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Rough solutions of the relativistic Euler equations

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Abstract. We prove that the time of classical existence of smooth solutions to the relativistic Euler equations can be bounded from below in terms of norms that measure the "(sound) wave-part" of the data in Sobolev space and "transport-part" in higher regularity Sobolev space and Hölder spaces. The solutions are allowed to have nontrivial vorticity and entropy. We use the geometric framework from [M. M. Disconzi and J. Speck, The relativistic Euler equations: Remarkable null structures and regularity properties, *Ann. Henri Poincaré* **20**(7) (2019) 2173–2270], where the relativistic Euler flow is decomposed into a "wave-part", that is, geometric wave equations for the velocity components, density and enthalpy, and a "transport-part", that is, transport-div-curl systems for the vorticity and entropy gradient. Our main result is that the Sobolev norm H^{2+} of the variables in the "wave-part" and the Hölder norm $C^{0,0+}$ of the variables in the "transport-part" can be controlled in terms of initial data for short times. We note that the Sobolev norm assumption H^{2+} is the optimal result for the variables in the "wave-part". Compared to low-regularity results for quasilinear wave equations and the three-dimensional (3D) non-relativistic compressible Euler equations, the main new challenge of the paper is that when controlling the acoustic geometry and bounding the wave equation energies, we must deal with the difficulty that the vorticity and entropy gradient are four-dimensional space-time vectors satisfying a space-time div-curl-transport system, where the space-time div-curl part is not elliptic. Due to lack of ellipticity, one cannot immediately rely on the approach taken in [M. M. Disconzi and J. Speck, The relativistic Euler equations: Remarkable null structures and regularity properties, *Ann. Henri Poincaré* **20**(7) (2019) 2173–2270] to control these terms. To overcome this difficulty, we show that the space-time div-curl systems imply elliptic div-curl-transport systems on constant-time hypersurfaces plus error terms that involve favorable differentiations and contractions with respect to the four-velocity. By using these structures, we are able to adequately control the vorticity and entropy gradient with the help of energy estimates for transport equations, elliptic estimates, Schauder estimates and Littlewood–Paley theory.

Keywords: Local well-posedness; low regularity; acoustic geometry; Schauder estimates; Strichartz estimates.

Mathematics Subject Classification 2020: 35Q31, 35Q75, 35Q35, 35L52, 35L72

1. Introduction

This paper is concerned with the special relativistic Euler equations on the Minkowski background $(\mathbb{R}^{1+3}, M)$, where M is the Minkowski metric. For use throughout the paper, we fix a coordinate system $\{x^\alpha\}_{\alpha=0,1,2,3}$, relative[a] to which $M_{\alpha\beta} := \mathrm{diag}(-1,1,1,1)$, where the speed of light is set to be 1. For these equations, there is considerable freedom in the choice of state-space variables, that is, the fundamental unknowns of the PDEs. In this work, we choose the logarithmic enthalpy h, the entropy s and the four-velocity v, which is a future-directed M-timelike vectorfield normalized as $M(v,v) = -1$. We allow for nontrivial vorticity. All other unknowns in the system can be considered as functions of the state-space variables. We denote the pressure as $\mathfrak{p} = \mathfrak{p}(h,s)$, the fluid density as $\varrho = \varrho(h,s)$ and speed of sound as $c := \sqrt{\frac{\partial \mathfrak{p}}{\partial \varrho}}$. In this coordinate system, the relativistic Euler equations can be expressed as[b]

$$v^\kappa \partial_\kappa h + c^2 \partial_\kappa v^\kappa = 0, \tag{1.1a}$$

$$v^\kappa \partial_\kappa (v_\flat)_\alpha + \partial_\alpha h + (v_\flat)_\alpha v^\kappa \partial_\kappa h - \mathfrak{q}\partial_\alpha s = 0, \tag{1.1b}$$

$$v^\kappa \partial_\kappa s = 0, \tag{1.1c}$$

where $\mathfrak{q} := \frac{\theta}{H}$ is temperature over enthalpy, which can be expressed as $\mathfrak{q} = \mathfrak{q}(h,s)$. Also see Secs. 2.2 and 2.3 for the details.

Our work intimately depends on a new formulation of the equations derived by Disconzi–Speck [12], where the authors found that the flow splits into a "sound-wave-part" ("wave-part" for short) for (h,s,v) and a "transport-div-curl-part" ("transport-part" for short) for the vorticity ω and the entropy gradient S. Schematically, the geometric formulation takes the following form[c]:

[a]Throughout this paper, we use the notation that Greek "space-time" indices take on the values $0,1,2,3$, while Latin "spatial indices" take on the values $1,2,3$. We use Einstein summation convention throughout the paper.
[b]For Greek and Latin indices, for any vectorfield or one-form V, we lower and raise indices with the Minkowski metric $M_{\alpha\beta} := \mathrm{diag}(-1,1,1,1)$ and its inverse by using the notation $(V_\flat)_\beta := M_{\alpha\beta}V^\alpha$ and $(V^\sharp)^\beta := (M^{-1})^{\alpha\beta}V_\alpha$.
[c]We denote schematic spatial partial derivatives and space-time partial derivatives by ∂ and $\boldsymbol{\partial}$, respectively. Also we use the following schematic notations throughout the paper where A, B, C are arrays of variables:

- $\mathscr{L}[A](B)$ denotes any scalar-valued function that is linear in the components of B with coefficients that are a function of the components of A.

Wave equations:

$$\square_{\mathbf{g}}\Psi = \mathscr{L}(\vec{\Psi})[\vec{\mathcal{C}},\mathcal{D}] + \mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\Psi}]. \tag{1.2}$$

Transport equations:

$$\mathbf{B}\omega^\alpha = \mathscr{L}(\vec{\Psi},\vec{\omega},\vec{S})[\boldsymbol{\partial}\vec{\Psi}], \tag{1.3a}$$

$$\mathbf{B}(S^\sharp)^\alpha = \mathscr{L}(\vec{\Psi},\vec{S})[\boldsymbol{\partial}\vec{\Psi}]. \tag{1.3b}$$

Transport-Div-Curl system:

$$\begin{aligned}\mathbf{B}\mathcal{C}^\alpha = \mathfrak{F}_{(\mathcal{C}^\alpha)} := {} & \mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi},(\boldsymbol{\partial}\omega,\boldsymbol{\partial}\vec{S},\boldsymbol{\partial}\vec{\Psi})] + \mathscr{Q}(\vec{S})[\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\Psi}] \\ & + \mathscr{L}(\vec{\Psi},\vec{\omega},\vec{S})[\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{S}],\end{aligned} \tag{1.4a}$$

$$\begin{aligned}\mathbf{B}\mathcal{D} = \mathfrak{F}_{(\mathcal{D})} := {} & \mathscr{L}(\vec{\Psi},\vec{S})[\boldsymbol{\partial}\omega] + \mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi},(\boldsymbol{\partial}\vec{S},\boldsymbol{\partial}\vec{\Psi})] \\ & + \mathscr{Q}(\vec{S})[\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\Psi}] + \mathscr{L}(\vec{\Psi},\vec{\omega},\vec{S})[\boldsymbol{\partial}\vec{\Psi}],\end{aligned} \tag{1.4b}$$

$$\mathrm{vort}^\alpha(S) = 0, \tag{1.4c}$$

$$\partial_\alpha\omega^\alpha = \mathscr{L}(\omega)[\boldsymbol{\partial}\vec{\Psi}], \tag{1.4d}$$

where $\vec{\Psi} := (v^0,v^1,v^2,v^3,h,s)$, $\mathbf{g} = \mathbf{g}(\vec{\Psi})$ is a solution-dependent Lorentzian metric which governs the geometry of sound waves, $\mathcal{C} \simeq \mathrm{vort}(\omega)$ and $\mathcal{D} \simeq \mathrm{div}S$ are special modified fluid variables, and $\mathbf{B}$ is the material derivative, which is parallel to v^α. See Definition 2.3 for the definition of ω, Definition 2.6 for the definition of S and Sec. 2.4 for the precise definitions of $\mathbf{g},\mathcal{C},\mathcal{D},\mathbf{B}$ and more details of the geometric formulation of the equations. This formulation reveals miraculous regularity and geometric properties of the flow, which is used in a fundamental way in the present work. These geometric properties are not visible in first-order equations (1.1). In this paper, we show that under low-regularity assumptions on the "wave-part" (see Sec. 3.5 for more details of "wave-part") of the initial data, the regularity of solutions of the relativistic Euler equations can be preserved for a short time. Specifically, we assume that the "wave-part" of the data belongs to H^{2+}, and that the "transport-part" $\mathcal{C}$ and $\mathcal{D}$ are in Hölder space $C^{0,0+}$. Our proof shows, in particular, that it is possible to avoid instantaneous shock formation, which in [19] was shown to occur in the irrotational case (i.e. for quasilinear wave equations) for initial data in H^2. *In particular, our regularity assumptions are optimal with respect to the "sound-wave-part" of the data.* One cannot hope to avoid singularities globally in time: it is known that, even in the irrotational and isentropic case, the compression of sound waves can cause shocks to develop from regular initial data in finite time. Moreover, in more than one space dimension and away from symmetry, these singularities are known to be stable as in [7].

- $\mathscr{Q}[A](B,C)$ denotes any scalar-valued function that is quadratic in the components of B and C with coefficients that are a function of the components of A.

For the irrotational and isentropic case in [7], the relativistic Euler equations reduce to a system of covariant quasilinear wave equations for the first derivatives $\psi_\alpha := \partial_\alpha \phi$ of a potential function ϕ of the following form:

$$\square_{\mathbf{g}(\psi)}(\psi_\alpha) = \mathscr{Q}(\psi)[\boldsymbol{\partial}\psi, \boldsymbol{\partial}\psi]. \tag{1.5}$$

Classical local well-posedness in $H^{(5/2)+}$ for the quasilinear wave system (1.5) can be obtained by applying energy estimates and Sobolev embedding, see Kato [14]. Starting in the late 90s, the regularity needed for local well-posedness for quasilinear wave equations was improved in a series of works by Bahouri–Chemin, Smith–Tataru and Klainerman–Rodnianski, see [4, 5, 16, 15, 26, 27]. The optimal result for low regularity H^{2+} of quasilinear wave equations was first achieved by Smith–Tataru in [23]. In [30], Wang reached the same result as in [23] by using a geometric approach. With the presence of vorticity, Disconzi–Luo–Mazzone–Speck, Wang and Zhang–Andersson proved low-regularity local well-posedness result for the three-dimensional (3D) compressible Euler equations in [11, 31, 32], respectively. In all three works, the regularity of "wave-part" is in the optimal level H^{2+}. We will discuss the details of the assumptions for the data in [11, 31] in Sec. 1.1.

Compared to the non-relativistic case, the first fundamental form of $\Sigma_t := \{t\} \times \mathbb{R}^3$ is no longer conformally flat in the relativistic case, leading to more complicated geometry. One of the main challenges of this paper is that (1.2)–(1.4) seemingly suffers from a loss of derivative. This is because at the level of regularity, $\mathcal{C}, \mathcal{D} \simeq \boldsymbol{\partial}^2 \vec{\Psi}$, which is an issue since $\mathcal{C}$ and $\mathcal{D}$ show up as the source terms in the right-hand side of the wave equation (1.2). In [11], this was solved by using Hodge theory on the spacelike hypersurfaces Σ_t. In our case, the transport-div-curl system (1.4) is a space-time (non-elliptic Hodge) system, from which we have to extract a quasilinear elliptic Hodge system on the spacelike hypersurfaces, that is, we rewrite (see Proposition 5.9) the space-time div-curl system (1.4) into a spatial elliptic div-curl system with source terms that can be controlled only due to the special structure of the equations

$$G^{ab}\partial_a((\omega_\flat), S)_b \ = F, \tag{1.6}$$

$$\partial_a((\omega_\flat), S)_b - \partial_b((\omega_\flat), S)_a = H_{ab}. \tag{1.7}$$

In (1.6), $G^{-1} := G^{-1}(v)$ is the inverse of a Riemannian metric on constant-time hypersurfaces (see Eq. (5.18c) for the definition of G^{-1}). By using these structures, we are able to adequately control the vorticity and entropy gradient by using energy estimates for transport equations, elliptic estimates, Schauder estimates and Littlewood–Paley theory.

We now state the main results of this paper.

Theorem 1.1 (Main theorem). *Consider a smooth*[d] *solution to the relativistic Euler equations whose initial data on the initial Cauchy hypersurface* $\Sigma_0 := \{0\} \times \mathbb{R}^3$

[d]By smooth we mean as smooth as necessary for the analysis arguments to go through. We note that all of our quantitative estimates depend only on the Sobolev and Hölder norms.

satisfies following assumptions for some real number $2 < N < 5/2$, $0 < \alpha < 1$, $c_1 > 0$ *and* D:

(1) "*Wave-part*": $\|h, v\|_{H^N(\Sigma_0)} \leq D$,
(2) "*Transport-part*": $\|\omega\|_{H^N(\Sigma_0)} + \|s\|_{H^{N+1}(\Sigma_0)} \leq D$, *In addition, modified fluid variables* $\mathcal{C}$ *and* $\mathcal{D}$ ($\mathcal{C} \simeq \mathrm{vort}(\omega)$ *and* $\mathcal{D} \simeq \mathrm{div}S$, *see Sec.* 2.2.2 *for the definition of operator* vort *and* ω, *Definition* 2.6 *for the definition of* S *and Sec.* 2.8 *for the definition of* $\mathcal{C}$ *and* $\mathcal{D}$) *satisfy the Hölder norm bound* $\|\mathcal{C}, \mathcal{D}\|_{C^{0,\alpha}(\Sigma_0)} \leq D$,
(3) *The image of data functions is contained in an interior of a compact subset* $\mathcal{R}$ (*defined in Sec.* 3.5) *and the enthalpy* H *is positive, i.e.* $H \geq c_1 > 0$.

Then the solution's time of classical existence $T := T(D, \mathcal{R}) > 0$ *can be controlled in terms of only* D *and* $\mathcal{R}$. *Moreover, the Sobolev and some Hölder regularities of the data are propagated by the solution.*

See Sec. 1.5 for the main ideas behind proving Theorem 1.1.

1.1. *Overview of previous low-regularity results*

There have been many developments on low-regularity problems for quasilinear wave equations and the non-relativistic 3D compressible Euler equations in past two decades. For quasilinear wave equations of the form (1.5), Bahouri–Chemin [5] and Tataru [26] independently showed local well-posedness with $H^{(2+\frac{1}{4})+}$ data. The improvements rely on Strichartz estimates based on Fourier integral parametrix representations. Bahouri–Chemin improved their earlier result to $H^{(2+\frac{1}{5})+}$ in [4]. Tataru pushed the results down to $H^{(2+\frac{1}{6})+}$ in [27] and Klainerman reached the same level in [15]. Klainerman–Rodnianski achieved $H^{(2+\frac{2-\sqrt{3}}{2})+}$ in [16]. The optimal low-regularity result H^{2+} for generic quasilinear wave equations was first achieved by Smith–Tataru in [23] by using wave-packets and properties of the geometry of characteristic light cones that were introduced in [16]. Besides the improvements over Sobolev exponents, a commuting vectorfield approach for Strichartz estimates was introduced by Klainerman in [15] and a fundamental decomposition of a Ricci component of $\mathbf{g}$ was used for improving the regularity in the causal geometry by Klainerman–Rodnianski in [16]. Recently, Wang gave a second proof of Smith–Tataru [23] by using this geometric approach. The proof in Wang [30] relied on an upgraded version of Klainerman–Rodnianski's vectorfield method with the help of conformal energy estimates. We again emphasize that, for the general quasilinear wave equation of the form (1.5), it is impossible to prove any well-posedness result with data in H^2. Specifically, Lindblad provided an example of ill-posedness for a quasilinear wave equation with H^2 initial data in [19]; see also [1–3] for more ill-posedness results for the compressible inviscid fluid. For the non-relativistic compressible Euler flow with vorticity and entropy, under the H^{2+} assumptions on "wave-part" and "transport-part" of the data, Disconzi–Luo–Mazzone–Speck [11] and Wang [31] proved local well-posedness result for 3D compressible Euler equations. By assuming the Hölder regularity $C^{0,\alpha}$ for the data of the modified fluid variables $\mathcal{C}$ and $\mathcal{D}$ (our analogue of $\mathcal{C}$ and $\mathcal{D}$ are defined in Definition 2.8), the authors are able to prove a Schauder estimate in [11] for a transport-div-curl system in order to propagate the vorticity and entropy gradient along the waves. A method of decomposing the velocity is given in [31] for the isentropic compressible Euler equations, which allowed Wang to remove the Hölder assumption on vorticity. In [32], Zhang–Andersson combined the methods in [23, 31] to give an alternate proof of the same result as in Wang [31].

1.2. *A brief overview of the strategy of the proof*

Klainerman [15], Klainerman–Rodnianski [16] and Wang [30] developed a geometric approach for proving the low-regularity well-posedness for the quasilinear wave equations. The new formulation (1.2)–(1.4) provided by Disconzi–Speck [12] makes it possible to import the geometric techniques from [16, 15, 30] to the "sound-wave-part" of compressible Euler flow. The main difference with the wave problem is the addition of another characteristic speed into the problem, namely, the "transport-part". These two parts of the equations and solutions interact with each other, which creates substantial difficulties for understanding the Euler flow. See Sec. 1.3 for further discussions of the geometric formulation. See also Luk–Speck [22] for similar formulations for 3D isentropic compressible Euler equation and Speck [24] for 3D compressible Euler equations with any equation of state. The main tool for controlling the solution in the low-regularity setting, by using energy estimate (see Christodoulou [7, Chap. 1] for the energy current and its properties) and Littlewood–Paley theory, is the following estimates[e]:

$$\begin{aligned}\|(h,s,v)\|_{H^{2+\varepsilon}(\Sigma_t)} \lesssim \|(h,s,v)\|_{H^{2+\varepsilon}(\Sigma_0)} + \int_0^t (\|\boldsymbol{\partial}(h,s,v)\|_{L^\infty_x(\Sigma_\tau)} + 1) \\ \times \|(h,s,v)\|_{H^{2+\varepsilon}(\Sigma_\tau)}\mathrm{d}\tau. \end{aligned} \tag{1.8}$$

In order to make (1.8) useful, one needs to control $\|\boldsymbol{\partial}(h,s,v)\|_{L^1_tL^\infty_x}$. Since one is not able to apply Sobolev embedding to recover the bound below $H^{5/2+}$, we instead use a geometric approach to show the following *Strichartz estimates*: $\|\boldsymbol{\partial}(h,s,v)\|_{L^2_tL^\infty_x} \lesssim T_*^{2\delta}$. This is done by a bootstrap argument with bootstrap assumptions $\|\boldsymbol{\partial}(h,s,v)\|_{L^2_tL^\infty_x} \leq 1$, where T_* is the bootstrap time. In order to prove the Strichartz estimates, we apply a series of reductions. We reduce the Strichartz estimates to decay estimates by using a $\mathcal{T}\mathcal{T}^*$ argument, then to a conformal energy estimate (see Definition 7.5 for the definition of conformal energy and Theorem 7.6 for boundness theorem for conformal energy) by Littlewood–Paley theory. See Sec. 1.5.4 for overview of the reduction, Sec. 4 for an extended overview of a global structure and Secs. 6 and 7 for details.

[e]We denote the constant-time hypersurface at time t by Σ_t. Moreover, $A \lesssim B$ means $A \leq C \cdot B$ for some universal constant C depending on region $\mathcal{R}$ and data D.

A crucial step in our geometric approach is the introduction of an acoustical function u satisfying the acoustical eikonal equation $(\mathbf{g}^{-1})^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$, where the acoustical metric $\mathbf{g} = \mathbf{g}(h, v, s)$ is a Lorentzian metric (see Definition 2.10 for the definition of $\mathbf{g}$) distinct from the Minkowski metric. With the help of u, we construct a null frame and control the acoustic geometry along acoustic null cones, which are the level sets of u (see the figure on p. 65). This allows us to derive suitable estimates for conformal energy. Disconzi–Luo–Mazzone–Speck [11] and Wang [31] showed that given good control over the "transport-part", we can run the machinery of Strichartz estimates for the "wave-part" where we treat the "transport-part" as a favorable source term. Thus, good control over the "transport-part" is a crucial component of our analysis. Inspired by the analysis of the 3D non-relativistic compressible Euler case in [11], we derive elliptic and Schauder estimates for the transport-div-curl systems to bound the $H^{1+\varepsilon}$ and $C^{0,\alpha}$ norms of $\mathcal{C}, \mathcal{D}$ when the wave-part is rough.

The main new difficulty that is not found in 3D non-relativistic compressible Euler is: due to the space-time structure of the relativistic Euler flow, we encounter space-time velocity, vorticity and the div-curl system where the ellipticity is not immediately apparent. To overcome this difficulty, we exploit two crucial aspects. We first note that the v-directional derivative of the vorticity and entropy gradient is favorable due to the transport phenomena. To obtain control of v-orthogonal directional derivatives, we reduce the space-time div-curl system of vorticity and entropy gradient to a dynamic div-curl system on the constant-time hypersurfaces. By combining these special structures of relativistic Euler equations with Littlewood–Paley decomposition and properties of pseudodifferential operators, we derive estimates for vorticity and entropy.

We present the logical graph of this paper in Sec. 1.5.

1.3. *Geometric formulation of the relativistic Euler equations*

Due to the coupling of sound waves with vorticity and entropy in Eqs. (1.2)–(1.4), when considering the relativistic Euler equations with an arbitrary equation of state, one needs to precisely and carefully split the dynamics into a "wave-part", which describes the propagation of sound waves, and a "transport-part", which describes the evolution of vorticity and entropy. For the 3D non-relativistic compressible Euler equations with any equation of state, Speck [24] derived a system consisting of geometric wave equations and transport-div-curl equations. This geometric formulation is used for the low-regularity problem in [11, 31]. See also Luk–Speck [21, 22] for the geometric formulation of the compressible Euler in the barotropic case and its application to the shock formation problem. Disconzi–Speck [12] derived the geometric formulation of the relativistic Euler equations with vorticity and dynamic entropy that we used in this paper. It allows us to describe the influence of transport phenomena on the wave-part of the system and the acoustic geometry with rough sound wave data given in the relativistic Euler flow. These geometric formulations have origins in Christodoulou and Christodoulou–Miao's proof of stable shock formation for the relativistic Euler equations and non-relativistic 3D compressible Euler equations in the irrotational and isentropic case [7, 9]. Also see [6, 20] for more results concerning shock formation problem of the compressible Euler equations.

The new geometric formulation from [12] (see (1.2)–(1.4) and Proposition 2.17 for the formulation) splits the dynamics into a "wave-part", which consists of geometric wave equations for the fluid variables h, v, s, and a "transport-div-curl-part", which governs the transport equations of special vorticity, entropy gradient and modified fluid variables $\mathcal{C}, \mathcal{D}$ ($\mathcal{C}, \mathcal{D}$ are special combinations of variables whose essential terms are the vorticity of vorticity and divergence of entropy gradient that are defined in Definition 2.8) and div-curl systems for the special vorticity and entropy gradient. The advantage of the geometric formulation is that one can do analysis on both "wave-part" and "transport-part", which are highly coupled. Here, we briefly summarize the new formulation and its connection to establishing the Strichartz estimate as follows:

- The "wave-part" of the formulation involves wave equations with principal part $\square_{\mathbf{g}}$. Properties of this operator are intimately related to the acoustic geometry, which is constructed via an acoustical function u. Here, u is a solution to the acoustical eikonal equation $(\mathbf{g}^{-1})^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$, where the acoustical metric $\mathbf{g} = \mathbf{g}(h, v, s)$ is the Lorentzian metric defined in Definition 2.10. With the help of u, we construct a null frame and derive some transport equations as well as div-curl systems for some particular connection coefficients along acoustic null cones, which are the level sets of u (see the figure on p. 65). We note that these equations for the connection coefficients are derived from basic geometry considerations and are independent of the relativistic Euler equations. By using a delicate decomposition of certain curvature components, which are highly tied to the geometric wave equations (2.29), we can control a large group of geometric quantities that are fundamental for deriving the conformal energy estimates. We emphasize already that achieving control of these geometric quantities is essential for controlling certain conformal energy for solutions to the linear wave equation corresponding to the acoustical metric $\mathbf{g}$, i.e. solutions φ to the PDE $\square_{\mathbf{g}(\vec{\Psi})}\varphi = 0$. It is crucial to control the conformal energy in order to derive the decay estimates, which we again emphasize are the main ingredient needed to obtain the desired Strichartz estimate. We will describe conformal energy and decay estimates with more details in Sec. 1.5.
- The "transport-div-curl-part" of the formulation allows one to control the vorticity and entropy at one derivative level above standard estimates. The analysis uses transport estimates as well as Hodge estimates at constant-time hypersurfaces. This is highly nontrivial and more complicated compared to non-relativistic 3D compressible Euler because the Hodge system that we encounter is a spacetime div-curl system. In total, we are able to show that the transport terms are "good" source terms in the wave equation estimates. We point out the vorticity and entropy gradient also appear in PDEs that we use to control the acoustic

geometry because of the geometric wave equations (1.2). This shows that there are interactions between the vorticity, entropy, sound waves and acoustic geometry.

1.4. *Comparison with low-regularity results for 3D non-relativistic compressible Euler equations*

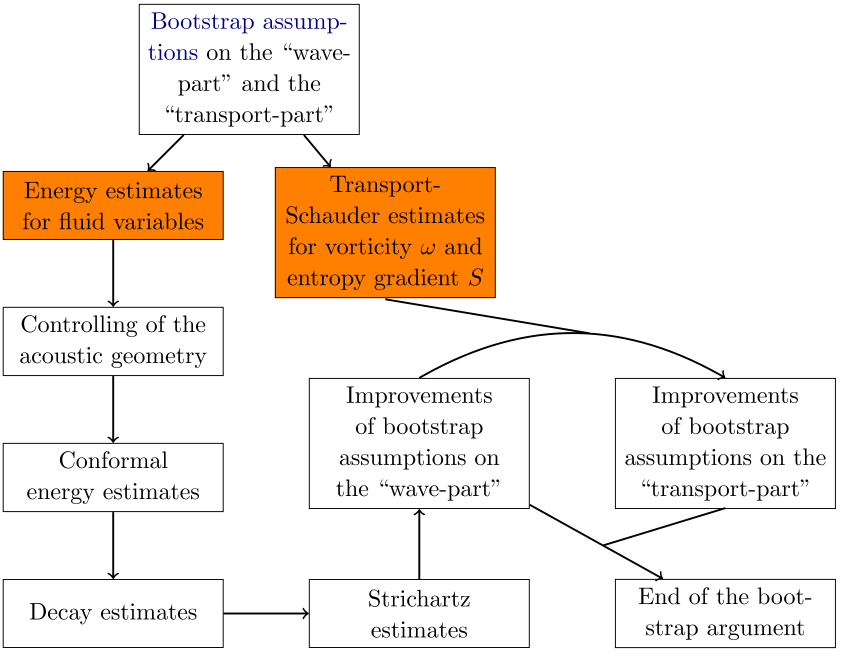
Recently, [11, 31] proved low-regularity results for the 3D non-relativistic compressible Euler equations with the help of the geometric formulation in [24]. In [11], Disconzi–Luo–Mazzone–Speck showed that if the "wave-part" of the data is initially in H^{2+}, and the "transport-part" $\mathcal{C}, \mathcal{D}$ are in $C^{0,\alpha}$, then the regularity of the solutions can be preserved for short times. For barotropic flow, Wang proved a similar result by removing the Hölder assumption for $\mathcal{C}, \mathcal{D}$ and assuming the H^{2+} assumptions on the "transport-part" in [31]. For the relativistic Euler equations, we prove a similar result as in [11] for the 3D compressible Euler equations. That is, we allow any equation of state and we have the same level of regularity assumptions on the initial data. Due to the geometric nature of the relativistic Euler flow, the vorticity ω^α in this paper is a space-time v-orthogonal vectorfield (see Definition 2.3) which solves a space-time transport-div-curl system. In the 3D non-relativistic compressible Euler case, the geometry of vorticity is much simpler: it is a Σ_t-tangent vectorfield and solves a div-curl system with constant coefficients on constant-time slices.

In the relativistic Euler equations, vorticity and entropy gradient satisfy transport equations in the v-direction. To control generic v-orthogonal (with respect to Minkowski metric) derivatives of vorticity and entropy gradient, we rely on a space-time div-curl system. Moreover, using the transport equations satisfied by the modified fluid variables $\mathcal{C}, \mathcal{D}$, we can control these quantities not only along constant-time slices, but also along null cones, which is fundamental in our work. To derive sufficient regularity for vorticity, entropy gradient and modified fluid variables along constant-time slices, we rewrite the space-time div-curl system into a spatial div-curl system with source terms that can be controlled only due to the special structure of the equations. A crucial ingredient in our analysis is that the spatial divergence equation has the form $(G^{-1})^{ij}\partial_i(\omega_\flat)_j = \cdots$ where $G^{-1} := G^{-1}(v)$ is the inverse of a Riemannian metric on constant-time hypersurfaces (see Eq. (5.18c) for the definition of G^{-1}). Because the coefficient metric G of the divergence equation is Riemannian, by using the technique of freezing the spatial points, we are able to derive a localized div-curl system with constant-coefficient principle terms, such that the Fourier transform of vorticity and entropy gradient is bounded in the frequency space by the source terms of the div-curl system. This allows us to control appropriate Hölder norms of $\partial\omega, \partial S$ in terms of the same Hölder norms of $\mathcal{C}, \mathcal{D}, \boldsymbol{\partial}\vec{\Psi}$. The analysis relies on the Littlewood–Paley theory as well as the standard theorem in pseudodifferential operators. We take a similar approach when deriving the elliptic div-curl estimates in L^2 space, where we need to control derivatives of vorticity and the entropy gradient by $\boldsymbol{\partial}\vec{\Psi}$, the modified fluid variables $\mathcal{C}$ and $\mathcal{D}$. Finally, we use the transport equations (1.4a) and (1.4b) and initial assumptions of $\mathcal{C}, \mathcal{D}$ to bound the Hölder norm of $\mathcal{C}, \mathcal{D}$ by $\partial\omega, \partial S$ to close the estimates for $\partial\omega, \partial S$.

1.5. *Main idea of the proof of Theorem* 1.1

Theorem 1.1 provides *a priori* estimates for smooth solutions, which is needed for a full proof of local well-posedness. The remaining aspects of a full proof of local well-posedness could be shown by deriving uniform estimates for sequences of smooth solutions and their differences. We refer readers to [23, Secs. 2 and 3] for the proof of local well-posedness based on *a priori* estimates.

In this section, we present the logic of proofs in this paper, that is, the bootstrap argument. The colored steps involve new ingredients, where we need to do analysis based on the special structure of the relativistic Euler equations (see Secs. 1.5.1 and 1.5.3 for a discussion of these steps). The uncolored steps are introduced in the previous low-regularity problem works (see Secs. 1.5.4–1.5.7 for a discussion of these steps). We emphasized that, with the estimates we derive in the colored steps, the proofs of the uncolored steps are essentially the same as in [4, 5, 11, 16, 15, 23, 27, 30]. Hence in this work, we provide all of the details for the colored steps, and give terse sketches for the uncolored steps with the appropriate citations.



1.5.1. *Overview of elliptic and energy estimates*

In this section, we provide an overview of how energy estimates work and are related to the bootstrap assumptions (1.15a) and (1.15b). We provide representative energy estimates for wave variables, vorticity and entropy gradient by using the basic energy estimates (see Sec. 5.1.1) and L^2 elliptic estimates (see Sec. 5.1.2). Then we leave the discussion of the key assumptions to future sections and detailed estimates in Sec. 5.1.

We first consider the energy estimates for the "wave-part". Given any $2 < N < 5/2$, $0 < t \leq T_*$ where $0 < T_* \ll 1$ denotes the bootstrap time. By the vectorfield multiplier method and Littlewood–Paley calculus applied to Eq. (1.2), we derive the following energy estimates for the "wave-part":

$$\|\boldsymbol{\partial}\vec{\Psi}\|^2_{H^{N-1}(\Sigma_t)} \lesssim \|\boldsymbol{\partial}\vec{\Psi}\|^2_{H^{N-1}(\Sigma_0)} + \int_0^t (\|\boldsymbol{\partial}\vec{\Psi}\|_{L^\infty_x(\Sigma_\tau)} + 1)\|\boldsymbol{\partial}\vec{\Psi}, \vec{\mathcal{C}}, \mathcal{D}\|^2_{H^{N-1}(\Sigma_\tau)} \mathrm{d}\tau, \tag{1.9}$$

which is an analogue of (1.8). We will provide a detailed expression and its proof in Sec. 5.1.3.

To control $\vec{\mathcal{C}}, \mathcal{D}$ on the right-hand side of (1.9), we then consider the energy estimates for the "transport-part". We first need an important L^2 elliptic div-curl estimate

$$\|(\partial\vec{\omega}, \partial\vec{S})\|_{L^2_x(\Sigma_t)} \lesssim \|\boldsymbol{\partial}\vec{\Psi}, \vec{\mathcal{C}}, \mathcal{D}\|_{L^2_x(\Sigma_t)}. \tag{1.10}$$

We note that the proof of (1.10) requires a rewriting div-curl system (5.17a) and (5.17b) for vorticity and entropy gradient, where one has to exploit the structure of the relativistic Euler equations. By splitting the space-time div-curl systems into time and spatial directions of derivatives, taking the advantage of the transport equations (1.3a) and (1.3b) for ω and S, we write time derivative of vorticity and entropy gradient components as a combination of spatial derivatives of ω and S. We obtain a new spatial div-curl system of the form

$$(G^{-1})^{ab}\partial_a((\omega_\flat), S)_b = F, \tag{1.11a}$$

$$\partial_a((\omega_\flat), S)_b - \partial_b((\omega_\flat), S)_a = H_{ab}, \tag{1.11b}$$

where $G^{-1} = G^{-1}(v)$ is Riemannian, $F = \mathcal{D} + \text{l.o.t.}$ and $H = \mathcal{C} + \text{l.o.t.}$ (1.11) is a PDE system on constant-time slices. Note that Eq. (1.11a) is a quasilinear divergence equation while the analogue in [11] is a constant-coefficient equation. Then by Littlewood–Paley estimates and a partition of unity argument, we prove (1.10) in Proposition 5.8.

As in Proposition 5.8, by (1.10) and Littlewood–Paley calculus, we also have, for $2 < N < 5/2$

$$\|(\partial\vec{\omega}, \partial\vec{S})\|_{H^{N-1}(\Sigma_t)} \lesssim \|\boldsymbol{\partial}\vec{\Psi}, \vec{\mathcal{C}}, \mathcal{D}\|_{H^{N-1}(\Sigma_t)}. \tag{1.12}$$

By applying energy estimates and Littlewood–Paley calculus on evolution equations (1.4a) and (1.4b) for $\vec{\mathcal{C}}, \mathcal{D}$, we have the following energy estimate for $\vec{\mathcal{C}}, \mathcal{D}$:

$$\begin{aligned} \|\vec{\mathcal{C}}, \mathcal{D}\|^2_{H^{N-1}(\Sigma_t)} \lesssim \|\vec{\mathcal{C}}, \mathcal{D}\|^2_{H^{N-1}(\Sigma_0)} + \int_0^t &(\|\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\omega}, \boldsymbol{\partial}\vec{S}\|_{L^\infty_x(\Sigma_\tau)} + 1) \\ &\times \|\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\omega}, \boldsymbol{\partial}\vec{S}, \vec{\mathcal{C}}, \mathcal{D}\|^2_{H^{N-1}(\Sigma_\tau)} \mathrm{d}\tau. \end{aligned} \tag{1.13}$$

By elliptic estimates (1.12), (1.13) and (1.9), we have

$$\begin{aligned}\|\boldsymbol{\partial}\vec{\omega}, \boldsymbol{\partial}\vec{S}\|^2_{H^{N-1}(\Sigma_t)} &\lesssim \|\boldsymbol{\partial}\vec{\Psi}, \vec{\mathcal{C}}, \mathcal{D}\|^2_{H^{N-1}(\Sigma_t)} \\ &\lesssim \|\boldsymbol{\partial}\vec{\Psi}, \vec{\mathcal{C}}, \mathcal{D}\|^2_{H^{N-1}(\Sigma_0)} + \int_0^t (\|\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\omega}, \boldsymbol{\partial}\vec{S}\|_{L^\infty_x(\Sigma_\tau)} + 1) \\ &\quad \times \|\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\omega}, \boldsymbol{\partial}\vec{S}, \vec{\mathcal{C}}, \mathcal{D}\|^2_{H^{N-1}\Sigma_\tau} \mathrm{d}\tau. \qquad (1.14)\end{aligned}$$

The results of energy estimates are obtained in Sec. 5.1.3.

From (1.9), (1.13), (1.14) and Grönwall's inequality, we see that if $\|\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\omega}, \boldsymbol{\partial}\vec{S}\|_{L^1_t L^\infty_x(\mathcal{M})}$ is bounded (note that $\mathcal{C} \simeq \mathrm{vort}(\omega) + \text{l.o.t.}, \mathcal{D} \simeq \mathrm{div} S + \text{l.o.t.}$), the Sobolev regularity of the data can be propagated by the solution. This drives us to setup a bootstrap argument with the bootstrap assumptions as introduced in the next section.

1.5.2. *Bootstrap assumptions*

As we made it clear in the previous section, our argument crucially relies on the boundness of the term $\|\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\omega}, \boldsymbol{\partial}\vec{S}\|_{L^1_t L^\infty_x(\mathcal{M})}$. We prove the boundness of this via a bootstrap argument that we now describe as follows.

Throughout the paper, $0 < T_* \ll 1$ denotes the bootstrap time. We assume that $\vec{\Psi}, \vec{\omega}, \vec{S}$ is a smooth solution to the relativistic Euler equations. For $\delta_0 > 0$ defined as in Sec. 3.4, we assume the following estimates hold:

$$\|\boldsymbol{\partial}\vec{\Psi}\|^2_{L^2_t L^\infty_x([0,T_*]\times\mathbb{R}^3)} + \sum_{\nu \geq 2} \nu^{2\delta_0} \|P_\nu \boldsymbol{\partial}\vec{\Psi}\|^2_{L^2_t L^\infty_x([0,T_*]\times\mathbb{R}^3)} \leq 1, \qquad (1.15a)$$

$$\|\partial\vec{\omega}, \partial\vec{S}\|^2_{L^2_t L^\infty_x([0,T_*]\times\mathbb{R}^3)} + \sum_{\nu \geq 2} \nu^{2\delta_0} \|P_\nu \partial\vec{\omega}, P_\nu \partial\vec{S}\|^2_{L^2_t L^\infty_x([0,T_*]\times\mathbb{R}^3)} \leq 1, \qquad (1.15b)$$

where P_ν is the Littlewood–Paley projection (see Sec. 3.2 for definition). The Littlewood–Paley terms in the assumptions are needed for establishing the dyadic Strichartz estimate in order to improve the bootstrap assumptions.

In the classical local well-posedness problem of the relativistic Euler equations, the regularity assumptions are $(h, v, s) \in H^{(5/2)+}(\Sigma_0)$. This gives the $\boldsymbol{\partial}\vec{\Psi} \in H^{(3/2)+}(\Sigma_0)$. One can recover the boundness assumption $\|\boldsymbol{\partial}\vec{\Psi}\|_{L^\infty_x(\Sigma_t)}$ by standard energy estimates and Sobolev embedding $H^{3/2+} \hookrightarrow L^\infty$ at any constant-time hypersurface Σ_t. The lack of Sobolev embedding in the low-regularity level forces one to find a new machinery to improve the reasonable bootstrap assumptions (see Sec. 1.1 for introduction of previous results). Recovering the bootstrap assumptions occupies a large part of this paper.

1.5.3. *Transport-schauder estimates for the transport-div-curl system*

In this section, we explain how to improve the bootstrap assumption (1.15b). In particular, by applying Hölder's inequality in time, this will show

$\|\boldsymbol{\partial}\vec{\omega}, \boldsymbol{\partial}\vec{S}\|_{L_t^1 L_x^\infty([0,T_*]\times\mathbb{R}^3)}$ is small. This improvement is conditional on (1.19), which we explain how to derive in Sec. 1.5.4.

Because of the lack of tools in Hodge estimates in L^∞ space, we have assumed a slight bit of extra regularity for the "transport-part". That is, we propagate the Hölder boundness of "transport-part" with given initial data $\|\vec{\mathcal{C}}, \mathcal{D}\|_{C_x^{0,\alpha}(\Sigma_0)} \leq D$ where $0 < \alpha < 1$ and $D \in \mathbb{R}$. Besides using the transport equations (1.3a) and (1.3b) which exhibit source terms with surprisingly good structures, we also rely on the following Schauder-type estimate:

$$\|\partial\vec{\omega}, \partial\vec{S}\|_{C_x^{0,\alpha}(\Sigma_t)} \lesssim \|\boldsymbol{\partial}\vec{\Psi}, \vec{\mathcal{C}}, \mathcal{D}\|_{C_x^{0,\alpha}(\Sigma_t)}. \tag{1.16}$$

In order to control $\mathcal{C}, \mathcal{D}$ on the right-hand side of (1.16), we use the transport equations (1.4a) and (1.4b) of $\mathcal{C}$ and $\mathcal{D}$, which are coupled to $\partial\omega, \partial S$ (see Eqs. (1.4a) and (1.4b)). By combining the two, under bootstrap assumptions (1.15a), we can apply Grönwall's inequality to bound the vorticity and entropy gradient of the relativistic Euler flow. We will discuss this approach in more details below.

To derive Schauder estimates, we split the derivative of the vorticity ω and entropy gradient S (see Definitions 2.3 and 2.6 for the definition of ω and S) into v-tangent direction and the Σ_t-tangent directions (v is transversal to Σ_t with respect to Minkowski metric). Now we highlight the following two features in our analysis:

- The v-tangent direction derivatives of vorticity and entropy gradient are favorable because of the transport phenomena. That is, by using transport equations (1.3a) and (1.3b), we are able to obtain the Hölder bound for v-tangent direction of vorticity and entropy gradient by using bootstrap assumptions.
- To control the Σ_t-tangent directional derivatives of vorticity and entropy gradient, we rely on a space-time transport-div-curl system for ω and S. We note that it is qualitatively distinct from the case in [11] for 3D non-relativistic compressible Euler equations where the div-curl equations are spatial with constant coefficients.

We now explain how we derive Schauder estimates (1.16). We use the same div-curl system (1.11a) and (1.11b) as in the L^2 elliptic estimates. By partition of unity, Fourier transform, Littlewood–Paley theory and properties of pseudodifferential operators, we are able to bound $\|\boldsymbol{\partial}\vec{\omega}, \boldsymbol{\partial}\vec{S}\|_{C_x^{0,\alpha}(\Sigma_t)}$ by $\|\boldsymbol{\partial}\vec{\Psi}, \vec{\mathcal{C}}, \mathcal{D}\|_{C_x^{0,\alpha}(\Sigma_t)}$ as in (1.16), see Lemma 5.20 for the detailed proof. Then we bound $\|\vec{\mathcal{C}}, \mathcal{D}\|_{C_x^{0,\alpha}(\Sigma_t)}$ by applying transport equations (1.4a) and (1.4b) as follows:

$$\|\vec{\mathcal{C}}, \mathcal{D}\|_{C_x^{0,\delta_1}(\Sigma_t)} \lesssim 1 + \int_0^t (\|\boldsymbol{\partial}\vec{\Psi}\|_{C_x^{0,\delta_1}(\Sigma_\tau)} + 1)\|\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\omega}, \boldsymbol{\partial}\vec{S}\|_{C_x^{0,\delta_1}(\Sigma_\tau)} \mathrm{d}\tau. \tag{1.17}$$

Finally, combining (1.16) and (1.17), we use Grönwall's inequality and bootstrap assumptions to close the transport-Schauder-type estimates

$$\|\boldsymbol{\partial}\vec{\omega}, \boldsymbol{\partial}\vec{S}\|_{C_x^{0,\alpha}(\Sigma_t)} \lesssim \|\boldsymbol{\partial}\vec{\Psi}\|_{C_x^{0,\alpha}(\Sigma_t)}. \tag{1.18}$$

We emphasize that later in the argument, we will integrate (1.18) in time and combine it with the improved Strichartz estimate (1.19) (which is obtained independently of (1.18)). These lead to a strict improvement of the bootstrap assumptions (1.15b). We provide full details of the Schauder estimates in Sec. 5.

1.5.4. *Reductions of the Strichartz-type estimates*

Our argument above crucially relies on bounding $\|\boldsymbol{\partial}\vec{\Psi}\|_{L^1_t L^\infty_x(\mathcal{M})}$. In this section, we explain how we derive strict improvements of bootstrap assumptions (1.15a), that is, we describe how to derive Strichartz estimates (1.19). By taking the advantage of the smallness of bootstrap time interval $[0, T_*]$, we will improve our bootstrap assumptions to the following Strichartz estimates:

$$\|\boldsymbol{\partial}\vec{\Psi}\|^2_{L^2_t L^\infty_x([0,T_*]\times\mathbb{R}^3)} + \sum_{\nu\geq 2} \nu^{2\delta_1} \|P_\nu \boldsymbol{\partial}\vec{\Psi}\|^2_{L^2_t L^\infty_x([0,T_*]\times\mathbb{R}^3)} \lesssim T_*^{2\delta}, \tag{1.19}$$

where $\delta > 0$ is sufficiently small as in Sec. 3.4 and $8\delta_0 < \delta_1 < N - 2$, where δ_0 is from the bootstrap assumptions (1.15a) and (1.15b). Note that if T_* is small, (1.19) is a strict improvement of (1.15a). We reduce the proof of (1.19) to the proof of estimates on the acoustic geometry by adopting the geometric approach of [30]. This reduction is done through the following steps: Improvement of bootstrap assumptions $\leftarrow$ Strichartz estimates $\leftarrow$ Decay estimates $\leftarrow$ Conformal energy estimates $\leftarrow$ Controlling of the acoustic geometry, where the left arrow indicates that the latter estimate implies the former. We remind readers of the logic diagram at the beginning of Sec. 1.5.

- *Reduction to dyadic Strichartz estimates.* The first step in the proof of (1.19) is to reduce Strichartz estimates to a dyadic Strichartz estimate. Specifically, for a fixed large frequency λ, we partition $[0, T_*]$ into disjoint union of sub-intervals $I_k := [t_{k-1}, t_k]$ of total number $\lesssim \lambda^{8\varepsilon_0}$ with $|I_k| \lesssim \lambda^{-8\varepsilon_0}$ (see Sec. 3.4 for the definition of ε_0). By Littlewood–Paley decomposition and Duhamel principle, the proof of (1.19) can be reduced to a dyadic Strichartz estimate

  $$\|P_\lambda \boldsymbol{\partial}\varphi\|_{L^q_t L^\infty_x([\tau,t_{k+1}]\times\mathbb{R}^3)} \lesssim \lambda^{\frac{3}{2}-\frac{1}{q}} \|\boldsymbol{\partial}\varphi\|_{L^2_x(\Sigma_\tau)}, \tag{1.20}$$

  where φ is a solution of geometric equation

  $$\square_{\mathbf{g}}\varphi = 0, \tag{1.21}$$

  on the time interval I_k. In (1.20), $q > 2$ is any real number which is sufficiently close to 2 and $\tau \in [t_k, t_{k+1}]$. Here, we focus on large frequencies since control of small frequency is easier due to Bernstein inequalities.
- *Reduction to decay estimates.* For a large frequency λ, by rescaling the coordinates (see Sec. 6.2 for the rescaling) and using an abstract $\mathcal{T}\mathcal{T}^*$ argument, we can reduce the dyadic Strichartz estimates (1.20) to $L^2 - L^\infty$ decay estimates at any

$t \in [0, T_{*;(\lambda)}]$ where $T_{*;(\lambda)}$ is the rescaled bootstrap time (see Sec. 6.2 for the definition of $T_{*;(\lambda)}$)

$$\|P_1 \mathbf{T}\varphi\|_{L^\infty_x(\Sigma_t)} \lesssim \left(\frac{1}{(1+|t-1|)^{\frac{2}{q}}} + d(t)\right)(\|\boldsymbol{\partial}\varphi\|_{L^2_x(\Sigma_1)} + \|\varphi\|_{L^2_x(\Sigma_1)}), \tag{1.22}$$

where the timelike vectorfield $\mathbf{T}$ is $\mathbf{g}$-unit normal to Σ_t (that is defined in Definition 2.11) and φ is an arbitrary solution to the equation $\square_{\mathbf{g}}\varphi = 0$ on the time interval $[0, T_{*;(\lambda)}] \times \mathbb{R}^3$ with $\varphi(1, x)$ supported in the Euclidean ball B_R (see Theorem 6.9 for detailed definition of R). Moreover, the function $d(t)$ satisfies

$$\|d\|_{L_t^{\frac{q}{2}}([0,T_{*;(\lambda)}])} \lesssim 1, \tag{1.23}$$

for $q > 2$ sufficiently close to 2.

- *Reduction to conformal energy estimates.* By product estimates and Littlewood–Paley theory, we reduce the proof of (1.22) to a proof of the following estimates for the conformal energy $\mathfrak{C}[\varphi](t)$ (see Sec. 7.5 for the definition of the conformal energy $\mathfrak{C}[\varphi](t)$) at time $t \in [1, T_{*;(\lambda)}]$

$$\mathfrak{C}[\varphi](t) \lesssim_\varepsilon (1+t)^{2\varepsilon}(\|\boldsymbol{\partial}\varphi\|^2_{L^2_x(\Sigma_1)} + \|\varphi\|^2_{L^2_x(\Sigma_1)}), \tag{1.24}$$

where φ is an arbitrary solution to the equation $\square_{\mathbf{g}}\varphi = 0$ on $[0, T_{*;(\lambda)}] \times \mathbb{R}^3$ with $\varphi(1)$ supported in $B_R \subset \mathcal{M}^{(\text{Int})} \cap \Sigma_1$ (see Sec. 7.1 for the definition of $\mathcal{M}^{(\text{Int})}$) where $\varepsilon > 0$ is an arbitrary small number.

We emphasize that both the reduction of (1.22)–(1.24) *and the very definition of* $\mathfrak{C}[\varphi]$ *require the acoustic geometry, where its sharp control is needed for deriving* (1.24). We describe how to obtain such control in Secs. 1.5.5–1.5.7. We provide an overview of the structure over the reductions in Sec. 4 and more detailed discussions in Sec. 6.

1.5.5. *Structures for the causal geometry of the acoustic space-time*

In order to reduce the decay estimates to conformal energy estimates (see Secs. 1.5.6 and 4.4 for introduction and Sec. 7.3 for details), one needs sharp information about the acoustic geometry. In this section, we discuss the geometric framework that is crucial for our analysis. This part of the result is well known and standard (see Sec. 1.1 for the introduction of the previous results). The central object of our geometric framework is the acoustical function u, which is defined as a solution of the acoustical eikonal equation $(\mathbf{g}^{-1})^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$, where $\mathbf{g}^{-1}$ is the inverse of the normalized acoustical metric. We denote the level sets of u by $\mathcal{C}_u$, which are forward truncated null cones (defined in Sec. 7.1).

We construct a null frame, which consists of a null pair $L, \underline{L}$ and two spherical vectorfields $\{e_A\}_{A=1,2}$ (see Sec. 7.2 for detailed definitions). We derive transport and Hodge-type equations for the Ricci coefficients. An important example is the

Raychaudhuri equation (see Sec. 8 for definitions of connection coefficients and PDEs verified by geometric quantities)

$$L\mathrm{tr}_{\not{g}}\chi + \frac{1}{2}(\mathrm{tr}_{\not{g}}\chi)^2 = -|\hat{\chi}|^2_{\not{g}} - k_{NN}\mathrm{tr}_{\not{g}}\chi - \mathbf{Ric}_{LL}. \tag{1.25}$$

With the help of a remarkable decomposition of the Ricci curvature tensor

$$\mathbf{Ric}_{\alpha\beta} = -\frac{1}{2}\Box_{\mathbf{g}}\mathbf{g}_{\alpha\beta}(\vec{\Psi}) + \frac{1}{2}(\mathbf{D}_\alpha\mathbf{\Gamma}_\beta + \mathbf{D}_\beta\mathbf{\Gamma}_\alpha) + \mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\Psi}], \tag{1.26}$$

and the Bianchi identities, where we can substitute $\Box_{\mathbf{g}}\mathbf{g}_{\alpha\beta}(\vec{\Psi})$ in (1.26) by using relativistic Euler equations (1.2), we are able to obtain some estimates for the Ricci coefficients, which we will utilize in the following sections. We emphasize that it is important to have exactly $\mathcal{C}, \mathcal{D}$ on the right-hand side of (1.2). $\mathcal{C}, \mathcal{D}$ satisfies the transport equations (1.4a) and (1.4b), which allows us to derive estimates along constant-time slices and null hypersurfaces (in Sec. 1.5.1) and control the acoustic geometry.

The advantage of using the acoustic geometry in this low-regularity setting is that it reveals the dispersive properties of solutions to the wave equations. That is, for a solution ϕ of wave equation $\Box_{\mathbf{g}}\phi = 0$, the derivatives which are tangent to the characteristic null cones $\mathcal{C}_u$ have better decay than the transversal derivatives parallel to the $\underline{L}$ direction. We have to control some geometric quantities for several reasons:

- The acoustic geometry that we setup must be well defined. In particular, we have to rule out short-time shock formation due to the intersection of distinct null cones.
- To bound a suitably constructed weighted energy in order to derive decay estimates, a multiplier vectorfield method needs to be introduced. The multipliers we use are related to L and Σ_t-tangent sphere normal vector N. Since L and N depend on the wave variables (h, s, v), the acoustical eikonal function u and their first derivatives, to control the weighted energy, one needs to control relative derivatives of the above quantities.

1.5.6. *Control of the conformal energy*

A crucial part in the reduction of Strichartz estimates is to derive the decay estimates. As we discussed in Sec. 1.5.4, we use the conformal energy method that was introduced by Wang in [30]. We need to consider both of equation $\Box_{\mathbf{g}}\varphi = 0$ and the conformal wave equation $\Box_{\widetilde{\mathbf{g}}}(e^{-\sigma}\varphi) = \cdots$ (see Definition 8.5 for the definition of σ and $\widetilde{\mathbf{g}}$) to control various terms via the energy method. We are interested in such equations because we have reduced the Strichartz estimates for solution φ of geometric equation $\Box_{\mathbf{g}}\varphi = 0$ (see Sec. 1.5.4 for the reduction).

Since the metric $\widetilde{\mathbf{g}}$ is only smoother along null hypersurfaces, we have to first use the original wave equation and choose $X = fN$ (see Sec. 7.6 for the definition of f and Sec. 7.2 for sphere normal vector N) as the multiplier. By using the divergence theorem for a modified current on an appropriate region, we get a Morawetz-type energy estimate where we obtain a uniform bound for the standard energy of φ along a union of a portion of the constant-time hypersurfaces and null cones.

Then we consider the conformal wave equation. We use the multiplier approach with $\tilde{r}^p L$-type vectorfields in the region $\{\tau_1 \leq u \leq \tau_2\} \cap \{\tilde{r} \geq R\}$ where $1 \leq \tau_1 < \tau_2 < T_{*;(\lambda)}$ to control the conformal energy in the exterior region and to provide energy decay for each null slice. Finally, we control the conformal energy in the interior region with the help of the argument in [10] by obtaining energy decay in each spatial-null slice.

The very definition of the conformal energy, as well as its analysis, requires delicate and precise control of the acoustic geometry. We state the boundness theorem of conformal energy in Theorem 7.6. By using our estimates on the Ricci coefficients, one could follow the steps listed in [11, Sec. 11] to prove Theorem 7.6. One could go through the details of the argument in [30, Sec. 7]. Also readers could look into [15, Sec. 3] for initial ideas. We omit these details because the exact same arguments hold in our case.

1.5.7. *Control of the acoustic geometry*

To control the conformal energy, we need to control the acoustic geometry. Klainerman–Rodnianski [16] and Wang [30] developed an approach of controlling the geometric quantities in the low-regularity setting. In this paper, we control the acoustic geometry by following the approach in Wang [30].

First, we provide the PDEs verified by the geometric quantities in [16, Sec. 2]. We write down the geometric transport equations and div-curl system for the connection coefficients. These equations depend on our geometric formalism and are independent of the relativistic Euler equations. Second, we use the estimates for certain Ricci and Riemann curvature tensor components by using the decomposition of the Ricci curvature (1.26) in [16, Lemma 2.1] and the Bianchi identities. It is at this step that the structure of the relativistic Euler equations is used. Specifically, we can substitute $\square_{\mathbf{g}} \mathbf{g}_{\alpha\beta}(\vec{\Psi})$ in (1.26) by using relativistic Euler equations (1.2). $\mathcal{C}, \mathcal{D}$ on the right-hand side of (1.2) satisfies the transport-div-curl system, which allows us to derive elliptic estimates and control the acoustic geometry.

Then by combining the geometric transport equations and the aforementioned Ricci and Riemann curvature tensor components estimates, one can derive and analyze the equations for many acoustic variables. These include the important mass aspect function $\upmu$ and the conformal factor $\upsigma$, which introduce the rough geometry. Finally, we derive mixed space-time norm estimates for all the quantities, which are needed in the conformal energy estimates. We omit the details of the proof of controlling the geometry since the argument follows the same as in [11, Sec. 10].

In the following few paragraphs, we show why the standard Morawetz energy estimates are insufficient. The deformation tensor ${}^{(K)}\boldsymbol{\pi} := \mathcal{L}_K \mathbf{g}$ is present in the standard Morawetz-type energy estimates, where $K := \frac{1}{2}\left(u^2 \underline{L} + (2t-u)^2 L\right)$ and

$\mathcal{L}$ is the Lie derivative. ${}^{(K)}\boldsymbol{\pi}$ can be expressed by the connection coefficients of the null frame. We need to control them with the help of the transport and Hodge-type equations for geometric quantities.

After integrating by parts to obtain the Morawetz-type energy identity, one needs to control various derivatives of ${}^{(K)}\boldsymbol{\pi}$. In particular, $\nabla\!\!\!/ \mathrm{tr}_{g\!\!\!/}\chi$, which is the angular derivative of the expansion scalar, as well as the mass aspect function $\mu := \underline{L}\mathrm{tr}_{g\!\!\!/}\chi + \frac{1}{2}\mathrm{tr}_{g\!\!\!/}\chi\mathrm{tr}_{g\!\!\!/}\underline{\chi}$ (see Definition 8.4 for the definition of these geometric quantities). In order to control $\nabla\!\!\!/ \mathrm{tr}_{g\!\!\!/}\chi$, we rely on the Raychaudhuri equation (1.25) commuted with $\nabla\!\!\!/$. To control the Ricci term in (1.25), we use (1.26) contracted with $L^\alpha L^\beta$. After commuting with $\nabla\!\!\!/$, we have to control the error term $L(\nabla\!\!\!/ \boldsymbol{\Gamma}_L)$. It $\nabla\!\!\!/ \mathrm{tr}_{g\!\!\!/}\chi$ and $\nabla\!\!\!/ \boldsymbol{\Gamma}_L$ separately. Therefore, standard Morawetz energy estimates are insufficient in our case.

Therefore, to control all terms at a consistent level of regularity, we use the approach of Wang [30], which relies on renormalized quantities and a metric that is conformal to the acoustical metric, where the conformal factor is carefully constructed so that the null expansion scalar associated to the conformal metric $\mathrm{tr}_{\tilde{g\!\!\!/}}\tilde{\chi}$ is precisely $\mathrm{tr}_{g\!\!\!/}\chi + \boldsymbol{\Gamma}_L$. We are able to obtain the regularity theory of $\mathrm{tr}_{g\!\!\!/}\chi + \boldsymbol{\Gamma}_L$, while it seems impossible to treat them independently.

The conformal wave equation attempts to resolve the issues described above, but introduces the difficult conformal factor σ. Thus, in order to obtain sufficient regularity for the conformal factor σ, we must in fact control the modified mass aspect function $\check{\mu}$

$$\check{\mu} := 2\Delta\!\!\!/\sigma + \mu - \mathrm{tr}_{g\!\!\!/}\chi k_{NN} + \frac{1}{2}\mathrm{tr}_{g\!\!\!/}\chi\boldsymbol{\Gamma}_{\underline{L}}, \tag{1.27}$$

as well as the modified torsion $\nabla\!\!\!/\sigma + \zeta$. These quantities satisfy favorable transport and div-curl systems, i.e. the source terms have sufficient regularity, moreover, they have good decay properties. We stress that this analysis relies on obtaining careful control over the top derivatives of the specific vorticity and entropy gradient, since the modified fluid variables $\mathcal{C}, \mathcal{D}$ enter as source terms in various geometric equations, such as the Raychaudhuri equation.

1.6. *Paper Outline*

The structure of this paper will follow the non-relativistic 3D compressible Euler in [11]. The logical graph of this paper is in Sec. 1.5.

- In Sec. 2, we first state the notations that we are going to use throughout the paper. Then we define the fluid variables and tensorfields, including the acoustical metric. We also introduce the geometric formulation of the relativistic Euler equations.
- In Sec. 3, we define the Littlewood–Paley projections, which are frequently used in our analysis. We provide the frequency-projected versions of the evolution

equations (i.e. the relativistic Euler equations) in Lemma 3.3. We state the main theorem and the bootstrap assumptions in Theorem 3.4 and Sec. 3.6.

- In Sec. 4, we discuss the structure of the proofs which we will follow in the rest of the paper.
- In Sec. 5, we use the bootstrap assumptions to derive the energy, L^2 elliptic and Schauder estimates for the fluid variables along constant-time hypersurfaces. Note that Schauder estimates will improve the bootstrap assumption (3.17c) in Sec. 3.6 after the bootstrap assumption (3.17b) is improved by Strichartz estimates.
- In Sec. 6, following the approach of Tataru [27] and Wang [30], we rescale the fluid solution and reduce the proof of the Strichartz estimates to the proof of a spatially localized decay estimate. The reduction is essentially the same as one used by Wang [30], which we refer to for various details.
- In Sec. 7, we implement nonlinear geometric optics by constructing an acoustical function u and setting up its geometry, including constructing an appropriate null frame. Finally, we define the conformal energy and state the boundness theorem of the conformal energy in Theorem 7.6, which plays a crucial role in deriving the decay estimates that were stated in Theorem 6.9.
- In Sec. 8, we prove the energy estimates for the fluid variables along the acoustical null hypersurfaces in Sec. 8.1. We define additional geometric quantities, including the connection coefficients of the null frame, conformal factors, mass aspect functions and curvature tensor components. Then we restate the estimates from [30] that yield control over geometry along the initial data hypersurface. We state the bootstrap assumptions satisfied by the rescaled fluid variables as well as the bootstrap assumptions for geometric quantities in Sec. 8.4.2. Then we state the main estimates for the geometric quantities in Proposition 8.10, followed by a discussion of its proof in Sec. 8.4.4.

2. The Relativistic Euler Equations and Its Geometric Formulation

In this section, we provide the standard first-order relativistic Euler equations and its geometric formulation. The latter will be used throughout our analysis.

2.1. *Notations*

Greek "space-time" indices take on the values $0, 1, 2, 3$, while Latin "spatial" indices take on the values $1, 2, 3$. In this paper, for Greek and Latin indices, for any vectorfield or one-form V, we lower and raise indices with the Minkowski metric $M_{\alpha\beta} := \mathrm{diag}(-1, 1, 1, 1)$ and its inverse by using the notation $(V_\flat)_\beta := M_{\alpha\beta}V^\alpha$ and $(V^\sharp)^\beta := (M^{-1})^{\alpha\beta}V_\alpha$. Similar notations apply to all tensorfields. Moreover, $\epsilon_{\alpha\beta\gamma\delta}$ denotes the fully antisymmetric symbol normalized by $\epsilon_{0123} = 1$. Note that $(\epsilon^\sharp)^{0123} = -1$. We use Einstein summation throughout the paper.

We denote $\Sigma_t := \{(t', x^1, x^2, x^3) \in \mathbb{R}^{1+3} | t' \equiv t\}$ as the standard constant-time slice.

We denote the spatial partial derivatives by ∂ and the space-time partial derivatives by $\boldsymbol{\partial}$.

Remark 2.1. Since $M_{\alpha\beta} := \mathrm{diag}(-1,1,1,1)$, for any vectorfield or one-form V, we have the identities $\boldsymbol{\partial}(V_\flat)_a = \boldsymbol{\partial}(V^\sharp)^a$ and $\boldsymbol{\partial}(V_\flat)_0 = -\boldsymbol{\partial}(V^\sharp)^0$.

2.2. *Definitions of the fluid variables and related quantities*

2.2.1. *The basic fluid variables*

The fluid velocity v^α is a future-directed four-vector and normalized by $(v_\flat)_\alpha v^\alpha = -1$. $\mathfrak{p}$ denotes the pressure, ϱ denotes the proper energy density, n denotes the proper number density, s denotes the entropy per particle, θ denotes the temperature and

$$H = (\varrho + \mathfrak{p})/n, \tag{2.1}$$

is the enthalpy per particle. Thermodynamics supplies the following laws:

$$H = \left.\frac{\partial \varrho}{\partial n}\right|_s, \quad \theta = \left.\frac{1}{n}\frac{\partial \varrho}{\partial s}\right|_n, \quad \mathrm{d}H = \frac{\mathrm{d}\mathfrak{p}}{n} + \theta \mathrm{d}s, \tag{2.2}$$

where $\frac{\partial}{\partial n}|_s$ denotes partial differentiation with respect to n at fixed s and $\frac{\partial}{\partial s}|_n$ denotes partial differentiation with respect to s at fixed n.

2.2.2. *v-orthogonal vorticity*

Definition 2.2 (The $\boldsymbol{v}$-orthogonal vorticity of a one-form). Given a spacetime one-form V, we define the corresponding v-orthogonal (with respect to Minkowski metric) vorticity vectorfield as follows:

$$\mathrm{vort}^\alpha(V) := -(\epsilon^\sharp)^{\alpha\beta\gamma\delta}(v_\flat)_\beta \partial_\gamma V_\delta. \tag{2.3}$$

Definition 2.3 (The $\boldsymbol{v}$-orthogonal vorticity vectorfield). We define the vorticity vectorfield ω^α as follows:

$$\omega^\alpha := \mathrm{vort}^\alpha(Hv) = -(\epsilon^\sharp)^{\alpha\beta\gamma\delta}(v_\flat)_\beta \partial_\gamma (H(v_\flat)_\delta). \tag{2.4}$$

2.2.3. *Auxiliary fluid variables*

Definition 2.4 (Logarithmic enthalpy). Let $\overline{H} > 0$ be a fixed constant value of the background enthalpy. We define the logarithmic enthalpy h as follows:

$$h := \ln(H/\overline{H}). \tag{2.5}$$

Definition 2.5 (Temperature over enthalpy). We define the quantity $\mathfrak{q}$ as follows:

$$\mathfrak{q} := \frac{\theta}{H}. \tag{2.6}$$

Definition 2.6 (Entropy gradient one-form). We define the entropy gradient one-form S_α as follows:

$$S_\alpha := \partial_\alpha s. \tag{2.7}$$

2.2.4. *Equation of state and speed of sound*

Definition 2.7 (Partial derivatives with respect to h and s). If Q is a quantity that can be expressed as a function of (h, s), then

$$Q_{;h} = Q_{;h}(h, s) := \left.\frac{\partial Q}{\partial h}\right|_{s}, \tag{2.8}$$

$$Q_{;s} = Q_{;s}(h, s) := \left.\frac{\partial Q}{\partial s}\right|_{h}. \tag{2.9}$$

We assume an equation of state of the form $\mathfrak{p} = \mathfrak{p}(\varrho, s)$. The speed of sound c is defined as follows:

$$c := \sqrt{\left.\frac{\partial \mathfrak{p}}{\partial \varrho}\right|_{s}}. \tag{2.10}$$

In the rest of the paper, we view the speed of sound be a function of h and s: $c = c(h, s)$.

We restrict to the physically relevant regime where the speed of sound does not exceed the speed of light

$$0 < c \leq 1. \tag{2.11}$$

2.3. *Standard first-order equations*

Considering s, h and $\{v^\alpha\}_{\alpha=0,1,2,3}$ to be the fundamental unknowns, as in [12, Sec. 3], the relativistic Euler equations take the form of a quasilinear hyperbolic system

$$v^\kappa \partial_\kappa h + c^2 \partial_\kappa v^\kappa = 0, \tag{2.12a}$$

$$v^\kappa \partial_\kappa (v_\flat)_\alpha + \partial_\alpha h + (v_\flat)_\alpha v^\kappa \partial_\kappa h - q \partial_\alpha s = 0, \tag{2.12b}$$

$$v^\kappa \partial_\kappa s = 0. \tag{2.12c}$$

2.4. *Modified fluid variables and the geometric wave–transport formulation*

In this section we define several variables followed by the new formulation of relativistic Euler equations.

Definition 2.8 ([12, Definition 2.8, Modified fluid variables]). We define the modified fluid variables as follows:

$$\begin{aligned} \mathcal{C}^\alpha &:= \mathrm{vort}^\alpha(\omega_\flat) + c^{-2}\epsilon^{\alpha\beta\gamma\delta}(v_\flat)_\beta(\partial_\gamma h)(\omega_\flat)_\delta + (\theta - \theta_{;h})(S^\sharp)^\alpha(\partial_\kappa v^\kappa) \\ &\quad + (\theta - \theta_{;h})v^\alpha\{(S^\sharp)^\kappa \partial_\kappa h\} - (\theta - \theta_{;h})(S^\sharp)^\kappa\{(M^{-1})^{\alpha\lambda}\partial_\lambda (v_\flat)_\kappa\}, \end{aligned} \tag{2.13}$$

$$\mathcal{D} := \frac{1}{n}\{\partial_\kappa (S^\sharp)^\kappa\} + \frac{1}{n}\{(S^\sharp)^\kappa \partial_\kappa h\} - \frac{1}{n}c^{-2}\{(S^\sharp)^\kappa \partial_\kappa h\}. \tag{2.14}$$

Definition 2.9 (Acoustical metric and its inverse). Let M be the Minkowski as defined in Sec. 2.1, we define the acoustical metric $\mathbf{g}_{\mathrm{Acou}_{\alpha\beta}}$ and its inverse $(\mathbf{g}^{-1}_{\mathrm{Acou}})^{\alpha\beta}$ as follows[f]:

$$\mathbf{g}_{\mathrm{Acou}_{\alpha\beta}} := c^{-2}M_{\alpha\beta} + (c^{-2}-1)(v_\flat)_\alpha(v_\flat)_\beta, \tag{2.15a}$$

$$(\mathbf{g}^{-1}_{\mathrm{Acou}})^{\alpha\beta} := c^2(M^{-1})^{\alpha\beta} + (c^2-1)v^\alpha v^\beta. \tag{2.15b}$$

Definition 2.10 (Adjusted acoustical metric and its inverse). We define the adjusted acoustical metric $\mathbf{g}_{\alpha\beta} = \mathbf{g}_{\alpha\beta}(\vec{\Psi})$ and its inverse $(\mathbf{g}^{-1})^{\alpha\beta} = (\mathbf{g}^{-1})^{\alpha\beta}(\vec{\Psi})$ as follows[g]:

$$\begin{aligned}\mathbf{g}_{\alpha\beta} &= \mathbf{g}_{\mathrm{Acou}_{\alpha\beta}}(-\mathbf{g}^{00}_{\mathrm{Acou}})\\ &= \{c^{-2}M_{\alpha\beta} + (c^{-2}-1)(v_\flat)_\alpha(v_\flat)_\beta\}\{c^2 - (c^2-1)(v^0)^2\},\end{aligned} \tag{2.16a}$$

$$(\mathbf{g}^{-1})^{\alpha\beta} = \frac{\mathbf{g}^{\alpha\beta}_{\mathrm{Acou}}}{-\mathbf{g}^{00}_{\mathrm{Acou}}} = \frac{c^2M^{\alpha\beta} + (c^2-1)v^\alpha v^\beta}{c^2 - (c^2-1)(v^0)^2}. \tag{2.16b}$$

We emphasize that $(\mathbf{g}^{-1})^{00} = -1$. This helps us to simplify some of our formulas. Our bootstrap assumption will be so that $0 < c \leq 1$, so $(\mathbf{g}^{-1}_{\mathrm{Acou}})^{00} < 0$.

We lower and raise indices with the acoustic metric $\mathbf{g}$ and its inverse by using the notation $V_\beta = \mathbf{g}_{\alpha\beta}V^\alpha$ and $V^\beta = (\mathbf{g}^{-1})^{\alpha\beta}V_\alpha$. Note the difference between with raising and lowering indices with $\mathbf{g}$ versus M, see Sec. 2.1.

Definition 2.11. We define the future-directed $\mathbf{g}$-timelike vectorfield

$$\mathbf{T}^\alpha := -\mathbf{g}^{\alpha 0}. \tag{2.17}$$

We note that $\mathbf{g}(\mathbf{T},\mathbf{T}) = -1$. We note that $\mathbf{T}_\alpha = -\delta^0_\alpha$ where δ^0_α is the Kronecker delta.

Definition 2.12. In Cartesian coordinates, the induced metric g and its inverse on constant-time hypersurface Σ_t from $\mathbf{g}$ are as follows:

$$g_{ab} := \mathbf{g}_{ab} + \mathbf{T}_a\mathbf{T}_b, \tag{2.18a}$$

$$\left(g^{-1}\right)^{ab} := \mathbf{g}^{ab} + \mathbf{T}^a\mathbf{T}^b. \tag{2.18b}$$

By (2.16) and (2.17), one can compute that $g_{ab}(g^{-1})^{bc} = \delta^c_a$. We note that g can be also viewed as a space-time tensor, that is

$$g_{\alpha\beta} := \mathbf{g}_{\alpha\beta} + \mathbf{T}_\alpha\mathbf{T}_\beta, \tag{2.19a}$$

$$(g^{-1})^{\alpha\beta} := \mathbf{g}^{\alpha\beta} + \mathbf{T}^\alpha\mathbf{T}^\beta. \tag{2.19b}$$

Note that by (2.17) and $(\mathbf{g}^{-1})^{00} = -1$, $\underline{\Pi}^\delta_\alpha := \mathbf{g}^{\beta\delta}g_{\alpha\beta}$ can be viewed as $\mathbf{g}$-orthogonal projection operator from whole space-time onto Σ_t.

[f] For convenience, we write $\mathbf{g}^{\alpha\beta}_{\mathrm{Acou}}$ instead of $(\mathbf{g}^{-1}_{\mathrm{Acou}})^{\alpha\beta}$ in this paper.

[g] For convenience, we write $\mathbf{g}^{\alpha\beta}$ instead of $(\mathbf{g}^{-1})^{\alpha\beta}$ in this paper.

Proposition 2.13. *The metric g and its inverse g^{-1} have the following spatial components relative to rectangular coordinates*:

$$(g^{-1})^{ab} = \{c^2 - (c^2-1)(v^0)^2\}^{-2}\{c^2\delta^{ab}[c^2 - (c^2-1)(v^0)^2] + c^2(c^2-1)v^a v^b\}, \tag{2.20a}$$

$$g_{ab} = \{c^{-2}\delta_{ab} + (c^{-2}-1)(v_\flat)_a(v_\flat)_b\}\{c^2 - (c^2-1)(v^0)^2\}. \tag{2.20b}$$

Using (2.16b), (2.17) *and* (2.18b), (2.20a) *is obtained by direct computation.* (2.20b) *is obtained by the fact that* $\mathbf{T}_a = 0$ *for* $a = 1, 2, 3$.

Definition 2.14 (Differential operators defined by g). $\mathbf{D}$ denotes the Levi-Civita connection of $\mathbf{g}$ and $\Box_{\mathbf{g}} := \mathbf{g}^{\alpha\beta}\mathbf{D}_\alpha\mathbf{D}_\beta$ denotes the corresponding covariant wave operator. In Cartesian coordinates, for scalar function φ, $\Box_{\mathbf{g}}\varphi = \frac{1}{\sqrt{|\mathbf{g}|}}\partial_\alpha(\sqrt{|\mathbf{g}|}\mathbf{g}^{\alpha\beta}\partial_\beta\varphi)$, where $|\mathbf{g}|$ is the determinant of $\mathbf{g}$.

Definition 2.15 (Arrays of variables). For convenience in presenting the formulations and analysis, we define the following arrays of solution variables:

$$\vec{v} := (v^0, v^1, v^2, v^3), \quad \vec{\omega} := (\omega^0, \omega^1, \omega^2, \omega^3), \tag{2.21a}$$

$$\vec{S} := ((S^\sharp)^0, (S^\sharp)^1, (S^\sharp)^2, (S^\sharp)^3), \quad \vec{\mathcal{C}} := (\mathcal{C}^0, \mathcal{C}^1, \mathcal{C}^2, \mathcal{C}^3). \tag{2.21b}$$

We also define the array $\vec{\Psi}$ of wave variables, as follows:

$$\vec{\Psi} := (v^0, v^1, v^2, v^3, h, s). \tag{2.22}$$

Definition 2.16 (Material derivative). We define the material derivative[h] $\mathbf{B} = \mathbf{B}(\vec{\Psi})$ as follows:

$$\mathbf{B} := \frac{v^\alpha}{v^0}\partial_\alpha. \tag{2.23}$$

2.4.1. *The geometric wave–transport formulation of the relativistic Euler equations*

We use the following schematic notations throughout the paper where A, B, C are arrays of variables:

- $\mathscr{L}[A](B)$ denotes any scalar-valued function that is linear in the components of B with coefficients that are a function of the components of A.
- $\mathscr{Q}[A](B, C)$ denotes any scalar-valued function that is quadratic in the components of B and C with coefficients that are a function of the components of A.

Proposition 2.17 ([12, (3.1)–(3.12b), The geometric wave–transport formulation of the relativistic Euler equations]). *If* $\Psi \in \{v^0, v^1, v^2, v^3, h, s\}$ *solves the relativistic Euler equations* (2.12), *then* $\Psi, \omega, S, \mathcal{C}, \mathcal{D}$ *also satisfy the*

[h] We note that in [11], $\mathbf{B} = \partial_t + v^i\partial_i$.

following:

Wave equations:

$$\square_{\mathbf{g}_{\mathrm{Acou}}}\Psi = \mathscr{L}(\vec{\Psi})[\vec{\mathcal{C}},\mathcal{D}] + \mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\Psi}]. \tag{2.24}$$

Transport equations:

$$\mathbf{B}\omega^\alpha = \mathscr{L}(\vec{\Psi},\vec{\omega},\vec{S})[\boldsymbol{\partial}\vec{\Psi}], \tag{2.25a}$$

$$\mathbf{B}(S^\sharp)^\alpha = \mathscr{L}(\vec{\Psi},\vec{S})[\boldsymbol{\partial}\vec{\Psi}]. \tag{2.25b}$$

Transport-Div-Curl system:

$$\begin{aligned}\mathbf{B}\mathcal{C}^\alpha = \mathfrak{F}_{(\mathcal{C}^\alpha)} &:= \mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi},(\boldsymbol{\partial}\omega,\boldsymbol{\partial}\vec{S},\boldsymbol{\partial}\vec{\Psi})] \\ &\quad + \mathscr{Q}(\vec{S})[\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\Psi}] + \mathscr{L}(\vec{\Psi},\vec{\omega},\vec{S})[\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{S}], \end{aligned}\tag{2.26a}$$

$$\begin{aligned}\mathbf{B}\mathcal{D} = \mathfrak{F}_{(\mathcal{D})} &:= \mathscr{L}(\vec{\Psi},\vec{S})[\boldsymbol{\partial}\omega] + \mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi},(\boldsymbol{\partial}\vec{S},\boldsymbol{\partial}\vec{\Psi})] \\ &\quad + \mathscr{Q}(\vec{S})[\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\Psi})] + \mathscr{L}(\vec{\Psi},\vec{\omega},\vec{S})[\boldsymbol{\partial}\vec{\Psi}], \end{aligned}\tag{2.26b}$$

$$\mathrm{vort}^\alpha(S) = 0, \tag{2.26c}$$

$$\partial_\alpha\omega^\alpha = \mathscr{L}(\vec{\omega})[\boldsymbol{\partial}\vec{\Psi}]. \tag{2.26d}$$

Remark 2.18. The div-curl system (2.26) in the geometric formulation of the relativistic Euler equations is a space-time div-curl system. This feature causes difficulties as we want to derive estimates for vorticity and entropy gradient along the constant-time hypersurface Σ_t. To solve this issue, we rewrite the div-curl system into a dynamic spatial system along constant-time slices and apply theory in Littlewood–Paley decomposition as well as pseudodifferential operators in Sec. 5. These difficulties are not present in the non-relativistic 3D compressible Euler equations because the analogue of (2.26c) and (2.26d) is already a spatial div-curl system.

We now provide some useful identities, which we are going to use throughout the rest of the paper.

Lemma 2.19 (Identities involving vorticity and entropy gradient). *We list some useful identities in* [12, Sec. 4] *as follows*:

$$\omega^\kappa(v_\flat)_\kappa = 0, \tag{2.27a}$$

$$v^\kappa\partial_\alpha(\omega_\flat)_\kappa = -\omega^\kappa\partial_\alpha(v_\flat)_\kappa, \tag{2.27b}$$

$$v^\kappa\partial_\alpha S_\kappa = -(S^\sharp)^\kappa\partial_\alpha(v_\flat)_\kappa, \tag{2.27c}$$

$$\begin{aligned}\partial_\gamma(\omega_\flat)_\delta - \partial_\delta(\omega_\flat)_\gamma &= \epsilon_{\gamma\delta\kappa\lambda}v^\kappa\mathrm{vort}^\lambda(\omega) - (v^\kappa\partial_\kappa(\omega_\flat)_\delta)(v_\flat)_\gamma + v^\kappa(\partial_\delta(\omega_\flat)_\kappa)(v_\flat)_\gamma \\ &\quad + (v^\kappa\partial_\kappa(\omega_\flat)_\gamma)(v_\flat)_\delta - v^\kappa(\partial_\delta(\omega_\flat)_\kappa)(v_\flat)_\delta, \end{aligned}\tag{2.27d}$$

$$\begin{aligned}\partial_\gamma S_\delta - \partial_\delta S_\gamma &= \epsilon_{\gamma\delta\kappa\lambda}v^\kappa\mathrm{vort}^\lambda(S) - (v^\kappa\partial_\kappa S_\delta)(v_\flat)_\gamma \\ &\quad + v^\kappa(\partial_\delta S_\kappa)(v_\flat)_\gamma + (v^\kappa\partial_\kappa S_\gamma)(v_\flat)_\delta - v^\kappa(\partial_\delta S_\kappa)(v_\flat)_\delta. \end{aligned}\tag{2.27e}$$

Discussion of the proof. (2.27a) follows from Definition 2.3. (2.27b) follows from taking ∂_α derivative of (2.27a). (2.27c) follows from taking ∂_α derivative of (2.12c).

To prove (2.27d) and (2.27e), we use Definition 2.3 to express $\mathrm{vort}^\lambda(V)$ for $V = \omega_\flat$ and $V = S$, respectively. Then using the fact that

$$-\epsilon_{\alpha\beta\gamma\delta}\epsilon^{\delta\theta\kappa\lambda} = \delta^\theta_\alpha\delta^\kappa_\gamma\delta^\lambda_\beta - \delta^\theta_\alpha\delta^\lambda_\gamma\delta^\kappa_\beta + \delta^\lambda_\alpha\delta^\theta_\gamma\delta^\kappa_\beta - \delta^\lambda_\alpha\delta^\kappa_\gamma\delta^\theta_\beta + \delta^\kappa_\alpha\delta^\lambda_\gamma\delta^\theta_\beta - \delta^\kappa_\alpha\delta^\theta_\gamma\delta^\lambda_\beta, \tag{2.28}$$

we rearrange the terms and obtain (2.27d) and (2.27e).

We refer readers to [12, Lemma 4.1] for detailed proofs. □

In the following proposition, we provide the geometric wave equation with respect to the rescaled acoustical metric $\mathbf{g}$, which is more convenient for us to derive energy estimates and construct geometry. We will use this equation in the rest of the paper.

Proposition 2.20 (Wave equations after rescaling the acoustical metric). *Let* $\mathbf{g}$ *be as defined in Definition* 2.10. *If* $\Psi \in \{v^0, v^1, v^2, v^3, h, s\}$ *solves the relativistic Euler equations* (2.12), *we have the following equation holds*:

$$\square_{\mathbf{g}}\Psi = \mathfrak{F}_{(\Psi)} := \mathscr{L}(\vec{\Psi})[\vec{\mathcal{C}}, \mathcal{D}] + \mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\Psi}]. \tag{2.29}$$

Proof of Proposition 2.20 using Proposition 2.17 given.

$$\begin{aligned}
\square_{\mathbf{g}}\Psi &= \frac{1}{\sqrt{|\mathbf{g}|}}\partial_\alpha(\sqrt{|\mathbf{g}|}\mathbf{g}^{\alpha\beta}\partial_\beta\Psi) \\
&= \frac{1}{\sqrt{|\mathbf{g}_{\mathrm{Acou}}|}}(-\mathbf{g}^{00}_{\mathrm{Acou}})^{-2}\partial_\alpha(\sqrt{|\mathbf{g}_{\mathrm{Acou}}|}(-\mathbf{g}^{00}_{\mathrm{Acou}})\mathbf{g}^{\alpha\beta}_{\mathrm{Acou}}\partial_\beta\Psi) \\
&= \frac{1}{\sqrt{|\mathbf{g}_{\mathrm{Acou}}|}}(-\mathbf{g}^{00}_{\mathrm{Acou}})^{-1}\partial_\alpha(\sqrt{|\mathbf{g}_{\mathrm{Acou}}|}\mathbf{g}^{\alpha\beta}_{\mathrm{Acou}}\partial_\beta\Psi) \\
&\quad + (-\mathbf{g}^{00}_{\mathrm{Acou}})^{-2}\mathbf{g}^{\alpha\beta}_{\mathrm{Acou}}\partial_\alpha(\mathbf{g}^{00}_{\mathrm{Acou}})\partial_\beta\Psi \\
&= (-\mathbf{g}^{00}_{\mathrm{Acou}})^{-1}\square_{\mathbf{g}_{\mathrm{Acou}}}\Psi + \mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\Psi}].
\end{aligned} \tag{2.30}$$

Note that $\mathbf{g}^{\alpha\beta}_{\mathrm{Acou}}$ is smooth function of $\vec{\Psi}$ and $\mathbf{g}^{00}_{\mathrm{Acou}} \neq 0$. Therefore by combining (2.24) and (2.30), we obtain the desired equation. □

3. Norms, Littlewood–Paley Projections, Statement of Main Results and Bootstrap Assumptions

In this section, we define the norms, define the standard Littlewood–Paley projections that we use in the analysis, state our main results of the paper and bootstrap assumptions.

3.1. *Norms*

In this paper, for functions f, g on a normed space $(X, \|\cdot\|_X)$, we use the notation $\|f, g\|_X := \|f\|_X + \|g\|_X$. Similarly, for an array of functions $\vec{U} = (U^1, U^2, \ldots, U^k)$, we have $\|\vec{U}\|_X := \sum_{a=1}^k \|U^a\|_X$. In particular, we use $|\vec{U}| := \sqrt{\sum_{i=1}^k (U^i)^2}$. For functions f and arrays $\vec{g}$, we also use $\|f, g\|_X := \|f\|_X + \|\vec{g}\|_X$.

Since the volume form on the constant-time hypersurface Σ_t induced by Minkowski metric M is $\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3$, and by identifying $(t, x^1, x^2, x^3) \in \Sigma_t$ with $(x^1, x^2, x^3) \in \mathbb{R}$, we define the standard Sobolev norm on Σ_t for $s \in \mathbb{R}$: $\|f\|_{H^s(\Sigma_t)} := \|\langle\xi\rangle^s \hat{f}(\xi)\|_{L^2_x(\Sigma_t)}$, where $\langle\xi\rangle := (1 + |\xi|^2)^{1/2}$.

We denote the standard Hölder seminorm $\dot{C}^{0,\beta}_x$ and Hölder norm $C^{0,\beta}_x$, where $0 < \beta < 1$, of a function F with respect to flat metric on constant-time hypersurface Σ_t by

$$\|F\|_{\dot{C}^{0,\beta}_x(\Sigma_t)} := \sup_{x \neq y \in \Sigma_t} \frac{|F(x) - F(y)|}{|x - y|^\beta}, \tag{3.1}$$

$$\|F\|_{C^{0,\beta}_x(\Sigma_t)} := \sup_{x \in \Sigma_t} |F(x)| + \sup_{x \neq y \in \Sigma_t} \frac{|F(x) - F(y)|}{|x - y|^\beta}. \tag{3.2}$$

We also use the following mixed norms for function $F : \mathbb{R}^3 \to \mathbb{R}$, where $1 \leq q_1 < \infty$, $1 \leq q_2 \leq \infty$ and I is an interval of time:

$$\|F\|_{L^{q_1}_t L^{q_2}_x(I \times \Sigma_t)} := \left\{ \int_I \|F\|^{q_1}_{L^{q_2}_x(\Sigma_\tau)} \mathrm{d}\tau \right\}^{1/q_1}, \tag{3.3a}$$

$$\|F\|_{L^\infty_t L^{q_2}_x(I \times \Sigma_t)} := \operatorname*{ess\,sup}_{\tau \in I} \|F\|_{L^{q_2}_x(\Sigma_\tau)}, \tag{3.3b}$$

$$\|F\|_{L^{q_1}_t C^{0,\beta}_x(I \times \Sigma_t)} := \left\{ \int_I \|F\|^{q_1}_{C^{0,q_2}_x(\Sigma_\tau)} \mathrm{d}\tau \right\}^{1/q_1}, \tag{3.3c}$$

$$\|F\|_{L^\infty_t C^{0,q_2}_x(I \times \Sigma_t)} := \operatorname*{ess\,sup}_{\tau \in I} \|F\|_{C^{0,q_2}_x(\Sigma_\tau)}. \tag{3.3d}$$

If $\{F_\lambda\}_{\lambda \in 2^{\mathbb{N}}}$ is a dyadic-indexed sequence of functions on Σ_t, we define

$$\|F_\nu\|_{l^2_\nu L^2_x(\Sigma_t)} := \left(\sum_{\nu \geq 1} \|F_\nu\|^2_{L^2_x(\Sigma_t)} \right)^{1/2}. \tag{3.4}$$

3.2. *Littlewood–Paley projections*

We fix a smooth function $\psi = \psi(|\xi|) : \mathbb{R}^3 \to [0, 1]$ supported on the frequency space annulus $\{\xi \in \mathbb{R}^3 | 1/2 \leq |\xi| \leq 2\}$ such that for $\xi \neq 0$, we have $\sum_{k \in \mathbb{Z}} \psi(2^k \xi) = 1$. For dyadic frequencies $\nu = 2^k$ with $k \in \mathbb{Z}$, we define the standard Littlewood–Paley projection P_ν, which acts on scalar functions $F : \mathbb{R} \to \mathbb{C}$, as follows:

$$P_\nu F(x) := \int_{\mathbb{R}^3} e^{2\pi i x\cdot\xi}\psi(\nu^{-1}\xi)\hat{F}(\xi)\mathrm{d}\xi, \tag{3.5}$$

where $\hat{F}(\xi) := \int_{\mathbb{R}^3} e^{-2\pi i x\cdot\xi}F(x)\mathrm{d}x$ is the Fourier transform of F. If $I \subset 2^{\mathbb{Z}}$ is an interval of dyadic frequencies, then $P_I F := \sum_{\nu\in I} P_\nu F$ and $P_{\leq\nu}F := P_{[-\infty,\nu]}F$. For functions f, g, we use the schematic notation that $P_\nu(f, g)$ as the linear combination of $P_\nu f$ and $P_\nu g$, namely, $P_\nu f + P_\nu g$.

Proposition 3.1. *For a function F, standard results in Littlewood–Paley theory give the following*:

$$\|F\|_{H^s(\Sigma_t)} \approx \|F\|_{L^2_x(\Sigma_t)} + \left(\sum_{\nu>1}\nu^{2s}\|P_\nu F\|_{L^2_x(\Sigma_t)}\right)^{1/2}, \tag{3.6}$$

$$\|F\|_{C^{0,s}_x(\Sigma_t)} \approx \|F\|_{L^\infty_x(\Sigma_t)} + \sup_{\nu\geq 2}\nu^s\|P_\nu F\|_{L^\infty_x(\Sigma_t)}, \tag{3.7}$$

where H^s is the standard Sobolev norm and $C^{0,s}$ is the standard Hölder norm. One can refer [18, Sec. 1; 28, A.1] *for the above results.*

The following two Lemmas consist of a commuted version of the equations. Lemma 3.2 commutes $\square_{\mathbf{g}}$ and $\mathbf{B}$ with $\boldsymbol{\partial}$ and is needed for below-top-order estimates. Lemma 3.3 commutes $\square_{\mathbf{g}}$ and $\mathbf{B}$ with $P_\nu\boldsymbol{\partial}$ and is needed for the top-order estimates.

Lemma 3.2 (Commuted equations satisfied by one derivative of the solution variables). *We consider the solutions to the equations of Proposition* 2.17, *that is, if $\Psi \in \{v^0, v^1, v^2, v^3, h, s\}$ solves the relativistic Euler equations* (2.12), *the following equations hold*:

$$\square_{\mathbf{g}}\boldsymbol{\partial}\Psi = \mathscr{L}(\vec{\Psi})[\boldsymbol{\partial}\vec{\mathcal{C}}, \boldsymbol{\partial}\mathcal{D}] + \mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}^2\vec{\Psi}, \boldsymbol{\partial}\vec{\Psi}] + \mathscr{L}(\vec{\Psi})[(\boldsymbol{\partial}\vec{\Psi})^3], \tag{3.8a}$$

$$\begin{aligned}\mathbf{B}\boldsymbol{\partial}\mathcal{C}^\alpha &= \mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi}, (\boldsymbol{\partial}^2\omega, \boldsymbol{\partial}^2\vec{S}, \boldsymbol{\partial}^2\vec{\Psi})] + \mathscr{Q}(\vec{\Psi})[(\boldsymbol{\partial}^2\vec{\Psi}, \boldsymbol{\partial}\vec{\omega}, \boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{S}), (\boldsymbol{\partial}\vec{\omega}, \boldsymbol{\partial}\vec{S}, \boldsymbol{\partial}\vec{\Psi})]\\ &\quad + \mathscr{L}(\vec{\Psi})[(\boldsymbol{\partial}\vec{\Psi})^2\cdot(\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{S}, \boldsymbol{\partial}\vec{\omega})],\end{aligned} \tag{3.8b}$$

$$\begin{aligned}\mathbf{B}\boldsymbol{\partial}\mathcal{D} &= \mathscr{L}(\vec{\Psi}, \vec{\omega}, \vec{S})[\boldsymbol{\partial}^2\omega, \boldsymbol{\partial}^2\vec{\Psi}, \boldsymbol{\partial}^2\vec{S}] + \mathscr{Q}(\vec{\Psi})[(\boldsymbol{\partial}^2\vec{\Psi}, \boldsymbol{\partial}\vec{\omega}, \boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{S}), (\boldsymbol{\partial}\vec{S}, \boldsymbol{\partial}\vec{\Psi})]\\ &\quad + \mathscr{L}(\vec{\Psi})[(\boldsymbol{\partial}\vec{\Psi})^2\cdot(\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{S})].\end{aligned} \tag{3.8c}$$

Sketch of the proof of Lemma 3.2. By commuting (2.29), (2.26a) and (2.26b) with $\boldsymbol{\partial}$, using relations (by Definition 2.8) $\mathcal{C} = \mathscr{L}(\vec{\Psi}, \vec{\omega}, \vec{S})[\boldsymbol{\partial}\vec{\omega}, \boldsymbol{\partial}\vec{\Psi}]$ and $\mathcal{D} = \mathscr{L}[\boldsymbol{\partial}\vec{S}] + \mathscr{L}(\vec{S})[\boldsymbol{\partial}\vec{\Psi}]$, (3.8a)–(3.8c) are derived by straightforward computations. $\square$

The following Lemma provides the commuted equations with the Littlewood–Paley projections.

Lemma 3.3 ([11, Lemmas 5.2, 5.4, Equations satisfied by the frequency-projected solution variables]). *For solutions to the equations of Proposition* 2.17, *the following equations hold*:

$$\square_{\mathbf{g}} P_{\nu} \boldsymbol{\partial} \Psi = \mathfrak{R}_{(\boldsymbol{\partial}\vec{\Psi});\nu}, \tag{3.9a}$$

$$\mathbf{B} P_{\nu} \boldsymbol{\partial} \mathcal{C}^{\alpha} = \mathfrak{R}_{(\boldsymbol{\partial}\mathcal{C}^{\alpha});\nu}, \tag{3.9b}$$

$$\mathbf{B} P_{\nu} \boldsymbol{\partial} \mathcal{D} = \mathfrak{R}_{(\boldsymbol{\partial}\mathcal{D});\nu}, \tag{3.9c}$$

where

$$\begin{aligned} \mathfrak{R}_{(\boldsymbol{\partial}\vec{\Psi});\nu} &= P_{\nu}\boldsymbol{\partial}\mathfrak{F}_{(\Psi)} - \sum_{(\alpha,\beta)\neq(0,0)} P_{\nu}\left[\boldsymbol{\partial}\mathbf{g}^{\alpha\beta}\partial_{\alpha}\partial_{\beta}\Psi\right] - \mathbf{\Gamma}^{\alpha} P_{\nu}\partial_{\alpha}\boldsymbol{\partial}\Psi \\ &\quad + \sum_{(\alpha,\beta)\neq(0,0)} \left[\mathbf{g}^{\alpha\beta} - P_{\leq\nu}\mathbf{g}^{\alpha\beta}\right] P_{\nu}\partial_{\alpha}\partial_{\beta}\boldsymbol{\partial}\Psi \\ &\quad + \sum_{(\alpha,\beta)\neq(0,0)} \left\{P_{\leq\nu}\mathbf{g}^{\alpha\beta} P_{\nu}\partial_{\alpha}\partial_{\beta}\boldsymbol{\partial}\Psi - P_{\nu}[\mathbf{g}^{\alpha\beta}\partial_{\alpha}\partial_{\beta}\boldsymbol{\partial}\Psi]\right\}, \end{aligned} \tag{3.10a}$$

$$\begin{aligned} \mathfrak{R}_{(\boldsymbol{\partial}\mathcal{C}^{\alpha});\nu} &= P_{\nu}\boldsymbol{\partial}\mathfrak{F}_{(\mathcal{C}^{\alpha})} - P_{\nu}\left[\boldsymbol{\partial}\left(\frac{v^a}{v^0}\right)\partial_a\mathcal{C}^{\alpha}\right] + \left[\frac{v^a}{v^0} - P_{\leq\nu}\left(\frac{v^a}{v^0}\right)\right] P_{\nu}\partial_a\boldsymbol{\partial}\mathcal{C}^{\alpha} \\ &\quad + P_{\leq\nu}\left(\frac{v^a}{v^0}\right) P_{\nu}\partial_a\boldsymbol{\partial}\mathcal{C}^{\alpha} - P_{\nu}\left[\frac{v^a}{v^0}\partial_a\boldsymbol{\partial}\mathcal{C}^{\alpha}\right], \end{aligned} \tag{3.10b}$$

$$\begin{aligned} \mathfrak{R}_{(\boldsymbol{\partial}\mathcal{D});\nu} &= P_{\nu}\boldsymbol{\partial}\mathfrak{F}_{(\mathcal{D})} - P_{\nu}\left[\boldsymbol{\partial}\left(\frac{v^a}{v^0}\right)\partial_a\mathcal{D}\right] + \left[\frac{v^a}{v^0} - P_{\leq\nu}\left(\frac{v^a}{v^0}\right)\right] P_{\nu}\partial_a\boldsymbol{\partial}\mathcal{D} \\ &\quad + P_{\leq\nu}\left(\frac{v^a}{v^0}\right) P_{\nu}\partial_a\boldsymbol{\partial}\mathcal{D} - P_{\nu}\left[\frac{v^a}{v^0}\partial_a\boldsymbol{\partial}\mathcal{D}\right]. \end{aligned} \tag{3.10c}$$

Moreover, *the following estimates hold for the remainders where* l^2_{ν}*-seminorm is taken over dyadic frequencies*:

$$\begin{aligned} &\|\nu^{N-2}(\mathfrak{R}_{(\boldsymbol{\partial}\vec{\Psi});\nu}, \mathfrak{R}_{(\boldsymbol{\partial}\mathcal{C}^{\alpha});\nu}, \mathfrak{R}_{(\boldsymbol{\partial}\mathcal{D});\nu})\|_{l^2_{\nu}L^2_x(\Sigma_t)} \\ &\quad \lesssim \|\partial(\vec{\mathcal{C}}, \mathcal{D})\|_{H^{N-2}(\Sigma_t)} + (\|\boldsymbol{\partial}(\vec{\Psi}, \vec{\omega}, \vec{S})\|_{L^{\infty}_x(\Sigma_t)} + 1) \\ &\quad\quad \times (\|\boldsymbol{\partial}(\vec{\Psi}, \vec{\omega}, \vec{S})\|_{H^{N-1}(\Sigma_t)} + 1). \end{aligned} \tag{3.11}$$

Discussion of the proof of Lemma 3.3. We omit the proof of Eqs. (3.10a)–(3.10c) since it follows from straightforward computations. We use bootstrap assumptions, product and commutator estimates for Littlewood–Paley calculus to prove estimates (3.11). We refer readers to [11, Lemma 5.4] for the detailed proofs where the structure of the equations are the same as in this paper. We note that we have $\mathbf{B}^a = \frac{v^a}{v^0}$ compared to $\mathbf{B}^a = v^a$ in [11], which doesn't change the proof or result in the estimates (3.11). $\square$

3.3. *Statement of main theorem*

Using the notations introduced in the previous sections, we precisely provide our assumptions on the data and the statement of the main theorem.

Theorem 3.4 (Main theorem). *Consider a solution to the relativistic Euler equations whose initial data satisfies following assumptions for some real number* $2 < N < 5/2$, $0 < \alpha < 1$, $c_1 > 0$ *and* D:

(1) $\|h, v, \omega\|_{H^N(\Sigma_0)} + \|s\|_{H^{N+1}(\Sigma_0)} \leq D$,
(2) $\|\mathcal{C}, \mathcal{D}\|_{C^{0,\alpha}(\Sigma_0)} \leq D$,
(3) *The data functions are contained in the interior of* $\mathcal{R}$ (*See Definition* 3.5 *for definition of* $\mathcal{R}$) *and the enthalpy* H *is strictly positive, i.e.* $H \geq c_1 > 0$.

Then the solution's time of classical existence $T > 0$ *can be bounded from below in terms of* D *and* $\mathcal{R}$. *Moreover, the Sobolev and some Hölder regularity*[i] *of the data are propagated by the solution on the slab of classical existence.*

3.4. *Choice of parameters*

In this section, we introduce several parameters that each of them either measures the regularity or plays a role in our analysis. We denote the assumed Sobolev regularity of the "wave-part" of the data and the Hölder regularity of the "transport-part" of the data by, respectively, $2 < N < 5/2$ and $0 < \alpha < 1$. For the purpose of analysis, we choose positive numbers $q, \varepsilon_0, \delta_0, \delta$ and δ_1 that satisfy the following conditions:

$$2 < q < \infty, \tag{3.12a}$$

$$< \varepsilon_0 := \frac{N-2}{10} < \frac{1}{10}, \tag{3.12b}$$

$$\delta_0 := \min\left\{\varepsilon_0^2, \frac{\alpha}{10}\right\}, \tag{3.12c}$$

$$0 < \delta := \frac{1}{2} - \frac{1}{q} < \varepsilon_0, \tag{3.12d}$$

$$\delta_1 := \min\{N - 2 - 4\varepsilon_0 - \delta(1 - 8\varepsilon_0), \alpha\} > 8\delta_0 > 0. \tag{3.12e}$$

More precisely, we consider N, α, ε_0 and δ_0 to be fixed throughout the paper, while q, δ and δ_1 will be treated as parameters. In particular, (3.12e) is related to a slightly (but enough) improvement of Strichartz estimates (4.1) over the bootstrap assumptions (3.17).

[i]We note that not entire Hölder regularity is propagated in the way that only $\|\vec{\Psi}, \mathcal{C}, \mathcal{D}\|_{C^{0,\delta_1}(\Sigma_t)}$ (see Sec. 3.4 for the definition of δ_1) is bounded by data.

3.5. *Assumptions on the initial data*

In this section, we provide the bootstrap assumptions that will be used in the proof of Theorem 3.4.

Definition 3.5 (Regime of hyperbolicity). We define $\mathcal{R}$ as follows:

$$\mathcal{R} := \{(h, s, \vec{v}, \vec{\omega}, \vec{S}) \in \mathbb{R}^{14} \mid 0 < c \leq 1\}. \tag{3.13}$$

With N and α as in Sec. 3.4, we assume that

$$\text{“Wave} - \text{part”}\ \|h, \vec{v}\|_{H^N(\Sigma_0)} < \infty, \tag{3.14}$$

$$\text{“Transport} - \text{part”}\ \|s\|_{H^{N+1}(\Sigma_0)} + \|\vec{\omega}\|_{H^N(\Sigma_0)} + \|\vec{\mathcal{C}}, \mathcal{D}\|_{C^{0,\alpha}(\Sigma_0)} < \infty. \tag{3.15}$$

Assumptions (3.14) and (3.15) correspond to regularity assumptions on the “wave-part” and “transport-part” of the data, respectively.

Let $\mathrm{int}\, U$ denote the interior of the set U. We assume that there is a compact subset $\breve{\mathcal{R}}$ such that

$$(\vec{\Psi}, \vec{\omega}, \vec{S})(\Sigma_0) \subset \mathrm{int}\breve{\mathcal{R}} \subset \breve{\mathcal{R}} \subset \mathrm{int}\mathcal{R}, \tag{3.16}$$

where $\mathcal{R}$ is defined in (3.13).

3.6. *Bootstrap assumptions*

Throughout the paper, $0 < T_* \ll 1$ denotes a bootstrap time that depends only on initial data. We assume that $\vec{\Psi}$ is a smooth[j] solution to the equation in Sec. 2.3 and the following estimates hold:

$$(\vec{\Psi}, \vec{\omega}, \vec{S})([0, T_*] \times \mathbb{R}^3) \subset \mathcal{R}, \tag{3.17a}$$

$$\|\boldsymbol{\partial}\vec{\Psi}\|^2_{L^2_t L^\infty_x([0,T_*]\times\mathbb{R}^3)} + \sum_{\nu\geq 2} \nu^{2\delta_0} \|P_\nu \boldsymbol{\partial}\vec{\Psi}\|^2_{L^2_t L^\infty_x([0,T_*]\times\mathbb{R}^3)} \leq 1, \tag{3.17b}$$

$$\|\partial\vec{\omega}, \partial\vec{S}\|^2_{L^2_t L^\infty_x([0,T_*]\times\mathbb{R}^3)} + \sum_{\nu\geq 2} \nu^{2\delta_0} \|P_\nu \partial\vec{\omega}, P_\nu \partial\vec{S}\|^2_{L^2_t L^\infty_x([0,T_*]\times\mathbb{R}^3)} \leq 1. \tag{3.17c}$$

In Theorem 5.18, we derive an improvement of (3.17c). In Theorem 6.1, we derive an improvement of (3.17b). By fundamental theorem of calculus, Eqs. (2.25a) and (2.25b), (3.17a) is a direct result of (3.17b) and (3.17c).

4. Structure of the Proofs in the Rest of the Paper

In this section, we provide the structure of the proofs in the paper. Our proofs rely on a bootstrap argument where the bootstrap assumptions are in Sec. 3.6. See Sec. 1.5 for the logic of the bootstrap argument. The main goal for us is to improve

[j] By smooth we mean as smooth as necessary for the analysis arguments to go through. Meanwhile, all of our quantitative estimates depend only on the Sobolev and Hölder norms.

the bootstrap assumptions to the following Strichartz-type estimates:

$$\|\boldsymbol{\partial}\vec{\Psi}\|^2_{L^2_tL^\infty_x([0,T_*]\times\mathbb{R}^3)} + \sum_{\nu\geq 2}\nu^{2\delta_1}\|P_\nu\boldsymbol{\partial}\vec{\Psi}\|^2_{L^2_tL^\infty_x([0,T_*]\times\mathbb{R}^3)} \lesssim T_*^{2\delta}, \tag{4.1a}$$

$$\|\boldsymbol{\partial}\vec{\omega},\boldsymbol{\partial}\vec{S}\|^2_{L^2_tL^\infty_x([0,T_*]\times\mathbb{R}^3)} + \sum_{\nu\geq 2}\nu^{2\delta_1}\|P_\nu\boldsymbol{\partial}\vec{\omega},P_\nu\boldsymbol{\partial}\vec{S}\|^2_{L^2_tL^\infty_x([0,T_*]\times\mathbb{R}^3)} \lesssim T_*^{2\delta}. \tag{4.1b}$$

We prove the (4.1a) through the following series of reductions, see Sec. 1.5.4 for an overview of the logic: Strichartz estimates $\leftarrow$ Decay estimates $\leftarrow$ Conformal energy estimates $\leftarrow$ Controlling of the acoustic null geometry.

To prove (4.1b), we prove a transport-Schauder-type estimate in Sec. 5, which is independent of the proof of (4.1a). In Theorem 5.18, we obtain (4.1b) by combining the transport-Schauder estimate and (4.1a).

4.1. *Similarities and differences compared to the 3D compressible Euler equations*

Broadly speaking, we use the same machinery as in [11] to reduce the proof of the Strichartz estimates to geometric quantities that have to be controlled in order to derive a conformal energy estimate. This reduction was first introduced and developed in the context of low-regularity problems for quasilinear wave equations, as we discussed in Sec. 1.1. Disconzi–Luo–Mazzone–Speck [11] and Wang [31] have exploited the remarkable structure of the non-relativistic 3D compressible Euler equations to derive similar low-regularity well-posedness results in the presence of vorticity and entropy. The main purpose of this paper is to derive similar results for the relativistic Euler flow by using the remarkable structure of the equations derived by Disconzi–Speck in [12].

Two main differences in the present paper compared to the non-relativistic case are (1) the first fundamental form of Σ_t is no longer conformally flat in the relativistic case, leading to more complicated geometry and (2) the L^2 elliptic and Schauder estimates that we need to handle the vorticity and entropy are more complicated because unlike in the non-relativistic case, the Hodge systems that we study are quasilinear (instead of constant coefficient).

4.2. *Energy, L^2 elliptic and Schauder estimates in Sec.* 5

In Sec. 5, first we prove the energy estimates for wave variables h, s, v and transport variables $\omega, S, \mathcal{C}, \mathcal{D}$ in Proposition 5.1. These estimates are essential to the local well-posedness Theorem 3.4. We also need these estimates for controlling the acoustic geometry. We control the $H^{2+\varepsilon}$ norm of wave variables under the bootstrap assumptions by using the geometric energy method in Sec. 5.1.1 and commuted equations in Lemmas 3.2 and 3.3. We refer readers to [29, Sec. 6] for the commutator estimates involving LP projections in fractional Sobolev spaces. We note that the L^2 elliptic estimates for transport variables $\omega, S, \mathcal{C}, \mathcal{D}$ in Proposition 5.8 is proven based on a rewritten dynamic div-curl system in Proposition 5.9.

We then prove the transport-Schauder estimates in the Hölder space C_x^{0,δ_1} for the transport variables $\omega, S, \mathcal{C}, \mathcal{D}$ in Theorem 5.18, which recovers the bootstrap assumptions (3.17c) in condition of (4.1a). To prove these estimates, we use the div-curl system (5.19) and the transport equations (2.25a) and (2.26b). We prove Schauder estimates by some standard results in pseudodifferential operators as well as the Littlewood–Paley decomposition with the help of the equivalence between Hölder spaces and frequency spaces (3.7).

4.3. *Reduction of Strichartz estimates to decay estimates in Sec. 6*

We state the Strichartz estimates for wave variables in Theorem 6.1, which improves the bootstrap assumptions (3.17b). Our reductions of the Strichartz estimates to the bounded conformal energy consists of several steps. In Sec. 6, we list several reductions from Strichartz estimates to a spatially localized decay estimate in Theorem 6.9. We first use Duhamel's principle to reduce Theorem 6.1 to a frequency-localized version of Strichartz estimates in Theorem 6.2. Then after rescaling all the quantities with respect to the frequency in Sec. 6.2, we run a $\mathcal{T}\mathcal{T}^*$ argument to reduce Theorem 6.2 to a decay estimate in Theorem 6.8. Finally, by Bernstein inequalities, partition of unity and Sobolev embedding, we obtain the spatially localized version of decay estimates in Theorem 6.9. The reductions are by now standard, therefore we only state the reductions without proof. We refer to Wang [30, Sec. 3] for the details. It is crucial to derive the decay estimates (6.18). To further reduce the decay estimates to conformal energy estimates, we need a geometric setup, which is in Sec. 7, as we will discuss in the next section.

4.4. *Geometric setup and conformal energy in Sec. 7*

To control the conformal energy, we use Wang's approach from [30], which relies on analysis on a conformal changed acoustic geometry. We reduce the decay estimates to the conformal energy estimates in Theorem 7.6 in Sec. 7.3 via product estimates and Bernstein inequality of Littlewood–Paley theory.

In order to define conformal energy and do analysis based on the geometric structure of the relativistic Euler equations, in Secs. 7.1 and 7.2, we construct the geometric null frame based on a solution u to the acoustical eikonal equation

$$(\mathbf{g}^{-1})^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0. \tag{4.2}$$

Then we control the acoustic geometry based on the null frame that we just constructed. Note that given the control over the acoustic geometry, we have favorable estimates for the conformal energy as we discuss in Sec. 7.3.2. We will discuss the control of the acoustic geometry in the next two sections.

4.5. *Energy along acoustic null hypersurfaces and control of the acoustic geometry in Sec.* 8

In Sec. 8, We prove the energy estimates for fluid variables along the acoustic null cones in Sec. 8.1. This is important since we need to control fluid variables along the null cones.

Then we control the acoustic geometry. To start with, we list the connection coefficients in Definition 8.4. We define conformal factor for the metric in Definition 8.5. We provide initial conditions for geometric cones in Propositions 8.8 and 8.9.

We setup the bootstrap assumptions for geometric quantities in Sec. 8.4.2 and list the main estimates for the geometric quantities in Proposition 8.10. We give a discussion of the proof of Proposition 8.10 in Sec. 8.4.4, which improves the assumptions in Sec. 8.4.2. Proposition 8.10 is proven via doing analysis on the transport equations and the div-curl systems of the geometric quantities. We only give a brief discussion of the proof since it follows the same in [11, Sec. 10].

5. Energy, L^2 Elliptic and Schauder Estimates

In this section, we first derive the energy and L^2 elliptic estimates along constant-time hypersurfaces. Note that we obtain the same results as in [11, Secs. 4 and 5], where ϱ in [11] plays the same role as h in this paper. Then we derive transport-Schauder-type estimates for the vorticity and entropy gradient.

5.1. *Energy and L^2 elliptic estimates*

The following Proposition is the main result of the energy estimates.

Proposition 5.1 (Energy and elliptic estimates). *Under the initial data and bootstrap assumptions of Sec.* 3, *smooth solutions to the relativistic Euler equations satisfy the following estimates for* $2 < N < 5/2$ *and* $t \in [0, T_*]$:

$$\begin{aligned}\sum_{k=0}^{2} \|\partial_t^k (h, \vec{v}, \vec{\omega})\|_{H^{N-k}(\Sigma_t)} + \sum_{k=0}^{2} \|\partial_t^k s\|_{H^{N+1-k}(\Sigma_t)} + \sum_{k=0}^{1} \|\partial_t^k (\vec{\mathcal{C}}, \mathcal{D})\|_{H^{N-1-k}(\Sigma_t)} \\ \lesssim \|(h, \vec{v}, \vec{\omega})\|_{H^N(\Sigma_0)} + \|s\|_{H^{N+1}(\Sigma_0)} + 1. \qquad (5.1)\end{aligned}$$

Remark 5.2. We note that 1 on the right-hand side of (5.1) is due to technical reasons. Specifically, as shown in Sec. 5.1.3, 1 can be replaced by $\int_0^t \|\boldsymbol{\partial}(\vec{\Psi}, \vec{\omega}, \vec{S})\|_{L^\infty_x(\Sigma_\tau)} \mathrm{d}\tau$. By bootstrap assumptions (3.17b) and (3.17c) and Hölder's inequality, we could actually bound this term by $T_*^{1/2}$.

We provide several key ingredients for proving Proposition 5.1 in the next two sections. We provide the basic energy inequality for wave equations and transport equations in Sec. 5.1.1. We prove a crucial elliptic div-curl estimate in Sec. 5.1.2.

We give the proof of Proposition 5.1 in Sec. 5.1.3. We refer readers to [11, Secs. 4 and 5; 30, Sec. 2] for the energy estimates in the non-relativistic 3D compressible Euler equations case and the quasilinear wave equations case, respectively.

5.1.1. *The basic energy inequality for wave equations and transport equations*

We provide the basic energy inequality for the wave equations in this section.

Definition 5.3 (Energy–momentum tensor, energy current and deformation tensor). We define the energy–momentum tensor $Q_{\mu\nu}[\varphi]$ associated to a scalar function φ to be the following tensorfield:

$$Q_{\mu\nu}[\varphi] := \partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}\mathbf{g}_{\mu\nu}(\mathbf{g}^{-1})^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi. \tag{5.2}$$

Given φ and any multiplier vectorfield $\mathbf{X}$, we define the corresponding energy current ${}^{(\mathbf{X})}\mathbf{J}^\alpha[\varphi]$ vectorfield as follows:

$${}^{(\mathbf{X})}\mathbf{J}^\alpha[\varphi] := Q^{\alpha\beta}[\varphi]\mathbf{X}_\beta - \varphi^2\mathbf{X}^\alpha. \tag{5.3}$$

We define the deformation tensor of $\mathbf{X}$ as follows:

$${}^{(\mathbf{X})}\boldsymbol{\pi}_{\alpha\beta} := \mathbf{D}_\alpha\mathbf{X}_\beta + \mathbf{D}_\beta\mathbf{X}_\alpha, \tag{5.4}$$

where $\mathbf{D}$ is the Levi-Civita connection with respect to $\mathbf{g}$.

We have the following well-known divergence identity:

$$\mathbf{D}_\alpha{}^{(\mathbf{X})}\mathbf{J}^\alpha[\varphi] = \square_{\mathbf{g}}\varphi(\mathbf{X}\varphi) + \frac{1}{2}Q^{\mu\nu}[\varphi]{}^{(\mathbf{X})}\boldsymbol{\pi}_{\mu\nu} - 2\varphi\mathbf{X}\varphi - \frac{1}{2}\varphi^2(\mathbf{g}^{-1})^{\mu\nu}{}^{(\mathbf{X})}\boldsymbol{\pi}_{\mu\nu}. \tag{5.5}$$

We define the energy $\mathbb{E}[\varphi](t)$ as follows where $\mathbf{T}^\alpha := -\mathbf{g}^{\alpha 0}$ is the future-directed $\mathbf{g}$-timelike vectorfield defined in Definition 2.11:

$$\mathbb{E}[\varphi](t) := \int_{\Sigma_t} {}^{(\mathbf{T})}\mathbf{J}^\alpha[\varphi]\mathbf{T}_\alpha \mathrm{d}\varpi_g = \int_{\Sigma_t} \left(Q^{00}[\varphi] + \varphi^2\right)\mathrm{d}\varpi_g, \tag{5.6}$$

where $\mathrm{d}\varpi_g$ is the volume form on Σ_t with respect to g induced by $\mathbf{g}$.

Lemma 5.4 (Coerciveness of $\mathbb{E}$). *Under the bootstrap assumptions of Sec.* 3.6, *the following estimate holds for $t \in [0, T_*]$:*

$$\mathbb{E}[\varphi](t) \approx \|\varphi\|^2_{H^1(\Sigma_t)} + \|\partial_t\varphi\|^2_{L^2_x(\Sigma_t)}. \tag{5.7}$$

Proof of Lemma 5.4. First recall that $\mathbf{T}_a = 0$, so $g_{ab} = \mathbf{g}_{ab}$. Note that since $0 < c(h,s) \le 1$ and $(v^0)^2 \ge 1$, by direct computation and the bootstrap assumption (3.17a), we have

$$\mathrm{d}\varpi_g = \sqrt{\det g}\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3$$

$$
\begin{aligned}
&= \{c^2 - (c^2-1)(v^0)^2\}^3\{c^{-6} + c^{-4}(c^{-2}-1)[(v^0)^2 - 1]\} \\
&\approx 1.
\end{aligned} \tag{5.8}
$$

Then we compute $Q^{00}[\varphi]$. By (2.18b) and (2.20a), we have

$$
\begin{aligned}
Q^{00}[\varphi] &= \frac{1}{2}\{(\mathbf{T}\varphi)^2 + (g^{-1})^{ab}\partial_a\varphi\partial_b\varphi\} \\
&= \frac{1}{2}\left\{(\mathbf{T}\varphi)^2 + \frac{c^2\delta^{ab}\left[c^2-(c^2-1)(v^0)^2\right] + c^2(c^2-1)v^a v^b}{\{c^2-(c^2-1)(v^0)^2\}^2}\partial_a\varphi\partial_b\varphi\right\},
\end{aligned} \tag{5.9}
$$

where δ^{ab} is the Kronecker delta. Note that

$$
\mathbf{T} = \partial_t + \frac{(c^2-1)v^a v^0}{c^2-(c^2-1)(v^0)^2}\partial_a. \tag{5.10}
$$

Then since the speed of sound satisfies $0 < c \le 1$, it follows that (5.9) is coercive in $|\partial\varphi|$, since $(v^0)^2 = 1 + \sum_{i=1,2,3}(v^i)^2$

$$
\begin{aligned}
&\{c^2\delta^{ab}[c^2-(c^2-1)(v^0)^2] + c^2(c^2-1)v^a v^b\}\partial_a\varphi\partial_b\varphi \\
&\quad = c^4|\partial\varphi|^2 - c^2(c^2-1)\{\delta^{ab}(v^0)^2 - v^a v^b\}\partial_a\varphi\partial_b\varphi \\
&\quad \ge c^4|\partial\varphi|^2.
\end{aligned} \tag{5.11}
$$

By bootstrap assumptions that $|v^\alpha|$ are uniformly bounded and Young's inequality, we derive that $Q^{00}[\varphi] \lesssim |\boldsymbol{\partial}\varphi|^2$. Combined with (5.11), the desired estimates (5.7) follows. $\square$

Lemma 5.5 (Basic energy inequality for the wave equations). *Let φ be smooth on $[0,T_*]\times\mathbb{R}^3$. Under the bootstrap assumptions of Sec.* 3.6, *the following inequality holds for $t\in[0,T_*]$:*

$$
\begin{aligned}
\mathbb{E}[\varphi](t) \lesssim \mathbb{E}[\varphi](0) &+ \int_0^t \|\boldsymbol{\partial}\vec{\Psi}\|_{L^\infty_x(\Sigma_\tau)}\mathbb{E}[\varphi](\tau)\mathrm{d}\tau \\
&+ \int_0^t \|\square_{\mathbf{g}}\varphi\|_{L^2_x(\Sigma_\tau)}\|\boldsymbol{\partial}\varphi\|_{L^2_x(\Sigma_\tau)}\mathrm{d}\tau.
\end{aligned} \tag{5.12}
$$

Proof of Lemma 5.5. We apply the divergence theorem on the space-time region $[0,t]\times\mathbb{R}^3$ relative to the volume form $\mathrm{d}\varpi_{\mathbf{g}} = \sqrt{\det g}\,\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3\mathrm{d}\tau = \mathrm{d}\varpi_g\mathrm{d}\tau$. Note that $\mathbf{T}$ is the future-directed $\mathbf{g}$-unit normal to Σ_t. By (5.4)–(5.7), with $\mathbf{X} := \mathbf{T}$, we have

$$
\begin{aligned}
\mathbb{E}[\varphi](t) = \mathbb{E}[\varphi](0) - \int_0^t\int_{\Sigma_\tau}\Big(&\square_{\mathbf{g}}\varphi(\mathbf{T}\varphi) + \frac{1}{2}Q^{\mu\nu}[\varphi]{}^{(\mathbf{T})}\boldsymbol{\pi}_{\mu\nu} \\
&-2\varphi\mathbf{T}\varphi - \frac{1}{2}\varphi^2(\mathbf{g}^{-1})^{\mu\nu}{}^{(\mathbf{T})}\boldsymbol{\pi}_{\mu\nu}\Big)\,\mathrm{d}\varpi_g\mathrm{d}\tau.
\end{aligned} \tag{5.13}
$$

By bootstrap assumptions, we have $|\mathbf{T}\varphi| \lesssim |\boldsymbol{\partial}\varphi|$, $|Q^{\mu\nu}[\varphi]| \lesssim |\boldsymbol{\partial}\varphi|^2$ and $|^{(\mathbf{T})}\boldsymbol{\pi}_{\mu\nu}| \lesssim |\boldsymbol{\partial}\vec{\Psi}|$. Thus by Cauchy–Schwarz inequality along Σ_τ, we get the desired estimate. □

Lemma 5.6 (Basic energy inequality for the transport equations). *Let φ be smooth on $[0, T_*] \times \mathbb{R}^3$. Under the bootstrap assumptions of Sec.* 3.6, *the following inequality holds for $t \in [0, T_*]$:*

$$\begin{aligned}\|\varphi\|^2_{L^2_x(\Sigma_t)} \lesssim \|\varphi\|^2_{L^2_x(\Sigma_0)} &+ \int_0^t \|\boldsymbol{\partial}\vec{\Psi}\|_{L^\infty_x(\Sigma_\tau)}\|\varphi\|^2_{L^2_x(\Sigma_\tau)}\mathrm{d}\tau \\ &+ \int_0^t \|\varphi\|_{L^2_x(\Sigma_\tau)}\|\mathbf{B}\varphi\|_{L^2_x(\Sigma_\tau)}\mathrm{d}\tau. \end{aligned} \tag{5.14}$$

Proof of Lemma 5.6. Let $\mathbf{J}^\alpha := \varphi^2\mathbf{B}^\alpha$, then $\partial_\alpha\mathbf{J}^\alpha = 2\varphi\mathbf{B}\varphi + (\partial_\alpha\mathbf{B}^\alpha)\varphi^2$. We apply the divergence theorem on the space-time region $[0, t] \times \mathbb{R}^3$ relative to the Cartesian coordinates. Note that $\mathbf{J}^0 = \varphi^2$. By Cauchy–Schwarz inequality along Σ_τ, we obtain the desired estimates. □

Remark 5.7. We remark that for our implementation of the geometric energy method for wave equations, the timelike vectorfield $\mathbf{T}$ (defined in Definition 2.11) plays the same role as $\mathbf{B}$ (Note that $\mathbf{B} = \partial_t + v^a\partial_a$ in [11] is not the same as $\mathbf{B} = \frac{v^\alpha}{v^0}\partial_\alpha$ in this paper.) in [11, Sec. 4.1]. All the arguments for geometric energy method for wave equations go through in the same fashion as in [11, Sec. 4.1].

5.1.2. *Elliptic div-curl estimates in L^2 space*

This section is dedicated to the proof of Proposition 5.8, which is a key ingredient in the proof of the energy estimates (5.1) for the $\vec{\omega}, \vec{S}, \vec{\mathcal{C}}, \mathcal{D}$.

Proposition 5.8 (Elliptic div-curl estimates in L^2 space). *Under the bootstrap assumptions in Sec.* 3.6, *the following estimates holds for ω and S:*

$$\|(\partial\vec{\omega}, \partial\vec{S})\|_{L^2_x(\Sigma_t)} \lesssim \|\vec{\omega}, \vec{S}, \vec{\mathcal{C}}, \mathcal{D}\|_{L^2_x(\Sigma_t)}. \tag{5.15}$$

Moreover, H^{N-k} elliptic estimate also holds true for $k = 1, 2$, that is

$$\|(\partial\vec{\omega}, \partial\vec{S})\|_{H^{N-k}(\Sigma_t)} \lesssim \|\vec{\omega}, \vec{S}, \vec{\mathcal{C}}, \mathcal{D}\|_{H^{N-k}(\Sigma_t)} \quad \textit{for } k = 1, 2. \tag{5.16}$$

Since we have to derive energy estimates on constant-time hypersurfaces, and the Hodge system (2.26) is a *space-time* div-curl system, we begin by deriving a **spatial** div-curl system for ω and S.

Proposition 5.9 (The div-curl system on constant-time hypersurfaces). *Given the div-curl system* (2.26), *the following equations hold on Σ_t for vorticity ω*

and entropy gradient S:

$$(G^{-1})^{ab}\partial_a(\omega_\flat)_b = F_\omega, \quad (G^{-1})^{ab}\partial_a S_b = F_S, \tag{5.17a}$$

$$\partial_a(\omega_\flat)_b - \partial_b(\omega_\flat)_a = {}^{(\omega)}H_{ab}, \quad \partial_a S_b - \partial_b S_a = {}^{(S)}H_{ab}, \tag{5.17b}$$

where

$$F_\omega = \mathscr{L}(\vec{\Psi}, \vec{\omega}, \vec{S})[\boldsymbol{\partial}\vec{\Psi}], \quad F_S = \mathscr{L}(\vec{\Psi})\mathcal{D} + \mathscr{L}(\vec{\Psi}, \vec{S})[\boldsymbol{\partial}\vec{\Psi}], \tag{5.18a}$$

$${}^{(\omega)}H_{ab} = \mathscr{L}(\vec{\Psi})\mathcal{C} + \mathscr{L}(\vec{\Psi}, \vec{\omega}, \vec{S})[\boldsymbol{\partial}\vec{\Psi}], \quad {}^{(S)}H_{ab} = \mathscr{L}(\vec{\Psi}, \vec{S})[\boldsymbol{\partial}\vec{\Psi}], \tag{5.18b}$$

$$(G^{-1})^{ab} = \delta^{ab} - \frac{v^a v^b}{(v^0)^2}. \tag{5.18c}$$

For convenience, we write the above two div-curl systems as follows where (η, F, H_{ab}) *which*[k] *is either* $(\omega_\flat, F_\omega, {}^{(\omega)}Hab)$ *or* $(S, F_S, {}^{(S)}Hab)$*:*

$$(G^{-1})^{ab}\partial_a\eta_b = F, \tag{5.19a}$$

$$\partial_a\eta_b - \partial_b\eta_a = H_{ab}. \tag{5.19b}$$

Proof of Proposition 5.9. For the div part (5.19a), by Eqs. (2.12c) and (2.27a), we write

$$\eta_0 = -\frac{\eta_b v^b}{v^0}. \tag{5.20}$$

Also by using transport equations (2.25a) and (2.25b), we have

$$\partial_0(\eta^\sharp)^b = -\frac{v^a\partial_a(\eta^\sharp)^b}{v^0} + \mathscr{L}(\vec{\Psi}, \vec{\omega}, S)[\boldsymbol{\partial}\vec{\Psi}]. \tag{5.21}$$

Using (2.26d) for $\eta = \omega_\flat$, we write $\partial_0\omega^0 + \partial_a\omega^a = \mathscr{L}(\vec{\omega})[\boldsymbol{\partial}\vec{\Psi}]$. By lowering the index $\partial_0\omega^0 = -\partial_0(\omega_\flat)_0$, Eqs. (5.20) and (5.21), we prove (5.17a) for ω. Similarly, for $\eta = S$, by definition of $\mathcal{D}$ (2.14), we write $\partial_0(S^\sharp)^0 + \partial_a(S^\sharp)^a = \mathscr{L}(\vec{\Psi})\mathcal{D} + \mathscr{L}(\vec{\Psi}, \vec{S})[\boldsymbol{\partial}\vec{\Psi}]$. Using Eqs. (5.20) and (5.21), we obtain Eq. (5.17a) for S.

Now we consider the curl part, note that we have the facts (2.27d) and (2.27e)

$$\begin{aligned}\partial_\gamma\eta_\delta - \partial_\delta\eta_\gamma &= \epsilon_{\gamma\delta\kappa\lambda}v^\kappa \mathrm{vort}^\lambda(\eta) - (v^\kappa\partial_\kappa\eta_\delta)(v_\flat)_\gamma + v^\kappa(\partial_\delta\eta_\kappa)(v_\flat)_\gamma \\ &\quad + (v^\kappa\partial_\kappa\eta_\gamma)(v_\flat)_\delta - v^\kappa(\partial_\delta\eta_\kappa)(v_\flat)_\delta.\end{aligned} \tag{5.22}$$

Recall that

$$\mathcal{C}^\alpha = \mathrm{vort}^\alpha(\omega_\flat) + \mathscr{L}(\vec{\Psi}, \vec{S})[\boldsymbol{\partial}\vec{\Psi}]. \tag{5.23}$$

Hence for $\eta = \omega$, the first term on the right-hand side of (5.22) for ω is manifestly in ${}^{(\omega)}Hab$. Next, using $v^\kappa\partial_\kappa(\omega_\flat)_\delta = v^0\mathbf{B}(\omega_\flat)_\delta$ and (2.25a), as well as (2.27b), we have that the right-hand side of (5.22) for ω is ${}^{(\omega)}Hab$. Similarly, by (2.26c), $v^\kappa\partial_\kappa S_\gamma = v^0\mathbf{B}S_\gamma$, (2.25b) and (2.27c), we obtain Eq. (5.17b) for S. □

[k] We use the same notation throughout the remainder of the paper.

Remark 5.10. In terms of elliptic estimates, there is a major difference in Proposition 5.8 compared to the Hodge system of the non-relativistic 3D compressible Euler equations. In the non-relativistic case, the analogous elliptic equations are constant-coefficient div-curl equations along flat hypersurfaces of constant Cartesian time and, for example, the basic L^2 theory can be derived with the simple Hodge identity for Σ_t vectorfields $V \in H^1(\mathbb{R}^3;\mathbb{R}^3)$

$$\sum_{a,b=1}^{3} \|\partial_a V^b\|^2_{L^2_x(\mathbb{R}^3)} = \|\mathrm{div}V\|^2_{L^2_x(\mathbb{R}^3)} + \|\mathrm{curl}V\|^2_{L^2_x(\mathbb{R}^3)}. \tag{5.24}$$

In contrast, the divergence equation (5.19a) has dynamic, solution-dependent coefficients.

Proposition 5.11 (The top-order div-curl system on constant-time hypersurfaces). *Using the same notation as in Proposition* 5.19, *we have*

$$(G^{-1})^{ab}\partial_a(\partial\eta)_b = F_{\partial\eta}, \tag{5.25a}$$

$$\partial_a(\partial\eta)_b - \partial_b(\partial\eta)_a = {}^{(\partial\eta)}H_{ab}, \tag{5.25b}$$

where

$$F_{\partial\eta} = \partial F_\eta - \partial\{(G^{-1})^{ab}\}\partial_a\eta, \tag{5.26a}$$

$${}^{(\partial\eta)}H_{ab} = \partial^{(\eta)}H_{ab}. \tag{5.26b}$$

Moreover, we have

$$(G^{-1})^{ab}\partial_a\partial P_\nu\eta_b = F_{\partial P_\nu\eta}, \tag{5.27a}$$

$$\partial_a\partial P_\nu\eta_b - \partial_b\partial P_\nu\eta_a = {}^{(\partial P_\nu\eta)}H_{ab}, \tag{5.27b}$$

where

$$F_{\partial P_\nu\eta} = P_\nu F_{\partial\eta} + [P_\nu,(G^{-1})^{ab}]\partial_a\partial\eta_b, \tag{5.28a}$$

$${}^{(\partial P_\nu\eta)}H_{ab} = P_\nu({}^{(\partial\eta)}H_{ab}). \tag{5.28b}$$

For convenience, we write the above three div-curl systems (5.19), (5.25) and (5.27) as follows, where (X, F, H_{ab}) is $(\eta, F_\eta, {}^{(\eta)}H_{ab})$ or $(\partial\eta, F_{\partial\eta}, {}^{(\partial\eta)}H_{ab})$ or $(\partial P_\nu\eta, F_{\partial P_\nu\eta}, {}^{(\partial P_\nu\eta)}H_{ab})$:

$$(G^{-1})^{ab}\partial_a X_b = F, \tag{5.29a}$$

$$\partial_a X_b - \partial_b X_a = H_{ab}. \tag{5.29b}$$

In order to derive elliptic estimates and Schauder estimates from the div-curl system (5.19), we need Lemma 5.12 provided below, which allows us to do estimates via Littlewood–Paley theory. We provide a partition of unity before Lemma 5.12.

We want to apply the Fourier transform to a localized version of the div-curl system in Proposition 5.8. We consider the lattice $\mathcal{A} := \delta_2\mathbb{Z}^3$, where δ_2 is assumed to be small and will be determined in future analysis. Note that $\{x_l\}_{l\in\mathbb{N}} := \mathcal{A} \subset \Sigma_t$ has points equally spread out, that is, for each x_l, there are 6 points in $\mathcal{A}$ such that the distance between x_l and any of them is δ_2. We define the family of functions $\{\psi_l\}_{l\in\mathbb{N}}$ as follows:

$$\psi_l(x) = \begin{cases} 1 & x \in B\left(x_l, \dfrac{1}{8}\delta_2\right), \\ \exp\left(\dfrac{4}{3\delta_2^2}\right)\exp\left(\dfrac{1}{|x-x_l|^2 - (\frac{7}{8}\delta_2)^2}\right) & x \in B\left(x_l, \dfrac{7}{8}\delta_2\right) - B\left(x_l, \dfrac{1}{8}\delta_2\right), \\ 0 & x \notin B\left(x_l, \dfrac{7}{8}\delta_2\right), \end{cases} \tag{5.30}$$

and set

$$\phi_l(x) := \frac{\psi_l(x)}{\sum_k \psi_k(x)}. \tag{5.31}$$

We note that $\|\phi_l\|_{L^\infty(\Sigma_t)}, \|\partial\phi_l\|_{L^\infty(\Sigma_t)} \lesssim 1$.

We have constructed cut-off functions $\{\phi_l\}_{l\in\mathbb{N}} \subset C_0^\infty(\Sigma_t)$ such that $\phi_l = 1$ in $B(x_l, \frac{1}{8}\delta_2)$, $\mathrm{supp}(\phi_l) \subset B(x_l, \frac{7}{8}\delta_2)$, $\sum_l \phi_l(x) = 1$ and for any $x_a, x_b \in \mathcal{A}$, $\phi_a(x) = \phi_b(x - x_a + x_b)$. We want to apply Fourier transform on a localized region, where $(G^{-1})^{ab}(x_l)$ is a constant and $(G^{-1})^{ab}(x) - (G^{-1})^{ab}(x_l)$ will be shown to be a controllable error term in the future analysis.

Lemma 5.12. *Given Propositions* 5.9 *and* 5.11 *with x_l and ϕ_l, $l \in \mathbb{N}$, defined as above, let X be the solution of Eqs.* (5.29)*. Then the following identity holds in frequency space for $i = 1, 2, 3$:*

$$(G^{-1})^{ab}(x_l)\xi_a\xi_b\widehat{(\phi_l X_i)} = C\xi_i \hat{\underline{F}}^l + \sum_{k\neq i} C(G^{-1})^{ak}(x_l)\xi_a \hat{\underline{H}}^l_{ki}, \tag{5.32}$$

where

$$\begin{aligned} \underline{F}^l &= (G^{-1})^{ab}(x_l)\partial_a(\phi_l X_b) \qquad &(5.33a) \\ &= \phi_l F + (G^{-1})^{ab}(x_l)(\partial_a\phi_l)X_b - [(G^{-1})^{ab}(x) - (G^{-1})^{ab}(x_l)]\partial_a(\phi_l X_b) \\ &\quad - [(G^{-1})^{ab}(x) - (G^{-1})^{ab}(x_l)](\partial_a\phi_l)X_b, \\ \underline{H}^l_{ab} &= \partial_a(\phi_l X_b) - \partial_b(\phi_l X_a) = (\partial_a\phi_l)X_b - (\partial_b\phi_l)X_a + \phi_l H_{ab}. \qquad &(5.33b) \end{aligned}$$

Proof of Lemma 5.12. By multiplying the div-curl system (5.19a) and (5.19b) by ϕ_l, we rewrite the system as follows:

$$\{(G^{-1})^{ab}(x_l) + (G^{-1})^{ab}(x) - (G^{-1})^{ab}(x_l)\}\{\partial_a(\phi_l X_b) - (\partial_a\phi_l)X_b\} = \phi_l F, \tag{5.34a}$$

$$\partial_a(\phi_l X_b) - (\partial_a\phi_l)X_b - \partial_b(\phi_l X_a) + (\partial_b\phi_l)X_a = \phi_l H_{ab}. \tag{5.34b}$$

Taking the Fourier transform of (5.34a) and multiplying by ξ_1, we have

$$\xi_1\{(G^{-1})^{a1}(x_l)\xi_a\widehat{(\phi_l X_1)} + (G^{-1})^{a2}(x_l)\xi_a\widehat{(\phi_l X_2)} + (G^{-1})^{a3}(x_l)\xi_a\widehat{(\phi_l X_3)}\}$$
$$= C\xi_1\hat{\underline{F}}^l, \tag{5.35}$$

where $C = \frac{1}{2\pi i}$ is a constant from Fourier transform, and

$$\begin{aligned}\underline{F}^l &= (G^{-1})^{ab}(x_l)\partial_a(\phi_l X_b)\\ &= \phi_l F + (G^{-1})^{ab}(x_l)(\partial_a\phi_l)X_b - [(G^{-1})^{ab}(x) - (G^{-1})^{ab}(x_l)]\partial_a(\phi_l X_b)\\ &\quad - [(G^{-1})^{ab}(x) - (G^{-1})^{ab}(x_l)](\partial_a\phi_l)X_b.\end{aligned} \tag{5.36}$$

Similarly, taking the Fourier transform of (5.34b) and multiplying by $(G^{-1})^{a2}(x_l)\xi_a$ and $(G^{-1})^{a3}(x_l)\xi_a$, we have

$$(G^{-1})^{a2}(x_l)\xi_a\{\xi_2\widehat{(\phi_l X_1)} - \xi_1\widehat{(\phi_l X_2)}\} = C(G^{-1})^{a2}(x_l)\xi_a\hat{\underline{H}}^l_{21}, \tag{5.37a}$$

$$(G^{-1})^{a3}(x_l)\xi_a\{\xi_3\widehat{(\phi_l X_1)} - \xi_1\widehat{(\phi_l X_3)}\} = C(G^{-1})^{a3}(x_l)\xi_a\hat{\underline{H}}^l_{31}, \tag{5.37b}$$

where $C = \frac{1}{2\pi i}$ is a constant from Fourier transform, and

$$\underline{H}^l_{ab} = \partial_a(\phi_l X_b) - \partial_b(\phi_l X_a) = (\partial_a\phi_l)X_b - (\partial_b\phi_l)X_a + \phi_l H_{ab}. \tag{5.38}$$

Adding (5.35), (5.37a) and (5.37b), we obtain

$$(G^{-1})^{ab}(x_l)\xi_a\xi_b\widehat{(\phi_l X_1)} = C\xi_1\hat{\underline{F}}^l + C(G^{-1})^{a2}(x_l)\xi_a\hat{\underline{H}}^l_{21} + C(G^{-1})^{a3}(x_l)\xi_a\hat{\underline{H}}^l_{31}. \tag{5.39}$$

We use the same argument for X_2 and X_3. Hence for $i = 1, 2, 3$, we obtain

$$(G^{-1})^{ab}(x_l)\xi_a\xi_b\widehat{(\phi_l X_i)} = C\xi_i\hat{\underline{F}}^l + \sum_{k\neq i} C(G^{-1})^{ak}(x_l)\xi_a\hat{\underline{H}}^l_{ki}. \tag{5.40}$$

□

Lemma 5.13 (Positive definiteness of G^{-1}). *For any Σ_t-tangent one-form ξ, that is, $\xi_0 = 0$, we denote $|\xi|^2 := \sum_{i=1,2,3}\xi_i^2$. Then the following estimate holds for any $x_l \in \Sigma_t$, where G is defined in Proposition* 5.9:

$$C|\xi|^2 \le (G^{-1})^{ab}(x_l)\xi_a\xi_b \le |\xi|^2, \tag{5.41}$$

where $0 < C < 1$ is a constant depends only on $\|v\|_{L^\infty_x(\Sigma_t)}$, which is in turn controlled by the bootstrap assumption (3.17a).

Proof of Lemma 5.13. Using the definition of $(G^{-1})^{ab}$, we have

$$(G^{-1})^{ab}(x_l)\xi_a\xi_b = \left(\delta^{ab} - \frac{v^a v^b}{(v^0)^2}\right)\xi_a\xi_b = |\xi|^2 - \left(\frac{v^i\xi_i}{v^0}\right)^2. \tag{5.42}$$

Hence by the normalization $(v_\flat)_\alpha v^\alpha = -1$ in Sec. 2.2.1, we have

$$C|\xi|^2 \le (G^{-1})^{ab}(x_l)\xi_a\xi_b \le |\xi|^2, \tag{5.43}$$

where $0 < C < 1$ is a constant depends only on $\|v\|_{L^\infty_x(\Sigma_t)}$. □

Lemma 5.14. *For G defined in Proposition* 5.9, *the following inequality holds*:

$$|(G^{-1})^{ab}(x) - (G^{-1})^{ab}(x_l)| \leq C_3 \|v\|^4_{C_0^{0,\delta_0}(\Sigma_t)} |x - x_l|^{\delta_0} \leq C_2^4 C_3 |x - x_l|^{\delta_0}, \tag{5.44}$$

where C_2, C_3 are constants (independent of l, x, a, b, δ_0).

Proof of Lemma 5.14. For Hölder continuous function $f, g \in C_0^{0,\delta_0}(\Sigma_t)$

$$\begin{aligned}\frac{|f(x)g(x) - f(y)g(y)|}{|x-y|^{\delta_0}} &= \frac{|f(x)g(x) - f(x)g(y)| + |f(x)g(y) - f(y)g(y)|}{|x-y|^{\delta_0}} \\ &\leq \|f\|_{L^\infty_x(\Sigma_t)} \|g\|_{\dot{C}_0^{0,\delta_0}(\Sigma_t)} + \|g\|_{L^\infty_x(\Sigma_t)} \|f\|_{\dot{C}_0^{0,\delta_0}(\Sigma_t)} \\ &\leq 2\|f\|_{C_0^{0,\delta_0}(\Sigma_t)} \|g\|_{C_0^{0,\delta_0}(\Sigma_t)}.\end{aligned} \tag{5.45}$$

Therefore, by definition of G^{-1} in (5.18c) and the fact that $v^0 \geq 1$, substitute (f, g) in (5.45) by (v^a, v^b) and again by $(v^a v^b, \frac{1}{(v^0)^2})$, we have the estimate

$$|(G^{-1})^{ab}(x) - (G^{-1})^{ab}(x_l)| \leq C_3 \|v\|^4_{C_0^{0,\delta_0}(\Sigma_t)} |x - x_l|^{\delta_0}. \tag{5.46}$$

By fundamental theorem of calculus, bootstrap assumptions (3.17) and (3.7), we have

$$\|v\|_{C_x^{0,\delta_0}(\Sigma_t)} \leq \|\boldsymbol{\partial}\vec{\Psi}\|_{L^1_t C_x^{0,\delta_0}(\Sigma_t)} + \|v\|_{C_x^{0,\delta_0}(\Sigma_0)} \lesssim \|\boldsymbol{\partial}\vec{\Psi}\|_{L^1_t C_x^{0,\delta_0}(\Sigma_t)} + 1 \leq C_2. \tag{5.47}$$

Combining (5.46) and (5.47), we have the desired result. □

Lemma 5.15 (Commutator estimates). *For scalar function F, G, δ_1 defined as in Sec.* 3.4, *Littlewood–Paley projection operator P_ν defined in Sec.* 3.2, *we have the following estimates*:

$$\|[P_\nu, G]F\|_{L^2(\Sigma_t)} \lesssim \nu^{-\delta_1} \|G\|_{C_x^{0,\delta_1}(\Sigma_t)} \|F\|_{L^2(\Sigma_t)}. \tag{5.48}$$

Proof of Lemma 5.15. For ψ defined as in Sec. 3.2, we define $M(x)$ as follows:

$$M_\nu(x) := \mathcal{F}^{-1}(\psi(\nu^{-1}\xi)) = \int e^{ix\xi} \psi(\nu^{-1}\xi) \mathrm{d}\xi. \tag{5.49}$$

Then we have

$$\begin{aligned}P_\nu, G]F &= \int M_\nu(x-y)\,(G(y) - G(x))\,F(y)\mathrm{d}y \\ &\leq \|G\|_{C_x^{0,\delta_1}(\Sigma_t)} \int M_\nu(x-y)|x-y|^{\delta_1} F(y)\mathrm{d}y.\end{aligned} \tag{5.50}$$

By Young's inequality, we have

$$\|[P_\nu, G]F\|_{L^2(\Sigma_t)} \lesssim \|G\|_{C_x^{0,\delta_1}(\Sigma_t)} \|F\|_{L^2(\Sigma_t)} \|M_\nu(x)|x|^{\delta_1}\|_{L^1(\Sigma_t)}. \tag{5.51}$$

By definition of Fourier transform and ψ in Sec. 3.2, we have

$$\int M_\nu(x)|x|^{\delta_1} \mathrm{d}x = \int \nu^3 \hat{\psi}(-\nu x)|x|^{\delta_1} \mathrm{d}x \lesssim \nu^{-\delta_1}. \tag{5.52}$$

Combining the above equations, we obtain the desired result. □

Lemma 5.16 (Control of the inhomogeneous terms). *For* $F_\eta, {}^{(\eta)}H, G^{-1}$ *defined as in Proposition* 5.9, *and* $F_{\partial\eta}, F_{\partial P_\nu \eta}, {}^{(\partial\eta)}H, {}^{(\partial P_\nu \eta)}H$ *defined as in Proposition* 5.11, *we have the following estimates*:

$$\|F_\eta\|_{L^2(\Sigma_t)}, \|{}^{(\eta)}H\|_{L^2(\Sigma_t)} \leq C\|\boldsymbol{\partial}\vec{\Psi}, \vec{\mathcal{C}}, \mathcal{D}\|_{L^2(\Sigma_t)}, \tag{5.53}$$

$$\begin{aligned}&\|F_{\partial\eta}\|_{L^2(\Sigma_t)}, \|{}^{(\partial\eta)}H\|_{L^2(\Sigma_t)}\\ &\quad \leq C\|\boldsymbol{\partial}^2\vec{\Psi}, \partial\vec{\mathcal{C}}, \partial\mathcal{D}\|_{L^2(\Sigma_t)} + C\alpha^{-1}\|\partial\vec{\Psi}\|^2_{H^1(\Sigma_t)}\|\partial\eta\|_{L^2(\Sigma_t)} + C\alpha\|\partial^2\eta\|_{L^2(\Sigma_t)},\end{aligned} \tag{5.54}$$

$$\begin{aligned}&\|\nu^{N-2}F_{\partial P_\nu \eta}\|_{l^2_\nu L^2(\Sigma_t)}, \|\nu^{N-2} \cdot {}^{(\partial P_\nu \eta)}H\|_{l^2_\nu L^2(\Sigma_t)}\\ &\quad \lesssim (\|\partial^2\vec{\Psi}\|_{L^2(\Sigma_t)} + 1)\|\partial^2\eta\|_{L^2(\Sigma_t)} + \|\boldsymbol{\partial}^2\vec{\Psi}, \partial\vec{\mathcal{C}}, \partial\mathcal{D}\|_{H^{N-2}(\Sigma_t)},\end{aligned} \tag{5.55}$$

where C, α *are constants*, C *is independent of* α, *and* $\alpha > 0$ (small), *which will be determined later.*

Proof of Lemma 5.16. (5.53) is the direct result of taking $L^2(\Sigma_t)$ for (5.18).

Taking $L^2(\Sigma_t)$ for (5.26), we have

$$\|\partial F_\eta, \|_{L^2(\Sigma_t)}, \|\partial^{(\eta)}H\|_{L^2(\Sigma_t)} \leq C\|\boldsymbol{\partial}^2\vec{\Psi}, \partial\vec{\mathcal{C}}, \partial\mathcal{D}\|_{L^2(\Sigma_t)}, \tag{5.56}$$

$$\begin{aligned}\|\partial\{(G^{-1})^{ab}\}\partial_a\eta\|_{L^2(\Sigma_t)} &\leq C\|\partial\vec{\Psi}\|_{H^1(\Sigma_t)}\|\partial\eta\|^{\frac{1}{2}}_{L^2(\Sigma_t)}\|\partial^2\eta\|^{\frac{1}{2}}_{L^2(\Sigma_t)}\\ &\leq C\alpha^{-1}\|\partial\vec{\Psi}\|^2_{H^1(\Sigma_t)}\|\partial\eta\|_{L^2(\Sigma_t)} + C\alpha\|\partial^2\eta\|_{L^2(\Sigma_t)},\end{aligned} \tag{5.57}$$

where for the first inequality in (5.57), we used the fact that $\|F \cdot G\|_{L^2(\Sigma_t)} \leq C\|F\|^{\frac{1}{2}}_{L^2(\Sigma_t)}\|F\|^{\frac{1}{2}}_{H^1(\Sigma_t)}\|G\|_{H^1(\Sigma_t)}$ (see [11, (79b)]), for the second inequality in (5.57), we used Young's inequality.

Now we consider the proof of (5.55). Taking $l^2_\nu L^2(\Sigma_t)$ norm of (5.28), we have

$$\begin{aligned}&\|\nu^{N-2}P_\nu F_{\partial\eta}, \|_{l^2_\nu L^2(\Sigma_t)}, \|\nu^{N-2}P_\nu({}^{(\partial\eta)}H)\|_{l^2_\nu L^2(\Sigma_t)}\\ &\quad \leq C\|\boldsymbol{\partial}^2\vec{\Psi}, \partial\vec{\mathcal{C}}, \partial\mathcal{D}\|_{H^{N-2}(\Sigma_t)} + \|\partial^2\vec{\Psi}\|_{L^2(\Sigma_t)}\|\partial^2\eta\|_{L^2(\Sigma_t)},\end{aligned} \tag{5.58}$$

where the second term on the right-hand side of (5.58) is from the fact that (see [11, (81b)])

$$\begin{aligned}\|\nu^{N-2}P_\nu(\partial\{(G^{-1})^{ab}\}\partial_a\eta)\|_{l^2_\nu L^2(\Sigma_t)} &\lesssim \|\partial\vec{\Psi}\|_{H^{N-\frac{3}{2}}(\Sigma_t)}\|\partial\eta\|_{H^1(\Sigma_t)}\\ &\quad + \|\partial\eta\|_{H^{N-\frac{3}{2}}(\Sigma_t)}\|\partial\vec{\Psi}\|_{H^1(\Sigma_t)}.\end{aligned} \tag{5.59}$$

What remains to be controlled is the commutator term $[P_\nu, (G^{-1})^{ab}]\partial_a\partial\eta_b$. By Lemma 5.15, where $G := (G^{-1})^{ab}$ and $F := \partial_a\partial\eta_b$, and (5.47), we have

$$\|\nu^{N-2}[P_\nu, (G^{-1})^{ab}]\partial_a\partial\eta_b\|_{l^2_\nu L^2(\Sigma_t)} \lesssim \|\partial^2\eta\|_{L^2(\Sigma_t)}. \tag{5.60}$$

Combining the above three estimates, we obtain (5.55). ◻

Lemma 5.17. *Let G^{-1} be defined as in* (5.18c). *F, H defined as in* (5.29). *For X the solution of the div-curl system* (5.29), *we have the following estimate*:

$$\|\partial X\|_{L^2_x(\Sigma_t)} \leq C\|X, F, H\|_{L^2_x(\Sigma_t)}, \tag{5.61}$$

where C is a constant, which is independent of X, F, H.

Proof of Lemma 5.17. Throughout, X, F, G, H are the same as in (5.29), and $\underline{F}, \underline{H}$ are defined in Lemma 5.12. Since $|\xi| \simeq 2\nu$ on support of $\widehat{P_\nu(\phi_l \eta_i)}$, by Littlewood–Paley estimate (3.6), (5.41) and (5.32), we remind ψ is defined in Sec. 3.2

$$\begin{aligned}
&\|\partial\{\phi_l X_i\}\|^2_{L^2_x(\Sigma_t)} \\
&\quad \leq C_1 \sum_{\nu>1} \|\nu P_\nu(\phi_l X_i)\|^2_{L^2_x(\Sigma_t)} \\
&\quad \leq C_1 \sum_{\nu>1} \left\| \mathcal{F}^{-1}\left(\psi\left(\frac{|\xi|}{\nu}\right)|\nu|^{-1}\left\{\xi_i \hat{\underline{F}}^l + \sum_{k\neq i}(G^{-1})^{jk}(x_l)\xi_j \hat{\underline{H}}^l_{ki}\right\}\right)\right\|^2_{L^2_x(\Sigma_t)} \\
&\quad \leq C_1(\|\underline{F}^l\|^2_{L^2_x(\Sigma_t)} + \|\underline{H}^l\|^2_{L^2_x(\Sigma_t)}),
\end{aligned} \tag{5.62}$$

where C_1 is a constant and $\mathcal{F}^{-1}$ is the Fourier inverse transform. By (5.33a), we have

$$\begin{aligned}
\|\underline{F}^l\|^2_{L^2_x(\Sigma_t)} &\leq \|\phi_l F\|^2_{L^2_x(\Sigma_t)} + \|(G^{-1})\|^2_{L^\infty_x(\Sigma_t)}\|(\partial\phi_l)X\|^2_{L^2_x(\Sigma_t)} \\
&\quad + \sup_{x\in B(x_l,\delta_2)} |(G^{-1})^{ab}(x) - (G^{-1})^{ab}(x_l)|^2 \|\partial(\phi_l X)\|^2_{L^2_x(\Sigma_t)}.
\end{aligned} \tag{5.63}$$

By (5.33b)

$$\|\underline{H}^l\|^2_{L^2_x(\Sigma_t)} \leq \|\phi_l H\|^2_{L^2_x(\Sigma_t)} + 2\|(\partial\phi_l)X\|^2_{L^2_x(\Sigma_t)}. \tag{5.64}$$

Let $C' := C_1 C_2^4 C_3$, where C_1, C_2, C_3 are constants from (5.62), (5.47), (5.44), respectively, and let δ_2 be small such that for all $x \in B(x_l, \delta_2)$

$$C'|x - x_l|^{\delta_0} < \frac{1}{4}. \tag{5.65}$$

Hence by (5.62)–(5.65), soaking the last term in the right-hand side of (5.63) to the left of (5.62) (by Lemma 5.14), we have

$$\|\partial\{\phi_l X_i\}\|^2_{L^2_x(\Sigma_t)} \leq C_4(\|\phi_l F\|^2_{L^2_x(\Sigma_t)} + \|\phi_l H\|^2_{L^2_x(\Sigma_t)} + \|(\partial\phi_l)X\|^2_{L^2_x(\Sigma_t)}), \tag{5.66}$$

where C_4 is a constant.

Now we consider the $\|\partial X_i\|^2_{L^2_x(\Sigma_t)}$. By (5.66), we have

$$\|\partial X_i\|^2_{L^2_x(\Sigma_t)} = \left\|\sum_l \partial\{\phi_l X_i\}\right\|^2_{L^2_x(\Sigma_t)}$$

$$
\begin{aligned}
&\leq C_5 \sum_l \|\partial\{\phi_l X_i\}\|^2_{L^2_x(\Sigma_t)} \\
&\leq C_4 C_5 \left(\sum_l \|\phi_l(F,H)\|^2_{L^2_x(\Sigma_t)} + \|(\partial\phi_l)X\|^2_{L^2_x(\Sigma_t)} \right).
\end{aligned}
\tag{5.67}
$$

For the first term on the right-hand side of (5.67), by (5.53), we have

$$
\begin{aligned}
\sum_l \|\phi_l(F,H)\|^2_{L^2_x(\Sigma_t)} &= \sum_l \int_{\Sigma_t} (\phi_l)^2(F^2+H^2)\mathrm{d}x \\
&\leq \sum_l \|\phi_l\|^2_{L^\infty(\Sigma_t)} \int_{\Sigma_t \cap \{\partial\phi_l \neq 0\}} (F^2+H^2)\mathrm{d}x \\
&\leq C_6 \|F,H\|^2_{L^2_x(\Sigma_t)}.
\end{aligned}
\tag{5.68}
$$

For the second term on the right-hand side of (5.67), we have

$$
\begin{aligned}
\sum_l \|(\partial\phi_l)X\|^2_{L^2_x(\Sigma_t)} &= \sum_l \int_{\Sigma_t} (\partial\phi_l)^2 X^2 \mathrm{d}x \\
&\leq C_7 \sum_l \|\partial\phi_l\|^2_{L^\infty(\Sigma_t)} \int_{\Sigma_t \cap \{\partial\phi_l \neq 0\}} X^2 \mathrm{d}x \\
&\leq C_8 \|X\|^2_{L^2_x(\Sigma_t)},
\end{aligned}
\tag{5.69}
$$

where the last inequality holds for both (5.68) and (5.69) since for each $x \in \Sigma_t$, there are finite many (at most 8) ϕ_l such that $\phi_l(x) \neq 0$.

Combining (5.67)–(5.69) and letting $C := C_4C_5(C_6 + C_8)$, we conclude the desired estimate. □

Proof of Proposition 5.8. (5.15) is a direct result by substituting (X, F, H) in (5.61) by $(\eta, F_\eta, {}^{(\eta)}H)$ and using the schematic definition $\eta = \partial\vec{\Psi}$, estimate (5.53).

Using (3.6), (5.16) can be proved in a similar fashion by using the following interpolation estimate:

$$
\|\partial\eta_i\|^2_{H^{N-1}(\Sigma_t)} \leq \|\partial^2\eta_i\|^2_{L^2_x(\Sigma_t)} + C_1 \sum_\nu \|\nu^{N-2} P_\nu(\partial^2\eta_i)\|^2_{L^2_x(\Sigma_t)},
\tag{5.70}
$$

where we bound the first term on the right-hand side of (5.70) as follows:

$$
\|\partial^2\eta_i\|_{L^2_x(\Sigma_t)} \lesssim \|\partial\boldsymbol{\partial}\vec{\Psi}, \partial\mathcal{C}, \partial\mathcal{D}\|_{L^2_x(\Sigma_t)} + \|\boldsymbol{\partial}\vec{\Psi}\|^3_{H^1(\Sigma_t)}.
\tag{5.71}
$$

(5.71) is obtained by exactly the same method as in the proof of (5.15). That is, substituting $(X, F, H, \mathcal{C}, \mathcal{D})$ in (5.61) by $(\partial\eta, F_{\partial_\eta}, {}^{(\partial_\eta)}H, \partial\mathcal{C}, \partial\mathcal{D})$ and using (5.54), we have

$$
\begin{aligned}
\|\partial^2\eta_i\|_{L^2_x(\Sigma_t)} \leq{}& C\|\boldsymbol{\partial}^2\vec{\Psi}, \partial\vec{\mathcal{C}}, \partial\mathcal{D}\|_{L^2(\Sigma_t)} + C\alpha^{-1}\|\partial\vec{\Psi}\|^2_{H^1(\Sigma_t)}\|\partial\eta\|_{L^2(\Sigma_t)} \\
&+ C\alpha\|\partial^2\eta\|_{L^2(\Sigma_t)}.
\end{aligned}
\tag{5.72}
$$

We pick α small ($C\alpha \leq \frac{1}{2}$) such that $C\alpha\|\partial^2\eta\|_{L^2(\Sigma_t)}$ can be soaked into the left-hand side of (5.72).

Now we consider the second term on the right-hand side of (5.70).

Substituting (X, F, H) in (5.61) by $(\partial P_\nu \eta, F_{\partial P_\nu \eta}, {}^{(\partial P_\nu \eta)}H)$, we have

$$\|\partial^2 P_\nu \eta_i\|_{L^2_x(\Sigma_t)} \lesssim \|\partial P_\nu \eta, F_{\partial P_\nu \eta}, {}^{(\partial P_\nu \eta)}H\|_{L^2_x(\Sigma_t)}. \tag{5.73}$$

Multiplying (5.73) by ν^{N-2} and taking the l^2_ν norm, we have

$$\|\nu^{N-2} P_\nu(\partial^2 \eta_i)\|_{l^2_\nu L^2_x(\Sigma_t)} \lesssim \|\nu^{N-2}(\partial P_\nu \eta, F_{\partial P_\nu \eta}, {}^{(\partial P_\nu \eta)}H)\|_{l^2_\nu L^2_x(\Sigma_t)}. \tag{5.74}$$

By (5.74) and (5.55), we have

$$\begin{aligned}\|\nu^{N-2} P_\nu(\partial^2 \eta_i)\|_{l^2_\nu L^2_x(\Sigma_t)} &\lesssim (\|\partial^2 \vec{\Psi}\|_{L^2(\Sigma_t)} + 1)\|\partial^2 \eta\|_{L^2(\Sigma_t)} \\ &\quad + \|\boldsymbol{\partial}^2 \vec{\Psi}, \partial \vec{\mathcal{C}}, \partial \mathcal{D}\|_{H^{N-2}(\Sigma_t)}.\end{aligned} \tag{5.75}$$

By (5.70), (5.71) and (5.75), we have proved (5.16). □

5.1.3. *Proof of Proposition* 5.1

Proof. For N defined as in Sec. 3.4, we let

$$P_N(t) := \sum_{k=0}^{2} \|\boldsymbol{\partial}^k(\vec{\Psi}, \vec{\omega}, \vec{S})\|^2_{H^{N-k}(\Sigma_t)} + \sum_{k=0}^{1} \|\boldsymbol{\partial}^k(\vec{\mathcal{C}}, \mathcal{D})\|^2_{H^{N-k-1}(\Sigma_t)}. \tag{5.76}$$

In this proof, we derive integral inequalities for $\sum_{k=0}^{2}\|\boldsymbol{\partial}^k \vec{\Psi}\|^2_{H^{N-k}(\Sigma_t)}$ and $\sum_{k=0}^{1}\|\boldsymbol{\partial}^k(\vec{\mathcal{C}}, \mathcal{D})\|^2_{H^{N-k-1}(\Sigma_t)}$ in $P_N(t)$, namely, (5.78), (5.82), (5.84), (5.85). We then use elliptic estimates (5.15) and apply Grönwall's inequality to all the terms in $P_N(t)$ collectively.

The proof of Proposition 5.1 for $\vec{\Psi}$ combines the vectorfield multiplier method and Littlewood–Paley theory. That is, to derive the energy estimates at the top order, one integrate (5.5) and applies the divergence theorem using the energy current ${}^{(\mathbf{T})}\mathbf{J}^\alpha[\boldsymbol{\partial}\vec{\Psi}] := Q^{\alpha\beta}[\boldsymbol{\partial}\vec{\Psi}]\mathbf{T}_\beta - \mathbf{T}^\alpha(\boldsymbol{\partial}\vec{\Psi})^2$ and ${}^{(\mathbf{T})}\mathbf{J}^\alpha[P_\nu\boldsymbol{\partial}\vec{\Psi}] := Q^{\alpha\beta}[P_\nu\boldsymbol{\partial}\vec{\Psi}]\mathbf{T}_\beta - \mathbf{T}^\alpha(P_\nu\boldsymbol{\partial}\vec{\Psi})^2$ on the space-time region bounded by Σ_0 and Σ_t. Then by Lemma 5.5 with $\boldsymbol{\partial}\vec{\Psi}$ and $P_\nu\boldsymbol{\partial}\vec{\Psi}$ in a role of φ, we have, respectively

$$\begin{aligned}\mathbb{E}[\boldsymbol{\partial}\vec{\Psi}](t) &\lesssim \mathbb{E}[\boldsymbol{\partial}\vec{\Psi}](0) + \int_0^t \|\boldsymbol{\partial}\vec{\Psi}\|_{L^\infty_x(\Sigma_\tau)}\mathbb{E}[\boldsymbol{\partial}\vec{\Psi}](\tau)\mathrm{d}\tau \\ &\quad + \int_0^t \|\square_{\mathbf{g}}\boldsymbol{\partial}\vec{\Psi}\|_{L^2_x(\Sigma_\tau)}[\mathbb{E}[\boldsymbol{\partial}\vec{\Psi}](\tau)]^{1/2}\mathrm{d}\tau,\end{aligned} \tag{5.77a}$$

$$\mathbb{E}[P_\nu \boldsymbol{\partial}\vec{\Psi}](t) \lesssim \mathbb{E}[P_\nu \boldsymbol{\partial}\vec{\Psi}](0) + \int_0^t \|\boldsymbol{\partial}\vec{\Psi}\|_{L_x^\infty(\Sigma_\tau)} \mathbb{E}[P_\nu \boldsymbol{\partial}\vec{\Psi}](\tau)\mathrm{d}\tau$$
$$+ \int_0^t \|\square_{\mathbf{g}} P_\nu \boldsymbol{\partial}\vec{\Psi}\|_{L_x^2(\Sigma_\tau)} [\mathbb{E}[P_\nu \boldsymbol{\partial}\vec{\Psi}](\tau)]^{1/2}\mathrm{d}\tau. \tag{5.77b}$$

Then we use Eq. (3.8a) to substitute for $\square_{\mathbf{g}}\boldsymbol{\partial}\vec{\Psi}$, and we use Eq. (3.9a) $\square_{\mathbf{g}} P_\nu \boldsymbol{\partial}\vec{\Psi}$ to substitute for the right-hand side of (5.77b). Multiplying (5.77b) by $\nu^{2(N-2)}$, summing over ν and using (3.6), estimates (3.11) and Hölder's inequality, we have

$$\begin{aligned}
\|\boldsymbol{\partial}^2\vec{\Psi}\|^2_{H^{N-2}(\Sigma_t)} &\lesssim \|\boldsymbol{\partial}^2\vec{\Psi}\|^2_{H^{N-2}(\Sigma_0)} + \int_0^t \|\boldsymbol{\partial}\vec{\Psi}\|_{L_x^\infty(\Sigma_\tau)} \|\boldsymbol{\partial}^2\vec{\Psi}\|^2_{H^{N-2}(\Sigma_\tau)}\mathrm{d}\tau \\
&\quad + \int_0^t \{\|\partial(\vec{\mathcal{C}},\mathcal{D})\|_{H^{N-2}(\Sigma_\tau)} \|\boldsymbol{\partial}^2\vec{\Psi}\|_{H^{N-2}(\Sigma_\tau)} \\
&\quad + (\|\boldsymbol{\partial}(\vec{\Psi},\vec{\omega},\vec{S})\|_{L_x^\infty(\Sigma_\tau)} + 1)(\|\boldsymbol{\partial}(\vec{\Psi},\vec{\omega},\vec{S})\|_{H^{N-1}(\Sigma_\tau)} + 1) \\
&\quad \times \|\boldsymbol{\partial}^2\vec{\Psi}\|_{H^{N-2}(\Sigma_\tau)}\}\mathrm{d}\tau \\
&\lesssim P_N(0) + \int_0^t \|\boldsymbol{\partial}(\vec{\Psi},\vec{\omega},\vec{S})\|_{L_x^\infty(\Sigma_\tau)}\mathrm{d}\tau \\
&\quad + \int_0^t (\|\boldsymbol{\partial}(\vec{\Psi},\vec{\omega},\vec{S})\|_{L_x^\infty(\Sigma_\tau)} + 1) P_N(\tau)\mathrm{d}\tau.
\end{aligned} \tag{5.78}$$

For $\|\boldsymbol{\partial}\vec{\Psi}\|_{H^{N-1}(\Sigma_t)}$, we first have

$$\|\boldsymbol{\partial}\vec{\Psi}\|^2_{H^{N-1}(\Sigma_t)} \lesssim \|\boldsymbol{\partial}\vec{\Psi}\|^2_{L_x^2(\Sigma_t)} + \|\boldsymbol{\partial}^2\vec{\Psi}\|^2_{H^{N-2}(\Sigma_t)}. \tag{5.79}$$

Then by the fundamental theorem of calculus in time, Minkowski integral inequality and smallness of T_*, we have

$$\begin{aligned}
\|\boldsymbol{\partial}\vec{\Psi}\|^2_{L_x^2(\Sigma_t)} &= \int_{\Sigma_t} \left\{\boldsymbol{\partial}\vec{\Psi}(0,x) + \int_0^t \partial_t \boldsymbol{\partial}\vec{\Psi}(\tau,x)\mathrm{d}\tau\right\}^2 \mathrm{d}x \\
&\lesssim P_N(0) + \|\boldsymbol{\partial}^2\vec{\Psi}\|^2_{H^{N-2}(\Sigma_t)}.
\end{aligned} \tag{5.80}$$

Similarly, we have

$$\|\vec{\Psi}\|_{H^N(\Sigma_t)} \lesssim P_N(0) + \|\boldsymbol{\partial}^2\vec{\Psi}\|^2_{H^{N-2}(\Sigma_t)}. \tag{5.81}$$

Therefore, by adding (5.78)–(5.80), we have

$$\begin{aligned}
\sum_{k=0}^2 \|\boldsymbol{\partial}^k\vec{\Psi}\|^2_{H^{N-k}(\Sigma_t)} &\lesssim P_N(0) + \int_0^t \|\boldsymbol{\partial}(\vec{\Psi},\vec{\omega},\vec{S})\|_{L_x^\infty(\Sigma_\tau)}\mathrm{d}\tau \\
&\quad + \int_0^t (\|\boldsymbol{\partial}(\vec{\Psi},\vec{\omega},\vec{S})\|_{L_x^\infty(\Sigma_\tau)} + 1) P_N(\tau)\mathrm{d}\tau.
\end{aligned} \tag{5.82}$$

Now we derive top-order estimates for $\vec{\mathcal{C}}$ and $\mathcal{D}$. We apply the energy estimates (5.14) with $\boldsymbol{\partial}(\vec{\mathcal{C}},\mathcal{D})$ and $P_{\nu}\boldsymbol{\partial}(\vec{\mathcal{C}},\mathcal{D})$ in a role of φ, respectively, to obtain

$$\begin{aligned}\|\boldsymbol{\partial}(\vec{\mathcal{C}},\mathcal{D})\|^2_{L^2_x(\Sigma_t)} &\lesssim \|\boldsymbol{\partial}(\vec{\mathcal{C}},\mathcal{D})\|^2_{L^2_x(\Sigma_0)} + \int_0^t \|\boldsymbol{\partial}\vec{\Psi}\|_{L^\infty_x(\Sigma_\tau)}\|\boldsymbol{\partial}(\vec{\mathcal{C}},\mathcal{D})\|^2_{L^2_x(\Sigma_\tau)}\mathrm{d}\tau \\ &\quad + \int_0^t \|\boldsymbol{\partial}(\vec{\mathcal{C}},\mathcal{D})\|_{L^2_x(\Sigma_\tau)}\|\mathbf{B}\boldsymbol{\partial}(\vec{\mathcal{C}},\mathcal{D})\|_{L^2_x(\Sigma_\tau)}\mathrm{d}\tau, \end{aligned} \tag{5.83a}$$

$$\begin{aligned}\|P_{\nu}\boldsymbol{\partial}(\vec{\mathcal{C}},\mathcal{D})\|^2_{L^2_x(\Sigma_t)} &\lesssim \|P_{\nu}\boldsymbol{\partial}(\vec{\mathcal{C}},\mathcal{D})\|^2_{L^2_x(\Sigma_0)} + \int_0^t \|\boldsymbol{\partial}\vec{\Psi}\|_{L^\infty_x(\Sigma_\tau)}\|P_{\nu}\boldsymbol{\partial}(\vec{\mathcal{C}},\mathcal{D})\|^2_{L^2_x(\Sigma_\tau)}\mathrm{d}\tau \\ &\quad + \int_0^t \|P_{\nu}\boldsymbol{\partial}(\vec{\mathcal{C}},\mathcal{D})\|_{L^2_x(\Sigma_\tau)}\|\mathbf{B}P_{\nu}\boldsymbol{\partial}(\vec{\mathcal{C}},\mathcal{D})\|_{L^2_x(\Sigma_\tau)}\mathrm{d}\tau. \end{aligned} \tag{5.83b}$$

We use Eqs. (3.8b) and (3.8c) to substitute for $\mathbf{B}\boldsymbol{\partial}(\vec{\mathcal{C}},\mathcal{D})$, and we use Eqs. (3.9b) and (3.9c) for $\mathbf{B}P_{\nu}\boldsymbol{\partial}(\vec{\mathcal{C}},\mathcal{D})$ to substitute for the right-hand side of (5.83b). Multiplying (5.83b) by $\nu^{2(N-2)}$, summing over ν and using (3.6), using estimates (3.11) and elliptic estimates (5.15) and (5.16), we have

$$\begin{aligned}\|\boldsymbol{\partial}(\vec{\mathcal{C}},\mathcal{D})\|^2_{H^{N-2}(\Sigma_t)} &\lesssim \|\boldsymbol{\partial}(\vec{\mathcal{C}},\mathcal{D})\|^2_{H^{N-2}(\Sigma_0)} + \int_0^t \|\boldsymbol{\partial}\vec{\Psi}\|_{L^\infty_x(\Sigma_\tau)}\|\boldsymbol{\partial}(\vec{\mathcal{C}},\mathcal{D})\|^2_{H^{N-2}(\Sigma_\tau)}\mathrm{d}\tau \\ &\quad + \int_0^t \{\|\partial(\vec{\mathcal{C}},\mathcal{D})\|_{H^{N-2}(\Sigma_\tau)}\|\boldsymbol{\partial}(\vec{\mathcal{C}},\mathcal{D})\|_{H^{N-2}(\Sigma_\tau)} \\ &\quad + (\|\boldsymbol{\partial}(\vec{\Psi},\vec{\omega},\vec{S})\|_{L^\infty_x(\Sigma_\tau)}+1)(\|\boldsymbol{\partial}(\vec{\Psi},\vec{\omega},\vec{S})\|_{H^{N-1}(\Sigma_\tau)}+1) \\ &\quad \times \|\boldsymbol{\partial}(\vec{\mathcal{C}},\mathcal{D})\|_{H^{N-2}(\Sigma_\tau)}\mathrm{d}\tau\} \\ &\lesssim P_N(0) + \int_0^t \|\boldsymbol{\partial}(\vec{\Psi},\vec{\omega},\vec{S})\|_{L^\infty_x(\Sigma_\tau)}\mathrm{d}\tau \\ &\quad + \int_0^t (\|\boldsymbol{\partial}(\vec{\Psi},\vec{\omega},\vec{S})\|_{L^\infty_x(\Sigma_\tau)}+1)P_N(\tau)\mathrm{d}\tau. \end{aligned} \tag{5.84}$$

For $\|\vec{\mathcal{C}},\mathcal{D}\|_{H^{N-1}(\Sigma_t)}$, using the same method as in (5.79)–(5.82), by (5.84), we have

$$\begin{aligned}\|\vec{\mathcal{C}},\mathcal{D}\|^2_{H^{N-1}(\Sigma_t)} &\lesssim P_N(0) + \int_0^t \|\boldsymbol{\partial}(\vec{\Psi},\vec{\omega},\vec{S})\|_{L^\infty_x(\Sigma_\tau)}\mathrm{d}\tau \\ &\quad + \int_0^t (\|\boldsymbol{\partial}(\vec{\Psi},\vec{\omega},\vec{S})\|_{L^\infty_x(\Sigma_\tau)}+1)P_N(\tau)\mathrm{d}\tau. \end{aligned} \tag{5.85}$$

Combining (5.78), (5.82), (5.84), (5.85) and elliptic estimates (5.15) and (5.16), we have

$$\begin{aligned}P_N(t) &\lesssim P_N(0) + \int_0^t \|\boldsymbol{\partial}(\vec{\Psi},\vec{\omega},\vec{S})\|_{L^\infty_x(\Sigma_\tau)}\mathrm{d}\tau \\ &\quad + \int_0^t (\|\boldsymbol{\partial}(\vec{\Psi},\vec{\omega},\vec{S})\|_{L^\infty_x(\Sigma_\tau)}+1)P_N(\tau)\mathrm{d}\tau. \end{aligned} \tag{5.86}$$

By bootstrap assumptions (3.17b) and (3.17c), Hölder inequality in time and Grönwall's inequality, we obtain the desired result. □

5.2. *Schauder estimates*

In this section, we bound the Hölder norms of the modified fluid variables $\mathcal{C}, \mathcal{D}$ and the derivatives of vorticity and entropy gradient. Moreover, in Proposition 5.18, we reduce the proof of the improvement of bootstrap assumption (3.17c) to the proof of the improvement of bootstrap assumption (3.17b).

Theorem 5.18 (Improvements of the bootstrap assumptions for the vorticity and entropy gradient). *Let δ and δ_1 be as in Sec.* 3.4. *Under the initial data and bootstrap assumptions of Sec.* 3, *assuming the improved estimates in Theorem* 6.1 *holds for $\boldsymbol{\partial}\vec{\Psi}$, that is*

$$\|\boldsymbol{\partial}\vec{\Psi}\|^2_{L^2_t L^\infty_x([0,T_*]\times\mathbb{R}^3)} + \sum_{\nu\geq 2} \nu^{2\delta_1} \|P_\nu \boldsymbol{\partial}\vec{\Psi}\|^2_{L^2_t L^\infty_x([0,T_*]\times\mathbb{R}^3)} \lesssim T_*^{2\delta}, \tag{5.87}$$

then the following estimates hold:

$$\sum_{\nu\geq 2} \nu^{\delta_1} \|P_\nu(\partial\vec{\varpi}, \partial\vec{S})\|^2_{L^2_t L^\infty_x([0,T_*]\times\mathbb{R}^3)} \lesssim T_*^{2\delta}. \tag{5.88}$$

Remark 5.19. We emphasize that the proof of the Strichartz estimates (6.1)/(5.87) is independent of (5.88). The additional frequency weights in the Strichartz estimates in (5.87) are crucial in the Schauder estimates for the "transport-part". In particular, by (3.7) and (5.87), we have the following Strichartz estimates in Hölder spaces:

$$\|\boldsymbol{\partial}\vec{\Psi}\|^2_{L^2_t C^{0,\delta_1}_x([0,T_*]\times\mathbb{R}^3)} \lesssim T_*^{2\delta}. \tag{5.89}$$

We will prove Theorem 5.18 in Sec. 5.3. In this section, we derive a Schauder-type estimate in the following lemma.

Lemma 5.20. *Let δ_1 be as in Sec.* 3.4. *Under the initial data and bootstrap assumptions of Sec.* 3, *the following estimates hold*:

$$\|\boldsymbol{\partial}\vec{\varpi}, \boldsymbol{\partial}\vec{S}\|_{C^{0,\delta_1}_x(\Sigma_t)} \lesssim \|\boldsymbol{\partial}\vec{\Psi}, \vec{\mathcal{C}}, \mathcal{D}\|_{C^{0,\delta_1}_x(\Sigma_t)} + 1. \tag{5.90}$$

Proof of Lemma 5.20. We define the smooth function $\psi_1 = \psi_1(|\xi|) : \mathbb{R}^3 \to [0,1]$ such that $\psi_1(\xi) = 0$ for $0 \leq |\xi| < \frac{1}{2}$ and $\psi_1(\xi) = 1$ for $|\xi| \geq 1$.

By Littlewood–Paley theory (3.7) and (5.32) where η, F, G, H are the same as in Proposition 5.9 and $\underline{F}, \underline{H}$ are defined in Lemma 5.12 with $X = \eta$

$$\begin{aligned}
&\|\partial_s(\phi_l \eta_i)\|_{\dot{C}^{0,\delta_1}_x(\Sigma_t)} \\
&\quad\approx \sup_{\nu\geq 2} \|\nu^{\delta_1} P_\nu(\partial_s \phi_l \eta_i)\|_{L^\infty_x} \\
&\quad= \sup_{\nu\geq 2} \left\| C_1 \nu^{\delta_1} \mathcal{F}^{-1} \left\{ \psi(\nu^{-1}\xi) \frac{\xi_s}{(G^{-1})^{ab}(x_l)\xi_a \xi_b} \right.\right.
\end{aligned}$$

$$
\begin{aligned}
&\times \left(\xi_i \underline{\hat{F}}^l + \sum_{k \neq i} (G^{-1})^{jk}(x_l) \xi_j \underline{\hat{H}}^l_{ki} \right) \Bigg\} \Bigg\|_{L^\infty_x} \\
&\le C_1 \left\| \mathcal{F}^{-1} \left\{ \psi_1(\xi) \frac{\xi_s}{(G^{-1})^{ab}(x_l)\xi_a \xi_b} \xi_i \underline{\hat{F}}^l \right\} \right\|_{C^{0,\delta_1}_x(\Sigma_t)} \\
&\quad + C_1 \sum_{k \neq i} \left\| \mathcal{F}^{-1} \left\{ \psi_1(\xi) \frac{\xi_s}{(G^{-1})^{ab}(x_l)\xi_a \xi_b} (G^{-1})^{jk}(x_l) \xi_j \underline{\hat{H}}^l_{ki} \right\} \right\|_{C^{0,\delta_1}_x(\Sigma_t)},
\end{aligned}
\tag{5.91}
$$

where C_1 is a constant, $\mathcal{F}^{-1}$ is the inverse Fourier transform operator and ψ is defined in Sec. 3.2.

Now let's consider the first term in the right-hand side of last line of (5.91). For each fixed $s, i = 1, 2, 3$, we define function $p_{s,i}(\xi)$ as follows:

$$
p_{s,i}(\xi) := \psi_1(\xi) \frac{\xi_s \xi_i}{(G^{-1})^{ab}(x_l)\xi_a \xi_b}. \tag{5.92}
$$

The associated pseudodifferential operator $p(D_\xi; s, i)$ is defined by using Fourier integral representation as follows:

$$
p_{s,i}(D_\xi) f(x) := \int p_{s,i}(\xi) \hat{f}(\xi) e^{ix\xi} \mathrm{d}\xi. \tag{5.93}
$$

By direct computation and positive definiteness of G which is showed in Lemma 5.13, we have

$$
|D^\alpha_\xi p_{s,i}(\xi)| \le C_{\alpha\beta} (1 + |\xi|^2)^{\frac{-|\alpha|}{2}}. \tag{5.94}
$$

So $p_{s,i}(\xi)$ is in the Hörmander class $S^0_{1,0}$ and $p_{s,i}(D)$ belongs to $OPS^0_{1,0}$. By the theory of pseudodifferential operators, we have $p_{s,i}(D_\xi) : C^{0,\delta_1}_x \to C^{0,\delta_1}_x$. We refer reader to [13, Chap. 18] for explicit definition of Hörmander class and [28, Proposition 2.1.D] for the bounds of the operator $p_{s,i}(D_\xi)$. Therefore

$$
\left\| \mathcal{F}^{-1} \left\{ \psi_1(\xi) \frac{\xi_s}{(G^{-1})^{ab}(x_l)\xi_a \xi_b} \xi_i \underline{\hat{F}}^l \right\} \right\|_{C^{0,\delta_1}_x(\Sigma_t)} \le C_4 (\|\underline{F}^l\|_{C^{0,\delta_1}_x(\Sigma_t)} + \|\underline{F}^l\|_{L^2_x(\Sigma_t)}), \tag{5.95}
$$

where C_4 is a constant (independent of s, i, j, k, l). Similarly, we have

$$
\begin{aligned}
&\sum_{k \neq i} \left\| \mathcal{F}^{-1} \left\{ \psi_1(\xi) \frac{\xi_s}{(G^{-1})^{ab}(x_l)\xi_a \xi_b} (G^{-1})^{jk}(x_l) \xi_j \underline{\hat{H}}^l_{ki} \right\} \right\|_{C^{0,\delta_1}_x(\Sigma_t)} \\
&\quad \le C_4 \sum_{k \neq i} (\|\underline{H}^l_{ki}\|_{C^{0,\delta_1}_x(\Sigma_t)} + \|\underline{H}^l_{ki}\|_{L^2_x(\Sigma_t)}).
\end{aligned}
\tag{5.96}
$$

Combining (5.91), (5.95) and (5.96), for any l, s, i, we have

$$\|\partial_s(\phi_l \eta_i)\|_{\dot{C}_x^{0,\delta_1}(\Sigma_t)} \leq C_1 C_4 \Bigg(\|\underline{F}^l\|_{C_x^{0,\delta_1}(\Sigma_t)} + \|\underline{F}^l\|_{L_x^2(\Sigma_t)} + \sum_{k \neq i} \|\underline{H}^l_{ki}\|_{C_x^{0,\delta_1}(\Sigma_t)} + \|\underline{H}^l_{ki}\|_{L_x^2(\Sigma_t)} \Bigg), \tag{5.97}$$

where the constant C_1, C_4 is independent of l, s, i. By (5.33a)

$$\|\underline{F}^l\|_{C_x^{0,\delta_1}(\Sigma_t)} \leq \|\phi_l F\|_{C_x^{0,\delta_1}(\Sigma_t)} + \|G^{-1}\|_{C_x^{0,\delta_1}(\Sigma_t)} \|\partial \phi_l\|_{C_x^{0,\delta_1}(\Sigma_t)} \|\eta\|_{C_x^{0,\delta_1}(\Sigma_t)} + \sup_{x \in B(x_l, \delta_2)} |(G^{-1})^{ab}(x) - (G^{-1})^{ab}(x_l)| \|\partial(\phi_l \eta)\|_{C_x^{0,\delta_1}(\Sigma_t)}. \tag{5.98}$$

By (5.33b)

$$\|\underline{H}^l\|_{C_x^{0,\delta_1}(\Sigma_t)} \leq \|\phi_l H\|_{C_x^{0,\delta_1}(\Sigma_t)} + 2\|\partial \phi_l\|_{C_x^{0,\delta_1}(\Sigma_t)} \|\eta\|_{C_x^{0,\delta_1}(\Sigma_t)}. \tag{5.99}$$

For $C_2 C_3$ as in (5.44), $C_1 C_4$ as in (5.97), by Lemma 5.14, we let δ_2 be small such that for all $x \in B(x_l, \delta_2)$

$$C_1 C_4 \sup_{x \in B(x_l, \delta_2)} |(G^{-1})^{ab}(x) - (G^{-1})^{ab}(x_l)| < \frac{1}{4}. \tag{5.100}$$

Combining (5.97)–(5.100), absorbing last term in the right-hand side of (5.98) by the left, using energy estimates (5.1), we have

$$\|\partial(\phi_l \eta_i)\|_{\dot{C}_x^{0,\delta_1}(\Sigma_t)} \lesssim \|\phi_l F\|_{C_x^{0,\delta_1}(\Sigma_t)} + \|G^{-1}\|_{C_x^{0,\delta_1}(\Sigma_t)} \|\partial \phi_l\|_{C_x^{0,\delta_1}(\Sigma_t)} \|\eta\|_{C_x^{0,\delta_1}(\Sigma_t)} + \|\phi_l H\|_{C_x^{0,\delta_1}(\Sigma_t)} + \|\partial \phi_l\|_{C_x^{0,\delta_1}(\Sigma_t)} \|\eta\|_{C_x^{0,\delta_1}(\Sigma_t)} + 1. \tag{5.101}$$

Now note that

$$\begin{aligned} \|\partial \eta\|_{\dot{C}_x^{0,\delta_1}(\Sigma_t)} &= \sup_{x,y \in \Sigma_t} \frac{|\partial \eta(x) - \partial \eta(y)|}{|x-y|^{\delta_1}} \\ &= \sup_{x,y \in \Sigma_t} \frac{|\partial \left(\sum_l \phi_l \eta\right)(x) - \partial \left(\sum_l \phi_l \eta\right)(y)|}{|x-y|^{\delta_1}} \\ &\leq \sup_{x,y \in \Sigma_t} \sum_{\phi_l(x) \neq 0 \text{ or } \phi_l(y) \neq 0} \|\partial(\phi_l \eta)\|_{\dot{C}_x^{0,\delta_1}(\Sigma_t)}. \end{aligned} \tag{5.102}$$

Note that for each $x \in \Sigma_t$, there are at most finitely many l's (at most 16) such that either $\phi_l(x)$ or $\phi_l(y)$ is nonzero. Hence by definition of G^{-1}, F, H in (5.18), (5.101) and (5.102)

$$\begin{aligned} \|\partial \eta\|_{\dot{C}_x^{0,\delta_1}(\Sigma_t)} &\lesssim \|F\|_{C_x^{0,\delta_1}(\Sigma_t)} + \|G^{-1}\|_{C_x^{0,\delta_1}(\Sigma_t)} \|\eta\|_{C_x^{0,\delta_1}(\Sigma_t)} \\ &\quad + \|H\|_{C_x^{0,\delta_1}(\Sigma_t)} + \|\eta\|_{C_x^{0,\delta_1}(\Sigma_t)} + 1 \\ &\lesssim \|\boldsymbol{\partial} \vec{\Psi}, \vec{\mathcal{C}}, \mathcal{D}\|_{C_x^{0,\delta_1}(\Sigma_t)} + 1. \end{aligned} \tag{5.103}$$

We now bound $\|\partial\eta\|_{L^\infty_x(\Sigma_t)}$. For any point $z \in \Sigma_t$, there is a $y \in B(z,1)$ such that $|\partial\eta(y)| \leq \|\partial\eta\|_{L^2_x(\Sigma_t)}$, thus

$$|\partial\eta(z)| \leq |\partial\eta(y)| + 2\|\partial\eta\|_{\dot{C}^{0,\delta_1}_x(\Sigma_t)} \lesssim \|\partial\eta\|_{L^2_x(\Sigma_t)} + \|\partial\eta\|_{\dot{C}^{0,\delta_1}_x(\Sigma_t)}, \tag{5.104}$$

that is

$$\|\partial\eta\|_{L^\infty_x(\Sigma_t)} \lesssim \|\partial\eta\|_{L^2_x(\Sigma_t)} + \|\partial\eta\|_{\dot{C}^{0,\delta_1}_x(\Sigma_t)}. \tag{5.105}$$

Combining with Proposition 5.1, we have

$$\|\partial\eta\|_{C^{0,\delta_1}_x(\Sigma_t)} \lesssim 1 + \|\boldsymbol{\partial}\vec{\Psi}, \vec{\mathcal{C}}, \mathcal{D}\|_{C^{0,\delta_1}_x(\Sigma_t)}. \tag{5.106}$$

□

5.3. *Estimates for $\|\mathcal{C}, \mathcal{D}\|_{C^{0,\delta_1}_x(\Sigma_t)}$ via transport equations and the Proof of Theorem* 5.18

In this section, we first estimate the Hölder norm of the modified fluid variables. We then prove Theorem 5.18. We will use the following lemma, which is a standard estimate for transport equations with Hölder data and Hölder inhomogeneities.

Lemma 5.21. *Let ϕ be a scalar function. If $F \in C^{0,\alpha}_x(\Sigma_\tau)$ with $\tau \in [0,t]$ and*

$$\mathbf{B}\phi = F, \tag{5.107}$$

then

$$\|\phi\|_{C^{0,\alpha}_x(\Sigma_t)} \lesssim \|\phi\|_{C^{0,\alpha}_x(\Sigma_0)} + \int_0^t \|F\|_{C^{0,\alpha}_x(\Sigma_\tau)} \mathrm{d}\tau. \tag{5.108}$$

Proof. Note that $\mathbf{B}^0 = 1$. Let γ be the integral curve of $\mathbf{B}$ such that

$$\gamma^i(0,x) = x^i, \tag{5.109}$$

$$\gamma^0(t,x) = t, \tag{5.110}$$

and

$$\partial_t\gamma^i(t,x) = \mathbf{B}^i(t,\gamma). \tag{5.111}$$

Then

$$\begin{aligned} |\gamma^i(t,x) - \gamma^i(t,y)| &\leq |x^i - y^i| + \int_0^t \mathbf{B}^i(\tau,\gamma(\tau,x)) - \mathbf{B}^i(\tau,\gamma(\tau,y))\mathrm{d}\tau \\ &\leq |x^i - y^i| + \int_0^t \|\partial\mathbf{B}^i\|_{L^\infty_x(\Sigma_\tau)}|\gamma(\tau,x) - \gamma(\tau,y)|\mathrm{d}\tau. \end{aligned} \tag{5.112}$$

By Grönwall's inequality, we have

$$\frac{|\gamma(t,x) - \gamma(t,y)|}{|x-y|} \lesssim \exp\left(\int_0^t \|\partial\mathbf{B}\|_{L^\infty_x(\Sigma_\tau)}\mathrm{d}\tau\right). \tag{5.113}$$

Note that $\|\partial \mathbf{B}\|_{L^\infty_x(\Sigma_\tau)} \approx \|\partial \vec{\Psi}\|_{L^\infty_x(\Sigma_\tau)}$. Similarly, considering $\bar{\gamma}(\tau) := \gamma(t-\tau)$ from Σ_t to Σ_0, we have

$$\frac{|x-y|}{|\gamma(t,x)-\gamma(t,y)|} \lesssim \exp\left(\int_0^t \|\partial \mathbf{B}\|_{L^\infty_x(\Sigma_\tau)} \mathrm{d}\tau\right). \tag{5.114}$$

Now we consider ϕ

$$\partial_t(\phi \circ \gamma) = F \circ \gamma, \tag{5.115}$$

then

$$\begin{aligned}&\phi \circ \gamma(t,x) - \phi \circ \gamma(t,y)\\ &\quad = \phi \circ \gamma(0,x) - \phi \circ \gamma(0,y) + \int_0^t F(\tau, \gamma(\tau,x)) - F(\tau,\gamma(\tau,y))\mathrm{d}\tau.\end{aligned} \tag{5.116}$$

By (5.113), (5.114) and bootstrap assumption in Sec. 3.6, we have

$$\begin{aligned}&\frac{|F(\tau,\gamma(\tau,x)) - F(\tau,\gamma(\tau,y))|}{|\gamma(t,x)-\gamma(t,y)|^\alpha}\\ &\quad = \frac{|F(\tau,\gamma(\tau,x)) - F(\tau,\gamma(\tau,y))|}{|\gamma(\tau,x)-\gamma(\tau,y)|^\alpha}\frac{|\gamma(\tau,x)-\gamma(\tau,y)|^\alpha}{|\gamma(t,x)-\gamma(t,y)|^\alpha}\\ &\quad \lesssim \|F\|_{C^{0,\alpha}_x(\Sigma_\tau)}.\end{aligned} \tag{5.117}$$

□

Proof of Theorem 5.18. Now we consider Eqs. (2.26a) and (2.26b)

$$\mathbf{B}\mathcal{C}^\alpha = \mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi}, (\boldsymbol{\partial}\omega, \boldsymbol{\partial}\vec{S}, \boldsymbol{\partial}\vec{\Psi})] + \mathscr{Q}(\vec{S})[\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\Psi}] + \mathscr{L}(\vec{\Psi}, \vec{\omega}, \vec{S})[\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{S}], \tag{5.118a}$$

$$\mathbf{B}\mathcal{D} = \mathscr{L}(\vec{\Psi}, \vec{S})[\boldsymbol{\partial}\omega] + \mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi}, (\boldsymbol{\partial}\vec{S}, \boldsymbol{\partial}\vec{\Psi})] + \mathscr{Q}(\vec{S})[\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\Psi}] + \mathscr{L}(\vec{\Psi}, \vec{\omega}, \vec{S})[\boldsymbol{\partial}\vec{\Psi}]. \tag{5.118b}$$

By Lemma 5.21, we have

$$\|\vec{\mathcal{C}}, \mathcal{D}\|_{C^{0,\delta_1}_x(\Sigma_t)} \lesssim 1 + \int_0^t (\|\boldsymbol{\partial}\vec{\Psi}\|_{C^{0,\delta_1}_x(\Sigma_\tau)} + 1)\|\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\omega}, \boldsymbol{\partial}\vec{S}\|_{C^{0,\delta_1}_x(\Sigma_\tau)}\mathrm{d}\tau. \tag{5.119}$$

By (5.90), bootstrap assumptions and Grönwall's inequality, we have

$$\|\vec{\mathcal{C}}, \mathcal{D}\|_{C^{0,\delta_1}_x(\Sigma_t)} \lesssim 1. \tag{5.120}$$

Integrating (5.90) in time, we have

$$\int_0^t \|\boldsymbol{\partial}\vec{\omega}, \boldsymbol{\partial}\vec{S}\|^2_{C^{0,\delta_1}_x(\Sigma_t)}\mathrm{d}t \lesssim \int_0^t 1 + \|\boldsymbol{\partial}\vec{\Psi}, \vec{\mathcal{C}}, \mathcal{D}\|^2_{C^{0,\delta_1}_x(\Sigma_t)}\mathrm{d}t. \tag{5.121}$$

If (5.87) holds, using (5.120), the bootstrap assumption (3.17b) and the standard results in Littlewood–Paley (3.7), (5.88) is obtained by following estimate:

$$\begin{aligned}\int_0^t \|\boldsymbol{\partial}\vec{\omega}, \boldsymbol{\partial}\vec{S}\|^2_{C^{0,\delta_1}_x(\Sigma_t)} \mathrm{d}t &\lesssim \int_0^t 1 + \|\boldsymbol{\partial}\vec{\Psi}, \vec{\mathcal{C}}, \mathcal{D}\|^2_{C^{0,\delta_1}_x(\Sigma_t)} \mathrm{d}t \\ &\leq T_*^{2\delta} + T_* \lesssim T_*^{2\delta}. \end{aligned} \tag{5.122}$$

□

6. Reduction of Strichartz Estimates and the Rescaled Solution

In this section, we state our main estimates as Theorem 6.1, which are the improvement of the bootstrap assumption (3.17b). We then provide a series of analytic reductions from the Strichartz estimates of Theorem 6.1 to the decay estimates of Theorem 6.9. We are quite terse in this section since the full proofs of these reductions are lengthy and difficult, yet standard. We refer readers to [11, 30, 16] for the detailed proofs.

Theorem 6.1 (Improvement of the Strichartz-type bootstrap assumptions for the wave variables). *If $\delta > 0$ is sufficiently small as in Sec.* 3.4, *then under the initial data and bootstrap assumptions of Sec.* 3, *the following estimates holds with a number* $8\delta_0 < \delta_1 < N - 2$:

$$\|\boldsymbol{\partial}\vec{\Psi}\|^2_{L^2_t L^\infty_x([0,T_*]\times\mathbb{R}^3)} + \sum_{\nu\geq 2} \nu^{2\delta_1} \|P_\nu \boldsymbol{\partial}\vec{\Psi}\|^2_{L^2_t L^\infty_x([0,T_*]\times\mathbb{R}^3)} \lesssim T_*^{2\delta}. \tag{6.1}$$

We first reduce the proof of Theorem 6.1 to the proof of Strichartz estimates on small time intervals.

6.1. *Partitioning of the bootstrap time interval*

Let λ be a fixed large number and let $0 < \varepsilon_0 < \frac{N-2}{5}$ be a fixed number as mentioned in Sec. 3.4. By the bootstrap assumptions, we can[1] partition $[0, T_*]$ into disjoint union of sub-intervals $I_k := [t_{k-1}, t_k]$ of total number $\lesssim \lambda^{8\varepsilon_0}$ with the properties that $|I_k| \leq \lambda^{-8\varepsilon_0} T_*$ and

$$\|\boldsymbol{\partial}\vec{\Psi}\|^2_{L^2_t L^\infty_x(I_k\times\mathbb{R}^3)} + \sum_{\nu\geq 2} \nu^{2\delta_0} \|P_\nu \boldsymbol{\partial}\vec{\Psi}\|^2_{L^2_t L^\infty_x(I_k\times\mathbb{R}^3)} \lesssim \lambda^{-8\varepsilon_0}, \tag{6.2a}$$

$$\|\partial\vec{\omega}, \partial\vec{S}\|^2_{L^2_t L^\infty_x(I_k\times\mathbb{R}^3)} + \sum_{\nu\geq 2} \nu^{2\delta_0} \|P_\nu (\partial\vec{\omega}, \partial\vec{S})\|^2_{L^2_t L^\infty_x(I_k\times\mathbb{R}^3)} \lesssim \lambda^{-8\varepsilon_0}. \tag{6.2b}$$

Now we reduce Theorem 6.1 to a frequency-localized estimate.

Theorem 6.2 (Frequency-localized Strichartz estimate). *Let φ be a solution of*

$$\square_{\mathbf{g}} \varphi = 0, \tag{6.3}$$

[1]The existence of such partition easily follows from the bootstrap assumptions, see [16, Remark 1.3].

on the time interval I_k. Then for any $q > 2$ sufficiently close to 2 and any $\tau \in [t_k, t_{k+1}]$, under the bootstrap assumptions, we have the following estimate:

$$\|P_\lambda \boldsymbol{\partial}\varphi\|_{L^q_t L^\infty_x([\tau,t_{k+1}]\times\mathbb{R}^3)} \lesssim \lambda^{\frac{3}{2}-\frac{1}{q}} \|\boldsymbol{\partial}\varphi\|_{L^2_x(\Sigma_\tau)}. \tag{6.4}$$

6.1.1. *Discussion of the reduction to Theorem* 6.2 *from Theorem* 6.1

The proof reducing Theorem 6.1 to Theorem 6.2 is exactly the same as the proof of [11, Theorem 7.2]. The reason is that the proof relies only on: (1) Duhamel's principle; (2) top-order energy estimates (5.1) which is the same as in [11] and (3) Littlewood–Paley estimates for the inhomogeneous terms in a frequency-projected version of the wave equations, and the wave equations in this paper have an identical schematic form to the ones in [11].

6.2. *Rescaled quantities and rescaled relativistic Euler equations*

In this section, in order to do further reductions, we consider the following coordinate change $(t, x) \mapsto (\lambda(t - t_k), \lambda x)$. Let

$$T_{*;(\lambda)} := \lambda(t_{k+1} - t_k). \tag{6.5}$$

Note that by construction

$$0 \leq T_{*;(\lambda)} \leq \lambda^{1-8\varepsilon_0} T_*. \tag{6.6}$$

Definition 6.3 (Rescaled quantities). First we define the following variables:

$$\vec{\Psi}_{(\lambda)}(t,x) := \vec{\Psi}(t_k + \lambda^{-1}t, \lambda^{-1}x), \tag{6.7a}$$

$$\vec{\omega}_{(\lambda)}(t,x) := \vec{\omega}(t_k + \lambda^{-1}t, \lambda^{-1}x), \tag{6.7b}$$

$$\vec{S}^\kappa_{(\lambda)}(t,x) := \vec{S}^\kappa(t_k + \lambda^{-1}t, \lambda^{-1}x), \tag{6.7c}$$

$$\theta_{(\lambda)} := \theta(t_k + \lambda^{-1}t, \lambda^{-1}x), \tag{6.7d}$$

$$c_{(\lambda)} := c(t_k + \lambda^{-1}t, \lambda^{-1}x), \tag{6.7e}$$

$$\begin{aligned}\mathcal{C}^\alpha_{(\lambda)} :={}& \mathrm{vort}^\alpha((\omega_\flat)_{(\lambda)}) + c^{-2}_{(\lambda)}\epsilon^{\alpha\beta\gamma\delta}((v_\flat)_{(\lambda)})_\beta(\partial_\gamma h_{(\lambda)})(\omega_\flat)_{(\lambda)\delta} + (\theta_{(\lambda)} - (\theta_{;h})_{(\lambda)})\\ &\times (S^\sharp)^\alpha_{(\lambda)}(\partial_\kappa(v_\flat)^\kappa_{(\lambda)}) + (\theta_{(\lambda)} - (\theta_{;h})_{(\lambda)})(v_\flat)^\alpha_{(\lambda)}((S^\sharp)^\kappa_{(\lambda)}\partial_\kappa h_{(\lambda)}) - (\theta_{(\lambda)}\\ &- (\theta_{;h})_{(\lambda)})(S^\sharp)^\kappa_{(\lambda)}((M^{-1})^{\alpha\lambda}\partial_\lambda(v_\flat)_{(\lambda)\kappa}),\end{aligned} \tag{6.7f}$$

$$\mathcal{D}_{(\lambda)} := \frac{1}{n}(\partial_\kappa(S^\sharp)^\kappa_{(\lambda)}) + \frac{1}{n}((S^\sharp)^\kappa_{(\lambda)}\partial_\kappa h_{(\lambda)}) - \frac{1}{n}c^{-2}((S^\sharp)^\kappa_{(\lambda)}\partial_\kappa h_{(\lambda)}). \tag{6.7g}$$

Then we define the following rescaled tensorfields:

$$(\mathbf{g}_{(\lambda)})_{\alpha\beta}(t,x) := \mathbf{g}_{\alpha\beta}(\vec{\Psi}_{(\lambda)}(t,x)), \tag{6.8a}$$

$$(g_{(\lambda)})_{\alpha\beta}(t,x) := g_{\alpha\beta}(\vec{\Psi}_{(\lambda)}(t,x)), \tag{6.8b}$$

$$\mathbf{B}^\alpha_{(\lambda)}(t,x) := \mathbf{B}^\alpha(\vec{\Psi}_{(\lambda)}(t,x)). \tag{6.8c}$$

Remark 6.4. We note that after rescaling, the new initial constant-time hypersurface Σ_0 corresponds to the constant-time hypersurface that was denoted by Σ_{t_k} for some k in Secs. 1–5.

The following proposition provides the equations satisfied by the rescaled quantities. We omit the straightforward proof.

Proposition 6.5 (The rescaled geometric wave–transport formulation of the relativistic Euler equations). *For* $\Psi \in \{v^0, v^1, v^2, v^3, h, s\}$, *after rescaling, we have the following equations inherited from Proposition* 2.17:

Wave equations:

$$\square_{\mathbf{g}_{(\lambda)}} \Psi_{(\lambda)} = \lambda^{-1} \mathscr{L}(\vec{\Psi}_{(\lambda)})[\vec{\mathcal{C}}_{(\lambda)}, \mathcal{D}_{(\lambda)}] + \mathscr{Q}(\vec{\Psi}_{(\lambda)})[\boldsymbol{\partial}\vec{\Psi}_{(\lambda)}, \boldsymbol{\partial}\vec{\Psi}_{(\lambda)}]. \tag{6.9}$$

Transport equations:

$$\mathbf{B}_{(\lambda)} \omega^\alpha_{(\lambda)} = \mathscr{L}(\vec{\Psi}_{(\lambda)}, \vec{\omega}_{(\lambda)}, \vec{S}_{(\lambda)})[\boldsymbol{\partial}\vec{\Psi}_{(\lambda)}], \tag{6.10a}$$

$$\mathbf{B}_{(\lambda)} (S^\sharp)^\alpha_{(\lambda)} = \mathscr{L}(\vec{\Psi}_{(\lambda)}, \vec{S}_{(\lambda)})[\boldsymbol{\partial}\vec{\Psi}_{(\lambda)}]. \tag{6.10b}$$

Transport-Div-Curl system:

$$\begin{aligned} \mathbf{B}_{(\lambda)} \mathcal{C}^\alpha_{(\lambda)} &= \mathscr{Q}(\vec{\Psi}_{(\lambda)})[\boldsymbol{\partial}\vec{\Psi}_{(\lambda)}, (\boldsymbol{\partial}\omega_{(\lambda)}, \boldsymbol{\partial}\vec{S}_{(\lambda)}, \boldsymbol{\partial}\vec{\Psi}_{(\lambda)})] \\ &\quad + \mathscr{L}(\vec{\Psi}_{(\lambda)}, \vec{\omega}_{(\lambda)}, \vec{S}_{(\lambda)})[\boldsymbol{\partial}\vec{\Psi}_{(\lambda)}, \boldsymbol{\partial}\vec{S}_{(\lambda)}], \end{aligned} \tag{6.11a}$$

$$\begin{aligned} \mathbf{B}_{(\lambda)} \mathcal{D}_{(\lambda)} &= \mathscr{L}(\vec{S}_{(\lambda)}, \vec{\Psi}_{(\lambda)})[\boldsymbol{\partial}\omega_{(\lambda)}] + \mathscr{Q}(\vec{\Psi}_{(\lambda)})[\boldsymbol{\partial}\vec{\Psi}_{(\lambda)}, (\boldsymbol{\partial}\vec{S}_{(\lambda)}, \boldsymbol{\partial}\vec{\Psi}_{(\lambda)})] \\ &\quad + \mathscr{L}(\vec{\Psi}_{(\lambda)}, \vec{\omega}_{(\lambda)}, \vec{S}_{(\lambda)})[\boldsymbol{\partial}\vec{\Psi}_{(\lambda)}, \boldsymbol{\partial}\vec{S}_{(\lambda)}], \end{aligned} \tag{6.11b}$$

$$\mathrm{vort}^\alpha(S^\sharp_{(\lambda)}) = 0, \tag{6.11c}$$

$$\partial_\alpha \omega^\alpha_{(\lambda)} = \mathscr{L}(\omega_{(\lambda)})[\boldsymbol{\partial}\vec{\Psi}_{(\lambda)}]. \tag{6.11d}$$

Remark 6.6. For notation convenience, in the remainder of the paper, we drop the sub and super scripts (λ) except for the rescaled time $T_{*;(\lambda)}$.

6.2.1. *Consequences of the bootstrap assumptions*

After rescaling in Sec. 6.2, assuming bootstrap assumptions (3.17b) and (3.17c), as in [11, Sec. 10.2.1], by standard computation based on Littlewood–Paley calculus, we have the following consequences of the bootstrap assumptions:

Estimates by using bootstrap assumptions of variables:

$$\begin{aligned} &\|\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\omega}, \boldsymbol{\partial}\vec{S}, \vec{\mathcal{C}}, \mathcal{D}\|_{L^2_t L^\infty_x(\mathcal{M})} \\ &\quad + \lambda^{\delta_0} \sqrt{\sum_{\nu > 2} \nu^{2\delta_0} \|P_\nu(\mathrm{f}(\vec{\Psi}, \vec{\omega}, \vec{S})(\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\omega}, \boldsymbol{\partial}\vec{S}, \vec{\mathcal{C}}, \mathcal{D}))\|^2_{L^2_t L^\infty_x(\mathcal{M})}} \\ &\lesssim \lambda^{-1/2 - 4\varepsilon_0}. \end{aligned} \tag{6.12}$$

6.3. *Further reduction of the Strichartz estimates*

By the rescaling in Sec. 6.2 and direct computation, to prove Theorem 6.2, it is equivalent to show the following Strichartz estimate on $[0, T_{*;(\lambda)}]$ with respect to LP projection on the frequency domain $\{1/2 \leq |\xi| \leq 2\}$.

Theorem 6.7. *Under the bootstrap assumptions. For any solution φ of $\square_{\mathbf{g}}\varphi = 0$ on the slab $[0, T_{*;(\lambda)}] \times \mathbb{R}^3$, the following estimates holds*:

$$\|P_1 \boldsymbol{\partial}\varphi\|_{L_t^q L_x^\infty([0,T_{*;(\lambda)}]\times\mathbb{R}^3)} \lesssim \|\boldsymbol{\partial}\varphi\|_{L_x^2(\Sigma_0)}, 1, \tag{6.13}$$

where $q > 2$ is sufficiently close to 2 and $\mathbf{g}$ is the rescaled metric $\mathbf{g}_{(\lambda)}$ defined in (6.8a).

The proof of Theorem 6.7 crucially relies on the following decay estimate.

Theorem 6.8 (Decay estimate). *There exists a large number Λ such that for any $\lambda \geq \Lambda$ and any solution φ of the equation $\square_{\mathbf{g}}\varphi = 0$ on $[0, T_{*;(\lambda)}] \times \mathbb{R}^3$, there is a function $d(t)$ satisfying*

$$\|d\|_{L_t^{\frac{q}{2}}([0,T_{*;(\lambda)}])} \lesssim 1, \tag{6.14}$$

for $q > 2$ sufficiently close to 2 such that for any $t \in [0, T_{*;(\lambda)}]$, the following decay estimate holds[m]:

$$\|P_1 \mathbf{T}\varphi\|_{L_x^\infty(\Sigma_t)} \lesssim \left(\frac{1}{(1+t)^{\frac{2}{q}}} + d(t)\right)\left(\sum_{m=0}^{3}\|\partial^m\varphi\|_{L_x^1(\Sigma_0)} + \sum_{m=0}^{2}\|\partial^m\partial_t\varphi\|_{L_x^1(\Sigma_0)}\right). \tag{6.15}$$

The proof of Theorem 6.7 using Theorem 6.8 is based on a $\mathcal{T}\mathcal{T}^*$ argument[n] (see [16, Sec. 8.6; 17, Sec. 8.30]).

Theorem 6.8 can be further reduced to the following spatially localized version.

Theorem 6.9 (Spatially localized version of decay estimate). *There exists a large number Λ such that for any $\lambda \geq \Lambda$ and any solution φ of the equation $\square_{\mathbf{g}}\varphi = 0$ on $[0, T_{*;(\lambda)}] \times \mathbb{R}^3$ with $\varphi(1, x)$ supported in the Euclidean ball B_R, where radius R is a fixed radius*[o] *such that*

$$B_R(p) \subset B_{\frac{1}{2}}(p, g_{(\lambda)}), \quad \forall\, p \in \Sigma_t,\ \ 0 \leq t \leq T_{*;(\lambda)}, \tag{6.16}$$

where $B_\rho(p, g_{(\lambda)})$ is the geodesic ball[p] *centered at p with radius ρ and $g_{(\lambda)}$ is the rescaled induced metric of $\mathbf{g}$ on Σ_t (defined in* (6.8b)*), there is a function $d(t)$*

[m] $\mathbf{T}_\flat = -\mathrm{d}t$, see Definition 2.11 for the definition of $\mathbf{T}$.
[n] This argument comes from functional analysis, which does not require the structure of the relativistic Euler equations.
[o] The radius R will be used in the following sections as well with the same definition. The existence of such an R is guaranteed by properties of g, namely, (2.20b), that ensures g is comparable to the Euclidean metric on Σ_t under the bootstrap assumptions.
[p] The notation $B_\rho(p, g_{(\lambda)})$ will be used in the remainder of the paper. This is consistent with the notation that is used in [11, 30].

satisfying

$$\|d\|_{L_t^{\frac{q}{2}}([0,T_{*;(\lambda)}])} \lesssim 1, \tag{6.17}$$

for $q > 2$ *sufficiently close to* 2 *such that for any* $t \in [0, T_{*;(\lambda)}]$, *there holds*

$$\|P_1 \mathbf{T}\varphi\|_{L_x^\infty(\Sigma_t)} \lesssim \left(\frac{1}{(1+|t-1|)^{\frac{2}{q}}} + d(t) \right) \left(\|\boldsymbol{\partial}\varphi\|_{L_x^2(\Sigma_1)} + \|\varphi\|_{L_x^2(\Sigma_1)} \right). \tag{6.18}$$

The proof of Theorem 6.8 using Theorem 6.9 can be done via an approach involving the Bernstein inequalities of LP projection, partition of unity[q] of φ and Sobolev embedding $W^{2,1} \hookrightarrow L^2$. We refer readers to [16, Sec. 8.5] for detailed proof.

To summarize, in this section we have reduced Theorem 6.1 to Theorem 6.9. To further reduce the estimates, we need to introduce the geometric setup. We will discuss the proof of Theorem 6.9 in Sec. 7.3.

7. Geometric Setup and Conformal Energy

In this section, we first construct acoustic geometry. It is deeply coupled with the relativistic Euler equations (via acoustic metric $\mathbf{g}$) and is crucial in our analysis. Then we provide the conformal energy and its estimates in Sec. 7.3.

Definition 7.1 (Christoffel symbols). We define the Christoffel symbols $\boldsymbol{\Gamma}_{\alpha\kappa\lambda}$ and $\boldsymbol{\Gamma}^{\beta}_{\kappa\lambda}$ with the rescaled metric $\mathbf{g}$ to be as follows:

$$\boldsymbol{\Gamma}_{\alpha\kappa\lambda} := \frac{1}{2}(\partial_\kappa \mathbf{g}_{\alpha\lambda} + \partial_\lambda \mathbf{g}_{\alpha\kappa} - \partial_\alpha \mathbf{g}_{\kappa\lambda}), \tag{7.1}$$

$$\boldsymbol{\Gamma}^{\beta}_{\kappa\lambda} := \mathbf{g}^{\alpha\beta}\boldsymbol{\Gamma}_{\alpha\kappa\lambda}. \tag{7.2}$$

7.1. *Construction of the acoustical function*

The goal of this section is to construct the geometry based on a solution u to the acoustical eikonal equation[r]

$$\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0. \tag{7.3}$$

7.1.1. *Point* $\mathbf{z}$ *and integral curve* $\gamma_{\mathbf{z}}(t)$

Let $\mathbf{z} \in \Sigma_0$ be an arbitrarily fixed point[s] in the rescaled space-time $[0, T_{*;(\lambda)}] \times \mathbb{R}^3$, where $T_{*;(\lambda)}$ is defined in (6.5). We let $\gamma_{\mathbf{z}}(t)$ denote the integral curve of the future-directed vectorfield $\mathbf{T}$ emanating from the point $\mathbf{z}$. We say $\gamma_{\mathbf{z}}(t)$ is the cone-tip axis. Specifically, the point $\mathbf{z}$ depends on the partition of unity of Σ_1 used in the proof of Theorem 6.8 using Theorem 6.9. More specifically, $\mathbf{z}$ is going to be the tip of the

[q] We take a sequence of Euclidean balls $\{B_I\}$ of radius R such that their union covers $\mathbb{R}^3$. For each ball B_R, it is centered at $\gamma_{\mathbf{z}}(1)$ (defined in Sec. 7.1.2) for some $\mathbf{z}$.
[r] For more details of the geometric construction of u, we refer reader to [8, Chaps. 9 and 14].
[s] Note that in the original space-time $[0, T_*] \times \mathbb{R}^3$, $\mathbf{z}$ is a point at Σ_{t_k} for some k.

cone (constructed in Sec. 7.1.2) such that $\gamma_{\mathbf{z}}(1)$ is the center of the Euclidean ball B_R (as in Theorem 6.9). We note that the estimates, constants and parameters in Secs. 7 and 8 are independent of $\mathbf{z}$.

7.1.2. *The interior and exterior solution u*

The interior solution *u*. We let $\{L_\omega|_{\mathbf{z}}\}_{\omega\in\mathbb{S}^2}$ be the family of null vectors (parametrized by $\mathbb{S}^2$, where the parametrization will be uniquely determined in the paragraph of the exterior solution u) in the tangent space $T_{\mathbf{z}}\mathcal{M}$ and $\langle L_\omega|_{\mathbf{z}}, \mathbf{T}\rangle = -1$. To propagate L_ω along the cone-tip axis $\gamma_{\mathbf{z}}(t)$, for any $\mathbf{p}\in\gamma_{\mathbf{z}}(t)$ (as defined in Sec. 7.1.1) and $\omega\in\mathbb{S}^2$, we define $L_\omega|_{\mathbf{p}}$ by solving the parallel transport equation $\mathbf{D_T}L_\omega = 0$. We note that for any $\mathbf{p}\in\gamma_{\mathbf{z}}(t)$, $\langle L_\omega|_{\mathbf{p}}, \mathbf{T}\rangle = -1$ since $\mathbf{T}$ is geodesic. Then for each $u\in[0,T_{*;(\lambda)}]$ and $\omega\in\mathbb{S}^2$, there exists a unique null geodesic $\Upsilon_{u,\omega}(t)$, where $t\in[u,T_{*;(\lambda)}]$, emanating from $\mathbf{p}=\gamma_{\mathbf{z}}(u)$ with $\frac{\mathrm{d}}{\mathrm{d}t}\Upsilon_{u,\omega}|_{t=u} = L_\omega$ and $\Upsilon^0_{u,\omega}(t) = t$. Specifically, $\Upsilon_{u,\omega}(t)$ is constructed by solving the following "geodesic" ODE system[t]:

$$\begin{aligned}\ddot{\Upsilon}^\alpha_{u,\omega}(t) &= -\mathbf{\Gamma}^\alpha_{\kappa\lambda}|_{\Upsilon_{u,\omega}(t)}\dot{\Upsilon}^\kappa_{u,\omega}(t)\dot{\Upsilon}^\lambda_{u,\omega}(t) + \frac{1}{2}[\mathcal{L}_{\mathbf{T}}\mathbf{g}]_{\kappa\lambda}|_{\Upsilon_{u,\omega}(t)}(\dot{\Upsilon}^\kappa_{u,\omega}(t)\\ &\quad - \mathbf{T}^\kappa|_{\Upsilon_{u,\omega}(t)})(\dot{\Upsilon}^\lambda_{u,\omega}(t) - \mathbf{T}^\lambda|_{\Upsilon_{u,\omega}(t)})\dot{\Upsilon}^\alpha_{u,\omega}(t), \end{aligned} \tag{7.4a}$$

$$\Upsilon^\alpha_{u,\omega}(u) = \gamma^\alpha_{\mathbf{z}}(u), \quad \dot{\Upsilon}^\alpha_{u,\omega}(u) = L^\alpha_\omega, \tag{7.4b}$$

where $\ddot{\Upsilon}^\alpha_{u,\omega} := \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Upsilon^\alpha_{u,\omega}$, $\dot{\Upsilon}^\alpha_{u,\omega} := \frac{\mathrm{d}}{\mathrm{d}t}\Upsilon^\alpha_{u,\omega}$, $\mathbf{\Gamma}$ is the Christoffel symbol of $\mathbf{g}$ and $\mathcal{L}_{\mathbf{T}}\mathbf{g}$ is the Lie derivative of $\mathbf{g}$ with respect to $\mathbf{T}$. The curve $t\to\Upsilon_{u,\omega}(t)$ is a non-affinely parametrized null geodesic such that the vectorfield $L^\alpha_{u,\omega} := \frac{\mathrm{d}}{\mathrm{d}t}\Upsilon^\alpha_{u,\omega}$ is null and normalized by $L^0_{u,\omega} = 1$. In fact, this vectorfield coincides (in the interior region) with the vectorfield L defined in (7.12). We define the truncated null cone $\mathcal{C}_u$ to be $\mathcal{C}_u := \bigcup_{\omega\in\mathbb{S}^2, t\in[u,T_{*;(\lambda)}]}\Upsilon_{u,\omega}(t)$. We define the acoustical function u by asserting that its level sets $\{u = u'\}$ are $\mathcal{C}_{u'}$. For $u\in[0,T_{*;(\lambda)}]$ and $t\in[u,T_{*;(\lambda)}]$, we let $S_{t,u} := \mathcal{C}_u\cap\Sigma_t$. For $u\neq t$, $S_{t,u}$ is a smooth surface diffeomorphic to $\mathbb{S}^2$. We define $\mathcal{M}^{(\mathrm{Int})} := \bigcup_{t\in[0,T_{*;(\lambda)}], 0\le u\le t} S_{t,u}$. For each $\omega\in\mathbb{S}^2$ and $u\in[0,T_{*;(\lambda)}]$, we define the angular coordinate functions $\{\omega^A\}_{A=1,2}$ to be constant along each fixed null geodesics $\Upsilon_{u,\omega}(t)$ and to coincide with standard angular coordinates on $\mathbb{S}^2$ at the tip $\mathbf{p}$, which corresponds to $t = u$.

The exterior solution *u*. Now we extend the foliation of space-time by null hypersurfaces to a neighborhood of $\bigcup_{t\in[0,T_{*;(\lambda)}], 0\le u\le t} S_{t,u}$ in $[0,T_{*;(\lambda)}]\times\mathbb{R}^3$. Let $w_* = \frac{4}{5}T_{*;(\lambda)}$. Using the arguments in [25], we can guarantee[u] that there is a

[t]See [11, Sec. 9.4.1] for detailed explanation of this ODE system where $\mathbf{T}$ coincide with $\mathbf{B}$.
[u]The existence of w-foliation for $w\in[0,\varepsilon]$ with a small $\varepsilon>0$ can be proven by Nash–Moser implicit function theorem and such foliation can be extended to w_* by an argument of continuity (see [25]).

neighborhood $\mathfrak{O} \in \Sigma_0$ contained in the geodesic ball $B_{T_{*;(\lambda)}}(\mathbf{z}, g_{(\lambda)})$, where $g_{(\lambda)}$ is the frequency rescaled induced metric of $\mathbf{g}$ on Σ_0 defined in (6.8b), such that $\mathfrak{O}$ can be foliated by the level sets $S_{0,-w}$ of a positive function w taking all values in $[0, w_*]$ with $w(\mathbf{z}) = 0$ and each $S_{0,-w}$ for positive w is diffeomorphic to $\mathbb{S}^2$. Fix a diffeomorphism $\omega \to \Phi_\omega(w_*)$ from $\mathbb{S}^2$ to $S_{0,-w_*}$. Then for each point $\mathbf{p} = \Phi_\omega(w_*)$, denoting the lapse $a := ((g^{-1})^{cd}\partial_c w \partial_d w)^{-\frac{1}{2}}$ with $a(\mathbf{z}) = 1$ and $a \approx 1$; see Proposition 8.8 (we note that the proof of Proposition 8.8 is independent of the construction of u), there is a unique integral curve[v] $w \to \Phi_\omega(w)$ of the vectorfield $a^2(g^{-1})^{ic}\partial_c w$ in Σ_0 with $\Phi_\omega(w_*) = \mathbf{p}$ and such that this integral curve can be extended to $\mathbf{z}$, i.e. $\Phi_\omega(0) = \mathbf{z}$ (the extendibility of Φ_ω to $\mathbf{z}$ follows by estimates (8.27h) and the fundamental theorem of calculus). Denoting $\dot{\Phi}_\omega := \frac{\mathrm{d}}{\mathrm{d}w}\Phi_\omega$, we then define $N_\omega|_{\mathbf{z}} := \dot{\Phi}_\omega(0)$ and $L_\omega|_{\mathbf{z}} := N_\omega|_{\mathbf{z}} + \mathbf{T}|_{\mathbf{z}}$. Note that the diffeomorphism $\omega \to L_\omega|_{\mathbf{z}}$ is uniquely determined by the vectorfield $a^2(g^{-1})^{ic}\partial_c w$, and it is precisely this diffeomorphism that appears in our construction of the interior solution described above, since $\langle N_\omega|_{\mathbf{z}} + \mathbf{T}|_{\mathbf{z}}, \mathbf{T}|_{\mathbf{z}}\rangle = -1$ and $\langle N_\omega|_{\mathbf{z}} + \mathbf{T}|_{\mathbf{z}}, N_\omega|_{\mathbf{z}} + \mathbf{T}|_{\mathbf{z}}\rangle = 0$ (because of $a(\mathbf{z}) = 1$). By such construction, for each $w \in (0, w_*]$ and any $\mathbf{p} \in S_{0,-w}$, there exists a $\omega \in \mathbb{S}^2$ such that $\mathbf{p} = \Phi_\omega(w)$ and such that the outward unit normal (in Σ_0) to $S_{0,-w}$ at $\mathbf{p}$ is $N_{w,\omega} := \dot{\Phi}_\omega(w)$. We set $L_{w,\omega} := N_{w,\omega} + \mathbf{T}|_{\Phi_\omega(w)}$, which is a null vector in $T_{\mathbf{p}}\mathcal{M}$. Then with $u = -w$, there is a unique (non-affinely parametrized) null geodesic $\Upsilon_{u,\omega}$ emanating from $\mathbf{p}$ and solving (7.4a) with the initial condition $\frac{\mathrm{d}}{\mathrm{d}t}\Upsilon_{u,\omega}|_{t=0} = L_{-u,\omega}$ and $\Upsilon_{u,\omega}(0) = \mathbf{p}$. We define the null cone $\mathcal{C}_u$ to be $\mathcal{C}_u := \bigcup_{\omega \in \mathbb{S}^2, t \in [0, T_{*;(\lambda)}]} \Upsilon_{u,\omega}(t)$. We define the acoustical function u by asserting that its level sets $\{u = u'\}$ are $\mathcal{C}_{u'}$. Let $S_{t,u} := \mathcal{C}_u \cap \Sigma_t$. We note that $\mathcal{C}_u$'s are the outgoing truncated null cones, that is, $\mathcal{C}_u := \bigcup_{t \in [0, T_{*;(\lambda)}]} S_{t,u}$. We define $\mathcal{M}^{(\mathrm{Ext})} := \bigcup_{t \in [0, T_{*;(\lambda)}], u \in [-w_*, 0)} S_{t,u}$. For each $\omega \in \mathbb{S}^2$ and $u \in [-w_*, 0)$, we define the angular coordinate functions $\{\omega^A\}_{A=1,2}$ to be constant along null geodesics $\Upsilon_{u,\omega}(t)$ and to coincide with the angular coordinates $\{\omega^A\}_{A=1,2}$ on Σ_0 provided by the above construction; note that on $\Sigma_0 \backslash \{\mathbf{z}\}$, by construction, the angular coordinate functions $\{\omega^A\}_{A=1,2}$ are constant along the integral curves of the vectorfield $a^2(g^{-1})^{ic}\partial_c w$.

We define the space-time region $\mathcal{M}$ as follows:

$$\mathcal{M} = \mathcal{M}^{(\mathrm{Int})} \cup \mathcal{M}^{(\mathrm{Ext})}. \tag{7.5}$$

By the constructions above, we have constructed the geometric coordinates (t, u, ω^A) in $\mathcal{M}$.

See Fig. 1 for the figure of the geometry.

7.2. *Geometric quantities*

Definition 7.2 (The radial variable). Recall that

$$0 \leq T_{*;(\lambda)} \leq \lambda^{1-8\varepsilon_0} T_*. \tag{7.6}$$

[v] The existence and uniqueness of such integral curve are ensured by the estimates on Σ_0 in Proposition 8.8. We refer readers to [11, Sec. 9.4.2] for details.

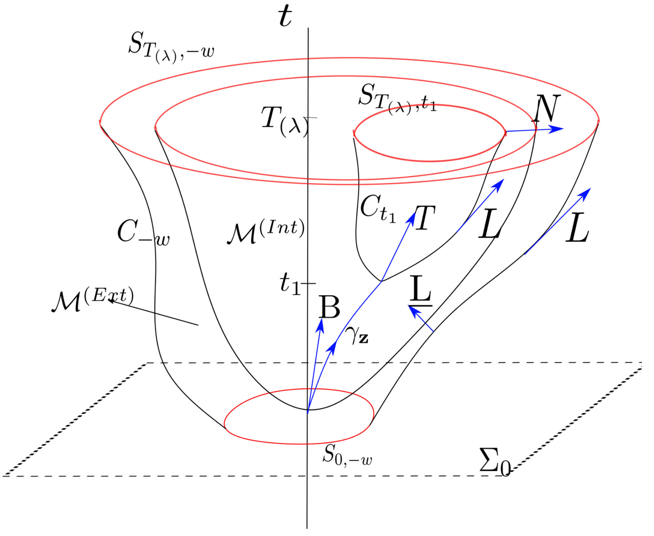


Fig. 1. The geometric construction out of acoustical function.

We define the geometric radial variable $\tilde{r}$ as follows:

$$\tilde{r} := t - u. \tag{7.7}$$

Since we have that $t \in [0, T_{*;(\lambda)}]$ and $u \in [-w_*, t]$ in $\mathcal{M}$, where $w_* := \frac{4}{5}T_{*;(\lambda)}$, we have

$$0 \leq \tilde{r} < 2T_{*;(\lambda)} = 2\lambda^{1-8\varepsilon_0}T_*, \quad -\frac{4}{5}\lambda^{1-8\varepsilon_0}T_* \leq u \leq \lambda^{1-8\varepsilon_0}T_*. \tag{7.8}$$

We will silently use estimates (7.8) throughout the paper.

Definition 7.3 (Acoustic vectorfields and scalar functions). We define the null vectorfield

$$L_{(\mathrm{Geo})} := -\mathbf{g}^{\alpha\beta}\partial_\beta u \partial_\alpha. \tag{7.9}$$

Note that by (7.3), we have $\mathbf{D}_{L_{(\mathrm{Geo})}} L_{(\mathrm{Geo})} = 0$.

We define the null lapse b as follows:

$$b := (\sqrt{g^{ij}\partial_i u \partial_j u})^{-1}. \tag{7.10}$$

We define the vectorfield N as follows:

$$N := -bg^{ij}\partial_i u \partial_j. \tag{7.11}$$

Note that $N|_{\Sigma_0} = N_\omega$, where N_ω is define in Sec. 7.1.

We define the principal null vectorfields L and $\underline{L}$ as follows:

$$L := \mathbf{T} + N, \quad \underline{L} := \mathbf{T} - N. \tag{7.12}$$

Note that by Definition 2.12 and (7.3), we have identity $\mathbf{T}u = |\nabla u|_g = \frac{1}{b}$. Then we have

$$L = bL_{(\mathrm{Geo})}. \tag{7.13}$$

We have following basic properties:

$$\langle \mathbf{T}, \mathbf{T} \rangle = -1, \quad \langle N, N \rangle = 1, \tag{7.14a}$$

$$\langle \mathbf{T}, N \rangle = 0, \quad \langle L, L \rangle = 0, \tag{7.14b}$$

$$\langle L, \underline{L} \rangle = -2, \quad \langle \underline{L}, \underline{L} \rangle = 0. \tag{7.14c}$$

We define $\not{g}$ to be as follows:

$$(\not{g}^{-1})^{\alpha\beta} := \mathbf{g}^{\alpha\beta} + \frac{1}{2} L^\alpha \underline{L}^\beta + \frac{1}{2} \underline{L}^\alpha L^\beta. \tag{7.15}$$

It is easy to check that $\not{g}$ is an induced metric of $\mathbf{g}$ on $S_{t,u}$. We denote a pair of unit orthogonal spherical vectorfields[w] on $S_{t,u}$ by $\{e_A\}_{A=1,2}$ such that $(\not{g}^{-1})^{\alpha\beta} = \sum_{A=1,2} e_A^\alpha e_A^\beta$. We call $L, \underline{L}, e_1, e_2$ a null frame for the geometry.

We denote the Levi-Civita connection on $S_{t,u}$ with respect to $\not{g}$ by $\not{\nabla}$.

As we discussed in Sec. 7.1, the angular coordinate functions $\{\omega^A\}_{A=1,2}$ satisfy the equation $L(\omega^A) = 0$ along null cones. By this construction of angular coordinates, with respect to geometric coordinates (t, u, ω^A), we have

$$L = \frac{\partial}{\partial t}, \quad N = -b^{-1} \frac{\partial}{\partial u} + Y^A \frac{\partial}{\partial \omega^A}. \tag{7.16}$$

By construction in Sec. 7.1.2, $Y^A = 0$ on Σ_0.

Definition 7.4 ($S_{t,u}$-tangent tensorfields). We define the $\mathbf{g}$-orthogonal projection $\not{\Pi}$ onto $S_{t,u}$, where δ^α_β is the Kronecker delta, as follows:

$$\not{\Pi}^\alpha_\beta := \delta^\alpha_\beta + \frac{1}{2} L^\alpha \underline{L}_\beta + \frac{1}{2} \underline{L}^\alpha L_\beta. \tag{7.17}$$

We use the notation $|\xi|_{\not{g}}$ to denote the norm of the $S_{t,u}$-tangent tensorfield ξ with respect to $\not{g}$, that is

$$|\xi|^2_{\not{g}} := \not{g}(\xi, \xi). \tag{7.18}$$

We use $\mathrm{tr}_{\not{g}} \xi$ to denote the trace of a $\binom{0}{2}$ $S_{t,u}$-tangent tensorfield ξ with respect to $\not{g}$

$$\mathrm{tr}_{\not{g}} \xi := \not{g}^{AB} \xi_{AB}. \tag{7.19}$$

We define $\hat{\xi}$ to be the trace-free part of the $\binom{0}{2}$ $S_{t,u}$-tangent tensorfield ξ

$$\hat{\xi} := \xi - \frac{1}{2} (\mathrm{tr}_{\not{g}} \xi) \not{g}. \tag{7.20}$$

[w]In the rest of the paper, we automatically sum over A if there are two A's in the expression.

7.3. *Conformal energy*

We setup the conformal energy method that was introduced by Wang in [30]. In order to give the definition of our conformal energy, we define two smooth cut-off functions $\underline{\varpi}$ and ϖ that depend only on t and u as follows:

$$\underline{\varpi} = \begin{cases} 1 & \text{on } 0 \le u \le t, \\ 0 & \text{on } u \le -\dfrac{t}{4}, \end{cases} \qquad \varpi = \begin{cases} 1 & \text{on } 0 \le u \le \dfrac{1}{2}t, \\ 0 & \text{if } u \ge \dfrac{3}{4}t \text{ or } u \le -\dfrac{t}{4}. \end{cases}$$

Definition 7.5 ([30, Sec. 4, Conformal energy]). For any scalar φ vanishing outside $\mathcal{M}^{(\mathrm{Int})}$, we define the conformal energy $\mathfrak{C}[\varphi]$as follows:

$$\mathfrak{C}[\varphi](t) = \mathfrak{C}^{(i)}[\varphi](t) + \mathfrak{C}^{(e)}[\varphi](t), \tag{7.21}$$

where

$$\mathfrak{C}^{(i)}[\varphi](t) = \int_{\Sigma_t} (\underline{\varpi} - \varpi) t^2 (|\mathbf{D}\varphi|^2 + |\tilde{r}^{-1}\varphi|^2) \mathrm{d}\varpi_g, \tag{7.22a}$$

$$\mathfrak{C}^{(e)}[\varphi](t) = \int_{\Sigma_t} \varpi (\tilde{r}^2 |\mathbf{D}_L \varphi|^2 + \tilde{r}^2 |\nabla\!\!\!/ \varphi|^2 + |\varphi|^2) \mathrm{d}\varpi_g. \tag{7.22b}$$

The main result of $\mathfrak{C}[\varphi](t)$ is the following.

Theorem 7.6 (Boundedness theorem). *Let φ be any solution of $\Box_{\mathbf{g}}\varphi = 0$ on $[0, T_{*;(\lambda)}] \times \mathbb{R}^3$ with $\varphi(1)$ supported in $B_R \subset \mathcal{M}^{(\mathrm{Int})} \cap \Sigma_1$. Then under bootstrap assumptions, for $t \in [1, T_{*;(\lambda)}]$, the conformal energy of φ satisfies the estimate*

$$\mathfrak{C}[\varphi](t) \lesssim (1+t)^{2\varepsilon} (\|\boldsymbol{\partial}\varphi\|^2_{L^2_x(\Sigma_1)} + \|\varphi\|^2_{L^2_x(\Sigma_1)}), \tag{7.23}$$

where $\varepsilon > 0$ is an arbitrary small number.

7.3.1. *Reduction of Theorem* 6.9

The proof of Theorem 6.9 by using Theorem 7.6 is done via product estimates and the Bernstein inequality of Littlewood–Paley theory. We refer reader to [16, Sec. 8; 30, Sec. 4] for the detailed proof.

7.3.2. *Discussion of the proof of Theorem* 7.6

Logically speaking, the proof of Theorem 7.6 relies on the control of the acoustic geometry derived in Proposition 8.10. However, Proposition 8.10 is logically independent of Theorem 7.6, so here we discuss the proof of Theorem 7.6, assuming that we have already controlled the acoustic geometry. In order to obtain an estimate for the conformal energy, we choose Θ to be as follows:

$$\Theta := \tilde{r}^{-1} f := \tilde{r}^{-1}\left(\beta - \frac{\beta}{(1+\tilde{r})^{\alpha}}\right), \tag{7.24}$$

where $\beta\alpha = 2$ and $\tilde{r}$ is as in (7.7). We introduce the modified weighted energy

$$\tilde{Q}[\varphi](t) = \int_{\Sigma_t} {}^{(X)}\widetilde{\mathbf{J}}_\mu[\varphi]\mathbf{T}^\mu \mathrm{d}x, \tag{7.25}$$

where ${}^{(X)}\widetilde{\mathbf{J}}_\mu[\varphi]$ is the modified energy current

$${}^{(X)}\widetilde{\mathbf{J}}_\mu[\varphi] = Q_{\mu\nu}[\varphi]X^\mu + \frac{1}{2}\Theta\partial_\mu(\varphi^2) - \frac{1}{2}\varphi^2\partial_\mu\Theta. \tag{7.26}$$

By choosing $X = fN$ and using the divergence theorem for the modified current on an appropriate region, one can derive a Morawetz-type energy estimate. This provides the uniform bounds for the standard energy of φ along a union of a portion of the constant-time hypersurfaces and null cones.

Then we consider the conformal wave equation $\square_{\widetilde{\mathbf{g}}}(e^{-\sigma}\varphi) = \cdots$, where φ satisfies $\square_{\mathbf{g}}\varphi = 0$, σ and $\widetilde{\mathbf{g}}$ are defined in Definition 8.5. We use the multiplier approach with $\tilde{r}^p L$-type vectorfields in the region $\{\tau_1 \leq u \leq \tau_2\} \cap \{\tilde{r} \geq R\}$ where $1 \leq \tau_1 < \tau_2 < T_{*;(\lambda)}$ to control the conformal energy in the exterior region and to provide energy decay along null slices. Finally we control the conformal energy in the interior region with the help of the argument in [10], where energy decay is obtained in each spatial-null slice.

The proof of Theorem 7.6 closes the reduction of the Strichartz estimate. One could follow the steps listed in [11, Sec. 11] to obtain the estimates of conformal energy with the control of Ricci coefficients given in Sec. 8. One could go through the details of the argument in [30, Sec. 7]. Also reader could look into [15, Sec. 3] for initial ideas.

8. Energy along Acoustic Null Hypersurfaces and Control of the Acoustic Geometry

In this section, we prove the energy estimates along acoustic null hypersurfaces, which is necessary for obtaining the mixed-norm estimates in Proposition 8.10. Then we introduce the notation for many geometric quantities, followed by their bootstrap assumptions. Their improved estimates are in Sec. 8.4. The proof of estimates for geometric quantities is obtained by transport equation and div-curl estimates for the acoustic quantities, decomposition of Ricci curvature components, trace and Sobolev inequalities. We omit the proof of these estimates since they are the same as in [11, Sec. 10].

8.1. *Energy estimates along acoustic null hypersurfaces*

In this section, we define acoustic null fluxes and derive energy estimates along acoustic null hypersurfaces. These estimates are necessary for deriving the mixed-norm estimates for the acoustical function quantities in Proposition 8.10.

Definition 8.1 (Acoustic null fluxes). For functions φ defined on $\mathcal{C}_u$, we define the acoustic null fluxes $\mathcal{F}_{(\text{wave})}[\varphi;\mathcal{C}_u]$ and $\mathcal{F}_{(\text{transport})}[\varphi;\mathcal{C}_u]$ as follows:

$$\mathcal{F}_{(\text{wave})}[\varphi;\mathcal{C}_u] := \int_{\mathcal{C}_u} ((L\varphi)^2 + |\nabla\!\!\!/\varphi|^2_{g\!\!\!/})\mathrm{d}\varpi_{g\!\!\!/}\mathrm{d}t, \tag{8.1}$$

$$\mathcal{F}_{(\text{transport})}[\varphi;\mathcal{C}_u] := \int_{\mathcal{C}_u} \varphi^2\mathrm{d}\varpi_{g\!\!\!/}\mathrm{d}t, \tag{8.2}$$

where $\mathrm{d}\varpi_{g\!\!\!/}$ is the volume form of reduced metric $g\!\!\!/$ on the $S_{t,u}$-sphere from $\mathbf{g}$.

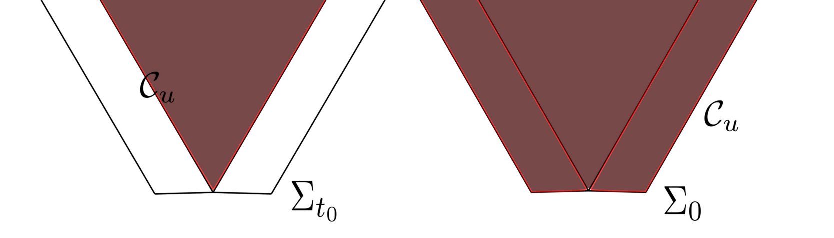
Proposition 8.2 (Energy estimates along acoustic null hypersurfaces). *Under the initial data, bootstrap assumptions and the standard energy estimates Proposition* 5.1, *we have the following estimates along null hypersurfaces* $\mathcal{C}_u$ *for* $u \in [-w_*, T_{*;(\lambda)}]$:

$$\mathcal{F}_{(\text{wave})}[\boldsymbol{\partial}\vec{\Psi};\mathcal{C}_u] + \sum_{\nu>1} \nu^{2(N-2)}\mathcal{F}_{(\text{wave})}[P_\nu\boldsymbol{\partial}\vec{\Psi};\mathcal{C}_u] \lesssim \lambda^{-1}, \tag{8.3}$$

$$\mathcal{F}_{(\text{transport})}[\boldsymbol{\partial}(\vec{\mathcal{C}},\mathcal{D});\mathcal{C}_u] + \sum_{\nu>1} \nu^{2(N-2)}\mathcal{F}_{(\text{transport})}[P_\nu\boldsymbol{\partial}(\vec{\mathcal{C}},\mathcal{D});\mathcal{C}_u] \lesssim \lambda^{-1}. \tag{8.4}$$

Proof of Proposition 8.2. We first prove (8.3). We apply the divergence theorem to the energy current ${}^{(\mathbf{T})}\mathbf{P}^\alpha[\varphi] := Q^{\alpha\beta}[\varphi]\mathbf{T}_\beta$, where $Q_{\alpha\beta}$ is the energy momentum tensor defined in (5.3), over the region bounded by $\mathcal{C}_u$, Σ_{t_0} and Σ_t where $t_0 := \max\{0, u\}$ (see Fig. 2 for the region that divergence theorem applied). Note that $\mathbf{D}_\alpha{}^{(\mathbf{T})}\mathbf{J}^\alpha[\varphi] = \square_{\mathbf{g}}\varphi(\mathbf{T}\varphi) + \frac{1}{2}Q^{\mu\nu}[\varphi]{}^{(\mathbf{T})}\boldsymbol{\pi}_{\mu\nu}$ where ${}^{(\mathbf{T})}\boldsymbol{\pi}_{\mu\nu}$ is define in (5.4). The same proof of Lemma 5.4 reveals the coercivity ${}^{(\mathbf{T})}\mathbf{P}^\alpha[\varphi] = Q^{00} \approx |\boldsymbol{\partial}\varphi|^2$. It is straightforward to check that ${}^{(\mathbf{T})}\mathbf{P}^\alpha[\varphi]L_\alpha = ((L\varphi)^2 + |\nabla\!\!\!/\varphi|^2_{g\!\!\!/})$. Thus we have

$$\begin{aligned}\mathcal{F}_{(\text{wave})}[\varphi;\mathcal{C}_u] = &\int_{\Sigma_t} |\boldsymbol{\partial}\varphi|^2\mathrm{d}\varpi_g - \int_{\Sigma_{t_0}} |\boldsymbol{\partial}\varphi|^2\mathrm{d}\varpi_g \\ &+ \int_{\bigcup_{u'\geq u}\mathcal{C}_{u'}} \left(\square_{\mathbf{g}}\varphi(\mathbf{T}\varphi) + \frac{1}{2}Q^{\mu\nu}[\varphi]{}^{(\mathbf{T})}\boldsymbol{\pi}_{\mu\nu}\right)\mathrm{d}\varpi_{\mathbf{g}}.\end{aligned} \tag{8.5}$$



(a) When $t_0 > 0$. (b) When $t_0 = 0$.

Fig. 2. The regions that the divergence theorem is applied on.

Note that $\vec{\Psi}$ are rescaled quantities defined in Definition 6.3. By bootstrap assumptions (3.17b), we have $|\mathbf{T}\varphi| \lesssim |\boldsymbol{\partial}\varphi|$, $|Q^{\mu\nu}[\varphi]| \lesssim |\boldsymbol{\partial}\varphi|^2$ and $|{}^{(\mathbf{T})}\boldsymbol{\pi}_{\mu\nu}| \lesssim |\boldsymbol{\partial}\vec{\Psi}| + 1$. Next we substitute $\boldsymbol{\partial}\vec{\Psi}$ and $P_{\mathrm{v}}\boldsymbol{\partial}\vec{\Psi}$ for φ, and use Eqs. (3.8a) and (3.9a) to substitute $\square_{\mathbf{g}}\boldsymbol{\partial}\vec{\Psi}$ and $\square_{\mathbf{g}}P_{\mathrm{v}}\boldsymbol{\partial}\vec{\Psi}$. By Cauchy–Schwarz inequality, estimate $\int_{\bigcup_{u'\geq u}\mathcal{C}_{u'}} \square_{\mathbf{g}}\varphi(\mathbf{T}\varphi)\mathrm{d}\varpi_{\mathbf{g}} \leq \int_0^{T_{*;(\lambda)}}\int_{\Sigma_\tau} |\square_{\mathbf{g}}\varphi||\mathbf{T}\varphi|\mathrm{d}\varpi_g\mathrm{d}\tau$, and energy estimates in Proposition 5.1, we obtain the desired estimates.

To prove (8.4), for each $\beta = 0, 1, 2, 3$, we apply divergence theorem for energy fluxes ${}^{(\mathbf{B})}\mathbf{J}^\alpha := |\boldsymbol{\partial}\mathcal{C}^\beta|^2\mathbf{B}^\alpha$ over the region bounded by $\mathcal{C}_u$, Σ_{t_0} and Σ_t as follows:

$$
\begin{aligned}
-\int_{\mathcal{C}_u} |\boldsymbol{\partial}\mathcal{C}^\beta|^2\mathbf{B}^\alpha L_\alpha \mathrm{d}\varpi_{\not g}\mathrm{d}t = \int_{\Sigma_{t_0}} |\boldsymbol{\partial}\mathcal{C}^\beta|^2\mathbf{B}^\alpha\mathbf{T}_\alpha\mathrm{d}\varpi_g - \int_{\Sigma_t} |\boldsymbol{\partial}\mathcal{C}^\beta|^2\mathbf{B}^\alpha\mathbf{T}_\alpha\mathrm{d}\varpi_g \\
- \int_{\bigcup_{u'\geq u}\mathcal{C}_{u'}} \mathbf{D}_\alpha(|\boldsymbol{\partial}\mathcal{C}^\beta|^2\mathbf{B}^\alpha)\mathrm{d}\varpi_{\mathbf{g}}.
\end{aligned} \tag{8.6}
$$

By the fact that $\mathbf{B}^\alpha = \mathbf{T}^\alpha + \mathrm{f}(\vec{\Psi})$ and $\mathbf{B}$ is timelike, we have $\mathbf{B}^\alpha L_\alpha \approx -1$ and $\mathbf{B}^\alpha\mathbf{T}_\alpha \approx -1$. By commuted equation (3.8b), we have

$$
\begin{aligned}
\mathbf{D}_\alpha(|\boldsymbol{\partial}\mathcal{C}^\beta|^2\mathbf{B}^\alpha) &= 2|\boldsymbol{\partial}\mathcal{C}^\beta|\mathbf{B}(\boldsymbol{\partial}\mathcal{C}^\beta) + |\boldsymbol{\partial}\mathcal{C}^\beta|^2(\partial_\alpha\mathbf{B}^\alpha - \boldsymbol{\Gamma}_\beta\mathbf{B}^\beta) \\
&\lesssim |\boldsymbol{\partial}\mathcal{C}^\beta|\mathbf{B}(\boldsymbol{\partial}\mathcal{C}^\beta) + |\boldsymbol{\partial}\mathcal{C}^\beta|^2\|\boldsymbol{\partial}\vec{\Psi}\|_{L^\infty_x(\Sigma_\tau)} \\
&\lesssim |\boldsymbol{\partial}\mathcal{C}^\beta|\boldsymbol{\partial}^2\vec{\Psi}(\boldsymbol{\partial}\omega, \boldsymbol{\partial}S) + |\boldsymbol{\partial}\mathcal{C}^\beta|\boldsymbol{\partial}\vec{\Psi}\cdot(\boldsymbol{\partial}^2\vec{\Psi}, \boldsymbol{\partial}^2\omega, \boldsymbol{\partial}^2 S) \\
&\quad + (|\boldsymbol{\partial}\mathcal{C}^\beta|^2 + |\boldsymbol{\partial}\vec{\Psi}|^2)\|\boldsymbol{\partial}\vec{\Psi}\|_{L^\infty_x(\Sigma_\tau)},
\end{aligned} \tag{8.7}
$$

where $\boldsymbol{\Gamma}$ is the contracted Christoffel symbols of $\mathbf{g}$ defined as $\boldsymbol{\Gamma}_\alpha := \mathbf{g}^{\kappa\lambda}\Gamma_{\alpha\kappa\lambda} = \mathbf{g}^{\kappa\lambda}\mathbf{g}_{\alpha\beta}\Gamma^\beta_{\kappa\lambda}$. Using the rescaled bootstrap assumptions (6.12), energy estimates in Proposition 5.1, note that $\mathcal{C}, \mathcal{D}$ are rescaled quantities defined in Definition 6.3, we have

$$
\mathcal{F}_{(\text{transport})}[\boldsymbol{\partial}\vec{\mathcal{C}}; \mathcal{C}_u] \lesssim \lambda^{-1}. \tag{8.8}
$$

The proofs for $\boldsymbol{\partial}\mathcal{D}$ and $P_{\mathrm{v}}\boldsymbol{\partial}(\vec{\mathcal{C}}, \mathcal{D})$ are of the same fashion. □

Remark 8.3. We refer readers to [11, Proposition 6.1] for the energy estimate along null cones for 3D non-relativistic compressible Euler case where $\mathbf{B}$ coincides with $\mathbf{T}$. We note that in [11, 30], authors derive energy estimate along null cones before rescaling (with respect to λ). However, the results coincide with ours when energy estimates along null cones are rescaled. That is, our estimate is sufficient to obtain the mix-norm estimates for fluid variables, which serves the same as in [11, 30].

8.2. *Connection coefficients*

In this section, we define connection coefficients. We will derive estimates for them in Proposition 8.10 as a part of controlling the acoustic geometry.

8.2.1. *Levi-Civita connections, angular operators and curvatures*

If ξ is a space-time tensor, then $\not{\Pi}\xi$ denotes its $\mathbf{g}$-orthogonal projection onto $S_{t,u}$. Then we define $\not{\mathbf{D}}_V\xi := \not{\Pi}\mathbf{D}_V\xi$ where V is a vector and $\mathbf{D}_V\xi$ is the covariant derivative of ξ in the V direction. Note that $\not{\mathbf{D}}_V\xi = \not{\nabla}_V\xi$ when both V and ξ are $S_{t,u}$-tangent. Also we define $\not{\Delta} := \not{\nabla}\cdot\not{\nabla}$

We let $\mathbf{Riem}_{\alpha\beta\gamma\delta}$ denote the Riemann curvature tensor of $\mathbf{g}$ and $\mathbf{Ric}_{\alpha\beta} := \mathbf{g}^{\gamma\delta}\mathbf{Riem}_{\gamma\alpha\delta\beta}$. We use the notation that $\langle\mathbf{D}_X\mathbf{D}_Y W - \mathbf{D}_Y\mathbf{D}_X W, Z\rangle = \mathbf{Riem}_{ZWXY} + \langle\mathbf{D}_{[X,Y]}W, Z\rangle$ where X, Y, W, Z are vectorfields, $[\cdot,\cdot]$ is the Lie bracket and $\langle\cdot,\cdot\rangle := \mathbf{g}(\cdot,\cdot)$.

Definition 8.4 (Connection coefficients). We define the second fundamental form k of Σ_t to be the $\binom{0}{2}$ Σ_t-tangent tensor such that the following relation holds for all Σ_t-tangent vectorfields X and Y:

$$k(X,Y) := -\mathbf{g}(\mathbf{D}_X\mathbf{T}, Y). \tag{8.9}$$

Let $\{e_A\}_{A=1,2}$ be the pair of unit orthogonal spherical vectorfields on $S_{t,u}$ from Definition 7.3 . We define the second fundamental form θ of $S_{t,u}$, the null second fundamental form χ of $S_{t,u}$, and $\underline{\chi}$ to be the following type $\binom{0}{2}$ $S_{t,u}$-tangent tensors:

$$\theta_{AB} := \mathbf{g}(\mathbf{D}_A N, e_B), \tag{8.10a}$$

$$\chi_{AB} := \mathbf{g}(\mathbf{D}_A L, e_B), \quad \underline{\chi}_{AB} := \mathbf{g}(\mathbf{D}_A \underline{L}, e_B). \tag{8.10b}$$

We define the torsion ζ and $\underline{\zeta}$ to be the following $S_{t,u}$-tangent one-forms:

$$\zeta_A := \frac{1}{2}\mathbf{g}(\mathbf{D}_{\underline{L}} L, e_A), \quad \underline{\zeta}_A := \frac{1}{2}\mathbf{g}(\mathbf{D}_L \underline{L}, e_A). \tag{8.11}$$

8.3. *Conformal metric, initial conditions on Σ_0 and on the cone-tip axis for the acoustical function u*

Definition 8.5 (Conformal factor and the modified null second fundamental form and acoustical quantities). In interior region $\mathcal{M}^{(\mathrm{Int})}$, we define σ to be the solution to the following transport initial value problem:

$$L\sigma(t,u,\omega) = \frac{1}{2}[\mathbf{\Gamma}_L](t,u,\omega), \quad u \in [0, T_{*;(\lambda)}], \ \ t \in [u, T_{*;(\lambda)}], \ \ \omega \in \mathbb{S}^2, \tag{8.12a}$$

$$\sigma(u,u,\omega) = 0, \quad u \in [0, T_{*;(\lambda)}], \omega \in \mathbb{S}^2, \tag{8.12b}$$

where $\mathbf{\Gamma}_L := \mathbf{\Gamma}_\alpha L^\alpha$ and $\mathbf{\Gamma}_\alpha := (\mathbf{g}^{-1})^{\kappa\lambda}\mathbf{\Gamma}_{\alpha\kappa\lambda} = (\mathbf{g}^{-1})^{\kappa\lambda}\mathbf{g}_{\alpha\beta}\mathbf{\Gamma}^{\beta}_{\kappa\lambda}$.

We define the conformal space-time metric and the Riemannian metric that induces on $S_{t,u}$ as follows:

$$\widetilde{\mathbf{g}} := e^{2\sigma}\mathbf{g}, \quad \widetilde{g\!\!/} := e^{2\sigma}g\!\!/. \tag{8.13}$$

We define the null second fundamental forms of the metric to be the following symmetric $S_{t,u}$-tangent tensors:

$$\tilde{\chi} := \frac{1}{2}\mathcal{L}\!\!\!/_L\widetilde{g\!\!/}, \quad \tilde{\underline{\chi}} := \frac{1}{2}\mathcal{L}\!\!\!/_{\underline{L}}\widetilde{g\!\!/}. \tag{8.14}$$

Using (8.13) and (8.14), it follows that $\chi, \underline{\chi}$ and $\tilde{\chi}, \tilde{\underline{\chi}}$ are related by

$$\tilde{\chi} = e^{2\sigma}(\chi + (L\sigma)g\!\!/), \quad \tilde{\underline{\chi}} = e^{2\sigma}(\underline{\chi} + (\underline{L}\sigma)g\!\!/), \tag{8.15a}$$

$$\mathrm{tr}_{\widetilde{g\!\!/}}\tilde{\chi} = \mathrm{tr}_{g\!\!/}\chi + 2L\sigma = \mathrm{tr}_{g\!\!/}\chi + \mathbf{\Gamma}_L, \quad \mathrm{tr}_{\widetilde{g\!\!/}}\tilde{\underline{\chi}} = \mathrm{tr}_{g\!\!/}\underline{\chi} + 2\underline{L}\sigma, \tag{8.15b}$$

$$\chi = \frac{1}{2}\left(\mathrm{tr}_{\widetilde{g\!\!/}}\tilde{\chi} - \mathbf{\Gamma}_L\right)g\!\!/ + \hat{\chi}, \quad \underline{\chi} = \frac{1}{2}(\mathrm{tr}_{\widetilde{g\!\!/}}\tilde{\underline{\chi}} - 2\underline{L}\sigma)g\!\!/ + \hat{\underline{\chi}}. \tag{8.15c}$$

We define the following:

$$\mathrm{tr}_{\widetilde{g\!\!/}}\tilde{\chi}^{(\mathrm{Small})} := \mathrm{tr}_{g\!\!/}\chi + \mathbf{\Gamma}_L - \frac{2}{\tilde{r}} = \mathrm{tr}_{\widetilde{g\!\!/}}\tilde{\chi} - \frac{2}{\tilde{r}}. \tag{8.16}$$

We note that the first equality in (8.16) holds in the whole region $\mathcal{M}$, while the second equality holds only in the interior region $\mathcal{M}^{(\mathrm{Int})}$.

In the following definition, we define mass aspect function μ and its modified version $\check{\mu}$, as well as modified torsion $\tilde{\zeta}$. These objects are defined to avoid loss of regularity of the acoustical eikonal function.

Definition 8.6 (Mass aspect function). We define the mass aspect function μ to be the following scalar function:

$$\mu := \underline{L}\mathrm{tr}_{g\!\!/}\chi + \frac{1}{2}\mathrm{tr}_{g\!\!/}\chi\mathrm{tr}_{g\!\!/}\underline{\chi}. \tag{8.17}$$

We now define the modified mass aspect function $\check{\mu}$ to be the following scalar function:

$$\check{\mu} := 2\Delta\!\!\!/\sigma + \underline{L}\mathrm{tr}_{g\!\!/}\chi + \frac{1}{2}\mathrm{tr}_{g\!\!/}\chi\mathrm{tr}_{g\!\!/}\underline{\chi} - \mathrm{tr}_{g\!\!/}\chi k_{NN} + \frac{1}{2}\mathrm{tr}_{g\!\!/}\chi\mathbf{\Gamma}_{\underline{L}}. \tag{8.18}$$

In $\mathcal{M}^{(\mathrm{Int})}$, we define $\mu\!\!\!/$ to be the $S_{t,u}$-tangent one-form that satisfies the following Hodge system on $S_{t,u}$:

$$\mathrm{di\!\!/v}\,\mu\!\!\!/ = \frac{1}{2}(\check{\mu} - \bar{\check{\mu}}), \quad \mathrm{cu\!\!/rl}\,\mu\!\!\!/ = 0. \tag{8.19}$$

In $\mathcal{M}^{(\mathrm{Int})}$, we define the modified torsion $\tilde{\zeta}$ to be the following $S_{t,u}$-tangent one-form:

$$\tilde{\zeta} := \zeta + \nabla\!\!\!/\sigma. \tag{8.20}$$

Definition 8.7 (Norms of $S_{t,u}$-tangent tensorfields). Let $e\!\!\!/ = e\!\!\!/(\omega)$ be the canonical metric on $\mathbb{S}^2$ and $\{\omega^A\}_{A=1,2}$ are local angular coordinates on $\mathbb{S}^2$. We can also view $\{\omega^A\}_{A=1,2}$ on $S_{t,u}$ as the image of the canonical isomorphism from $\mathbb{S}^2$ to $S_{t,u}$. For $p \in [1, \infty)$, We define the Lebesgue norms L^p_ω and $L^p_{\not g}$ for $S_{t,u}$-tangent tensorfields ξ as follows:

$$\|\xi\|_{L^p_\omega(S_{t,u})} := \left(\int_{\omega \in \mathbb{S}^2} |\xi|^p_{\not g} \mathrm{d}\varpi_{\not e}\right)^{1/p}, \quad \|\xi\|_{L^p_{\not g}(S_{t,u})} := \left(\int_{\omega \in \mathbb{S}^2} |\xi|^p_{\not g} \mathrm{d}\varpi_{\not g}\right)^{1/p}. \tag{8.21}$$

Since we can have a parallel transport along $\mathbb{S}^2$ that preserves the tensor products and contractions, that is, $\Phi^m_n(\omega_{(2)}; \omega_{(1)})$ is the parallel transport with respect to $\not e$ from the vector space of type $\binom{m}{n}$ tensors at $\omega_{(2)} \in \mathbb{S}^2$ to the vector space of type $\binom{m}{n}$ tensors at $\omega_{(1)} \in \mathbb{S}^2$, for $\xi = \xi(\omega)$ a type $\binom{m}{n}$ tensorfield on $\mathbb{S}^2$, we define the L^∞ norm L^∞_ω and Hölder norm $C^{0,\alpha}_\omega$ of ξ as follows:

$$\|\xi\|_{L^\infty_\omega(S_{t,u})} := \operatorname*{ess\,sup}_{\omega \in \mathbb{S}^2} |\xi|_{\not g}, \tag{8.22}$$

$$\begin{aligned}\|\xi\|_{C^{0,\alpha}_\omega(S_{t,u})} := \|\xi\|_{L^\infty_\omega(S_{t,u})} &+ \sup_{0 < d_{\not e}(\omega_{(1)}, \omega_{(2)}) < \frac{\pi}{2}} \\ &\times \frac{|\xi(t,u,\omega_{(1)}) - \Phi^m_n(\omega_{(2)}; \omega_{(1)}) \circ \xi(t,u,\omega_{(2)})|_{\not g(t,u,\omega_{(1)})}}{d^\alpha_{\not e}(\omega_{(1)}, \omega_{(2)})}.\end{aligned} \tag{8.23}$$

In the following two propositions, we list the estimates of the initial foliation. These estimates are crucial for the well-defined geometric setup in Sec. 7.1 and controlling the acoustic geometry.

Proposition 8.8 ([11, Proposition 9.8; 30, Proposition 4.3, Existence and properties of the initial foliation]). *On Σ_0, there exists a function $w = w(x)$ on the domain $0 \le w \le w_* := \frac{4}{5} T_{*;(\lambda)}$, such that $w(\mathbf{z}) = 0$ and each level set $S_{0,w}$ is diffeomorphic to $\mathbb{S}^2$ and*

$$\mathrm{tr}_{\not g}\theta + k_{NN} = \frac{2}{aw} + \mathrm{tr}_g k - \mathbf{\Gamma}_L, \quad a(\mathbf{z}) = 1. \tag{8.24}$$

By (8.16), $\tilde{r}(0, -u) = w$, *and the fact that* $\chi_{AB} = \theta_{AB} - k_{AB}$, (8.24) *is equivalent to*

$$\mathrm{tr}_{\tilde{\not g}} \tilde{\chi}^{(\mathrm{Small})}|_{\Sigma_0} = \frac{2(1-a)}{aw} \quad \textit{for } 0 \le w \le w_*, \tag{8.25}$$

where we define the lapse a as follows:

$$a = (\sqrt{g^{cd}\partial_c w \partial_d w})^{-1}. \tag{8.26}$$

Note that $\frac{\partial}{\partial w} = aN|_{\Sigma_0}$. *Then on* $\Sigma_0^{w(\lambda)} := \bigcup_{0 \le w \le w_*} S_{0,w}$, *for* $0 < 1 - \frac{2}{q_*} < N - 2$, *there hold*

$$|a-1| \lesssim \lambda^{-4\varepsilon_0} \le \frac{1}{4}, \quad \|w^{-1/2}(a-1)\|_{L_w^\infty C_\omega^{0,1-\frac{2}{q_*}}(\Sigma_0^{w(\lambda)})} \lesssim \lambda^{-1/2},$$
$$\upsilon := \frac{\sqrt{\det \not{g}}}{\sqrt{\det \not{e}}} \approx w^2. \tag{8.27a}$$

$$\|w^{\frac{1}{2}-\frac{2}{q_*}}(\hat{\theta}, \nabla\!\!\!/ \ln a)\|_{L_w^\infty L_{\not{g}}^{q_*}(\Sigma_0^{w(\lambda)})}, \|\nabla\!\!\!/ \ln a\|_{L_w^2 L_\omega^\infty(\Sigma_0^{w(\lambda)})}, \|\hat{\chi}\|_{L_w^2 L_\omega^\infty(\Sigma_0^{w(\lambda)})} \lesssim \lambda^{-1/2}. \tag{8.27b}$$

$$\max_{A,B=1,2} \left\| w^{-2}\not{g}\left(\frac{\partial}{\partial \omega^A}, \frac{\partial}{\partial \omega^B}\right) - \not{e}\left(\frac{\partial}{\partial \omega^A}, \frac{\partial}{\partial \omega^B}\right) \right\|_{L_x^\infty(\Sigma_0^{w(\lambda)})} \lesssim \lambda^{-4\varepsilon_0}, \tag{8.27c}$$

$$\max_{A,B,C=1,2} \left\| \frac{\partial}{\partial \omega^C}\left(w^{-2}\not{g}\left(\frac{\partial}{\partial \omega^A}, \frac{\partial}{\partial \omega^B}\right) - \not{e}\left(\frac{\partial}{\partial \omega^A}, \frac{\partial}{\partial \omega^B}\right)\right) \right\|_{L_\omega^{q_*}(S_{0,w})} \lesssim \lambda^{-4\varepsilon_0}, \tag{8.27d}$$

$$\|w^{\frac{1}{2}-\frac{2}{q_*}} \nabla\!\!\!/ \ln\left(\tilde{r}^{-2}\upsilon\right)\|_{L_{\not{g}}^{q_*}(S_{0,w})} \lesssim \lambda^{-1/2}. \tag{8.27e}$$

In addition

$$\|w \mathrm{tr}_{\tilde{\not{g}}}\tilde{\chi}^{(\mathrm{Small})}\|_{L_x^\infty(\Sigma_0^{w(\lambda)})} \lesssim \lambda^{-4\varepsilon_0}, \tag{8.27f}$$

$$\|w^{3/2}\nabla\!\!\!/ \mathrm{tr}_{\tilde{\not{g}}}\tilde{\chi}^{(\mathrm{Small})}\|_{L_w^\infty L_\omega^p(\Sigma_0^{w(\lambda)})} + \|w^{1/2}\mathrm{tr}_{\tilde{\not{g}}}\tilde{\chi}^{(\mathrm{Small})}\|_{L_w^\infty C_\omega^{0,1-\frac{2}{p}}(\Sigma_0^{w(\lambda)})} \lesssim \lambda^{1/2}. \tag{8.27g}$$

Finally

$$\sum_{i,j=1,2,3} |w \Pi\!\!\!/_j^a \partial_a N^i - \Pi\!\!\!/_j^i| = \mathcal{O}(w) \quad \text{as } w \downarrow 0. \tag{8.27h}$$

Proposition 8.9 ([11, Lemma 9.9, Initial conditions on the cone-tip axis tied to the acoustical function]). *On any null cone* $\mathcal{C}_u$ *initiating from a point on the time axis* $0 \le t = u \le T_{*;(\lambda)}$ *there hold*

$$\begin{aligned}
&\mathrm{tr}_{\not{g}}\chi - \frac{2}{\tilde{r}}, \tilde{r}\mathrm{tr}_{\tilde{\not{g}}}\tilde{\chi}^{(\mathrm{Small})}, |\hat{\chi}|_{\not{g}}, |\tilde{r}\Pi\!\!\!/_j^a \partial_a L^i - \Pi\!\!\!/_j^i|, b-1, |\zeta|_{\not{g}}, \sigma, \\
&\tilde{r}|\nabla\!\!\!/ \mathrm{tr}_{\not{g}}\chi|_{\not{g}}, \tilde{r}^2|\nabla\!\!\!/ \mathrm{tr}_{\tilde{\not{g}}}\tilde{\chi}^{(\mathrm{Small})}|_{\not{g}}, \tilde{r}|\nabla\!\!\!/ \hat{\chi}|_{\not{g}}, \tilde{r}|\nabla\!\!\!/ b|_{\not{g}}, \tilde{r}|\nabla\!\!\!/ \zeta|_{\not{g}}, \tilde{r}|\nabla\!\!\!/ \sigma|_{\not{g}}, \qquad (8.28a) \\
&\tilde{r}^2 \Delta\!\!\!/ b, \tilde{r}^2 \Delta\!\!\!/ \sigma, \tilde{r}^2\mu, \tilde{r}^2\check{\mu} = \mathcal{O}(\tilde{r}) \quad \text{as } t \downarrow u,
\end{aligned}$$

$$\lim_{t \downarrow u} \|\underline{\zeta}, k\|_{L^\infty(S_{t,u})} < \infty. \tag{8.28b}$$

Moreover

$$\lim_{t\downarrow u} \tilde{r}^{-2} \not{g}\left(\frac{\partial}{\partial \omega^A}, \frac{\partial}{\partial \omega^B}\right) = \not{e}\left(\frac{\partial}{\partial \omega^A}, \frac{\partial}{\partial \omega^B}\right), \tag{8.28c}$$

$$\lim_{t\downarrow u} \tilde{r}^{-2} \frac{\partial}{\partial \omega^C} \not{g}\left(\frac{\partial}{\partial \omega^A}, \frac{\partial}{\partial \omega^B}\right) = \frac{\partial}{\partial \omega^C} \not{e}\left(\frac{\partial}{\partial \omega^A}, \frac{\partial}{\partial \omega^B}\right). \tag{8.28d}$$

Discussion of the proof of Propositions 8.8 and 8.9. The existence of such initial foliation can be proved by Nash–Moser implicit function theorem (see [25]). The proof of the estimates in Proposition 8.8 relies on the energy estimates (5.1). The reason is that Proposition 8.8 yields a foliation and estimates on the hypersurface Σ_0 with respect to the rescaled coordinates, and this hypersurface corresponds to the hyperfaces Σ_{t_k} for each k with respect to original space-time (see Remark 6.4 and Sec. 7.1.2 for the description of Σ_0 in rescaled space-time). The point is that we need the energy estimates of (5.1) to control the fluid along the "old" Σ_{t_k}. We refer the reader to [30, Appendix C] for the proof of the estimates in Propositions 8.8 and 8.9. □

8.4. *Restatement of bootstrap assumptions and estimates for quantities constructed out of the acoustical Eikonal equation*

In this section, we restate the consequence of bootstrap assumptions of fluid variables, vorticity and entropy gradient that we obtained in (6.12), followed by the bootstrap assumptions for acoustic geometry. Then we state the main estimates for the acoustical function quantities in Proposition 8.10. The estimates in Proposition 8.10 are required by the conformal energy method to close the whole bootstrap argument in this paper. We provide a discussion of the proof of Proposition 8.10 in Sec. 8.4.4 via a bootstrap argument where the bootstrap assumptions are listed in Sec. 8.4.2. For the details of the proof, we refer readers to [11, Sec. 10; 30, Secs. 5 and 6].

8.4.1. *The fixed number p*

In the rest of the paper, p denotes a fixed number with

$$0 < \delta_0 < 1 - \frac{2}{p} < N - 2, \tag{8.29}$$

where δ_0 is defined in Sec. 3.4.

8.4.2. *Bootstrap assumptions for geometric quantities*

After rescaling in Sec. 6.2, we make several bootstrap assumptions for the quantities in acoustic geometry. These assumptions will be recovered and improved by estimates in Proposition 8.10.

First we recall (6.12) as follows.

Estimates by using bootstrap assumptions of variables:

$$\begin{aligned}
&\|\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\omega}, \boldsymbol{\partial}\vec{S}, \vec{\mathcal{C}}, \mathcal{D}\|_{L_t^2 L_x^\infty(\mathcal{M})} \\
&\qquad + \lambda^{\delta_0} \sqrt{\sum_{\nu>2} \nu^{2\delta_0} \|P_\nu(\mathrm{f}(\vec{\Psi}, \vec{\omega}, \vec{S})(\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\omega}, \boldsymbol{\partial}\vec{S}, \vec{\mathcal{C}}, \mathcal{D}))\|^2_{L_t^2 L_x^\infty(\mathcal{M})}} \\
&\quad \lesssim \lambda^{-1/2-4\varepsilon_0}. \qquad (8.30)
\end{aligned}$$

Then we make few more assumptions for the quantities of acoustical geometry as follows.

Bootstrap assumptions for the acoustical function quantities:

$$\max_{A,B=1,2} \left\| \tilde{r}^{-2} \not{g}\left(\frac{\partial}{\partial \omega^A}, \frac{\partial}{\partial \omega^B}\right) - \not{e}\left(\frac{\partial}{\partial \omega^A}, \frac{\partial}{\partial \omega^B}\right) \right\|_{L_x^\infty(\mathcal{M})} \lesssim \lambda^{-\varepsilon_0}, \qquad (8.31a)$$

$$\max_{A,B,C=1,2} \left\| \frac{\partial}{\partial \omega^C} \left(\tilde{r}^{-2} \not{g}\left(\frac{\partial}{\partial \omega^A}, \frac{\partial}{\partial \omega^B}\right) - \not{e}\left(\frac{\partial}{\partial \omega^A}, \frac{\partial}{\partial \omega^B}\right)\right) \right\|_{L_t^\infty L_\omega^p(\mathcal{C}_u)} \lesssim \lambda^{-\varepsilon_0}. \qquad (8.31b)$$

Also

$$\|\mathrm{tr}_{\tilde{\not{g}}} \tilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \zeta\|_{L_t^2 C_\omega^{0,\delta_0}(\mathcal{C}_u)} \lesssim \lambda^{-1/2+2\varepsilon_0}. \qquad (8.32)$$

Moreover

$$\|\tilde{r}(\mathrm{tr}_{\tilde{\not{g}}} \tilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \zeta)\|_{L_\omega^p(S_{t,u})} \leq 1, \qquad (8.33a)$$

$$\|b-1\|_{L_\omega^\infty(S_{t,u})} \leq \frac{1}{2}. \qquad (8.33b)$$

Finally, we assume that the following estimates hold in the interior region:

$$\|\mathrm{tr}_{\tilde{\not{g}}} \tilde{\chi}^{(\mathrm{Small})}, \hat{\chi}\|_{L_t^2 L_u^\infty C_\omega^{0,\delta_0}(\mathcal{M}^{(\mathrm{Int})})} \leq \lambda^{-1/2}, \quad \|\zeta\|_{L_t^2 L_x^\infty(\mathcal{M}^{(\mathrm{Int})})} \leq \lambda^{-1/2}, \qquad (8.34a)$$

$$\|\not{\nabla}\sigma\|_{L_u^2 L_t^2 L_\omega^\infty(\mathcal{M}^{(\mathrm{Int})})} \leq 1. \qquad (8.34b)$$

8.4.3. *Main estimates for the acoustical function quantities*

In the following proposition, we derive the estimates for the various acoustical function quantities. These estimates are sufficient to derive the boundness theorem of the conformal energy in Theorem 7.6 and to recover and improve the bootstrap assumptions in Sec. 8.4.2.

Proposition 8.10 ([11, Sec. 10; 30, Secs. 5 and 6. The main estimates for the acoustical function quantities]). *Under the bootstrap assumptions we have the following estimates where* $2 < q \leq 4$*:*

Estimates for connection coefficients:

$$\|\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \zeta\|_{L_t^2 L_\omega^p(\mathcal{C}_u)}, \|\tilde{r}\not{\mathbf{D}}_L(\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \zeta)\|_{L_t^2 L_\omega^p(\mathcal{C}_u)} \lesssim \lambda^{-1/2}, \tag{8.35a}$$

$$\|\tilde{r}^{1/2}(\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \zeta)\|_{L_t^\infty L_\omega^p(\mathcal{C}_u)} \lesssim \lambda^{-1/2}, \tag{8.35b}$$

$$\|\tilde{r}(\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \zeta)\|_{L_t^\infty L_\omega^p(\mathcal{C}_u)} \lesssim \lambda^{-4\varepsilon_0}, \tag{8.35c}$$

$$\tilde{r}\mathrm{tr}_{\tilde{\not g}}\tilde{\chi} \approx 1, \tag{8.36a}$$

$$\|\tilde{r}^{1/2}\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})}\|_{L^\infty(\mathcal{M})} \lesssim \lambda^{-1/2}, \tag{8.36b}$$

$$\|\tilde{r}^{3/2}\tilde{\not\nabla}\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})}\|_{L_t^\infty L_u^\infty L_\omega^p(\mathcal{M})} \lesssim \lambda^{-1/2}, \tag{8.36c}$$

$$\|\tilde{r}(\tilde{\not\nabla}\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})}, \tilde{\not\nabla}\hat{\chi})\|_{L_t^2 L_\omega^p(\mathcal{C}_u)} \lesssim \lambda^{-1/2}, \tag{8.36d}$$

$$\|\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \zeta\|_{L_t^2 C_\omega^{0,\delta_0}(\mathcal{C}_u)} \lesssim \lambda^{-1/2}. \tag{8.36e}$$

In addition, the null lapse b verifies the following:

$$\begin{aligned} &\left\|\frac{b^{-1}-1}{\tilde{r}}\right\|_{L_t^2 L_x^\infty(\mathcal{M})}, \quad \left\|\frac{b^{-1}-1}{\tilde{r}^{1/2}}\right\|_{L_t^\infty L_u^\infty L_\omega^{2p}(\mathcal{M})}, \\ &\left\|\tilde{r}(\not{\mathbf{D}}_L, \tilde{\not\nabla})\left(\frac{b^{-1}-1}{\tilde{r}}\right)\right\|_{L_t^2 L_\omega^p(\mathcal{C}_u)} \lesssim \lambda^{-1/2}. \end{aligned} \tag{8.37}$$

Moreover, we have

$$\|\mathrm{f}_{(\vec{L})}\|_{L_t^\infty L_u^\infty C_\omega^{0,\delta_0}(\mathcal{M})} \lesssim 1. \tag{8.38}$$

Furthermore

$$\left\|\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \mathrm{tr}_{\not g}\chi - \frac{2}{\tilde{r}}\right\|_{L_t^{\frac{q}{2}} L_u^\infty C_\omega^{0,\delta_0}(\mathcal{M})} \lesssim \lambda^{\frac{2}{q}-1-4\varepsilon_0\left(\frac{4}{q}-1\right)}, \tag{8.39}$$

$$\|\zeta\|_{L_t^2 L_x^\infty(\mathcal{M})} \lesssim \lambda^{-1/2-3\varepsilon_0}, \tag{8.40}$$

$$\|\zeta\|_{L_t^{\frac{q}{2}} L_x^\infty(\mathcal{M})} \lesssim \lambda^{\frac{2}{q}-1-4\varepsilon_0\left(\frac{4}{q}-1\right)}.$$

Improved estimates in the interior region:

$$\left\|\frac{b^{-1}-1}{\tilde{r}}\right\|_{L_t^2 L_x^\infty(\mathcal{M}^{(\mathrm{Int})})} \lesssim \lambda^{-1/2-4\varepsilon_0}, \tag{8.41}$$

$$\|\tilde{r}^{1/2}(\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \zeta)\|_{L_\omega^{2p} L_t^\infty(\mathcal{C}_u)} \lesssim \lambda^{-1/2} \quad \textit{if } \mathcal{C}_u \subset \mathcal{M}^{(\mathrm{Int})}, \tag{8.42}$$

$$\left\|\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \mathrm{tr}_{\not g}\chi - \frac{2}{\tilde{r}}\right\|_{L_t^2 L_u^\infty C_\omega^{0,\delta_0}(\mathcal{M}^{(\mathrm{Int})})} \lesssim \lambda^{-1/2-3\varepsilon_0}, \tag{8.43}$$

$$\|\zeta\|_{L_t^2 L_x^\infty(\mathcal{M}^{(\mathrm{Int})})} \lesssim \lambda^{-1/2-3\varepsilon_0}. \tag{8.44}$$

Estimates for the geometric angular coordinate components of $\gamma\!\!\!/$:

$$\max_{A,B=1,2}\left\|\tilde{r}^{-2}\gamma\!\!\!/\left(\frac{\partial}{\partial\omega^A},\frac{\partial}{\partial\omega^B}\right)-e\!\!\!/\left(\frac{\partial}{\partial\omega^A},\frac{\partial}{\partial\omega^B}\right)\right\|_{L^\infty(\mathcal{M})}\lesssim\lambda^{-4\varepsilon_0}, \tag{8.45a}$$

$$\max_{A,B,C=1,2}\left\|\frac{\partial}{\partial\omega^C}\left(\tilde{r}^{-2}\gamma\!\!\!/\left(\frac{\partial}{\partial\omega^A},\frac{\partial}{\partial\omega^B}\right)-e\!\!\!/\left(\frac{\partial}{\partial\omega^A},\frac{\partial}{\partial\omega^B}\right)\right)\right\|_{L_t^\infty L_\omega^p(\mathcal{C}_u)}\lesssim\lambda^{-4\varepsilon_0}. \tag{8.45b}$$

Estimates for υ *and* b:

$$\upsilon:=\frac{\sqrt{\det\gamma\!\!\!/}}{\sqrt{\det e\!\!\!/}}\approx\tilde{r}^2, \tag{8.46a}$$

$$\|b-1\|_{L^\infty(\mathcal{M})}\lesssim\lambda^{-4\varepsilon_0}<\frac{1}{4}. \tag{8.46b}$$

Furthermore

$$\begin{aligned}&\|\tilde{r}^{1/2}\nabla\!\!\!/\ln\left(\tilde{r}^{-2}\upsilon\right)\|_{L_t^\infty L_u^\infty L_\omega^p(\mathcal{M})},\|\nabla\!\!\!/\ln\left(\tilde{r}^{-2}\upsilon\right)\|_{L_t^2L_\omega^p(\mathcal{C}_u)},\\&\|\tilde{r}L\nabla\!\!\!/\ln\left(\tilde{r}^{-2}\upsilon\right)\|_{L_t^2L_\omega^p(\mathcal{C}_u)}\lesssim\lambda^{-1/2}.\end{aligned} \tag{8.47}$$

Estimates for $\upmu$ *and* $\nabla\!\!\!/\zeta$:

$$\|\tilde{r}\upmu,\tilde{r}\nabla\!\!\!/\zeta\|_{L_t^2L_\omega^p(\mathcal{C}_u)}\lesssim\lambda^{-1/2}. \tag{8.48}$$

Interior region estimates for σ ($\mathcal{C}_u\subset\mathcal{M}^{(\mathrm{Int})}$):

$$\|\tilde{r}^{1/2}L\sigma\|_{L_t^\infty L_\omega^{2p}(\mathcal{C}_u)}\lesssim\lambda^{-1/2-2\varepsilon_0},\|\tilde{r}^{1/2}\nabla\!\!\!/\sigma\|_{L_\omega^pL_t^\infty(\mathcal{C}_u)},\quad\|\nabla\!\!\!/\sigma\|_{L_t^2L_\omega^p(\mathcal{C}_u)}\lesssim\lambda^{-1/2}, \tag{8.49a}$$

$$\|\sigma\|_{L^\infty(\mathcal{M}^{(\mathrm{Int})})}\lesssim\lambda^{-8\varepsilon_0}, \tag{8.49b}$$

$$\|\tilde{r}^{-1/2}\sigma\|_{L^\infty(\mathcal{M}^{(\mathrm{Int})})}\lesssim\lambda^{-1/2-4\varepsilon_0}. \tag{8.49c}$$

Interior region estimates for σ, $\upmu$, $\tilde{\zeta}$ *and* $\mu\!\!\!/$:

$$\|\nabla\!\!\!/\sigma\|_{L_u^2L_t^2C_\omega^{0,\delta_0}(\mathcal{M}^{(\mathrm{Int})})},\|\tilde{r}\check{\upmu},\tilde{r}\nabla\!\!\!/\tilde{\zeta}\|_{L_u^2L_t^2L_\omega^p(\mathcal{M}^{(\mathrm{Int})})}\lesssim\lambda^{-4\varepsilon_0}, \tag{8.50a}$$

$$\|\tilde{r}^{\frac{3}{2}}\check{\upmu}\|_{L_u^2L_t^\infty L_\omega^p(\mathcal{M}^{(\mathrm{Int})})}\lesssim\lambda^{-4\varepsilon_0}. \tag{8.50b}$$

In addition

$$\|\tilde{r}\nabla\!\!\!/\mu\!\!\!/,\mu\!\!\!/\|_{L_t^2L_u^2L_\omega^p(\mathcal{M}^{(\mathrm{Int})})},\|\mu\!\!\!/\|_{L_t^2L_u^2C_\omega^{0,\delta_0}(\mathcal{M}^{(\mathrm{Int})})}\lesssim\lambda^{-4\varepsilon_0}. \tag{8.51}$$

Decomposition of $\nabla\!\!\!/\sigma$ *and corresponding estimates in the interior region*: *In* $\mathcal{M}^{(\mathrm{Int})}$, *we can decompose* $\nabla\!\!\!/\sigma$ *as follows*:

$$\nabla\!\!\!/\sigma=-\zeta+(\tilde{\zeta}-\mu\!\!\!/)+\mu\!\!\!/_{(1)}+\mu\!\!\!/_{(2)}, \tag{8.52}$$

where the following asymptotic conditions near the cone-tip axis are satisfied:

$$\tilde{r}\mu\!\!\!/_{(1)}(t,u,\omega),\tilde{r}\mu\!\!\!/_{(2)}(t,u,\omega)=\mathcal{O}(\tilde{r})\quad\text{as } t\downarrow u. \tag{8.53}$$

Moreover

$$\|\tilde{\zeta} - \not\mu\|_{L^2_t L^\infty_x(\mathcal{M}^{(\mathrm{Int})})}, \|\not\mu_{(1)}\|_{L^2_t L^\infty_x(\mathcal{M}^{(\mathrm{Int})})} \lesssim \lambda^{-1/2-3\varepsilon_0}, \tag{8.54a}$$

$$\|\not\mu_{(2)}\|_{L^2_u L^\infty_t L^\infty_\omega(\mathcal{M}^{(\mathrm{Int})})} \lesssim \lambda^{-1/2-4\varepsilon_0}. \tag{8.54b}$$

8.4.4. *Discussion of the proof of Proposition* 8.10

In this section, we discuss the proof of Proposition 8.10. For the details of the proof, we refer readers to [11, Sec. 10; 30, Secs. 5 and 6].

The structure of the proof consists of three major steps:

(1) First of all, we derive the transport equations along null hypersurfaces and div-curl systems[x] on $S_{t,u}$ verified by the geometric quantities. These equations are derived using basic differential geometry and, at the appropriate spots, using the relativistic Euler equations for substitution for reasons further described in Step (2). The key point is that *all of the equations we obtain have the exact same schematic structure as the equations in* [11]. We refer readers to [16, Sec. 2] for the PDEs satisfied by connection coefficients and mass aspect function $\upmu$. We refer readers to [30, Sec. 6] for the PDEs of conformal factor σ, modified torsion $\tilde{\zeta}$ and modified mass aspect function $\check{\upmu}, \not\mu$.

(2) Second, certain Ricci and Riemann curvature tensor components, which appear as source terms in the PDEs that we just obtained in the previous step, are rewritten by using Bianchi identities and the decomposition of the following Ricci curvature [16, Lemma 2.1]:

$$\mathbf{Ric}_{\alpha\beta} = -\frac{1}{2}\square_{\mathbf{g}}\mathbf{g}_{\alpha\beta}(\vec{\Psi}) + \frac{1}{2}\left(\mathbf{D}_\alpha\mathbf{\Gamma}_\beta + \mathbf{D}_\beta\mathbf{\Gamma}_\alpha\right) + \mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\Psi}]. \tag{8.55}$$

It is at this step that the wave equation (2.29) of the geometric formulation of the relativistic Euler equations is used to substitute for the term $\square_{\mathbf{g}}\mathbf{g}_{\alpha\beta}(\vec{\Psi})$, as we alluded to in the previous step. We again emphasize that following this substitution, one obtains equations of the exact same schematic form as in [11].

After the substitution, one is faced with controlling the source terms in the geometric equations in mixed space-time norms. The source terms depend on $\boldsymbol{\partial}\vec{\Psi}, \vec{\omega}, \vec{S}, \vec{\mathcal{C}}, \mathcal{D}$ and have the same schematic structure as in [11]. This step requires trace inequalities and Sobolev inequalities, which are provided by [11, Proposition 10.2; 30, (5.34)–(5.39)]; the proofs of these inequalities are the same as in [11], and rely on bootstrap assumptions (3.17b) and (3.17c), energy estimates on constant-time hypersurfaces (5.1) and energy estimates on null hypersurfaces (8.3) and (8.4).

(3) After one has controlled the source terms in the geometric PDEs for the acoustic geometry from Step (1), one uses a transport lemma and div-curl estimates

[x]These div-curl systems depend on the acoustic geometry and are independent of the structure of the relativistic Euler equations. Therefore, these div-curl systems are completely unrelated to the ones that we derived for the vorticity and entropy gradient in Sec. 5.

to obtain various mixed space-time norms estimates for $\tilde{r}^p$ weighted acoustic quantities in Proposition 8.10. We refer readers to [11, Sec. 10.9; 30, Secs. 5 and 6] for the detailed proof. We emphasize that the proof is the same as in [11] because it relies only on bootstrap assumptions (3.17b) and (3.17c) and source term bounds that one obtained in Step (2), ingredients which are already available to us at this step in the proof.

We now point out some differences between the non-relativistic 3D compressible Euler equations and the relativistic Euler equations in terms of the control of acoustic geometry. Besides the different acoustic metric $\mathbf{g}$ in this paper compared to [11], the $\mathbf{g}$-Σ_t-normal vectorfield is $\mathbf{T}$ (see (2.17)) in this paper, while in the non-relativistic case [11], it is $\mathbf{B} = \partial_t + v^a \partial_a$. Although these differences have necessitated changes to some of the proofs earlier in the paper (such as the proof of the energy estimates on constant-time hypersurfaces and the acoustic null hypersurfaces), these changes do not have any effect on the proofs of the estimate for the acoustic geometry; it is for this reason that we refer to [11, Sec. 10] for the details behind the proof of Proposition 8.10.

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Appendix A. Notations

In this appendix, we gather the notations that we use throughout the paper.

Symbol	Ref.
$M, M^{-1}, \epsilon_{\gamma\delta\kappa\lambda}, \Sigma_t, \partial, \boldsymbol{\partial}$	Sec. 2.1
$v, \mathfrak{p}, n, s, \theta, H$	Sec. 2.2.1
$\mathrm{vort}^{\alpha}(\cdot)$	Definition 2.2
ω	Definition 2.3
h	Definition 2.4
$\mathfrak{q}$	Definition 2.5
S	Definition 2.6
c	Sec. 2.2.4
$\mathcal{C}, \mathcal{D}$	Definition 2.8
$\mathbf{g}_{\mathrm{Acou}}, \mathbf{g}_{\mathrm{Acou}}^{-1}$	Definition 2.9
$\mathbf{g}, \mathbf{g}^{-1}$	Definition 2.10
$\mathbf{T}$	Definition 2.11

Symbol	Ref.
g, g^{-1}	Definition 2.12
$\mathbf{D}, \square_{\mathbf{g}}$	Definition 2.14
$\vec{v}, \vec{\omega}, \vec{S}, \vec{\mathcal{C}}, \vec{\Psi}, \mathbf{B}$	Definition 2.15
$\mathscr{L}, \mathscr{Q}$	Sec. 2.4.1
$\mathfrak{F}_{(\mathcal{C}^{\alpha})}, \mathfrak{F}_{(\mathcal{D})}$	Proposition 2.17
$\mathfrak{F}_{(\Psi)}$	Proposition 2.20
$\nu, P_{\nu}, P_I, P_{\leq \nu}$	Sec. 3.2
$\mathfrak{R}_{(\boldsymbol{\partial}\Psi);\nu}, \mathfrak{R}_{(\partial\mathcal{C}^{\alpha});\nu}, \mathfrak{R}_{(\partial\mathcal{D});\nu}$	Lemma 3.3
$q, \varepsilon_0, \delta_0, \delta, \delta_1$	Sec. 3.4
$\mathcal{R}, \breve{\mathcal{R}}$	Sec. 3.5
T_*	Sec. 3.6
$Q, {}^{(X)}\mathbf{J}, {}^{(X)}\boldsymbol{\pi}, \mathbb{E}, \mathrm{d}\varpi_g$	Definition 5.3
λ	Sec. 6.1
$T_{*;(\lambda)}$	Sec. 6.2
$u, \mathcal{C}_u, S_{t,u}, \mathcal{M}^{(\mathrm{Int})}, \mathcal{M}^{(\mathrm{Ext})}, \mathcal{M}$	Sec. 7.1
$\tilde{r}, w_*$	Definition 7.2
$L_{(\mathrm{Geo})}, b, N, L, \underline{L}, \not{g}, e_A, \omega^A$	Definition 7.3
$\not{\Pi}, \lvert\xi\rvert_{\not{g}}, \mathrm{tr}_{\not{g}}\xi, \hat{\xi}$	Definition 7.4
$\mathcal{F}_{(\mathrm{wave})}, \mathcal{F}_{(\mathrm{transport})}, \mathrm{d}\varpi_{\not{g}}$	Definition 8.1
$\mathfrak{C}, \mathfrak{C}^{(i)}, \mathfrak{C}^{(e)}$	Definition 7.5
$\not{\mathbf{D}}, \not{\nabla}, \not{\Delta}, \mathbf{Ric}, \mathbf{Riem}$	Sec. 8.2.1
$k, \theta, \chi, \underline{\chi}, \zeta, \underline{\zeta}, \mathcal{L}, \not{\mathcal{L}}$	Definition 8.4
$\sigma, \boldsymbol{\Gamma}_L, \widetilde{\mathbf{g}}, \tilde{\chi}, \underline{\tilde{\chi}}, \mathrm{tr}_{\tilde{\not{g}}}, \tilde{\chi}^{(\mathrm{Small})}$	Definition 8.5
μ	Definition 8.6
$\not{e}$	Definition 8.7
$a, w, \frac{\partial}{\partial w}, \frac{\partial}{\partial \omega^A}, \upsilon$	Proposition 8.8
$\breve{\mu}, \not{\mu}, \breve{\zeta}$	Definition 8.6
p	Sec. 8.4.1

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