), we have

$$\|M_{Q_\alpha}^1 g\|_{L^s(\mathbb{K})} \lesssim \|g\|_{L^{r'}(\mathbb{K})},$$

uniformly in α whenever $1/s = 1/r - 1/d$, completing the proof of the lemma. $\square$

We now come to the proof of Theorem 1.6.

As in the set up for Theorem 7.1, we fix a smooth function φ with compact Fourier support. When $\mathbb{K}$ is Archimedean, let φ be a Schwartz function on $\mathbb{K}$ so that

$$\mathbb{1}_{[1]}(\xi) \leq \widehat{\varphi}(\xi) \leq \mathbb{1}_{[2]}(\xi), \quad \xi \in \mathbb{K}.$$

When $\mathbb{K} = \mathbb{R}$, we set $\varphi_N(x) = N^{-1}\varphi(N^{-1}x)$ for any $N > 0$ and when $\mathbb{K} = \mathbb{C}$, we set $\varphi_N(z) = N^{-1}\varphi(N^{-1/2}z)$ for any $N > 0$. When $\mathbb{K}$ is non-Archimedean, we set $\varphi(x) = \mathbb{1}_{B_1(0)}(x)$ so that $\widehat{\varphi}(\xi) = \mathbb{1}_{B_1(0)}(\xi)$ and we set $\varphi_N(x) = N^{-1}\mathbb{1}_{[N]}(x)$ for any scale N . We restate Theorem 1.6 in a more formal, precise way.

Theorem 7.23 (A Sobolev inequality for $A_N^{\mathcal{P}}$). *Let $\mathcal{P} := \{P_1, \dots, P_m\}$ be a collection of polynomials such that $1 \leq \deg P_1 < \dots < \deg P_m$. Let $N \gg_{\mathcal{P}} 1$ be a scale, $m \in \mathbb{Z}_+$ and $0 < \delta \leq 1$ be given. Let $1 < p_1, \dots, p_m < \infty$ satisfying $\frac{1}{p_1} + \dots + \frac{1}{p_m} = 1$ be given. Suppose $N \gtrsim \delta^{-O_{\mathcal{P}}(1)}$. Then for all $f_1 \in L^{p_1}(\mathbb{K}), \dots, f_m \in L^{p_m}(\mathbb{K})$ we have*

$$\|A_N^{\mathcal{P}}(f_1, \dots, f_{j-1}, (\delta_0 - \varphi_{N_j}) * f_j, f_{j+1}, \dots, f_m)\|_{L^1(\mathbb{K})} \lesssim \delta^{1/8} \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{K})}, \quad (7.24)$$

where $N_j \simeq \delta^{C_j} N^{\deg P_j}$ and C_j is the parameter from Theorem 7.1. Here, $\widehat{\delta_0} \equiv 1$.

Remark. The proof of Theorem 7.23 (and its statement) implicitly assumes that $m \geq 2$, but there is a version when $m = 1$, which will be given in Section 8 where it is needed.

Proof. We fix $j \in \llbracket m-1 \rrbracket$ and recall $N_j \simeq \delta^{O(1)} N^{\deg(P_j)}$. We first prove that for every functions $f_1, \dots, f_{j-1}, f_{j+1}, \dots, f_{m-1} \in L^\infty(\mathbb{K})$ and $f_j, f_m \in L^2(\mathbb{K})$, we have

$$\begin{aligned} & \|A_N^{\mathcal{P}}(f_1, \dots, f_{j-1}, (\delta_0 - \varphi_{N_j}) * f_j, f_{j+1}, \dots, f_m)\|_{L^1(\mathbb{K})} \\ & \lesssim \delta^{1/8} \left(\prod_{\substack{i=1 \\ i \neq j}}^{m-1} \|f_i\|_{L^\infty(\mathbb{K})} \right) \|f_j\|_{L^2(\mathbb{K})} \|f_m\|_{L^2(\mathbb{K})}. \end{aligned} \quad (7.25)$$

Choose $f_0 \in L^\infty(\mathbb{K})$ so that $\|f_0\|_{L^\infty(\mathbb{K})} = 1$ and

$$\begin{aligned} & \|A_N^{\mathcal{P}}(f_1, \dots, f_{j-1}, (\delta_0 - \varphi_{N_j}) * f_j, f_{j+1}, \dots, f_m)\|_{L^1(\mathbb{K})} \\ & \simeq |\langle A_N^{\mathcal{P}}(f_1, \dots, f_{j-1}, (\delta_0 - \varphi_{N_j}) * f_j, f_{j+1}, \dots, f_m), f_0 \rangle| \\ & = |\langle (\delta_0 - \varphi_{N_j}) * (A_N^{\mathcal{P}})^{*j}(f_1, \dots, f_0, \dots, f_m), f_j \rangle|. \end{aligned}$$

By the Cauchy–Schwarz inequality, it will suffice to prove

$$\begin{aligned} & \|(\delta_0 - \varphi_{N_j}) * (A_N^{\mathcal{P}})^{*j}(f_1, \dots, f_0, \dots, f_m)\|_{L^2(\mathbb{K})} \\ & \lesssim \delta^{1/8} \|f_0\|_{L^\infty(\mathbb{K})} \left(\prod_{\substack{i=1 \\ i \neq j}}^{m-1} \|f_i\|_{L^\infty(\mathbb{K})} \right) \|f_m\|_{L^2(\mathbb{K})}. \end{aligned} \quad (7.26)$$

By multilinear interpolation, the bounds (7.25) imply Equation (7.24) and so the proof of Theorem 7.23 is reduced to establishing Equation (7.26) which will be divided into three steps. In the first two steps, we will assume that f_m is supported in some interval of measure N_0 where $N_0 \simeq N^{\deg(P_m)}$. $\square$

Step 1

In this step, we will establish the bound

$$\begin{aligned} & \|(\delta_0 - \varphi_{N_j}) * (A_N^{\mathcal{P}})^{*j}(f_1, \dots, f_0, \dots, f_m)\|_{L^2(\mathbb{K})} \\ & \lesssim \delta^{1/2} N_0^{1/2} \|f_0\|_{L^\infty(\mathbb{K})} \left(\prod_{\substack{i=1 \\ i \neq j}}^{m-1} \|f_i\|_{L^\infty(\mathbb{K})} \right) \|f_m\|_{L^\infty(\mathbb{K})} \end{aligned} \quad (7.27)$$

under the assumption that f_m is supported in an interval of measure N_0 (when $\mathbb{K} = \mathbb{C}$, this implies in particular that f_m is supported in a square with measure about N_0 , which in Step 3 will be a helpful observation). When f_m has this support condition,

$$(A_N^{\mathcal{P}})^{*j}(f_1, \dots, f_0, \dots, f_m) = (A_N^{\mathcal{P}})^{*j}(f'_1, \dots, f'_0, \dots, f'_m),$$

where $f'_i(x) = f_i(x)\mathbb{1}_{I_0}(x)$ for some interval I_0 of measure $O(N_0)$. To prove Equation (7.27), it suffices to assume $\|f_i\|_{L^\infty(\mathbb{K})} = 1$ for $i = 0, 1, \dots, j-1, j+1, \dots, m$ and so Equation (7.27) takes the form

$$\|(\delta_0 - \varphi_{N_j}) * (A_N^{\mathcal{P}})^{*j}(f_1, \dots, f_0, \dots, f_m)\|_{L^2(\mathbb{K})} \lesssim \delta^{1/2} N_0^{1/2}. \quad (7.28)$$

We apply the decomposition (7.14) to $(A_N^{\mathcal{P}})^{*j}(f'_1, \dots, f'_0, \dots, f'_m)$ to write

$$(A_N^{\mathcal{P}})^{*j}(f_1, \dots, f_0, \dots, f_m)(x) = H_j(x) + E_j(x),$$

where H_j satisfies Equation (7.15) and E_j satisfies Equation (7.16). Using the fact that $\widehat{H}_j \subseteq [(N_j)^{-1}]$, we conclude that $(\delta_0 - \varphi_{N_j}) * H_j = 0$. Thus,

$$(\delta_0 - \varphi_{N_j}) * (A_N^{\mathcal{P}})^{*j}(f_1, \dots, f_0, \dots, f_m) = (\delta_0 - \varphi_{N_j}) * E_j.$$

From Equation (7.16) and the 1-boundedness of $(A_N^{\mathcal{P}})^{*j}(f_1, \dots, f_0, \dots, f_m)$, we have

$$\|(\delta_0 - \varphi_{N_j}) * E_j\|_{L^1(\mathbb{K})} \lesssim \delta N_0, \quad \text{and} \quad \|(\delta_0 - \varphi_{N_j}) * E_j\|_{L^\infty(\mathbb{K})} \lesssim 1,$$

respectively. Therefore,

$$\|(\delta_0 - \varphi_{N_j}) * E_j\|_{L^2(\mathbb{K})} \lesssim \delta^{1/2} N_0^{1/2},$$

establishing (7.28) and hence Equation (7.27). This completes Step 1.

Step 2.

We continue with our assumption that f_m is supported in an interval of measure N_0 , but now we relax the $L^\infty(\mathbb{K})$ control on f_m to $L^2(\mathbb{K})$ control and show that

$$\begin{aligned} & \|(\delta_0 - \varphi_{N_j}) * (A_N^{\mathcal{P}})^{*j}(f_1, \dots, f_0, \dots, f_m)\|_{L^2(\mathbb{K})} \\ & \lesssim \delta^{1/4} \|f_0\|_{L^\infty(\mathbb{K})} \left(\prod_{\substack{i=1 \\ i \neq j}}^{m-1} \|f_i\|_{L^\infty(\mathbb{K})} \right) \|f_m\|_{L^2(\mathbb{K})}. \end{aligned} \quad (7.29)$$

The main tool for this will be the L^p -improving estimate (7.22) for the polynomial average M_N^Q . We have a pointwise bound

$$|(A_N^{\mathcal{P}})^{*j}(f_1, \dots, f_0, \dots, f_m)(x)| \leq M_N^{P_m - P_j} |f_m|(x),$$

which combined with Equation (7.22) (for $Q = P_m - P_j$, $d = \deg(P_m)$, $s = 2$ and $r = (d+2)/2d$) yields

$$\begin{aligned} & \|(\delta_0 - \varphi_{N_j}) * (A_N^{\mathcal{P}})^{*j}(f_1, \dots, f_0, \dots, f_m)\|_{L^2(\mathbb{K})} \\ & \lesssim N_0^{-1/d} \|f_0\|_{L^\infty(\mathbb{K})} \left(\prod_{\substack{i=1 \\ i \neq j}}^{m-1} \|f_i\|_{L^\infty(\mathbb{K})} \right) \|f_m\|_{L^r(\mathbb{K})}. \end{aligned} \quad (7.30)$$

Interpolating Equations (7.27) and (7.30), we obtain Equation (7.29) as desired.

Step 3.

In this final step, we remove the support condition on f_m and establish (7.26). To prove Equation (7.26), we may assume that $\|f_i\|_{L^\infty(\mathbb{K})} = 1$ for $i = 0, 1, \dots, j-1, j+1, \dots, m-1$. We split $f_m = \sum_{I \in \mathcal{I}} f_m \mathbb{1}_I$, where I ranges over a partition $\mathcal{I}$ of $\mathbb{K}$ into intervals I of measure N_0 . We have seen this is possible when $\mathbb{K}$ is non-Archimedean or when $\mathbb{K} = \mathbb{R}$. This is not possible when $\mathbb{K} = \mathbb{C}$, but in this case, we can find a partition $\mathcal{I}$ of squares. By Step 1 and Step 2, the local dual function $D_I := (A_N^{\mathcal{P}})^{*j}(f_1, \dots, f_0, \dots, f_m \mathbb{1}_I)$ obeys the bound

$$\|(\delta_0 - \varphi_{N_j}) * D_I\|_{L^2(\mathbb{K})} \lesssim \delta^{1/4} \|f_m\|_{L^2(I)} \quad (7.31)$$

for each interval I , and we wish to establish

$$\left\| \sum_{I \in \mathcal{I}} (\delta_0 - \varphi_{N_j}) * D_I \right\|_{L^2(\mathbb{K})} \lesssim \delta^{1/8} \|f_m\|_{L^2(\mathbb{K})}.$$

We will square out the sum. To handle the off-diagonal terms, we observe that for finite intervals $I, J \subset \mathbb{K}$ (squares when $\mathbb{K} = \mathbb{C}$) of measure N_0 and $M > 0$ and $1 \leq p < \infty$, we have

$$\|\varphi_{N_j} * (f \mathbb{1}_I)\|_{L^p(J)} \lesssim_{M,p} (1 + N_0^{-1} \text{dist}(I, J))^{-M} \|f\|_{L^p(I)}. \quad (7.32)$$

By squaring and applying Schur's test, it suffices to obtain the decay bound

$$|\langle (\delta_0 - \varphi_{N_j}) * D_I, (1 - \varphi_{N_j}) * D_J \rangle| \lesssim \delta^{1/4} (1 + N_0^{-1} \text{dist}(I, J))^{-2} \|f_m\|_{L^2(I)} \|f_m\|_{L^2(J)}$$

for all intervals I, J of measure N_0 . By Cauchy–Schwarz and Equation (7.31), we know

$$\langle (\delta_0 - \varphi_{N_j}) * D_I, (1 - \varphi_{N_j}) * D_J \rangle \lesssim \delta^{1/2} \|f_m\|_{L^2(I)} \|f_m\|_{L^2(J)}.$$

On the other hand, D_I is supported in a $O(N_0)$ -neighborhood of I , and similarly for D_J . From Equation (7.32) and Cauchy–Schwarz, we thus have

$$\begin{aligned} \langle (\delta_0 - \varphi_{N_j}) * D_I, (1 - \varphi_{N_j}) * D_J \rangle &\lesssim (1 + N_0^{-1} \text{dist}(I, J))^{-10} \|D_I\|_{L^2(\mathbb{K})} \|D_J\|_{L^2(\mathbb{K})} \\ &\lesssim (1 + N_0^{-1} \text{dist}(I, J))^{-10} \|f_m\|_{L^2(I)} \|f_m\|_{L^2(J)}. \end{aligned}$$

Taking the geometric mean of the two estimates, we obtain the claim in Equation (7.26). This completes the proof of Theorem 7.23.

8. The implication Theorem 1.6 $\implies$ Theorem 1.3

Here, we give the details of Bourgain’s argument in [3] which allow us to pass from Theorem 1.6 to Theorem 1.3 on polynomial progressions. Let $\mathcal{P} = \{P_1, \dots, P_m\}$ be a sequence of polynomials in $\mathbb{K}[y]$ with distinct degrees and no constant terms. Without loss of generality, we may assume

$$\deg P_1 < \deg P_2 < \dots < \deg P_m,$$

and we set $d_{m-j} := \deg P_j$ and $d := d_0 = \deg P_m$ so that $d_{m-1} < \dots < d_1 < d$.

Since the argument showing how Theorem 1.6 implies Theorem 1.3 has been given in [3], [11] and [8] in the Euclidean setting (albeit for shorter polynomial progressions), we will only give the details for non-Archimedean fields $\mathbb{K}$ where uniform notation can be employed.

We will proceed in several steps.

Step 1

When $\mathbb{K}$ is non-Archimedean, the family $(Q_t)_{t>0}$ of convolution operators defined by

$$Q_t f(x) = f * \mu_{[t]}(x) = \frac{1}{t} \int_{|y| \leq t} f(x - y) d\mu(y) \quad \text{for scales } t > 0$$

gives us a natural approximation of the identity and form the analogue of the Poisson semigroup in the non-Archimedean setting. They also give us Fourier localization since

$$\widehat{Q_t f}(\xi) = \widehat{Q_t}(\xi) \widehat{f}(\xi) = \mathbb{1}_{[t^{-1}]}(\xi) \widehat{f}(\xi). \quad (8.1)$$

We will need the following bound for $(Q_t)_{t>0}$ (see Lemma 6 in [3] or Lemma 2.1 in [11]): For $f \geq 0$ and scales $0 < t_1, \dots, t_m \leq 1$,

$$\int_{B_1(0)} f(x) Q_{t_1} f(x) \cdots Q_{t_m} f(x) d\mu(x) \geq \left(\int_{B_1(0)} f(x) d\mu(x) \right)^{m+1}. \quad (8.2)$$

The proof in the euclidean setting given in [11] established Equation (8.2) for general approximations of the identity, but the first step is to show Equation (8.2) for martingales $(E_k)_{k \in \mathbb{N}}$ defined with respect to dyadic intervals. However, a small scale t in a non-Archimedean field $\mathbb{K}$ is the form $t = q^{-k}$ and

$$Q_t f(x) = q^k \int_{|y| \leq q^{-k}} f(x - y) d\mu(y) = \sum_{\underline{x} \in \mathcal{C}_k} A_{k, \underline{x}} f \mathbb{1}_{B_{q^{-k}}}(\underline{x}), \quad \text{where}$$

$$\mathcal{C}_k = \{\underline{x} = x_0 + x_1\pi + \dots + x_{k-1}\pi^{k-1} : x_j \in o_{\mathbb{K}}/m_{\mathbb{K}}\} \quad \text{and} \quad A_{k, \underline{x}} f = q^k \int_{B_{q^{-k}}(\underline{x})} f(u) d\mu(u).$$

Hence, $(Q_t)_{t>0}$ is a martingale with respect to the dyadic structure of non-Archimedean fields and so the argument in [11] extends without change to establish Equation (8.2).

Step 2

Fix $\varepsilon > 0$. Our goal is to find a $\delta(\varepsilon, \mathcal{P}) > 0$ and $N(\varepsilon, \mathcal{P}) \geq 1$ such that for any scale $N \geq N(\varepsilon, \mathcal{P})$ and $f \in L^0(\mathbb{K})$ with $0 \leq f \leq 1$ satisfying $\int_{\mathbb{K}} f d\mu \geq \varepsilon N^d$, we have

$$I := \frac{1}{N^d} \iint_{\mathbb{K}^2} f(x) f(x + P_1(y)) \cdots f(x + P_m(y)) d\mu_{[N]}(y) d\mu(x) \geq \delta. \quad (8.3)$$

Taking $f = \mathbb{1}_S$ with $S \subseteq \mathbb{K}$ in Theorem 1.3 implies Equation (1.4), the desired conclusion. We may assume the f is supported in the interval $[N^d]$.

Let $\alpha, \beta \in \mathbb{K}$ satisfy $|\alpha| = N^d$ and $|\beta| = N$, and write

$$I = \iint_{\mathbb{K}^2} g(x) g(x + R_1(y)) \cdots g(x + R_m(y)) d\mu_{[1]}(y) d\mu(x),$$

where $g(x) = f(\alpha x)$ and $R_j(y) = \alpha^{-1} P_j(\beta y)$. In particular, we have $\int_{\mathbb{K}} g \geq \varepsilon$. We note that g is supported in $[1] = B_1(0)$. Fix three small scales $0 < t_0 \ll t_1 \ll t \ll 1$ and decompose

$$t_1^{-1} I \geq \iint_{\mathbb{K}^2} g(x) g(x + R_1(y)) \cdots g(x + R_m(y)) d\mu_{[t_1]}(y) d\mu(y) =: I_1 + I_2 + I_3, \quad (8.4)$$

where

$$\begin{aligned} I_1 &= \iint_{\mathbb{K}^2} g(x) \prod_{j=1}^{m-1} g(x + R_j(y)) Q_t g(x + R_m(y)) d\mu_{[t_1]}(y) d\mu(x), \\ I_2 &= \iint_{\mathbb{K}^2} g(x) \prod_{j=1}^{m-1} g(x + R_j(y)) [Q_{t_0} - Q_t] g(x + R_m(y)) d\mu_{[t_1]}(y) d\mu(x) \text{ and} \\ I_3 &= \iint_{\mathbb{K}^2} g(x) \prod_{j=1}^{m-1} g(x + R_j(y)) [\text{Id} - Q_{t_0}] g(x + R_m(y)) d\mu_{[t_1]}(y) d\mu(x). \end{aligned}$$

For I_1 , we note that for $t_1 \ll_{P_m} t$,

$$Q_t g(x + R_m(y)) = \frac{1}{t} \int_{|u| \leq t} g(x + R_m(y) - u) d\mu(u) = \frac{1}{t} \int_{|u| \leq t} g(x - u) d\mu(u) = Q_t g(x)$$

whenever $|y| \leq t_1$. For the final equality, we made the change of variables $u \rightarrow u - R_m(y)$, noting that when $|y| \leq t_1$, then $|R_m(y)| \leq C_{P_m} t_1 \leq t$. Hence,

$$I_1 = \iint_{\mathbb{K}^2} g(x) \prod_{j=1}^{m-1} g(x + R_j(y)) Q_t g(x) d\mu_{[t_1]}(y) d\mu(x).$$

For I_2 , we use the Cauchy–Schwarz inequality to see that

$$I_2 \leq \|Q_{t_0} g - Q_t g\|_{L^2(\mathbb{K})}. \quad (8.5)$$

For I_3 , we will use the more precise formulation of Theorem 1.6 given in Theorem 7.23. We rescale I_3 , moving from g, R_j back to f, P_j and write

$$I_3 = \frac{1}{N^d} \iint_{\mathbb{K}^2} f(x) \prod_{j=1}^{m-1} f(x + P_j(y)) [\text{Id} - Q_{t_0 N^d}] f(x + P_m(y)) d\mu_{[t_1 N]}(y) d\mu(x),$$

where the function $h(x) = [\text{Id} - Q_{t_0 N^d}]f(x)$ has the property that $\widehat{h}(\xi) = 0$ whenever $|\xi| \leq (t_0 N^d)^{-1}$; See (8.1). Hence,

$$I_3 \leq N^{-d} \|A_{t_1 N}^{\mathcal{P}}(f, f, \dots, f, [\text{Id} - Q_{t_0 N^d}]f)\|_{L^1(\mathbb{K})},$$

and we will want to apply Theorem 7.23 to the expression on the right with N replaced by $t_1 N$ and $0 < \delta \leq 1$ defined by $\delta^{C_m} (N t_1)^d = N^d t_0$ or $\delta = (t_0/t_1^d)^{1/C_m}$. In order to apply Theorem 7.23, we will need to ensure

$$N \geq t_1^{-1} (t_1^{d_{m-1}}/t_0)^{C'} \geq \dots \geq t_1^{-1} (t_1^d/t_0)^{C'} \quad (8.6)$$

for some appropriate large $C' = C'_{\mathcal{P}}$. If Equation (8.6) holds, then Theorem 7.23 implies there exists a constant $b = b_{\mathcal{P}} > 0$ such that

$$\|A_{t_1 N}^{\mathcal{P}}(f, f, \dots, f, h)\|_{L^1(\mathbb{K})} \lesssim_{\mathcal{P}} (t_0/t_1^d)^b \prod_{j=1}^m \|f\|_{L^{p_i}(\mathbb{K})} \leq (t_0/t_1^d)^b N^d$$

since $1/p_1 + \dots + 1/p_m = 1$ and $\|f\|_{L^{p_i}(\mathbb{K})} \leq N^{d/p_i}$ for $i \in \llbracket m \rrbracket$ (which follows since f is 1-bounded and supported in $[N^d]$). Hence,

$$I_3 \lesssim_{\mathcal{P}} (t_0/t_1^d)^b \quad \text{if} \quad (8.6) \quad \text{holds.}$$

Step 3

Next we decompose $I_1 = I_1^1 + I_2^1 + I_3^1$, where

$$I_1^1 = \iint_{\mathbb{K}^2} g(x) \prod_{j=1}^{m-2} g(x + R_j(y)) Q_{t/N^{d-d_1}} g(x + R_{m-1}(y)) Q_t g(x) d\mu_{[t_1]}(y) d\mu(x),$$

$$I_2^1 = \iint_{\mathbb{K}^2} g(x) \prod_{j=1}^{m-2} g(x + R_j(y)) [Q_{t_0/N^{d-d_1}} - Q_{t/N^{d-d_1}}] g(x + R_{m-1}(y)) Q_t g(x) d\mu_{[t_1]}(y) d\mu(x) \quad \text{and}$$

$$I_3^1 = \iint_{\mathbb{K}^2} g(x) \prod_{j=1}^{m-2} g(x + R_j(y)) [\text{Id} - Q_{t_0/N^{d-d_1}}] g(x + R_{m-1}(y)) Q_t g(x) d\mu_{[t_1]}(y) d\mu(x).$$

For I_1^1 , we set $s = t/N^{d-d_1}$ and note that for $t_1 \ll_{\mathcal{P}} t$,

$$Q_s g(x + R_{m-1}(y)) = \frac{1}{s} \int_{|u| \leq s} g(x + R_{m-1}(y) - u) d\mu(u) = \frac{1}{s} \int_{|u| \leq s} g(x - u) d\mu(u) = Q_s g(x)$$

whenever $|y| \leq t_1$. For the final equality we made the change of variables $u \rightarrow u - R_{m-1}(y)$, noting that when $|y| \leq t_1$, then $|R_{m-1}(y)| \leq C_{P_{m-1}} N^{-(d-d_1)} t_1 \leq s$ since $t_1 \ll_{\mathcal{P}} t$. Hence,

$$I_1^1 = \iint_{\mathbb{K}^2} g(x) \prod_{j=1}^{m-2} g(x + R_j(y)) Q_{t/N^{d-d_1}} g(x) Q_t g(x) d\mu_{[t_1]}(y) d\mu(x).$$

As in Equation (8.5), we have

$$I_2^1 \leq \|Q_{t_0/N^{d-d_1}} g - Q_{t/N^{d-d_1}} g\|_{L^2(\mathbb{K})}.$$

For I_3^1 , we will use Theorem 7.23. We rescale I_3^1 , moving from g, R_j back to f, P_j and write

$$I_3^1 = \frac{1}{N^d} \iint_{\mathbb{K}^2} f(x) \prod_{j=1}^{m-2} f(x + P_j(y)) [\text{Id} - Q_{t_0 N^{d_1}}] f(x + P_{m-1}(y)) Q_{t N^d} f(x) d\mu_{[t_1 N]}(y) d\mu(x),$$

where the function $h'(x) = [\text{Id} - Q_{t_0 N^{d_1}}] f(x)$ has the property that $\widehat{h'}(\xi) = 0$ whenever $|\xi| \leq (t_0 N^{d_1})^{-1}$. Hence, for $\mathcal{P}' = \{P_1, \dots, P_{m-1}\}$,

$$I_3^1 \leq N^{-d} \|A_{t_1 N}^{\mathcal{P}'}(f Q_{t N^d} f, f, \dots, f, [\text{Id} - Q_{t_0 N^{d_1}}] f)\|_{L^1(\mathbb{K})}$$

and so, as long as Equation (8.6) holds, Theorem 7.23 implies there exists a constant $b' = b_{\mathcal{P}'} > 0$ such that

$$\|A_{t_1 N}^{\mathcal{P}'}(f Q_{t N^d} f, f, \dots, f, h')\|_{L^1(\mathbb{K})} \lesssim_{\mathcal{P}'} (t_0/t_1^d)^{b'} \prod_{j=1}^m \|f\|_{L^{p_i}(\mathbb{K})} \leq (t_0/t_1^d)^{b'} N^d$$

since $1/p_1 + \dots + 1/p_{m-1} = 1$ and $\|f\|_{L^{p_i}(\mathbb{K})} \leq N^{d/p_i}$ for $i \in \llbracket m-1 \rrbracket$ (which follows since f is 1-bounded and supported in $[N^d]$). Hence,

$$I_3^1 \lesssim_{\mathcal{P}'} (t_0/t_1^d)^{b'} \quad \text{if} \quad (8.6) \quad \text{holds.}$$

Step 4

We iterate, decomposing $I_1^1 = I_1^2 + I_2^2 + I_3^2$, followed by decomposing $I_1^2 = I_1^3 + I_2^3 + I_3^3$ and so on. For each $0 \leq j \leq m-1$, we have

$$I_1^j = \iint_{\mathbb{K}^2} g(x) \left(\prod_{i=1}^{m-j-1} g(x + R_i(y)) \right) \left(\prod_{i=0}^j Q_{t/N^{d-d_i}} g(x) \right) d\mu_{[t_1]}(y) d\mu(x), \quad (8.7)$$

$$I_2^j \leq \|Q_{t_0/N^{d-d_j}} g - Q_{t/N^{d-d_j}} g\|_{L^2(\mathbb{K})} \quad \text{and} \quad I_3^j \lesssim_{\mathcal{P}} (t_0/t_1^d)^b \quad \text{for some} \quad b = b_{\mathcal{P}} > 0, \quad (8.8)$$

again if Equation (8.6) holds. Strictly speaking, the estimate (8.8) for I_3^j does not follow from Theorem 7.23 when $j = m-1$ since the proof of Theorem 7.23 assumed that the collection $\mathcal{P}$ of polynomials consisted of at least two polynomials. Nevertheless, the bound (8.8) holds when $j = m-1$. To see this, we apply the Cauchy–Schwarz inequality and Plancherel’s theorem to see that

$$\begin{aligned} |I_3^{m-1}|^2 &\leq \frac{1}{N^d} \int_{\mathbb{K}} \left| \int_{\mathbb{K}} [\text{Id} - Q_{t_0 N^{d_{m-1}}}] f(x + P_1(y)) d\mu_{[t_1 N]}(y) \right|^2 d\mu(x) \\ &= \frac{1}{N^d} \int_{|\xi| \geq (N^{d_{m-1}} t_0)^{-1}} |\widehat{f}(\xi)|^2 |m_{N, t_1}(\xi)|^2 d\mu(\xi), \quad \text{where} \quad m_{N, t_1}(\xi) := \int_{B_1(0)} e(P_1(t_1 N y) \xi) d\mu(y). \end{aligned}$$

The oscillatory integral bound (3.1) implies that $|m_{N, t_1}(\xi)| \lesssim_{\mathcal{P}} (t_0/t_1)^b$ whenever $|\xi| \geq (N^{d_{m-1}} t_0)^{-1}$, and so Equation (8.8) for I_3^j follows when $j = m-1$ since $\|f\|_{L^2(\mathbb{K})}^2 \leq N^d$.

Step 5

From Equation (8.4) and the iterated decomposition of I_1 , we see that $t_1^{-1} I \geq A + B + C$, where

$$A = \int_{\mathbb{K}} g(x) \prod_{j=0}^{m-1} Q_{t/N^{d-d_j}} g(x) d\mu(x) \geq \varepsilon^{m+1}$$

by Equation (8.2), and for some $C_{\mathcal{P}} > 0$, we have

$$|B| \leq C_{\mathcal{P}} \sum_{j=0}^{m-1} \|Q_{t_0/N^{d-d_j}} g - Q_{t_1/N^{d-d_j}} g\|_{L^2(\mathbb{K})} \quad \text{and} \quad |C| \leq C_{\mathcal{P}} (t_0/t_1^d)^b \leq \varepsilon^{m+1}/4$$

if $t_0 \leq c_0 \varepsilon^{(m+1)/b} t_1^d$ and $c_0^b C_{\mathcal{P}} < 1/4$ and Equation (8.6) holds.

Finally, we claim that we can find a triple $t_0 \ll t_1 \ll t$ of small scales such that $|B| \leq \varepsilon^{m+1}/4$. If we are able to do this, then $I \geq \varepsilon^{m+1} t_1/2$ and the proof is complete.

Define $v := -C_0 \log_q(c_0 \varepsilon^{(m+1)/b})$ for some large constant $C_0 \gg d$. Choose a sequence of small scales $t_0 = q^{-\ell_j}$ and $t_1 = q^{-k_j}$ and $t = q^{-u_j}$ satisfying

$$0 \leq u_1 < dk_1 + v < \ell_1 < u_2 < dk_2 + v < \ell_2 < \dots < u_n < dk_n + v < \ell_n < \dots \quad (8.9)$$

$$\text{and} \quad \ell_{n+1} \leq \ell_n - C_0 \log_q(c_0 \varepsilon^{(m+1)/b}).$$

Taking $L \in \mathbb{N}$ such that $L = \lfloor 16C_{\mathcal{P}} m^2 \varepsilon^{-2(m+1)} \rfloor + 1$ we claim that there exists $j \in \llbracket L \rrbracket$ such that

$$C_{\mathcal{P}} \sum_{n=0}^{m-1} \|Q_{q^{-\ell_j} N^{-(d-d_n)}} g - Q_{q^{-u_j} N^{-(d-d_n)}} g\|_{L^2(\mathbb{K})} < \varepsilon^{m+1}/4. \quad (8.10)$$

Indeed, suppose for a contradiction that Equation (8.10) does not hold. Then for all $j \in \llbracket L \rrbracket$ by the Cauchy–Schwarz inequality, we have

$$\varepsilon^{2(m+1)} \leq 16C_{\mathcal{P}}^2 m \sum_{n=0}^{m-1} \|Q_{q^{-\ell_j} N^{-(d-d_n)}} g - Q_{q^{-u_j} N^{-(d-d_n)}} g\|_{L^2(\mathbb{K})}^2.$$

Then

$$L \varepsilon^{2(m+1)} \leq 16C_{\mathcal{P}}^2 m \sum_{j=1}^L \sum_{n=0}^{m-1} \|Q_{q^{-\ell_j} N^{-(d-d_n)}} g - Q_{q^{-u_j} N^{-(d-d_n)}} g\|_{L^2(\mathbb{K})}^2$$

$$= 16C_{\mathcal{P}}^2 m \sum_{n=0}^{m-1} \int_{\mathbb{K}} |\widehat{g}(\xi)|^2 \sum_{j=1}^L |\mathbb{1}_{[q^{\ell_j} N^{d-d_n}]}(\xi) - \mathbb{1}_{[q^{u_j} N^{d-d_n}]}(\xi)|^2 d\mu(\xi) \leq 16C_{\mathcal{P}}^2 m^2 \|g\|_{L^2(\mathbb{K})}^2,$$

and this implies $L \leq 16C_{\mathcal{P}}^2 m^2 \varepsilon^{-2(m+1)}$ since $\|g\|_{L^2(\mathbb{K})} \leq 1$, which is impossible by our choice of L .

Therefore, there exists $j \in \llbracket L \rrbracket$ and a corresponding triple of scales $t_0 = q^{-\ell_j} \ll t_1 = q^{-k_j} \ll t = q^{-u_j}$ satisfying the desired properties for which Equation (8.10) is true. In particular, $|B| \leq \varepsilon^{m+1}/4$ holds.

Step 6

Furthermore, with these scales by Equation (8.9), we have $t_0 = q^{-\ell_j} \gtrsim (c_0 \varepsilon^{m+1})^{O_{\mathcal{P}}(m^2 \varepsilon^{-2(m+1)})}$. In order to ensure that Equation (8.6) holds for every iteration in the decomposition, we set

$$N(\varepsilon, \mathcal{P}) := (c_0 \varepsilon^{m+1})^{-O_{\mathcal{P}}(m^2 \varepsilon^{-2(m+1)})}$$

so that for every $N \geq N(\varepsilon, \mathcal{P})$ condition (8.6) holds. Hence,

$$I \gtrsim \varepsilon^{m+1} t_1 \gtrsim \varepsilon^{m+1} t_0 \gtrsim \varepsilon^{m+1} (c_0 \varepsilon^{m+1})^{O_{\mathcal{P}}(m^2 \varepsilon^{-2(m+1)})},$$

establishing the desired bound (8.3) with $\delta = \varepsilon^{C_1 \varepsilon^{-2m-2}}$ for some $C_1 > 0$ depending only on $\mathcal{P}$.

This completes the proof of Theorem 1.3.

Acknowledgements. We thank the referees for careful reading of the manuscript and useful remarks that led to the improvement of the presentation.

Competing interest. The authors have no competing interest to declare.

Financial support. Mariusz Mirek is supported by the NSF grant DMS-2154712 and by the NSF CAREER grant DMS-2236493. Sarah Peluse is supported by the NSF grant DMS-2401117 and was supported by the NSF Mathematical Sciences Postdoctoral Research Fellowship Program under grant DMS-1903038. James Wright is supported by a Leverhulme Research Fellowship RF-2023-709\9

References

- [1] G. I. Arkhipov, V. N. Chubarikov and A. A. Karatsuba, *Trigonometric Sums in Number Theory and Analysis*, de Gruyter Expositions in Mathematics 39 (Walter de Gruyter, 2004).
- [2] J. Bourgain, ‘A Szemerédi type theorem for sets of positive density in $\mathbb{R}^k$ ’, *Israel J. Math.* **54**(3) (1986), 307–316.
- [3] J. Bourgain, ‘A nonlinear version of Roth’s theorem for sets of positive density in the real line’, *J. Analyse Math.* **50** (1988), 169–181.
- [4] J. Bourgain and M.-C. Chang, ‘Nonlinear Roth type theorems in finite fields’, *Israel J. Math.* **221**(2) (2017), 853–867.
- [5] V. Bergelson, H. Furstenberg and B. Weiss, Seminar talk at Stanford University, August 1986.
- [6] V. Bergelson and A. Leibman, ‘Polynomial extensions of van der Waerden’s and Szemerédi’s theorems’, *J. Amer. Math. Soc.* **9**(3) (1996), 725–753.
- [7] T. F. Bloom and O. Sisask, ‘An improvement to the Kelley-Meka bounds on three-term arithmetic progressions’, Available at [arXiv:2309.02353](https://arxiv.org/abs/2309.02353), (2023).
- [8] X. Chen, J. Guo and X. Li, ‘Two bipolynomial Roth theorems in $\mathbb{R}$ ’, *J. Funct. Anal.* **281**(2) (2021), 109024.
- [9] V. Chang, I. Laba, M. Pramanik, ‘Finite configurations in sparse sets’, *J. Anal. Math.* **128** (2016), 289–335.
- [10] D. Dong, X. Li and W. Sawin, ‘Improved estimates for polynomial Roth type theorems in finite fields’, *J. Anal. Math.* **141**(2) (2020), 689–705.
- [11] P. Durcik, S. Guo and J. Roos, ‘A polynomial Roth theorem on the real line’, *Trans. Amer. Math. Soc.* **371**(10) (2019), 6973–6993.
- [12] S. N. Evans, ‘The expected number of zeros of a random system of p -adic polynomials’, *Electron. Comm. Probab.* **11** (2006), 278–290.
- [13] L. K. Hua, ‘On exponential sums’, *J. Chinese Math. Soc.* **20** (1940), 301–312.
- [14] L. Huckaba, N. Lyall and A. Magyar, ‘Simplexes and sets of positive upper density in $\mathbb{R}^{d^+}$ ’, *Proc. Amer. Math. Soc.* **145**(6) (2017), 2335–2347.
- [15] A. Gaitan and V. Lie, ‘The boundedness of the (sub-)bilinear maximal function along “non-flat” smooth curves’, *J. Fourier Anal. Appl.* (2020), 26–69.
- [16] B. Green and T. Tao, ‘Linear equations in primes’, *Ann. of Math.* **171**(3) (2010), 1753–1850.
- [17] B. Green and T. Tao, ‘New bounds for Szemerédi’s theorem, III: A polylogarithmic bound for $r_4(N)$ ’, *Mathematika* **63**(3) (2017), 944–1040.
- [18] W. T. Gowers, ‘A new proof of Szemerédi’s theorem’, *Geom. Funct. Anal.* **11**(3) (2001), 465–588.
- [19] T. Keleti, ‘A 1-dimensional subset of the reals that intersects each of its translates in at most a single point’, *Real Anal. Exchange* **24**(2) (1998/99), 843–844.
- [20] T. Keleti, ‘Construction of one-dimensional subsets of the reals not containing similar copies of given patterns’, *Anal. PDE* **1**(1) (2008), 29–33.
- [21] Z. Kelley and R. Meka, ‘Strong bounds for 3-progressions’, in *IEEE 64th Annual Symposium on Foundations of Computer Science—FOCS* (2023), 933–973.
- [22] B. Krause, ‘A non-linear Roth theorem for fractals of sufficiently large dimension’, Available at [arXiv:1904.10562](https://arxiv.org/abs/1904.10562), (2019).
- [23] B. Krause, M. Mirek and T. Tao, ‘Pointwise ergodic theorems for non-conventional bilinear polynomial averages’, *Ann. Math.* **195**(3) (2022), 997–1109.
- [24] I. Laba and M. Pramanik, ‘Arithmetic progressions in sets of fractional dimension’, *Geom. Funct. Anal.* **19**(2) (2009), 429–456.
- [25] M. Lacey, ‘The bilinear maximal functions map into L^p for $2/3 < p \leq 1$ ’, *Ann. of Math.* (2) **151**(1) (2000), 35–57.
- [26] J. Leng, A. Sah and M. Sawhney, ‘Improved bounds for Szemerédi’s theorem’, Available at [arXiv:2402.17995](https://arxiv.org/abs/2402.17995), (2024).
- [27] X. Li, ‘Bilinear Hilbert transforms along curves, I’, *Analysis & PDE* **6**(1) (2013), 197–220.
- [28] X. Li and L. Xiao, ‘Uniform estimates for bilinear Hilbert transforms and bilinear maximal functions associated to polynomials’, *Amer. J. Math.* **138**(4) (2016), 907–962.
- [29] V. Lie, ‘On the boundedness of the bilinear Hilbert transform along “non-flat” smooth curves’, *American Journal of Mathematics* **137**(2) (2015), 313–363.
- [30] N. Lyall and A. Magyar, ‘An optimal version of Sarközy’s theorem’, Available at [arXiv:1010.3451](https://arxiv.org/abs/1010.3451), (2010).
- [31] N. Lyall and A. Magyar, ‘Optimal polynomial recurrence’, *Canad. J. Math.* **65**(1) (2013), 171–194.

- [32] N. Lyall and A. Magyar, ‘Distances and trees in dense subsets of $\mathbb{Z}^d$ ’, *Israel J. Math.* **240** (2020), 769–790.
- [33] N. Lyall and A. Magyar, ‘Product of simplices and sets of positive upper density in $\mathbb{R}^d$ ’, *Math. Proc. Cambridge Philos. Soc.* **165**(1) (2018), 25–51.
- [34] L. J. Mordell, ‘On a sum analogous to a Gauss sum’, *Quart. J. Math.* **3** (1932), 161–162.
- [35] J. Neukirch, *Algebraic Number Theory* (Springer-Verlag, 1999).
- [36] S. Peluse, ‘Three-term polynomial progressions in subsets of finite fields’, *Israel J. Math.* **228**(1) (2018) 379–405.
- [37] S. Peluse, ‘On the polynomial Szemerédi theorem in finite fields’, *Duke Math. J.* **168**(5) (2019), 749–774.
- [38] S. Peluse, ‘Bounds for sets with no polynomial progressions’, *Forum Math. Pi* (2020), e16.
- [39] S. Peluse and S. Prendiville, ‘Quantitative bounds in the non-linear Roth theorem’, *Invent. Math.* (2024). <https://doi.org/10.1007/s00222-024-01293-x>.
- [40] K. F. Roth, ‘On certain sets of integers’, *J. London Math. Soc.* **28** (1953), 104–109.
- [41] E. Stein, *E Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Mathematical Series, 43. Monographs in Harmonic Analysis, III. (Princeton University Press, Princeton, NJ, 1993).
- [42] E. Stein and S. Wainger, ‘Oscillatory integrals related to Carleson’s theorem’, *Math. Res. Lett.* **8** (2001) 789–800.
- [43] E. Szemerédi, ‘On sets of integers containing no k elements in arithmetic progression’, *Acta Arith.* **27** (1975), 199–247.
- [44] T. Wolff, *Lectures on Harmonic Analysis*, edited by Izabella Laba and Carol Shubin, University Lecture Series, vol. 29 (American Mathematical Society, Providence, RI, 2003). *With a foreword by Charles Fefferman and a preface by Izabella Laba.*
- [45] J. Wright, ‘From oscillatory integrals and sublevel sets to polynomial congruences and character sums’, *J. Geom. Anal.* **21**(1) (2011), 224–240.
- [46] J. Wright, ‘On a conjecture of Igusa in two dimensions’, *Amer. J. Math.* **142**(4) (2020), 1193–1238.