

Chapter 12 Competition of Many Searchers



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Abstract First passage times (FPTs) are often used to estimate timescales in physical, chemical, and biological processes. FPTs generically describe the time it takes a random “searcher” to find a “target.” In many systems the important timescale is not the time it takes a single searcher to find a target, but rather the time it takes the fastest searcher out of many searchers to find a target. Such fastest FPTs or extreme FPTs result from many searchers competing to find the target and differ markedly from FPTs of single searchers. In this chapter, we review recent results on fastest FPTs. We show how fastest FPTs depend on the mode of stochastic search (including search by diffusion, subdiffusion, superdiffusion and discrete jumps), the initial searched distribution, and properties of the spatial domain.

12. Introduction

Many events in physical, chemical, and biological systems are initiated when a “searcher” finds a “target,” which is called a first passage time (FPT) [1]. Depending on the system, the searcher could be, for instance, an ion, protein, cell, or predator, and the target could be a receptor, ligand, cell, or prey. The majority of prior work in this area studies the FPT of a given single searcher. However, the relevant timescale in many systems is the time it takes the fastest searcher out of many searchers to find a target, which is called a fastest or extreme FPT [2–20].

Concretely, let τ denote the FPT to a target U_{target} of a single searcher whose stochastic position at time $t \geq 0$ is denoted by $X(t)$. Mathematically,

$$\tau := \inf\{t \geq 0 : X(t) \in U_{\text{target}}\}. \quad (12.1)$$

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Note that U_{target} may be a disjoint union of multiple sets (i.e., the “target” U_{target} may consist of multiple targets). If independent and identically distributed (iid) searchers with respective FPTs τ_1, \dots, τ_N , then the fastest FPT is

$$T_N := \min\{\tau_1, \dots, \tau_N\}, \quad (12.2)$$

where τ_1, \dots, τ_N are N iid realizations of τ in (12.1). More generally, if we set $T_{1,N} := T_N$, then the k th fastest FPTs

$$T_{k,N} := \min \left\{ \tau_1, \dots, \tau_N \setminus \cup_{j=1}^{k-1} \{T_{j,N}\} \right\}, \quad k \in \{1, \dots, N\}. \quad (12.3)$$

In this chapter we review the results of this distributional statistics of T_N and $T_{k,N}$ for $1 \leq k \leq N$. We consider researchers which move by diffusion in Sect. 12.2, superdiffusion in Sect. 12.3, subdiffusion in Sect. 12.4, and jumps on a discrete network in Sect. 12.5. We conclude in Sect. 12.6 by discussing some closely related problems.

12.2 Diffusion

We now study fastest FPTs for diffusing searchers. We start with a simple one-dimensional example.

12.2.1 A n n o t o r y Example: Diffusion in One Dimension

Suppose $X = \{X(t)\}_{t \geq 0}$ is a one-dimensional pure diffusion process with diffusivity $D > 0$. Let τ denote the FPT of X to the origin (i.e., τ is in (12.1) with $U_{\text{target}} = 0$). Assuming $X(0) = L > 0$, the distribution of τ is given in terms of the error function,

$$P(\tau > t) = \text{erf} \left(\frac{L}{\sqrt{4Dt}} \right). \quad (12.4)$$

Now, the mean of any nonnegative random variable $Z \geq 0$ is given by the integral of its survival probability, $E[Z] = \int_0^\infty P(Z > z) dz$. Hence, the mean time for a single searcher to reach the origin is infinite,

$$E[\tau] = \int_0^\infty P(\tau > t) dt = \infty,$$

due to the slow large-time decay of $P(\tau > t)$ in (12.4),

$$P(\tau > t) \sim (L/\sqrt{\pi D})t^{1/2} \quad \text{as } t \rightarrow \infty, \quad (12.5)$$

where throughout this chapter, $f \sim g$ denotes $f/g \rightarrow 1$. Further,

$$E[T_N] = \int_0^\infty [P(\tau > t)]^N dt, \quad (12.6)$$

since $P(\tau_N > t) = P(\min\{\tau_1, \dots, \tau_N\} > t) = [P(\tau > t)]^N$, where we have used that τ_1, \dots, τ_N are iid. Therefore, (12.5) and (12.6) imply that $E[T_N]$ is infinite if $N = 2$, but finite for $N \geq 3$ [21].

To obtain the large N asymptotics of $E[T_N]$, we combine (12.4) and (12.6) to yield

$$E[T_N] = \int_0^\infty \operatorname{erf}^N \left(\frac{L}{\sqrt{4Dt}} \right) dt \sim \int_0^\infty \operatorname{erf}^N \left(\frac{L}{\sqrt{4D\epsilon}} \right) dt \quad \text{as } N \rightarrow \infty \quad (12.7)$$

for any $\epsilon > 0$, since the part of integral from $t = \epsilon$ to $t = \infty$ vanishes exponentially fast as $N \rightarrow \infty$. Using the large ϵ asymptotics of $\operatorname{erf}(z)$, it follows that [2]

$$E[T_N] \sim \frac{L^2}{4D \ln N} \quad \text{as } N \rightarrow \infty. \quad (12.8)$$

In the next two subsections we review how the result in (12.8) for this simple example extends to more general cases.

12.2.2 Higher Spatial Dimensions

The analysis in (12.8) has been extended to bounded two-dimensional and three-dimensional spatial domains using probabilistic methods to prove that the fastest searcher takes a direct path to the target [17]. This result is a simple case that the spatial domain is a cylinder. Specifically, let the domain $U \subset \mathbb{R}^3$ be a cylinder of radius $r > 0$ and height $h > 0$,

$$U := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq r^2, z \in (0, h)\}.$$

Suppose the target is a disk of radius $a \in (0, r)$ at the bottom of the cylinder,

$$U_{\text{target}} := \{(x, y, 0) \in \mathbb{R}^3 : x^2 + y^2 \leq a^2\}.$$

Suppose that $N + 1$ searchers are initially placed at $(0, 0, L)$ with $L \in (0, h)$ and the diffusion coefficient $D > 0$ and reflecting boundary conditions.

Let $Z_n(t) \in [0, h]$ and $R_n(t) \in [0, r]$ denote the height and radial position of the n -th searcher at time $t \geq 0$. The first time that the n -th searcher hits the target is

$$\tau_n^{3d} := \inf\{t > 0 : Z_n(t) = 0, R_n(t) < a\}, \quad n \in \{1, \dots, N\}.$$

The first time that the n -th searcher hits the bottom of the cylinder (regardless of the radial position) is

$$\tau_n^{1d} := \inf\{t > 0 : Z_n(t) = 0\}, \quad n \in \{1, \dots, N\}.$$

Hence the first time that any searcher hits the target and the first time that any searcher hits the bottom of the cylinder are, respectively given by

$$T_N^{3d} := \min_n \{\tau_n^{3d}\}, \quad T_N^{1d} := \min_n \{\tau_n^{1d}\}.$$

The next results show that the moments of T_N^{3d} and T_N^{1d} become identical as N grows.

Theorem 1 (From Ref. [17]) *For any moment $m \geq 1$, we have that*

$$E[(T_N^{3d})^m] \sim E[(T_N^{1d})^m] \sim \frac{\#_m}{\frac{L^2}{4D \ln N}} \quad \text{as } N \rightarrow \infty. \quad (12.9)$$

We make four comments on Theorem 1. First, the proof of Theorem 1 relies on proving that the path of the first searcher to reach the target is almost a straightline from the initial position to the target. In particular, the fastest searcher out of $N+1$ searchers never leaves a tube of radius $a > 0$ connecting the starting location to the target, and therefore $T_N^{3d} = T_N^{1d}$, and the large N behavior of the moments of T_N^{1d} is already known since T_N^{1d} concerns diffusion in one space dimension [5].

Second, Theorem 1 holds for any fixed target size $a > 0$.

Third, the cylinder is finite (i.e., $r < \infty$ and $h < \infty$). In fact, if $r = h = \infty$, then (12.9) cannot hold since $E[T_N^{3d}] = \infty$ for all $N \geq 1$. To prove this, note that each searcher has a strictly positive probability of never reaching the target (i.e., $P(\tau^{3d} = \infty) > 0$) since three-dimensional Brownian motion is transient. Therefore, for any $N \geq 1$, there is a strictly positive probability that no searcher hits the target,

$$P(T_N^{3d} = \infty) = [P(\tau^{3d} = \infty)]^N > 0,$$

and therefore $E[T_N^{3d}] = \infty$ for all $N \geq 1$. This is distinct from the phenomenon where the mean FPT of a single searcher is infinite while the mean FPT of the fastest searcher is finite (see Sect. 12.2.1).

Finally, while Theorem 1 concerns a specialized spatial domain (i.e. a cylinder), the argument can be extended to prove that (12.9) holds in much more general two- and three-dimensional domains by considering a thin tube of length L connecting the starting location to the target (see Theorems 5 and 10 in [17] for a precise statement). However, three restrictive assumptions of this argument are that (i) the searchers all start at a single point, (ii) the domain contains a straight-line path from this single starting location to the target, and (iii) the searchers move by pure diffusion. The next sections show that (12.9) still holds without these three assumptions.

12.2.3 A Universal Moment Formula

Looking back to the one-dimensional problem in Sect. 12.2.1, the formula for $P(\tau > t)$ in (12.4) yields the following short-time behavior of $P(\tau \leq t)$ on a logarithmic scale,

$$\lim_{t \rightarrow 0^+} t \ln P(\tau \leq t) = -C < 0, \quad \text{where } C := \frac{L^2}{4D} > 0. \quad (12.10)$$

Due to Varadhan's formula from large deviation theory [22, 23] the short-time behavior of the distribution of τ in (12.10) holds in much greater generality [18].

In these more general scenarios, $D > 0$ is a characteristic length, $L > 0$ is a certain geodesic distance between the searcher's starting location(s) (possibly a distribution on a set of starting locations) and the target that (i) avoids any reflecting obstacles, (ii) includes any spatial variation or anisotropy in diffusivity, and (iii) incorporates any geometry in the case of diffusion on a curved manifold [18]. Further, L is unaffected by deterministic forces on the diffusive searcher (i.e., a drift) or a partially absorbing target [18].

Since the large N behavior of $E[T_N]$ depends only on the short-time behavior of $P(\tau \leq t)$ (see (12.7)), the general formula in (12.10) for the short-time behavior of $P(\tau \leq t)$ for diffusive searchers suggests that the large N asymptotics of $E[T_N]$ in (12.8) should extend to the aforementioned general scenarios in which (12.10) holds. The following theorem shows that this is indeed the case, and in fact the result extends to the m th moment of the k th fastest FPT $T_{k,N}$ for $1 \leq k \leq N$.

Theorem 2 (From Ref. [18]) Let $\{\tau_n\}_{n=1}^\infty$ be a sequence of iid nonnegative random variables. Assume that

$$\int_0^\infty P(\tau > t)^{1/N} dt < \infty \quad \text{for some } N \geq 1, \quad (12.11)$$

and assume that there exists a constant $C > 0$ such that

$$\lim_{t \rightarrow 0^+} t \ln P(\tau \leq t) = -C < 0. \quad (12.12)$$

Then for any $m \geq 1$ and $k \geq 1$, the m th moment of $T_{k,N}$ in (12.3) satisfies

$$E[(T_{k,N})^m] \sim \frac{C}{\ln N} \quad \text{as } N \rightarrow \infty. \quad (12.13)$$

Theorem 2 is a general result that applies to order statistics of any random variable τ satisfying (12.11)–(12.12). Therefore, Theorem 2 and the universal behavior in (12.10) for diffusive searchers show that (12.13) holds with $C = L^2/(4D)$ and the var-

ious details in the problem (spatial dimension drift, domains size, target size, searcher number, etc.) are irrelevant to the leading order statistics of $T_{k,N}$ as $N \rightarrow \infty$.

However, the convergence rate of (12.13) is generally quite slow. Hence, it is important to identify how these various details affect the fastest FPT statistics at higher order. The next subsection computes these corrections using extreme value theory.

12.2.4 Extreme Value Theory Approach

Extreme value theory is a branch of probability theory and statistics dealing with extreme events in the field of reliability, extreme value theory, and risk analysis. It is concerned with determining the distribution of the minima (or maxima) of a large sequence of iid random variables in terms of the short-time distribution of a single random variable. We now apply this theory to FPTs of diffusion.

Looking back again to the simple example in Sect. 12.2.1, the formula in (12.4) yields the following short-time behavior of $P(\tau \leq t)$ on a linear scale (i.e., more information than the logarithmic scale in (12.10)),

$$P(\tau \leq t) \sim At^p e^{-C/t} \quad \text{as } t \rightarrow 0+, \quad (12.14)$$

where $C = L^2/(4D) > 0$, $A = \sqrt{4D/(L^2\pi)}$, and $p = 1/2$. It turns out that this short-time behavior of the survival probability of diffusion is useful for extending to much more general scenarios where $C = L^2/(4D)$ is the diffusion timescale in (12.10) and $A > 0$ and $p \in \mathbb{R}$ depend on the details of the problem (spatial dimension, drift, target reactivity, etc.). The following theorem gives the distribution and statistics of T_N using the information in (12.14).

Theorem 12.14 (From Ref. [19]) Let $\{\tau_n\}_{n \geq 1}$ be iid and assume that there exist constants $C > 0$, $A > 0$, and $p \in \mathbb{R}$ so that (12.14) holds. Then $T_N := \min\{\tau_1, \dots, \tau_N\}$ satisfies

$$\frac{T_N - b_N}{a_N} \xrightarrow{d} X = \text{Gumbel}(0, 1) \quad \text{as } N \rightarrow \infty,$$

where

$$a_N = \frac{C}{(\ln N)^2}, \quad b_N = \frac{C}{\ln N} \# 1 + \frac{p \ln(\ln(N))}{\ln N} - \frac{\ln(AC^p)}{\ln N} \$. \quad (12.15)$$

If we assume further that $E[T_N] < \infty$ for some $N \geq 1$, then for each $m \in (0, \infty)$,

$$E \left[\frac{T_N - b_N}{a_N} \right] \xrightarrow{m/l} E[X^m] \quad \text{as } N \rightarrow \infty, \quad \text{where } X \stackrel{d}{=} \text{Gumbel}(0, 1).$$

We note that Theorem 3 can be generalized to describe $T_{k,N}$ for $1 \leq k \leq N$ (see Theorems 4 and 5 in [19]).

In Theorem 3 and throughout this chapter, \xrightarrow{d} denotes convergence in distribution [26] and $\stackrel{d}{=}$ denotes equality in distribution. Further, Gumbel(0, 1) denotes a random variable with a Gumbel distribution with location parameter $b = 0$ and scale parameter $a = 1$. Generally, a random variable X has a Gumbel distribution with location parameter $b \in \mathbb{R}$ and scale parameter $a > 0$ if¹

$$P(X > x) = \exp \left[- \exp \left(\frac{x - b}{a} \right) \right], \quad \text{for all } x \in \mathbb{R}. \quad (12.16)$$

Note that if $X \stackrel{d}{=} \text{Gumbel}(b, a)$, then $E[X] = b - \gamma a$ where $\gamma \approx 0.5772$ is the Euler–Mascheroni constant and $\text{Variance}(X) = \frac{\pi^2}{6} a^2$. Theorem 3 thus yields higher order estimates of statistics of T_N . In particular,

$$\begin{aligned} E[T_N] &= b_N - \gamma a_N + o(a_N), \\ \text{Variance}(T_N) &= \frac{\pi^2}{6} a_N^2 + o(a_N^2). \end{aligned}$$

Roughly speaking, Theorem 3 implies that T_N is approximately Gumbel with shape b_N and scale a_N .

To illustrate Theorem 3, consider one-dimensional diffusions using search as in Sect. 12.2.1, but now suppose there is a constant drift $V > 0$ pushing the searchers toward the target at $x = 0$. That is, the position of a searcher evolves according to the stochastic differential equation,

$$dX = -V dt + \sqrt{2D} dW, \quad X(0) = L > 0, \quad (12.17)$$

where W is a standard Brownian motion. If τ is the first time to reach the origin, (12.17) with $U_{\text{target}} = 0$, then the survival probability is

$$P(\tau > t) = \frac{1}{2} \left[1 + \operatorname{erf} \left(\frac{L - Vt}{\sqrt{4Dt}} \right) - e^{\frac{Vt}{D}} \operatorname{erfc} \left(\frac{L + Vt}{\sqrt{4Dt}} \right) \right], \quad t > 0. \quad (12.18)$$

Using (12.18), one can check that (12.14) holds with $C = L^2/(4D)$, $A = 1/(C\pi) e^{LV/(2D)}$, and $p = 1/2$. Hence, Theorem 3 implies that as $N \rightarrow \infty$,

¹ A Gumbel distribution is sometimes defined differently by saying that $-X$ has a Gumbel distribution with shape $-b$ and scale a if (12.16) holds.

$$E[T_N] = \frac{L^2}{4D \ln N} \left[1 + \frac{\ln(\ln(N))}{2 \ln N} - \frac{\#_{LV/D - \ln \pi + 2\gamma}}{\ln N} \right] + o((\ln N)^{-1}). \quad (12.19)$$

This result shows that increasing V (i.e., increasing the drift toward the target) decreases $E[T_N]$. This is to be expected but (12.19) shows that the effect of increasing V only affects $E[T_N]$ at third order as $N \rightarrow \infty$.

We note that the analysis above of how T_N varies is based on $N \rightarrow \infty$ for a fixed drift, fixed target size, etc. However, it is clear that T_N must diverge if we fix $N \neq 1$ (even a large value of N) and take, for example, the size of the target to zero (similarly, T_N diverges if we take the target reactivity to zero or take a repelling drift pushing the searchers away from the target to infinity). For an analysis of the “competing limits” of $N \rightarrow \infty$ versus V (as in Fig. 12.1) (and a more detailed target or a large repulsive drift), we refer the reader to Reference [27].

12.2.5 Uniform Initial Conditions

The analysis above all assumes that the diffusive searchers cannot start arbitrarily close to the target. That is, even in the case where the searchers may have a continuum of possible starting locations (as in Sect. 12.2.3), the support of this set of starting locations is a strictly positive distance away from the closest part of the target. For example, in the one-dimensional case a problem in Sect. 12.2.1 could be modified so that the initial distribution of X is uniform on the interval $[a, b]$ as long as $0 < a < b < \infty$. In this case, the moment formula in (12.13) holds with $C = L^2/(4D)$ and $L = b - a > 0$.

How does the fastest FPT change if searchers can start arbitrarily close to the target? For example, suppose X is initially uniform on the interval $[0, l]$ with targets at both $x = 0$ and $x = l > 0$ (i.e., τ is in (12.1) with $U_{\text{target}} = \{0, l\}$). In this case, it was shown back in the first work on fastest FPTs of diffusion that [2]

$$E[T_N] \sim \frac{\pi l^2}{8D} \frac{1}{N^2} \quad \text{as } N \rightarrow \infty, \quad (12.20)$$

which is of course much faster than the $1/\ln N$ decay seen in the section above for the case that searchers cannot start arbitrarily close to the target.

The much faster decay of $E[T_N]$ for this problem stems from the following short-time behavior of the distribution of τ [27],

$$P(\tau \leq t) \sim At^p \quad \text{as } t \rightarrow 0+, \quad (12.21)$$

where $p = 1/2$ and $A = \frac{1}{4^2 D/(l^2 \pi)}$, which is much slower than the short-time decay of $P(\tau \leq t)$ in (12.10). In fact, rather than just the mean in (12.20), it follows from (12.21) and Theorem 4 that $(AN)^{1/p} T_N$ converges in distribution to a Weibull random variable with unit scale and shape as $N \rightarrow \infty$.

The decay of $E[T_N]$ for this example slows from $1/N^2$ in (12.20) to $1/N$ if the targets at $x = 0$ and $x = l > 0$ are partially absorbing within the interval $(0, \infty)$. Specifically, it was shown in [27, 28] that

$$E[T_N] \sim \frac{l}{2\kappa} \frac{1}{N} \quad \text{as } N \rightarrow \infty. \quad (12.22)$$

The behavior in (12.22) follows from the fact that the short-time behavior satisfies (12.21) with $p = 1$ and $A = 2\kappa/l$ [27]. In fact, rather than just the mean in (12.22), it follows from (12.21) and Theorem 4 that $(2\kappa/l)N T_N$ converges in distribution to a unit rate exponential random variable as $N \rightarrow \infty$.

Though we illustrated (12.20) and (12.22) in the case of diffusion in an interval, we emphasize that the N^{-2} decay in (12.20) and the N^{-1} decay in (12.22) are generic results which apply to diffusive searchers which are initially uniformly distributed in the entire domain with either perfectly absorbing targets (the case of (12.20)) or partially absorbing targets (the case of (12.22)) [27, 28].

12. Superdiffusion

Section 12.2 concerns fastest FPTs of normal diffusion processes. Normal diffusion is marked by a squared displacement that grows linearly in time. However, rather than normal diffusion, an anomalous form of diffusion called superdiffusion has been observed in a variety of physical and biological systems [29]. In this section, we investigate fastest FPTs for a model of superdiffusion called a Lévy flight [30]. We note that a Lévy flight is superdiffusive because its squared displacement grows up rapidly in time (see (12.26)), even though its mean squared displacement is infinite.

12.3.1 Lévy flights

A Lévy flight with stability index $\alpha \in (0, 2)$ can be obtained from the continuous-time random walk model [29, 31] assuming that the waiting time between jumps of the walker has a finite mean $\tau_0 \in (0, \infty)$ and the probability density of the jump distance has the following slow power law decay:

$$f(y) \sim \frac{(\tau_0)^\alpha}{y^{1+\alpha}} \quad \text{as } y \rightarrow \infty \quad \text{for some lengthscale } \tau_0 > 0. \quad (12.23)$$

The probability density $p(x, t)$ for the position of a Lévy flight satisfies the following space-fractional Fokker–Planck equation [32],

$$\frac{\partial}{\partial t} p = -K(-)^{\alpha/2} p, \quad (12.24)$$

where $K = (I_0)^\alpha / t_0 > 0$ is the generalized diffusivity and $(-)^{\alpha/2}$ denotes the fractional Laplacian [33].

A Lévy flight X in \mathbb{R}^d can also be defined as the following random time change (i.e., as a subordination) of a Brownian motion [32]:

$$X(t) := B(S(t)) + X(0) \quad t \geq 0, \quad (12.25)$$

where $B = \{B(s)\}_{s \geq 0}$ is a standard Brownian motion with unit diffusivity (i.e., $E[B(s)^2] = 2ds$ for all $s \geq 0$) and $S = \{S(t)\}_{t \geq 0}$ is an independent $(\alpha/2)$ -stable subordinator with Laplace exponent $(\beta) = -K\beta^{\alpha/2}$ (i.e., $E[e^{-\beta S(t)}] = e^{-t(\beta)}$ for all $t \geq 0$, $\beta \geq 0$). Observe that the Brownian scaling where $=_d$ denotes equality in distribution),

$$B(s) =_d s^{1/2} B(1) \quad \text{for all } s > 0,$$

and the $(\alpha/2)$ -stable subordinator scaling,

$$S(t) =_d t^{2/\alpha} S(1) \quad \text{for all } t > 0,$$

imply that the Lévy flight defined in (12.25) satisfies the following scaling:

$$X(t) - X(0) =_d t^{1/\alpha} (X(1) - X(0)) \quad \text{for all } t > 0. \quad (12.26)$$

Squaring (12.26) implies that the squared displacement of a Lévy flight grows super-linearly in time.

The search efficiency of a Lévy flight has previously been investigated in terms of the FHT of a single searcher to a target in a variety of one-dimensional [34–36] or two-dimensional settings [37]. In the following section, we study the FHT of the fastest searcher of many Lévy flight searchers.

12.3.2 Fastest Hit of Lévy Flights

Define the first hitting time (FHT) τ of the Lévy flight X in (12.25) to some target $U_{\text{target}} \in \mathbb{R}^d$ as in (12.1). We use the term FHT rather than FPT since these two concepts can be distinct for Lévy flights due to their discontinuous sample paths [35, 38–40]. As above, determining the statistics and distribution of $T_{k,N}$ for $1 \leq k \leq N$ in (12.3) requires the short-time distribution,

Assuming that $X(0)$ cannot be in the target U_{target} (meaning that the intersection of the support of $X(0)$ and the closure of U_{target} is empty), it was proven in [41] that τ has the universal short-time distribution,

$$P(\tau \leq t) \sim \rho t \quad \text{as } t \rightarrow 0+, \quad (12.27)$$

where $\rho \in (0, \infty)$ is the rate,

$$\rho := \int_0^\infty P(B(s) + X(0) \in U_{\text{target}}) \frac{\alpha/2}{(1 - \alpha/2)} \frac{K}{s^{1+\alpha/2}} ds. \quad (12.28)$$

If $X(0) = 0$ then the Gaussianity of $B(s)$ means that ρ can be written as

$$\begin{aligned} \rho &= \int_0^\infty \frac{1}{(4\pi s)^{d/2}} \exp \left(-\frac{\|X(0)\|^2}{4s} \right) dx \frac{\alpha/2}{(1 - \alpha/2)} \frac{K}{s^{1+\alpha/2}} ds \\ &= 2^\alpha \frac{\#_{d+\alpha}}{2} \frac{K}{\pi^{d/2}} \frac{\alpha/2}{(1 - \alpha/2)} \int_{U_{\text{target}}} \frac{1}{x^{\alpha+d}} dx. \end{aligned}$$

The following theorem yields the distribution and statistics of T_N in (12.2) from the short-time behavior in (12.27). Analogous results hold for $T_{k,N}$ in (12.3) with $1 \leq k < N$ (see Theorems 5 and 6 in [27]).

Theorem 4 (From Ref. [27]) Let $\{\tau_n\}_{n \geq 1}$ be an iid sequence of random variables and assume that for some $A > 0$ and $p > 0$, we have that

$$P(\tau_n \leq t) \sim At^p \quad \text{as } t \rightarrow 0+. \quad (12.29)$$

Then, the following rescaling of $T_N := \min\{\tau_1, \dots, \tau_N\}$ converges in distribution,

$$(AN)^{1/p} T_N \rightarrow_d \text{Weibull}(1, p) \quad \text{as } N \rightarrow \infty.$$

If we assume further that $E[T_N] < \infty$ for some $N \geq 1$, then for each $m \in (0, \infty)$,

$$E[(T_N)^m] \sim \frac{(1 + m/p)}{(AN)^{m/p}} \quad \text{as } N \rightarrow \infty.$$

Throughout this chapter, $X =_d \text{Weibull}(t, p)$ means that X is a Weibull random variable with scale $t > 0$ and shape $p > 0$, which means

$$P(X > x) = \exp(-(x/t)^p), \quad x \geq 0. \quad (12.30)$$

Note that $\text{Weibull}(1, p)$ is exponential with unit rate if $p = 1$. Therefore, applying Theorem 4 to (12.27) implies that T_N converges in distribution to a unit rate exponential variable as N grows [41]. Hence, T_N is approximately exponentially distributed with rate N . Furthermore, if $E[T_N] < \infty$ for some $N \geq 1$, then Theorem 4 yields all the moments of T_N for large N . In particular,

$$E[T_N] \sim \sqrt{Variance[T_N]} \sim \frac{1}{\rho N} \quad \text{as } N \rightarrow \infty. \quad (12.31)$$

Analogous result hold for (i) $T_{k,N}$ in (12.3) for any $1 \leq k \leq N$ and (ii) the case that S is any nondeterministic Lévy subordinator [41].

We now contrast (12.31) with the ease of normal diffusion in (12.31) with the $1/N$ decay in (12.31) is much faster than the $1/\ln N$ decay for normal diffusion processes in Sect. 12.2. Further, fastest FPTs of diffusion depend on the shortest path to the target since the fastest diffusive searchers follow this geodesic [18]. Indeed, fastest FPTs of diffusion are unaffected by changes to the problem outside of this path, such as altering the domain or target size or even the spatial dimension. In contrast, the expression for the rate in (12.28) implies that fastest FPTs of Lévy flights depend on these global properties of the problem. Indeed, the fastest Lévy flights do not take a direct path to the closest part of the target [41].

12.8 Subdiffusion

Another form of anomalous diffusion seen in a variety of physical and biological systems is subdiffusion [42–45]. Subdiffusion is marked by a mean-squared displacement that grows according to the following sublinear power law:

$$\mathbb{E} |X(t) - X(0)|^2 \propto t^\gamma, \quad \gamma \in (0, 1).$$

12.4.1 Subordination

One common way to model subdiffusion is via a time-fractional Fokker–Planck equation [46], which derives from the continuous-time random walk model with infinite mean waiting times between jumps [31, 47]. Such a time-fractional Fokker–Planck equation describes the distribution of a subdiffusive process $\{X(t)\}_{t \geq 0}$ defined by subordinating a normal diffusive process $\{Y(s)\}_{s \geq 0}$ [48]. Precisely, if $\{Y(s)\}_{s \geq 0}$ is a normal diffusion process satisfying an Itô stochastic differential equation, then

$$X(t) := Y(S(t)), \quad t \geq 0, \quad (12.32)$$

where $S = \{S(t)\}_{t \geq 0}$ is an independent inverse γ -stable subordinator. We emphasize that S is an inverse stable subordinator in this section (in contrast with Sect. 12.3), which is defined by

$$S(t) := \inf\{s > 0 : T(s) > t\},$$

where $\{T(s)\}_{s \geq 0}$ is a γ -stable subordinator.

The definition of X in (12.32) implies that if τ and σ are respectively FPTs of X and Y to some target, then the distribution of τ is

$$P(\tau \leq t) = E[F_\sigma(S(t))], \quad (12.33)$$

where $F_\sigma(s) := P(\sigma \leq s)$ is the cumulative distribution function of the FPT σ of the normal diffusion process Y . It follows immediately from (12.33) that

$$P(\tau \leq t) = \int_0^\infty P(\sigma \leq s) \frac{t}{\gamma s^{1+\gamma}} I_\gamma \left(\frac{t}{s^{\gamma}} \right) ds, \quad (12.34)$$

since the probability density that $S(t) = s$ is

$$\frac{d}{ds} P(S(t) \leq s) = \frac{t}{\gamma s^{1+\gamma}} I_\gamma \left(\frac{t}{s^{\gamma}} \right), \quad (12.35)$$

where $I_\gamma(z)$ is defined by its Laplace transform,

$$\int_0^\infty e^{-rz} I_\gamma(z) dz = e^{-r^\gamma}, \quad \gamma \in (0, 1), \quad r \neq 0.$$

12.4.2 Fastest FPTs of subdiffusion

By using (12.34), the behavior of $P(\sigma \leq s)$, and the small z behavior of $I_\gamma(z)$, we can develop a theory of fastest FPTs of subdiffusion that is analogous to the theory for normal diffusion in Sect. 12.2. In particular, since σ is an FPT of a normal diffusion process, then we have that under very general conditions (see Sect. 12.2.3),

$$\lim_{s \rightarrow 0^+} s \ln P(\sigma \leq s) = -L^2/(4D) < 0,$$

where D and L are as in Sect. 12.2.3 for the normal diffusion process Y (note that $Y = Y(s)$ is indexed by “internal time” $s \geq 0$, which is not physical time, but rather has dimension (time) $^\gamma$ and thus D has dimension (length) 2 (time) $^{-\gamma}$). Then, using (12.34) and the fact that [49, 50]

$$\lim_{z \rightarrow 0^+} z^{\gamma/(1-\gamma)} \ln I_\gamma(z) = -(1-\gamma) \gamma^{-\gamma/(1-\gamma)} < 0, \quad (12.36)$$

we obtain the following universal behavior of FPTs of subdiffusion modeled by a time-fractional Fokker–Planck equation,

$$\lim_{t \rightarrow 0^+} t^{\gamma/(2-\gamma)} P(\tau \leq t) = -C := -(2-\gamma) \gamma^{\frac{\gamma}{2-\gamma}} (L^2/(4D))^{\frac{1}{2-\gamma}} < 0. \quad (12.37)$$

By a result analogous to Theorem 2 (see Theorem 6 in [51]), the short-time behavior on a logarithmic scale (12.37) implies that the m th moment of $T_{k,N}$ in

(12.3) for subdiffusions satisfies, assuming $E[T_N] < \infty$ for some $N \geq 1$,

$$E[(T_{k,N})^m] \sim \frac{t_\gamma}{(\ln N)^{2\gamma-1}} \quad \text{as } N \rightarrow \infty, \quad (12.38)$$

where $t_\gamma > 0$ is the characteristic subdiffusion timescale,

$$t_\gamma := \frac{\gamma^\gamma (2-\gamma)^{2-\gamma}}{4D} \frac{L^2}{\$_{1/\gamma}} > 0, \quad \gamma \in (0, 1].$$

We can also generalize Theorem 3 to determine higher order corrections to (12.38) and the Gumbel probability distribution of $T_{k,N}$, but we omit these details for brevity (see Theorems 9, 10, and 11 in [51]). Note also that we focused here on the case when researchers cannot start arbitrarily close to the target, but one can also determine the fastest FPTs of subdiffusive searches which start uniformly in a bounded spatial domain (see the discussion in Section 6 of [51]).

12.4.3 Extreme Statistics in Subdiffusive Searches in a Normal Diffusion

Comparing (12.38) with (12.13) in Sect. 12.2 for normal diffusion yields the counterintuitive result that fastest FPTs of subdiffusion are faster than fastest FPTs of normal diffusion. Concretely let σ^{norm} and τ^{sub} denote FPTs to a target in a bounded domain for normal diffusive and subdiffusive searches respectively. We generally have

$$E[\sigma^{\text{norm}}] < E[\tau^{\text{sub}}] = \infty,$$

where $E[\tau^{\text{sub}}] = \infty$ because the slow decay of $P(\tau^{\text{sub}} > t)$ as $t \rightarrow \infty$ [52]. However, if $P(\sigma^{\text{norm}} > s)$ decays at least as fast as a power law as $s \rightarrow \infty$, then (12.34) implies that (see Theorem 8 in [51])

$$E[T_N^{\text{sub}}] < \infty \text{ for sufficiently large } N,$$

where $T_N^{\text{sub}} := \min\{\tau_1^{\text{sub}}, \dots, \tau_N^{\text{sub}}\}$ and $\{\tau_n^{\text{sub}}\}_n$ are iid. Further, (12.38) and (12.13) imply

$$E[T_N^{\text{sub}}] \approx E[\sigma_N^{\text{norm}}] \quad \text{for sufficiently large } N,$$

where $\sigma_N^{\text{norm}} := \min\{\sigma_1^{\text{norm}}, \dots, \sigma_N^{\text{norm}}\}$ and $\{\sigma_n^{\text{norm}}\}_n$ are iid.

12.2. Random Walks on networks

In Sects 12.2–12.4, we considered search processes on a continuous statespace. We now follow [53] and consider searchers which move on a “network” of discrete states.

12.5.1 Random Walk Setup

Each searcher moves on the network according to a continuous-time Markov chain. Hence the waiting time between jumps is exponentially distributed, but this assumption is often violated (see e.g. in [53] for many searchers and also [54] for the case of a single searcher).

Concretely, let $X = \{X(t)\}_{t \geq 0}$ be a continuous-time Markov chain on a finite or countably infinite statespace I . The process X is a single searcher, and its dynamics are described by its infinitesimal generator matrix $Q = \{q(i, j)\}_{i, j \in I}$ [55]. The off-diagonal entries of Q (i.e., $q(i, j) \neq 0$ for $i \neq j$) are nonnegative and give the rate that X jumps from $i \in I$ to $j \in I$. The diagonal entries of Q are nonpositive (i.e., $q(i, i) \leq 0$ for all $i \in I$) and are chosen so that Q has zero rowsums. Assume that $\sup_{i \in I} |q(i, i)| < \infty$ so that X cannot take infinitely many jumps in finite time.

12.5.2 Single P TS

Let τ in (12.1) be the FPT of X to some target set $U_{\text{target}} \subset I$. Let $\rho = \{\rho(i)\}_{i \in I} = \{P(X(0) = i)\}_{i \in I}$ denote the initial distribution of X . To avoid visiting the same states as sum that X cannot start on the target, which means that $U_{\text{target}} \cap \text{supp}(\rho) = \emptyset$ where $\text{supp}(\rho) + I$ denotes the support of ρ (i.e., $\text{supp}(\rho) := \{i \in I : \rho(i) > 0\}$).

As in the section above, determining the distribution of T_N in (12.2) and $T_{k, N}$ in (12.3) requires determining the distribution of τ . Theorem 5 shows that

$$P(\tau \leq t) \sim \frac{1}{d!} t^d \quad \text{as } t \rightarrow 0+, \quad (12.39)$$

where $d \geq 1$ is the smallest number of jumps that X must take to hit the target and $d > 0$ is the sum of the products of the jump rates along the shortest path to the target.

To make and in (12.39) precise, define a path P of length $d \in \mathbb{Z}_{\geq 0}$ from $i_0 \in I$ to $i_d \in I$ to be a sequence of $d + 1$ states in I ,

$$P = (P(0), \dots, P(d)) = (i_0, i_1, \dots, i_d) \in I^{d+1}, \quad (12.40)$$

so that

$$q(P(k), P(k+1)) > 0, \quad \text{for } k \in \{0, 1, \dots, d-1\}. \quad (12.41)$$

The condition in (12.41) means that X has a transition probability of following the path P . Assume that there is a path from the support of ρ to the target (otherwise, $P(\tau = \infty) = 1$ and the problem is trivial).

For a path $P \in I^{d+1}$, let $\lambda(P)$ be the product of the rates along the path,

$$\lambda(P) := \prod_{i=0}^{d-1} q(P(i), P(i+1)) > 0. \quad (12.42)$$

Let $d_{\min}(I_0, I_1) \in \mathbb{Z}_{\geq 0}$ denote the length of the shortest path from $I_0 + I$ to $I_1 + I$,

$$d_{\min}(I_0, I_1) := \inf\{d : P \in I^{d+1}, P(0) \in I_0, P(d) \in I_1\}.$$

That is, $d_{\min}(I_0, I_1)$ is the smallest number of jumps that X must take from I_0 to I_1 . Define the set of all paths from I_0 to I_1 with the length $d_{\min}(I_0, I_1)$,

$$S(I_0, I_1) := \{P \in I^{d+1} : P(0) \in I_0, P(d) \in I_1, d = d_{\min}(I_0, I_1)\}. \quad (12.43)$$

Define

$$/(P, I_1) := \frac{1}{\sum_{P \in S(\text{supp}(\rho), I_1)} \rho(P(0)) \lambda(P)}. \quad (12.44)$$

To explain $/(P, I_1)$, suppose first that $\rho(i_0) = 1$ for some $i_0 \in I$ (i.e., $\text{supp}(\rho) = i_0 = X(0)$). If there is only one path with the minimum number of jumps $d_{\min}(I_0, I_1)$, then $/(P, I_1)$ is the product of the jump rates along this path (i.e., $\lambda(P)$ in (12.42)). If there is more than one shortest path, then $/(P, I_1)$ is the sum of the products of the jump rates along these paths. Finally, if ρ is not concentrated at a single point, then $/(P, I_1)$ merely sums the products of the jump rates along all the shortest paths, where the sum is weighted according to ρ . With these definitions in place, we can now give the short-time behavior of the distribution of τ .

Theorem 5 (From Ref. [53]) The short-time distribution satisfies (12.39) where

$$d = d_{\min}(\text{supp}(\rho), U_{\text{target}}) \in \mathbb{Z}_{>0},$$

$$I = /(P, U_{\text{target}}) > 0.$$

12.5.3 Fast FPTs

Having determined the short-time probability distribution of a single FPT in Theorem 5, we can immediately find the distribution statistics of fastest

FPTs for random walks on networks. In particular, define

$$A = \frac{1}{d!} > 0,$$

where $d! 1$ and $1/d! > 0$ are as in Theorem 5. Theorem 4 implies that the following rescaling of T_N converges in distribution to a Weibull random variable,

$$(AN)^{1/d} T_N \xrightarrow{d} \text{Weibull}(1, d) \quad \text{as } N \rightarrow \infty. \quad (12.45)$$

Furthermore if $E[T_N] < \infty$ for some $N \geq 1$, then for each moment $m \in (0, \infty)$,

$$E[(T_N)^m] \sim \frac{(1 + m/d)}{(AN)^{m/d}} \quad \text{as } N \rightarrow \infty. \quad (12.46)$$

Compared to a single FPT τ , fastest FPTs T_N are faster and less variable since (12.46) implies that the mean and variance of T_N vanish as $N \rightarrow \infty$. Furthermore, fastest FPTs are independent of much of the properties of the network. In particular, (12.45) implies that the large N distribution of T_N is fully determined by N , $1/d$, and d , which depend only on network features along the shortest path(s) from the initial distribution to the target. Similar conclusions hold for $T_{k,N}$ for $1 \leq k \leq N$ (see Theorems 2 and 4 in [53]).

12. Discussion

In this chapter, we reviewed fastest FPTs out of many searchers for a variety of search processes. We found that such extreme FPTs tend to differ drastically from the behavior of a single FPT. We saw how that the statistics and distribution of fastest FPTs are determined by the short-time distribution of a single FPT. Hence, details in the problem (spatial dimension, domains, boundary geometry, some aspects of searcher dynamics, etc.) can become irrelevant to fastest FPTs in the many searcher limit. These details become irrelevant because the fastest FPTs typically correspond to searchers which take a direct path to the target.

We now briefly mention a few closely related problems. The behavior of fastest searchers mimics so-called “mortal” or “evanescent” searchers [7, 56–61] which have been conditioned to reach the target before a fast inactivation time [53, 62, 63]. Conditioning on not being inactivated effectively filters out searchers which do not take a direct path to the target.

In addition to FPTs, cover times of multiple searchers have also been investigated [64–68]. Cover times measure the speed of exhaustive search and are defined as the first time a searcher (s) comes within a specified detection radius of every point in the target region (often the entire spatial domain). For diffusive search, it was recently proven that the asymptotic $L^2/(4D \ln N)$ also holds for the time it takes $N \geq 1$ iid

searchers to collectively cover a target if L is the shortest distance from the searcher starting locations to the *first* part of the target [67] (see [68] for results on cover times of many discrete random walks on a network). Another statistic which is related to fastest FPTs was discussed in [69] in terms of first-crossing times for boundary local time processes of individual searchers.

In the case of multiple targets, one can consider the probability that the fastest searcher hits a particular target first (the so-called splitting probability or hitting probability). In the limit of many diffusing searchers, the fastest searcher always hits the nearest target, regardless of target size, domain geometry, drift, etc. [70] This phenomenon has been shown to offer a way for biological cells to detect the location of the source of a diffusing signal [71, 72] and also offers insight into the choices made by the first deciders in large groups of decision-makers [73, 74].

Slowest FPTs have recently found application in ovarian aging and menopause timing [75, 76]. Women are born with $N \in [10^5, 10^6]$ primordial follicles in their ovaries and these are lost over time to the nearest $k = 10^3$ follicles remaining which is thought to trigger menopause [77]. Further, there is a long history [78–83] of models that assume that follicles leave the reserve at random times T_1, \dots, T_N . Hence, the menopause age is $T_{N-k, N}$ (using the notation in (12.3)), which means that menopause age is determined by the $k/N \approx 0.2\%$ slowest follicles to leave the reserve. This analysis has been used to offer an explanation for the so-called “waste-fu” over-supply [75] and also to analyze proposed surgical procedures for delaying menopause [84]. Other instances of apparent “redundancy” in biological systems have been understood in terms of fastest FPTs [7, 10–16].

A possible problem with analysis of first FPTs differs at a certain time, in some scenarios, that the vanishing mean fastest search time,

$$E[T_N] \sim \frac{L^2}{4D \ln N} \quad \text{as } N \rightarrow \infty, \quad (12.47)$$

becomes unphysical. Specifically, if the searchers have a finite maximum speed $v > 0$, then no searcher can reach a target that is distance $L > 0$ from the starting location before time $L/v > 0$. That is, we must have $T_N \geq L/v > 0$, which contradicts (12.47) for sufficiently large N . This contradicts a theorem from the well-known [85] problem of approximating finite speed random walks by diffusion due to the infinite speed of propagation of solutions to the diffusion equation [86, 87]. This discrepancy between finite speed random walks and infinite speed diffusion can often be ignored in many applications, including the discrepancy that occurs in the tails of the distribution. However, these details determine fastest FPTs. For a discussion and analysis of (i) when (12.47) becomes unphysical and (ii) fastest FPTs of random tumble processes (i.e., finite speed random walks), we refer the reader to Reference [88].

In terms of practical application, an important question concerns the convergence rate of the asymptotic results reviewed in this chapter. Put simply, how large does the number of searchers N need to be for the asymptotic to be accurate? The convergence in (12.47) is generally quite slow, though the “sufficiently large” value of N for which the approximation $E[T_N] \approx \frac{L^2}{4D \ln N}$ is accurate depends strongly on the specifics of

the system to illustrate in a concrete example consider three-dimensional diffusive search confined in a sphere of radius $R > 0$ with reflecting boundaries. Suppose the searchers start at the boundary of the sphere and the target is a concentric sphere with radius εR . If we fix $\varepsilon \in (0, 1)$ (even a small value of ε) and take $N \rightarrow \infty$, then the mean fastest search time vanishes according to (12.47) where $L = (1 - \varepsilon)R$. If we instead fix N (even a large value of N) and take $\varepsilon \rightarrow 0$, then the mean fastest search time diverges according to [27]

$$E[T_N] \sim \frac{E[\tau]}{N} \sim \frac{1}{N} \frac{R^2}{3D\varepsilon} \quad \text{as } \varepsilon \rightarrow 0, \quad (12.48)$$

where $E[\tau] \sim R^2/(3D\varepsilon)$ is the diverging mean FPT of a single searcher. Hence, smaller targets require more researchers in order for (12.47) to be accurate. When is the system in the fastest escape regime of (12.47) versus the slow escape regime of (12.48)? If we fix $\varepsilon < 1$ and increase N , (12.48) implies that $E[T_N]$ initially decreases like $E[\tau]/N$. However, this rather rapid $1/N$ decay cannot continue indefinitely since it must eventually follow the $1/\ln N$ decay in (12.47). Since $E[T_N]$ is monotonically decreasing in N , this argument suggests the following simple approximation:

$$E[T_N] \approx \max \left(\frac{E[\tau]}{N}, \frac{L^2}{4D \ln N} \right)^3, \quad (12.49)$$

and thus

$$E[T_N] \approx \frac{L^2}{4D \ln N} \quad \text{if } \frac{E[\tau]}{N} & \frac{L^2}{4D \ln N}. \quad (12.50)$$

The estimates (12.49)–(12.50) can be extended to more general scenarios assuming that $L^2/D \ll E[\tau] < \infty$ [27]. A more detailed investigation of when various approximations of fastest FPTs of diffusion are valid can be found in Ref. [27].

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