

# Non-invertible symmetries in $S_N$ orbifold CFTs and holography

Michael Gutperle<sup></sup>,<sup>a</sup> Yan-Yan Li,<sup>a</sup> Dikshant Rathore<sup>a</sup> and Konstantinos Roumpedakis<sup></sup><sup>b</sup>

<sup>a</sup>*Mani L. Bhaumik Institute for Theoretical Physics,  
Department of Physics and Astronomy, University of California,  
Los Angeles, CA 90095, U.S.A.*

<sup>b</sup>*William H. Miller III Department of Physics and Astronomy, Johns Hopkins University,  
3400 North Charles Street, Baltimore, MD 21218, U.S.A.*

*E-mail:* [gutperle@ucla.edu](mailto:gutperle@ucla.edu), [yanyanli@ucla.edu](mailto:yanyanli@ucla.edu), [drathore@physics.ucla.edu](mailto:drathore@physics.ucla.edu),  
[kroumpe1@jh.edu](mailto:kroumpe1@jh.edu)

**ABSTRACT:** We study non-invertible defects in two-dimensional  $S_N$  orbifold CFTs. We construct universal defects which do not depend on the details of the seed CFT and hence exist in any orbifold CFT. Additionally, we investigate non-universal defects arising from the topological defects of the seed CFT. We argue that there exist universal defects that are non-trivial in the large- $N$  limit, making them relevant for the  $\text{AdS}_3/\text{CFT}_2$  correspondence. We then focus on  $\text{AdS}_3 \times S^3 \times \mathcal{M}_4$  with one unit of NS-NS flux and propose an explicit realization of these defects on the worldsheet.

**KEYWORDS:** AdS-CFT Correspondence, Global Symmetries

**ARXIV EPRINT:** [2405.15693](https://arxiv.org/abs/2405.15693)

---

## Contents

|          |                                                                                     |           |
|----------|-------------------------------------------------------------------------------------|-----------|
| <b>1</b> | <b>Introduction</b>                                                                 | <b>1</b>  |
| <b>2</b> | <b>Review</b>                                                                       | <b>3</b>  |
| 2.1      | Non-invertible symmetries in rational CFTs                                          | 3         |
| 2.2      | Symmetric orbifold CFT                                                              | 5         |
| <b>3</b> | <b>Universal defects in symmetric orbifolds</b>                                     | <b>7</b>  |
| 3.1      | Construction                                                                        | 8         |
| 3.2      | Modular invariance and defect operators                                             | 9         |
| <b>4</b> | <b>Example: <math>S_3</math> symmetric orbifold</b>                                 | <b>10</b> |
| <b>5</b> | <b>Non-universal defects</b>                                                        | <b>15</b> |
| <b>6</b> | <b>Large-<math>N</math> symmetric orbifold</b>                                      | <b>16</b> |
| <b>7</b> | <b>Non-invertible symmetries and <math>\text{AdS}_3/\text{CFT}_2</math> duality</b> | <b>18</b> |
| 7.1      | Tensionless limit                                                                   | 18        |
| 7.2      | Deformation away from the $k = 1$ and the supergravity limit                        | 21        |
| <b>8</b> | <b>Discussion</b>                                                                   | <b>21</b> |
| <b>A</b> | <b>Defect Hilbert spaces in <math>S^3</math> orbifold</b>                           | <b>23</b> |

---

## 1 Introduction

In recent years, generalized and non-invertible symmetries in various dimensions have been a very active field of investigation [1–57] providing valuable insight into the dynamics of quantum field theories. At the core of these generalizations is the realization of symmetries as (extended) topological operators and vice versa. In this picture, a natural question is to find the holographic dual of these symmetry operators in the bulk. In theories with weakly coupled holographic duals, it has been proposed that boundary topological operators correspond to D-branes in the bulk [58–72] which, in certain limits, can become topological. These proposals are based on anomaly inflow arguments. The goal of this paper is to put forward a proposal for the holographic dual of topological operators in a setup with a strongly coupled bulk theory using the  $\text{AdS}_3/\text{CFT}_2$  correspondence.

In two-dimensional conformal field theories (CFT), non-invertible symmetries have a long history starting with the work of Verlinde in rational CFTs [73], the construction of topological defects in terms of projectors [74, 75], the construction of topological duality defects [76, 77], and the topological field theory formulation of RCFTs [78] to name a few. Orbifolds are a powerful way to obtain new CFTs from old ones [79, 80]. In this paper, we focus on the so-called symmetric orbifold CFT, where one takes the tensor product of  $N$

copies of a seed CFT  $\mathcal{M}$  and gauges the  $S_N$  symmetric group [81] that permutes the different copies. A remarkable feature of these theories is that they observe a large- $N$  factorization of correlation functions [82–84], similar to the case of single-trace operators in large- $N$  gauge theories. Proposals for the holographic dual of symmetric orbifold CFTs go back to [85], relating a type IIB  $AdS_3 \times S^3 \times \mathcal{M}_4$  background, with  $\mathcal{M}_4$  being either  $K_3$  or  $T^4$ , to the  $\mathcal{N} = (4, 4)$  supersymmetric orbifold CFT  $\mathcal{M}_4^{\otimes N}/S_N$ . This background allows for an exact worldsheet description with only NS-NS flux. However, it has been a challenge to identify the particular background which corresponds to the undeformed  $\mathcal{M}_4^{\otimes N}/S_N$  orbifold [86]. Recently, such a dual was found for the  $AdS_3 \times S^3 \times \mathcal{M}_4$  background with the minimal  $k = 1$  unit of NS-fivebrane flux [87–91].

It is well known that gauging a group-like symmetry leads to a quantum symmetry in the gauged theory [92, 93]. Hence, by construction, an orbifold CFT exhibits a universal set of symmetries arising by the gauging of a discrete group. In the case of interest, the gauging of the non-abelian  $S_N$  group leads to a universal set of non-invertible symmetries. In this work, we look at these universal defects in two-dimensional symmetric orbifold CFTs. We construct them by applying the projector construction of Petkova and Zuber [74], and use the modular transformations of the partition function to identify the defect Hilbert spaces of operators on which line-defects can end. Additionally, we propose a realization of the defect operators in terms of non-gauge invariant operators under  $S_N$ . These defects, being universal, depend solely on the group theory structure of  $S_N$  and not on the specifics of the seed CFT. To illustrate their properties, we examine the simplest example that exhibits non-invertible symmetries, i.e.  $N = 3$ .

In symmetric orbifold CFTs, universal defects are labeled by the representations of  $S_N$ . We study these defects in the large- $N$  limit and analyze which representations remain non-trivial. We argue that these defects correspond to topological defects on the worldsheet and propose an explicit realization. While we verify some of their properties, further investigation is required for a complete realization.

The structure of this paper is as follows: in section 2, we review some background material, in particular non-invertible symmetries in RCFTs and the construction of symmetric orbifold CFTs. In section 3, we construct the universal topological line-defect operators in the symmetric orbifold CFTs that realize the  $\text{Rep}(S_N)$  fusion algebra. We use the modular transformation of the torus partition function with a defect inserted to determine the corresponding defect Hilbert spaces, and we construct the vacuum-state operators in terms of non-gauge invariant twist fields. In section 4, we present a detailed example of the  $S_3$  symmetric orbifold. This is one of the simplest examples where non-invertible symmetries arise in CFTs. We determine the lines and defect operators, as well as the Ward identities for two and three-point correlators of twist operators. In section 5, we discuss non-universal topological defects, starting with a nontrivial topological defect operator in the seed theory and constructing a “maximally fractional” defect, where all sectors are built on the same seed CFT defect in an  $S_N$  invariant manner. We show that such defects are consistent, in the sense that they satisfy the Cardy-Petkova-Zuber [74, 75, 94] conditions. In section 6, we discuss some aspects of the large- $N$  limit of these defects, particularly whether a non-invertible defect survives this limit. In section 7, we discuss the AdS/CFT dual of the symmetric orbifold [89],

which states that the AdS dual is a type IIB string theory on  $AdS_3 \times S^3 \times \mathcal{M}_4$  with one unit of NS-NS fivebrane flux. We propose a worldsheet dual of the universal defects and discuss some of their properties. In section 8, we close with some discussion of our results and list some open questions and speculations. Some technical details of the  $S_3$  example are relegated to appendix A.

## 2 Review

In this section, we review well-known material which will play an important role in the main part of the paper. First, we discuss the construction of Verlinde lines in rational conformal field theories, introducing key concepts of non-invertible symmetries in two-dimensional CFTs. Second, we review the construction of the  $\mathcal{M}^{\otimes N}/S_N$  symmetric orbifold CFT, for which we will construct the non-invertible symmetries in following sections.

### 2.1 Non-invertible symmetries in rational CFTs

Let us first look at the construction of topological line-defects in rational conformal field theories (RCFTs). We will restrict our attention to the case of diagonal RCFTs and the Verlinde lines [73] which are in one-to-one correspondence with the chiral primaries. The torus partition function of such an RCFT is given by

$$Z(\tau, \bar{\tau}) = \sum_{i,j} n_{ij} \chi_i(\tau) \bar{\chi}_j(\bar{\tau}), \quad (2.1)$$

where  $n_{ij} = \delta_{ij}$  and  $\chi_i(\tau)$  denotes the character of a finite irreducible chiral algebra labeled by the representation  $i$ . Under the modular  $S$ -transformation, the characters transform as  $\chi_i(-1/\tau) = \sum_j S_{ij} \chi_j(\tau)$ . The operator corresponding to a Verlinde line  $\mathcal{I}_a$  associated to the chiral primary  $|\Phi_a\rangle$  is given by [74, 75]

$$\mathcal{I}_a = \sum_i \frac{S_{ai}}{S_{0i}} P_{i\bar{i}}, \quad (2.2)$$

where  $P_{i\bar{i}}$  is a projector acting on the space spanned by the  $i$ -th primary and its descendants:

$$P_{i\bar{i}} = \sum_{n\bar{n}} |i, n\rangle \otimes |i, \bar{n}\rangle \langle i, n| \otimes \langle i, \bar{n}|. \quad (2.3)$$

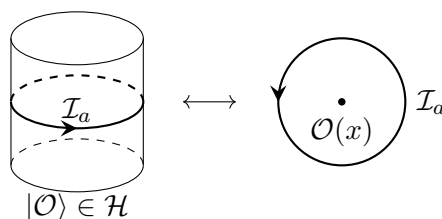
Using the Verlinde formula [73],

$$n_{ab}^c = \sum_i \frac{S_{ai} S_{bi} S_{ci}^*}{S_{0i}}, \quad (2.4)$$

we can verify that the fusion algebra of the Verlinde lines is isomorphic to that of the chiral primaries,

$$\mathcal{I}_a \mathcal{I}_b = \sum_c n_{ab}^c \mathcal{I}_c, \quad (2.5)$$

where  $n_{ab}^c$  is the non-negative integer fusion coefficient, which denotes the multiplicity with which  $\mathcal{I}_c$  appears in the fusion product of the defects. Given that the Verlinde lines act as



**Figure 1.** Using the state-operator correspondence to depict the action of the line-defect on an operator in the untwisted Hilbert space.

projectors on the modules of the chiral algebra, which includes the Virasoro algebra, they commute with both the holomorphic and anti-holomorphic modes of the stress tensor,

$$L_n \mathcal{I}_a = \mathcal{I}_a L_n, \quad \bar{L}_n \mathcal{I}_a = \mathcal{I}_a \bar{L}_n, \quad (2.6)$$

and are therefore topological. Using the folding trick of [95, 96], the topological defects have a realization as Ishibashi boundary states [94, 97] which characterize  $\mathbb{Z}_2$  permutation branes [98–100]. If we start with the folded theory  $\text{CFT}_{(1)} \times \overline{\text{CFT}}_{(2)}$ , the boundary state defining a  $\mathbb{Z}_2$  permutation brane is given by

$$|B_a\rangle = \sum_i \frac{S_{ai}}{S_{0i}} \sum_n |i, n\rangle_{(1)} \otimes \overline{|i, n\rangle}_{(2)} \sum_m |i, m\rangle_{(2)} \otimes \overline{|i, m\rangle}_{(1)}, \quad (2.7)$$

such that they obey the following Virasoro gluing conditions:

$$\begin{aligned} (L_k^{(1)} - \bar{L}_{-k}^{(2)}) \sum_n |i, n\rangle_{(1)} \otimes \overline{|i, n\rangle}_{(2)} &= 0, \\ (L_k^{(2)} - \bar{L}_{-k}^{(1)}) \sum_m |i, m\rangle_{(2)} \otimes \overline{|i, m\rangle}_{(1)} &= 0. \end{aligned}$$

The quantum dimension of the topological defect operator is the eigenvalue of the defect acting on the CFT vacuum state,

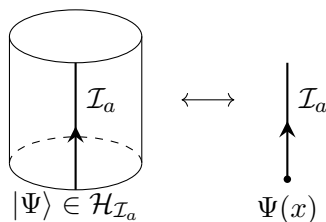
$$\mathcal{I}_a |0\rangle = \langle \mathcal{I}_a \rangle |0\rangle. \quad (2.8)$$

For the Verlinde lines we obtain  $\langle \mathcal{I}_a \rangle = S_{a0}/S_{00}$ . It was pointed out in [1] that defects that have a quantum dimension not equal to one are in general associated with non-invertible symmetries. Note that in the boundary state representation (2.7) the quantum dimension is obtained by  $\langle \mathcal{I}_a \rangle = \langle 0 | B_a \rangle$ .

Operators like (2.2) satisfying the Cardy-Petkova-Zuber [74, 75, 94] conditions can be viewed as topological line-defects. The action of these line-defects can be visualized on the complex plane mapped from the cylinder as in figure 1, where we insert the defect at a particular point in time. We refer to the Hilbert space defined on  $S^1$  for every point in time as the *untwisted* Hilbert space.

The corresponding torus partition function is given by inserting the defect inside the trace over the Hilbert space (see figure 2),

$$Z^a(\tau, \bar{\tau}) = \text{tr}_{\mathcal{H}} [\mathcal{I}_a q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}}], \quad (2.9)$$



**Figure 2.** States in the defect Hilbert space  $\mathcal{H}_{\mathcal{I}_a}$  correspond to non-local operators attached to the line-defect  $\mathcal{I}_a$ .

$$\mathcal{O} \uparrow \mathcal{I} = \frac{1}{\dim \mathcal{I}} \left( \mathcal{I} \uparrow \mathcal{I} \cdot \mathcal{O} \right) + \mathcal{I} \uparrow \mathcal{I} \rightarrow \mathcal{O}_{\text{defect}}$$

**Figure 3.** Lasso action of a topological line-defect.

where  $q = e^{2\pi i \tau}$  with  $\tau$  being the torus modular parameter. After performing a modular  $S$ -transformation, the line-defect now extends in the time direction, and at every point in time we obtain the *defect Hilbert space*,  $\mathcal{H}_{\mathcal{I}_a}$ . The torus partition function is now modified to trace over this defect Hilbert space corresponding to the defect  $\mathcal{I}_a$ ,

$$Z_a(-\frac{1}{\bar{\tau}}, -\frac{1}{\bar{\tau}}) = \text{tr}_{\mathcal{H}_{\mathcal{I}_a}} \left[ \tilde{q}^{L_0 - \frac{c}{24}} \bar{\tilde{q}}^{\bar{L}_0 - \frac{c}{24}} \right], \quad (2.10)$$

where  $\tilde{q} = e^{-2\pi i/\tau}$ . Using the  $S$ -transformation of the characters, and the fact that the defect commutes with the chiral vertex algebra, we can expand the defect Hilbert space in terms of the characters. The defect partition function can thus be written as

$$Z_a(\tau, \bar{\tau}) = \sum_{b,c} N_{ab}^c \chi_b(\tau) \bar{\chi}_c(\bar{\tau}), \quad (2.11)$$

where the non-negative integers  $N_{ab}^c$  encode the degeneracy of the states in  $\mathcal{H}_{\mathcal{I}_a}$ . Note that even though we started with a diagonal theory, the defect partition function is not necessarily diagonal, i.e.  $N_{ab}^c$  need not be a diagonal matrix. This tells us that states in  $\mathcal{H}_{\mathcal{I}_a}$  have non-integer spins and correspond to non-local operators attached to the corresponding line  $\mathcal{I}_a$ .

An important feature of non-invertible line-defects is the “lasso” effect [1] where, when a topological line-defect is moved past a local operator, one obtains in addition to the symmetry action on the local operator another topological line ending on a defect operator and a junction of topological line operators (see figure 3). The mathematical structure describing symmetry line-defects, defect operators, and their manipulations is called a tube algebra [27].

## 2.2 Symmetric orbifold CFT

In this section, we review the symmetric orbifold CFT. Consider  $N$  copies of a CFT  $\mathcal{M}$  (not necessarily rational). The tensor product CFT has an  $S_N$  symmetry that permutes the  $N$  copies of the theory. We construct the  $S_N$  orbifold CFT by gauging this permutation symmetry.

Let's denote the primary operators of the tensor product theory by  $\mathcal{O}_I$  where  $I = 1, 2, \dots, N$ . The spectrum of the orbifold theory consists of the untwisted sector constructed by taking gauge invariant combinations of operators in the tensor product theory such as

$$\sum_{I=1}^N \mathcal{O}_I, \quad \sum_{I,J=1}^N \mathcal{O}_I \mathcal{O}_J, \dots \quad (2.12)$$

Modular invariance additionally requires twisted sectors. We introduce the twist operators  $\sigma_g$  labeled by the elements of  $S_N$  by

$$\mathcal{O}_I(e^{2\pi i} z) \sigma_g(0) = \mathcal{O}_{g \cdot I}(z) \sigma_g(0). \quad (2.13)$$

These operators introduce non-trivial holonomies for the operators of the seed theory. However, the operators  $\sigma_g$  are not gauge invariant. To define gauge invariant operators we define [83, 84, 101]

$$\sigma_{[g]} \propto \sum_{h \in S_N} \sigma_{h^{-1}gh}, \quad (2.14)$$

labeled by the conjugacy classes  $[g]$  of  $S_N$ . The proportionality constant can be fixed by normalizing the two-point function of these operators [84]. A conjugacy class  $[g]$  is fixed by giving the number  $l_i$  of cycles of length  $i = 1, 2, \dots, N$ .

$$[g] = 1^{l_1} \cdot 2^{l_2} \dots N^{l_N}, \quad \sum_k k l_k = N. \quad (2.15)$$

Excited states are created by acting with operators of the seed theory on twisted states. For example, consider the non-gauge invariant twist operator  $\sigma_g$  where  $g$  is a cycle of length  $n$ . Given (2.13), the operators of the seed theory have fractional modes

$$\mathcal{O}_I(z) = \frac{1}{n} \sum_m (\mathcal{O}_I)_{\frac{m}{n}} z^{-\frac{m}{n} - h} e^{-2\pi i \frac{m}{n} (I-1)}. \quad (2.16)$$

The modes with fractional indices correspond to excited twist operators. Hence, in the twisted sector corresponding to an element of  $S_N$  with a single cycle of length  $n$ , there are  $n$  subsectors (the vacuum twist operator  $\sigma_n$  and excited states dressed with fractional modes of the operators of the seed theory). More generally, in a twisted sector created by  $\sigma_g$ , twist operators are labeled by the centralizer of  $g$ . Thus, the torus partition function can be written as follows [101, 102]

$$Z = \frac{1}{|G|} \sum_{gh=hg} h \begin{array}{|c|} \hline \square \\ \hline g \end{array} (\tau, \bar{\tau}), \quad (2.17)$$

where  $g$  denotes the twisted sector and  $h$  an element of its centralizer. Each term in the sum is equal to

$$h \begin{array}{|c|} \hline \square \\ \hline g \end{array} (\tau, \bar{\tau}) = \text{tr}_{\mathcal{H}_g} \left( h e^{2\pi i \tau (L_0 - \frac{c}{24})} e^{-2\pi i \bar{\tau} (\bar{L}_0 - \frac{c}{24})} \right). \quad (2.18)$$

The summation of elements  $h \in S_N$  that commute with  $g$  imposes the projection onto states that are invariant under the centralizer  $C[g]$ . Here  $|G| = N!$  is the order of  $S_N$ . Using the fact that the torus partition function in the individual sectors transforms as follows under modular transformations  $\tau \rightarrow -\frac{1}{\tau}$

$$h \square_g(\tau, \bar{\tau}) \rightarrow g^{-1} \square_h \left( -\frac{1}{\tau}, -\frac{1}{\bar{\tau}} \right), \quad (2.19)$$

it is straightforward to check that the orbifold partition function is modular invariant.

The spectrum of the twisted sectors can be obtained by expanding the partition function. For a twisted sector labelled by a cycle  $[g]$  in (2.15), the conformal dimensions  $h_{\text{CFT}}$  of the universal twisted sector ground state is given by

$$h_{\text{CFT}} = \sum_{k=1}^N l_k \frac{c_{\text{seed}}}{24} \left( k - \frac{1}{k} \right). \quad (2.20)$$

These operators do not depend on the details of the seed CFT. In addition, there are twisted sector states built on primaries in the seed theory with  $h_{\text{seed}} > 0$  as well as (fractional) Virasoro descendants, but these states will not play an important role in the following.

Another way [81, 103] to look at the partition function is via a grand canonical generating function. The modular invariant partition function of the seed theory is denoted as  $Z(\tau, \bar{\tau})$

$$\sum_N p^N Z_N = \exp \left[ \sum_{k=1}^{\infty} p^k T_k Z(\tau, \bar{\tau}) \right], \quad (2.21)$$

where  $T_k$  is the Hecke operator

$$T_k Z(\tau, \bar{\tau}) = \frac{1}{k} \sum_{i|k} \sum_{j=0}^{i-1} Z \left( \frac{k\tau}{i^2} + \frac{j}{i}, \frac{k\bar{\tau}}{i^2} + \frac{j}{i} \right). \quad (2.22)$$

Correlation function of twist operators can be calculated by going to the covering space [80, 83, 84, 104, 105]. Suppose we want to calculate a correlation function of twist operators. Due to (2.13), correlation functions of non-gauge invariant operators have branch cuts. The union of the different sheets forms the covering space. Although we consider the orbifold theory on  $S^2$ , the covering space does not necessarily have zero genus; its genus is determined by the Riemann-Hurwitz formula [83].

### 3 Universal defects in symmetric orbifolds

We construct the orbifold CFT starting with a tensor product of  $N$  copies of a seed theory  $\mathcal{M}$  and gauge the  $S_N$  symmetry that permutes the  $N$  copies. The gauged theory exhibits a quantum symmetry  $\text{Rep}(S_N)$ ,<sup>1</sup> [5, 92, 93]. In this section, we construct these universal topological defects using the projector construction of Petkova and Zuber [74]. In particular, the details of the seed CFT will not play a role in the sense that the defect can be thought of as being built on the trivial topological defect in the seed CFT, which always exists. In addition to these defects, there may be other topological defects coming from the defects of the seed theory itself. We discuss non-universal defects, which are based on nontrivial topological defects in the seed theory, in section 5.

---

<sup>1</sup>Intuitively, this symmetry is generated by the Wilson lines of the  $S_N$  gauge field.

### 3.1 Construction

Utilizing the folded permutation brane construction for a topological defect, a universal defect corresponds to the following boundary state in the folded  $\mathcal{M}^{\otimes N}/S_N \times \mathcal{M}^{\otimes N}/S_N$  CFT

$$|B_R\rangle\rangle = \sum_{[g]} \chi_R([g]) \sum_n |n\rangle_{(1),[g]} \otimes |\bar{n}\rangle_{(2),[g]} \sum_m |m\rangle_{(2),[g]} \otimes |\bar{m}\rangle_{(1),[g]}. \quad (3.1)$$

Here  $[g]$  denotes the twisted sector in the  $\mathcal{M}^{\otimes N}/S_N$  orbifold labeled by the conjugacy class of  $S_N$ . The summation over  $n, m$  denotes the sum over all states in the  $[g]$ -twisted sector including the sum over primaries of the seed theory as well as Virasoro descendants producing an Ishibashi boundary state [97]. The folded boundary state, and thus the defect, is labeled by the choice of representation  $R$  of  $S_N$  with  $\chi_R([g])$  being the character of the conjugacy class  $[g]$  in the representation  $R$ . Consequently, the boundary state satisfies

$$\left(L_n^{(1)} - \bar{L}_{-n}^{(2)}\right) |B_R\rangle\rangle = \left(L_n^{(2)} - \bar{L}_{-n}^{(1)}\right) |B_R\rangle\rangle = 0, \quad (3.2)$$

where  $L_n^{(i)}, i = 1, 2$  are the modes of the stress tensor of the two copies of the  $\mathcal{M}^{\otimes N}/S_N$  orbifold after folding. After unfolding, the two copies of the CFT living on half-spaces are separated by the defect operator

$$\mathcal{I}_R = \sum_{[g]} \chi_R([g]) P_{[g]} \bar{P}_{[g]}. \quad (3.3)$$

The defect is expressed in terms of the identity projectors in the  $[g]$ -twisted sector satisfying  $P_{[g]}^2 = P_{[g]}$  and  $P_{[g]} P_{[g']} = 0$  if  $[g] \neq [g']$ . It is clear from the property of the projectors or the unfolding of the boundary state condition (3.2) that the defect is topological, i.e. it commutes with the holomorphic and anti-holomorphic modes of the stress tensor separately.

$$L_n \mathcal{I}_R = \mathcal{I}_R L_n, \quad \bar{L}_n \mathcal{I}_R = \mathcal{I}_R \bar{L}_n. \quad (3.4)$$

Using the properties of the projection operators, one can calculate the fusion product of two defect operators labeled by two representations  $R_1, R_2$

$$\begin{aligned} \mathcal{I}_{R_1} \mathcal{I}_{R_2} &= \sum_{[g]} \chi_{R_1}([g]) \chi_{R_2}([g]) P_{[g]} \bar{P}_{[g]} \\ &= \sum_{R_3} \frac{1}{|G|} \sum_g \chi_{R_1}(g) \chi_{R_2}(g) \chi_{R_3}(g)^* \mathcal{I}_{R_3} \\ &= \sum_{R_3} N_{R_1 R_2}^{R_3} \mathcal{I}_{R_3}. \end{aligned} \quad (3.5)$$

Here  $N_{R_1 R_2}^{R_3}$  are the Kronecker coefficients of the decomposition of the tensor product  $R_1 \otimes R_2$  into representations  $R_3$ . Since  $S_N$  for  $N > 2$  is a non-abelian group, the topological line-defects include non-invertible ones. This can also be seen from the fact that acting on a state in the  $[g]$ -twisted sector one has

$$\mathcal{I}_R |\Phi_{[g]}\rangle = \chi_R([g]) |\Phi_{[g]}\rangle. \quad (3.6)$$

In particular, this implies that the quantum dimension of a defect  $\mathcal{I}_R$  is given by  $\langle \mathcal{I}_R \rangle = \chi_R(1) = \dim_R$ , i.e. the dimension of the representation. In general, for  $S_N$  with  $N > 2$ , apart from the trivial and the alternating (or sign) representation, all other representations have dimensions greater than one, hence giving rise to non-invertible defects.

### 3.2 Modular invariance and defect operators

In this section, we use modular invariance to identify the operators in the defect Hilbert space which can end on topological lines. The starting point is to insert a defect  $\mathcal{I}_R$  along the spatial cycle in the torus partition function

$$\begin{aligned} Z^R(\tau, \bar{\tau}) &= \frac{1}{|G|} \sum_{gh=hg} \text{tr}_{\mathcal{H}_g} \left( h e^{2\pi i \tau (L_0 - \frac{c}{24})} e^{-2\pi i \bar{\tau} (\bar{L}_0 - \frac{c}{24})} \mathcal{I}_R \right) \\ &= \frac{1}{|G|} \sum_{gh=hg} \chi_R([g]) h \square_g(\tau, \bar{\tau}). \end{aligned} \quad (3.7)$$

The defect Hilbert space can be obtained by performing a modular transformation  $\tau \rightarrow -\frac{1}{\tau} = \tau'$ , which exchanges the spatial and temporal cycles. Using the modular transformation (2.19), the defect partition function is given by

$$Z_R(\tau', \bar{\tau}') = \frac{1}{|G|} \sum_{gh=hg} \chi_R([g]) g \square_h(\tau', \bar{\tau}'), \quad (3.8)$$

where we used the fact that the characters of  $S_N$  are real.

In the orbifold partition function, we decompose the elements  $h, g$  which commute  $gh = hg$  by labeling the element  $g$  by its conjugacy class and  $h$  by the element of the centralizer. However, one can do this the other way around, where  $h$  labels conjugacy classes and  $g$  labels the centralizer of  $h$ .

The simplest and most relevant case to the large- $N$  limit is when  $h$  corresponds to a single cycle of length  $[w]$ . The centralizer of such an element is  $\mathbb{Z}_w \times S_{N-w}$ . The group  $\mathbb{Z}_w$  contains elements  $1, \omega, \omega^2, \dots, \omega^{w-1}$ , where  $\omega$  is the  $w$ -th root of unity. The defect partition function corresponding to the single  $w$ -twisted cycle can be written as

$$Z_{R,[w]} = \frac{1}{\dim C[w]} \chi_R(C[w]) C[w] \square_{[w]}. \quad (3.9)$$

Using the decomposition of  $C[w] = \mathbb{Z}_w \times S_{N-w}$  and  $\dim C[w] = w(N-w)!$ , one gets

$$Z_{R,[w]} = \frac{1}{(N-w)!} \sum_{g \in S_{N-w}} \frac{1}{w} \left( \chi_R([1_w \times g]) 1_w \times g \square_{[w]} + \sum_{k=1}^{w-1} \chi_R([\omega^k \times g]) \omega^k \times g \square_{[w]} \right). \quad (3.10)$$

To determine the number of defect operators, we want to pick out the ground state contribution of the  $w$ -twisted sector from (3.10), which has the conformal dimension

$$h = \bar{h} = \frac{c_{\text{seed}}}{24} \left( w - \frac{1}{w} \right). \quad (3.11)$$

As far as the  $S_{N-w}$  permutation is concerned, the sector  $[w]$  is untwisted. Consequently, the insertion of  $g \in S_{N-w}$  contributes the same factor to the ground state in the defect partition function (3.10). The same holds for the  $1_w$  and  $\omega^k, k = 1, 2, \dots, w-1$  insertions since these enforce the  $h - \bar{h} \in \mathbb{Z}$  projection, which is automatically satisfied for the twisted

ground state (3.11). This implies that the number of  $w$ -twisted defect ground states is simply obtained by setting  $\square = 1$  for all terms in the partition function (3.10), giving

$$n_w^R = \frac{1}{(N-w)!} \sum_{h' \in S_{N-w}} \frac{1}{w} \sum_{k=0}^{w-1} \chi_R([\omega^k \times h']), \quad (3.12)$$

where we included  $\omega^0 = 1_w$ . A vanishing  $n_w^R$  implies that the  $\mathcal{I}_R$  defect cannot end on a bare twist operator in the  $w$ -twisted sector.

One way to understand the defect operators at the endpoints of topological line-defects is to view these line-defects as open Wilson lines ending on non-gauge invariant operators [93]. A natural proposal for such operators involves using characters to project onto the representation  $R$

$$\sigma_g^R \equiv N_R \sum_{h \in S_N} \chi_R(h) \sigma_{h^{-1}gh}, \quad (3.13)$$

where we pick an element  $g$  in the conjugacy class that labels the twisted sector and we focus again on a single cycle of length  $w$ . We propose that the  $\mathcal{I}_R$  line-defect can end on a defect operator in the  $[g]$ -twisted sector if (3.13) does not vanish. We can decompose the sum over  $h \in S_N$  in (3.13) into a sum over the centralizer of  $g$  and the rest

$$\sigma_g^R \equiv N_R \left( \sum_{h' \in S_{N-w}} \sum_{k=0}^{w-1} \chi_R([\omega^k \times h']) \sigma_g + \cdots \right), \quad (3.14)$$

where the dots denote summation over other group elements in the conjugacy class of  $g$ . Note that the coefficient multiplying  $\sigma_g$  in (3.14) is proportional to (3.12). Consequently, the vanishing of one implies the vanishing of the other and the criterion for the existence of a defect operator ending on a line-defect is the same. The vanishing of all the terms on the right-hand side of (3.13) can be established by conjugating the group element  $g$  and using the fact that the characters take the same value on all elements of the same conjugacy class.

It should be noted that for  $S_N$  with  $N > 4$ , the degeneracy of the twisted sector ground states (3.12) can be larger than one. For the non-gauge invariant operators in (3.13), this can be understood by the fact that the action of  $S_N$  on all elements  $g$  in a fixed conjugacy class by conjugation  $g \rightarrow h^{-1}gh$  determines a (generally) reducible representation. The value  $n_w^R$  gives the number of times the irreducible representation  $R$  appears in this representation.

In the next section we will further test this construction by evaluating correlation functions of defect operators and check the Ward identities following from the non-invertible symmetry  $\text{Rep}(S_3)$ .

## 4 Example: $S_3$ symmetric orbifold

We now consider a simple example, the  $S_3$  orbifold theory of a seed CFT  $\mathcal{M}$  with central charge  $c$ . Upon gauging the  $S_3$  symmetry, a  $\text{Rep}(S_3)$  quantum symmetry emerges. The objects of  $\text{Rep}(S_3)$  symmetry are topological lines which are labeled by the irreducible representations of  $S_3$ .

The symmetric group  $S_3$  is a non-abelian group consisting of permutations of three elements. The group-elements of  $S_3$  are

$$S_3 = \{e, (12), (23), (31), (123), (132)\}, \quad (4.1)$$

where  $e$  is the identity permutation,  $(12)$  is a 2-cycle representing the permutation  $1 \rightarrow 2 \rightarrow 1$ ,  $(123)$  is a 3-cycle representing the permutation  $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$  and so on. There are three conjugacy classes

$$[e] = \{e\}, \quad [2] = \{(12), (13), (23)\}, \quad [3] = \{(123), (132)\}. \quad (4.2)$$

The character table of  $S_3$  is given below

|          | $[e]$ | $[2]$ | $[3]$ |
|----------|-------|-------|-------|
| $\chi_e$ | 1     | 1     | 1     |
| $\chi_A$ | 1     | -1    | 1     |
| $\chi_S$ | 2     | 0     | -1    |

(4.3)

where  $e$  denotes the trivial representation,  $A$  denotes the alternating representation, and  $S$  denotes the standard representation.

In this case, the fusion rules of the universal defects defined in (3.3) follow from the above character table and read

$$\begin{aligned} \mathcal{I}_A \times \mathcal{I}_A &= 1, \\ \mathcal{I}_S \times \mathcal{I}_A &= \mathcal{I}_A \times \mathcal{I}_S = \mathcal{I}_S, \\ \mathcal{I}_S \times \mathcal{I}_S &= 1 + \mathcal{I}_A + \mathcal{I}_S. \end{aligned} \quad (4.4)$$

We see that the line-defect associated with the standard representation is non-invertible. Using (2.14), the gauge invariant ground state operators are

$$\begin{aligned} \sigma_{[2]} &= \frac{1}{\sqrt{3}} \left( \sigma_{(12)} + \sigma_{(23)} + \sigma_{(13)} \right), \\ \sigma_{[3]} &= \frac{1}{\sqrt{2}} \left( \sigma_{(123)} + \sigma_{(132)} \right), \end{aligned} \quad (4.5)$$

where the proportionality constants were fixed by requiring that the two-point functions be normalized to 1. The action of the defects on these operators is summarized in figure 4.

Next, let us discuss the defect Hilbert spaces. From (3.13), we observe that there are no two-cycle ground states in  $\mathcal{H}_{\mathcal{I}_A}$  and no three-cycle ground states in  $\mathcal{H}_{\mathcal{I}_S}$ . In the appendix, we verify this counting by analyzing the partition function. The twisted ground states in the defect Hilbert space are

$$\begin{aligned} \sigma_{[2]}^S &= N_S \left( 2\sigma_{(12)} - \sigma_{(23)} - \sigma_{(13)} \right), \\ \sigma_{[3]}^A &= N_A \left( \sigma_{(123)} - \sigma_{(132)} \right). \end{aligned} \quad (4.6)$$

Here, without loss of generality, we choose  $\sigma_{(12)}$  and  $\sigma_{(123)}$  as representatives of  $[2]$  and  $[3]$  respectively. The constants  $N_A, N_S$  can be fixed by normalizing the two-point correlators of the

$$\begin{array}{cc}
 \text{Red dashed circle with } \sigma_{[2]} = -\sigma_{[2]} & \text{Red dashed circle with } \sigma_{[3]} = \sigma_{[3]} \\
 \text{Black solid circle with } \sigma_{[2]} = 0 & \text{Black solid circle with } \sigma_{[3]} = -\sigma_{[3]}
 \end{array}$$

**Figure 4.** Action of the universal line-defects. The red line denotes the invertible sign defect  $\mathcal{I}_A$  while the black line denotes the standard defect  $\mathcal{I}_S$ .

$$\begin{array}{cc}
 \text{Black circle with } \sigma_{[2]} \text{ and line} = \alpha \text{ } \sigma_{[2]}^S \text{ and line} \\
 \text{Black circle with } \sigma_{[3]} \text{ and red dashed line} = \beta \text{ } \sigma_{[3]}^A \text{ and red dashed line}
 \end{array}$$

**Figure 5.** Lasso diagrams where the right-hand side is obtained after shrinking the standard defect loop which produces a defect operator.

non-gauge invariant operators with the line-defects inserted to 1. For example, demanding that

$$\langle \sigma_{[2]}^S \bullet \text{---} \bullet \sigma_{[2]}^S \rangle = N_S^2 \left( 4\langle \sigma_{(12)} \sigma_{(12)} \rangle + \langle \sigma_{(23)} \sigma_{(23)} \rangle + \langle \sigma_{(13)} \sigma_{(13)} \rangle \right) = 6N_S^2 = 1, \quad (4.7)$$

gives us  $N_S^2 = 1/6$ . Similarly, we get  $N_A^2 = -1/2$ . To show this, we used that the two-point correlator vanishes unless the permutation corresponding to the second twist operator is the inverse of the first [83]. In the above correlator, we implicitly assumed that insertions of the two operators are at the poles of the sphere.

Next, we determine the coefficients in the lasso diagrams. Since the defects are topological, the scaling dimension does not change and the two possible diagrams are shown in figure 5. In the following, we fix the coefficients  $\alpha$  and  $\beta$ .

Consider the two-point correlator of the two-cycles encircled by the  $\mathcal{I}_S$  line. We can shrink the line-defect to a point by avoiding both operators, which gives us the quantum dimension of the line  $\langle \mathcal{I}_S \rangle$ . Alternatively, we can make use of the F-symbols [106, 107] in figure 6 corresponding to the crossing relations of  $\text{Rep}(S_3)$  to pull the line-defect through the

$$\begin{aligned}
 \rangle \langle &= \frac{1}{2} \text{ (two arcs meeting at a vertex) } + \frac{1}{2} \text{ (two arcs meeting at a vertex with a dashed red line) } + \frac{1}{\sqrt{2}} \text{ (three arcs meeting at a vertex) } \\
 \rangle \text{---} \langle &= \frac{1}{2} \text{ (two arcs meeting at a vertex) } + \frac{1}{2} \text{ (two arcs meeting at a vertex with a dashed red line) } - \frac{1}{\sqrt{2}} \text{ (three arcs meeting at a vertex) } \\
 \text{---} \langle &= \frac{1}{\sqrt{2}} \text{ (two arcs meeting at a vertex) } - \frac{1}{\sqrt{2}} \text{ (two arcs meeting at a vertex with a dashed red line) }
 \end{aligned}$$

**Figure 6.** F-symbols of  $\text{Rep}(S_3)$ .

operators and then shrink it to a point. Pictorially, the two ways of shrinking the defect are

$$\langle \sigma_{[2]} \sigma_{[2]} \rangle = \frac{1}{\dim S} \langle \text{ (two circles connected by a horizontal line) } \rangle = \frac{1}{2\sqrt{2}} \langle \text{ (two circles connected by a horizontal line) } \rangle = \frac{\alpha^2}{2\sqrt{2}} \langle \sigma_{[2]}^S \bullet \text{---} \bullet \sigma_{[2]}^S \rangle, \quad (4.8)$$

Requiring that the correlators on both sides are normalized to 1, we get  $\alpha^2 = 2\sqrt{2}$ . Similarly, for the two-point correlator of the three-cycle ground states,

$$\begin{aligned}
 \langle \sigma_{[3]} \sigma_{[3]} \rangle &= \frac{1}{\dim S} \langle \text{ (two circles connected by a horizontal line) } \rangle = \frac{1}{4} \langle \text{ (two circles connected by a horizontal line) } \rangle + \frac{1}{4} \langle \text{ (two circles connected by a dashed red line) } \rangle \\
 &= \frac{1}{4} \langle \sigma_{[3]} \sigma_{[3]} \rangle + \frac{\beta^2}{4} \langle \sigma_{[3]}^A \bullet \text{---} \bullet \sigma_{[3]}^A \rangle, \quad (4.9)
 \end{aligned}$$

leads to  $\beta^2 = 3$ .

We can now obtain selection rules for the three-point correlators in the presence of the line-defects. The only non-trivial three-point correlators of bare twist operators are  $\langle \sigma_{[2]} \sigma_{[2]} \sigma_{[3]} \rangle$  and  $\langle \sigma_{[3]} \sigma_{[3]} \sigma_{[3]} \rangle$ . To illustrate the procedure we focus on the former. As in the case of two-point functions, the locations of the operators in the three-point correlators are fixed using the  $\text{SL}(2, \mathbb{R})$  symmetry. Consider a line-defect  $\mathcal{I}_S$  encircling only  $\sigma_{[3]}$ . This is equivalent to encircling both the  $\sigma_{[2]}$  operators on the sphere as illustrated in figure 7. The second line in the figure is obtained by using the F-symbols in figure 6 and the lasso diagram in figure 5. The correlator  $\langle \sigma_{[2]} \sigma_{[2]} \sigma_{[3]} \rangle$  can be expanded using (4.5) as follows

$$\begin{aligned}
 -\langle \sigma_{[2]} \sigma_{[2]} \sigma_{[3]} \rangle &= \frac{-1}{3\sqrt{2}} \langle (\sigma_{(12)} + \sigma_{(23)} + \sigma_{(13)})(\sigma_{(12)} + \sigma_{(23)} + \sigma_{(13)})(\sigma_{(123)} + \sigma_{(132)}) \rangle \\
 &= -\sqrt{2} C_{223}, \quad (4.10)
 \end{aligned}$$

$$\begin{aligned}
 - \left\langle \begin{array}{cc} \sigma_{[2]} & \sigma_{[2]} \\ & \sigma_{[3]} \end{array} \right\rangle &= \left\langle \begin{array}{cc} \sigma_{[2]} & \sigma_{[2]} \\ & \textcircled{\sigma_{[3]}} \end{array} \right\rangle = \left\langle \begin{array}{cc} \textcircled{\sigma_{[2]}} & \textcircled{\sigma_{[2]}} \\ & \sigma_{[3]} \end{array} \right\rangle \\
 &= \frac{1}{\sqrt{2}} \left\langle \begin{array}{cc} \textcircled{\sigma_{[2]}} & \textcircled{\sigma_{[2]}} \\ & \sigma_{[3]} \end{array} \right\rangle = 2 \left\langle \begin{array}{cc} \sigma_{[2]}^S & \sigma_{[2]}^S \\ & \sigma_{[3]} \end{array} \right\rangle
 \end{aligned}$$

**Figure 7.** The Ward identity associated with  $\mathcal{I}_S$  wrapping  $\sigma_{[3]}$ . We can shrink the line-defect the other way on the sphere which gives the correlator with both the  $\sigma_{[2]}$  encircled.

$$\begin{aligned}
 \left\langle \begin{array}{cc} \sigma_{[2]} & \sigma_{[2]} \\ & \sigma_{[3]} \end{array} \right\rangle &= \frac{1}{2} \left\langle \begin{array}{cc} \textcircled{\sigma_{[2]}} & \textcircled{\sigma_{[2]}} \\ & \textcircled{\sigma_{[3]}} \end{array} \right\rangle = \frac{\alpha^2}{4} \left\langle \begin{array}{cc} \sigma_{[2]}^S & \sigma_{[2]}^S \\ & \textcircled{\sigma_{[3]}} \end{array} \right\rangle \\
 &= -\frac{\alpha^2}{4\sqrt{2}} \left\langle \begin{array}{cc} \sigma_{[2]}^S & \sigma_{[2]}^S \\ & \sigma_{[3]} \end{array} \right\rangle - \frac{\alpha^2\beta}{4\sqrt{2}} \left\langle \begin{array}{cc} \sigma_{[2]}^S & \sigma_{[2]}^S \\ & \textcircled{\sigma_{[3]}^A} \end{array} \right\rangle
 \end{aligned}$$

**Figure 8.** The Ward identity associated with  $\mathcal{I}_S$  encircling all three operators giving rise to a junction. Shrinking the line-defect by avoiding the operator insertions gives us  $\langle \mathcal{I}_S \rangle = 2$ .

where we have defined  $C_{223} \equiv \langle \sigma_{(12)}\sigma_{(23)}\sigma_{(123)} \rangle = \langle \sigma_{(12)}\sigma_{(13)}\sigma_{(132)} \rangle = \langle \sigma_{(23)}\sigma_{(13)}\sigma_{(123)} \rangle$ . Meanwhile, the rightmost correlator in the second line of figure 7 can be computed using (4.6) as

$$\begin{aligned}
 2\langle \sigma_{[2]}^S \bullet \bullet \sigma_{[2]}^S \sigma_{[3]} \rangle &= 2 \frac{1}{6} \frac{1}{\sqrt{2}} \langle (2\sigma_{(12)} - \sigma_{(23)} - \sigma_{(13)})(2\sigma_{(12)} - \sigma_{(23)} - \sigma_{(13)})(\sigma_{(123)} + \sigma_{(132)}) \rangle \\
 &= -\sqrt{2} C_{223} = -\langle \sigma_{[2]}\sigma_{[2]}\sigma_{[3]} \rangle.
 \end{aligned} \tag{4.11}$$

Hence, using the explicit expressions of non-gauge invariant twist fields, we can check the Ward identity of  $\mathcal{I}_S$ . Similarly, from figure 8, we can also use these defects to calculate the following correlator

$$\left\langle \begin{array}{cc} \sigma_{[2]}^S & \sigma_{[2]}^S \\ & \textcircled{\sigma_{[3]}^A} \end{array} \right\rangle = -\frac{\sqrt{3}}{2} \langle \sigma_{[2]}\sigma_{[2]}\sigma_{[3]} \rangle. \tag{4.12}$$

## 5 Non-universal defects

In section 3, we constructed universal defects labeled by an irreducible representation  $R$  of the symmetric group  $S_N$ . As can be seen from the expression (3.3), the defect operator in each twisted sector is proportional to the identity operator and hence does not depend on the details of the seed CFT. Another way to look at this is that the topological defect one starts with, in the seed theory, is the identity or trivial defect. It is an interesting question whether one can start with a nontrivial defect in the seed theory and construct non-universal defects which depend on the details of the seed theory and the topological defects present in it. Similar questions arise in the construction of D-brane states in symmetric orbifolds, as discussed in [108, 109], which we will build upon in the following section.

Here we only construct a simple example of a non-universal defect that is analogous to the “maximally fractional” D-brane of [108]. A twisted sector (2.15) in the  $S_N$  orbifold theory is labeled by a conjugacy class  $[g]$  which is fixed by giving the number  $l_i$  of cycles of given length  $i = 1, 2, \dots, N$ . We focus on diagonal RCFTs discussed in section 2.1 as the seed CFT. In particular, we use the defects  $\mathcal{I}_a$  in the RCFT defined in (2.2). In the tensor product states, we will use the same boundary state labeled by  $a$  for all factors to construct a “maximally fractional” defect, which is automatically  $S_N$  invariant.<sup>2</sup> In a twisted sector corresponding to a single  $w$  cycle, the doubled boundary state is given by

$$|B_a\rangle\rangle_{(w)} = \sum_i \frac{S_{ai}}{S_{0i}} \sum_n |i, n\rangle_{(w)}^{(1)} \otimes \overline{|i, n\rangle_{(w)}^{(2)}} \sum_m |i, m\rangle_{(w)}^{(2)} \otimes \overline{|i, m\rangle_{(w)}^{(1)}}, \quad (5.1)$$

where the sum over  $n, m$  denotes the sum over all the descendants (including fractional Virasoro modes) of the  $w$ -twisted sector primary  $|i\rangle_{(w)}$ . Using these boundary states for single cycles, we can construct a defect boundary state which is labeled by the representation  $R$  of  $S_N$  as well as the choice of primary  $a$  of the RCFT as

$$|B_a^R\rangle\rangle = \sum_{[g] \in S_N} \chi_R([g]) \prod_{j=1}^N \prod_{k_j=1}^{l_j} |B_a\rangle\rangle_{(j)}. \quad (5.2)$$

The consistency condition for the topological defects derived by Petkova and Zuber in [74] is equivalent to the Cardy condition for the doubled boundary state (5.2). The cylinder amplitude for two defect-boundary states is given by

$$\begin{aligned} Z_{ab}^{R_1 R_2}(q) &= \langle\langle B_a^{R_1} | q^{L_0^{(1)} + \bar{L}_0^{(1)} - \frac{c}{12}} q^{L_0^{(2)} + \bar{L}_0^{(2)} - \frac{c}{12}} | B_b^{R_2} \rangle\rangle \\ &= \sum_{[g] \in S_N} \bar{\chi}_{R_1}([g]) \chi_{R_2}([g]) \prod_{j=1}^N \left( \sum_i \frac{S_{ai}^*}{S_{0i}} \frac{S_{bi}}{S_{0i}} \chi_i(t/j) \chi_i(t/j) \right)^{l_j}. \end{aligned} \quad (5.3)$$

The Cardy condition states that under modular transformation to the annulus channel, the partition function has a consistent interpretation as an open string partition function, i.e. the primary representations appear with integer multiplicities. Using the modular transformation of the RCFT characters and the Verlinde formula, one obtains

$$Z_{ab}^{R_1 R_2}(\tilde{q}) = \sum_{[g] \in S_N} \bar{\chi}_{R_1}([g]) \chi_{R_2}([g]) \prod_{j=1}^N \left( \sum_{rs} N_{ab}^{rs} \chi_r(j\tilde{t}) \chi_s(j\tilde{t}) \right)^{l_j}, \quad (5.4)$$

<sup>2</sup>For discussions on a more general construction of D-branes in symmetric orbifold CFTs see [109].

where  $N_{ab}^{rs}$  can be expressed in terms of the fusion coefficients  $n_{bc}^a$  of the seed RCFT

$$\begin{aligned} N_{ab}^{rs} &= \sum_{i,k,j} \frac{S_{ai}^* S_{bi} S_{ik}}{S_{0i}} \frac{S_{kj}^* S_{jr} S_{is}}{S_{0j}} \\ &= \sum_k n_{ab}^k n_k^{rs}. \end{aligned} \quad (5.5)$$

Consequently, the  $N_{ab}^{rs}$  are non-negative integers. The partition function appearing in (5.4)

$$Z_{ab}(\tilde{t}) = \sum_{rs} N_{ab}^{rs} \chi_r(\tilde{t}) \chi_s(\tilde{t}), \quad (5.6)$$

contains the characters of the folded RCFT with integer multiplicities and can be written as a trace over a Hilbert space  $\mathcal{H}_{ab}$  in the doubled tensor product of the seed RCFT. Hence, (5.4) can be written as

$$Z_{ab}^{R_1 R_2}(\tilde{q}) = \sum_{[g] \in S_N} \bar{\chi}^{R_1}([g]) \chi^{R_2}([g]) \text{tr}_{(\mathcal{H}_{ab})^{\otimes N}} \left( g \tilde{q}^{L_0 - \frac{c}{24}} \right), \quad (5.7)$$

and the character sum projects onto the representations of  $S_N$  in the  $N$ -fold tensor product of the folded CFT of  $R_1$  and  $R_2$ . It follows that the open string partition function will be a sum over states with integer multiplicity. The maximal fractional boundary state (5.1) satisfies the Cardy condition and, therefore, defines a consistent topological defect.

It follows from unfolding the boundary state (5.2) that the quantum dimension of the maximally fractional defect  $\mathcal{I}_{a,R}$  labelled by  $a, R$  is given by

$$\langle \mathcal{I}_{a,R} \rangle = \langle 0 | B_a^R \rangle = \chi_R([1]) \left( \frac{S_{a0}}{S_{00}} \right)^N, \quad (5.8)$$

which is the product of the quantum dimension of the universal defect and the quantum dimensions of the  $N$  copies of the topological defect in the RCFT present in the untwisted sector. Similar considerations as presented above determine the fusion coefficients of the defects as well as their defect Hilbert spaces.

The construction of the maximally fractional defects can straightforwardly be extended to other seed CFTs including compact bosons,  $T^4$ , and  $K_3$  sigma models (see e.g. [77, 100, 110–114]), where the RCFT defects in (5.2) are replaced by topological defects of the seed CFT, which realize more general non-invertible symmetries. We leave the discussion of these defects for future work.

## 6 Large- $N$ symmetric orbifold

In this section, we discuss the universal defects of section 3 in the large- $N$  limit. We provide a criterion for a defect to be non-trivial in this limit. Then, we show that there exist representations that pass this criterion. In the next section, we will argue that in the  $\text{AdS}_3/\text{CFT}_2$  correspondence, these non-trivial defects correspond to topological defects on the worldsheet.

Consider the topological defect  $\mathcal{I}_R$ , the general action on twist operators is shown in figure 9. This depends on the Kronecker coefficients  $N_{RR'}^{R'}$  as well the constants  $c_{RR'}^i([g])$ .

$$\left. \begin{array}{c} \sigma[g] \\ \bullet \end{array} \right|_{\mathcal{I}_R} = \frac{\chi_R([g])}{\dim_R} \left. \begin{array}{c} \sigma[g] \\ \bullet \end{array} \right|_{\mathcal{I}_R} + \sum_{R' \neq 1} \sum_{i,v} c_{RR'}^i([g]) \left. \begin{array}{c} v \\ \bullet \end{array} \right|_{\mathcal{I}_{R'}} \begin{array}{c} \sigma^{R',i} \\ \bullet \end{array}$$

**Figure 9.** Action of the topological defect  $\mathcal{I}_R$  on twist operators. The vector  $v$  is an element in  $\text{Hom}(R \otimes R, R')$ . The sum on the right-hand side is over  $R'$  with  $N_{RR}^{R'} \neq 0$ . We have omitted the orientation of the defect since all the representations of  $S_N$  are real.

The former determines the representations  $R'$  that appear in the sum. In principle, to determine the constants  $c_{RR'}^i([g])$  we need to know both the Kronecker coefficients as well as the F-symbols. At present, it is an open problem in mathematics to find explicit expressions for the Kronecker coefficients for arbitrary representations. In fact, it is known that the Kronecker coefficients in general are unbounded for large- $N$  [115].

For a defect to be non-trivial in the large- $N$  limit, it needs to satisfy<sup>3</sup>

$$\lim_{N \rightarrow \infty} \frac{\chi_R([g])}{\dim_R} < 1. \quad (6.1)$$

If this is not the case, in the limit  $N \rightarrow \infty$  the defect commutes with all the operators and therefore the contributions of non-local operators are zero (see section 2.2.4 of [1]). We conclude that for finite  $N$  the contributions of non-local operators on the right-hand side of figure 9 are subleading.

Let us now comment on what representations lead to non-trivial Ward identities in the large- $N$  limit. Recall that as is the case for the conjugacy classes, representations of  $S_N$  are also labeled by partitions of  $N$ ,<sup>4</sup>

$$R: (r_1, r_2, \dots), \quad r_1 + r_2 + \dots = N. \quad (6.2)$$

We can associate a Young tableau to each representation where the number of boxes in the row  $i$  is equal to  $r_i$ . For a representation  $R$ ,<sup>5</sup> the character of a conjugacy class with a single cycle has the following limit [116, 117]

$$\lim_{N \rightarrow \infty} \frac{\chi_R([n])}{\dim_R} = \sum_i \alpha_i^n - (-1)^n \sum_i \beta_i^n, \quad (6.3)$$

where  $\alpha_i = \lim_{N \rightarrow \infty} \frac{r_i}{N}$  is the number of boxes in row  $i$  divided by  $N$ . Similarly,  $\beta_i$  is the number of boxes in the column  $i$  divided by  $N$ . As an example, let us look at the standard representation of  $S_N$ , which corresponds to the Young tableau

|  |  |  |     |  |
|--|--|--|-----|--|
|  |  |  | ... |  |
|  |  |  |     |  |

<sup>3</sup>Note the characters satisfy  $\chi_R([g]) \leq \dim_R$  for any representation.

<sup>4</sup>Here we are using a different notation to label partitions of  $N$  compared to section 3 where  $l_i$  is the number of  $r_i$ 's that satisfy  $r_i = i$ .

<sup>5</sup>The representation needs to have a well-defined limit, see for example [116].

with the corresponding partition being  $(N - 1, 1)$ . We see that

$$\alpha = \lim_{N \rightarrow \infty} \left( \frac{N-1}{N}, \frac{1}{N}, 0, 0, \dots \right) = (1, 0, 0, \dots), \quad (6.4)$$

and

$$\beta = \lim_{N \rightarrow \infty} \left( \frac{2}{N}, \frac{1}{N}, \frac{1}{N}, \dots \right) = (0, 0, 0, \dots). \quad (6.5)$$

Hence, in the large- $N$  limit  $\frac{\chi_S([g])}{\dim_S} \rightarrow 1$ . This can also be derived by calculating the character of the standard representation for finite  $N$  using the Frobenius formula. Therefore, we conclude that the Ward identities of the standard representations are trivial in the large- $N$  limit and only constrain subleading contributions coming from higher genus covering spaces.

Examples of representations with nontrivial ratio  $\frac{\chi_R([g])}{\dim_R}$  in the large- $N$ -limit are Young diagrams with two rows and diagrams of hook-shape. For both of the above, the Kronecker coefficients  $N_{RR}{}^{R'}$  are known [118–120]. For example, for the Young diagram with two rows

|  |  |  |     |  |
|--|--|--|-----|--|
|  |  |  | ... |  |
|  |  |  | ... |  |

we have

$$\alpha = \left( \frac{1}{2}, \frac{1}{2}, 0, \dots \right), \quad \beta = (0, 0, 0, \dots), \quad (6.6)$$

and therefore

$$\frac{\chi_R([n])}{\dim_R} \rightarrow \frac{1}{2^{n-1}}. \quad (6.7)$$

Thus, we expect the defect corresponding to this representation to be non-trivial in the large- $N$  limit.

## 7 Non-invertible symmetries and $\text{AdS}_3/\text{CFT}_2$ duality

The tensionless limit of the type II  $\text{AdS}_3 \times S^3 \times \mathcal{M}_4$  background with NS-NS flux is a holographic setup where the non-invertible symmetries of the  $S_N$  orbifold CFT are realized. In this section, we take some first steps in constructing the holographic duals of these defects.<sup>6</sup>

### 7.1 Tensionless limit

In recent years, an example of an exact AdS/CFT duality has been established, relating type II string theory on  $\text{AdS}_3 \times S^3 \times \mathcal{M}_4$  with  $k = 1$  units of NS-NS flux to the symmetric orbifold CFT  $\mathcal{M}_4^{\otimes N}/S_N$  in the large- $N$  limit. Here,  $\mathcal{M}_4$  is the CFT defined by the supersymmetric sigma-model with target space  $T^4$  or  $K_3$ . The characteristic feature of this duality is that the string theory is defined on highly curved spaces far away from the supergravity limit, yet a tractable worldsheet CFT exists. The worldsheet description is based on a product of a PSU(1,1|2) WZW model times a twisted  $\mathcal{M}_4$  sigma model [88, 122]. The agreement of the spectra on the AdS and CFT side was demonstrated in [88, 89], correlation functions

---

<sup>6</sup>See the recent paper [121] for related discussions.

were identified in [105, 123], the torus partition function was related to thermal AdS in [124], and D-branes on both sides were related in [108]. The identification of states on both sides relates the vertex operators  $\mathcal{O}_h^{(w)}(x)$  for a CFT state in a single cycle twisted sector labeled by  $w$  and a seed CFT state labeled by  $h$ , to a worldsheet vertex operator  $V_h^w(x; z)$  where  $w$  now denotes the  $w$ -spectral flow sector. This duality relates string amplitudes to the orbifold worldsheet correlation function

$$\begin{aligned} & \langle \mathcal{O}_{h_1}^{(w_1)}(x_1) \mathcal{O}_{h_2}^{(w_2)}(x_2) \cdots \mathcal{O}_{h_n}^{(w_n)}(x_n) \rangle_{S^2} \\ &= \int_{\mathcal{M}_{g,n}} d\mu \langle V_{h_1}^{w_1}(x_1; z_1) V_{h_2}^{w_2}(x_2; z_2) \cdots V_{h_n}^{w_n}(x_n; z_n) \rangle_{\Sigma_{g,n}}. \end{aligned} \quad (7.1)$$

The equality of both sides hinges on a remarkable localization property of the string path integral. The integral over the moduli space  $\mathcal{M}_{g,n}$  only contributes if a holomorphic covering map exists from the worldsheet  $\Sigma_{g,n}$  to the spacetime  $S^2$ . In this map, the vertex operator locations  $z_i$  on the worldsheet are mapped to the locations  $x_i$  of the CFT operators. These points have ramification indices  $w_i$  for  $i = 1, 2, \dots, n$ . The covering map behaves like

$$\Gamma(z) = x_i + a_i^\Gamma (z - z_i)_i^w + \mathcal{O}((z - z_i)^{w_i+1}), \quad (7.2)$$

near  $z = z_i$  and hence close to  $z_i$  describes a  $w_i$ -fold cover. The fact that the string worldsheet path integral localizes was shown using worldsheet Ward identities in [91] (see [125–129] for further discussions). In the following, we will focus on correlation functions at leading order in large- $N$ , where both the string worldsheet and the target space are two spheres. This map allows us to reverse engineer the topological defect on the string worldsheet.

We recall that the topological defect operators associated with  $\text{Rep}(S_N)$  given in (3.3) are projectors on the twisted sectors of the  $S_N$  orbifold, weighted by the character of the representation

$$\mathcal{I}_R^{\text{CFT}} = \sum_{[g]} \chi_R([g]) P_{[g]} \bar{P}_{[g]}. \quad (7.3)$$

Single string states created by the vertex operators  $V_h^w(x, z)$  in the spectral flow sector  $w$  correspond to single cycle twisted states of length  $w$ . Limiting ourselves to the single cycle twisted sector, a natural proposal for a worldsheet is the projector

$$\mathcal{I}_R^{\text{ws}} = \sum_w \chi_R([w]) P_w \bar{P}_w, \quad (7.4)$$

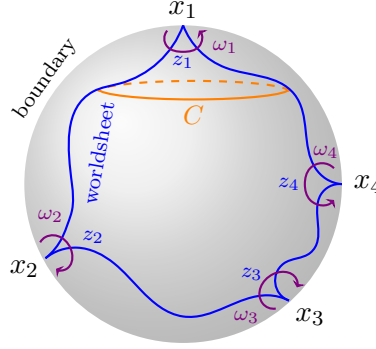
where  $P_w$  is a projector on the  $w$ -spectral flow sector and  $\chi_R([w])$  is the  $S_N$  character of a single cycle conjugacy class of length  $w$  in representation  $R$ .

Such a projector<sup>7</sup> can be constructed as an operator on the worldsheet using the free field Wakimoto construction [135] of the  $\text{SL}(2, \mathbb{R})$  part of the worldsheet sigma model. As an initial step, we define  $I[C]$  as a contour integral over the worldsheet bosonic field  $\Phi(z)$ ,<sup>8</sup>

$$I[C] = -2 \oint_C \frac{dz}{2\pi i} \partial_z \Phi, \quad (7.5)$$

<sup>7</sup>Another proposal for  $I[C]$  is the boundary central term  $\mathcal{I} = \oint \frac{da}{2\pi i} \frac{\partial_z \gamma}{\gamma}$  [130–134]. This operator gives the winding number of the covering map  $\Gamma(z)$ , but the winding number is only defined with respect to a base point ( $z = 0$  in the given formula) and does not lead to a viable worldsheet version of the topological defect.

<sup>8</sup>It has been shown in appendix C of [91] that the commutator of  $\partial_z \Phi$  with the spacetime Virasoro generators is a total derivative, hence  $I(C)$  is topological from the spacetime CFT perspective.



**Figure 10.** Contour  $C$  of operator  $I$  enclosing a vertex operator localized at  $x_1$ .

where the contour  $C$  is always oriented counterclockwise. The localization of the worldsheet path integral [91, 125, 126], pictorially shown in figure 10, leads to a very special property of the  $\partial_z \Phi(z)$  field when inserted in string correlation functions [91]

$$\langle (-2\partial\Phi(z))V_{h_1}^{w_1}(x_1; z_1) \cdots V_{h_n}^{w_n}(x_n; z_n) \rangle = \frac{\partial_z^2 \Gamma(z)}{\partial_z \Gamma(z)} \langle V_{h_1}^{w_1}(x_1; z_1) \cdots V_{h_n}^{w_n}(x_n; z_n) \rangle. \quad (7.6)$$

The expectation value of  $I(C)$  can then be evaluated using the following identity for the covering map  $\Gamma$

$$\frac{\partial_z^2 \Gamma(z)}{\partial_z \Gamma(z)} = \sum_{i=1}^n \frac{w_i - 1}{z - z_i} - \sum_{a=1}^{N-1} \frac{2}{z - z_a^*}. \quad (7.7)$$

Here,  $z_a$  are the location of the “secret poles,” which in general depend on the location of the CFT operators  $x_i$  and are necessary for the existence of the covering map. We have placed the  $N$ -th secret pole at  $z = \infty$ , which is always possible and doing so cancels the background charge of  $\Phi$  at infinity as discussed in [91]. It is easy to verify that the residue of  $\partial_z^2 \Gamma / \partial_z \Gamma$  at infinity is zero using the Riemann-Hurwitz formula. The value of  $I[C]$  for a given contour can now be determined using the residue theorem.

We can now give a proposal for  $\mathcal{I}_R^{\text{ws}}[C]$ , i.e. the worldsheet version of the topological defect for contour  $C$

$$\begin{aligned} \mathcal{I}_R^{\text{ws}}[C] &= \sum_w \chi_R(w) \delta_{w-I[C]} \\ &= \sum_{w=1}^{\infty} \chi_R(w) \int_0^{2\pi} \frac{d\alpha}{2\pi} \exp\left(i\alpha(w - 1 - I[C])\right). \end{aligned} \quad (7.8)$$

Thanks to the localization formula (7.6) and (7.7), it is easy to see that (7.5) evaluates to  $w_i - 1$  for a contour which encircles only  $z_i$  and is independent of small deformations of the contour. Hence, (7.8) is acting like the projector (7.4) when encircling a single vertex operator. In particular, if the contour is inserted around a vertex operator in the untwisted sector (this can be the ground state itself), we have  $w_i = 1$  and  $I[C]$  evaluates to zero and hence  $\mathcal{I}_R^{\text{ws}}[C]$  evaluates to  $\chi^R([1])$ , i.e. the quantum dimension of the operator.

Consider now a two-point function of twist operators on the boundary. The dual correlator is a two-point function of spectrally flowed vertex operators. It is not too difficult to check

that the value of  $\mathcal{I}_R^{\text{ws}}[C]$  around any of the two insertions is the same. This is consistent with inserting the corresponding operator  $\mathcal{I}_R^{\text{CFT}}$  on the boundary around either operators which gives the same value. The more interesting case is when we insert the operator  $\mathcal{I}_R^{\text{CFT}}$  around a line that does not enclose any operator. This gives the dimension of the representation. On the worldsheet, the corresponding operator  $\mathcal{I}_R^{\text{ws}}[C]$  will be along a curve with no vertex operator inside giving again the dimension of the representation. However, in this case, the contour integral in  $I[C]$  with no poles inside is not equal to the sum of the contour integrals around each of the vertex operators. This is because of the presence of the secret poles in (7.7). On the worldsheet, we interpret this as the result of the lasso effect described in figure 3.

The sign (alternating) defect is an example of an invertible defect for which  $\chi^A(w) = (-1)^{w+1}$ . When inserted on the boundary, the defect enforces the selection rule that  $\sum_i w_i$  is odd. Since the character of the sign representation on single cycles of length  $w$  satisfies  $\chi_A(w+2) = \chi_A(w)$ , we see that  $\mathcal{I}_R^{\text{ws}}[C]$  is blind to the secret poles (they contribute  $-2$  to the contour integral) and, therefore, the contour integral can be written as a sum of contour integrals around the ramification points reproducing the correct Ward identity on the worldsheet. The same line of reasoning can be applied to the case of three-point functions reproducing the corresponding Ward identities on the worldsheet.

## 7.2 Deformation away from the $k = 1$ and the supergravity limit

Turning on RR-fluxes in the bulk corresponds to deforming the symmetric orbifold CFT by an exactly marginal deformation [129, 136–145]. This deformation is proportional to a two-cycle twist field which we denote by  $\sigma_{[2]}$ . In the following, we argue that this deformation breaks all the non-invertible universal defects associated with the representations of  $S_N$ .

Consider the action of a topological defect  $\mathcal{I}_R$  on the twisted state created by  $\sigma_{[2]}$

$$\mathcal{I}_R |\sigma_{[2]}\rangle = \chi_R([2]) |\sigma_{[2]}\rangle. \quad (7.9)$$

The deformation breaks all the defects if and only if  $\chi_R([2]) \neq \dim_R$ . Below we prove that no defect satisfies this and, therefore, all the universal defects are broken in the supergravity limit.

Consider an element  $g$  in some representation  $R$ . Since all finite groups are unitarizable, this is a  $\dim_R \times \dim_R$  matrix whose eigenvalues satisfy  $|\lambda_i| = 1$ . Hence,

$$\chi_R(g) = \sum_{i=1}^{\dim_R} \lambda_i, \quad |\lambda_i| = 1. \quad (7.10)$$

We see that  $\chi_R(g) = \dim_R$  if and only if  $\lambda_i = 1 \ \forall i$  which is true only for the trivial representation. Hence, we conclude that any marginal deformation involving twist operators breaks all the universal defects. This means that all nontrivial symmetries do not survive the deformation away from the  $k = 1$  point.

## 8 Discussion

In this paper, we investigated certain aspects of the universal topological defects of the symmetric orbifold CFT  $\mathcal{M}^{\otimes N}/S_N$ . These defects are universal in the sense that they do

not depend on the details of the seed CFT, and exist in any  $S_N$  orbifold theory. Using modular transformations of torus partition function, we determined universal defect operators on which the topological lines can end. These operators are built on the twisted sector ground states. These states are derived from the vacuum of the seed CFT and hence do not depend on the details of the seed CFT other than the central charge and the conjugacy class labeling the twisted sector.

The non-invertibility of the symmetry leads to Ward identities which relate correlation functions of local operators to (sums of) correlation functions of defect and local operators with lines as well as junctions inserted. We illustrated this in the simplest example which exhibits non-invertible symmetries, namely the  $S_3$  symmetric orbifold.

In addition, we constructed an example of a non-universal defect which is built from a topological defect in the seed CFT (which we took to be diagonal RCFTs with Verlinde lines). This “maximally” fractional defect satisfies the Cardy constraints and it would be interesting to see whether more general solutions (maybe along the lines of similar constructions for D-branes in symmetric orbifolds in [109]) are possible. The maximally fractional defect we constructed in the present paper can be, in some sense, viewed as a product of the RCFT and symmetric orbifold defects.

We provided a criterion that determined which universal defects are non-trivial in the large- $N$  limit. This was possible due to the known asymptotic behavior of  $S_N$  characters. It would be particularly interesting to study the Ward identities of non-trivial defects in the large- $N$  limit. At this moment, this task looks daunting since there are no explicit expressions for both the F-symbols as well as the Kronecker coefficients of  $S_N$  to our knowledge.

The universal defects of the orbifold CFT are interesting in the context of  $\text{AdS}_3/\text{CFT}_2$  correspondence. We argued that these defects correspond to defects on the worldsheet, a case distinct from previous examples where boundary topological defects correspond to  $D$ -branes in the bulk. This is a peculiarity of the tensionless limit. This idea has appeared recently in other works [121, 146]. On the boundary, passing an operator through a topological defect generally produces non-local operators. It would be interesting to understand the holographic dual of these operators.

Away from the orbifold point, we showed that the universal defects are broken explicitly by a marginal deformation. Hence, these defects do not generally exist in the supergravity limit. It would be interesting to understand what happens when the deformation parameter is infinitesimal and what role these softly broken defects might play in holography.

In this work, we focused on the tensionless limit with  $k = 1$ . This corresponds to the smallest possible radius for  $\text{AdS}_3$ . For higher values of  $k$ , it is believed that the boundary theory is tensored with a Liouville factor but remains an orbifold CFT [134]. Hence, we expect our universal defects to be present for higher values of  $k$  as well. It would be interesting to understand the holographic dual of the universal defects in the large  $k$  limit where semi-classical supergravity computations are more reliable.

The entanglement entropy in the presence of topological defects is an interesting observable that has been calculated in some examples for topological defects [147, 148]. We plan to generalize these calculations to the topological defects constructed in the present paper [149] and it would be interesting to understand their implications for holography.

## Acknowledgments

The work of M.G. was supported, in part, by the National Science Foundation under grant PHY-2209700. M.G is grateful to the Department of Physics and Astronomy, Johns Hopkins University for hospitality while this paper was finalized. M.G., Y.L. and D.R. are grateful to the Bhaumik Institute for support. K.R. is supported by the Simons Collaboration on Global Categorical Symmetries and also by the NSF grant PHY-2112699.

## A Defect Hilbert spaces in $S^3$ orbifold

In this appendix, we verify the counting of defect operators in section 4. From (2.21), the torus partition function of the  $S_3$  symmetric orbifold is

$$Z(\tau) = \frac{1}{6}Z(\tau)^3 + \frac{1}{3}Z(3\tau) + \frac{1}{2}Z(\tau)Z(2\tau) + \frac{1}{2}Z(\tau) \left( Z\left(\frac{\tau}{2}\right) + Z\left(\frac{\tau+1}{2}\right) \right) + \frac{1}{3} \left( Z\left(\frac{\tau}{3}\right) + Z\left(\frac{\tau+1}{3}\right) + Z\left(\frac{\tau+2}{3}\right) \right), \quad (\text{A.1})$$

where for simplicity we only denoted the dependence on  $\tau$ , and the dependence on its complex conjugate  $\bar{\tau}$  is implied. The partition function of the seed CFT can be divided into the vacuum Virasoro module and the excited states

$$Z(\tau) = \frac{q^{-\frac{c}{24}} \bar{q}^{-\frac{c}{24}}}{\prod_{n=2}^{\infty} (1 - q^n) \prod_{n=2}^{\infty} (1 - \bar{q}^n)} + \text{excited states}, \quad (\text{A.2})$$

where  $q(\tau) = e^{2\pi i \tau}$ . The vacuum states lead to the universal sector of the symmetric orbifold CFT created by the bare twist operators  $\sigma_{[g]}$ .

Let us denote the scaling dimension of primary operators in the seed CFT by  $h_{\text{seed}}$ . From the partition function, it can be seen that the bare twist operators for a single cycle with length  $k$  have scaling dimensions,

$$h_k^{(bt)} = \frac{c}{24} \left( k - \frac{1}{k} \right), \quad (\text{A.3})$$

where ‘bt’ is the abbreviation for bare twists. In the  $S^3$  orbifold,  $k = 1, 2, 3$ . More generally, the scaling dimension in the single  $k$ -cycle twisted sector is

$$h = \frac{c}{24} \left( k - \frac{1}{k} \right) + \frac{h_{\text{seed}}}{k}. \quad (\text{A.4})$$

Let us now consider the defect Hilbert spaces  $\mathcal{H}_A, \mathcal{H}_S$  associated to the defects of section 4. Using (3.7) and (4.3), we can write down the torus partition function with a defect  $\mathcal{I}_A$  inserted along the spacial cycle,

$$Z_A(\tau) = \frac{1}{6}Z(\tau)^3 + \frac{1}{3}Z(3\tau) + \frac{1}{2}Z(\tau)Z(2\tau) - \frac{1}{2}Z(\tau) \left( Z\left(\frac{\tau}{2}\right) + Z\left(\frac{\tau+1}{2}\right) \right) + \frac{1}{3} \left( Z\left(\frac{\tau}{3}\right) + Z\left(\frac{\tau+1}{3}\right) + Z\left(\frac{\tau+2}{3}\right) \right). \quad (\text{A.5})$$

Under modular transformation (2.19), the defect partition function becomes

$$Z^A(\tau') = \frac{1}{6}Z(\tau)^3 - \frac{1}{2}Z(\tau)Z(2\tau) + \frac{1}{3}Z(3\tau) + \frac{1}{2}Z(\tau) \left( Z\left(\frac{\tau}{2}\right) - Z\left(\frac{\tau+1}{2}\right) \right) + \frac{1}{3} \left( Z\left(\frac{\tau}{3}\right) + Z\left(\frac{\tau+1}{3}\right) + Z\left(\frac{\tau+2}{3}\right) \right). \quad (\text{A.6})$$

Using (A.2) we can expand the above in powers of  $q$  and  $\bar{q}$ . For the 2-cycle twisted sector,

$$\frac{1}{2}Z(\tau) \left( Z\left(\frac{\tau}{2}\right) - Z\left(\frac{\tau+1}{2}\right) \right) = (q\bar{q})^{-\frac{c}{16}} \left( q^{\frac{3}{2}} + \bar{q}^{\frac{3}{2}} + q^{\frac{3}{2}}\bar{q} + q\bar{q}^{\frac{3}{2}} + 3q^2\bar{q}^{3/2} + 3q^{3/2}\bar{q}^2 \dots \right), \quad (\text{A.7})$$

and hence there is no bare twist-operator in  $\mathcal{H}_A$  verifying (4.6). Similarly for the 3-cycle we have that

$$\frac{1}{3} \left( Z\left(\frac{\tau}{3}\right) + Z\left(\frac{\tau+1}{3}\right) + Z\left(\frac{\tau+2}{3}\right) \right) = (q\bar{q})^{-\frac{c}{72}} (1 + q + q^{2/3}\bar{q}^{2/3} + \bar{q} + q\bar{q} + \dots). \quad (\text{A.8})$$

Finally for the 1-cycle sector,

$$\frac{1}{6}Z(\tau)^3 + \frac{1}{3}Z(3\tau) - \frac{1}{2}Z(\tau)Z(2\tau) = (q\bar{q})^{-\frac{c}{8}} (q^2\bar{q}^2 + q^3\bar{q}^2 + q^2\bar{q}^3 + q^3\bar{q}^3 + 3q^2\bar{q}^4 + 3q^4\bar{q}^2 \dots), \quad (\text{A.9})$$

which means that there is no operator with zero scaling dimensions and therefore  $\text{Hom}(\mathcal{I}_A, \mathcal{I}_A)$  is trivial.

Similarly, we can study the  $\mathcal{I}_S$  defect Hilbert space. There is no identity operator in the  $\mathcal{I}_S$  defect Hilbert space. There is a bare 2-cycle and no bare 3-cycle twist operators, in agreement with (4.6).

**Open Access.** This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

## References

- [1] C.-M. Chang et al., *Topological Defect Lines and Renormalization Group Flows in Two Dimensions*, *JHEP* **01** (2019) 026 [[arXiv:1802.04445](https://arxiv.org/abs/1802.04445)] [[INSPIRE](#)].
- [2] Y.-H. Lin and S.-H. Shao, *Duality Defect of the Monster CFT*, *J. Phys. A* **54** (2021) 065201 [[arXiv:1911.00042](https://arxiv.org/abs/1911.00042)] [[INSPIRE](#)].
- [3] R. Thorngren and Y. Wang, *Fusion category symmetry. Part I. Anomaly in-flow and gapped phases*, *JHEP* **04** (2024) 132 [[arXiv:1912.02817](https://arxiv.org/abs/1912.02817)] [[INSPIRE](#)].
- [4] Z. Komargodski, K. Ohmori, K. Roumpedakis and S. Seifnashri, *Symmetries and strings of adjoint QCD<sub>2</sub>*, *JHEP* **03** (2021) 103 [[arXiv:2008.07567](https://arxiv.org/abs/2008.07567)] [[INSPIRE](#)].
- [5] R. Thorngren and Y. Wang, *Fusion category symmetry. Part II. Categoriosities at  $c = 1$  and beyond*, *JHEP* **07** (2024) 051 [[arXiv:2106.12577](https://arxiv.org/abs/2106.12577)] [[INSPIRE](#)].
- [6] M. Nguyen, Y. Tanizaki and M. Ünsal, *Semi-Abelian gauge theories, non-invertible symmetries, and string tensions beyond N-ality*, *JHEP* **03** (2021) 238 [[arXiv:2101.02227](https://arxiv.org/abs/2101.02227)] [[INSPIRE](#)].

- [7] Y. Choi et al., *Noninvertible duality defects in 3+1 dimensions*, *Phys. Rev. D* **105** (2022) 125016 [[arXiv:2111.01139](#)] [[INSPIRE](#)].
- [8] J. Kaidi, K. Ohmori and Y. Zheng, *Kramers-Wannier-like Duality Defects in  $(3+1)D$  Gauge Theories*, *Phys. Rev. Lett.* **128** (2022) 111601 [[arXiv:2111.01141](#)] [[INSPIRE](#)].
- [9] Y. Choi et al., *Non-invertible Condensation, Duality, and Triality Defects in  $3+1$  Dimensions*, *Commun. Math. Phys.* **402** (2023) 489 [[arXiv:2204.09025](#)] [[INSPIRE](#)].
- [10] C. Cordova and K. Ohmori, *Noninvertible Chiral Symmetry and Exponential Hierarchies*, *Phys. Rev. X* **13** (2023) 011034 [[arXiv:2205.06243](#)] [[INSPIRE](#)].
- [11] K. Roumpedakis, S. Seifnashri and S.-H. Shao, *Higher Gauging and Non-invertible Condensation Defects*, *Commun. Math. Phys.* **401** (2023) 3043 [[arXiv:2204.02407](#)] [[INSPIRE](#)].
- [12] L. Bhardwaj, L.E. Bottini, S. Schäfer-Nameki and A. Tiwari, *Non-invertible higher-categorical symmetries*, *SciPost Phys.* **14** (2023) 007 [[arXiv:2204.06564](#)] [[INSPIRE](#)].
- [13] J.A. Damia, R. Argurio and L. Tizzano, *Continuous Generalized Symmetries in Three Dimensions*, *JHEP* **05** (2023) 164 [[arXiv:2206.14093](#)] [[INSPIRE](#)].
- [14] G. Arias-Tamargo and D. Rodriguez-Gomez, *Non-invertible symmetries from discrete gauging and completeness of the spectrum*, *JHEP* **04** (2023) 093 [[arXiv:2204.07523](#)] [[INSPIRE](#)].
- [15] Y. Hayashi and Y. Tanizaki, *Non-invertible self-duality defects of Cardy-Rabinovici model and mixed gravitational anomaly*, *JHEP* **08** (2022) 036 [[arXiv:2204.07440](#)] [[INSPIRE](#)].
- [16] J. Kaidi, G. Zafrir and Y. Zheng, *Non-invertible symmetries of  $\mathcal{N} = 4$  SYM and twisted compactification*, *JHEP* **08** (2022) 053 [[arXiv:2205.01104](#)] [[INSPIRE](#)].
- [17] A. Antinucci, G. Galati and G. Rizi, *On continuous 2-category symmetries and Yang-Mills theory*, *JHEP* **12** (2022) 061 [[arXiv:2206.05646](#)] [[INSPIRE](#)].
- [18] Y. Choi, H.T. Lam and S.-H. Shao, *Noninvertible Time-Reversal Symmetry*, *Phys. Rev. Lett.* **130** (2023) 131602 [[arXiv:2208.04331](#)] [[INSPIRE](#)].
- [19] Y. Choi, H.T. Lam and S.-H. Shao, *Noninvertible Global Symmetries in the Standard Model*, *Phys. Rev. Lett.* **129** (2022) 161601 [[arXiv:2205.05086](#)] [[INSPIRE](#)].
- [20] T. Bartsch, M. Bullimore, A.E.V. Ferrari and J. Pearson, *Non-invertible Symmetries and Higher Representation Theory I*, *SciPost Phys.* **17** (2024) 015 [[arXiv:2208.05993](#)] [[INSPIRE](#)].
- [21] B. Heidenreich et al., *Non-invertible global symmetries and completeness of the spectrum*, *JHEP* **09** (2021) 203 [[arXiv:2104.07036](#)] [[INSPIRE](#)].
- [22] C. Cordova, K. Ohmori and T. Rudelius, *Generalized symmetry breaking scales and weak gravity conjectures*, *JHEP* **11** (2022) 154 [[arXiv:2202.05866](#)] [[INSPIRE](#)].
- [23] T.-C. Huang, Y.-H. Lin and S. Seifnashri, *Construction of two-dimensional topological field theories with non-invertible symmetries*, *JHEP* **12** (2021) 028 [[arXiv:2110.02958](#)] [[INSPIRE](#)].
- [24] V. Bashmakov, M. Del Zotto and A. Hasan, *On the 6d origin of non-invertible symmetries in 4d*, *JHEP* **09** (2023) 161 [[arXiv:2206.07073](#)] [[INSPIRE](#)].
- [25] F. Benini, C. Copetti and L. Di Pietro, *Factorization and global symmetries in holography*, *SciPost Phys.* **14** (2023) 019 [[arXiv:2203.09537](#)] [[INSPIRE](#)].
- [26] J.A. Damia, R. Argurio and E. Garcia-Valdecasas, *Non-invertible defects in 5d, boundaries and holography*, *SciPost Phys.* **14** (2023) 067 [[arXiv:2207.02831](#)] [[INSPIRE](#)].
- [27] Y.-H. Lin, M. Okada, S. Seifnashri and Y. Tachikawa, *Asymptotic density of states in 2d CFTs with non-invertible symmetries*, *JHEP* **03** (2023) 094 [[arXiv:2208.05495](#)] [[INSPIRE](#)].

- [28] P. Niro, K. Roumpedakis and O. Sela, *Exploring non-invertible symmetries in free theories*, *JHEP* **03** (2023) 005 [[arXiv:2209.11166](#)] [[INSPIRE](#)].
- [29] C. Cordova and K. Ohmori, *Quantum duality in electromagnetism and the fine structure constant*, *Phys. Rev. D* **109** (2024) 105019 [[arXiv:2307.12927](#)] [[INSPIRE](#)].
- [30] T. Rudelius and S.-H. Shao, *Topological Operators and Completeness of Spectrum in Discrete Gauge Theories*, *JHEP* **12** (2020) 172 [[arXiv:2006.10052](#)] [[INSPIRE](#)].
- [31] K. Inamura, *Topological field theories and symmetry protected topological phases with fusion category symmetries*, *JHEP* **05** (2021) 204 [[arXiv:2103.15588](#)] [[INSPIRE](#)].
- [32] M. Nguyen, Y. Tanizaki and M. Ünsal, *Noninvertible 1-form symmetry and Casimir scaling in 2D Yang-Mills theory*, *Phys. Rev. D* **104** (2021) 065003 [[arXiv:2104.01824](#)] [[INSPIRE](#)].
- [33] L. Bhardwaj, S. Schäfer-Nameki and J. Wu, *Universal Non-Invertible Symmetries*, *Fortsch. Phys.* **70** (2022) 2200143 [[arXiv:2208.05973](#)] [[INSPIRE](#)].
- [34] J. Kaidi, K. Ohmori and Y. Zheng, *Symmetry TFTs for Non-invertible Defects*, *Commun. Math. Phys.* **404** (2023) 1021 [[arXiv:2209.11062](#)] [[INSPIRE](#)].
- [35] S. Chen and Y. Tanizaki, *Solitonic Symmetry beyond Homotopy: Invertibility from Bordism and Noninvertibility from Topological Quantum Field Theory*, *Phys. Rev. Lett.* **131** (2023) 011602 [[arXiv:2210.13780](#)] [[INSPIRE](#)].
- [36] V. Bashmakov, M. Del Zotto, A. Hasan and J. Kaidi, *Non-invertible symmetries of class S theories*, *JHEP* **05** (2023) 225 [[arXiv:2211.05138](#)] [[INSPIRE](#)].
- [37] C. Cordova, S. Hong, S. Koren and K. Ohmori, *Neutrino Masses from Generalized Symmetry Breaking*, *Phys. Rev. X* **14** (2024) 031033 [[arXiv:2211.07639](#)] [[INSPIRE](#)].
- [38] Y. Choi, H.T. Lam and S.-H. Shao, *Non-invertible Gauss law and axions*, *JHEP* **09** (2023) 067 [[arXiv:2212.04499](#)] [[INSPIRE](#)].
- [39] R. Yokokura, *Non-invertible symmetries in axion electrodynamics*, [arXiv:2212.05001](#) [[INSPIRE](#)].
- [40] L. Bhardwaj, S. Schäfer-Nameki and A. Tiwari, *Unifying constructions of non-invertible symmetries*, *SciPost Phys.* **15** (2023) 122 [[arXiv:2212.06159](#)] [[INSPIRE](#)].
- [41] L. Bhardwaj, L.E. Bottini, S. Schäfer-Nameki and A. Tiwari, *Non-invertible symmetry webs*, *SciPost Phys.* **15** (2023) 160 [[arXiv:2212.06842](#)] [[INSPIRE](#)].
- [42] T. Bartsch, M. Bullimore, A.E.V. Ferrari and J. Pearson, *Non-invertible Symmetries and Higher Representation Theory II*, *SciPost Phys.* **17** (2024) 067 [[arXiv:2212.07393](#)] [[INSPIRE](#)].
- [43] A. Apte, C. Cordova and H.T. Lam, *Obstructions to gapped phases from noninvertible symmetries*, *Phys. Rev. B* **108** (2023) 045134 [[arXiv:2212.14605](#)] [[INSPIRE](#)].
- [44] J. Kaidi, E. Nardoni, G. Zafrir and Y. Zheng, *Symmetry TFTs and anomalies of non-invertible symmetries*, *JHEP* **10** (2023) 053 [[arXiv:2301.07112](#)] [[INSPIRE](#)].
- [45] M. Koide, Y. Nagoya and S. Yamaguchi, *Noninvertible symmetries and boundaries in four dimensions*, *Phys. Rev. D* **108** (2023) 065009 [[arXiv:2304.01550](#)] [[INSPIRE](#)].
- [46] J.A. Damia et al., *Non-invertible symmetries along 4d RG flows*, *JHEP* **02** (2024) 084 [[arXiv:2305.17084](#)] [[INSPIRE](#)].
- [47] L. Bhardwaj and S. Schäfer-Nameki, *Generalized Charges, Part II: Non-Invertible Symmetries and the Symmetry TFT*, [arXiv:2305.17159](#) [[INSPIRE](#)].

- [48] M. van Beest et al., *Monopoles, Scattering, and Generalized Symmetries*, [arXiv:2306.07318](#) [[INSPIRE](#)].
- [49] C. Lawrie, X. Yu and H.Y. Zhang, *Intermediate defect groups, polarization pairs, and noninvertible duality defects*, *Phys. Rev. D* **109** (2024) 026005 [[arXiv:2306.11783](#)] [[INSPIRE](#)].
- [50] S. Chen and Y. Tanizaki, *Solitonic symmetry as non-invertible symmetry: cohomology theories with TQFT coefficients*, [arXiv:2307.00939](#) [[INSPIRE](#)].
- [51] C. Cordova, P.-S. Hsin and C. Zhang, *Anomalies of Non-Invertible Symmetries in  $(3+1)d$* , [arXiv:2308.11706](#) [[INSPIRE](#)].
- [52] A. Antinucci et al., *Anomalies of non-invertible self-duality symmetries: fractionalization and gauging*, [arXiv:2308.11707](#) [[INSPIRE](#)].
- [53] V. Benedetti, H. Casini and J.M. Magan, *ABJ anomaly as a  $U(1)$  symmetry and Noether's theorem*, [arXiv:2309.03264](#) [[INSPIRE](#)].
- [54] Y. Choi, M. Forslund, H.T. Lam and S.-H. Shao, *Quantization of Axion-Gauge Couplings and Noninvertible Higher Symmetries*, *Phys. Rev. Lett.* **132** (2024) 121601 [[arXiv:2309.03937](#)] [[INSPIRE](#)].
- [55] Y. Nagoya and S. Shimamori, *Non-invertible duality defect and non-commutative fusion algebra*, *JHEP* **12** (2023) 062 [[arXiv:2309.05294](#)] [[INSPIRE](#)].
- [56] M. Okada and Y. Tachikawa, *Non-invertible symmetries act locally by quantum operations*, [arXiv:2403.20062](#) [[INSPIRE](#)].
- [57] N. Kan, K. Kawabata and H. Wada, *Symmetry fractionalization and duality defects in Maxwell theory*, [arXiv:2404.14481](#) [[INSPIRE](#)].
- [58] F. Apruzzi, I. Bah, F. Bonetti and S. Schäfer-Nameki, *Noninvertible Symmetries from Holography and Branes*, *Phys. Rev. Lett.* **130** (2023) 121601 [[arXiv:2208.07373](#)] [[INSPIRE](#)].
- [59] I. García Etxebarria, *Branes and Non-Invertible Symmetries*, *Fortsch. Phys.* **70** (2022) 2200154 [[arXiv:2208.07508](#)] [[INSPIRE](#)].
- [60] J.J. Heckman et al., *Top down approach to topological duality defects*, *Phys. Rev. D* **108** (2023) 046015 [[arXiv:2212.09743](#)] [[INSPIRE](#)].
- [61] J.J. Heckman, M. Hübner, E. Torres and H.Y. Zhang, *The Branes Behind Generalized Symmetry Operators*, *Fortsch. Phys.* **71** (2023) 2200180 [[arXiv:2209.03343](#)] [[INSPIRE](#)].
- [62] M. van Beest, D.S.W. Gould, S. Schäfer-Nameki and Y.-N. Wang, *Symmetry TFTs for 3d QFTs from M-theory*, *JHEP* **02** (2023) 226 [[arXiv:2210.03703](#)] [[INSPIRE](#)].
- [63] M. Etheredge, I. García Etxebarria, B. Heidenreich and S. Rauch, *Branes and symmetries for  $\mathcal{N}=3$  S-folds*, *JHEP* **09** (2023) 005 [[arXiv:2302.14068](#)] [[INSPIRE](#)].
- [64] I. Bah, E. Leung and T. Waddleton, *Non-invertible symmetries, brane dynamics, and tachyon condensation*, *JHEP* **01** (2024) 117 [[arXiv:2306.15783](#)] [[INSPIRE](#)].
- [65] M. Dierigl, J.J. Heckman, M. Montero and E. Torres, *R7-branes as charge conjugation operators*, *Phys. Rev. D* **109** (2024) 046004 [[arXiv:2305.05689](#)] [[INSPIRE](#)].
- [66] A. Bourget, A. Collinucci and S. Schäfer-Nameki, *Generalized Toric Polygons, T-branes, and 5d SCFTs*, [arXiv:2301.05239](#) [[INSPIRE](#)].
- [67] F. Apruzzi, F. Bonetti, D.S.W. Gould and S. Schäfer-Nameki, *Aspects of Categorical Symmetries from Branes: SymTFTs and Generalized Charges*, [arXiv:2306.16405](#) [[INSPIRE](#)].

- [68] F. Baume et al., *SymTrees and Multi-Sector QFTs*, *Phys. Rev. D* **109** (2024) 106013 [[arXiv:2310.12980](#)] [[INSPIRE](#)].
- [69] M. Cvetič, J.J. Heckman, M. Hübner and E. Torres, *Generalized symmetries, gravity, and the swampland*, *Phys. Rev. D* **109** (2024) 026012 [[arXiv:2307.13027](#)] [[INSPIRE](#)].
- [70] J.J. Heckman, M. Hübner and C. Murdia, *On the holographic dual of a topological symmetry operator*, *Phys. Rev. D* **110** (2024) 046007 [[arXiv:2401.09538](#)] [[INSPIRE](#)].
- [71] R. Argurio et al., *On the symmetry TFT of Yang-Mills-Chern-Simons theory*, *JHEP* **07** (2024) 130 [[arXiv:2404.06601](#)] [[INSPIRE](#)].
- [72] H.Y. Zhang, *K-theoretic Global Symmetry in String-constructed QFT and T-duality*, [arXiv:2404.16097](#) [[INSPIRE](#)].
- [73] E.P. Verlinde, *Fusion Rules and Modular Transformations in 2D Conformal Field Theory*, *Nucl. Phys. B* **300** (1988) 360 [[INSPIRE](#)].
- [74] V.B. Petkova and J.B. Zuber, *Generalized twisted partition functions*, *Phys. Lett. B* **504** (2001) 157 [[hep-th/0011021](#)] [[INSPIRE](#)].
- [75] V.B. Petkova and J.B. Zuber, *The Many faces of Ocneanu cells*, *Nucl. Phys. B* **603** (2001) 449 [[hep-th/0101151](#)] [[INSPIRE](#)].
- [76] J. Frohlich, J. Fuchs, I. Runkel and C. Schweigert, *Kramers-Wannier duality from conformal defects*, *Phys. Rev. Lett.* **93** (2004) 070601 [[cond-mat/0404051](#)] [[INSPIRE](#)].
- [77] J. Frohlich, J. Fuchs, I. Runkel and C. Schweigert, *Duality and defects in rational conformal field theory*, *Nucl. Phys. B* **763** (2007) 354 [[hep-th/0607247](#)] [[INSPIRE](#)].
- [78] J. Fuchs, I. Runkel and C. Schweigert, *TFT construction of RCFT correlators 1. Partition functions*, *Nucl. Phys. B* **646** (2002) 353 [[hep-th/0204148](#)] [[INSPIRE](#)].
- [79] L.J. Dixon, J.A. Harvey, C. Vafa and E. Witten, *Strings on Orbifolds*, *Nucl. Phys. B* **261** (1985) 678 [[INSPIRE](#)].
- [80] L.J. Dixon, D. Friedan, E.J. Martinec and S.H. Shenker, *The Conformal Field Theory of Orbifolds*, *Nucl. Phys. B* **282** (1987) 13 [[INSPIRE](#)].
- [81] R. Dijkgraaf, G.W. Moore, E.P. Verlinde and H.L. Verlinde, *Elliptic genera of symmetric products and second quantized strings*, *Commun. Math. Phys.* **185** (1997) 197 [[hep-th/9608096](#)] [[INSPIRE](#)].
- [82] A. Jevicki, M. Mihailescu and S. Ramgoolam, *Gravity from CFT on  $S^{**}N(X)$ : Symmetries and interactions*, *Nucl. Phys. B* **577** (2000) 47 [[hep-th/9907144](#)] [[INSPIRE](#)].
- [83] O. Lunin and S.D. Mathur, *Correlation functions for  $M^{**}N / S(N)$  orbifolds*, *Commun. Math. Phys.* **219** (2001) 399 [[hep-th/0006196](#)] [[INSPIRE](#)].
- [84] A. Pakman, L. Rastelli and S.S. Razamat, *Diagrams for Symmetric Product Orbifolds*, *JHEP* **10** (2009) 034 [[arXiv:0905.3448](#)] [[INSPIRE](#)].
- [85] J.M. Maldacena, *The large  $N$  limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [[hep-th/9711200](#)] [[INSPIRE](#)].
- [86] N. Seiberg and E. Witten, *The  $D1/D5$  system and singular CFT*, *JHEP* **04** (1999) 017 [[hep-th/9903224](#)] [[INSPIRE](#)].
- [87] M.R. Gaberdiel, R. Gopakumar and C. Hull, *Stringy  $AdS_3$  from the worldsheet*, *JHEP* **07** (2017) 090 [[arXiv:1704.08665](#)] [[INSPIRE](#)].

- [88] M.R. Gaberdiel and R. Gopakumar, *Tensionless string spectra on  $AdS_3$* , *JHEP* **05** (2018) 085 [[arXiv:1803.04423](#)] [[INSPIRE](#)].
- [89] L. Eberhardt, M.R. Gaberdiel and R. Gopakumar, *The Worldsheet Dual of the Symmetric Product CFT*, *JHEP* **04** (2019) 103 [[arXiv:1812.01007](#)] [[INSPIRE](#)].
- [90] G. Giribet et al., *Superstrings on  $AdS_3$  at  $k = 1$* , *JHEP* **08** (2018) 204 [[arXiv:1803.04420](#)] [[INSPIRE](#)].
- [91] L. Eberhardt, M.R. Gaberdiel and R. Gopakumar, *Deriving the  $AdS_3/CFT_2$  correspondence*, *JHEP* **02** (2020) 136 [[arXiv:1911.00378](#)] [[INSPIRE](#)].
- [92] C. Vafa, *Quantum Symmetries of String Vacua*, *Mod. Phys. Lett. A* **4** (1989) 1615 [[INSPIRE](#)].
- [93] Y. Tachikawa, *On gauging finite subgroups*, *SciPost Phys.* **8** (2020) 015 [[arXiv:1712.09542](#)] [[INSPIRE](#)].
- [94] J.L. Cardy, *Boundary Conditions, Fusion Rules and the Verlinde Formula*, *Nucl. Phys. B* **324** (1989) 581 [[INSPIRE](#)].
- [95] M. Oshikawa and I. Affleck, *Boundary conformal field theory approach to the critical two-dimensional Ising model with a defect line*, *Nucl. Phys. B* **495** (1997) 533 [[cond-mat/9612187](#)] [[INSPIRE](#)].
- [96] C. Bachas, J. de Boer, R. Dijkgraaf and H. Ooguri, *Permeable conformal walls and holography*, *JHEP* **06** (2002) 027 [[hep-th/0111210](#)] [[INSPIRE](#)].
- [97] N. Ishibashi, *The Boundary and Crosscap States in Conformal Field Theories*, *Mod. Phys. Lett. A* **4** (1989) 251 [[INSPIRE](#)].
- [98] A. Recknagel, *Permutation branes*, *JHEP* **04** (2003) 041 [[hep-th/0208119](#)] [[INSPIRE](#)].
- [99] N. Drukker, D. Gaiotto and J. Gomis, *The Virtue of Defects in 4D Gauge Theories and 2D CFTs*, *JHEP* **06** (2011) 025 [[arXiv:1003.1112](#)] [[INSPIRE](#)].
- [100] C. Cordova and G. Rizi, *Non-Invertible Symmetry in Calabi-Yau Conformal Field Theories*, [arXiv:2312.17308](#) [[INSPIRE](#)].
- [101] R. Dijkgraaf, C. Vafa, E.P. Verlinde and H.L. Verlinde, *The Operator Algebra of Orbifold Models*, *Commun. Math. Phys.* **123** (1989) 485 [[INSPIRE](#)].
- [102] A. Klemm and M.G. Schmidt, *Orbifolds by Cyclic Permutations of Tensor Product Conformal Field Theories*, *Phys. Lett. B* **245** (1990) 53 [[INSPIRE](#)].
- [103] J.M. Maldacena, G.W. Moore and A. Strominger, *Counting BPS black holes in toroidal Type II string theory*, [hep-th/9903163](#) [[INSPIRE](#)].
- [104] K. Roumpedakis, *Comments on the  $S_N$  orbifold CFT in the large  $N$ -limit*, *JHEP* **07** (2018) 038 [[arXiv:1804.03207](#)] [[INSPIRE](#)].
- [105] A. Dei and L. Eberhardt, *Correlators of the symmetric product orbifold*, *JHEP* **01** (2020) 108 [[arXiv:1911.08485](#)] [[INSPIRE](#)].
- [106] D. Barter, J.C. Bridgeman and R. Wolf, *Computing associators of endomorphism fusion categories*, *SciPost Phys.* **13** (2022) 029 [[arXiv:2110.03644](#)].
- [107] L. Bhardwaj, L.E. Bottini, D. Pajer and S. Schäfer-Nameki, *Gapped Phases with Non-Invertible Symmetries:  $(1+1)d$* , [arXiv:2310.03784](#) [[INSPIRE](#)].
- [108] M.R. Gaberdiel, B. Knighton and J. Vošmera, *D-branes in  $AdS_3 \times S^3 \times \mathbb{T}^4$  at  $k = 1$  and their holographic duals*, *JHEP* **12** (2021) 149 [[arXiv:2110.05509](#)] [[INSPIRE](#)].

- [109] A. Belin, S. Biswas and J. Sully, *The spectrum of boundary states in symmetric orbifolds*, *JHEP* **01** (2022) 123 [[arXiv:2110.05491](#)] [[INSPIRE](#)].
- [110] J. Fuchs, M.R. Gaberdiel, I. Runkel and C. Schweigert, *Topological defects for the free boson CFT*, *J. Phys. A* **40** (2007) 11403 [[arXiv:0705.3129](#)] [[INSPIRE](#)].
- [111] C.P. Bachas, *On the Symmetries of Classical String Theory*, in the proceedings of the *workshop on Quantum Mechanics of Fundamental Systems: the Quest for Beauty and Simplicity: Dedicated to Claudio Bunster on the occasion of his 60th birthday*, Valdivia, Chile, 10–11 January 2008 [[DOI:10.1007/978-0-387-87499-9\\_3](#)] [[arXiv:0808.2777](#)] [[INSPIRE](#)].
- [112] C. Bachas, I. Brunner and D. Roggenkamp, *A worldsheet extension of  $O(d, d : Z)$* , *JHEP* **10** (2012) 039 [[arXiv:1205.4647](#)] [[INSPIRE](#)].
- [113] I. Brunner, N. Carqueville and D. Plencner, *Orbifolds and topological defects*, *Commun. Math. Phys.* **332** (2014) 669 [[arXiv:1307.3141](#)] [[INSPIRE](#)].
- [114] S. Bharadwaj, P. Niro and K. Roumpedakis, *Non-invertible defects on the worldsheet*, [arXiv:2408.14556](#) [[INSPIRE](#)].
- [115] I. Pak and G. Panova, *On the complexity of computing Kronecker coefficients*, *Comp. Complexity* **26** (2017) 1 [[arXiv:1404.0653](#)] [[INSPIRE](#)].
- [116] A. Borodin and G. Olshanski, *Representations of the Infinite Symmetric Group*, Cambridge University Press (2016) [[DOI:10.1017/cbo9781316798577](#)].
- [117] A. Okounkov, *On the representations of the infinite symmetric group*, [math/9803037](#).
- [118] J.B. Remmel, *A formula for the Kronecker products of Schur functions of hook shapes*, *J. Algebra* **120** (1989) 100.
- [119] J.B. Remmel and T. Whitehead, *On the Kronecker product of Schur functions of two row shapes*, *Bull. Belg. Math. Soc. Simon Stevin* **1** (1994) 649.
- [120] M.H. Rosas, *The Kronecker product of Schur functions indexed by two-row shapes or hook shapes*, [math/0001084](#).
- [121] J.J. Heckman et al., *On the Fate of Stringy Non-Invertible Symmetries*, [arXiv:2402.00118](#) [[INSPIRE](#)].
- [122] N. Berkovits, C. Vafa and E. Witten, *Conformal field theory of AdS background with Ramond-Ramond flux*, *JHEP* **03** (1999) 018 [[hep-th/9902098](#)] [[INSPIRE](#)].
- [123] A. Dei, M.R. Gaberdiel, R. Gopakumar and B. Knighton, *Free field world-sheet correlators for AdS<sub>3</sub>*, *JHEP* **02** (2021) 081 [[arXiv:2009.11306](#)] [[INSPIRE](#)].
- [124] L. Eberhardt, *Partition functions of the tensionless string*, *JHEP* **03** (2021) 176 [[arXiv:2008.07533](#)] [[INSPIRE](#)].
- [125] B. Knighton, S. Seet and V. Sriprachyakul, *Spectral flow and localisation in AdS<sub>3</sub> string theory*, *JHEP* **05** (2024) 113 [[arXiv:2312.08429](#)] [[INSPIRE](#)].
- [126] N.M. McStay and R.A. Reid-Edwards, *Symmetries and covering maps for the minimal tension string on AdS<sub>3</sub> × S<sup>3</sup> × T<sup>4</sup>*, *JHEP* **04** (2024) 048 [[arXiv:2306.16280](#)] [[INSPIRE](#)].
- [127] Y. Hikida and V. Schomerus, *Engineering perturbative string duals for symmetric product orbifold CFTs*, *JHEP* **06** (2024) 071 [[arXiv:2312.05317](#)] [[INSPIRE](#)].
- [128] W. Lerche, *Gromov-Witten/Hilbert versus AdS<sub>3</sub>/CFT<sub>2</sub> Correspondence*, [arXiv:2310.15237](#) [[INSPIRE](#)].

- [129] B. Knighton and V. Sriprachyakul, *Unravelling  $AdS_3/CFT_2$  near the boundary*, [arXiv:2404.07296](#) [[INSPIRE](#)].
- [130] A. Giveon, D. Kutasov and N. Seiberg, *Comments on string theory on  $AdS_3$* , *Adv. Theor. Math. Phys.* **2** (1998) 733 [[hep-th/9806194](#)] [[INSPIRE](#)].
- [131] D. Kutasov and N. Seiberg, *More comments on string theory on  $AdS_3$* , *JHEP* **04** (1999) 008 [[hep-th/9903219](#)] [[INSPIRE](#)].
- [132] A. Giveon and D. Kutasov, *Notes on  $AdS_3$* , *Nucl. Phys. B* **621** (2002) 303 [[hep-th/0106004](#)] [[INSPIRE](#)].
- [133] J. Kim and M. Porrati, *On the central charge of spacetime current algebras and correlators in string theory on  $AdS_3$* , *JHEP* **05** (2015) 076 [[arXiv:1503.07186](#)] [[INSPIRE](#)].
- [134] L. Eberhardt and M.R. Gaberdiel, *String theory on  $AdS_3$  and the symmetric orbifold of Liouville theory*, *Nucl. Phys. B* **948** (2019) 114774 [[arXiv:1903.00421](#)] [[INSPIRE](#)].
- [135] M. Wakimoto, *Fock representations of the affine lie algebra  $A_1^{(1)}$* , *Commun. Math. Phys.* **104** (1986) 605 [[INSPIRE](#)].
- [136] B.A. Burrington, A.W. Peet and I.G. Zadeh, *Operator mixing for string states in the  $D1$ - $D5$  CFT near the orbifold point*, *Phys. Rev. D* **87** (2013) 106001 [[arXiv:1211.6699](#)] [[INSPIRE](#)].
- [137] M.R. Gaberdiel, C. Peng and I.G. Zadeh, *Higgsing the stringy higher spin symmetry*, *JHEP* **10** (2015) 101 [[arXiv:1506.02045](#)] [[INSPIRE](#)].
- [138] B. Guo and S.D. Mathur, *Lifting of level-1 states in the  $D1D5$  CFT*, *JHEP* **03** (2020) 028 [[arXiv:1912.05567](#)] [[INSPIRE](#)].
- [139] C.A. Keller and I.G. Zadeh, *Conformal Perturbation Theory for Twisted Fields*, *J. Phys. A* **53** (2020) 095401 [[arXiv:1907.08207](#)] [[INSPIRE](#)].
- [140] L. Eberhardt, *A perturbative CFT dual for pure NS-NS  $AdS_3$  strings*, *J. Phys. A* **55** (2022) 064001 [[arXiv:2110.07535](#)] [[INSPIRE](#)].
- [141] B. Balthazar, A. Giveon, D. Kutasov and E.J. Martinec, *Asymptotically free  $AdS_3/CFT_2$* , *JHEP* **01** (2022) 008 [[arXiv:2109.00065](#)] [[INSPIRE](#)].
- [142] L. Apolo et al., *Deforming symmetric product orbifolds: a tale of moduli and higher spin currents*, *JHEP* **08** (2022) 159 [[arXiv:2204.07590](#)] [[INSPIRE](#)].
- [143] M.-A. Fiset, M.R. Gaberdiel, K. Naderi and V. Sriprachyakul, *Perturbing the symmetric orbifold from the worldsheet*, *JHEP* **07** (2023) 093 [[arXiv:2212.12342](#)] [[INSPIRE](#)].
- [144] M.R. Gaberdiel, R. Gopakumar and B. Nairz, *Beyond the tensionless limit: integrability in the symmetric orbifold*, *JHEP* **06** (2024) 030 [[arXiv:2312.13288](#)] [[INSPIRE](#)].
- [145] S. Frolov and A. Sfondrini, *Comments on integrability in the symmetric orbifold*, *JHEP* **08** (2024) 179 [[arXiv:2312.14114](#)] [[INSPIRE](#)].
- [146] J. Kaidi, Y. Tachikawa and H.Y. Zhang, *On a class of selection rules without group actions in field theory and string theory*, [arXiv:2402.00105](#) [[INSPIRE](#)].
- [147] E.M. Brehm, I. Brunner, D. Jaud and C. Schmidt-Colinet, *Entanglement and topological interfaces*, *Fortsch. Phys.* **64** (2016) 516 [[arXiv:1512.05945](#)] [[INSPIRE](#)].
- [148] M. Gutperle and J.D. Miller, *A note on entanglement entropy for topological interfaces in RCFTs*, *JHEP* **04** (2016) 176 [[arXiv:1512.07241](#)] [[INSPIRE](#)].
- [149] M. Gutperle, Y.-Y. Li, D. Rathore and K. Roumpedakis, *A note on entanglement entropy and topological defects in symmetric orbifold CFTs*, [arXiv:2406.10967](#) [[INSPIRE](#)].