Topological entropy of Hamiltonian diffeomorphisms: a persistence homology and Floer theory perspective

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Abstract

We study topological entropy of compactly supported Hamiltonian diffeomorphisms from a perspective of persistent homology and Floer theory. We introduce barcode entropy, a Floer-theoretic invariant of a Hamiltonian diffeomorphism, measuring exponential growth under iterations of the number of not-too-short bars in the barcode of the Floer complex. We prove that the barcode entropy is bounded from above by the topological entropy and, conversely, that the barcode entropy is bounded from below by the topological entropy of any hyperbolic invariant set, e.g., a hyperbolic horseshoe. As a consequence, we conclude that for Hamiltonian diffeomorphisms of surfaces the barcode entropy is equal to the topological entropy.

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1 Introduction

In this work we study topological entropy of compactly supported Hamiltonian diffeomorphisms from a perspective of Floer theory, using the machinery of persistent homology. We introduce a Floer-theoretic invariant of a Hamiltonian diffeomorphism, which we call barcode entropy, measuring roughly speaking the rate of exponential growth under iterations of the number of bars (of length greater than $\epsilon > 0$) in the barcode of the Floer complex. This invariant comes in two forms: the absolute barcode entropy associated with the Hamiltonian Floer complexes of the iterates of the diffeomorphism and the relative barcode entropy arising from the Lagrangian Floer complex of a fixed Lagrangian submanifold and its iterated images.

Barcode entropy can be thought of as a Floer theory counterpart of topological entropy and the two invariants are closely related. We show that the barcode entropy (absolute and relative) is bounded from above by the topological entropy (Theorem A and Corollary A) and, conversely, that the absolute barcode entropy is bounded from below by the topological entropy of any (uniformly) hyperbolic invariant set, e.g., a horseshoe; see Theorem B. In particular, as a consequence of these two bounds and a work of Katok, [68], we conclude in Theorem C that the absolute barcode entropy is equal to the topological entropy for Hamiltonian diffeomorphisms of closed surfaces.

The crux of the paper lies in the definition of barcode entropy and its connection to topological entropy. The very existence of an invariant relating features of the Hamiltonian or Lagrangian Floer complex to topological entropy is not obvious. The only prior indication known to us that this is indeed possible in dimension two for relative barcode entropy comes from the main theorem in [74] and also [63, Prop. 3.1.10] discussed in more detail in Remark 2.8.

To detect topological entropy, one has to extract from the Floer complexes of the iterates an amount of information, up to an ϵ-error, growing exponentially with the order of iteration. *A priori* it is unclear if the complexes carry this much information and, if so, how to extract it. (The ϵ-error clause is essential). For instance, it is not obvious how and if a high-entropy horseshoe localized to a small ball would register on the level of Floer complexes. One apparent difficulty is that in most cases the effective diameter of the action spectrum grows sub-exponentially with the order of iteration; see Remark 3.4. One can think of Floer theory as a filter or an intermediate device between a dynamical system and the observer, and it is not clear if it lets through enough information to detect topological entropy in the Hamiltonian setting. (See also Remark 2.14 for a different perspective).

To the best of our knowledge, there are only two other settings where connections between topological entropy–type invariants and symplectic topology have been studied. The first setting concerns topological entropy of Reeb flows and the growth of various flavors of contact homology and, in a similar vein, the second one deals with topological entropy of symplectomorphisms (or contactomorphisms) and again the growth of Floer homology. (We do not touch upon slow entropy, for this is ultimately an invariant of a very different nature; see, however, Remark 2.15). This is an extensively studied subject and we will elaborate on the results in Remark 2.17. Here we only mention that the underlying theme is that positivity of topological entropy is obtained as a consequence of exponential growth of some variant of Floer or contact homology. By contrast, for Hamiltonian diffeomorphisms, the Floer homology is independent of the order of iteration and there is no homology growth. In fact, in the relative case, the Floer homology can even be zero. In our setting, topological entropy is related to a non-robust (i.e., depending on the map) invariant of the Floer complex.

Barcode entropy, the key notion introduced in the paper, relies in a crucial way on the language of persistent homology. Following [91], this machinery has become one of the standard tools in studying the dynamics of Hamiltonian diffeomorphisms by symplectic topological methods and, more generally, in symplectic dynamics, although the class of problems it has been applied to is quite different from the exponential growth questions we focus on here. The barcode of a Floer complex encompasses completely robust invariants of the system, such as spectral invariants and Floer homology, and also more fragile features via finite bars. We refer the reader to [90, 99] for a general introduction and to [9, 22, 27, 60, 75, 78, 94–96] for an admittedly incomplete collection of sample results. One key point in some of these works and also in, e.g., [49, 51, 59, 86, 97, 98], not using barcodes directly, is that finite bars, i.e., relatively fragile features of a barcode, carry information related to interesting dynamical properties of the system. The present paper builds on this point.

The paper is organized as follows. In Sect. 2 we state and extensively discuss the main definitions and results. In Sect. 3 we set our conventions and notation and briefly review relevant facts about filtered Lagrangian and Hamiltonian Floer homology and barcodes following mainly [98, 99]. We return to the definition of relative barcode entropy in Sect. 4, where we state its minor generalization and also some of its properties. In Sect. 5 we generalize and prove Theorem A. Finally, Theorems B and C are proved in Sect. 6, where we also touch upon an *a priori* lower bound on the γ-norm of the iterates in the presence of a hyperbolic set; see Proposition 6.7.

2 Key definitions and results

2.1 Definitions

The key new notion introduced in the paper—barcode entropy—comes in two versions: absolute and relative.

Let us start by briefly describing the setting in which these variants of barcode entropy are defined. Consider a symplectic manifold M and a closed monotone Lagrangian submanifold $L \subset M$ and a second Lagrangian submanifold L' Hamiltonian isotopic to L. Denote by Λ the universal Novikov field over the ground field $\mathbb{F} = \mathbb{F}_2$. Informally, Λ can be thought of as the field of Laurent series with coefficients in $\mathbb{F}$ with real (rather than integer) exponents. The ambient manifold M is not required to be closed, but it has to have a sufficiently nice structure at infinity, e.g., to be "tame" or convex. In addition, the minimal Maslov number N_L of L needs to be at least 2. We refer the reader to Sect. 3 for our conventions and notation, precise definitions and further details.

Assuming first that L and L' are transverse, we have the filtered Floer complex $\mathrm{CF}(L, L')$ which is a finite-dimensional vector space over Λ generated by $L \cap L'$. (The grading of the Floer complex and homology is immaterial for our purposes). Denote by $\mathcal{B}(L, L')$ the barcode of $\mathrm{CF}(L, L')$ over Λ; see Sect. 3.3. Note that in this case (i.e., whenever L, L' are transverse) the barcode $\mathcal{B}(L, L')$ is a finite set. For $\epsilon > 0$, let $b_\epsilon(L, L')$ be the number of bars of length greater than ϵ in this barcode:

$$b_\epsilon(L, L') := \big|\{ \text{ bars of length greater than } \epsilon \text{ in } \mathcal{B}(L, L')\}\big|. \tag{2.1}$$

This definition extends in a straightforward way, essentially by continuity, to the case when the manifolds are not necessarily transverse. For instance, when ϵ is outside the closure $\bar{\mathcal{D}}(L, L')$ of the action difference set (see Sect. 3.2.3) we can set

$$b_\epsilon(L, L') = b_\epsilon(L, \tilde{L}'),$$

where $\tilde{L}' \pitchfork L$ and $\tilde{L}'$ is C^∞-close and Hamiltonian isotopic to L'; see (4.2). We refer the reader to Sect. 4.1 and, in particular, (4.1) for the definition in the general case, and to [90, Chap. 6] for other appearances of b_ϵ. Here we note that in the non-transverse case the barcode $\mathcal{B}(L, L')$ may contain infinitely many bars. However, $b_\epsilon(L, L') < \infty$ for all $\epsilon > 0$.

Let $\varphi = \varphi_H \colon M \to M$ be a compactly supported Hamiltonian diffeomorphism. Set $L^k = \varphi^k(L)$.

Definition 2.1 (*Relative Barcode Entropy, I*) The ϵ-*barcode entropy of φ relative to L* is

$$\hbar_\epsilon(\varphi; L) := \limsup_{k \to \infty} \frac{\log^+ b_\epsilon\big(L, L^k\big)}{k}$$

and the *barcode entropy of φ relative to L* is

$$\hbar(\varphi; L) := \lim_{\epsilon \searrow 0} \hbar_\epsilon(\varphi, L) \in [0, \infty].$$

Here and throughout the paper the logarithm is taken base 2 and $\log^+ := \max\{\log, 0\}$. Observe that $\hbar_\epsilon(\varphi, L)$ is increasing as $\epsilon \searrow 0$, and hence the limit in the definition of $\hbar(\varphi, L)$ exists although *a priori* it can be infinite. We also emphasize again that the Floer homology $\mathrm{HF}(L)$ is immaterial for this construction beyond the fact that it is defined. For instance, L can be a small circle in a surface with $\mathrm{HF}(L) = 0$. We will extend this definition to pairs of Lagrangian submanifolds in Sect. 4.1.

Remark 2.2 Definition 2.1 might feel somewhat counterintuitive. The underlying idea is that the barcode counting function $b_\epsilon(L, L')$ gives a lower bound on the number of transverse intersections, which is in some sense stable under small perturbations with respect to the Lagrangian Hofer distanse d_H. For instance, assume that Lagrangian submanifolds L, L' and L'' are Hamiltonian isotopic, $L'' \pitchfork L$ and $d_H(L', L'') < \delta/2$. Then, regardless of whether L and L' are transverse or not, we have

$$|L \cap L''| \geq b_{\epsilon+\delta}(L, L')$$

for any $\epsilon \geq 0$; see Sects. 3.3 and 4.1. This would not be true if we replaced $b_{\epsilon+\delta}(L, L')$ by $|L \cap L'|$: nearby intersections can be eliminated by a C^∞-small perturbation.

Let now M be a closed monotone symplectic manifold and again let $\varphi = \varphi_H \colon M \to M$ be a Hamiltonian diffeomorphism. Then we can apply the above constructions to $L = \Delta$, the diagonal in the symplectic square $(M \times M, -\omega \oplus \omega)$, with φ replaced by $id \times \varphi$, or directly to the Floer complex $\mathrm{CF}(\varphi)$ of φ *for all free homotopy classes of loops in M*. For instance, denoting by $\mathcal{B}(\varphi)$ the barcode of $\mathrm{CF}(\varphi)$ over Λ, we have

$$b_\epsilon(\varphi^k) = \left| \{ \text{bars of length greater than } \epsilon \text{ in the barcode } \mathcal{B}(\varphi^k) \} \right|$$
$$= b_\epsilon(L, L^k),$$

where $L = \Delta$ and L^k is the graph of φ^k.

Definition 2.3 (*Absolute Barcode Entropy*) The ϵ-barcode entropy of φ is

$$\hbar_\epsilon(\varphi) := \limsup_{k \to \infty} \frac{\log^+ b_\epsilon(\varphi^k)}{k}$$

and the *(absolute) barcode entropy* of φ is

$$\hbar(\varphi) := \lim_{\epsilon \searrow 0} \hbar_\epsilon(\varphi) \in [0, \infty]$$

or, in other words,

$$\hbar(\varphi) := \hbar(id \times \varphi; \Delta).$$

Here again $\hbar_\epsilon(\varphi)$ is increasing as $\epsilon \searrow 0$, and hence the limit in the definition of $\hbar(\varphi)$ exists. Note that in this definition, in contrast with the relative barcode entropy, we can work with any ground field $\mathbb{F}$ as long as M is monotone.

In this paper we are primarily interested in absolute barcode entropy while relative entropy plays a purely technical role, arising naturally in our approach to the proof of Corollary A via Theorem A. We will revisit the definitions and briefly touch upon general properties of barcode entropy in Sect. 4.

2.2 Main results

With the definition of barcode entropy in place, we are ready to state the main results of the paper, which ultimately justify the definition.

Theorem A *Let L be a closed monotone Lagrangian submanifold with minimal Chern number $N_L \geq 2$ in a symplectic manifold M and let $\varphi \colon M \to M$ be a compactly supported Hamiltonian C^∞-diffeomorphism. Then*

$$\hbar(\varphi; L) \leq h_{top}(\varphi).$$

Note that since φ is compactly supported, the lack of compactness of M, provided that it is "tame" at infinity, causes no additional problems. We can set $h_{top}(\varphi) := h_{top}(\varphi|_{\text{supp}(\varphi)})$ or equivalently $h_{top}(\varphi) := h_{top}(\varphi|_X)$ for any compact set $X \supset \text{supp}(\varphi)$. Variants of Theorem A also hold in some other cases; see, e.g., Remark 5.2.

Since $h_{top}(id \times \varphi) = h_{top}(\varphi)$, as an immediate consequence of Theorem A, we have the following.

Corollary A *Let $\varphi \colon M \to M$ be a Hamiltonian C^∞-diffeomorphism of a closed monotone symplectic manifold M. Then*

$$\hbar(\varphi) \leq h_{top}(\varphi).$$

Another interesting consequence of Theorem A, not obvious from the definitions, is that $\hbar(\varphi; L) < \infty$ and, in particular, $\hbar(\varphi) < \infty$. In contrast with Theorem A, here and in Theorems B and C the ground field $\mathbb{F}$ can have any characteristic.

Remark 2.4 (*Growth of periodic points*) One cannot replace the number of bars $b_\epsilon(\varphi^k)$ or $b_\epsilon(L, L^k)$ in the definition of barcode entropy by the total number of k-periodic points or Lagrangian intersections, while keeping Theorem A and Corollary A. Indeed, in dimension two, the number of periodic points can grow arbitrarily fast, and moreover super-exponential growth is in some sense typical; see [15]. In higher dimensions, a smooth zero-entropy map may have super-exponential orbit growth; [67]. Hence, in both cases the exponential growth rate of the number of periodic points could in general be infinite.

Remark 2.5 (*Idea of the proof of* Theorem A) As in many results of this type (see, e.g., [83]), Theorem A is ultimately based on Yomdin's theorem relating topological entropy to the rate of exponential volume growth; see [101] and also [55]. (Hence, the requirement that φ is C^∞-smooth is essential). Let us briefly outline the idea of the proof assuming that all intersections are transverse. For a small $\epsilon > 0$, the barcode entropy is roughly the rate of exponential growth of $b_\epsilon(L, L^k)$ as $k \to \infty$. By Remark 2.2, this rate bounds from below the rate of exponential growth of $N_k(\tilde{L}) := |\tilde{L} \cap L^k|$ for any Lagrangian submanifold $\tilde{L}$ Hamiltonian isotopic and d_H-close to L with the upper bound on $d_H(L, \tilde{L})$ completely determined by ϵ. We construct a *Lagrangian tomograph*: a family of such Lagrangian submanifolds $\tilde{L} = L_s$, independent of k and parametrized by s in some ball B, so that

$$\int_B N_k(L_s) \, ds \leq const \cdot \text{vol}\left(L^k\right)$$

by a variant of Crofton's inequality; cf. [13, 14] where a similar construction is used. Now Yomdin's theorem gives a lower bound on $h_{top}(\varphi)$. Note that in contrast with some other arguments of this type (see, e.g., [7, 43] and references therein), the ball B can possibly have very large dimension and the map $\Psi \colon B \times L \to M$ sending (s, x) to the image of x on L_s need not be a fibration but only a submersion onto its image.

Remark 2.6 Instead of working with the class of monotone Lagrangian submanifolds L one can require L to be oriented, relatively spin and weakly unobstructed after bulk deformation as in [45] and replaced the coefficient field $\mathbb{F}_2$ by a field of zero characteristic. We expect Theorem A to still hold in this setting and the proof to go through word-for-word.

We do not view Theorem A or Corollary A as an effective method to calculate $h_{top}(\varphi)$, but rather as a result connecting two notions of entropy lying in completely disparate domains. Yet, this result would be meaningless and hold trivially if $\hbar(\varphi, L)$ were always zero. (As

an example, $\hslash(\varphi_H) = 0$ when H is an autonomous Hamiltonian on a surface. This fact is a consequence of Theorem C below and is not directly obvious. An alternative approach would be to show that in this case the volume of the graph of φ^k grows sub-exponentially and then invoke the proof of Theorem A or, when H is real analytic or Morse, one can verify directly that $b_\epsilon(\varphi^k)$ grows at most polynomially).

Here we focus on absolute barcode entropy, and the next two theorems show that often $\hslash(\varphi) \neq 0$ and, in fact, the two notions of entropy are perhaps (numerically) closer to each other than one might expect.

In the next theorem we are concerned with (uniformly) *hyperbolic* invariant sets. Recall that an invariant set $K \subset M$ is said to be hyperbolic if for some Riemannian metric on M there exist positive constants $\lambda_- < 1 < \lambda_+$ and a splitting $T_x M = E_x^- \oplus E_x^+$ for every $x \in K$, invariant under $D\varphi$, such that

$$\|D\varphi|_{E_x^-}\| \leq \lambda_- \text{ and } \|D\varphi|_{E_x^+}\| \geq \lambda_+$$

for all $x \in K$. We refer the reader to, e.g., [70, Sect. 6] for a detailed discussion of hyperbolicity. Here such sets are required to be compact by definition.

Theorem B *Let $\varphi\colon M \to M$ be a Hamiltonian diffeomorphism of a closed monotone symplectic manifold M and let $K \subset M$ be a hyperbolic invariant subset. Then*

$$\hslash(\varphi) \geq h_{top}(\varphi|_K).$$

The key to the proof of this theorem is the fact that a Floer trajectory u asymptotic to a periodic orbit x of any period with $x(0) \in K$ must have energy bounded away from zero by a certain constant $\epsilon_K > 0$ independent of u; see Proposition 6.3. The proof of the proposition is based on the Crossing Energy Theorem, [52, Thm. 6.1], and the Anosov Closing Lemma, [70, Thm. 6.4.15]. In contrast with other results of this paper, it would be sufficient in Theorem B to assume that φ is only C^2-smooth.

Among the examples of sets K meeting the requirements of Theorem B are hyperbolic horseshoes and then $h_{top}(\varphi|_K) > 0$; see the discussion in Sect. 6.2. Hence, whenever φ has such a horseshoe, which is a common occurrence, $\hslash(\varphi) > 0$.

In dimension two, $h_{top}(\varphi)$ is the supremum of $h_{top}(\varphi|_K)$ over all K as in Theorem B; see [68] and also [70, Suppl. S by Katok and Mendoza]. Hence, by Theorem A (or Corollary A) and Theorem B, we have the following.

Theorem C *Let $\varphi\colon M \to M$ be a Hamiltonian C^∞-diffeomorphism of a closed surface M. Then*

$$\hslash(\varphi) = h_{top}(\varphi). \tag{2.2}$$

We will prove and further discuss Theorem C in Sect. 6.

A bonus consequence of the proof of Theorem B is that the γ-norm, and hence the Hofer norm, of the iterates φ^k is bounded away from zero when, for instance, φ has a horseshoe or sufficiently many hyperbolic fixed points and also in some other cases; see Sect. 6.1.5 and Proposition 6.7.

Another consequence of the proof of Theorem A and Theorem C is a relation between the barcode and topological entropy of φ and the exponential growth rate of the volume of the graph of φ^k; see Sect. 5.3.

Remark 2.7 Since this work appeared as a preprint, a variant of Theorem B for Lagrangian intersections has been proved in [84].

2.3 Discussion and further remarks

Our definition of barcode entropy is quite general and can be applied to any sequence of persistence modules. (See, e.g., [24] and also [90] for an introduction to persistent homology theory and further references). Then barcode entropy can be thought of as the rate of exponential growth of information carried by this sequence, provided that we take b_ϵ as a measure of the amount of information. With this in mind, we expect the notion to be useful in some other settings. However, from a purely algebraic perspective the definition is also quite primitive: in contrast with topological entropy, one cannot infer any interesting properties of barcode entropy directly from the definition; cf. Sect. 4.2.

Yet, Theorems A, B and C show that this is the "right" definition in the context of Floer theory and that in this setting barcode entropy has numerous properties not formally following from the definition. The reason probably is that Floer complexes have rich additional features, and it would be interesting to understand what properties of Floer complexes "make this definition work"—the proofs do not answer this question.

In the rest of this section we will further comment on our main results.

Remark 2.8 (Theorem A when dim $M = 2$) The assertion of Theorem A is already nontrivial when L is an embedded loop (a Lagrangian submanifold) on a surface M, even when $L = S^1$ is the zero section in $M = T^*S^1 = S^1 \times \mathbb{R}$ or L is a meridian or a parallel in $M = S^1 \times S^1$. However, in this situation, one might expect to have a more general criterion. Namely, the results from [74] building on [21] indicate that when dim $M = 2$, for a broad class of loops L, there could be a sufficient condition, aka a criterion, for $h_{top}(\varphi) > 0$ expressed solely in terms of $\mathrm{CF}\big(L, \varphi(L)\big)$ without using the iterates φ^k, similarly to the forced entropy results. (See, e.g., [12] and references therein). However, no such a criterion is known; nor if this is a purely low-dimensional phenomenon. Furthermore, for a fixed L, such an entropy positivity condition, even if it exists, could not be necessary, i.e., it cannot possibly always detect positive entropy; see Example 2.9 below. Also note that [63, Prop. 3.1.10], based on [73], gives a lower bound for the topological entropy of a Hamiltonian diffeomorphism $\varphi \colon S^2 \to S^2$ in terms of the linear growth rate of the Hofer distance between L and $\varphi^k(L)$, where L is the equator. (To the best of our knowledge, there are no examples where this distance is shown to grow linearly, although conjecturally such examples exist).

Example 2.9 (*Strict inequality in* Theorem A) In contrast with Theorem C, in the setting of Theorem A the inequality can be strict even when dim $M = 2$ (and hence dim $L = 1$). Indeed, assume that L is contained in a region where $\varphi = id$, but φ has a hyperbolic horseshoe elsewhere. Then $\hbar(\varphi; L) = 0$, but $h_{top}(\varphi) > 0$. Also note that in dimension two $\hbar(\varphi; L) \leq \hbar(\varphi)$ as a consequence of Theorem C, and it would be very illuminating to understand if this is true in general.

Remark 2.10 (*Non-compact version of* Corollary A) The corollary readily extends to compactly supported Hamiltonian diffeomorphisms $\varphi = \varphi_H$ of symplectic manifolds M sufficiently "tame" at infinity. For instance, M can be convex at infinity or wide in the sense of [58]. To define the barcode entropy of φ, we fix a proper autonomous Hamiltonian Q without 1-periodic orbits at infinity and vanishing on supp H. (For instance, Q can be a small positive definite quadratic form outside a large ball when $M = \mathbb{R}^{2n}$). Next, let ψ be a small non-degenerate perturbation of $\varphi_Q \varphi^k$ coinciding with φ_Q at infinity. The filtered Floer complex of ψ is defined and can be used in place of $\mathrm{CF}\big(\varphi^k\big)$ to define the barcode entropy $\hbar(\varphi)$ of φ. (This complex depends on Q and the perturbation, but $b_\epsilon(\psi)$ is independent of the perturbation, although it might still depend on Q). Now the proof of Corollary A via

Theorem 5.1 goes through, establishing the inequality $\hbar(\varphi) \leq h_{top}(\varphi)$, when we let L_0 be the diagonal in $M \times M$ and replace L^k by the graph of ψ; see Sect. 5.2. In a similar vein, Theorems B and C also hold with the definition of barcode entropy suitably adjusted when M is "open" but φ is compactly supported.

Remark 2.11 (*Free homotopy classes of loops, I*) The requirement that we work with the Floer complex for the entire set $\tilde{\pi}_1(M)$ of free homotopy classes of loops in M is absolutely essential and without it the theorems would not hold; see Example 2.12. In the Lagrangian setting, its counterpart is commonly assumed: the Lagrangian Floer complex $\mathrm{CF}(L_0, L_1)$ is usually defined for all free homotopy classes of paths from L_0 and L_1. On the other hand, the Hamiltonian Floer complex $\mathrm{CF}(\varphi)$ is traditionally defined only for contractible loops. This is not the case in Definition 2.3 where we use the Floer complex associated with the entire collection $\tilde{\pi}_1(M)$ of free homotopy classes of loops in M. In fact, one can associate a variant of barcode entropy $\hbar_X$ to any subset $X \subset \tilde{\pi}_1(M)$, and clearly, $\hbar_Y \leq \hbar_X$ when $Y \subset X$. Then Theorem A would still obviously hold, for $\hbar_X \leq \hbar$. However, Theorems B and C would in general fail when $X \neq \tilde{\pi}_1(M)$; see Example 2.12. This phenomenon is unrelated to algebraic topological conditions for entropy positivity (e.g., the Entropy Conjecture or the fundamental group growth; see [6, 69, 70, 101] and references therein): Theorem C holds for $M = S^2$ and for $M = \mathbb{T}^2$. We will elaborate on this in Remark 2.13.

Example 2.12 (*Failure of* Theorems B *and* C *for restricted free homotopy classes of loops*) Let U be a small disk in $\mathbb{T}^2$ and let K be a time-dependent Hamiltonian with supp $K \subset U$ generating a positive-entropy time-one map φ_K. Next, let F be an autonomous Hamiltonian on $\mathbb{T}^2$ such that under the flow φ_F^t for time $t \in [0, 1]$ every point in U traces a non-contractible loop in a class $\mathfrak{c} \in \tilde{\pi}_1(\mathbb{T}^2) = \mathbb{Z}^2$ and $\varphi_F^1|_U = id$. Define H by concatenating the Hamiltonians F and K, and set $\varphi = \varphi_H^1$. Thus $\varphi|_U = \varphi_K$ and $\varphi|_{\mathbb{T}^2 \setminus U} = \varphi_F|_{\mathbb{T}^2 \setminus U}$. It follows that $h_{top}(\varphi) = h_{top}(\varphi_K) > 0$. However, $\hbar_X(\varphi) = 0$ whenever $|X \cap \mathfrak{c}\mathbb{N}| < \infty$. Indeed, pick $\mathfrak{x} \in \tilde{\pi}_1(\mathbb{T}^2)$ and assume that k is so large that $\mathfrak{x} \neq k\mathfrak{c}$. Then, since every k-periodic orbit of φ_H starting in U is in the class $k\mathfrak{c}$, the maps φ_H and φ_F have the same k-periodic orbits in the class $\mathfrak{x}$ and the actions are equal. A homotopy from H to F which phases K out does not affect k-periodic orbits in the class $\mathfrak{x}$. As a consequence, φ_H^k and φ_F^k have the same filtered Floer homology in this class and the same barcode; see, e.g., [48, Sect. 3.2.3] and references therein for similar arguments. Therefore, $\hbar_X(\varphi_H) = \hbar_X(\varphi_F)$ whenever $X \cap \mathfrak{c}\mathbb{N}$ is finite. By Theorem C, $\hbar(\varphi_F) = 0$ and hence $\hbar_X(\varphi_F) = 0$.

Remark 2.13 (*Free homotopy classes of loops, II*) All k-periodic orbits of φ_H^t have length bounded from above by $k\|\nabla H\|_{C^0}$. Therefore, for a fixed $k \in \mathbb{N}$ only finitely many free homotopy classes can be represented by periodic orbits with period up to k. By the Svarč–Milnor lemma, [37], these classes are contained in the ball of radius $R = O(k)$, with respect to the word length metric, in $\tilde{\pi}_1(M)$ centered at the origin. The number of conjugacy classes is known to have exponential growth for many hyperbolic-type groups, e.g., for $\pi_1(M)$ when M has negative sectional curvature; see [34, 62] and references therein. (We expect this to also be true when M a symplectically hyperbolic manifold in the sense of [72, 89], but this is not obvious). Hence, in this case, the number of classes populated by periodic orbits of φ with period less than k can grow exponentially. When φ is such that this is indeed the case, we have $h_{top}(\varphi) > 0$; see [65, 66] and also [6] and references therein. However, in general this is not the source of positive entropy in Theorems B and C as Example 2.12 and the case of $M = S^2$ show.

Remark 2.14 (Theorem C, *periodic orbits and topological entropy*) As stated, Theorem C fails when dim $M > 4$; see [26]. Our admittedly very optimistic conjecture is that (2.2) holds in all dimensions C^∞-generically for Hamiltonian diffeomorphisms.

Remark 2.15 (*Slow entropy*) Theorem A and Corollary A have analogues for polynomial growth, slow entropy-type invariants (see, e.g., [43, 71, 89]) which can be proved essentially by the same method with natural modifications. To be more specific, the polynomial growth rate of $b_\epsilon(L, L^k)$, i.e., $\limsup \log^+ b_\epsilon(L, L^k)/\log k$, gives a lower bound on the polynomial growth rate of the volume of L^k and, likewise, the polynomial growth rate of $b_\epsilon(\varphi^k)$ provides a lower bound on the polynomial growth rate of the volume of the graph of φ^k.

Remark 2.16 (*Lower barcode entropy*) If we replaced the upper limit by the lower limit in the definition of barcode entropy, Theorem A and Corollary A would obviously still hold for the so-defined lower barcode entropy. However, the proof of Theorem B would break down because of the upper limit in (6.7) and, as a consequence, we would not be able to establish Theorem C.

Theorems A, B and C lead to several other questions completely left out in this paper. One of them, alluded to in Example 2.9, is if there is a relation between absolute and relative entropy. Another is a construction of a Floer theoretic analogue of dynamical Morse entropy introduced in [19]. Also, there are several possible alternatives to our definition of barcode entropy, some of which are examined in [29]. Furthermore, the definition of barcode entropy, Corollary A and Theorems B and C generalize to symplectomorphisms symplectically isotopic to the identity with the Floer complex defined as in [79]; see [87]. Finally, it would also be interesting to extend the definition of relative barcode entropy to a broader class of Lagrangian submanifolds beyond the case when Lagrangian Floer homology is defined. Such a generalization may have applications to dynamics.

Remark 2.17 (*Topological entropy of Reeb flows and symplectomorphisms*) As was mentioned in Sect. 1, connections between topological entropy of Reeb flows or symplectomorphisms and exponential growth of contact or Floer homology of various flavors have been extensively studied. In the Reeb setting, these connections generalize well-known relations between the topological entropy of the geodesic flow and the geometry or topology of the underlying manifold; see [69, 70] for further references. This study was originally initiated in [81] and since then the subject has been extensively developed in a series of papers by Alves and his collaborators (see, e.g., [1, 2, 6–8, 10, 12] and also [83]). In many of these results, positivity of topological entropy follows from exponential growth of contact or Legendrian contact homology. For instance, in [10], the authors construct a contact structure on $S^{2n-1 \geq 7}$ such that every Reeb flow has positive entropy due to exponential growth of wrapped Floer homology, which is a robust feature independent of the contact from. This approach is reminiscent of the Entropy Conjecture and other results where algebraic topological features of a map give a lower bound on the topological entropy; see [69, 70, 101] for further references. The case of symplectomorphisms or contactomorphisms is very similar in spirit and closely related to the Reeb setting, and the results usually rely again on exponential growth of a variant of Floer homology; see, e.g., [35, 36, 44]. On the other hand, for compactly supported Hamiltonian diffeomorphisms, there is no Floer homology growth. Topological entropy is instead related to barcode entropy, a non-robust invariant of the Floer complex.

Remark 2.18 Since the first version of this work appeared as a preprint, significant progress has been made in extending our results and constructions to Reeb flows. In particular, barcode

entropy is defined and the analogues Theorems A, B and C are proved for geodesic flows in [54]. More generally, for Reeb flows on the boundary of a Liouville domain, the barcode entropy is defined in [39], where a version of Corollary A is also proved, and Theorems B and C are established in [32]. Moreover, Theorem A for wrapped Floer homology is proved in [40].

Remark 2.19 (*Forcing and connections with Hofer's geometry in dimension two*) Connections of topological entropy in dimension two with Hofer's geometry have been recently explored in [10, 33]. The main theme of the latter work is closely related to forcing of topological entropy; see [77] and references therein. The key result of the former is Hofer lower semicontinuity of topological entropy for Hamiltonian diffeomorphisms of closed surfaces.

3 Preliminaries

3.1 Conventions and notation

For the reader's convenience, we set here our conventions and notation and briefly recall some basic definitions; see, e.g., [98] for further details and references. The reader may want to consult this section only as needed.

Throughout the paper we assume that the underlying symplectic manifold (M, ω) is either closed or "tame" at infinity (e.g., convex) so that the Gromov compactness theorem holds; see, e.g., [82]. All Lagrangian submanifolds are assumed to be closed unless explicitly stated otherwise, and *monotone*, i.e., for some $\kappa \geq 0$, we have $\langle \omega, A \rangle = \kappa \langle \mu_L, A \rangle$ for all $A \in \pi_2(M, L)$, where $\mu_L \in \mathrm{H}^2(M, L; \mathbb{Z})$ is the Maslov class. Then M is also monotone with monotonicity constant 2κ, i.e., $\langle \omega, A \rangle = 2\kappa \langle c_1(TM), A \rangle$ for all $A \in \pi_2(M)$. As in [98], we define the *minimal Maslov number* of L as the positive generator N_L of the subgroup of $\mathbb{Z}$ generated by $\langle \mu_L, w \rangle$ for all maps $w \colon \mathbb{A} \to M$ from the cylinder $\mathbb{A} := S^1 \times [0, 1]$ to M sending the boundary $\partial \mathbb{A}$ to L. When this group is trivial, we set $N_L = \infty$. In what follows, we require that $N_L \geq 2$ unless $\kappa = 0$. Note that this definition allows $[\omega]$ to vanish on $\pi_2(M, L)$ and thus includes weakly exact Lagrangian submanifolds.

Alternatively, as has already been pointed out, one can require L to be oriented, relatively spin and weakly unobstructed after bulk deformation as in [45] and replaced the coefficient field $\mathbb{F} = \mathbb{F}_2$ by a field of zero characteristic. We expect that Theorem A extends to this setting.

A *Hamiltonian diffeomorphism* $\varphi = \varphi_H = \varphi_H^1$ is the time-one map of the time-dependent flow (i.e., a *Hamiltonian isotopy*) $\varphi^t = \varphi_H^t$ of a 1-periodic in time Hamiltonian $H \colon S^1 \times M \to \mathbb{R}$, where $S^1 = \mathbb{R}/\mathbb{Z}$. The Hamiltonian vector field X_H of H is defined by $i_{X_H}\omega = -dH$. All Hamiltonians are assumed to be compactly supported.

The k-th iterate φ^k is viewed as the time-k map of φ_H^t. The k-periodic *points* of φ are in one-to-one correspondence with the k-periodic *orbits* of H, i.e., of the time-dependent flow φ_H^t. A k-periodic orbit x of H is said to be *non-degenerate* if the linearized return map $D_{x(0)}\varphi^k \colon T_{x(0)}M \to T_{x(0)}M$ has no eigenvalues equal to one. A Hamiltonian H is *non-degenerate* if all of its 1-periodic orbits are non-degenerate and *strongly non-degenerate* when all of its periodic orbits are non-degenerate.

Recall that the *Hofer norm* of φ is defined as

$$\|\varphi\|_H = \inf_H \int_{S^1} \left(\max_M H_t - \min_M H_t \right) dt,$$

where the infimum is taken over all 1-periodic in time Hamiltonians H generating φ, i.e., $\varphi = \varphi_H$; see [61, 76, 88, 100]. The *Hofer distance* between two Hamiltonian isotopic Lagrangian submanifolds L and L' is

$$d_H(L, L') = \inf \left\{ \|\varphi\|_H \mid \varphi(L) = L' \right\};$$

see [25].

3.2 Filtered Lagrangian Floer complex

In this section we very briefly spell out the definition of the filtered Lagrangian Floer complex we use in this paper. There are various settings and levels of generality one could work with here. For the sake of simplicity, we focus on the case of Hamiltonian isotopic closed monotone Lagrangian submanifolds. (We will touch upon other settings in Remark 3.2). For such Lagrangian submanifolds, Floer homology was originally defined in [85], albeit in a somewhat different algebraic setting. In this short narrative we adopt the framework from [98] with only minor modifications. We refer the reader to this work and, of course, to [45] for a much more detailed treatment and further references.

3.2.1 Floer complex

Let M be a symplectic manifold which is supposed to be sufficiently "tame" at infinity (e.g., compact or convex) to guarantee that compactness theorems hold, and let L and L' be closed monotone Lagrangian submanifolds intersecting transversely. We assume that L and L' are Hamiltonian isotopic to each other, i.e., there exists a Hamiltonian isotopy φ_F^t, $t \in [0, 1]$, such that $L' = \varphi_F(L)$. For the time being we will treat the isotopy φ_F^t as a part of the data. Furthermore, we require that $N_L \geq 2$, where $N_L = N_{L'}$ is the minimal Chern number as defined in Sect. 3.1.

Let $\mathcal{P}(L, L')$ be the space of smooth paths in M from L to L' and $\pi_1(M; L, L')$ be the set of its connected components. For instance, the intersection points of L and L' are elements of $\mathcal{P}(L, L')$; however, these elements might be in different connected components of $\mathcal{P}(L, L')$. Fix a reference path $x_{\mathfrak{c}}$ in each $\mathfrak{c} \in \pi_1(M; L, L')$. A *capping* w of a path $x \in \mathfrak{c}$ is a homotopy of x to $x_{\mathfrak{c}}$ (with end-points on L and L'), taken up to a certain equivalence relation. Namely, two cappings w and w' are equivalent if and only if the cylinder v obtained by attaching w' to w with the reversed orientation has zero symplectic area and zero Maslov number. Furthermore, we say that two such cylinders are equivalent when their symplectic areas and Maslov numbers are equal. When w and w' are not equivalent, we call v, taken up to this equivalence relation, and also (x, w') a *recapping* of (x, w) and write $(x, w') =: (x, w)\#v$. We usually suppress a capping in the notation. One can assign a well-defined Maslov index to a capped path from L to L' by fixing also a trivialization of TM along $x_{\mathfrak{c}}$.

For the sake of simplicity, the ground field $\mathbb{F}$ throughout the paper is $\mathbb{F}_2$. (However, in the Hamiltonian setting of Corollary A and Theorems B and C we can work with any ground field $\mathbb{F}$). The Floer complexes we mainly consider are finite-dimensional vector spaces over the "universal" *Novikov field* Λ formed by formal sums

$$\lambda = \sum_{j \geq 0} f_j T^{a_j}, \tag{3.1}$$

where $f_j \in \mathbb{F}$ and $a_j \in \mathbb{R}$ and the sequence a_j (with $f_j \neq 0$) is either finite or $a_j \to \infty$.

The Floer complex $\mathrm{CF}(L, L')$, or just $\mathcal{C}$ for brevity, is generated by the intersections $L \cap L'$ with arbitrarily fixed cappings. The differential ∂_{Fl} is defined by the standard formula counting holomorphic strips u with boundary components on L and L', asymptotic to the intersections with now all possible cappings and the symplectic area $\omega(v)$ of the recapping v contributing the term $T^{\omega(v)}$ to the differential.

To be more precise, denote the generators by x_i. Thus

$$\mathcal{C} = \bigoplus_i \Lambda x_i. \tag{3.2}$$

Then, assuming that the underlying almost complex structure is regular, we have

$$\partial_{Fl} x_i = \sum_j \lambda_{ij} x_j, \tag{3.3}$$

where

$$\lambda_{ij} = \sum_v f_v T^{\omega(v)} \in \Lambda \tag{3.4}$$

Here the sum extends over all recappings v of x_j such that the Maslov index difference of x_i and $x_j \# v$ is 1 and $f_v \in \mathbb{F}$ is the parity of the number of holomorphic strips u asymptotic to x_i and x_j, and such that the original fixed capping of x_i is equivalent to the capping obtained by attaching $x_j \# v$ to these strips.

This complex is not graded, due to our choice of the Novikov field, or only $\mathbb{Z}_2$-graded. We denote the Floer homology, i.e., the homology of $\mathcal{C}$, by $\mathrm{HF}(L, L')$ or just $\mathrm{HF}(L)$. This is also a finite-dimensional vector space over Λ.

The complex $\mathcal{C}$ and its homology $\mathrm{HF}(L, L')$ split into a direct sum of complexes $\mathcal{C}_\mathfrak{c} = \mathrm{CF}_\mathfrak{c}(L, L')$ and homology groups $\mathrm{HF}_\mathfrak{c}(L, L')$ over $\mathfrak{c} \in \pi_1(M; L, L')$.

3.2.2 Action filtration

To define the action filtration on $\mathcal{C} = \mathrm{CF}(L, L')$ it is beneficial to look at this complex from a different perspective and this is where the condition that L and L' are Hamiltonian isotopic, which has not been explicitly used so far, becomes essential. Namely, applying the inverse Hamiltonian isotopy $(\varphi_F^t)^{-1}$ to paths from L to L' we obtain paths from L to itself and thus a homeomorphism between $\mathcal{P}(L, L')$ and $\mathcal{P}(L, L)$ and a bijection between $\pi_1(M; L, L')$ and $\pi_1(M; L) := \pi_1(M; L, L)$. The intersections $L \cap L'$ turn into Hamiltonian chords from L to itself for $(\varphi_F^t)^{-1}$. Likewise, the reference paths $x_\mathfrak{c}$ become the reference paths $y_\mathfrak{c}$ from L to L and a capping of $y \in \mathcal{P}(L, L)$ is a homotopy from $y \in \mathfrak{c}$ to $y_\mathfrak{c}$ with end-points on L up to the same equivalence relation.

This procedure turns the Cauchy–Riemann equation into the Floer equation

$$\frac{\partial u}{\partial s} + J \frac{\partial u}{\partial t} = -\nabla H_t(u)$$

where $(s, t) \in \mathbb{R} \times [0, 1]$ and $u: \mathbb{R} \times [0, 1] \to M$, and $H_t = -F_t \circ \varphi_F^t$ is a Hamiltonian generating the inverse of the isotopy $\varphi_F^t: M \to M$ from L to L'. Holomorphic strips with boundaries on L and L' and asymptotic to $L \cap L'$ become solutions u of the Floer equation with boundary on L asymptotic to Hamiltonian chords. The Floer differential now counts solutions of the Floer equation. The Hamiltonian action of a capped path $\bar{y} = (y, w)$ is given

by the standard formula

$$\mathcal{A}(\bar{y}) = -\int_w \omega + \int_0^1 H_t(y(t)) \, dt. \tag{3.5}$$

The Floer differential is strictly action-decreasing, and we obtain the required action filtration on $\mathcal{C}$ with the convention that for $\lambda \in \Lambda$ given by (3.1), we have

$$\nu(\lambda) := \min\{a_j\}, \text{ where } f_j \neq 0, \tag{3.6}$$

and

$$\mathcal{A}\left(\sum \lambda_i y_i\right) := \max_i \mathcal{A}(\lambda_i y_i), \text{ where } \mathcal{A}(\lambda_i y_i) := \mathcal{A}(y_i) - \nu(\lambda_i). \tag{3.7}$$

A few remarks are due at this point. First of all, the action filtration depends on the choice of the reference paths $x_{\mathfrak{c}}$ (or $y_{\mathfrak{c}}$). Namely, on every direct summand $\mathcal{C}_{\mathfrak{c}}$ a change of the reference path shifts the filtration by a constant. Thus the filtration is well-defined only up to independent shifts on these summands. Clearly, this ambiguity does not affect the barcode of $\mathcal{C}$ introduced in Sect. 3.3.1. In particular, the bar length and the number of bars of a given length and $b_\epsilon(L, L')$ are independent of the choice of reference paths.

Furthermore, while the generators of $\mathcal{C}$ are (capped) intersections, the differential genuinely depends on the background almost complex structure or, in the second interpretation, on the almost complex structure and the Hamiltonian F. However, the resulting complexes are chain homotopy equivalent with homotopy preserving the filtration up to a shift. Thus, the number of bars of a given length either and $b_\epsilon(L, L')$ are again well-defined.

The Floer homology $\mathrm{HF}(L, L')$, as a vector space over $\mathbb{F}$, inherits the action filtration from $\mathcal{C}$. To be more specific, we have a family of vector spaces $\mathrm{HF}^a(L, L')$ over $\mathbb{F}$, parametrized by $a \in \mathbb{R}$, where $\mathrm{HF}^a(L, L')$ is the homology of the subcomplex of $\mathcal{C}$ comprising the chains $\sum \lambda_i y_i$ with action less than a. The inclusion of subcomplexes for $a < a'$ gives rise to natural maps $\mathrm{HF}^a(L, L') \to \mathrm{HF}^{a'}(L, L')$.

Essentially by construction, we have the "Poincaré duality". Namely, for a suitable choice of the auxiliary data,

$$\mathrm{CF}(L', L) = \mathrm{CF}(L, L')^* \tag{3.8}$$

over Λ with "inverted" filtration. (Here we used the fixed basis $\{x_i\}$ to identify the vector spaces $\mathrm{CF}(L', L)$ and $\mathrm{CF}(L, L')$ and their duals, and then the Floer differential in $\mathrm{CF}(L', L)$ turns into the adjoint of the Floer differential in $\mathrm{CF}(L,' L)$). As a consequence, $\mathrm{HF}(L', L) = \mathrm{HF}(L, L')^*$.

We also note that whether or not L and L' are transverse, only a finite collection of elements in $\pi_1(M; L, L') \cong \pi_1(M; L)$ is represented by Hamiltonian chords. Thus $\mathcal{C}_{\mathfrak{c}} = 0$ for all but a finite collection of $\mathfrak{c} \in \pi_1(M; L, L') \cong \pi_1(M; L)$; this collection might however depend on the Hamiltonian isotopy; cf. [98, Prop. 6.2].

Remark 3.1 In the situation we are interested in there is usually a natural choice of the Hamiltonian isotopy. Namely, when $L' = \varphi^k(L)$ as in Sect. 2.1, the isotopy comes from φ_H^t with time interval $[0, k]$ scaled to $[0, 1]$. In Sect. 4.1, $L = L_0$ and $L' = \varphi^k(L_1)$, where L_0 and L_1 are assumed to be Hamiltonian isotopic, and the isotopy from L to L' is obtained by concatenating the two isotopies.

Remark 3.2 (*Exact Lagrangians, I*) Finally, note that there are of course other settings where the filtration on $\mathcal{C}$ is defined and has the desired properties. One is when (M, ω) is an exact

symplectic manifold (i.e., ω is exact) meeting certain additional requirements at infinity, and L and L' are exact Lagrangian submanifolds (i.e., the restrictions of a primitive of ω to L and L' are exact), not necessarily Hamiltonian isotopic. Then $\Lambda = \mathbb{F}$. In this case, sometimes we can even allow one of the manifolds to be non-compact. For instance, L can be Hamiltonian isotopic to the zero section in a cotangent bundle of a closed manifold and L' can be a fiber.

3.2.3 Period group, the action spectrum and the action difference set

In this subsection we do not assume that L and L' are transverse unless explicitly stated otherwise.

For every class $\mathfrak{c} \in \pi_1(M; L)$, denote by $\Gamma_\mathfrak{c}$ the subgroup of $\mathbb{R}$ formed by the integrals $\omega(v)$ for all maps $v \colon \mathbb{A} = S^1 \times [0, 1] \to M$ sending $\partial\mathbb{A}$ to L and with the meridian $v(\{0\} \times [0, 1])$ in $\mathfrak{c}$. (Such annuli v, with $\mathfrak{c}$ fixed, taken up to homotopy or the equivalence relation discussed above, form a group of recappings of paths in $\mathfrak{c}$ and $\Gamma_\mathfrak{c}$ is the group of action changes resulting from recappings). The monotonicity condition forces this group to be discrete for the "unit" class $\mathfrak{c} = 1$ of the constant path. In general, $\Gamma_\mathfrak{c}$ depends on $\mathfrak{c}$ and can be discrete for some classes $\mathfrak{c}$ and dense for other classes; see Example 3.3. In the former case we denote by $\lambda_\mathfrak{c}$ its positive generator and call $\mathfrak{c}$ *toroidally rational*. As usual, we set $\lambda_\mathfrak{c} = \infty$ when $\mathfrak{c}$ is *atoroidal*, i.e., $\Gamma_\mathfrak{c} = \{0\}$. Finally, let $\Gamma \subset \mathbb{R}$ be the subgroup generated by the union of all $\Gamma_\mathfrak{c}$. We will refer to $\Gamma_\mathfrak{c}$ as a *period group* and to Γ as the *complete period group*. (Hypothetically it is possible that Γ is dense even when all $\Gamma_\mathfrak{c}$ are discrete, but we do not have an example when this happens).

The *action spectrum* $\mathcal{S}_\mathfrak{c}(L, L') \subset \mathbb{R}$ (for a class $\mathfrak{c}$) is the collection of actions for all capped chords in $\mathfrak{c}$ or, equivalently, capped intersections $L \cap L'$ in the class $\mathfrak{c}$. Clearly, a change of a reference chord $y_\mathfrak{c}$ results in a shift of $\mathcal{S}_\mathfrak{c}(L, L')$ by a constant. Furthermore, as we have already mentioned, only a finite collection of free homotopy classes can contain Hamiltonian chords, and hence $\mathcal{S}_\mathfrak{c}(L, L') = \emptyset$ for all but a finite set of the classes $\mathfrak{c}$. In general, this collection of classes depends on the Hamiltonian isotopy from L to L', and a change of the isotopy results in action shifts and "relabeling" of the action spectra; cf. [98, Prop. 6.2]. We denote by $\mathcal{S}(L, L')$ the union of the sets $\mathcal{S}_\mathfrak{c}(L, L')$ for all $\mathfrak{c}$.

The following is a list of standard properties of the action spectrum which are relevant to our purposes although not directly used in the proofs:

- The sets $\mathcal{S}_\mathfrak{c}(L, L')$ are non-empty for only a finite collection of elements $\mathfrak{c} \in \pi_1(M; L)$.
- The sets $\mathcal{S}_\mathfrak{c}(L, L')$, and hence their union $\mathcal{S}(L, L')$, have zero measure.
- The set $\mathcal{S}_\mathfrak{c}(L, L')$ is countable whenever the intersections $L \cap L'$ in the class $\mathfrak{c}$ are transverse. (The converse is not true).
- The set $\mathcal{S}_\mathfrak{c}(L, L')$ is invariant under translations by elements of $\Gamma_\mathfrak{c}$.
- The set $\mathcal{S}_\mathfrak{c}(L, L') \neq \emptyset$ is compact if and only if $\Gamma_\mathfrak{c} = \{0\}$.
- The set $\mathcal{S}_\mathfrak{c}(L, L') \neq \emptyset$ is closed if and only if $\Gamma_\mathfrak{c}$ is discrete.
- The set $\mathcal{S}_\mathfrak{c}(L, L') \neq \emptyset$ is dense if and only if $\Gamma_\mathfrak{c}$ is dense.

We denote by $\mathcal{D}_\mathfrak{c}(L, L') \subset \mathbb{R}$ the set comprising all action differences between the capped intersections in $\mathfrak{c}$, i.e.,

$$\mathcal{D}_\mathfrak{c}(L, L') := \mathcal{S}_\mathfrak{c}(L, L') - \mathcal{S}_\mathfrak{c}(L, L') = \{a - b \mid a, b \in \mathcal{S}_\mathfrak{c}(L, L')\}.$$

This set is countable whenever the intersections $L \cap L'$ in the class $\mathfrak{c}$ are transverse, and dense when $\Gamma_\mathfrak{c}$ is dense. (The converse is not true in both cases). Furthermore, $\mathcal{D}_\mathfrak{c}(L, L')$ is closed if and only if $\mathcal{S}_\mathfrak{c}(L, L')$ is closed. An important point is that in general $\mathcal{D}_\mathfrak{c}(L, L')$ need not

be a zero measure set. (For instance, $\mathcal{D}_{\mathfrak{c}}(L, L') = [-1, 1]$ when $\mathcal{S}_{\mathfrak{c}}(L, L')$ is the standard Cantor set; see, e.g., [47, p. 87]). We also let $\bar{\mathcal{D}}_{\mathfrak{c}}(L, L')$ to be the closure of $\mathcal{D}_{\mathfrak{c}}(L, L')$ and $\bar{\mathcal{D}}(L, L')$ stand for the union of the closures $\bar{\mathcal{D}}_{\mathfrak{c}}(L, L')$ for all $\mathfrak{c} \in \pi_1(M; L)$. This union is also closed since $\mathcal{D}_{\mathfrak{c}}(L, L') \neq \emptyset$ only for a finite collection of classes $\mathfrak{c}$.

Applying these constructions to the Hamiltonian diffeomorphism $id \times \varphi$ of the symplectic square $(M \times M, -\omega \oplus \omega)$, we obtain the filtered Floer complex $\mathcal{C} := \mathrm{CF}(\varphi)$ of φ. This complex is isomorphic to the standard $\mathbb{Z}_2$-graded Floer complex over Λ generated by 1-periodic orbits in all free homotopy classes $\mathfrak{c} \in \tilde{\pi}_1(M)$. This distinction from the more customary construction limited to contractible orbits is essential; see Sect. 2.3. The free homotopy class of a k-periodic orbit of φ_H^t is completely determined by the Hamiltonian diffeomorphism φ_H and is independent of the isotopy. This is a consequence of the proof of Arnold's conjecture which in the monotone case was established in [42]. As we have already pointed out, in the Lagrangian setting the situation is more complicated. The period group $\Gamma_{\mathfrak{c}} \subset \mathbb{R}$ is formed by the integrals $\omega(v)$ where now $v \colon \mathbb{T}^2 = S^1 \times S^1 \to M$ with the meridian $v(\{0\} \times S^1)$ in $\mathfrak{c}$, and $\lambda_{\mathfrak{c}}$ is the postive generator of $\Gamma_{\mathfrak{c}}$ when this groups is non-trivial and $\lambda_{\mathfrak{c}} = 0$ otherwise. (The homotopy classes of such tori form a group). As above, we denote by Γ the complete period group, i.e., the group generated by the union of the groups $\Gamma_{\mathfrak{c}}$.

Example 3.3 Assume first that $M = \mathbb{T}^2$ with area a and let as above $\tilde{\pi}_1(M)$ be the set of free homotopy classes of loops in M, i.e., conjugacy classes in $\pi_1(M)$. Then $\tilde{\pi}_1(M) = \pi_1(M) = \mathbb{Z}^2$ and $\lambda_{\mathfrak{c}} = a$ for every primitive class $\mathfrak{c} \neq 0$ and, using additive notation, $\lambda_{k\mathfrak{c}} = k\lambda_{\mathfrak{c}}$ for every $k \geq 0$; see [86]. A similar description applies to $\mathbb{T}^{2n} = \mathbb{T}^2 \times \cdots \times \mathbb{T}^2$ equipped with the standard symplectic structure for which all factors $\mathbb{T}^2$ have the same area a, although now $\lambda_{\mathfrak{c}} \in a\mathbb{Z}$ may depend on $\mathfrak{c}$ even when $\mathfrak{c}$ is primitive. A surface Σ of genus $g \geq 2$ or, more generally, a symplectically hyperbolic manifold (see, e.g., [72, 89]) is atoroidal, i.e,. $\Gamma_{\mathfrak{c}} = 0$ for all $\mathfrak{c}$. Next, let $M = \mathbb{T}^2 \times \Sigma$. Then $\lambda_{\mathfrak{c}}$ is completely determined by the projection of $\mathfrak{c}$ to $\mathbb{T}^2$ and then calculated as in the previous example. Finally, let $M = \mathbb{T}_1^2 \times \mathbb{T}_a^2$, where the first torus has area 1 and the second has an irrational area a. Then classes $\mathfrak{c}$ lying in $\pi_1(\mathbb{T}_1^2)$ and $\pi_1(\mathbb{T}_a^2)$ are toroidally rational and $\lambda_{\mathfrak{c}}$ is calculated as above. For other classes, the group $\Gamma_{\mathfrak{c}}$ is dense in $\mathbb{R}$.

Remark 3.4 (*Growth of the action spectrum*) As we have pointed out in the introduction, one difficulty in thinking about topological entropy in terms of Floer theory is that in many cases the "effective" diameter of the action spectrum $\mathcal{S}(\varphi^k)$ grows sub-exponentially with k even when the cardinality of $\mathcal{S}(\varphi^k)$ grows super-exponentially (see Remark 2.4). For instance, assume that ω is atoroidal. Then it is not hard to show that for a suitable choice of reference loops in $\tilde{\pi}_1(M)$ the diameter of $\mathcal{S}(\varphi^k)$ grows sub-exponentially when the pullback of ω to the universal covering of M has a primitive with sub-exponential growth. For instance, when M is a surface of genus $g \geq 2$ or, more generally, M is symplectically hyperbolic, the diameter grows linearly.

When M is simply connected and monotone, one can pin every finite bar to be contained in the interval $\left[0, \lambda_M + \|\varphi^k\|_H\right] \subset \left[0, \lambda_M + k\|\varphi\|_H\right]$, where λ_M is the rationality constant; see [98] and Remark 3.6. (Moreover, under these assumptions on M, we can replace the Hofer norm by the γ-norm resulting in a bounded interval in some instances, e.g., for $\mathbb{CP}^n$; see [38, 75]). A similar upper bound holds for $M = \mathbb{T}^{2n}$, but now one has to also use the result from [86] mentioned in Example 3.3.

3.2.4 Floer package and shrinking the Novikov field

Purely formally, the above constructions can be summarized as the following "Floer package":

- The finite-dimensional vector space $\mathcal{C}$ over Λ with a fixed set of generators x_i; see (3.2).
- The differential ∂_{Fl}; see (3.3).
- The action filtration on $\mathcal{C}$ given by (3.6) and (3.7) such that ∂_{Fl} is strictly action decreasing.

We emphasize that in this package there are no algebraic constraints on the actions $\mathcal{A}(x_i)$ other than that ∂_{Fl} is *required* to be strictly action decreasing.

Next, observe that all exponents occurring in λ_{ij} in (3.4) are in Γ or in $\Gamma_{\mathfrak{c}}$ when we focus on $\mathcal{C}_{\mathfrak{c}}$. Thus we could have replaced in the construction the universal Novikov field Λ by the field Λ^{Γ} defined in a similar fashion, but with all exponents a_j in (3.1) in Γ (or in $\Gamma_{\mathfrak{c}}$). Moreover, we could have worked with the field Λ^G for any subgroup G of $\mathbb{R}$ containing Γ, resulting in the same barcode. (For instance, $\Lambda = \Lambda^{\mathbb{R}}$). When essential, we will write $\mathcal{C}(\Lambda^{\Gamma})$ to indicate the Novikov field.

On a purely formal level of a Floer package, we can define Γ as a countable subgroup of $\mathbb{R}$ generated by all exponents occurring in λ_{ij} and then use the field Λ^G whenever $\Gamma \subset G$. (This group Γ can potentially be smaller that the period group Γ defined in Sect. 3.2.3 geometrically).

Our choice to mainly work with Λ rather than Λ^{Γ} is dictated by expository considerations: it is convenient to have the Novikov ring independent of the geometrical setting.

3.3 Persistent homology and barcodes

3.3.1 Barcodes

In this section we briefly recall a few basic facts and definitions concerning persistent homology and barcodes in the context of Floer theory. We refer the reader to [90] for a very detailed introduction and a discussion in much broader context. Here, treating barcodes in the framework of Lagrangian Floer complexes, we closely follow [99], with some minor simplifications. (A different although equivalent approach to barcodes in this context is put forth in [75]. That approach is, however, slightly less convenient for our purposes).

The key difference between the settings used here and in [99] is that Floer complexes in [99] are $\mathbb{Z}$-graded, while in this paper, as in [27], the complexes are ungraded (or $\mathbb{Z}_2$-graded), i.e., in [99] the differential comprises maps between different spaces over Λ, while here ∂_{Fl} is a map from $\mathcal{C}$ to itself. However, the constructions, results and proofs from [99] carry over to our framework. For instance, one can apply these results to the "two-storey" $\mathbb{Z}$-graded complex

$$0 \to \mathcal{C}/\ker \partial_{Fl} \xrightarrow{\partial_{Fl}} \ker \partial_{Fl} \to 0$$

over Λ.

As in Sect. 3.2, consider the filtered Lagrangian Floer complex $\mathcal{C} := CF(L, L')$ of a transverse pair of closed monotone Lagrangian submanifolds L and L'. A finite set of vectors $\xi_i \in \mathcal{C}$ is said to be *orthogonal* if for any collection of coefficients $\lambda_i \in \Lambda$ we have

$$\mathcal{A}\left(\sum \lambda_i \xi_i\right) = \max \mathcal{A}(\lambda_i \xi_i),$$

where $\mathcal{A}$ is defined by (3.6) and (3.7). It is not hard to show that an orthogonal set is necessarily linearly independent over Λ.

Definition 3.5 A basis $\Sigma = \{\alpha_i, \eta_j, \gamma_j\}$ of $\mathcal{C}$ over Λ is said to be a *singular value decomposition* if

- $\partial_{Fl}\alpha_i = 0$,
- $\partial_{Fl}\gamma_j = \eta_j$,
- the basis is orthogonal.

It is shown in [99, Sects. 2 and 3] that $\mathcal{C}$ admits a singular value decomposition. Ordering the pairs (η_j, γ_j) by the action difference, we have

$$\mathcal{A}(\gamma_1) - \mathcal{A}(\eta_1) \leq \mathcal{A}(\gamma_2) - \mathcal{A}(\eta_2) \leq \cdots$$

This increasing sequence together with $\dim_\Lambda \mathrm{HF}(L, L')$ infinite bars (corresponding to the basis elements α_i) is referred to as the *barcode* of $\mathcal{C}$ and denoted by $\mathcal{B}(\mathcal{C})$ or $\mathcal{B}(L, L')$. Fixing a class $\mathfrak{c}$ we also obtain the barcode $\mathcal{B}_\mathfrak{c}(L, L')$. This is a multiset. As a consequence of the discussion in Sect. 3.2.4, $\mathcal{B}_\mathfrak{c}(L, L') \subset \mathcal{D}_\mathfrak{c}(L, L') \cup \{\infty\}$ if we ignore multiplicities. The barcode is independent of the choice of a singular value decomposition and auxiliary data involved in the construction of $\mathcal{C}$; see, e.g., [99]. As in (2.1), we set

$$b_\epsilon(L, L') = b_\epsilon(\mathcal{C}) := \left|\{\beta \in \mathcal{B}(\mathcal{C}) \mid \beta > \epsilon\}\right|. \tag{3.9}$$

The complexes $\mathrm{CF}(L', L) = \mathrm{CF}(L, L')^*$ and $\mathrm{CF}(L, L')$ have the same barcode:

$$\mathcal{B}(L, L') = \mathcal{B}(L', L);$$

cf. [99]. (This does not follow directly from (3.8) because a singular value decomposition for $\mathrm{CF}(L, L')$ need not be a singular value decomposition for $\mathrm{CF}(L', L)$ when the two spaces are identified by fixing a basis of capped intersections). Hence,

$$b_\epsilon(L, L') = b_\epsilon(L', L). \tag{3.10}$$

We also set $b(\mathcal{C}) = b(L, L') := |\mathcal{B}(L, L')|$ to be the total number of bars in the barcode. Then

$$|L \cap L'| = \dim_\Lambda \mathrm{CF}(L, L') = 2b(L, L') - \dim_\Lambda \mathrm{HF}(L, L') \geq b(L, L').$$

In particular, $b_\epsilon(L, L')$ gives a lower bound for the number of intersections:

$$|L \cap L'| \geq b_\epsilon(L, L'). \tag{3.11}$$

We emphasize that here and throughout this section we have assumed that $L \pitchfork L'$.
    Clearly,

$$\mathcal{B}\big(\psi(L), \psi(L')\big) = \mathcal{B}(L, L')$$

for any symplectomorphism $\psi : M \to M$, and hence

$$b_\epsilon\big(\psi(L), \psi(L')\big) = b_\epsilon(L, L'). \tag{3.12}$$

Furthermore, $b_\epsilon(L, L')$ is constant as a function of ϵ on any interval in the complement of $\bar{\mathcal{D}}(L, L')$.
    The longest finite bar $\beta_{\max}(L, L')$ in $\mathcal{B}(L, L')$ is called the *boundary depth*. As shown in [98], $\beta_{\max}(L, L') \leq d_H(L, L') + \beta_{\max}(L, L)$ and, in particular, $\beta_{\max}(\varphi) \leq \|\varphi\|_H$. (In general, $\beta_{\max}(L, L)$ can be non-zero. For instance, for a displaceable circle L bounding a disk of area a in a surface, $\mathcal{B}(L, L)$ has only one bar, which has length a, and hence $\beta_{\max}(L, L) = a > 0$).

Remark 3.6 (*Pinned bars*) Our definition of a barcode is a simplification of the standard one in which, when $\Gamma = \{0\}$, a barcode is the collection of the intervals $[\mathcal{A}(\eta_j), \mathcal{A}(\gamma_j)]$ and $[\mathcal{A}(\alpha_i), \infty)$ rather than just their length; see, e.g., [90]. Hence, a bar is "pinned" to its

beginning in $\mathbb{R}$. When working over the universal Novikov field Λ, the situation is similar, but now $\mathbb{R}$ is replaced by $\mathbb{R}/\Gamma$. Namely, with more care in the choice of a singular value decomposition, a finite bar is determined by $\mathcal{A}(\gamma_i)$, viewed as an element of $\mathbb{R}/\Gamma$, and the difference $\mathcal{A}(\gamma_i) - \mathcal{A}(\eta_i)$, and an infinite bar is the pair $\mathcal{A}(\alpha_i) \in \mathbb{R}/\Gamma$ and ∞; see [99]. Thus a bar is "pinned" to a point in $\mathcal{S}(L, L')/\Gamma$. However, then the position of each bar depends on the auxiliary data (reference paths and the Hamiltonian isotopy from L to L'), and the shift resulting from a change of the data also depends on the connected component $\mathfrak{c} \in \pi_1(M; L, L')$. Since here we are mainly concerned with counting bars of length greater than ϵ, the simplified definition is more convenient for our purposes.

The barcodes are fairly insensitive to small perturbations of the Lagrangian submanifolds with respect to the Hofer metric. Namely, assume that $d_H(L', L'') < \delta/2$. Then

$$b_{\epsilon+\delta}(L, L') \leq b_\epsilon(L, L'') \leq b_{\epsilon-\delta}(L, L'). \tag{3.13}$$

We refer the reader to [75, 90, 99] for the proof.

These constructions apply *verbatim* to the direct summand $\mathrm{CF}_\mathfrak{c}(L, L')$ for $\mathfrak{c} \in \pi_1(M; L)$.

It is not hard to extend the notion of barcode to the situation where L and L' are not transverse. However, here we are interested only in counting bars of length above a fixed treshold and we extend its definition to the non-transverse case in an *ad hoc* manner in the next section.

Specializing to the case where L is the diagonal in $M \times M$ and L' is the graph of a strongly non-degenerate Hamiltonian diffeomorphism $\varphi : M \to M$, we obtain the barcode $\mathcal{B}(\varphi)$ of $\mathrm{CF}(\varphi)$ (for all free homotopy classes of loops $\mathfrak{c} \in \tilde{\pi}_1(M)$), which is again insensitive to small perturbations of φ with respect to the Hofer metric and hence to C^∞-small perturbations, and $\mathcal{B}_\mathfrak{c}(\varphi) \subset \mathcal{D}_\mathfrak{c}(\varphi) \cup \{\infty\}$. Likewise, $\bar{\mathcal{D}}(\varphi)$ will stand for the union of the closures $\bar{\mathcal{D}}_\mathfrak{c}(\varphi)$, etc.

Throughout this section we could have replaced the universal Novikov field Λ by a smaller field Λ^Γ as in Sect. 3.2.4 with exactly the same resulting barcode with the same properties. In particular, this change would not affect $b_\epsilon(L, L')$. Moreover, it would lead to a small technical advantage. Namely, assume that all intersections $L \cap L'$ have actions distinct modulo Γ, i.e., $\mathcal{A}(x_i) \neq \mathcal{A}(x_j)$, $i \neq j$, in $\mathcal{S}(L, L')/\Gamma$. (This is a C^∞-generic condition). For $\xi \in \mathcal{C}(\Lambda^\Gamma)$, write

$$\xi = fT^a x + \dots,$$

where the dots stand for lower-action terms and x is one of the capped intersections x_i (and $f \neq 0$). It is easy to see that since the intersections have distinct actions modulo Γ and $a \in \Gamma$, the term $T^a x$ and the intersection x are unique. We will refer to $T^a x$, or sometimes just x, as the *leading action term* in ξ.

Example 3.7 Assume that the intersections have distinct actions modulo Γ. Then it is easy to see that the set $\xi_i \in \mathcal{C}(\Lambda^\Gamma)$ is orthogonal if and only if the leading action terms x_i are distinct.

3.3.2 Bounding $b_\epsilon(\mathcal{C})$ from below

The proof of Theorem B hinges on a lower bound on $b_\epsilon(\varphi)$ via the number of periodic orbits which are in a certain sense energy-isolated. In this section we will deal with the algebraic aspect of the argument.

It is convenient, although strictly speaking not necessary, to introduce the *Floer graph* G associated with a Floer package $\mathcal{C}$; cf. [27]. The vertices of G are the generators x_i. For each non-zero term fT^a in λ_{ij} (see (3.4)) we connect x_i to x_j by an arrow and label that arrow by the exponent a. (Thus G is a directed graph with finitely many vertices but possibly infinitely many edges). The length of an arrow is the action difference $\mathcal{A}(x_i) - \mathcal{A}(T^a x_j) = \mathcal{A}(x_i) - \mathcal{A}(x_j) + a$, i.e., the energy of any underlying Floer trajectory. We say that $x = x_i$ is ϵ-isolated if every arrow from or to x has length strictly greater that ϵ. For instance, x is ϵ-isolated if every Floer trajectory asymptotic to x at $\pm\infty$ has energy strictly greater than ϵ. (The converse need not be true).

Proposition 3.8 *Assume that G has p ϵ-isolated vertices. Then $b_\epsilon(\mathcal{C}) \geq p/2$.*

Proof Let us switch from the universal field Λ to Λ^Γ. Thus, throughout the proof, $\mathcal{C} := \mathcal{C}(\Lambda^\Gamma)$. By continuity, we can assume in addition that all generators x_i have distinct actions modulo Γ. Indeed, a small change of actions $\mathcal{A}(x_i)$ does not affect $b_\epsilon(\mathcal{C})$ and ϵ-isolation, as a consequence of (3.13) and the fact that in the definition of b_ϵ, (4.2), the bars are required to be strictly greater than ϵ. Furthermore, as has been pointed out in Sect. 3.3.1 this can be achieved by a C^∞-small perturbation. (In fact, since we are working in a purely algebraic setting we can simply change the filtration of $\mathcal{C}$ by altering $\mathcal{A}(x_i)$ in the formal framework of a Floer package; see Sect. 3.2.4). Now Example 3.7 applies.

Next, recall from [99] that a non-zero element $\zeta \in \text{im } \partial_{Fl}$ is said to be ϵ-*robust* if $\mathcal{A}(\xi) - \mathcal{A}(\zeta) > \epsilon$ for every ξ with $\partial_{Fl}\xi = \zeta$. A subspace $W \subset \text{im } \partial_{Fl}$ is said to be ϵ-robust if every (non-zero) vector $\zeta \in W$ is ϵ-robust. The key fact established in [99] that we will use in this proof is that the number of finite bars of length greater than ϵ is equal to the maximal dimension of an ϵ-robust subspace. Thus it suffices to find an ϵ-robust subspace W with $\dim W \geq p/2 - \dim H(\mathcal{C})$.

Let $x_1, \ldots, x_p$ be the ϵ-isolated vertices and let V be their span. We will need the following two observations:

(i) For every linear combination ξ of ϵ-isolated vertices x_i, we have $\mathcal{A}(\xi) - \mathcal{A}(\partial_{Fl}\xi) > \epsilon$.
(ii) Let $w_j = x_j + \ldots$, where the dots stand for lower action terms, be any collection of vectors with x_j distinct, i.e., w_j are orthogonal. Then any exact (i.e., in the image of ∂_{Fl}) linear combination of w_j is ϵ-robust.

Next, from the short exact sequence

$$0 \rightarrow \ker(\partial_{Fl}|_V) \rightarrow V \rightarrow \partial_{Fl}(V) \rightarrow 0,$$

we have

$$\dim V = \dim \ker(\partial_{Fl}|_V) + \dim \partial_{Fl}(V).$$

Let Y be an orthogonal complement of $\ker(\partial_{Fl}|_V)$ in V. (We refer the reader to [99, Sect. 2] for an extensive discussion of orthogonality in the nonarchimedean setting and further references). Thus $V = \ker(\partial_{Fl}|_V) \oplus Y$ and ∂_{Fl} induces an isomorphism between Y and $\partial_{Fl}(V)$.

Let us now modify Y. If all elements of $\partial_{Fl}(V)$ are ϵ-robust we do nothing. Assume not: there is a vector $\xi_1 \in Y$ such that $\partial_{Fl}\xi_1$ is not ϵ-robust, but $\mathcal{A}(\xi_1) - \mathcal{A}(\partial_{Fl}\xi_1) > \epsilon$ by Observation (i). Therefore, there exists ζ_1 with $\partial_{Fl}\zeta_1 = \partial_{Fl}\xi_1$ and

$$\mathcal{A}(\zeta_1) - \mathcal{A}(\partial_{Fl}\xi_1) < \epsilon.$$

Then $w_1 := \xi_1 - \zeta_1$ is closed and $\mathcal{A}(w_1) = \mathcal{A}(\xi_1)$, i.e., the leading term in w_1 is the leading term in ξ_1. Let $Y_1 \subset Y$ be an orthogonal complement to ξ_1. If every element of $\partial_{Fl}(Y_1)$ is

ϵ-robust the process stops. If not, we pick ξ_2 such that $\partial_{Fl}\xi_2$ is not ϵ-robust, etc. Proceeding, we will construct, for some $s \geq 1$, vectors $\xi_1, \ldots, \xi_s$ in Y as above together with vectors w_i, and an orthogonal complement Y_s to $\text{span}(\xi_1, \ldots, \xi_s)$ in Y such that $\partial_{Fl}(Y_s)$ is ϵ-robust and $\dim \partial_{Fl}(Y_s) = \dim Y_s$.

Each vector w_i has the same leading term as ξ_i, which we can assume to be one of ϵ-isolated generators x_j. The vectors w_i are still orthogonal to $\ker(\partial_{Fl}|_V)$ and to each other since so are the vectors ξ_i. Hence, by Example 3.7, the leading terms x_j of these vectors are distinct and also distinct from the leading terms in any orthogonal basis in $\ker(\partial_{Fl}|_V)$. Hence, Observation (ii) applies to a an orthogonal basis in

$$W_0 = \ker(\partial_{Fl}|_V) \oplus \text{span}(w_1, \ldots, w_s).$$

Also set

$$W_1 = \partial_{Fl}(Y_s).$$

Every element in W_0 is closed, and every vector in W_1 is exact and ϵ-robust by construction. Furthermore,

$$\dim W_0 + \dim W_1 = \dim V = p$$

and W_0 has an orthogonal basis with distinct leading terms x_j.

Finally, let W_{00} be the subspace of exact vectors in W_0, i.e., the kernel of the natural map $W_0 \to \text{H}(\mathcal{C})$. Every element in W_{00} is ϵ-robust by Observation (ii). Clearly,

$$\dim W_{00} \geq \dim W_0 - \dim \text{H}(\mathcal{C}).$$

One of the spaces W_0 or W_1 has dimension at least $p/2$. If this is W_1, we set $W = W_1$ and the proof is finished. If this is W_0, we set $W = W_{00}$, and then $\dim W \geq p/2 - \dim \text{H}(\mathcal{C})$ and every element in W is ϵ-robust. □

4 Definitions revisited and general properties

4.1 Definitions: pairs of Lagrangian submanifolds

In this section we slightly generalize the definition of barcode entropy from Sect. 2.1, extending it to pairs of Hamiltonian isotopic Lagrangian submanifolds. Thus let M, L and L' be as in Sect. 3: M is a symplectic manifold, compact or "tame" at infinity (e.g., convex), and L and L' are closed monotone Lagrangian submanifolds Hamiltonian isotopic to each other.

To extend the barcode counting function $b_\epsilon(L, L')$ to the situation where L and L' need not be transverse, set

$$b_\epsilon(L, L') := \liminf_{\tilde{L}' \to L'} b_\epsilon(L, \tilde{L}') \in \mathbb{Z}. \qquad (4.1)$$

Here the limit is taken over Lagrangian submanifolds $\tilde{L}' \pitchfork L$ which are Hamiltonian isotopic to L' and converge to L' in the C^∞-topology (or at least in the C^1-topology). Note that, as a consequence, $d_H(\tilde{L}', L') \to 0$. By (3.10), we could alternatively require that $\tilde{L}'$ is Hamiltonian isotopic to L, transverse to L' and converges to L. Furthermore, since $b_\epsilon(L, L') \in \mathbb{Z}$, the limit in (4.1) is necessarily attained, i.e., there exists $\tilde{L}'$ arbitrarily close to L' such that $b_\epsilon(L, L') = b_\epsilon(L, \tilde{L}')$.

Moreover, when $\epsilon \notin \bar{\mathcal{D}}(L, L')$, the right-hand side of (4.1) stabilizes before the limit, i.e.,

$$b_\epsilon(L, L') = b_\epsilon(L, \tilde{L}'), \tag{4.2}$$

when $\tilde{L}' \pitchfork L$ and $\tilde{L}'$ is C^∞-close and Hamiltonian isotopic to L', as is easy to see from (3.13).

With this definition, $b_\epsilon(L, L')$ is monotone increasing as $\epsilon \searrow 0$, and (3.10), (3.12) and (3.13) continue to hold. For instance, to prove the first inequality in (3.13), note first that in (4.1) we could have replaced the lower limit over $\tilde{L}' \to L'$ by the lower limit over $\tilde{L} \to L$. Then

$$b_{\epsilon+\delta}(L, L') := \liminf_{\tilde{L} \to L} b_{\epsilon+\delta}(\tilde{L}, L') \leq \liminf_{\tilde{L} \to L} b_\epsilon(\tilde{L}, L'') =: b_\epsilon(L, L''),$$

as desired.

Furthermore, as has been mentioned in the introduction, $b_\epsilon(L, L')$ gives a lower bound on the number of transverse intersections which is in some sense stable under small perturbations with respect to the Hofer distance; cf. Remark 2.2. To be more precise, assume that Lagrangian submanifolds L, L' and L'' are Hamiltonian isotopic, $L'' \pitchfork L$ and $d_H(L', L'') < \delta/2$. Then, regardless of whether L and L' are transverse or not, we have

$$|L \cap L''| \geq b_\epsilon(L, L'') \geq b_{\epsilon+\delta}(L, L'). \tag{4.3}$$

Here the first inequality follows from (3.11), and in the second we use (3.13) and the fact that $d_H(L', L'') < \delta/2$. These inequalities play a central role in the proof of Theorem A.

Let now $\varphi = \varphi_H \colon M \to M$ be a compactly supported Hamiltonian diffeomorphism. Similarly to Definition 2.1, we have

Definition 4.1 (*Relative Barcode Entropy, II*) The *barcode entropy of φ relative to* (L, L') is

$$\hbar(\varphi; L, L') := \lim_{\epsilon \searrow 0} \hbar_\epsilon(\varphi; L, L') \in [0, \infty],$$

where

$$\hbar_\epsilon(\varphi; L, L') := \limsup_{k \to \infty} \frac{\log^+ b_\epsilon(L, L^k)}{k} \text{ with } L^k := \varphi^k(L').$$

Here, as in Definition 2.1, $\hbar_\epsilon(\varphi; L, L')$ is increasing as $\epsilon \searrow 0$, and hence the limit exists, although *a priori* it could be infinite.

Remark 4.2 The key issue we have to deal with in these definitions is that the difference set $\mathcal{D}(L, L^k)$ can be dense, where as above $L^k := \varphi^k(L')$. When the closure $\bar{\mathcal{D}}(L, L^k)$ is nowhere dense for all k, a simpler approach is available. Namely, then we can require $\epsilon \searrow 0$ not to be in the union of $\bar{\mathcal{D}}(L, L^k)$ for $k \in \mathbb{N}$ and for each k set $b_\epsilon(L, L^k) := b_\epsilon(L, \tilde{L}_k)$, where $\tilde{L}_k$ is a C^∞-small perturbation of L^k, as in (4.2). The resulting definition of the barcode entropy would be literally equivalent to the more general one given above.

Remark 4.3 (*Exact Lagrangians, II*) Continuing Remark 3.2, we note that these definitions and constructions extend word-for-word to the case where L and L' are exact Lagrangian submanifolds.

4.2 Basic properties

In this section we list, for the sake of completeness, some basic properties of barcode entropy.

Proposition 4.4 (Properties of Barcode Entropy) *In the notation and conventions from Sects. 3 and 4.1, we have the following:*

(i) *For every $k \in \mathbb{N}$, we have $\hbar(\varphi^k; L, L') \leq k\hbar(\varphi; L, L')$. In particular, $\hbar(\varphi^k) \leq k\hbar(\varphi)$.*

(ii) *Assume that the products $L_0 \times L_1$ and $L_0' \times L_1'$ in $M_0 \times M_1$ are monotone. Then for Hamiltonian diffeomorphisms $\varphi_0 \colon M_0 \to M_0$ and $\varphi_1 \colon M_1 \to M_1$, we have*

$$\hbar(\varphi_0 \times \varphi_1; L_0 \times L_1, L_0' \times L_1') \leq \hbar(\varphi_0; L_0, L_0') + \hbar(\varphi_1; L_1, L_1').$$

In particular, $\hbar(\varphi_0 \times \varphi_1) \leq \hbar(\varphi_0) + \hbar(\varphi_1)$.

(iii) *We have $\hbar(\varphi; L, L') = \hbar(\varphi^{-1}; L', L)$ and, in particular, $\hbar(\varphi^{-1}) = \hbar(\varphi)$.*

(iv) *For any symplectomorphism $\psi \colon M \to M$,*

$$\hbar(\varphi; L, L') = \hbar(\psi\varphi\psi^{-1}; \psi(L), \psi(L')) \text{ and } \hbar(\varphi) = \hbar(\psi\varphi\psi^{-1}).$$

As a consequence, for every $k \in \mathbb{N}$,

$$\hbar(\varphi; L, L') = \hbar(\varphi; \varphi^k(L), \varphi^k(L')).$$

(v) *For a fixed Hamiltonian diffeomorphism φ, the barcode entropy $\hbar(\varphi; L, L')$ is lower semicontinuous in the pair (L, L') with respect to the Hofer distance. In particular, $\hbar(\varphi; L)$ is lower semicontinuous in L.*

Proof Assertion (i) is a direct consequence of the definition. To prove (ii) recall that the Floer complex of the pair $(L_0 \times L_1, L_0' \times L_1')$ is the tensor product of the Floer complexes of (L_0, L_0') and (L_1, L_1') over Λ; see, e.g., [64, Sect. 2.6] and [80]. Then a singular value decomposition for the product is obtained by taking the "product" of singular value decompositions of the factors in a self-evident way. As a consequence, every pair of bars $\beta_0 \in \mathcal{B}(L_0, L_0')$ and $\beta_1 \in \mathcal{B}(L_1, L_1')$ gives rise to two bars of length $\min\{\beta_0, \beta_1\}$ in $\mathcal{B}(L_0 \times L_1, L_0' \times L_1')$ when both bars are finite. If one or both bars in a pair are infinite, the pair gives rise to one bar of length $\min\{\beta_0, \beta_1\}$. Therefore,

$$b_\epsilon(L_0 \times L_1, L_0' \times L_1') \leq 2b_\epsilon(L_0, L_0') \cdot b_\epsilon(L_1, L_1'),$$

which proves (ii).

Assertion (iii) follows from (3.12) and the Poincaré duality, (3.10). The first identity in (iv) also follows from (3.12). The second identity is a consequence of the fact that when φ is non-degenerate φ and $\psi\varphi\psi^{-1}$ have isomorphic Floer complexes and hence the same barcode. By continuity, $\mathcal{B}(\varphi) = \mathcal{B}(\psi\varphi\psi^{-1})$ even when φ is degenerate. The last one follows from the first identity by setting $\psi = \varphi^k$ and using the fact that φ commutes with φ^k.

To prove (v), it suffices to show that

$$\hbar(\varphi; \tilde{L}, \tilde{L}') \geq \hbar_{4\delta}(\varphi; L, L'), \tag{4.4}$$

whenever $d_H(L, \tilde{L}) < \delta$ and $d_H(L', \tilde{L}') < \delta$. Thus assume that $\tilde{L}' = \psi(L')$, where $\|\psi\|_H < \delta$. Then $d_H(\varphi^k, \varphi^k\psi) < \delta$, and setting $\tilde{L}^k = \varphi^k(\tilde{L}')$ and $L^k = \varphi^k(L')$ as above, we have $d_H(L^k, \tilde{L}^k) < \delta$. Therefore, by (3.13),

$$\hbar_\epsilon(\varphi; \tilde{L}, \tilde{L}') \geq \hbar_{\epsilon+4\delta}(\varphi; L, L').$$

Passing to the limit as $\epsilon \searrow 0$, we obtain (4.4). $\qquad\qquad\square$

Remark 4.5 By analogy with topological entropy, we would expect (i) and (ii) to actually be equalities. Moreover, by Theorem C, $\hbar(\varphi^k) = k\hbar(\varphi)$ when M is a surface. Furthermore, in (v) whenever L is wide one can replace the Hofer norm by the γ-norm by the results from [75]. Also recall that, as was shown in [11], $h_{top}(\varphi)$ is lower semicontinuous in φ with respect to the Hofer metric when $\dim M = 2$. Hence, $\hbar(\varphi)$ is also Hofer lower semicontinuous for Hamiltonian diffeomorphisms of surfaces by Theorem C. This observation leads to the question/conjecture if/that this is also true in all dimensions.

We also note that $\hbar(\varphi; L, L')$, with φ fixed, is quite sensitive to deformations of L and L' even by a Hamiltonian isotopy; cf. Example 2.9.

5 From barcode entropy to topological entropy

Our goal in this section is to prove Theorem A and further explore some of its consequences. We will establish a slightly more general result.

5.1 Generalization of Theorem A

With the notation from Sects. 2.1 and 4.1, we have the following result generalizing Theorem A to pairs of Lagrangian submanifolds.

Theorem 5.1 *Let L_0 and L_1 be closed Lagrangian submanifolds of a symplectic manifold M and let $\varphi \colon M \to M$ be a compactly supported Hamiltonian C^∞-diffeomorphism. Assume that L_0 and L_1 are monotone, Hamiltonian isotopic and $N_{L_0} \geq 2$. Then*

$$\hbar(\varphi; L_0, L_1) \leq h_{top}(\varphi). \tag{5.1}$$

Taking $L = L_0 = L_1$, we obtain Theorem A. We emphasize that here M need not be compact, but then it has to have sufficiently "tame" structure at infinity (e.g., convex) so that the Gromov compactness theorem holds and φ is required to be compactly supported; see, e.g., [82]. Note also that, as a consequence of Theorem 5.1, $\hbar(\varphi; L_0, L_1) < \infty$ which is otherwise not obvious.

Remark 5.2 (*Exact Lagrangians, III*) Theorem 5.1 holds in some other situations. For instance, one of them is when L_0 and L_1 are closed exact Lagrangian submanifolds in an exact convex symplectic manifold; cf. Remarks 3.2 and 4.3.

5.2 Proof of Theorem 5.1

We break down the proof into three subsections. The first two of them—Sects. 5.2.1 and 5.2.2—focus on the machinery of Lagrangian tomographs which the proof relies on; the actual proof is given in Sect. 5.2.3. Throughout the argument we have an auxiliary Riemannian metric on M fixed.

5.2.1 Lagrangian tomographs and Crofton's inequality

The notion of Lagrangian tomograph and a variant of Crofton's inequality, originating in integral geometry, are the key tools used in the proof of the theorem. The framework described

in this section is essentially contained in [13, 14] in the setting very close to ours, and we include the proofs only for the sake of completeness. (See also [17, 28, 93] for other applications in the context of symplectic dynamics). Furthermore, our setting is similar to double fibrations utilized in integral geometry for generalizing and proving Crofton's formula; see [4, 5, 46].

Let L be a closed manifold, B a compact manifold possibly with boundary and ds a smooth measure on B. In the situation we are interested in, B is the closed d-dimensional ball B^d and ds is the Lebesgue measure, and L will be the Lagrangian submanifold L_0. Denote by $\pi : E = B \times L \to B$ the projection to the first factor. We denote a point in B by s. Furthermore, let

$$\Psi : E \to M$$

be a submersion onto its image where M is a Riemannian manifold (see Remark 5.4). This manifold need not be compact but, of course, $\Psi(E) \subset M$ is, since E is compact. We require $\Psi_s := \Psi|_{s \times L}$ to be an embedding for all s, and hence $L_s := \Psi_s(L)$ is a smooth closed submanifold of M.

In the spirit of integral geometry we will refer to Ψ as a *tomograph* and call L_0 the *core* of the tomograph. We say that Ψ is a *Lagrangian tomograph* if all submanifolds L_s are Lagrangian and Hamiltonian isotopic to each other.

Finally, let L' be a closed submanifold of M with

$$\operatorname{codim} L' = \dim L.$$

Since Ψ is a submersion, $\Psi_s \pitchfork L'$ for almost all $s \in B$. Hence,

$$N(s) := |L_s \cap L'|$$

is a locally constant function on the complement to a zero measure closed subset of B. As a consequence, N is an integrable function on B.

Lemma 5.3 *(Crofton's inequality) We have*

$$\int_B N(s)\, ds \leq const \cdot \operatorname{vol}(L'), \tag{5.2}$$

where the constant depends only on ds, Ψ and the metric on M, but not on L'.

Remark 5.4 Perhaps a clarification is due on how the submersion condition is to be interpreted at ∂B. A way, which is sufficient and convenient for our purposes, is to assume that Ψ is defined on a slightly larger space $B' \times L$ (or just $E' \supset E$) where B' is an open enlargement of B (or E' is an open enlargement of E).

Lemma 5.3 is proved in [13, 14]. The argument is simple and short, and we include it below for the sake of completeness. This general framework and Lemma 5.3 are also very much in the spirit of the Gelfand transform in integral geometry and various versions of Crofton's formula; see, e.g., [4, 5] and references therein. The key difference is that here and in [13, 14] Ψ is required to be only a submersion, not a fibration. Furthermore, we note that in this generality, in contrast with the actual Crofton formula, one cannot expect an inequality going in the direction opposite of (5.2). Indeed, the graph of a smooth map or even of a diffeomorphism between two closed manifolds can have arbitrarily large volume but it intersects every vertical slice at only one point.

Proof of Lemma 5.3 Set $\Sigma = \Psi^{-1}(L')$. This is a smooth submanifold of E and

$$\text{codim } \Sigma = \text{codim } L' = \dim L, \text{ i.e., } \dim \Sigma = \dim B.$$

By construction,

$$|(s \times L) \cap \Sigma| = |L_s \cap L'| = N(s). \tag{5.3}$$

In the proof it will be convenient to equip E with two different auxiliary metrics: the first metric adapted to π and the second metric to Ψ.

We begin by fixing some metrics on B and L, and assuming first that $E = B \times L$ carries the product metric and ds is the Riemannian volume form or, to be more precise, the volume density. (It would be sufficient to require $D\pi$ to be an isometry on the normals to the fibers). Then, by (5.3),

$$\int_B N(s) \, ds = \int_\Sigma \pi^* ds \le \text{vol}(\Sigma).$$

Here, $\pi^* ds$ is the pull-back measure or the pull-back density, but not the pull-back differential form. The last inequality is a consequence of the fact that $D\pi_x : T_x E \to T_{\pi(x)} B$, $x \in E$, is an orthogonal projection along the fiber, and hence, when restricted to $T_x \Sigma$, it can only decrease the dim B-dimensional volume. As a consequence, for an arbitrary metric on E and an arbitrary smooth measure ds on B, we have

$$\int_B N(s) \, ds \le const \cdot \text{vol}(\Sigma). \tag{5.4}$$

Next, let us equip E with a metric such that the restriction of $D\Psi$ to the normals to the fibers of Ψ (i.e., the inverse images $\Psi^{-1}(y)$, $y \in M$) is an isometry. Then, by Fubini's theorem or, more specifically, the coarea formula (see [23, Sect. 13.4.3]), we have

$$\text{vol}(\Sigma) = \int_{L'} \text{vol}\left(\Psi^{-1}(y)\right) dy|_{L'} \le \max_{y \in \Psi(E)} \text{vol}\left(\Psi^{-1}(y)\right) \cdot \text{vol}(L'), \tag{5.5}$$

where in the first equality $dy|_{L'}$ stands for the induced volume form on L'. Thus

$$\text{vol}(\Sigma) \le const \cdot \text{vol}(L'). \tag{5.6}$$

For an arbitrary metric on E, (5.6) still holds, albeit with a different constant. Combining (5.4) and (5.6), we obtain (5.2). □

Remark 5.5 Note that by (5.5) and since the constant in (5.4) is independent of Ψ, the constant in (5.2) is continuous in Ψ in the C^1-topology. In our application $L' = L^k$, i.e., it ranges through a countable collection of submanifolds of M. Then, by C^1-perturbing Ψ slightly, we can ensure that $N(s)$ is finite for all s. (This fact is inessential for our purposes, and we omit a proof). Hence, the functions $N_k(s)$ from Sect. 5.2.3 can also be made finite for all s.

5.2.2 Existence of Lagrangian tomographs

As the second step of the proof we establish in this section the existence of Lagrangian tomographs. Let $L = L_0$ be a closed Lagrangian submanifold of M.

Lemma 5.6 *A Lagrangian tomograph with core L and $\dim B = d$ exists if and only if L admits an immersion into $\mathbb{R}^d$.*

Proof By the Weinstein tubular neighborhood theorem it is sufficient to prove the lemma when L is the zero section in $M = T^*L$.

Let $\iota \colon L \to \mathbb{R}^d$ be an immersion, and

$$f_s = s_1 g_1 + \cdots + s_d g_d,$$

where $s = (s_1, \ldots, s_d) \in \mathbb{R}^d$ are the coordinate functions on $\mathbb{R}^d$ and $g_i := s_i \circ \iota$ are the restrictions of the coordinate functions to L. Let us now require s to be in a ball $B \subset \mathbb{R}^d$ centered at the origin. Then, setting $\Psi_s(x) = df_s(x)$, $x \in L$, we obtain a map $\Psi \colon B \times L \to T^*L$. It is easy to see that the condition that ι is an immersion is equivalent to that Ψ is a desired tomograph.

Indeed, the immersion condition is equivalent to that $dg_1, \ldots, dg_d$ generate T_x^*L at every point $x \in L$. In other words, the map

$$D\Psi_{(0,x)} \colon T_0 B \oplus T_x L \to T_{(0,x)} T^*L = T_x^*L \oplus T_x L$$

is onto. Therefore, Ψ is a submersion when B is sufficiently small if and only if $(g_1, \ldots, g_d) \colon L \to \mathbb{R}^d$ is an immersion. (This observation is already contained in, e.g., [57]). It is clear from the construction that the submanifolds L_s are embedded.

Conversely, assume that $\Psi \colon L \times B \to T^*L$ is a tomograph with $\Psi_0 = id$. Then the linearization $A = \partial \Psi_s / \partial s$ is a linear map from $T_0 B$ to the space of exact sections of T^*L. Pick some functions $\{g_\ell\}$ so that $dg_\ell = A(e_\ell)$ where $e_1, \ldots, e_d$ is a basis in $T_0 B$. Then $\iota = (g_1, \ldots, g_d)$ is an immersion $L \to \mathbb{R}^d$; cf. [57]. This completes the proof of the lemma. □

5.2.3 From barcodes to Lagrangian volume to entropy

Without loss of generality, we may require that $\hbar(\varphi; L_0, L_1) > 0$—otherwise there is nothing to prove. We denote the Riemannian volume of $L^k := \varphi^k(L_1)$ by $\mathrm{vol}(L^k)$.

Fix $\epsilon > 0$ and $\alpha < \hbar_{2\epsilon}(\varphi; L_0, L_1)$. Then

$$b_{2\epsilon}\big(L_0, L^{k_i}\big) \geq const \cdot 2^{\alpha k_i} \tag{5.7}$$

for some sequence $k_i \to \infty$. (Here and in what follows, the value of the constant *const* can change from one formula to another and even in different parts of the same formula). By Yomdin's theorem (see [101] and also the survey [55]), it is sufficient to show that

$$\mathrm{vol}(L^{k_i}) \geq const \cdot 2^{\alpha k_i} \tag{5.8}$$

with the constant independent of i, but possibly depending on α and ϵ. Indeed, then $\alpha \leq h_{top}(\varphi)$. Passing to the limit as $\epsilon \to 0$ and $\alpha \to \hbar(\varphi; L_0, L_1)$, we obtain (5.1).

To prove (5.8), pick a Lagrangian tomograph with core L_0. Thus we have a family of Lagrangian submanifolds L_s smoothly parametrized by the closed d-dimensional ball $B^d = B^d(r)$ of radius $r > 0$ (for some large d). By shrinking B if necessary, we can ensure that these submanifolds have the following properties:

(i) The Lagrangian submanifolds L_s are Hamiltonian isotopic to L_0 and the Hofer distance between L_0 and L_s is small:

$$d_H(L_0, L_s) < \epsilon/2 \tag{5.9}$$

(In fact, L_s can even be taken C^∞-close to L_0).

(ii) The Lagrangian submanifold L_s is transverse to L^k for all k and almost all $s \in B^d$. Let

$$N_k(s) := |L_s \cap L^k|$$

be the number of intersections of L_s and L^k. Then $N_k(s)$ is a measurable function, finite for almost all s. (In fact, we can even have $N_k(s)$ finite for all $s \in B^d$; see Remark 5.5). Furthermore, by Crofton's inequality (Lemma 5.3), we have

$$\int_{B^d} N_k(s)\,ds \leq const \cdot \mathrm{vol}(L^k), \tag{5.10}$$

where ds is the standard Lebesgue measure on B^d.

Note that all conditions in (i) and (ii), other than (5.9), are satisfied automatically; see Sect. 5.2.1. As we have already pointed out, to guarantee (5.9) we can simply shrink B.

Then, whenever $L_s \pitchfork L^k$, we have

$$N_k(s) \geq b_\epsilon(L_s, L^k) \geq b_{2\epsilon}(L_0, L^k).$$

This is an immediate consequence of (4.3) with L replaced by L^k, L' replaced by L_0 and L'' replaced by L_s. Hence, by (5.7),

$$N_{k_i}(s) \geq const \cdot 2^{\alpha k_i}$$

for almost all $s \in B^d$, and (5.8) follows from (5.10), which finishes the proof of Theorem 5.1.

We emphasize that this argument does not require L_0 and L^k to be transverse, but only that $L_s \pitchfork L^k$ which holds automatically for almost all s regardless of whether $L_0 \pitchfork L^k$ or not.

5.3 Entropy and the graph volume growth

A consequence of the proof of Theorem 5.1 and Theorem C is a relation between the barcode and topological entropy of φ and the exponential growth rate of the volume of the graph of φ^k. To be more precise, assume that M is compact and fix a Riemannian metric on M. Denote by $\Gamma_k \subset M \times M$ the graph of φ^k and by $\mathrm{vol}(\Gamma_k)$ its volume. Set

$$h_{vol}(\varphi) = \limsup_{k \to \infty} \frac{\log^+ \mathrm{vol}(\Gamma_k)}{k}.$$

Corollary 5.7 *Let φ be a Hamiltonian C^∞-diffeomorphism of a compact monotone symplectic manifold. Then*

$$\hbar(\varphi) \leq h_{vol}(\varphi) \leq h_{top}(\varphi).$$

Here the first inequality is an immediate consequence of the proof of Theorem 5.1, and the second one follows from Yomdin's theorem, [101], and holds for general C^∞-diffeomorphisms. Combining these inequalities with Theorem C, we obtain

Corollary 5.8 *Let φ be a Hamiltonian C^∞-diffeomorphism of a closed surface. Then*

$$h_{vol}(\varphi) = h_{top}(\varphi).$$

Surprisingly, this equality appears to be new. There are however similar results in the holomorphic setting; see [56] and the comments therein by S. Cantat for further references.

6 From horseshoes to barcode entropy

Our goal in this section is to prove Theorems B and C. Throughout the proofs, $\varphi^t := \varphi^t_H$ will stand for the time-dependent Hamiltonian flow of $H\colon S^1 \times M \to \mathbb{R}$, $S^1 = \mathbb{R}/\mathbb{Z}$, on a (monotone) symplectic manifold M and $\varphi := \varphi^1_H$ will be the Hamiltonian diffeomorphism generated by H.

6.1 Crossing energy, the proof of Theorem B and the γ-norm

In this section we prove Theorem B and also briefly touch upon an application of the proof to establishing a lower bound on the γ-norm of the iterates in the presence of a hyperbolic set. We break down the proof of the theorem in a few simple steps, some of which might be of independent interest.

6.1.1 Generalities and terminology

The iterate Hamiltonian diffeomorphism φ^k is the time-k map in the Hamiltonian isotopy φ^t_H generated by H. In what follows, when working with the Floer equation for this iterate, it is convenient to denote the Hamiltonian by $H^{\sharp k}$ and refer to it, somewhat abusing terminology, as the *iterated Hamiltonian*; cf. [50, 52]. We emphasize that $H^{\sharp k}$ is the same Hamiltonian as H, but now viewed as k-periodic in time. Likewise, the solutions of the Floer equation, (6.1), are allowed to be k-periodic in time rather than 1-periodic or, more generally, defined on a closed domain $\Sigma \subset \mathbb{R} \times S^1_k$, where $S^1_k = \mathbb{R}/k\mathbb{Z}$, rather than a domain in $\mathbb{R} \times S^1$. There are, of course, other Hamiltonians, with easily adjustable period, generating φ^k and giving rise to the same filtered Floer complex, but this is a natural and convenient choice from the dynamics perspective. Moreover, this choice becomes essential for the proof of Theorem 6.1; see [52].

Thus, consider solutions $u\colon \Sigma \to M$ of the Floer equation

$$J\partial_s u = \partial_t u - J\nabla H^{\sharp k} \tag{6.1}$$

for the iterated Hamiltonian $H^{\sharp k}\colon S^1_k \times M \to \mathbb{R}$ with $S^1_k = \mathbb{R}/k\mathbb{Z}$, where $\Sigma \subset \mathbb{R} \times S^1_k$ is a closed domain, i.e., a closed subset with non-empty interior and J is a background k-periodic in time almost-complex structure. By definition, the energy of u is

$$E(u) = \int_\Sigma \|\partial_s u\|^2 \, ds dt.$$

where $\|\cdot\|$ stands for the norm with respect to $\langle\cdot,\cdot\rangle = \omega(\cdot, J\cdot)$, and hence $\|\cdot\|$ depends on J. Recall that when $\Sigma = \mathbb{R} \times S^1_k$ and u is asymptotic to k-periodic orbits x at $-\infty$ and y at ∞, we have

$$E(u) = \mathcal{A}(x) - \mathcal{A}(y),$$

i.e., $E(u)$ is the action difference between x and y. Here we treat x and y as capped k-periodic orbits of H with the capping of y obtained by "attaching" u to the capping of x; see Sect. 3.2.1.

Throughout the proof, it will be convenient to work with the extended phase space $\tilde{M} = S^1 \times M$ with $S^1 = \mathbb{R}/\mathbb{Z}$. The time-dependent flow φ^t lifts as the genuine flow $\tilde{\varphi}^t$ on $\tilde{M}$ given by

$$\tilde{\varphi}^t(\theta, p) = \big(\theta + t, \varphi^t(p)\big)$$

generated by the vector field $\partial_\theta + X_H$, where in the first term t is viewed as an element of $S^1 = \mathbb{R}/\mathbb{Z}$. Likewise, any map $z\colon \mathbb{R} \to M$ or $z\colon S_k^1 \to M$ lifts to the map $\tilde{z}(t) = (t, z(t))$, and in a similar vein a solution u of the Floer equation lifts to a map $\tilde{u}\colon \Sigma \to \tilde{M}$. If u is asymptotic to x and y, the lift $\tilde{u}$ is asymptotic to $\tilde{x}$ and $\tilde{y}$ in the natural sense. In what follows, a lift from M to $\tilde{M}$ will always be indicated by the tilde and we will identify M with $\{0\} \times M$.

A loop $x\colon S_k^1 \to M$ is a k-periodic orbit of φ^t if and only if its lift $\tilde{x}$ is a k-periodic orbit of $\tilde{\varphi}^t$ and if and only if the sequence $\hat{x} = \{x_i := x(i) \mid i \in \mathbb{Z}_k \subset S_k^1\}$ formed by the intersections of $\tilde{x}$ with the cross-section M is a k-periodic orbit of φ.

6.1.2 Crossing energy

Next let us recall the crossing energy theorem, [52, Thm. 6.1] (see also [50]), which is crucial to the proof. Let $K \subset M$ be a compact invariant set of a Hamiltonian diffeomorphism φ of M. Recall that K is said to be *locally maximal* or isolated (as an invariant set) or basic if there exists a neighborhood $U \supset K$ such that for no initial condition $p \in U \setminus K$ the orbit through p is contained in U, i.e., there exists $k \in \mathbb{Z}$, possibly depending on p, such that $\varphi^k(p) \notin U$. The neighborhood U is called an isolating neighborhood of K. Then any neighborhood of K contained in U is also isolating, and hence such neighborhoods can be made arbitrarily small. In other words, whenever $U \supset V \supset K$ and U is an isolating neighborhood and V is open, V is also an isolating neighborhood. For a flow, a locally maximal set is defined in a similar fashion.

The set K naturally lifts to an invariant set $\tilde{K} \subset \tilde{M}$ of the flow $\tilde{\varphi}^t$, which is the union of the integral curves through $K = \{0\} \times K$. (Since K is invariant it suffices to take only $t \in [0, 1]$.) The set K is locally maximal for φ if and only if $\tilde{K}$ is locally maximal for the flow.

As in Sect. 6.1.1, let $u\colon \Sigma \to M$ be a solution of the Floer equation, (6.1), where $\Sigma \subset \mathbb{R} \times S_k^1$ is a closed domain. We say that u is *asymptotic* to K at ∞ (or at $-\infty$) if for any neighborhood $\tilde{U}$ of $\tilde{K}$ there is a half-cylinder $[s_{\tilde{U}}, \infty) \times S_k^1$ (or $(-\infty, s_{\tilde{U}}] \times S_k^1$) in Σ which is mapped into $\tilde{U}$ by $\tilde{u}$. For instance, u is asymptotic to K whenever $u(s, \cdot)$ uniformly converges as $s \to \infty$ or $s \to -\infty$ to a k-periodic orbit x with $x(0) \in K$ (but not necessarily with $x(t) \in K$ for all $t \in S_k^1$). In this case, abusing terminology, we will also say that u is asymptotic to the k-periodic orbit $\hat{x} := \{x_i := x(i) \mid i \in \mathbb{Z}_k\}$ of φ. We emphasize that here the domain Σ of u need not be a cylinder, although to be asymptotic to K it must contain a half-cylinder.

Finally, fix a (sufficiently small) isolating neighborhood $\tilde{U}$ of $\tilde{K}$. Set $\partial\tilde{U} := \mathrm{closure}(\tilde{U}) \setminus \tilde{U}$.

Theorem 6.1 (Crossing Energy Theorem, Thm. 6.1 in [52]) *Fix a 1-periodic in time almost complex structure J on M. Let J' be a k-periodic in time almost complex structure on M which is sufficiently C^∞-close to J, depending on k, uniformly on U. Furthermore, let $u\colon \Sigma \to M$, where $\Sigma \subset \mathbb{R} \times S_k^1$, be a solution of the Floer equation for J' and $H^{\sharp k}$, asymptotic to K as $s \to \infty$ or $s \to -\infty$, and such that*

(a) either $\partial\Sigma \neq \emptyset$ and $\tilde{u}(\partial\Sigma) \subset \partial\tilde{U}$
(b) or $\Sigma = \mathbb{R} \times S_k^1$ and $\tilde{u}(\Sigma) \not\subset \tilde{U}$.

Then there exists a constant $c_\infty > 0$, independent of k, J', u and Σ such that

$$E(u) > c_\infty. \tag{6.2}$$

Here we are mainly interested in Case (b) of the theorem. It is easy to see that this case is a consequence of the more general Case (a) which was actually established in [52]. However, Case (b) can also be proved directly by an argument which is considerably simpler than the original proof therein; cf. Remark 6.5.

Remark 6.2 It is worth keeping in mind that the lower bound c_∞ depends on the choice of an isolating neighborhood $\tilde{U}$ of $\widetilde{K}$: a smaller neighborhood might necessitate a smaller lower bound. The threshold on how close J' and J need to be for (6.2) to hold depends on k. Finally, as readily follows from the proof, the lower bound c_∞ can also be chosen to be stable with respect to C^∞-small perturbations of $H^{\sharp k}$, i.e., so that (6.2) holds for solutions of the Floer equation for all k-periodic Hamiltonians C^∞-close to $H^{\sharp k}$.

6.1.3 Energy bound for Floer trajectories asymptotic to *K*

Consider Floer trajectories $u\colon \Sigma = \mathbb{R} \times S_k^1 \to M$ for some k-periodic almost complex structure J' sufficiently close to a fixed 1-periodic almost complex structure J as in Theorem 6.1 and asymptotic to k-periodic orbits x and y of φ^t with $x(0) \in K$. (It does not matter if u is a asymptotic to x at ∞ or $-\infty$, and whether $y(0)$ is in K or not). The key to the proof of Theorem B is the following result which is an easy consequence of Theorem 6.1 and the Anosov Closing Lemma, [70, Thm. 6.4.15].

Proposition 6.3 *Assume that K is a locally maximal hyperbolic invariant set of φ. Then*

$$E(u) = |\mathcal{A}(x) - \mathcal{A}(y)| > \epsilon_K, \;\; unless \; E(u) = 0, \tag{6.3}$$

for some constant $\epsilon_K > 0$, independent of u and x and y, and also of k and J' as long as J' is sufficiently C^∞-close to J in the class of k-periodic in time almost complex structures on M.

Proof Pick an almost complex structure J' which is sufficiently close to a 1-periodic almost-complex structure J and denote by d the distance on M with respect to an arbitrary Riemannian metric. We will need the following standard fact:

Lemma 6.4 *Let $u\colon \mathbb{R} \times S_k^1 \to M$ be a solution of the Floer equation, (6.1). Assume that $E(u)$ is sufficiently small, i.e., $E(u) < e$ with an upper bound $e > 0$ depending only on (M, ω, J) and H. Then for every $s \in \mathbb{R}$ the set*

$$\hat{z} := \{z_i := u(s, i) \mid i \in \mathbb{Z}_k\}$$

is a periodic η-pseudo-orbit of φ, i.e.,

$$d\big(\varphi(z_i), z_{i+1}\big) < \eta \; for \; all \; i \in \mathbb{Z}_k, \tag{6.4}$$

where we can take

$$\eta = O\big(E(u)^{1/4}\big). \tag{6.5}$$

The key point of this lemma is that every "circle" in a Floer cylinder u for $H^{\#k}$ is in essense an η-pseudo-orbit with $\eta = O\big(E(u)^{1/4}\big)$, provided that $E(u)$ is below a certain threshold $e > 0$ which depends only on (M, ω, J) and H, but not u or k.

Proof Recall that when $E(u)$ is sufficiently small (with an upper bound e depending on M and H but not u and k), we have the pointwise upper bound

$$\|\partial_s u\| \leq const \cdot E(u)^{1/4} = O\big(E(u)^{1/4}\big), \qquad (6.6)$$

where the constant is again independent of u and k and of J' when J' is close to J; see [92, Sect. 1.5] or [53, pp. 542–543] or, for a different proof, [20]. (Note that it is essential here that the domain of u is the entire cylinder $\mathbb{R} \times S_k^1$). Now, (6.4) and (6.5) follow from the Floer equation, (6.1), and the triangle inequality. □

To prove the proposition, we consider two cases depending on the location of the orbit y.

The first case is when $y(0) \notin K$. Then $\tilde{y}$ is not entirely contained in any isolating neighborhood of $\tilde{K}$. Applying Theorem 6.1 (Case b), we obtain (6.3) with $\epsilon_K = c_\infty$.

The second case is when $y(0) \in K$. Then both $\hat{x}$ and $\hat{y} := \{y_i := y(i) \mid i \in \mathbb{Z}_k\}$ are k-periodic orbits of φ in K. Let us assume that u is asymptotic to x at $-\infty$; the other case is handled similarly. We will show that then $x = y$ and $E(u) = 0$ when $E(u)$ is below a certain treshold which depends only on M, J, K and H, but neither on u nor x nor y nor k.

To this end, fix a sufficiently small isolating neighborhood U of K. In particular, we may assume that the Anosov Closing Lemma applies to φ on U; see [70, Thm. 6.4.15]. Then, by Theorem 6.1, $\tilde{u}$ is entirely contained in $\tilde{U}$. Hence, $\hat{z}$ is contained in $\tilde{U} \cap M = U$ for all $s \in \mathbb{R}$. Assume furthermore that $E(u)$ is so small that Lemma 6.4 applies and thus $\hat{z} = \{u(s, i) \mid i \in \mathbb{Z}_k\}$ is a periodic η-pseudo-orbit in U for every s.

Therefore, by the Anosov Closing Lemma, there exists a true periodic orbit $\hat{w}$ in K shadowing $\hat{z}$. Namely, we have $d(z_i, w_i) < C\eta$, $i \in \mathbb{Z}_k$, for some constant $C > 0$, which depends only on U and φ. By [70, Cor. 6.4.10], $\varphi|_K$ is expansive: there is a constant $\delta > 0$ such that any two distinct orbits $\{v_i\}$ and $\{v_i'\}$ of φ in K are at least δ apart, i.e., $d(v_j, v_j') > \delta$ for some $j \in \mathbb{Z}$. It follows that when $E(u)$ and hence η are small enough (e.g., $2C\eta < \delta$), the orbit $\hat{w}$ is unique and depends continuously on $\hat{z}$ and thus on $s \in \mathbb{R}$. Therefore, again since $\varphi|_K$ is expansive, $\hat{w}$ is independent of $s \in \mathbb{R}$. Clearly, when s is close to $-\infty$, we have $\hat{w} = \hat{x}$, and $\hat{w} = \hat{y}$ when s is close to ∞. Thus, $x = y$, and setting $u(\infty, t) = x(t) = y(t)$, we can view u as a C^0-map from $\mathbb{T}^2 = \big(\mathbb{R} \cup \{\infty\}\big) \times S_k^1$ to M. This map is smooth on the complement to $\{\infty\} \times S_k^1$. Furthermore, it is easy to see from (6.6) and Lemma 6.4 that for every $s \in \mathbb{R}$ the loop $t \mapsto u(s, t)$ is C^0-close to the loop $x = y$ pointwise uniformly in s. Hence, the loop $s \mapsto u(s, t)$ lies in a small neighborhood of $x(t)$. As a consequence, u contracts to x in M, and hence $E(u) = 0$. Indeed, $E(u)$ is the difference of actions of capped periodic orbits. Since $x = y$, this difference is the integral of ω over u. The cycle represented by u is homologous to zero, and hence the integral is zero. □

Remark 6.5 The proof of Case (a) of Theorem 6.1 in [52] relies on a variant of the Gromov Compactness Theorem from [41]. As has already been mentioned, Case (b) used here follows from Case (a), but it can also be proved directly with somewhat simpler tools under the slightly more restrictive requirement that, as in Lemma 6.4, J' is sufficiently C^∞-close to J in the class of k-periodic in time almost complex structures on M. Namely, arguing by contradiction, assume that $E(u)$ can be arbitrarily small, i.e., there exists a sequence $u_k \colon \mathbb{R} \times S_k^1 \to M$, where $k = k_i \to \infty$ with $E(u_k) \to 0$ such that the image of $\tilde{u}_k$ is not entirely contained in the closure of $\tilde{U}$. Consider the largest half-cylinder in $\mathbb{R} \times S_k^1$ whose image is contained in the closure. By Lemma 6.4, the restriction of u_k to the boundary of this cylinder gives rise to an η-pseudo-orbit with $\eta = O\big(E(u_k)\big)$ passing through a point close to ∂U. Thus we obtain longer and longer two-directional η-pseudo-orbits with $\eta \to 0$ passing through a point close to ∂U. Passing to the limit as $E(u_k) \to 0$, we obtain an entire orbit of φ which is contained in

the closure of U, but not in K. The argument is spelled out in detail in a very similar context in [31, 32].

Remark 6.6 Since this work appeared as a preprint, variants of Proposition 6.3 for geodesic and Reeb flows were proved in [54] and in [31, 32], leading to Reeb analogues to Theorems B and C and also of multiplicity results along the lines of [50]. A version of Proposition 6.3 for Lagrangian intersections has been recently proved in [84].

6.1.4 Proof of Theorem B

Recall that a compact invariant set K of φ is said to be *locally maximal* (or basic) if there exists a neighborhood $U \supset K$, called an *isolating neighborhood*, such that K is the maximal invariant subset of U or, in other words, $x \in K$ whenever the entire orbit $\{\varphi^k(x) \mid k \in \mathbb{Z}\}$ through x is contained in U or, equivalently,

$$K = \bigcap_{k \in \mathbb{Z}} \varphi^k(U).$$

By [16, Thm. 3.3] and the Variational Principle, [70, Thm. 4.5.3], for every hyperbolic set K there exists a locally maximal hyperbolic set K' with nearly the same entropy. (In fact, the hyperbolicity condition is essential for [16, Thm. 3.3] but not for the Variational Principle). To be more precise, for every $\delta > 0$, one can find K' such that $h_{top}(\varphi|_{K'}) \geq h_{top}(\varphi|_K) - \delta$. As a consequence, we can assume the hyperbolic set K in the theorem to be locally maximal.

Denote by $p(k)$ the number of k-periodic points of $\varphi|_K$. Since K is hyperbolic, by [70, Thm. 18.5.1], we have

$$h_{top}(\varphi|_K) = \limsup_{k \to \infty} \frac{\log^+ p(k)}{k}. \tag{6.7}$$

Hence, to prove the theorem, it is sufficient to show that

$$b_\epsilon(\varphi^k) \geq p(k)/2 \tag{6.8}$$

when $\epsilon > 0$ is small. We will use Proposition 6.3 to prove this for $\epsilon < \epsilon_K$.

Fix $k \geq 1$ and recall from Sect. 4.1 that the limit (4.1) in the definition of $b_\epsilon(\varphi^k)$ is attained, i.e, there exists non-degenerate, arbitrarily C^∞-small perturbations ψ of φ^k such that

$$b_\epsilon(\varphi^k) = b_\epsilon(\psi). \tag{6.9}$$

All k-periodic points of φ in K are non-degenerate and hence persist as fixed points of ψ or, equivalently, as k-periodic orbits $S_k^1 \to M$ of the Hamiltonian flow ψ^t. Moreover, by Proposition 6.3, $E(u) > \epsilon_K$ for any Floer trajectory u asymptotic to such an orbit x of ψ when ψ is sufficiently C^∞-close to φ^k.

Let $\mathcal{K}$ be the collection of fixed points of ψ corresponding to the k-periodic points of φ in K. There are exactly $p(k)$ of them: $|\mathcal{K}| = p(k)$. By Proposition 6.3 every such fixed point is ϵ-isolated in the sense of Sect. 3.3.2 and, by Proposition 3.8 and (6.9),

$$b_\epsilon(\varphi^k) = b_\epsilon(\psi) \geq |\mathcal{K}|/2 = p(k)/2,$$

which proves (6.8) and completes the proof of the theorem. □

6.1.5 Application to the γ-norm of the iterates

The proof of Theorem B has an application to symplectic topology, relating the behavior of the γ-norm $\gamma\left(\varphi^k\right)$ and the Hofer norm $\|\varphi^k\|_H$ as $k \to \infty$ to the existence of hyperbolic locally maximal invariant subsets. Namely, recall from [75, Thm. A] that

$$\beta_{\max}(\varphi) \leq \gamma(\varphi) \leq \|\varphi\|_H,$$

where $\beta_{\max}$ is the boundary depth; see Sect. 3.3 and [97, 98]. (Strictly speaking, the Floer complex in [75] is restricted to the contractible free homotopy class by a background assumption, but the inequality still holds without this restriction). Therefore, $\gamma\left(\varphi^k\right) > \epsilon$ whenever $b_\epsilon\left(\varphi^k\right)$ is large enough to guarantee that there is a finite bar of length greater than ϵ. For instance, since the Floer persistence module has exactly $\dim \mathrm{H}_*(M; \mathbb{F})$ infinite bars, to have the sequence $\gamma\left(\varphi^k\right)$ bounded away from zero it suffices to ensure that $b_\epsilon\left(\varphi^k\right) > \dim \mathrm{H}_*(M; \mathbb{F})$ for some $\epsilon > 0$ and all large k.

This is the case, for instance, when φ has a locally maximal hyperbolic set K such that for each large k the set K contains more than $\dim \mathrm{H}_*(M; \mathbb{F})$ periodic points. For instance, K can be a horseshoe in the sense of [70, Sect. 6.5] or, more generally, a hyperbolic locally maximal invariant subset such that $p(k) \to \infty$ in the notation of Sect. 6.1.4 or just a collection of more than $\dim \mathrm{H}_*(M; \mathbb{F})$ hyperbolic fixed points. Alternatively, it is enough to require that $\dim \mathrm{H}_{odd}(M; \mathbb{F}) = 0$ and K contains a periodic orbit of index $m(k)$ such that $m(k) - n$ is odd. (For instance, this is so when K is a positive hyperbolic fixed point and $M = S^2$. To prove Proposition 6.7 below in this case, it is useful to (re)introduce the $\mathbb{Z}_2$-grading on CF(φ)). Summarizing these observations we obtain the following.

Proposition 6.7 *Assume that φ has a locally maximal hyperbolic set K such that for every sufficiently large k, one of the following two conditions is met:*

(i) *K contains more than $\dim \mathrm{H}_*(M; \mathbb{F})$ k-periodic points; or*

(ii) *$\dim \mathrm{H}_{odd}(M; \mathbb{F}) = 0$ and K contains a k-periodic orbit of index $m(k)$ such that $m(k) - n$ is odd.*

Then the sequences $\gamma\left(\varphi^k\right)$ and $\left\|\varphi^k\right\|_H$ are bounded away from zero.

This result is new, although not entirely unexpected to the authors. Of course, the assertion of Theorem B is much stronger than this proposition; for it guarantees exponential growth of $b_\epsilon\left(\varphi^k\right)$ when $h_{top}(K) > 0$. However, the proposition is more general. We also note that $\gamma\left(\varphi^k\right)$ is not bounded away from zero unconditionally when $\varphi^k \neq id$ for all $k \in \mathbb{N}$. For instance, $\gamma\left(\varphi^k\right)$ can be arbitrarily small for pseudo-rotations of $\mathbb{CP}^n$; see [52]. We revisit connections between the γ-norm and dynamics and refine Proposition 6.7 in [29, 30]. Overall, little seems to be known about the behavior of the sequence $\gamma\left(\varphi^k\right)$.

6.2 Proof of Theorem C

It suffices to show that

$$\hbar(\varphi) \geq h_{top}(\varphi) \tag{6.10}$$

whenever $\dim M = 2$; for $\hbar(\varphi) \leq h_{top}(\varphi)$ by Theorem A.

When M is a closed surface,

$$h_{top}(\varphi) = \sup \left\{h_{top}(\varphi|_K) \mid K \text{ is hyperbolic}\right\}, \tag{6.11}$$

as a consequence of the results in [68], and (6.10) follows from Theorem B. $\square$

Remark 6.8 Note that in (6.11) it is enough to assume that φ is only $C^{1+\alpha}$-smooth. Furthermore, we can also require K to be locally maximal already by the results from [68]. To see this, first recall from [18, Sect. 15.4] that in dimension two

$$h_{top}(\varphi) = \sup \left\{ h_{top}(\varphi|_K) \mid K \text{ is a hyperbolic horseshoe} \right\}. \tag{6.12}$$

(This is a consequence of [70, Thm. S.5.9], based on [68], and two standard results: the Variational Principle, [70, Thm. 4.5.3], and the Ergodic Decomposition Theorem, [70, Thm. 4.1.12], which allows one to restrict the supremum in the Variational Principle to ergodic measures only).

Thus, to obtain (6.11) with K locally maximal from (6.12), we just need to make sure that in this context hyperbolic horseshoes are locally maximal. The definition used in these results is that a horseshoe is a closed invariant set K such that $\varphi|_K$ is conjugate to a subshift of finite type (aka, a topological Markov chain). To be more precise, there is a decomposition $K = K_0 \cup \ldots \cup K_{r-1}$ such that $\varphi(K_i) = K_{i+1}$, $i \in \mathbb{Z}_r$, and $\varphi^r|_{K_i}$ is conjugate to a full shift in s-symbols; see [70, Sect. S.5.d]. (In addition, here K is hyperbolic). Then K has a local product structure: for any two nearby points the intersection of the local stable manifold through one of them with the local unstable manifold through the other is transverse and comprises exactly one point. This is equivalent to local maximality, see, e.g., [3, Sect. 5].

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