A MORAWETZ INEQUALITY FOR WATER WAVES

THOMAS ALAZARD, MIHAELA IFRIM AND DANIEL TATARU

Abstract. We consider gravity water waves in two space dimensions, with finite or infinite depth. Assuming some uniform scale invariant Sobolev bounds for the solutions, we prove local energy decay (Morawetz) estimates globally in time. Our result is uniform in the infinite depth limit.

1. Introduction

Our aim in this paper is to initiate the study of Morawetz inequalities for water waves. The water-wave equations describe the dynamics of the interface separating air from a perfect fluid. This is a system of two coupled equations: the incompressible Euler equation inside the fluid domain, and a kinematic equation describing the evolution of the interface. Assuming that the flow is irrotational, we thus have two unknowns: the velocity potential ϕ, whose gradient gives the velocity, and the free surface elevation η, whose graph is the free surface.

We consider the 2D-gravity equations, without surface tension, and assume that the fluid domain has a flat bottom. Then, at time t, the fluid domain is of the form

$$\Omega(t) = \{\, (x,y) \in \mathbb{R}\times\mathbb{R} \,:\, -h < y < \eta(t,x) \,\},$$

where $h > 0$ is the depth. Given a compactly supported bump function $\chi = \chi(x)$, we want to estimate the local energy

$$\frac{g}{2}\int_0^T\!\!\int_{\mathbb{R}} \chi(x-x_0)\eta^2(t,x)\,dxdt + \frac{1}{2}\int_0^T\!\!\int_{\mathbb{R}}\int_{-h}^{\eta(t,x)} \chi(x-x_0)\,|\nabla_{x,y}\phi(t,x,y)|^2\,dydxdt,$$

uniformly in time T and space location x_0.

In the infinite depth case ($h=\infty$), neglecting all nonlinearities, the gravity water-wave equations can be written as a fractional Schrödinger equation

$$\partial_t u + i\,|D_x|^{\frac{1}{2}}\,u = 0, \quad \text{with} \quad u = \eta + i\,|D|^{\frac{1}{2}}\,\psi.$$

For this equation, one can obtain a Morawetz inequality by using some standard dispersive tools. The first goal of this paper is to extend this linear analysis to the finite depth case, and prove an estimate which is uniform with respect to $h \geq 1$. This problem exhibits some

very interesting difficulties at low frequencies, whose analysis requires a careful study of harmonic functions in a strip.

The second and main task of this paper is to obtain a Morawetz inequality for the nonlinear equations. Our main result extends the linear inequality; it holds provided that some scale invariant norms remain small enough uniformly in time. Our nonlinear analysis is highly non-perturbative, since it is a very delicate problem to estimate the nonlinearities by scale invariant norms (this can be seen by recalling that one does not even know the existence of weak-solutions in such scale invariant spaces).

The proofs combine multiple methods and ideas in a novel way: *i*) local conservation laws for the momentum conservation (inspired by Morawetz, and introducing a new momentum density for the water waves equations); *ii*) a systematic use of conformal transformations, *iii*) appropriate Littlewood-Paley decompositions and multilinear estimates to analyze the low-frequency component, *iv*) a fully nonlinear normal form type modification of the momentum density to handle the worst nonlinearities.

In the Appendix, we also complement this analysis by showing a Morawetz inequality for possibly large solutions, but at the expense of loosing the uniformity in the depth as well as the control of the low-frequency component of the velocity potential.

1.1. **Morawetz estimates.** Also known as *local energy decay*, they were originally introduced in Morawetz's paper [35]. In their original form they assert that, for solutions to the linear wave equation, the local energy of the solutions is bounded, globally in time, by the initial energy. One may view this as a statement about the local decay of solutions which is invariant with respect to time translations.

Another interesting example is the Schrödinger equation. Unlike the wave equation, where one has a finite speed of propagation, in this case the group velocity increases to infinity in the high frequency limit. Because of this, the natural local energy measures a higher regularity (1/2 derivative more to be precise) than the initial data energy of the solutions; for this reason the Morawetz estimates for the Schrödinger equation have been originally called *local smoothing*, see [18], [40].

Up to the present time, the Morawetz estimates have had a rich and complex history, which is too extensive to try to describe here. For further references we direct the reader to [33] for the wave equation, [32] for the Schrödinger equation and [37] for other models. Morawetz estimates have been proved for linear and nonlinear models, and have been used as a key ingredient in many results concerning the long time

behavior of solutions in nonlinear dispersive flows. One other key development was the introduction of *interaction Morawetz estimates* in [17], which has played a major role in the study of nonlinear Schrödinger equations.

We turn our attention now to Morawetz estimates for water waves. Here additional challenges arise due to the fact that the equations are not only fully nonlinear, but also nonlocal. Another striking difference is due to the fact that in the high frequency limit the dispersive part of the group velocity goes to zero. Because of this, here we have the opposite phenomena to local smoothing, namely a loss of 1/4 derivative in the local energy. Combined with the nonlinear and nonlocal character of the equations, this brings substantial difficulties in the low frequency analysis.

1.2. **The water wave equations.** Consider the incompressible Euler equations for a potential flow in a $2D$-domain located between with a free surface and a flat bottom. At time t the fluid domain is of the form

$$\Omega(t) = \{\, (x,y) \in \mathbb{R}\times\mathbb{R} \,:\, -h < y < \eta(t,x) \,\},$$

where $h > 0$ is the depth and η, the free surface elevation, is an unknown function. The velocity field is the gradient of a harmonic potential function $\phi = \phi(t,x,y)$, satisfying the Bernoulli equation,

$$\begin{cases} \Delta_{x,y}\phi = 0 \quad \text{in } \Omega(t) \\ \partial_t\phi + \dfrac{1}{2}\left|\nabla_{x,y}\phi\right|^2 + P + gy = 0 \quad \text{in } \Omega(t) \\ \phi_y = 0 \quad \text{on } y = -h, \end{cases} \tag{1.1}$$

where $g > 0$ is the acceleration of gravity, $P\colon \Omega \to \mathbb{R}$ is the pressure, $\nabla_{x,y} = (\partial_x, \partial_y)$ and $\Delta_{x,y} = \partial_x^2 + \partial_y^2$. Partial differentiations in space are denoted by suffixes so that $\phi_x = \partial_x\phi$ and $\phi_y = \partial_y\phi$.

The water-wave equations are given by two boundary conditions on the free surface: firstly an equation describing the deformations of the domain,

$$\partial_t\eta = \sqrt{1+\eta_x^2}\,\phi_n|_{y=\eta} = \phi_y(t,x,\eta(t,x)) - \eta_x(t,x)\phi_x(t,x,\eta(t,x)), \tag{1.2}$$

and secondly an equation expressing the balance of forces at the free surface. In the present article we consider pure gravity waves, so that this balance of forces reads

$$P|_{y=\eta} = 0. \tag{1.3}$$

One can give more explicit evolution equations by introducing the trace of the velocity potential at the free surface,

$$\psi(t,x) = \phi(t,x,\eta(t,x)),$$

as well as the Dirichlet to Neumann operator associated to the fluid domain $\Omega(t)$, defined by

$$G(\eta)\psi = \sqrt{1+\eta_x^2}\,\phi_n|_{y=\eta} = (\phi_y - \eta_x\phi_x)_{|_{y=\eta}}.$$

Then (see [46]), with the above notations, the water-wave system reads

$$\begin{cases} \partial_t \eta = G(\eta)\psi \\ \partial_t \psi + g\eta + \dfrac{1}{2}\psi_x^2 - \dfrac{1}{2}\dfrac{(G(\eta)\psi + \eta_x\psi_x)^2}{1+\eta_x^2} = 0. \end{cases} \tag{1.4}$$

1.3. **Symmetries and conservation laws.** Introduce the energy $\mathcal{H}$, defined by

$$\mathcal{H} = \frac{g}{2}\int_{\mathbb{R}} \eta^2\,dx + \frac{1}{2}\int_{\mathbb{R}}\int_{-h}^{\eta(t,x)} |\nabla_{x,y}\phi|^2\,dydx. \tag{1.5}$$

The energy is conserved. Furthermore, it is known since Zakharov ([46]) that the water-wave system is Hamiltonian. Precisely, we have

$$\frac{\partial \eta}{\partial t} = \frac{\delta \mathcal{H}}{\delta \psi}, \qquad \frac{\partial \psi}{\partial t} = -\frac{\delta \mathcal{H}}{\delta \eta}.$$

A second conservation law arises by Noether's theorem from the invariance with respect to horizontal translations. This is the horizontal momentum, which has the form

$$\mathcal{M} = \int_{\mathbb{R}} \eta\psi_x\,dx. \tag{1.6}$$

Together with the energy, this will play a key role in what follows.

Another symmetry is given by the scaling invariance which holds in the infinite depth case (that is when $h = \infty$). If ψ and η are solutions of the gravity water waves equations (1.4), then ψ_λ and η_λ defined by

$$\psi_\lambda(t,x) = \lambda^{-3/2}\psi(\sqrt{\lambda}t, \lambda x), \quad \eta_\lambda(t,x) = \lambda^{-1}\eta(\sqrt{\lambda}t, \lambda x),$$

solve the same system of equations. The (homogeneous) Sobolev spaces invariant by this scaling correspond to η in $\dot{H}^{3/2}(\mathbb{R})$ and ψ in $\dot{H}^{2}(\mathbb{R})$.

1.4. **The Cauchy problem.** The energy, the momentum and the scale invariant norms are super-critical for the current local well-posedness results about the Cauchy problem. One does not even know the existence of weak-solutions for initial data such that these three quantities are finite.

The local well-posedness for the Cauchy problem with initial data in Sobolev spaces has been extensively studied; we refer the reader to [36, 41, 42, 16, 30, 19, 28, 38, 4, 25, 29, 5, 1]. The water wave equations are now known to be locally well-posed in suitable function spaces

which are 1/2-derivative[1] more regular than the scaling invariance, e.g. when initially

$$\eta \in H^s(\mathbb{R}), \qquad \psi - T_{\phi_y|_{y=\eta}}\eta \in H^{s+\frac{1}{2}}(\mathbb{R}), \qquad s \geq 2,$$

where $T_a b$ is the paraproduct decomposition of the product of two functions a and b; it represents the portion which favours the "low-high" interaction when a low-frequency component of a is multiplied with a high-frequency component of b. Here the expression $\psi - T_{\phi_y|_{y=\eta}}\eta$ represents the so called *good unknown* of Alinhac ([10, 9, 28, 8]), and is imposed by the non-diagonal quasilinear structure of the equations. Alternatively, one can re-express the second condition in terms of the gradient of the velocity potential, namely by requiring that $\nabla\phi|_{y=\eta}$ belongs to $H^{s-\frac{1}{2}}(\mathbb{R})$.

Since we are interested in uniform in time estimates, let us recall that much less is known concerning the long time dynamics. For data of size ϵ it is known that solutions persist for at least a cubic lifespan $O(\epsilon^{-2})$, see[2] [6, 25] for the deep water case and [24] for the finite depth case (see also [45] for the 3D problem). For longer times it is not at all clear what happens to the solutions, and the blow-up scenario in particular has not been excluded (see [15, 20] for large data blow-up). An exception to this is the case when the initial data is not only small but also localized, where there solutions are known to be global, see [43, 27, 7, 25, 26] and also similar results in three dimensions [22, 44].

Rather than trying to study the size of the solutions for longer time, in the present article we take a different track, and assume that we have a solution which stays bounded (small) in a reasonable Sobolev norm on a time interval $[0, T]$, with no a-priori bound on T, and ask what can be said about the dispersive properties of the solutions. More precisely, our goal here is to initiate the study of Morawetz inequalities for water waves. We consider the case of gravity waves in the present article, and the case of gravity-capillary waves in a second article.

1.5. **Function spaces.** In this paragraph we introduce three spaces: a space E^0 associated to the energy, a space $E^{\frac{1}{4}}$ associated to the momentum, and a uniform in time control norm $\|\cdot\|_X$ which respects the scaling invariance.

The above energy $\mathcal{H}$ (Hamiltonian) corresponds to the energy space for (η, ψ),

$$E^0 = g^{-\frac{1}{2}} L^2(\mathbb{R}) \times \dot{H}_h^{\frac{1}{2}}(\mathbb{R}),$$

[1] Even slightly below that, see [5, 1].

[2] As a historical note, the question of obtaining cubic lifespan bounds first arose in the work of Zakharov [46] in the context of the NLS approximation for deep water waves; see also [39] for more recent results in this direction.

with the depth dependent $H_h^{\frac{1}{2}}(\mathbb{R})$ space defined as

$$\dot{H}_h^{\frac{1}{2}}(\mathbb{R}) = \dot{H}^{\frac{1}{2}}(\mathbb{R}) + h^{-\frac{1}{2}}\dot{H}^1(\mathbb{R}).$$

Similarly, in order to measure the momentum, we use the space $E^{\frac{1}{4}}$, which is the h-adapted linear $H^{\frac{1}{4}}$-type norm for (η, ψ) (which corresponds to the momentum),

$$E^{\frac{1}{4}} := g^{-\frac{1}{4}} H_h^{\frac{1}{4}}(\mathbb{R}) \times g^{\frac{1}{4}} \dot{H}_h^{\frac{3}{4}}(\mathbb{R})$$

with

$$H_h^{\frac{1}{4}}(\mathbb{R}) := \dot{H}^{\frac{1}{4}}(\mathbb{R}) \cap h^{\frac{1}{4}} L^2(\mathbb{R}), \qquad \dot{H}_h^{\frac{3}{4}}(\mathbb{R}) = \dot{H}^{\frac{3}{4}}(\mathbb{R}) + h^{-\frac{1}{4}}\dot{H}^1(\mathbb{R}).$$

For our uniform a-priori bounds for the solutions, ideally one would like to use a scale invariant norm, which would correspond to the following Sobolev bounds:

$$\eta \in H_h^{\frac{3}{2}}(\mathbb{R}), \qquad \nabla\phi|_{y=\eta} \in H_h^1(\mathbb{R}).$$

Our uniform control norm, denoted by X, nearly matches the above ideal scenario. Precisely, we define the homogeneous norm X_0 by

$$X_0 := L_t^\infty H_h^{\frac{3}{2}} \times g^{-\frac{1}{2}} L_t^\infty H_h^1,$$

and then set

$$\|(\eta, \psi)\|_X := \|P_{\leq h^{-1}}(\eta, \nabla\phi|_{y=\eta})\|_{X_0} + \sum_{h^{-1} \leq \lambda \in 2^{\mathbb{Z}}} \|P_\lambda(\eta, \nabla\phi|_{y=\eta})\|_{X_0}.$$

Here we use a standard Littlewood-Paley decomposition beginning at frequency $1/h$,

$$1 = P_{<1/h} + \sum_{1/h < \lambda \in 2^{\mathbb{Z}}} P_\lambda.$$

Based on the expression (1.5) for the energy, we introduce the following notations for the local energy. Fix an arbitrary compactly supported nonnegative function χ. Then, the local energy centered around a point x_0 is

$$\|(\eta, \psi)\|_{LE_{x_0}}^2 := g \int_0^T \int_{\mathbb{R}} \chi(x - x_0)\eta^2 \, dx \, dt + \int_0^T \int_{\mathbb{R}} \int_{-h}^{\eta(t,x)} \chi(x - x_0) \left|\nabla_{x,y}\phi\right|^2 \, dy \, dx \, dt.$$

It is also of interest to take the supremum over x_0,

$$\|(\eta, \psi)\|_{LE}^2 := \sup_{x_0 \in \mathbb{R}} \|(\eta, \psi)\|_{LE_{x_0}}^2.$$

1.6. **Main result.** Our main result for gravity water waves is as follows:

Theorem 1.1 (Local energy decay for gravity waves)**.** *Let $s > 5/2$. There exist ϵ_0 and C_0 such that the following result holds. For all $T \in (0, +\infty)$, all $g \in (0, +\infty)$, all $h \in [1, +\infty)$ and all solutions $(\eta, \psi) \in C^0([0, T]; H^s(\mathbb{R}) \times H^s(\mathbb{R}))$ of the water-wave system (1.4) satisfying*

$$\|(\eta, \psi)\|_X \leq \epsilon_0 \tag{1.7}$$

the following estimate holds

$$\|(\eta, \psi)\|_{LE}^2 \leq C_0(\|(\eta, \psi)(0)\|_{E^{\frac{1}{4}}}^2 + \|(\eta, \psi)(T)\|_{E^{\frac{1}{4}}}^2). \tag{1.8}$$

We continue with several remarks concerning the choices of parameters/norms in the theorem.

Remark 1.2. One key feature of our result is that it is global in time (uniform in T) and uniform in $h \geq 1$. In particular our estimate is uniform in the infinite depth limit.

Remark 1.3. Another feature of our result is that the statement of Theorem 1.1 is invariant with respect to the following scaling law (time associated scaling)

$$(\eta(t, x), \psi(t, x)) \to (\eta(\lambda t, x), \lambda\psi(\lambda t, x))$$
$$(g, h) \to (\lambda^2 g, h).$$

This implies that the value of g is not important. By scaling one could simply set it to 1 in all the proofs. We do not do that in order to improve the readability of the article.

Remark 1.4. As already explained, the uniform control norms in (1.7) are below the current local well-posedness threshold for this problem, and are instead what one might view as the critical, scale invariant norms for this problem. The dependence on h is natural as spatial scaling will also alter the depth h. In the infinite depth limit one recovers exactly the homogeneous Sobolev norms. We also note that, by Sobolev embeddings, our smallness assumption guarantees that

$$|\eta| \lesssim \epsilon_0 h, \qquad |\eta_x| \lesssim \epsilon_0.$$

Remark 1.5. The constraint $h \geq 1$ is due to the window size of 1 in the local energy norm. Of course, once a local energy estimate is obtained for a window size, the similar bound for all larger window sizes also follow. Then bounds for $h < 1$ or for smaller window sizes can also be achieved by scaling; however, the uniformity in h will be lost.

Remark 1.6. In Appendix A, we complement this result by showing a similar estimate for possibly large solutions (satisfying a smallness assumption which is milder than (1.7)), but at the expense of loosing the uniformity in the depth as well as the control of the low-frequency component of the velocity potential.

As in Morawetz' s original paper ([35]), we will obtain these results by using the multiplier method, based on the momentum conservation law. When doing this, we encounter two difficulties:

- **High frequency issues** which are due to the fact that our equations are quasilinear.
- **Low frequency issues** which are due both to the fact that the equations are nonlocal, and that they have quadratic nonlinearities.

Of these, the low frequency issues are far more delicate. To approach them we use both the Eulerian coordinates and the holomorphic coordinates. The latter will provide a better setting to understand the fine bilinear and multilinear structure of the equations.

1.7. **Plan of the paper.** In the next section, we review density flux pairs for the momentum. The density $\eta\psi_x$ implicit in (1.6) only allows one to control the local potential energy, while for the local kinetic energy we introduce an alternate density and the associated flux.

To exploit the density flux identities we need a good understanding of the Dirichlet problem in a strip, which in turn leads us to the holomorphic coordinates. This is discussed in the following section, which also provides the formulation of the equations in holomorphic coordinates and reviews the correspondence between the two settings.

In Section 4 we use the quadratic versions of the above density flux pairs in order to prove the local energy bounds for the corresponding linear flow. This will be later used to handle the leading, quadratic part of the nonlinear identities.

Finally, in the last sections we use the nonlinear density flux pairs to prove the local energy decay bounds in the theorem. Here we use the linear analysis for the main quadratic terms, and the bulk of the work is devoted to estimates for the cubic and higher terms. It is there that a delicate analysis is required in order to handle both the low and high frequency contributions. In particular, the worst such contribution turns out to be unbounded. However, we discovered that this error term can be balanced using a carefully chosen nonlinear normal form type correction to the momentum density.

Acknowledgements. This research was initiated at IHÉS in the spring 2016 during a Trimester on nonlinear waves. The authors thank the

organizers of this semester and the IHÉS for their hospitality. The first author thanks the University of California-Berkeley and the last two authors thank the École Normale Supérieure de Cachan and the University of Orsay, where part of this research was carried out. The first author was partially supported by the grant ÒANAÉÓ ANR-13-BS01-0010-03. The second author was partially supported by a Clare Boothe Luce Professorship. The third author was partially supported by the NSF grant DMS-1266182 as well as by a Simons Investigator grant from the Simons Foundation.

2. Conservation of momentum and local conservation laws

This section contains several formal identities related to the conservation of momentum (these formal computations will be made rigorous later on). We prove that there are certain momentum densities that one can use in defining the horizontal momentum. Clever manipulations of such quantities will lead to control of the kinetic and potential components of the local energy.

The fact that the momentum is a conserved quantity comes from the fact that the problem is invariant with respect to horizontal translation (see Benjamin and Olver [13] for studies of the invariants and symmetries of the water-wave equations). To exploit the conservation of the momentum we will use density flux pairs (I, S) which by definition must satisfy

$$\mathcal{M} = \int I \, dx, \tag{2.1}$$

and also the conservation law

$$\partial_t I + \partial_x S = 0. \tag{2.2}$$

In what follows $m(x)$ is a positive increasing function. Multiplying the identity (2.2) by $m = m(x)$, integrating over $[0, T] \times \mathbb{R}$ and then integrating by parts yields

$$\iint_{[0,T]\times\mathbb{R}} S(t,x) m_x \, dxdt = \int_{\mathbb{R}} m(x) I(T,x) \, dx - \int_{\mathbb{R}} m(x) I(0,x) \, dx.$$

Since m_x is nonnegative, the above identity is favorable provided that S is also nonnegative.

We begin by writing the momentum as an integral over the whole water domain

$$\mathcal{M}(t) = \int_{\mathbb{R}} \int_{-h}^{\eta(t,x)} \phi_x(t,x,y) \, dydx.$$

The next lemma shows that the momentum is an invariant. As already mentioned, this is a well-known result. For the sake of completeness, we

give, following Longuet-Higgins [31], a formal computational proof of the conservation of momentum which is linked to other computations made below. We also give three different expressions for a possible choice of the momentum density.

Lemma 2.1. *We have*

$$\frac{d}{dt}\mathcal{M} = 0.$$

Proof. Set $\Omega(t) = \{(x, y) \ : \ -h < y < \eta(t, x)\}$. To prove this result we first check that, for any function $f = f(t, x, y)$, one has

$$\frac{d}{dt}\iint_{\Omega(t)} f(t, x, y)\, dydx = \iint_{\Omega(t)} (\partial_t + \nabla_{x,y}\phi \cdot \nabla_{x,y}) f\, dydx.$$

Indeed,

$$\begin{aligned}
\int_{\mathbb{R}} (\partial_t \eta) f(t, x, \eta)\, dx &= \int_{\mathbb{R}} (\partial_n \phi) f(t, x, \eta) \sqrt{1 + \eta_x^2}\, dx \\
&= \int_{\partial\Omega(t)} n \cdot (f \nabla_{x,y}\phi)\, d\sigma \\
&= \iint_{\Omega(t)} \mathrm{div}_{x,y}(f \nabla_{x,y}\phi)\, dydx \\
&= \iint_{\Omega(t)} \nabla_{x,y}\phi \cdot \nabla_{x,y} f\, dydx.
\end{aligned}$$

By applying the previous identity with $f = \phi_x$, we deduce that

$$\frac{d}{dt}\mathcal{M} = \iint_{\Omega(t)} (\partial_t + \nabla_{x,y}\phi \cdot \nabla_{x,y})\phi_x\, dydx,$$

so

$$\frac{d}{dt}\mathcal{M} = \iint_{\Omega(t)} \partial_x(\partial_t \phi + \frac{1}{2}\,|\nabla_{x,y}\phi|^2)\, dydx = -\iint_{\Omega(t)} \partial_x P\, dydx.$$

Now, we have

$$\iint_{\Omega(t)} \partial_x P\, dydx = \int_{\mathbb{R}} \partial_x \Big(\int_{-h}^{\eta(t,x)} P\, dx \Big)\, dx - \int \eta_x P|_{y=\eta}\, dx = 0,$$

where we used the boundary condition $P|_{y=\eta} = 0$. This gives the wanted result. □

In addition to the conservation of momentum, one has local conservation laws of the form (2.2), which imply the conservation of momentum. The study of these conservation laws for water waves was initiated by Benjamin and we refer to his broad survey paper about impulse conservation in [12]. Here we discuss density-flux pairs (I, S) for the momentum. These are not unique, and in effect there are three such pairs that play a role in our work. The first two pairs are well

known in fluid mechanics, but the third one is new to the best of our knowledge.

Lemma 2.2. *The expression*

$$I_1(t,x) = \int_{-h}^{\eta(t,x)} \phi_x(t,x,y)\, dy,$$

is a density for the momentum, with associated density flux

$$S_1(t,x) := -\int_{-h}^{\eta(t,x)} \partial_t\phi\, dy - \frac{g}{2}\eta^2 + \frac{1}{2}\int_{-h}^{\eta(t,x)} (\phi_x^2 - \phi_y^2)\, dy.$$

Proof. This result follows from the study by Benjamin in [12]. We give here another proof.

From now on, given a function $f = f(t,x,y)$, we denote by $\widetilde{f}$ the function

$$\widetilde{f}(t,x) = f(t,x,\eta(t,x)).$$

With this notation, one has

$$\partial_t I_1 = \partial_t \int_{-h}^{\eta} \phi_x\, dy = (\partial_t\eta)\widetilde{\phi_x} + \int_{-h}^{\eta} \partial_t\phi_x\, dy.$$

Using the equations for η and the velocity ϕ_x this gives that

$$\partial_t I_1 = (\widetilde{\phi_y} - \eta_x\widetilde{\phi_x})\widetilde{\phi_x} - \int_{-h}^{\eta} \partial_x\left(\frac{1}{2}\left|\nabla_{x,y}\phi\right|^2 + P\right) dy.$$

We deduce that

$$\partial_t I_1 = \widetilde{\phi_y}\widetilde{\phi_x} + \frac{1}{2}\eta_x\widetilde{\phi_y}^2 - \frac{1}{2}\eta_x\widetilde{\phi_x}^2 + \eta_x\widetilde{P} - \partial_x\int_{-h}^{\eta}\left(\frac{1}{2}\left|\nabla_{x,y}\phi\right|^2 + P\right) dy.$$

Using the two equations for the pressure (in the fluid domain and at the free surface), we conclude that

$$\partial_t I_1 = \widetilde{\phi_y}\widetilde{\phi_x} + \frac{1}{2}\eta_x\widetilde{\phi_y}^2 - \frac{1}{2}\eta_x\widetilde{\phi_x}^2 + \partial_x\int_{-h}^{\eta}(\partial_t\phi + gy)\, dy.$$

Since $\partial_x\int_{-h}^{\eta} gy\, dy = \partial_x(g\eta^2/2)$, to conclude the proof, it remains only to check that

$$\widetilde{\phi_y}\widetilde{\phi_x} + \frac{1}{2}\eta_x\widetilde{\phi_y}^2 - \frac{1}{2}\eta_x\widetilde{\phi_x}^2 = \frac{1}{2}\partial_x\int_{-h}^{\eta}(\phi_y^2 - \phi_x^2)\, dy.$$

This can be verified by a direct computation, noticing that

$$\int_{-h}^{\eta}(\phi_x\phi_{yx} - \phi_x\phi_{xx})\, dy = \int_{-h}^{\eta}(\phi_x\phi_{yx} + \phi_x\phi_{yy})\, dy = \int_{-h}^{\eta}\partial_y(\phi_x\phi_y)\, dy = \widetilde{\phi_x}\widetilde{\phi_y},$$

where we used the equations for ϕ to get $\phi_{xx} = -\phi_{yy}$ and $\phi_y(t,x,-h) = 0$. □

Lemma 2.3. *The expression*

$$I_2(t,x) = \eta(t,x)\psi_x(t,x)$$

is a density for the momentum, with associated density flux

$$S_2(t,x) := -\eta\psi_t - \frac{g}{2}\eta^2 + \frac{1}{2}\int_{-h}^{\eta(t,x)} (\phi_x^2 - \phi_y^2)\, dy.$$

Proof. We write that

$$\begin{aligned} I_1 &= \int_{-h}^{\eta(t,x)} \phi_x(t,x,y)\, dy = \partial_x \int_{-h}^{\eta(t,x)} \phi\, dy - \eta_x\psi \\ &= \partial_x \left(\int_{-h}^{\eta(t,x)} \phi\, dy - \eta\psi \right) + \eta\psi_x \\ &= \partial_x \left(\int_{-h}^{\eta(t,x)} \phi\, dy - \eta\psi \right) + I_2. \end{aligned}$$

This immediately implies that $\mathcal{M} = \int I_1\, dx = \int I_2\, dx$ and

$$\partial_t I_2 = \partial_t I_1 - \partial_x \left(\int_{-h}^{\eta(t,x)} \partial_t\phi\, dy - \eta\psi_t \right) = -\partial_x \left(S_1 + \int_{-h}^{\eta(t,x)} \partial_t\phi\, dy - \eta\psi_t \right),$$

so that the wanted expression for S_2 can be deduced from the previous lemma. □

To define the third pair we introduce two auxiliary functions as follows (we shall later rigorously justify that these functions are well-defined). The function q, defined inside the fluid domain, is the stream function, or the harmonic conjugate of ϕ, and satisfies

$$\begin{cases} q_x = -\phi_y, & \text{in } -h < y < \eta(t,x), \\ q_y = \phi_x, & \text{in } -h < y < \eta(t,x), \\ q(t,x,-h) = 0. \end{cases} \tag{2.3}$$

The function θ is the harmonic extension of η with Dirichlet boundary condition on the bottom:

$$\begin{cases} \Delta_{x,y}\theta = 0 & \text{in } -h < y < \eta(t,x), \\ \theta(t,x,\eta(t,x)) = \eta(t,x), \\ \theta(t,x,-h) = 0. \end{cases} \tag{2.4}$$

Now the following lemma states that there is another natural density/flux pair for the momentum.

Lemma 2.4. *The expression*

$$I_3(t,x) = \int_{-h}^{\eta} \nabla\theta(t,x,y) \cdot \nabla q(t,x,y)\, dy$$

is a density for the momentum, with associated density flux

$$S_3(t,x) := -\frac{g}{2}\eta^2 - \int_{-h}^{\eta(t,x)} \theta_y \phi_t \, dy + \int_{-h}^{\eta(t,x)} \Big(\frac{1}{2}(\phi_x^2 - \phi_y^2) + \theta_t \phi_y \Big) \, dy.$$

Proof. We write

$$\nabla\theta \cdot \nabla q = \partial_x(\theta q_x) + \partial_y(\theta q_y),$$

and integrate in y,

$$\begin{aligned} \int_{-h}^{\eta(x,t)} \nabla\theta \cdot \nabla q \, dy &= \int_{-h}^{\eta(x,t)} \big(\partial_x(\theta q_x) + \partial_y(\theta q_y) \big) \, dy \\ &= \partial_x \left(\int_{-h}^{\eta(x,t)} \theta q_x \, dy \right) - \eta_x \widetilde{\theta q_x} + \widetilde{\theta q_y}, \end{aligned}$$

where we recall that, given $f = f(t,x,y)$, we set $\widetilde{f}(t,x) = f(t,x,\eta(t,x))$.

Now we notice that

$$-\eta_x \widetilde{q_x} + \widetilde{q_y} = \eta_x \widetilde{\phi_y} + \widetilde{\phi_x} = \partial_x \widetilde{\phi} = \psi_x,$$

so, recalling that $\widetilde{\theta} = \eta$, we conclude that

$$I_3 = I_2 + \partial_x \int \theta q_x \, dy.$$

Hence I_3 is also a momentum density. Further, its flux is

$$S_3 = S_2 - \partial_t \int_{-h}^{\eta(t,x)} \theta q_x \, dy.$$

We further expand the last time derivative,

$$\begin{aligned} \partial_t \Big(\int_{-h}^{\eta(t,x)} \theta q_x \, dy \Big) &= \eta_t \widetilde{\theta q_x} + \int_{-h}^{\eta(t,x)} \big(-\theta_t \phi_y + \theta q_{xt} \big) \, dy \\ &= -\eta_t \eta \widetilde{\phi_y} + \int_{-h}^{\eta(t,x)} \big(-\theta_t \phi_y - \theta \phi_{yt} \big) \, dy \\ &= -\eta_t \eta \widetilde{\phi_y} - \eta \widetilde{\phi_t} + \int_{-h}^{\eta(t,x)} \big(-\theta_t \phi_y + \theta_y \phi_t \big) \, dy \\ &= -\eta \psi_t + \int_{-h}^{\eta(t,x)} \big(-\theta_t \phi_y + \theta_y \phi_t \big) \, dy. \end{aligned}$$

The conclusion of the lemma easily follows. □

2.1. **The expressions ϕ_t and θ_t.** Here we provide a better description of the functions θ_t and ϕ_t arising in the last momentum flux S_3. For that we introduce two bounded operators, H_D and H_N, which act on functions on the top and produce their harmonic extension within the fluid domain with zero Dirichlet, respectively Neumann boundary condition on the bottom (we shall explain later on that these operators are defined on a space large enough to contain all the functions that

we shall encounter, namely they are well defined on uniformly local L^2 spaces). As an example of the usage of these notations, we have

$$\theta = H_D(\eta), \qquad \phi = H_N(\psi),$$

which means that

$$\begin{cases} \Delta\theta = 0 \quad \text{in } -h < y < \eta, \\ \theta|_{y=\eta} = \eta, \\ \theta|_{y=h} = 0, \end{cases} \qquad \begin{cases} \Delta\phi = 0 \quad \text{in } -h < y < \eta, \\ \phi|_{y=\eta} = \psi, \\ \partial_y\phi|_{y=h} = 0. \end{cases}$$

Recall that, given a function $f = f(t, x, y)$, we set $\widetilde{f}(t, x) := f(t, x, \eta(t, x))$.

Lemma 2.5. *The function ϕ_t is harmonic in the fluid domain, with homogeneous Neumann boundary condition on the bottom, and can be represented as*

$$\phi_t = -gH_N(\eta) - H_N\left(\widetilde{|\nabla\phi|^2}\right). \tag{2.5}$$

The function θ_t is harmonic in the fluid domain, with homogeneous Dirichlet boundary condition on the bottom, and can be represented as

$$\theta_t = \phi_y - H_D\left(\widetilde{\nabla\theta\cdot\nabla}\phi\right). \tag{2.6}$$

Proof. The equation for ϕ follows directly from the Bernoulli equation. To compute the equation for θ, we write that, on the top $\{y = \eta(t, x)\}$,

$$\widetilde{\theta}_t = \eta_t\big(1 - \widetilde{\theta}_y\big), \qquad \widetilde{\theta}_x = \eta_x\big(1 - \widetilde{\theta}_y\big).$$

Then we deduce that

$$\widetilde{\theta_t - \phi_y} = -\eta_x\widetilde{\phi_x}\big(1 - \widetilde{\theta}_y\big) - \widetilde{\phi_y\theta_y} = -\widetilde{\theta_x\phi_x} - \widetilde{\theta_y\phi_y} = -\widetilde{\nabla\theta\cdot\nabla}\phi. \tag{2.7}$$

Since $\theta_t - \phi_y$ vanishes on the bottom $\{y = -h\}$, this implies that $\theta_t - \phi_y$ is the harmonic extension with the Dirichlet boundary condition of $\nabla\theta\cdot\nabla\phi$. □

3. Holomorphic coordinates

3.1. **Harmonic functions in the canonical domain.** Here we discuss two classes of harmonic functions in the horizontal strip $S = \mathbb{R}\times(-h, 0)$.

We start by considering solutions to the homogeneous Laplace equation with homogeneous Neumann boundary condition on the bottom,

$$\begin{cases} \Delta u = 0 \qquad \text{in } \ S \\ u(\alpha, 0) = f \\ \partial_\beta u(\alpha, -h) = 0. \end{cases} \tag{3.1}$$

The solution may be written in the form

$$u(\alpha,\beta) = P_N(\beta, D)f(\alpha) := \frac{1}{2\pi}\int p_N(\xi,\beta)\hat{f}(\xi)e^{i\alpha\xi}\,d\xi,$$

where the Fourier multiplier symbol p_N is given by

$$p_N(\xi,\beta) = \frac{\cosh((\beta+h)\xi)}{\cosh(h\xi)}.$$

We are also interested in the Dirichlet to Neumann map $\mathcal{D}_N$ defined by

$$\mathcal{D}_{\mathcal{N}} f = u_\beta(\cdot, 0).$$

This is closely related to the Tilbert transform, defined by the formula

$$\mathcal{T}_h f(\alpha) = -\frac{1}{2h}\lim_{\epsilon\downarrow 0}\int_{|\alpha-\alpha'|>\epsilon} \operatorname{cosech}\left(\frac{\pi}{2h}(\alpha-\alpha')\right) f(\alpha')\,d\alpha', \tag{3.2}$$

or equivalently, given by the Fourier multiplier

$$\mathcal{T}_h = -i\tanh(hD).$$

We remark that it takes real-valued functions to real-valued functions. We denote the inverse Tilbert transform by $\mathcal{T}_h^{-1}$; a-priori this is only defined modulo constants.

With this notation the Dirichlet to Neumann map for the problem (3.1) is given by

$$\mathcal{D}_N f = \mathcal{T}_h \partial_\alpha f.$$

We will also need to consider to similar problem with the Dirichlet boundary condition on the bottom

$$\begin{cases} \Delta v = 0 \qquad \text{in} \quad S \\ v(\alpha, 0) = g \\ v(\alpha, -h) = 0. \end{cases} \tag{3.3}$$

The solution may be written in the form

$$v(\alpha,\beta) = P_D(\beta, D)g(\alpha) := \frac{1}{2\pi}\int p_D(\xi,\beta)\hat{g}(\xi)e^{i\alpha\xi}\,d\xi,$$

where the Fourier multiplier symbol p_D is given by

$$p_D(\xi,\beta) = \frac{\sinh((\beta+h)\xi)}{\sinh(h\xi)}.$$

The Dirichlet to Neumann map $\mathcal{D}_D$ for this problem is given by

$$v_\beta(\alpha, 0) = \mathcal{D}_D g = -\mathcal{T}_h^{-1}\partial_\alpha g.$$

The solutions to the two problems (3.1) and (3.3) can be related via harmonic conjugates. Precisely, given a real-valued solution u to (3.1),

there exists a unique solution v to (3.3), which is harmonic conjugate to u, i.e., satisfying the Cauchy-Riemmann equations

$$\begin{cases} u_\alpha = -v_\beta \\ u_\beta = v_\alpha \\ \partial_\beta u(\alpha, -h) = 0. \end{cases}$$

The Dirichlet data g for v on the top is determined by the Dirichlet data f for u on the top via the relation

$$g = -\mathcal{T}_h f.$$

Conversely, given v we could seek a corresponding harmonic conjugate u. The difference in this case is that u will only be uniquely determined modulo real constants.

3.2. **A parabolic estimate for harmonic functions.** We are interested in estimates of harmonic functions on vertical lines in terms of the Dirichlet data on the top. These are parabolic type estimates for solutions of these elliptic equations. To introduce these estimates, let us consider the Laplace equation in the half space:

$$\Delta v = 0 \text{ in } \beta < 0, \quad v|_{\beta=0} = g.$$

By considering the Fourier transform in α, one obtains that

$$\partial_\beta^2 \hat{v} - |\xi|^2 \hat{v} = 0,$$

so

$$\hat{v} = A(\xi) e^{\beta|\xi|} + B(\xi) e^{-\beta|\xi|}.$$

Since $\beta < 0$, one has necessarily $B(\xi) = 0$ so we deduce that v solves a parabolic equation (we see β as a time variable)

$$\partial_\beta v - |D_\alpha|\, v = 0 \quad \text{in } \beta < 0, \quad v|_{\beta=0} = g.$$

This is a backward parabolic equation. Namely, the function $w(\alpha, \beta) = v(\alpha, -\beta)$ satisfies $\partial_\beta w + |D_\alpha|\, w = 0$. Now, if we perform a standard energy estimate, multiplying the equation by v, one obtains that

$$\|v(\cdot, \beta)\|_{L^2_\alpha}^2 + 2 \int_\beta^0 \left\| |D_\alpha|^{1/2} v(\cdot, \beta') \right\|_{L^2_\alpha}^2 d\beta' = \|g\|_{L^2_\alpha}^2 .$$

By letting β go to $-\infty$, we conclude that

$$2 \|v\|_{L^2_\beta \dot{H}^{\frac{1}{2}}_\alpha}^2 \leq \|g\|_{L^2_\alpha}^2 .$$

The following lemma improves this inequality in several directions: it allows to control the L^∞-norm instead of the $\dot{H}^{1/2}$-norm, it allows to consider initial data in H^s, and it gives a result that is uniform with respect to the depth. Our main estimate is as follows

Proposition 3.1. *i) Let $s \in \left(-\infty, \frac{1}{2}\right)$. Then the solutions to the equation (3.3) satisfy the following bound:*

$$\|\beta^{-s} v(\alpha, \beta)\|_{L^2_\beta L^\infty_\alpha} \lesssim \|g\|_{H^s_h}. \tag{3.4}$$

ii) The same result also holds for the equation (3.1).

This will transfer easily later on to a similar bound for the Laplace equation in the fluid domain.

Proof. As already mentioned, the solution to (3.3) is of the form

$$v(\alpha, \beta) = \frac{1}{2\pi} \int p_D(\xi, \beta) \hat{g}(\xi) e^{i\alpha\xi} \, d\xi,$$

where

$$p_D(\xi, \beta) = \frac{\sinh((\beta + h)\xi)}{\sinh(h\xi)}.$$

Notice that $|p_D(\xi, \beta)| \leq e^{c\beta|\xi|}$ for some positive constant c.

We now consider a Littlewood-Paley decomposition of g,

$$g = g_{\leq 1/h} + \sum_{1/h < \lambda \in 2^{\mathbb{Z}}} g_\lambda.$$

By the triangle inequality and Bernstein's inequality applied to each corresponding dyadic piece of v we obtain

$$\|v(\beta)\|_{L^\infty_\alpha} \lesssim h^{-\frac{1}{2}} \|g_{\leq 1/h}\|_{L^2} + \sum_{\lambda > 1/h} \lambda^{\frac{1}{2}} e^{c\beta\lambda} \|g_\lambda\|_{L^2}.$$

For $s < \frac{1}{2}$ the functions $\beta^{-s} e^{c\beta\lambda}$ are easily seen to be almost orthogonal in $L^2(-h, 0)$. Then it follows that

$$\|\beta^{-s} v\|^2_{L^2_\beta L^\infty_\alpha} \lesssim h^{-2s} \|g_{\leq 1/h}\|^2_{L^2} + \sum_{\lambda > 1/h} \lambda^{2s} \|g_\lambda\|^2_{L^2},$$

which completes the proof of i).

To prove ii), we remark that we above we have only used the fact that $|p_D(\xi, \beta)|$ is bounded from above by $e^{c\beta|\xi|}$. Since the symbol p_N satisfies the same bound, the same conclusion holds for the solution to (3.1). □

3.3. **Holomorphic functions in the canonical domain.** Here we consider holomorphic functions w in the canonical domain $S := \{\alpha + i\beta \ : \ \alpha \in \mathbb{R}, \ -h \leq \beta \leq 0\}$, which are real on the bottom $\{\mathbb{R} - ih\}$. These functions form a real algebra. Such functions are uniquely determined by their values on the real line $\{\beta = 0\}$, and can be expressed as

$$w = u + iv,$$

where u and v are harmonic conjugate functions which solve the equations (3.1), respectively (3.3).

By extension we will call functions on the real line holomorphic if they are the restriction on the real line of holomorphic functions in the strip and satisfy the above boundary condition on the bottom. This consists of functions $w\colon \mathbb{R} \to \mathbb{C}$ so that there is an holomorphic function, still denoted by $w\colon S \to \mathbb{C}$, which satisfies

$$\operatorname{Im} w = -\mathcal{T}_h \operatorname{Re} w$$

on the top.

The complex conjugates of holomorphic functions are called anti-holomorphic.

3.4. **Holomorphic coordinates and water waves.** Given the fluid domain Ω at some time t we introduce holomorphic coordinates $z = \alpha + i\beta$, via conformal maps

$$Z\colon S \to \Omega(t),$$

which associate the top to the top, and the bottom to the bottom.

These maps are uniquely defined up to horizontal translations in S. Restricted to the real axis this provides a parametrization for the water surface Γ. Because of the boundary condition on the bottom of the fluid domain the function W is holomorphic when $\alpha \in \mathbb{R}$.

Such a conformal transformation exists by the Riemann mapping theorem, and can be constructed as follows:

- construct the harmonic function β in the fluid domain, which takes values 0 on the top, and $-h$ on the bottom.
- construct the function α in the fluid domain as a harmonic conjugate of β. This is uniquely determined modulo real constants.
- invert the holomorphic map $x + iy \to \alpha + i\beta$ to obtain the desired conformal map Z.

Given such a map Z, we denote by

$$W := Z - \alpha,$$

where $W = 0$ if the fluid surface is flat i.e., $\eta = 0$.

Turning our attention to the velocity potential ϕ, we consider its harmonic conjugate q and then the function $Q := \phi + iq$ taken in conformal coordinates is the holomorphic counterpart of ϕ. Here q is exactly the stream function also used in the previous section.

One can model the water wave equations in holomorphic coordinates as an evolution for (W, Q) within the space of holomorphic functions defined on the surface. This is described in detail in the papers [25] for the infinite depth case, respectively [24] for the finite depth case (see also [21]). We recall the equations:

$$\begin{cases} W_t + F(1+W_\alpha) = 0 \\ Q_t + FQ_\alpha - g\mathcal{T}_h[W] + \mathbf{P}_h\left[\dfrac{|Q_\alpha|^2}{J}\right] = 0, \end{cases} \tag{3.5}$$

where

$$J = |1+W_\alpha|^2, \qquad F = \mathbf{P}_h\left[\frac{Q_\alpha - \bar{Q}_\alpha}{J}\right].$$

Here $\mathbf{P}_h$ represents the orthogonal projection on the space of holomorphic functions with respect with the inner product in the Hilbert space $\mathfrak{H}_h$ introduced in [24]. This has the form

$$\langle u, v\rangle_{\mathfrak{H}_h} := \int (\mathcal{T}_h \operatorname{Re} u \cdot \mathcal{T}_h \operatorname{Re} v + \operatorname{Im} u \cdot \operatorname{Im} v)\, d\alpha,$$

and coincides with the L^2 inner product in the infinite depth case. Written in terms of the real and imaginary parts of u, the projection $\mathbf{P}_h$ takes the form

$$\mathbf{P}_h u = \frac{1}{2}\left[(1 - i\mathcal{T}_h)\operatorname{Re} u + i(1 + i\mathcal{T}_h^{-1})\operatorname{Im} u\right]. \tag{3.6}$$

Since all the functions in the system (3.5) are holomorphic, it follows that these relations also hold in the full strip S for the holomorphic extensions of each term.

We also remark that in the finite depth case there is an additional gauge freedom in the above form of the equations, in that $\operatorname{Re} F$ is a-priori only uniquely determined up to constants. This corresponds to the similar degree of freedom in the choice of the conformal coordinates, and will be discussed in the last subsection.

A very useful function in the holomorphic setting is

$$R = \frac{Q_\alpha}{1+W_\alpha},$$

which represents the "good variable" in this setting, and corresponds to the Eulerian function

$$R = \phi_x + i\phi_y.$$

We also remark that the function θ introduced in the previous section is described in holomorphic coordinates by

$$\theta = \operatorname{Im} W.$$

Also related to W, we will use the auxiliary holomorphic function

$$Y = \frac{W_\alpha}{1+W_\alpha}.$$

Another important auxiliary function here is the advection velocity

$$b = \operatorname{Re} F,$$

which represents the velocity of the particles on the fluid surface in the holomorphic setting.

It is also interesting to provide the form of the conservation laws in holomorphic coordinates. We begin with the energy (Hamiltonian), which has the form

$$\mathcal{H} = \frac{g}{2}\int |\operatorname{Im} W|^2(1+\operatorname{Re} W_\alpha)\,d\alpha - \frac{1}{4}\langle Q, \mathcal{T}_h^{-1}[Q_\alpha]\rangle_{\mathfrak{H}_h}.$$

The momentum on the other has the form

$$\mathcal{M} = \frac{1}{2}\langle W, \mathcal{T}_h^{-1}Q_\alpha\rangle_{\mathfrak{H}_h} = \int_{\mathbb{R}} \mathcal{T}_h \operatorname{Re} W \cdot \operatorname{Re} Q_\alpha\,d\alpha = \int_{\mathbb{R}} \operatorname{Im} W \cdot \operatorname{Re} Q_\alpha\,d\alpha.$$

3.5. **Uniform bounds for the conformal map.** In order to freely switch computations between the Eulerian and holomorphic setting it is very useful to verify that our Eulerian uniform smallness assumption also has an identical interpretation in the holomorphic setting.

To account for the uniformity in time in the X norm it is very convenient to use the language of frequency envelopes. We define a *frequency envelope* for $(\eta, \nabla\phi_{|y=\eta})$ in X to be any positive sequence

$$\left\{c_\lambda;\quad h^{-1} < \lambda \in 2^{\mathbb{Z}}\right\}$$

with the following two properties:

(1) Dyadic bound from above,

$$\|P_\lambda(\eta, \nabla\phi_{|y=\eta})\|_{X_0} \le c_\lambda.$$

(2) Slowly varying,

$$\frac{c_\lambda}{c_\mu} \le \max\left\{\left(\frac{\lambda}{\mu}\right)^\delta, \left(\frac{\mu}{\lambda}\right)^\delta\right\}.$$

Here $\delta \ll 1$ is a small universal constant. Among all such frequency envelopes there exists a *minimal frequency envelope*. In particular, this envelope has the property that

$$\|(\eta, \nabla\phi_{|y=\eta})\|_X \approx \|c\|_{\ell^1}.$$

This will play an important role in our analysis:

Definition 3.2. *By $\{c_\lambda\}_{\lambda\ge 1/h}$ we denote the minimal frequency envelope for $(\eta, \nabla\phi|_{y=\eta})$ in X_0. We call $\{c_\lambda\}$ the* control frequency envelope.

Since in solving the Laplace equation on the strip, solutions at depth β are localized at frequencies $\le \lambda$ where $\lambda \approx |\beta|^{-1}$, we will also use the notation

$$c_\beta = c_\lambda, \qquad \lambda \approx |\beta|^{-1}.$$

This uniquely determines c_β up to a small multiplicative constant, which suffices for our purposes.

We now use the control envelope to transfer the control norm bound for $(\eta, \nabla\phi_{|y=\eta})$ to their counterpart $(\operatorname{Im} W, R)$ in the holomorphic coordinates.

Proposition 3.3. *Assume the smallness condition (1.7), and let $\{c_\lambda\}$ be the control envelope as above. Then we have*

$$\|P_\lambda(\operatorname{Im} W, R)\|_{X_0} \lesssim c_\lambda. \tag{3.7}$$

Remark 3.4. We remark that this in particular implies the X bound

$$\|(\operatorname{Im} W, R)\|_X \lesssim \epsilon_0, \tag{3.8}$$

and also, by Bernstein's inequality, the pointwise bound

$$\|W_\alpha\|_{L^\infty} \lesssim \epsilon_0. \tag{3.9}$$

This in turn implies that the Jacobian matrix for the change of coordinates stays close to the identity.

Proof. By a continuity argument, it suffices to prove the desired bounds under the additional bootstrap assumption

$$\|(\operatorname{Im} W, R)\|_X \leq \epsilon_1, \qquad \epsilon_0 \ll \epsilon_1 \ll 1. \tag{3.10}$$

We caution the reader that the two X norms and their associated frequency envelopes for $(\eta, \nabla\phi_{|y=\eta})$, respectively $(\operatorname{Im} W, R)$ are relative to different coordinate systems, Eulerian vs. holomorphic.

To prove the proposition we first compare the regularity of $\operatorname{Im} W$ with the regularity of η, since (either of) these functions determine the conformal map. Let $\{c_\lambda\}$, $\{d_\lambda\}$ be minimal frequency envelopes for $(\eta, \nabla\phi_{|y=\eta})$, respectively $(\operatorname{Im} W, R)$ in X, so that we have

$$\|d\|_{\ell^1} \leq \epsilon_1.$$

Then we will show that for each $\lambda \geq 1/h$ we have the equivalence

$$c_\lambda \approx d_\lambda. \tag{3.11}$$

Our bootstrap assumption insures that $\operatorname{Re} W_\alpha$ is pointwise small, which implies that the change of coordinates $x = \alpha + \operatorname{Re} W(\alpha)$ is biLipschitz, so we easily have the norm equivalence

$$\|f\|_{L^2_\alpha} \approx \|f\|_{L^2_x}, \qquad \|f\|_{\dot H^1_\alpha} \approx \|f\|_{\dot H^1_x}. \tag{3.12}$$

The L^2 bound allows us to easily compare the L^2 norms of η and $\operatorname{Im} W$, which accounts for the case $\lambda = 1/h$, namely

$$\|\operatorname{Im} W_{<1/h}\|_{L^2} \lesssim h^{\frac{3}{2}} c_{1/h}, \qquad \|\eta\|_{L^2} \lesssim h^{\frac{3}{2}} d_{1/h}.$$

For higher frequencies, it remains to compare minimal frequency envelopes for their derivatives η_x and $\operatorname{Im} W_\alpha$ in $H_h^{\frac{1}{2}}$, which are also comparable to c_λ, respectively d_λ. Here we also need bounds for

$$\operatorname{Re} W_\alpha = -\mathcal{T}_h^{-1}\partial_\alpha \operatorname{Im} W.$$

But it is easily seen that d_λ is also an envelope for $\operatorname{Re} W_\alpha$ in $H_h^{\frac{1}{2}}$.

To begin with, we note that by interpolation, the bound (3.12) insures the equivalence of all intermediate $l^p(H_h^s)$ norms and envelopes for all $1 \le p \le \infty$ and $0 < s < 1$, with uniform frequency envelope bounds. We will use this property for the norm $\ell^1 H_h^{\frac{1}{2}}$, in order to harmlessly switch the function η_x to holomorphic coordinates. Hence it remains to compare the $H_h^{\frac{1}{2}}$ frequency envelopes for the functions η_x and W_α both measured in the holomorphic coordinates. This is convenient since by chain rule we have the relation

$$\eta_x = \frac{\operatorname{Im} W_\alpha}{1 + \operatorname{Re} W_\alpha}.$$

To deal with the nonlinear expression we use the algebra property of $\ell^1 H_h^{\frac{1}{2}}$, expressed in a frequency envelope fashion. For convenience, we state this as

Lemma 3.5. *a) The space $\ell^1 H_h^{\frac{1}{2}}$ is an algebra[3]. Furthermore, if $u, v \in \ell^1 H_h^{\frac{1}{2}}$ have frequency envelopes c_λ^u, c_λ^v then an envelope for uv is given by*

$$c_\lambda^{uv} = c_\lambda^u \|c^v\|_{\ell^1} + c_\lambda^v \|c^u\|_{\ell^1}.$$

b) Let $u \in L^2$ and $v \in \ell^1 H_h^{\frac{1}{2}}$ have frequency envelopes c_λ^u, c_λ^v then an envelope for uv in L^2 is given by

$$c^{uv}(\lambda) = c_\lambda^u \|c^v\|_{\ell^1}.$$

The proof of the lemma is relatively simple and is omitted.

The smallness of ϵ_1 in our bootstrap assumption allows us to use the lemma in order to estimate the difference

$$\eta_x - \operatorname{Im} W_\alpha = -\frac{\operatorname{Im} W_\alpha \cdot \operatorname{Re} W_\alpha}{1 + \operatorname{Re} W_\alpha}.$$

Precisely, a frequency envelope for $\eta_x - \operatorname{Im} W_\alpha$ will be given by $\epsilon_1 d_\lambda$. Then, by the triangle inequality for minimal frequency envelopes, we must have

$$|c_\lambda - d_\lambda| \lesssim \epsilon_1 d_\lambda.$$

[3]This property suffices in the present paper since W_α is small in L^∞. However, even if W_α were large, then bounds as in the lemma would still be valid. However, proving that would require corresponding Moser estimates in $\ell^1 H_h^{\frac{1}{2}}$. For that we refer the reader to the similar analysis in [25].

Since $\epsilon_1 \ll 1$, this implies that $c_\lambda \approx d_\lambda$. This concludes the proof of (3.11) restricted to the η and $\operatorname{Im} W$ components.

Next we consider the equivalence of the frequency envelopes for $\nabla\phi_{|y=\eta}$ respectively $R = (\phi_x + i\phi_y)_{|y=\eta}$ in H^1_h. These are one and the same function, and the only difficulty is that the H^1_h norms and frequency envelopes are measured in different frames, Eulerian vs. holomorphic. The L^2 part of the H^1_h norm is easily dealt with using (3.12), so it remains to compare the frequency envelopes for their derivatives in L^2.

As before, we compute using the chain rule

$$g := \partial_x(\phi_x + i\phi_y)_{|y=\eta} = \frac{R_\alpha}{1 + \operatorname{Re} W_\alpha}.$$

Using part (b) of the last lemma, it is easily seen that in holomorphic coordinates the function g has a minimal frequency envelope comparable to that of R_α. Thus it only remains to see that the function g has equivalent L^2 minimal frequency envelopes in Eulerian and holomorphic coordinates.

This follows if we show the following off-diagonal decay:

$$\|P^E_\lambda P_\mu g\|_{L^2} \lesssim \left\{\frac{\lambda}{\mu}, \frac{\mu}{\lambda}\right\}, \tag{3.13}$$

where P^E_λ and P_μ are Littlewood-Paley projectors in the Eulerian, respectively holomorphic frame.

To prove (3.13) we consider two cases:

a) $\lambda \geq \mu$. Then we write

$$\|P^E_\lambda P_\mu g\|_{L^2} \lesssim \lambda^{-1}\|\partial_x P_\mu g\|_{L^2} \lesssim \lambda^{-1}\|\partial_\alpha P_\mu g\|_{L^2} \lesssim \mu/\lambda.$$

b) $\lambda \leq \mu$. Then we use duality to interchange the two projections, and then argue exactly in the same way.

The proof of the Proposition 3.3 is complete. □

As a consequence of the last proposition we can further extend the range of our frequency envelope estimates:

Remark 3.6. The previous proposition and its proof show that $\{c_\lambda\}$ is also a frequency envelope for

- $(\operatorname{Im} W, R)$ in X_0.
- W_α in $H^{\frac{1}{2}}_h$ and L^∞.
- Y in $H^{\frac{1}{2}}_h$.

Here the last property is a direct consequence of Lemma 3.5.

3.6. **Vertical strips in Eulerian vs holomorphic coordinates.** In our main result, we define local energy functionals using vertical strips in Eulerian coordinates. On the other hand, for the multilinear analysis in our error estimates in the last two sections, we would like to use vertical strips in holomorphic coordinates. Of course these two types of vertical strips do not perfectly match. To switch from one to the other we need to estimate the horizontal drift between the two strips in depth.

As the conformal map is biLipschitz, it suffices to compare the centers of the two strips. It is more convenient to do this in the reverse order, and compare the Eulerian image of the holomorphic vertical section with the Eulerian vertical section:

Proposition 3.7. *Let $(x_0, \eta(x_0)) = Z(\alpha_0, 0)$, respectively $(\alpha_0, 0)$ be the coordinates of a point on the free surface in Eulerian, respectively holomorphic coordinates. Assume that (1.7) holds, and let $\{c_\lambda\}$ be the control frequency envelope in Definition 3.2. Then we have the uniform bounds:*

$$|\operatorname{Re} Z(\alpha_0, \beta) - x_0 + \beta \operatorname{Im} W_\alpha(\alpha_0, \beta)| \lesssim c_\lambda, \qquad |\beta| \approx \lambda^{-1}. \tag{3.14}$$

As a corollary, we see that the distance between the two strip centers grows at most linearly:

Corollary 3.8. *Under the same assumptions as in the above proposition we have*

$$|\operatorname{Re} Z(\alpha_0, \beta) - x_0| \lesssim \epsilon_0 |\beta|. \tag{3.15}$$

Proof. We consider the expression

$$D = \operatorname{Re} Z(\alpha, \beta) - x_0 + \beta \operatorname{Im} W_\alpha(\alpha_0, \beta) = \operatorname{Re} W(\alpha, \beta) - \operatorname{Re} W(\alpha, 0) + \beta \operatorname{Im} W_\alpha(\alpha_0, \beta).$$

We can express this in terms of $\operatorname{Im} W$ on the top as follows:

$$\begin{aligned} D &= (P_N(D, \beta) - 1) \operatorname{Re} W(\alpha, 0) + \beta \partial_\alpha P_D(D, \beta) \operatorname{Im} W(\alpha, 0) \\ &= \big(\mathcal{T}_h^{-1}(P_N(D, \beta) - 1) - i\beta D P_D(D, \beta)\big) \operatorname{Im} W(\alpha, 0). \end{aligned}$$

The symbol for the multiplier

$$M(D, \beta) = \mathcal{T}_h^{-1}(P_N(D, \beta) - 1) - i\beta D P_D(D, \beta)$$

is

$$\begin{aligned} m(\xi, \beta) &= \frac{i \cosh\left((\beta + h)\xi\right) - \cosh(h\xi)}{\sinh(h\xi)} - \frac{i\beta\xi \sinh\left((\beta + h)\xi\right)}{\sinh(h\xi)} \\ &= \frac{2 \sinh\left(\beta\xi/2\right) \sinh\left((h + \beta/2)\xi\right) - \beta\xi \sinh\left((\beta + h)\xi\right)}{\sinh(h\xi)}. \end{aligned}$$

This is easily seen to be smooth and satisfy the bound

$$|m(\xi, \beta)| \lesssim \min\{1, |\beta\xi|^2\}.$$

Given this symbol bound, the conclusion of the proposition follows by applying Bernstein's inequality for each dyadic frequency, and then summing up. □

3.7. **The horizontal gauge invariance.** Here we briefly discuss the gauge freedom due to the fact that $\operatorname{Re} F$ is a-priori only uniquely determined up to constants. In the infinite depth case this gauge freedom is removed by making the assumption $F \in L^2$. In the finite depth case (see [24]) instead this is more arbitrarily removed by setting $F(\alpha = -\infty) = 0$.

In the present paper no choice is necessary for our main result, as well as for most of the proof. However, in the choice of the normal form momentum density correction in Section 5 it is convenient to make such a choice, which is discussed next. This choice is used in the very last step in Section 7.

Assume first that we have a finite depth. We start with a point $x_0 \in \mathbb{R}$ where our local energy estimate is centered. Then we resolve the gauge invariance with respect to horizontal translations by setting $\alpha(x_0) = x_0$, which corresponds to setting $\operatorname{Re} W(x_0) = 0$. In dynamical terms, this implies that the real part of F is uniquely determined by

$$0 = \operatorname{Re} W_t(x_0) = \operatorname{Re}(F(1 + W_\alpha))(x_0),$$

which yields

$$\operatorname{Re} F(x_0) = \operatorname{Im} F(x, 0) \frac{\operatorname{Im} W_\alpha(x_0)}{1 + \operatorname{Re} W_\alpha(x_0)}.$$

In the infinite depth case, the canonical choice for F is the one vanishing at infinity. This corresponds to a moving location in the α variable. We can still rectify this following the finite depth model, at the expense of introducing a constant component in both $\operatorname{Re} W$ and in F. We will follow this convention in the paper, in order to insure that our infinite depth computation is an exact limit of the finite depth case.

4. Local energy decay for linear gravity waves

4.1. **Linearized equations in Eulerian coordinates.** In Eulerian coordinates the linearized equations around the zero solution are

$$\begin{cases} \partial_t \eta = \mathcal{D}_N \psi \\ \partial_t \psi = -g\eta, \end{cases} \tag{4.1}$$

where $\mathcal{D}_N$ is the Dirichlet to Neumann map associated to depth $h > 0$, given by

$$D_N \psi = \partial_y \phi_{|top} = \mathcal{T}_h \partial_x \psi,$$

where recall that $\mathcal{T}_h$ is the Tilbert operator given by (3.2) and ϕ is the harmonic extension of ψ in the flat strip $S = \{(x, y) \in \mathbb{R}^2 \ : \ -h < y < 0\}$, so that

$$\begin{cases} \Delta\phi = 0 \quad \text{in } S, \\ \phi|_{y=0} = \psi, \\ \partial_y\phi|_{y=-h} = 0. \end{cases}$$

For such η and ψ we define the (conserved) energy as

$$E_{lin}(\eta, \psi) := \frac{g}{2}\|\eta\|_{L^2}^2 + \frac{1}{2}\langle \mathcal{T}_h\partial_x\psi, \psi\rangle.$$

We can express the energy in a more symmetric fashion by using the harmonic extension ϕ of ψ in the strip S with Neumann boundary condition on the bottom. Then

$$E_{lin}(\eta, \psi) = \frac{g}{2}\|\eta\|_{L^2}^2 + \frac{1}{2}\|\nabla\phi\|_{L^2(S)}^2.$$

We also introduce higher energies

$$E_{lin}^s(\eta, \psi) := \frac{g^{1-2s}}{2}\|(\mathcal{T}_h^{-1}\partial_x)^s\eta\|_{L^2}^2 + \frac{g^{-2s}}{2}\|(\mathcal{T}_h^{-1}\partial_x)^s\nabla\phi\|_{L^2(S)}^2.$$

These are homogeneous norms in the infinite depth case, but the homogeneity is broken in the finite depth case.

The local energy for the linearized equation is given by

$$\|(\eta, \psi)\|_{LE}^2 = \|\eta\|_{LE^0}^2 + \|\nabla\phi\|_{LE^{-\frac{1}{2}}}^2,$$

where

$$\|\eta\|_{LE^0} := \sup_{x_0\in\mathbb{R}} \|\eta\|_{LE^0_{x_0}}, \qquad \|\eta\|_{LE^0_{x_0}}^2 = \int_0^T \int \chi(x - x_0)\eta^2 \, dx \, dt$$

while

$$\|\nabla\phi\|_{LE^{-\frac{1}{2}}} := \sup_{x_0\in\mathbb{R}} \|\nabla\phi\|_{LE^{-\frac{1}{2}}_{x_0}}, \qquad \|\nabla\phi\|_{LE^{-\frac{1}{2}}_{x_0}}^2 = \int_0^T \iint_S \chi(x-x_0)|\nabla\phi|^2 \, dxdydt.$$

With these notations, the local energy decay estimate for the linearized equation is as follows:

Theorem 4.1. *There exists a constant C such that, for all $h \in [1, +\infty)$ and all $T \in (0, +\infty)$, solutions (η, ψ) to the above system (4.1) satisfy the local energy bound*

$$\|(\eta, \psi)\|_{LE} \le C\left(\|(\eta, \psi)(0)\|_{E^{\frac{1}{4}}_{lin}} + \|(\eta, \psi)(T)\|_{E^{\frac{1}{4}}_{lin}}\right). \tag{4.2}$$

The rest of the section is devoted to the proof of the theorem. By scaling we can and will assume without any loss of generality that $h \gg 1$. Precisely, in the following proof h will play the role of an (inverse) semiclassical parameter.

The proof is based on Morawetz' identities starting from the momentum conservation, and more precisely from the linear counterparts of the momentum densities I_2 and I_3 in section 2. We define the momentum as

$$\mathcal{M} = \int \eta\psi_x \, dx$$

with $I_2(x) = \eta\psi_x$ as the first momentum density.

For this proof, given a function $f = f(t, x, y)$ with $(x, y) \in \overline{S}$, we set

$$\widetilde{f}(t, x) := f(t, x, 0).$$

Now, using the equations for η, ψ, given a bounded increasing function m, we compute

$$\partial_t \int m(x) I_2(t, x) \, dx = \int m\widetilde{\phi_y}\psi_x \, dx - \int gm\eta\eta_x \, dx.$$

The second term in the right-hand side gives

$$\frac{g}{2} \int m_x \eta^2 \, dx.$$

The first term can be written as

$$\begin{aligned}
\int m\widetilde{\phi_y}\widetilde{\phi_x} \, dx &= \iint m\partial_y(\phi_y\phi_x) \, dydx \\
&= \iint m(\phi_{yy}\phi_x + \phi_y\phi_{xy}) \, dydx \\
&= \iint m(-\phi_{xx}\phi_x + \phi_y\phi_{xy}) \, dydx \\
&= \frac{1}{2} \iint m_x(\phi_x^2 - \phi_y^2) \, dydx.
\end{aligned}$$

Thus we conclude that

$$\partial_t \int mI_2 \, dx = \frac{g}{2} \int m_x\eta^2 \, dx + \frac{1}{2} \iint m_x(\phi_x^2 - \phi_y^2) \, dydx. \tag{4.3}$$

The first term on the right is a component of the local energy, whereas the second is nonnegative when m_x is replaced by 1 (see Lemma A.3 in the appendix, applied with $w = 1$ and $\eta = 0$).

We now continue by using a second momentum density I_3, which in addition to the functions η, ψ and ϕ, depends on the functions θ and q introduced in the previous sections (see (2.3) and (2.4)):

- θ is the harmonic extension of η with Dirichlet boundary condition on the bottom;
- q is the harmonic conjugate of ϕ with Dirichlet boundary condition on the bottom.

With these notations, one has

$$\mathcal{M} = \int I_3\, dx \quad \text{with} \quad I_3 = \int_{-h}^{0} \nabla\theta \cdot \nabla q\, dy.$$

Although it is natural to define I_3 in terms of (θ, q), for the computations it is convenient to express I_3 in terms of (θ, ϕ). It follows immediately from the equations $q_x = -\phi_y$ and $q_y = \phi_x$ that

$$I_3(t,x) = \int_{-h}^{0} \big(\theta_y\phi_x - \theta_x\phi_y\big)\, dy.$$

Notice that

$$\partial_t\theta = \phi_y$$

(this is the simplified version of (2.6) for the linearized equation). As a result, we get for any weight m,

$$\frac{d}{dt}\int mI_3\, dx = -\iint m\big(\phi_{yy}\phi_x - \phi_{yx}\phi_y\big)\, dydx + \iint m\big(\theta_y\partial_t\phi_x - \theta_x\partial_t\phi_y\big)\, dydx. \tag{4.4}$$

Since $\phi_{yy} = -\phi_{xx}$, integrating by parts, the first term gives the expression

$$-\iint m\big(\phi_{yy}\phi_x - \phi_{yx}\phi_y\big)\, dydx = \frac{1}{2}\iint m_x|\nabla\phi|^2\, dydx,$$

which is the second part of the local energy. Our second observation is that the second term depends only on m and η. To see this, we use the operator H_D (respectively H_N) introduced in the previous section, which maps a function $f = f(x)$ to its harmonic extension in the strip S with Dirchlet (respectively Neumann) boundary condition on the bottom. Then, by definition, one has $\theta = H_D(\eta)$. On the other hand, since $\partial_t\phi|_{y=0} = -g\eta$, it follows that $\partial_t\phi = -gH_N(\eta)$. Consequently, one has

$$\iint m\big(\theta_y\partial_t\phi_x - \theta_x\partial_t\phi_y\big)\, dydx = gQ_m(\eta),$$

where

$$Q_m(\eta) := \iint m\big(H_N(\eta)_y H_D(\eta)_x - H_N(\eta)_x H_D(\eta)_y\big)\, dydx. \tag{4.5}$$

Thus, we conclude that

$$\frac{d}{dt}\int mI_3\, dx = \frac{1}{2}\iint m_x|\nabla\phi|^2\, dydx + g\iint mQ_m(\eta)\, dydx. \tag{4.6}$$

Notice that in the infinite depth case, one has $H_N(\eta) = H_D(\eta)$ so $Q_m(\eta) = 0$, which greatly simplifies the proof of the theorem. To prove a result that holds uniformly in the finite depth case, the idea here is

now to try to combine the two local energies in a more balanced way. Given a parameter $\sigma \in [0,1]$, we define

$$\mathcal{I}_m^\sigma(t) = \sigma \int mI_2\, dx + (1-\sigma) \int mI_3\, dx.$$

Then we have the following:

Proposition 4.2. *Let $h \gg 1$. Then*

a) For each $\sigma \in [0,1]$ we have

$$|\mathcal{I}_m^\sigma(t)| \lesssim \|(\eta,\psi)\|^2_{E^{\frac14}_{lin}}. \tag{4.7}$$

b) There exist $\sigma < \frac12$ close to $\frac12$ and $c < 1$ independent of h so that

$$\mathcal{I}_m^\sigma(T) - \mathcal{I}_m^\sigma(0) \geq \|(\eta,\psi)\|^2_{LE_{x_0=0}} - c\|(\eta,\psi)\|^2_{LE} \tag{4.8}$$

holds for all solutions (η,ψ) of the equation (4.1).

The conclusion of the theorem follows by taking supremum over all translates of (4.8). The remainder of the section is devoted to the proof of the proposition.

We begin with part (a). We need to consider the the two momentum densities I_2 and I_3. The contribution of I_2 has the form

$$\int m\eta\psi_x\, dx.$$

We estimate this as follows

$$\left| \int m\eta\psi_x\, dx \right| \lesssim \|m\eta\|_{H_h^{\frac14}} \|\psi_x\|_{H_h^{-\frac14}},$$

and conclude using the fact that m is a bounded multiplication operator in $H_h^{\frac14}$,

$$\|m\eta\|_{H_h^{\frac14}} \lesssim \|m_x\|_{L^1} \|\eta\|_{H_h^{\frac14}}. \tag{4.9}$$

Now we consider the contribution of I_3. To do so, we integrate by parts to arrive at

$$\begin{aligned} I_3 &= \iint_S m(\theta_y\phi_x - \theta_x\phi_y)\, dydx = \iint_S m\big(\partial_y(\theta\phi_x) - \partial_x(\theta\phi_y)\big)\, dydx \\ &= \int mI_2\, dx - \iint_S m_x\theta q_x\, dydx. \end{aligned}$$

It remains to estimate the second part for which we will use the x-localized L^2 bounds for harmonic extensions in Proposition 3.1. This yields

$$\big\|y^{-\frac14}\theta\big\|_{L^\infty_x(L^2_y)} \leq \big\|y^{-\frac14}\theta\big\|_{L^2_y(L^\infty_x)} \lesssim \big\|\eta\big\|_{H_h^{\frac14}},$$

and similarly

$$\big\|y^{+\frac14}\phi_y\big\|_{L^\infty_x(L^2_y)} \lesssim \big\|\phi_y|_{y=0}\big\|_{H_h^{-\frac14}}.$$

Since m_x is a positive function with integral 1, we conclude that

$$\left|\iint_S m_x \theta \phi_y \, dydx\right| \le \Big(\int m_x \, dx\Big) \big\|y^{-\frac{1}{4}}\theta\big\|_{L^\infty_x(L^2_y)} \big\|y^{+\frac{1}{4}}\phi_y\big\|_{L^\infty_x(L^2_y)} \lesssim \|\eta\|_{H^{\frac{1}{4}}_h} \|\phi_y\|_{H^{-\frac{1}{4}}_h}.$$

Since $\phi_y|_{y=0} = \mathcal{D}\psi$, this gives the wanted estimate (4.7).

We now prove part (b). We have

$$\mathcal{I}^\sigma_m(T) - \mathcal{I}^\sigma_m(0) = LE_\phi + LE_\eta, \tag{4.10}$$

where

$$LE_\phi = \int_0^T \iint \Big(\frac{1-\sigma}{2} m_x |\nabla\phi|^2 + \frac{\sigma}{2} m_x(\phi_x^2 - \phi_y^2)\Big)\, dydxdt,$$

and

$$LE_\eta = \frac{\sigma g}{2}\int_0^T \int m_x \eta^2 \, dxdt + (1-\sigma) g \int_0^T Q_m(\eta)\, dt,$$

where $Q_m(\eta)$ is defined by (4.5). We first observe that the second term in LE_ϕ is clearly positive if $\sigma < \frac{1}{2}$. So, to conclude the proof, it is sufficient to prove that the LE_η component controls the potential energy. This in turn is straightforward in the infinite depth case, since then, $Q_m(\eta) = 0$. Hence from here on we focus on the finite depth case where the challenge is in part to gain the uniformity as $h \to \infty$.

So the goal is to prove that for some $\sigma \in (0, 1/2)$, the expression LE_η is positive definite, either directly or after taking a supremum over all translations of m. For that we need to write it in terms of η and m_x.

Notation 4.3. Given a complex-valued function $b = b(\xi_1, \xi_2)$, we define the bilinear Fourier multiplier B with symbol b by

$$B(f,g)(x) := \frac{1}{2\pi} \iint_{\mathbb{R}^2} e^{ix(\xi_1+\xi_2)} b(\xi_1,\xi_2) \hat{f}(\xi_1)\hat{g}(\xi_2)\, d\xi_1\, d\xi_2.$$

Lemma 4.4. *The bilinear form Q_m admits the representation*

$$Q_m(\eta) = \int m_x B^h(\eta,\eta)\, dx,$$

where $B^h(\eta,\eta)$ is a bilinear Fourier multiplier with symbol

$$b^h(\xi,\zeta) = \frac{\xi\zeta}{\sinh 2\xi h \sinh 2\zeta h} \frac{\cosh 2h\xi - \cosh 2h\zeta}{(\xi+\zeta)(\xi-\zeta)}.$$

Proof. Recall that

$$\widehat{H_D(\eta)}(\xi, y) = \frac{\sinh \xi(y+h)}{\sinh \xi h}\hat{\eta}(\xi), \quad \text{resp.} \quad \widehat{H_N(\eta)}(\xi, y) = \frac{\cosh((h+y)\xi)}{\cosh(h\xi)}\hat{\eta}(\xi).$$

Consequently,

$$\begin{aligned}&\big(H_N(\eta)_y H_D(\eta)_x - H_N(\eta)_x H_D(\eta)_y\big)(t,x,y)\\&\quad= \frac{1}{2\pi}\iint_{\mathbb{R}^2} e^{ix(\zeta+\xi)} b_0^h(y,\xi,\zeta)\hat{\eta}(t,\xi)\hat{\eta}(t,\zeta)\,d\xi\,d\alpha,\end{aligned}$$

where

$$\begin{aligned}b_0^h(y,\xi,\zeta) &= i\xi\zeta\left(\frac{\sinh\xi(y+h)}{\sinh\xi h}\frac{\sinh\zeta(y+h)}{\cosh\zeta h} - \frac{\cosh\xi(y+h)}{\sinh\xi h}\frac{\cosh\zeta(y+h)}{\cosh\zeta h}\right)\\&= \frac{-i\xi\zeta}{\sinh\xi h\cosh\zeta h}\cosh((y+h)(\xi-\zeta)).\end{aligned}$$

Integrate in y to get

$$\int_{-h}^{0} b_0^h(y,\xi,\zeta)\,dy = \frac{-i\xi\zeta}{\sinh\xi h\cosh\zeta h}\frac{\sinh h(\xi-\zeta)}{\xi-\zeta}.$$

Notice that for any bilinear Fourier multiplier B with symbol b, one has

$$B(f,f) = B^{sym}(f,f) \quad \text{with} \quad b^{sym}(\xi_1,\xi_2) = \frac{1}{2}(b(\xi_1,\xi_2)+b(\xi_2,\xi_1)).$$

By so doing, we obtain that

$$\int \big(H_N(\eta)_y H_D(\eta)_x - H_N(\eta)_x H_D(\eta)_y\big)(t,x,y)\,dx = B_1^h(\eta,\eta),$$

where B_1^h is the bilinear Fourier multiplier with symbol

$$b_1^h(\xi,\zeta) = \frac{-2i\xi\zeta}{\sinh 2\xi\sinh 2\zeta h}\frac{\sinh h(\xi+\zeta)\sinh h(\xi-\zeta)}{2(\xi-\zeta)}.$$

Integrating by parts we obtain

$$\int m(x)B_1^h(\eta,\eta)\,dx = \int m_x(x)B^h(\eta,\eta)\,dx,$$

where the symbol of B^h is given by

$$b^h(\xi,\zeta) = \frac{i}{\xi+\zeta}b_1^h(\xi,\zeta) = \frac{2\xi\zeta}{\sinh 2\xi h\sinh 2\zeta h}\frac{\sinh h(\xi+\zeta)\sinh h(\xi-\zeta)}{2(\xi+\zeta)(\xi-\zeta)},$$

which gives the desired result. □

To conclude the proof of (4.8), in light of (4.10) and the previous lemma, it remains only to prove the following result.

Proposition 4.5. *For the bilinear form B^h above there exists $c < \frac{1}{2}$ so that we have*

$$\int_0^T\int m_x B^h(\eta,\eta)\,dxdt \geq -c\sup_{x_0\in\mathbb{R}}\int_0^T\int m_x(x-x_0)\eta^2\,dxdt. \quad (4.11)$$

This concludes the proof of the Proposition 4.2. It now remains to prove this proposition. We remark that we have written this proposition as a separate result in order to be able to apply it directly also for the nonlinear problem.

Our first task is to understand the properties of the symbols B^h and of their kernels K^h. The first observation concerning the symbols b^h is that they are all obtained by scaling from a single symbol

$$b(\xi,\zeta) = \frac{2\xi\zeta}{\sinh 2\xi \cosh 2\zeta} \frac{\sinh(\xi+\zeta)\sinh(\xi-\zeta)}{(\xi+\zeta)(\xi-\zeta)},$$

as follows,

$$b^h(\xi,\zeta) = b(h\xi, h\zeta).$$

Then the kernels K^h are related to the kernel K of B by

$$K_h(x_1,x_2) = h^{-2} K(h^{-1}x_1, h^{-1}x_2).$$

Concerning the symbol b, one easily sees that it has the following properties:

- It is real, even and symmetric.
- It is uniformly smooth.
- It decays exponentially away from the axes $\xi = 0$, $\zeta = 0$,

$$|b(\xi,\zeta)| \leq \frac{1}{1+|\xi|+|\zeta|} e^{-c\min\{|\xi|,|\zeta|\}}.$$

- Near $\xi = 0$ it has the expansion

$$b(\xi,\zeta) = \frac{1}{|\zeta|}\frac{2\xi}{\sinh 2\xi} + O(|\zeta|^{-3}), \qquad |\zeta| \to \infty,$$

  and symmetrically near $\zeta = 0$.

Next, we consider the kernel K of B, which is the inverse Fourier transform of the symbol $b(\xi,\zeta)$:

$$K(x_1,x_2) = \frac{1}{(2\pi)^2} \iint e^{ix_1\xi + ix_2\zeta} b(\xi,\zeta)\, d\xi\, d\alpha.$$

From the above properties of b we the corresponding properties of K, which for later reference are collected in the following lemma:

Lemma 4.6. *The kernel K has the following properties:*

(1) K is real, even in each variable and symmetric.

(2) K is smooth and rapidly decreasing away from the axes $x_1 = 0$, $x_2 = 0$.

(3) Near the axes $x_1 = 0$, $x_2 = 0$ we can expand

$$K(x_1,x_2) = -\ln|x_1| \operatorname{sech}^2 x_2 - \ln|x_2| \operatorname{sech}^2 x_1 + K^{lip}(x_1,x_2)$$

where K^{lip} is C^1 and decays rapidly, together with its derivatives.

We now use these properties to carry out a preliminary step in the proof of the Proposition. This is based on the observation that B^h is primarily localized at frequency $1/h$, which should allow us to discard the high frequencies of η from $B^h(\eta,\eta)$. Here to fix the meaning of "high frequencies" we need to choose a frequency threshold λ_0 so that $1/h \ll \lambda_0 \ll 1$. Then we seek to replace η with $\eta_{\le\lambda_0} = P_{\le\lambda_0}\eta$.

Here rather than choosing a sharp frequency localization operator $P_{\le\lambda_0}$, we instead choose a localization operator with a nonnegative kernel; the price to pay for this is to allow harmless rapidly decreasing tails at higher frequency. Then we claim that

Lemma 4.7. *If $1/h \ll \lambda_0 \ll 1$ then*

$$\int_0^T \int m_x B^h(\eta,\eta)\,dxdt = \int_0^T \int m_x B^h(\eta_{\le\lambda_0},\eta_{\le\lambda_0})\,dxdt + O(\frac{1}{\lambda_0 h})\|\eta\|^2_{LE^0}.$$

Proof. Indeed, consider two dyadic frequencies $h^{-1} \le \mu \le \lambda \lesssim 1$. We will estimate the contribution of $B(\eta_\lambda,\eta_\mu)$ in terms of the local energy of η. For $|\xi| \approx \lambda$ and $|\zeta| \approx \mu$ we have

$$|b^h(\xi,\eta)| \lesssim \frac{1}{1+h\lambda} e^{-ch\mu}$$

with matching regularity on the same dyadic scale. Then we have

$$\left|\int_0^T \int m_x B^h(\eta_\mu,\eta_\lambda)\,dxdt\right| \lesssim \frac{1}{1+h\lambda} e^{-ch\mu}\|\eta_\mu\|_{LE^0}\|\eta_\lambda\|_{LE^0} \lesssim \frac{1}{1+h\lambda} e^{-ch\mu}\|\eta\|^2_{LE^0}.$$

Then the conclusion of the lemma follows after summation over $\mu > 1/h$, $\lambda > \lambda_0$.

□

The last Lemma allows us to localize η to low frequencies on the left in (4.11). We now investigate the effect of such a change on the right in (4.11). The idea here is that averaging η over a large scale allows us to replace the local L^2 norm in x by the L^∞ norm. Precisely, we have

Lemma 4.8. *For $\lambda_0 \le 1$ we have*

$$\|\eta_{\le\lambda_0}\|^2_{L^\infty_x L^2_t} \le (1 + C\lambda_0)\|\eta\|^2_{LE^0}.$$

Proof. Here we take advantage of the fact that the kernel of $P_{\le\lambda_0}$ is nonnegative and has integral 1. Then by the triangle inequality we have

$$\|\eta_{\le\lambda_0}\|_{LE^0} \le \|\eta\|_{LE^0}.$$

On the other hand differentiating yields another λ_0 factor,

$$\|\partial_x \eta_{\le\lambda_0}\|_{LE^0} \lesssim \lambda_0\|\eta\|_{LE^0}.$$

Then by the fundamental theorem of calculus and by the Cauchy-Schwarz's inequality, we compute

$$\left| \int_0^T \eta^2_{\leq \lambda_0}(x,t)\, dt - \int_0^T \int m_x(x) \eta^2_{\leq \lambda_0}(x,t)\, dxdt \right| \lesssim \|\eta_{\leq \lambda_0}\|_{LE^0} \|\partial_x \eta_{\leq \lambda_0}\|_{LE^0},$$

which implies that

$$\begin{aligned} \int_0^T \eta^2_{\leq \lambda_0}(x,t) dt &\leq \|\eta_{\leq \lambda_0}\|^2_{LE^0_0} + \|\eta_{\leq \lambda_0}\|_{LE^0} \|\partial_x \eta_{\leq \lambda_0}\|_{LE^0} \\ &\leq (1 + C\lambda_0) \|\eta\|^2_{LE^0} \end{aligned}$$

as needed. □

As a consequence of the last two lemmas, by choosing $1/h \ll \lambda_0 \ll 1$ and using the fact that $\int m_x\, dx = 1$, we can replace the bound (4.11) with

$$\int_0^T \int B^h(\eta_{<\lambda_0}, \eta_{<\lambda_0})(0)\, dt \geq -cg \|\eta_{<\lambda_0}\|^2_{L^\infty_x L^2_t}, \qquad 0 < c < \frac{1}{2}.$$

Now we discard the frequency localization; then h becomes a scaling parameter and we can freely set it to 1. Hence, we have reduced Proposition 4.5 to the following:

Proposition 4.9. *The following bound holds:*

$$\int_0^T \int B(\eta, \eta)\, dxdt \geq -c \|\eta\|^2_{L^\infty_x L^2_t}, \qquad 0 < c < \frac{1}{2}.$$

We first observe that $B(0,0) = \dfrac{1}{2}$. This implies that

$$\int K(x_1, x_2)\, dx_1 dx_2 = \frac{1}{2}.$$

The key step in the proof of the proposition is the following

Lemma 4.10. *The kernel K is positive.*

Before proving this result, let us explain how to conclude the proof of Proposition 4.9 with this lemma. Firstly, notice that if K is nonnegative, then

$$\int |K(x_1, x_2)|\, dx_1 dx_2 = \frac{1}{2},$$

and then it is obvious that the proposition holds with $c = \frac{1}{2}$. But if K is actually positive, there is a little trick to get a small extra gain. Precisely, we can write

$$K(x_1, x_2) = K_1(x_1, x_2) + L(x_1) L(x_2),$$

where L is nonnegative and K_1 is still positive. Then the contribution of the L term is nonnegative, while K_1 has integral $c < \frac{1}{2}$. Then the

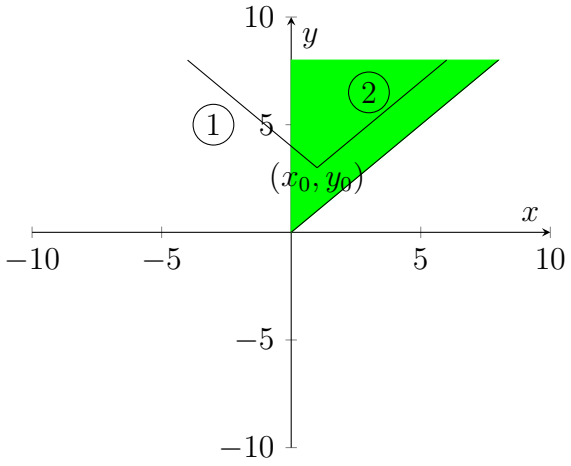

FIGURE 1. Integration regions for K

conclusion of the proposition follows for this c. We now have to prove the lemma.

Proof of Lemma 4.10. By the symmetries of K, it is sufficient to consider the case $0 < x \leq y$ (shaded region in the picture). To compute K we view the symbol b as a product of

$$C_1 = \coth 2\xi \operatorname{csch} 2\zeta - \operatorname{csch} 2\xi \coth 2\zeta,$$

and

$$D_1 = \frac{\zeta\xi}{\zeta^2 - \xi^2}.$$

The Fourier transforms of $\coth \xi$ and $\operatorname{csch} \xi$ are $F = \coth x$ respectively $G = \tanh$, so the Fourier transform of C_1 is (up to positive constants)

$$F(x)G(y) - G(x)F(y).$$

On the other hand for the Fourier transform of D_1 we use the backward fundamental solution for the wave equation, and then differentiate it in x and y. We get

$$\partial_x \delta_{y+|x|=0},$$

which is supported on a $\pi/2$ degree angle downward from 0. Taking the convolution of the two we get

$$K(x_0, y_0) = \int_{y-y_0=|x-x_0|} G'(x)F(y) - F'(x)G(y)\, dx,$$

where the region of integration $\textcircled{1} \cup \textcircled{2}$ is the upward $\pi/2$ degree angle from (x_0, y_0). (see picture). Here F is singular at $x = 0$, so the second term is interpreted in the principal value sense.

Based on previous computations, we know that K blows up logarithmically on the axes and decays exponentially on the diagonals. Then the positivity of K would be a consequence of the bounds

$$(\partial_y - \partial_x)K > 0, \qquad 0 < x_0 \leq y_0, \tag{4.12}$$

within the shaded area of the picture, respectively

$$(\partial_y + \partial_x)K < 0, \qquad 0 < x_0 = y_0. \tag{4.13}$$

Indeed, we can compute

$$(\partial_y - \partial_x)K(x_0, y_0) = \int_{\textcircled{2}} F'(y)G'(x) - G'(y)F'(x)\, dx,$$

respectively

$$(\partial_y - \partial_x)K(x_0, y_0) = \int_{\textcircled{1}} F'(y)G'(x) - G'(y)F'(x)\, dx.$$

Here the first integrand is nonsingular, but the second is again interpreted in the principal value sense at $x = 0$.

We remark that $G' > 0$, $F' < 0$ and

$$\frac{F'(x)}{G'(x)} = -\coth^2(x),$$

which immediately shows that the above integral over $\textcircled{2}$ is positive and thus (4.12) holds. Then it remains to establish (4.13) over the positive half-line $x_0 = y_0$. While the integrand over $\textcircled{1}$ is also positive pointwise, it has the distributional x^{-2} type singularity at $x = 0$, which we expect makes the outcome negative !

To summarize, we need to prove that the following integral is negative,

$$I(x_0) = \int_{-\infty}^{x_0} F'(y)G'(x) - G'(y)F'(x)\, dx, \qquad y = 2x_0 - x.$$

We separate the analysis into three cases:

i) Large x_0, $x_0 > 5$. There $y > 5$, so it is natural to expand in powers of e^{-y}. Since $F'(y), G'(y) \approx e^{-2y}$, the leading term in the integrand is e^{-4y_0} (here we take $x_0 = y_0$).

For F' and G' we have the asymptotic expressions at infinity

$$F'(x) = -\frac{1}{\sinh^2 x} = -\frac{4e^{-2x}}{(1 - e^{-2x})^2} \approx -4e^{-2x} - 8e^{-4x},$$

$$G'(x) = \frac{1}{\cosh^2 x} = \frac{4e^{-2x}}{(1 + e^{-2x})^2} \approx 4e^{-2x} - 8e^{-4x}.$$

Then for our integral we have the expansion up to e^{-8y_0} terms

$$I \approx \int_{-\infty}^{y_0} \frac{(4e^{-2(2y_0-x)} - 8e^{-4(2y_0-x)})}{\sinh^2 x} - \frac{(4e^{-2(2y_0-x)} + 8e^{-4(2y_0-x)})}{\cos^2 x}\, dx$$
$$= -4e^{-4y_0} \int_{-\infty}^{y_0} e^{2x}(\frac{1}{\cosh^2 x} - \frac{1}{\sinh^2 x})\, dx - 8e^{-8y_0} \int_{-\infty}^{y_0} e^{4x}(\frac{1}{\cosh^2 x} + \frac{1}{\sinh^2 x})\, dx.$$

By direct computation the first integral gives zero when taken all the way to $+\infty$. Thus, we get

$$I = 4e^{-4y_0} \int_{y_0}^{\infty} e^{2x}(\frac{1}{\cosh^2 x} - \frac{1}{\sinh^2 x})\, dx - 8e^{-8y_0} \int_{-\infty}^{y_0} e^{4x}(\frac{1}{\cosh^2 x} + \frac{1}{\sinh^2 x})\, dx + O(e^{-8y_0}).$$

Now in both integrals the leading contribution comes from $x = y_0$, and has size e^{-6y_0}. To compute it we write

$$I = -4e^{-4y_0} \int_{y_0}^{\infty} e^{2x} \frac{1}{\cosh^2 x \sinh^2 x}\, dx - 16e^{-8y_0} \int_{-\infty}^{y_0} 4e^{2x}\, dx + O(e^{-8y_0})$$
$$= -4e^{-4y_0} \int_{y_0}^{\infty} 16e^{-2x} \frac{1}{\cosh^2 x \sinh^2 x}\, dx - 16e^{-8y_0} \int_{-\infty}^{y_0} 4e^{2x}\, dx + O(e^{-8y_0})$$
$$= -32e^{-6y_0} - 32e^{-6y_0} + O(e^{-8y_0}) = -64e^{-6y_0} + O(e^{-8y_0}).$$

i) Small x_0, $x_0 < 0.1$. In this range we have

$$I = \int_{-1}^{x_0} -\frac{1}{\sinh^2(2x_0 - x)} + \frac{1}{\sinh^2 x}\, dx + O(1)$$
$$= -2\int_{x_0}^{1} \frac{1}{\sinh^2 x}\, dx + O(1) = -2\coth x_0 + O(1),$$

as desired. Here in the first line the expression $\dfrac{1}{\sinh^2 x}$ is interpreted as the distributional derivative $\partial_x(p.v.\operatorname{cosech} x)$.

i) Medium x_0, $0.1 < x_0 < 5$. For the intermediate range we do not have an algebraic proof, but a direct MATLAB computation easily confirms the result. □

5. Local energy decay for gravity waves

In this section we prove our main result in Theorem 1.1. We begin by emulating the computation in the previous section for the linear case. We define the functional

$$\mathcal{I}_m^\sigma(t) = \int m(x)(\sigma I_2(x,t) + (1-\sigma)I_3(x,t))\, dx.$$

Using the density-flux pairs for the momentum, we have

$$\partial_t \mathcal{I}_m^\sigma(t) = \int m_x(\sigma S_2(x,t) + (1-\sigma)S_3(x,t))\, dx.$$

Hence, in order to prove the theorem we need to establish the following bounds:

(i): Fixed time bounds,

$$\left| \int m(x) I_2 \, dx \right| \lesssim \|\eta\|_{H_h^{\frac{1}{4}}} \|\psi_x\|_{H_h^{-\frac{1}{4}}}, \tag{5.1}$$

$$\left| \int m(x) I_3 \, dx \right| \lesssim \|\eta\|_{H_h^{\frac{1}{4}}} \|\psi_x\|_{H_h^{-\frac{1}{4}}}. \tag{5.2}$$

(ii): Time integrated bound; for some $\sigma \in (0,1)$ and $c < 1$, we have

$$\int_0^T \int m_x(\sigma S_2(t) + (1-\sigma) S_3(t)) \, dxdt \gtrsim LE_0(\eta,\psi) - cLE(\eta,\psi). \tag{5.3}$$

5.1. **Fixed time bounds.** The bound for the contribution of I_2 is identical to the one in the linear model. For the contribution of I_3 there is a slight difference, which is due to the fact that the domain of integration is no longer a strip. Hence in order to apply Proposition 3.1 we need to switch to holomorphic coordinates, and to use Proposition 3.7 in order to relate vertical strips in holomorphic vs. euclidean coordinates.

5.2. **Time integrated bounds.** As before, here we take $\sigma < \frac{1}{2}$, but close to $\frac{1}{2}$. Using the expressions in Lemmas 2.3, 2.4 as well as the relations (2.5) and (2.6) we write the integral in (5.3) as a combination of two leading order terms plus error terms

$$\int_0^T \int m_x(\sigma S_2(t) + (1-\sigma) S_3(t)) \, dxdt = LE_\psi + gLE_\eta + Err_1 + gErr_2 + Err_3,$$

where

$$LE_\psi := \frac{1}{2} \int_0^T \iint m_x[\sigma(\phi_x^2 - \phi_y^2) + (1-\sigma)|\nabla\phi|^2] \, dxdydt$$

$$LE_\eta := \int_0^T \frac{\sigma}{2} \int m_x \eta^2 \, dx - (1-\sigma) \iint m_x \theta_y (\theta - H_N(\eta)) \, dxdydt,$$

and finally

$$Err_1 := \sigma \int_0^T \int m_x \eta \mathcal{N}(\eta) \psi \, dxdt,$$

$$Err_2 := \frac{1-\sigma}{2} \int_0^T \iint m_x \theta_y H_N(|\nabla\phi|^2) \, dxdydt,$$

$$Err_3 := \frac{1-\sigma}{2} \int_0^T \iint m_x \phi_y H_D(\nabla\theta \nabla\phi) \, dxdydt.$$

Our strategy in what follows will be to peel off a leading quadratic part, which we interpret using our bounds for the linear equation. The

remaining cubic and higher order expressions will be viewed as error terms. All but one of the the cubic error terms will be estimated perturbatively.

Finally, the last error term turns out to be unbounded both due to low and to high high frequencies. For this term we instead apply a partial normal form correction, which replaces it with bounded terms, both time integrated, and at the endpoints of the time intervals. The latter correspond to a nonlinear normal form modification of the momentum density.

For many of the nonlinear estimates it is useful to switch to holomorphic coordinates. That greatly facilitates multilinear analysis. There is a price to pay for that, as our m_x cutoff is vertical in the Eulerian frame, but not in the holomorphic frame.

For the remainder of this section we reduce the nonlinear estimate to the linear estimates in Section 4, plus a number of error terms, which need to be estimated perturbatively. The last two sections are devoted to the proof of the error estimates. In Section 6 we show that the Eulerian local energy norms admit equivalent counterparts in the holomorphic setting, and use this equivalence and multilinear analysis to estimate some of the error terms. Finally, in Section 7 we deal with the more difficult error terms which involve the function F, and arise out of the normal form analysis.

5.3. **The LE_θ term.** Here we need to compare the contribution of $H_N(\eta)$,

$$I_1 = \int_0^T \iint m_x \theta_y (H_N(\eta) - \theta)\, dxdy = \iint m_x \theta (\theta - H_N(\eta))_y \, dxdydt,$$

with the expression

$$\int_0^T \int \frac{1}{2} m_x \eta^2 \, dxdt$$

from the first term in LE_θ.

We remark that $H_N(\eta)$ and θ solve the same equation and have the same boundary condition on the top, but different boundary conditions on the bottom (Dirichlet, respectively Neumann). Thus they cancel in the infinite depth case, but not in the finite depth case.

To estimate this we move to conformal coordinates $z = \alpha + i\beta$. This does not change the equations for $H_N(\eta)$ and θ. Precisely, if α_0 is the image of x_0 in the conformal setting, then we seek to compare the integral I_1 with its conformal counterpart

$$I_1^{hol} = \int_0^T \iint m_\alpha \, (\alpha - \alpha_0) \theta (\theta - H_N(\eta))_\beta \, d\alpha d\beta dt.$$

We will view the difference between the two integrals as an error term,

$$Err_4 = I_1 - I_1^{hol}$$

to be estimated later.

The expression I_1^{hol} can be rewritten as

$$\begin{aligned} I_1^{hol} &= \int_0^T \iint m_\alpha\, \theta(\theta - H_N(\eta))_\beta \, d\alpha d\beta dt \\ &= \int_0^T \iint m\, \theta_\alpha (H_N(\eta) - \theta)_\beta + m\, \theta (H_N(\eta) - \theta)_{\alpha\beta} \, d\alpha d\beta dt \\ &= \int_0^T \iint m\, \theta_\alpha (H_N(\eta) - \theta)_\beta - m\, \theta_\beta (H_N(\eta) - \theta)_\alpha \, d\alpha d\beta \\ &= \int_0^T \iint m\, (\theta_\alpha H_N(\eta)_\beta - \theta_\beta H_N(\eta)_\alpha) \, d\alpha d\beta dt. \end{aligned}$$

Recalling that $\theta = H_D(\eta)$, the above integral becomes

$$I_1^{hol} = \int_0^T \iint m H_D(\eta)_\alpha H_N(\eta)_\beta - H_D(\eta)_\beta H_N(\eta)_\alpha) \, d\alpha d\beta dt,$$

which is identical to the corresponding expression obtained in the analysis of the linearized problem in Section 4. Hence, as there, it can be further represented as

$$I_1^{hol} = \int_0^T \int m_\alpha (\alpha - \alpha_0)\, B^h(\eta, \eta)\, d\alpha dt.$$

On the other hand, as a consequence of the bound $|W_\alpha| \lesssim \epsilon$ we obtain the relation

$$\int_0^T \int \frac{1}{2} m_x \eta^2 \, dx = \int_0^T \int \frac{1}{2} m_\alpha \eta^2 \, d\alpha dt + O(\epsilon) \|\eta\|_{LE}^2.$$

Combining the two terms, we have established that

$$LE_\theta - Err_4 = \int_0^T \frac{\sigma}{2} \int m_\alpha \eta^2 \, d\alpha - (1-\sigma) \int_0^T \int m_\alpha (\alpha - \alpha_0) B_2^h(\eta, \eta) \, d\alpha dt + O(\epsilon) \|\eta\|_{LE}^2.$$

We conclude the argument here by showing that for $\sigma > \frac{1}{2}$ close to $\frac{1}{2}$ we have the bound

$$LE_\theta - Err_4 \gtrsim \frac{1}{2} \|\eta\|_{LE_{x_0}}^2 - c \|\eta\|_{LE}^2, \tag{5.4}$$

where $c < \frac{1}{2}$ is a universal constant. This in turn is a consequence of

Proposition 5.1. *For the bilinear form B_2^h above there exists $c < \frac{1}{2}$ so that we have*

$$\int_0^T \int m_\alpha (\alpha - \alpha_0) B_2^h(\eta, \eta) \, d\alpha dt \geq -cg \sup_{x_1 \in \mathbb{R}} \int_0^T \int m_\alpha (\alpha - \alpha_1) \eta^2 \, d\alpha dt. \tag{5.5}$$

This is a direct counterpart of Proposition 4.5 from the linear analysis. The only difference is that on the right, α_1 is not constant in time but instead we have that $\alpha_1 = \alpha_1(t, x_1)$. Because of this we cannot directly cite Proposition 4.5 here. However, it will be easy to reduce the above proposition to Proposition 4.5.

Proof. To reduce to Proposition 4.5 we simply change coordinates back into Eulerian coordinates. The Jacobian is $1 + \operatorname{Re} W_\alpha = 1 + O(\epsilon)$ so it only yields negligible $O(\epsilon)$ errors. The same applies for the changes in the argument of m,

$$m_\alpha(\alpha - \alpha_0) = m_x(x - x_0) + O(\epsilon).$$

It remains to consider the change in the operator B^h. We consider this at the level of the kernel K^h of B^h. Referring back to Section 4, the kernel of K^h in the holomorphic coordinates is

$$K^h(\alpha_1, \alpha_2; \alpha) = K\Big(\frac{\alpha_1 - \alpha}{h}, \frac{\alpha_2 - \alpha}{h}\Big).$$

After the change of coordinates this becomes

$$\tilde{K}_h(x_1, x_2; x) := K\Big(\frac{\alpha(x_1, t) - \alpha(x, t)}{h}, \frac{\alpha(x_2, t) - \alpha(x, t)}{h}\Big).$$

We would like to replace this with $K^h(x_1, x_2, x)$ at the expense of $O(\epsilon)$ errors. For this we use the relations

$$\alpha(x_i, t) - \alpha(x, t) = (x_i - x)(1 + O(\epsilon).$$

Then we compute using the properties of K in Lemma 4.6:

$$|\tilde{K}_h(x_1, x_2; x) - K_2^h(x_1, x_2, x)| \lesssim \epsilon h^{-2}(1 + h^{-1}(|x - x_1| + |x - x_2|)^{-N}).$$

This easily gives $O(\epsilon)$ errors, and finally allows us to reduce the proposition to Proposition 4.5. □

5.4. **The error terms.** At this point we have four error terms to deal with, Err_1, Err_2, Err_3 and Err_4. Three of them will be directly estimated in a perturbative fashion:

Proposition 5.2. *We have the following estimates:*

$$|Err_1| + |Err_2| + |Err_4| \lesssim \epsilon \|(\eta, \psi)\|_{LE}^2. \tag{5.6}$$

This proposition is proved in the following section.

The difficult term is Err_3, which turns out to be unbounded both because of low frequency contributions and high frequency contributions. We will address this difficulty in two steps. The first is to switch to the holomorphic coordinates counterpart of Err_3. The second is to apply a nonlinear normal form type correction to the momentum density.

For the first step, the holomorphic counterpart of Err_3 is

$$Err_3^{hol} := \int_0^T \iint m_\alpha(\alpha - \alpha_0)\phi_y H_D(\nabla\theta\nabla\phi)\, d\alpha d\beta dt.$$

On the top we have $\phi_y = \operatorname{Im} R$, while for $\nabla\theta\nabla\phi$ we compute its value as

$$\nabla\theta\nabla\phi = J^{-1}(\nabla_h\theta\nabla_h\phi) = J^{-1}\operatorname{Im}(\bar W_\alpha Q_\alpha) = \operatorname{Im}(\frac{\bar W_\alpha}{1+\bar W_\alpha}R).$$

Therefore we obtain

$$Err_3^{hol} = \int_0^T \iint m_\alpha(\alpha - \alpha_0)\operatorname{Im} R H_D\left(\frac{\bar W_\alpha}{1+\bar W_\alpha}R\right) d\alpha d\beta dt.$$

The transition between Err_3 and Err_3^{hol} is harmless:

Proposition 5.3. *We have the following estimate:*

$$|Err_3 - Err_3^{hol}| \lesssim \epsilon\|(\eta,\psi)\|_{LE}. \tag{5.7}$$

Next we turn our attention to the remaining unbounded error term Err_3^{hol}. Here we will borrow an idea from normal forms, and rectify this error via a normal form type correction. Since we are trying to address both low and high frequencies, our correction will be genuinely nonlinear as opposed to the traditional cubic one, which would only address the low frequencies.

Our correction is based on the following computation, which uses the equations (3.5):

$$\begin{aligned}
\frac{d}{dt}(\operatorname{Im} W \operatorname{Re} W_\alpha) &= \partial_\alpha(\operatorname{Im} W \operatorname{Re} W_t) + \operatorname{Im}(W_t\bar W_\alpha)\\
&= \partial_\alpha(\operatorname{Im} W \operatorname{Re} W_t) - \operatorname{Im}(F(1+W_\alpha)\bar W_\alpha)\\
&= \partial_\alpha(\operatorname{Im} W \operatorname{Re} W_t) - \operatorname{Im} F|W_\alpha|^2 - \operatorname{Im}(F\bar W_\alpha)\\
&= \partial_\alpha(\operatorname{Im} W \operatorname{Re} W_t) - \operatorname{Im} F(|W_\alpha|^2 + 2\operatorname{Re} W_\alpha) + \operatorname{Im}(FW_\alpha)\\
&= \partial_\alpha(\operatorname{Im} W \operatorname{Re} W_t) - \operatorname{Im} Q_\alpha J^{-1}(|W_\alpha|^2 + 2\operatorname{Re} W_\alpha) + \operatorname{Im}(FW_\alpha)\\
&= \partial_\alpha(\operatorname{Im} W \operatorname{Re} W_t) - \operatorname{Im}\left(R\frac{\bar W_\alpha}{1+\bar W_\alpha}\right) - \operatorname{Im}(RW_\alpha) + \operatorname{Im}(FW_\alpha)\\
&= \partial_\alpha(\operatorname{Im} W \operatorname{Re} W_t) - \operatorname{Im}\left(R\frac{\bar W_\alpha}{1+\bar W_\alpha}\right) + \operatorname{Im}((F-R)W_\alpha).
\end{aligned}$$

This allows us to express $\operatorname{Im}(\frac{\bar W_\alpha}{1+\bar W_\alpha}R)$ on the top as

$$2\operatorname{Im}(\frac{\bar W_\alpha}{1+\bar W_\alpha}R) = -\frac{d}{dt}(\operatorname{Im} W \operatorname{Re} W_\alpha) + \partial_\alpha(\operatorname{Im} W \operatorname{Re} W_t) + \operatorname{Im}((F-R)W_\alpha).$$

The first expression on the right will correspond to our (partial) normal form correction to the Morawetz 's identity. The second has an α

derivative, and thus better low frequency decay. Finally, the third is the imaginary part of a holomorphic function, so it has a trivial holomorphic extension.

Correspondingly, we can write Err_3^{hol} in the form

$$2Err_3^{hol} = \iint m_\alpha \operatorname{Im} RH_D \left(\operatorname{Im} W \operatorname{Re} W_\alpha\right) d\alpha d\beta \Big|_0^T + Err_5 + Err_6 + Err_7, \tag{5.8}$$

where

$$\begin{aligned} Err_5 &:= \int_0^T \iint m_\alpha \operatorname{Im} R_t \, H_D \left(\operatorname{Im} W \operatorname{Re} W_\alpha\right) \, d\alpha d\beta dt, \\ Err_6 &:= \int_0^T \iint m_\alpha \operatorname{Im} R \, \operatorname{Im}((F-R)W_\alpha) \, d\alpha d\beta dt, \\ Err_7 &:= \int_0^T \iint m_\alpha \operatorname{Im} R \, \partial_\alpha \, H_D \left(\operatorname{Im} W \operatorname{Re} W_t\right)) \, d\alpha d\beta dt. \end{aligned}$$

The first term in (5.8) can be estimated directly using Proposition 3.1,

$$\begin{aligned} \left| \iint m_\alpha \operatorname{Im} R \, H_D \left(\operatorname{Im} W \operatorname{Re} W_\alpha\right) \, d\alpha d\beta \right| &\lesssim \|R\|_{H^{-\frac{1}{4}}} \| \operatorname{Im} W \operatorname{Re} W_\alpha \|_{H^{\frac{1}{4}}} \\ &\lesssim \|R\|_{H^{-\frac{1}{4}}} \| \operatorname{Im} W \|_{H^{\frac{1}{4}}} \lesssim E^{\frac{1}{4}}, \end{aligned}$$

since $\operatorname{Re} W_\alpha \in l^1 H_h^{\frac{1}{2}}$, due to the multiplicative estimate

$$\|fg\|_{H_h^{\frac{1}{4}}} \lesssim \|f\|_{H_h^{\frac{1}{4}}} \|g\|_{l^1 H_h^{\frac{1}{2}}}.$$

Then it remains to estimate the error terms:

Proposition 5.4. *We have the following estimates:*

$$|Err_5| + |Err_6| + |Err_7| \lesssim \epsilon \|(\eta, \psi)\|_{LE}^2. \tag{5.9}$$

All of these errors involve the expression F, since in the fluid domain we have

$$W_t = F(1 + W_\alpha),$$

for W, respectively

$$\begin{aligned} R_t &= \frac{1}{1+W_\alpha}(Q_{\alpha t} - RW_{\alpha t}) \\ &= \frac{1}{1+W_\alpha}((-FQ_\alpha)_\alpha + R(F(1+W_\alpha))_\alpha + g\mathcal{T}W_\alpha + P[|R|^2]_\alpha) \\ &= -FR_\alpha + \frac{1}{1+W_\alpha}(-g\mathcal{T}W_\alpha + P[|R|^2]_\alpha) \end{aligned}$$

for R. Corresponding to the last relation, we split

$$Err_5 = Err_5^1 + Err_5^2 + Err_5^3.$$

Thus, we have proved Theorem 1.1 modulo the results in Propositions 5.2, 5.3 and 5.4.

6. Local energy bounds in holomorphic coordinates

As a first step in the proof of the error estimates needed for our main theorem, in this section we seek to understand how to transfer the local energy bounds to the holomorphic setting. Then we will also consider some bilinear expressions, and use them to estimate the simpler error terms.

6.1. **Notations.** Our starting point here is represented by the local energy norms in the Eulerian setting, which, are equivalently defined as

$$\|(\eta,\psi)\|_{LE} = \|\eta\|_{LE^0} + \|\nabla\phi\|_{LE^{-\frac12}},$$

where

$$\|\eta\|_{LE^0} := \sup_{x_0\in\mathbb{R}} \|\eta\|_{L^2(S(x_0))}, \qquad \|\nabla\phi\|_{LE^{-\frac12}} := \sup_{x_0\in\mathbb{R}} \|\nabla\phi\|_{L^2(\mathbf{S}(x_0))}.$$

Here $S(x_0)$, respectively $\mathbf{S}(x_0)$ represent the Eulerian strips

$$S(x_0) := \{[0,T]\times[x_0-1,x_0+1]\}, \qquad \mathbf{S}(x_0) := \{[0,T]\times[x_0-1,x_0+1]\times[-h,0]\}.$$

Our first objective will be to prove that these norms are equivalent to their counterparts in the holomorphic setting. In holomorphic coordinates the functions η and $\nabla\phi$ are represented by $\operatorname{Im} W$ and R. Thus we will seek to replace the above local energy norm with

$$\|(W,R)\|_{LE} := \|\operatorname{Im} W\|_{LE^0} + \|R\|_{LE^{-\frac12}},$$

where

$$\|\operatorname{Im} W\|_{LE^0} := \sup_{x_0\in\mathbb{R}} \|\operatorname{Im} W\|_{L^2(S_h(x_0))}, \qquad \|R\|_{LE^{-\frac12}} := \sup_{x_0\in\mathbb{R}} \|R\|_{L^2(\mathbf{S}_h(x_0))}.$$

Here $S_h(x_0)$, respectively $\mathbf{S}_h(x_0)$, represent the holomorphic strips

$$S_h(x_0) := \{(t,\alpha) : t\in[0,T],\ \alpha\in[\alpha_0-1,\alpha_0+1]\}, \qquad \mathbf{S}_h(x_0) := S(x_0)\times[-h,0],$$

where $\alpha_0 = \alpha_0(t,x_0)$ represents the holomorphic coordinate of x_0, which in general will depend on t.

We remark that while the strips $S_h(x_0)$ on the top roughly correspond to the image of $S(x_0)$ in holomorphic coordinates, this is no longer the case for the strips $\mathbf{S}_h(x_0)$ relative to $\mathbf{S}(x_0)$. While these are well matched on the top, in depth there may be a horizontal drift, which has been estimated in Proposition 3.7.

The first main outcome of this section will be the equivalence

Proposition 6.1. *Assuming the uniform bound (1.7), we have the equivalence:*

$$\|(\eta,\psi)\|_{LE} \approx \|(W,R)\|_{LE}. \tag{6.1}$$

Here the correspondance between the LE^0 norms of η and $\operatorname{Im} W$ is straightforward due to the bilipschitz property of the conformal map. However, the correspondence between the $LE^{-\frac{1}{2}}$ norms of $\nabla\phi$ and R is less obvious, and is proved in Proposition 6.9 below.

One difference between the norms for $\operatorname{Im} W$ and for R is that they are expressed in terms of the size of the function on the top, respectively in depth. For the purpose of multilinear estimates later on we will need access to both types of norms for both $\operatorname{Im} W$ and for R. Since the local energy norms are defined using the unit spatial scale, in order to describe the behavior of functions in these spaces we will differentiate between high frequencies and low frequencies. We begin with functions on the top:

a) **High frequency characterization on top.** Here we will use local norms on the top, for which we will use the abbreviated notation

$$\|u\|_{L^2_t H^s_{loc}} := \sup_{x_0\in\mathbb{R}} \|u\|_{L^2_t H^s_\alpha([\alpha_0-1,\alpha_0+1])},$$

where again $\alpha_0 = \alpha_0(x_0, t)$.

b) **Low frequency characterization on top.** Here we will use local norms on the top to describe the frequency λ or $\leq \lambda$ part of functions, where $\lambda < 1$ is a dyadic frequency. By the uncertainty principle such bounds should be uniform on the λ^{-1} spatial scale. Then it is natural to use the following norms:

$$\|u\|_{L^2_t L^\infty_{loc}(B_\lambda)} := \sup_{x_0\in\mathbb{R}} \|u\|_{L^2_t L^\infty_\alpha(B_\lambda(x_0))},$$

where

$$B_\lambda(x_0) := \{\alpha \in \mathbb{R} : \ |\alpha - \alpha_0| \lesssim \lambda^{-1}\}.$$

We remark that the local norms in a) correspond exactly to the $B_\lambda(x_0)$ norms with $\lambda = 1$.

Next we consider functions in the strip which are harmonic extensions of functions on the top.

a1) **High frequency characterization in strip.** Here we will use local norms on regions with depth at most 1, for which we will use the abbreviated notation

$$\|u\|_{L^2_t X_{loc}(A_1)} := \sup_{x_0\in\mathbb{R}} \|u\|_{L^2_t X(A_1(x_0))},$$

where X will represent various Sobolev norms and

$$A_1(x_0) := \{(\alpha, \beta) : |\beta| \lesssim 1, \ |\alpha - \alpha_0| \lesssim 1\}.$$

b1) **Low frequency characterization in strip.** Here a frequency $\lambda < 1$ is associated with depths $|\beta| \approx \lambda^{-1}$. Thus, we define the regions

$$A_\lambda(x_0) = \{(\alpha,\beta) : |\beta| \approx \lambda^{-1},\ |\alpha - \alpha_0| \lesssim \lambda^{-1}\}, \qquad \lambda < 1,$$

and in these regions we use the uniform norms,

$$\|u\|_{L^2_t L^\infty_{loc}(A_\lambda)} := \sup_{x_0\in\mathbb{R}} \|u\|_{L^2_t L^\infty_{\alpha,\beta}(A_\lambda(x_0))}.$$

We will also denote

$$\mathbf{B}^1(x_0) := \{(\alpha,\beta);\ |\alpha-\alpha_0| \le 1,\ \beta\in[-1,0]\},$$
$$\mathbf{B}^\lambda(x_0) := \{(\alpha,\beta);\ |\alpha-\alpha_0| \le \lambda^{-1},\ \beta\in[-\lambda^{-1},0]\},\ \text{for } \lambda<1.$$

To simplify the notations in the following analysis, we will also denote

$$\|(\eta,\psi)\|_{LE} := M, \qquad \|(\eta,\psi)\|_X := \epsilon \le \epsilon_0 \ll 1. \tag{6.2}$$

Given the equivalence of the X norms in Proposition 3.8, as well as the equivalence of the LE norms in the next subsection, these bounds also transfer to the holomorphic setting as follows:

$$\|(\operatorname{Im} W, R)\|_{LE} \lesssim M, \qquad \|(\operatorname{Im} W, R)\|_X \lesssim \epsilon \ll 1. \tag{6.3}$$

Furthermore, we recall that the frequency envelopes $\{c_\lambda\}$ for (η,ψ) in X also transfer to $(\operatorname{Im} W, R)$ in X.

6.2. **Multipliers and Bernstein's inequality in uniform norms.** Here we aim to understand how multipliers act on the uniform spaces defined above.

We will work with a multiplier $M_{\lambda_2}(D)$ associated to a dyadic frequency λ_2. In order to be able to use the bounds in several circumstances, we make a weak assumption on their (Lipschitz) symbols $m_{\lambda_2}(\xi)$:

$$|m_{\lambda_2}(\xi)| \lesssim\ (1+\lambda_2^{-1}|\xi|)^{-3},\ \text{ and }\ |\partial_\xi^{k+1} m_{\lambda_2}(\xi)| \lesssim c_k\ |\xi|^{-k}(1+\lambda_2^{-1}|\xi|)^{-4}. \tag{6.4}$$

Examples of such symbols include

- Littlewood-Paley localization operators P_{λ_2}, $P_{\le\lambda_2}$.
- The multipliers $p_D(\beta,D)$ and $p_N(\beta,D)$ in subsection 3.1 with $|\beta| \approx \lambda_2^{-1}$.

We will separately consider high frequencies, where we work with the spaces $L^2_t L^p_{loc}$, and low frequencies, where we work with the spaces $L^2_t L^p_{loc}(B_\lambda)$ associated with a dyadic frequency $1/h \le \lambda \le 1$.

A. High frequencies. Here we consider a dyadic high frequency $1/h \le \lambda_2 \le 1$, and seek to understand how multipliers $M_{\lambda_2}(D)$ associated to frequency λ_2 act on the spaces $L^2_t L^p_{loc}$.

Lemma 6.2. *Let $1/h \le \lambda_1, \lambda_2 \le 1$ and $1 \le p \le q \le \infty$. Then*

$$\|M_{\lambda_2}(D)\|_{L^2_t L^p_{loc} \to L^2_t L^q_{loc}} \lesssim \lambda_2^{\frac{1}{p}-\frac{1}{q}}. \tag{6.5}$$

B. Low frequencies. Here we consider two dyadic low frequencies $1/h \le \lambda_1, \lambda_2 \le 1$, and seek to understand how multipliers $M_{\lambda_2}(D)$ associated to frequency λ_2 act on the spaces $L^2_t L^p_{loc}(B_{\lambda_1})$. For such multipliers we have:

Lemma 6.3. *Let $1/h \le \lambda_1, \lambda_2 \le 1$ and $1 \le p \le q \le \infty$.*

a) Assume that $\lambda_1 \le \lambda_2$. Then

$$\|M_{\lambda_2}(D)\|_{L^2_t L^p_{loc}(B_{\lambda_1}) \to L^2_t L^q_{loc}(B_{\lambda_1})} \lesssim \lambda_2^{\frac{1}{p}-\frac{1}{q}}. \tag{6.6}$$

b) Assume that $\lambda_2 \le \lambda_1$. Then

$$\|M_{\lambda_2}(D)\|_{L^2_t L^p_{loc}(B_{\lambda_1}) \to L^2_t L^q_{loc}(B_{\lambda_2})} \lesssim \lambda_1^{\frac{1}{p}} \lambda_2^{-\frac{1}{q}}. \tag{6.7}$$

We remark that part (a) is nothing but the classical Bernstein's inequality in disguise, as the multiplier M_{λ_2} does not mix λ_1^{-1} intervals. Part (b) is the more interesting one, where the λ_1^{-1} intervals are mixed.

Proof of Lemmas 6.2,6.3. We first note that Lemma 6.2 can be viewed a a particular case of Lemma 6.3 (a) with $\lambda_1 = 1$. So in what follows we will only prove Lemma 6.3.

A direct consequence of the symbol bounds (6.4) is the fact that the kernel K_{λ_2} of $M_{\lambda_2}(D)$ satisfies the bound

$$|K_{\lambda_2}(\alpha)| \lesssim \frac{\lambda_2}{1+\lambda_2^2\alpha^2}. \tag{6.8}$$

We will show that (6.8) yields the conclusion of the Lemma.

a) We fix $x_0 \in \mathbb{R}$ and seek to estimate

$$\|M_{\lambda_2}(D)u\|_{L^2_t L^q_{loc}(B_{\lambda_1}(x_0))}.$$

For that we cover $S \times [0,T]$ with width λ_1^{-1} strips,

$$S \times [0,T] = \bigcup_{j \in \mathbb{Z}} S_{\lambda_1}(x_0 + j\lambda_1^{-1}).$$

For $(t,\alpha) \in S_{\lambda_1}(x_0)$ we write

$$|M_{\lambda_2}(D)u(t,\alpha)| \lesssim |u| * \frac{\lambda_2}{1+\lambda_2^2\alpha^2} \lesssim \sum_j (1_{S_{\lambda_1}(x_0+j\lambda_1^{-1})}|u|) * \frac{\lambda_2}{1+\lambda_2^2\alpha^2}.$$

Now we consider two cases. If $|j| \le 2$ then we simply use Young's inequality. This no longer suffices for all j because of the need for summation in j. However, for such j we can use the kernel decay instead. If $(t,\alpha_1) \in S_{\lambda_1}(x_0 + j\lambda_1^{-1})$ then

$$|\alpha - \alpha_1| \approx j\lambda_1^{-1}.$$

Therefore

$$\frac{\lambda_2}{1+\lambda_2^2(\alpha-\alpha_1)^2} \approx \lambda_2^{-1} j^{-2} \lambda_1^2.$$

Using Young's inequality yields

$$\|(1_{S_{\lambda_1}(x_0+j\lambda_1^{-1})}|u|)*\frac{\lambda_2}{1+\lambda_2^2\alpha^2}\|_{L_t^2 L^q(B_{\lambda_1}(x_0))} \lesssim \lambda_2^{\frac{1}{p}-\frac{1}{q}} \frac{\lambda_1^2}{\lambda_2^2 j^2} \|1_{S_{\lambda_1}(x_0+j\lambda_1^{-1})} u\|_{L_t^2 L^p(B_{\lambda_1}(x_0+j\lambda_1^{-1}))}.$$

Now the j summation is straightforward.

b) It suffices to consider the case $q = \infty$, and then use Hölder's inequality. Here we seek to estimate

$$\|M_{\lambda_2}(D)u\|_{L_t^2 L^q_{loc}(B_{\lambda_2}(x_0))}.$$

We use the same covering as above, and for $(t,\alpha) \in S_{\lambda_1}(x_0)$ we write

$$|M_{\lambda_2}(D)u(t,\alpha)| \lesssim |u| * \frac{\lambda_2}{1+\lambda_2^2\alpha^2} \lesssim \sum_j (1_{S_{\lambda_1}(x_0+j\lambda^{-1})}|u|) * \frac{\lambda_2}{1+\lambda_2^2\alpha^2}.$$

This time Hölder's inequality yields

$$\|(1_{S_{\lambda_1}(x_0+j\lambda_1^{-1})}|u|)*\frac{\lambda_2}{1+\lambda_2^2\alpha^2}\|_{L_t^2 L^\infty(B_{\lambda_2}(x_0))} \lesssim \frac{\lambda_1^{\frac{1}{p}-1}\lambda_2}{1+\lambda_2^2\lambda_1^{-2}j^2} \|1_{S_{\lambda_1}(x_0+j\lambda_1^{-1})} u\|_{L_t^2 L^p(B_{\lambda_1}(x_0+j\lambda_1^{-1}))},$$

and the result follows again after j summation. □

6.3. **Switching strips.** At several points in our analysis we need to switch local energy type integrals from the Euclidean to the holomorphic setting. Here we compute this transition systematically, establishing bounds that will be repeatedly used in the sequel.

The set-up is as follows. We consider some smooth function Ψ in the fluid domain, which can be viewed either in the Eulerian or the holomorphic coordinates. For such a function, we seek to compare the following two integrals:

$$\mathcal{I}_E := \int_0^T \iint_{\Omega(t)} m'(x-x_0)\Psi(x,y)\, dydxdt,$$

respectively

$$\mathcal{I}_H := \int_0^T \iint_{\Omega(t)} m'(\alpha-\alpha_0)\Psi(\alpha,\beta)\, d\beta d\alpha dt.$$

To fix the notations, the Eulerian strip is centered at $x = x_0$, which on the top corresponds to $\alpha = \alpha_0$. However, in depth the line $x = x_0$ corresponds to a curve $\alpha = \alpha_0(t,\beta)$. We will need to account for this difference. Our result is as follows:

Proposition 6.4. *We have*

$$|\mathcal{I}_E - \mathcal{I}_H| \lesssim \int_0^T \iint_{A_1(x_0)} (|W_\alpha| + |\operatorname{Re} W(\alpha,\beta) - \operatorname{Re} W(\alpha_0,0)|)\,|\Psi(\alpha,\beta)|\,d\beta d\alpha dt + \int_0^T \int_{\beta\in[-h,-1]} (c_\beta + \sup_{|\alpha-\alpha_0|} |W_\alpha|) \sup_{|\alpha-\alpha_0|<|\beta|} (|\Psi| + |\beta||\Psi_\alpha|)\,d\beta dt, \tag{6.9}$$

Proof. We switch $\mathcal{I}_E$ to holomorphic coordinates by changing variables. This yields

$$\mathcal{I}_E = \int_0^T \iint_{\Omega(t)} Jm'(x-x_0)\Psi(\alpha,\beta)\,d\beta d\alpha dt.$$

Since $|J-1| \lesssim |W_\alpha|$, we can harmlessly replace J by 1, and then we are left with the difference

$$\int_0^T \iint_{\Omega(t)} (m'(x-x_0) - m'(\alpha-\alpha_0))\Psi(\alpha,\beta)\,d\beta d\alpha dt.$$

Here we have

$$x - x_0 = \alpha - \alpha_0 + \operatorname{Re} W(\alpha,\beta) - \operatorname{Re} W(\alpha_0,0).$$

The function m' is supported in the unit interval, and W has an ϵ small Lipschitz constant (ϵ is the control norm defined in the Introduction). Then, within the support of $m'(x-x_0)$ we must have

$$|\alpha - \alpha_0| \lesssim \epsilon|\beta| + 1. \tag{6.10}$$

We now divide the analysis in two cases depending on the size of β.

a) Small depth, $-1 < \beta < 0$. Here we simply use the Lipschitz property of m' to get

$$|m'(x-x_0) - m'(\alpha-\alpha_0)| \lesssim |\operatorname{Re} W(\alpha,\beta) - \operatorname{Re} W(\alpha_0,0)|.$$

b) Large depth, $-h < \beta < -1$. Here we continuously switch between the two bumps $m'(x-x_0)$ and $m'(\alpha-\alpha_0)$. Denoting

$$d(k,\alpha) := \alpha - \alpha_0 + k(\operatorname{Re} W(\alpha,\beta) - \operatorname{Re} W(\alpha_0,0)),$$

we consider the family of bump functions $m'(d(k,\alpha))$ with $k \in [0,1]$. Within the support of these bump functions we still have $|\alpha-\alpha_0| \le |\beta|$, therefore, using also Proposition 3.7

$$|\operatorname{Re} W(\alpha,\beta) - \operatorname{Re} W(\alpha_0,0)| \lesssim |\beta|(c_\beta + \sup_{|\alpha-\alpha_0|\le|\beta|} |W_\alpha|). \tag{6.11}$$

Using the functions $m'(d(k,\alpha))$ we have

$$\begin{aligned} m'(x-x_0)-m'(\alpha-\alpha_0) &= \int_0^1 \frac{d}{dk} m'(d(k,\alpha))\,dk \\ &= \int_0^1 m''(d(k,\alpha))(\operatorname{Re} W(\alpha,\beta)-\operatorname{Re} W(\alpha_0,0))\,dk \\ &= \int_0^1 \partial_\alpha m'(d(k,\alpha))\frac{\operatorname{Re} W(\alpha,\beta)-\operatorname{Re} W(\alpha_0,0)}{1+\operatorname{Re} W_\alpha}\,dk. \end{aligned}$$

Hence, integrating by parts we get

$$\begin{aligned} D(\beta,t) &:= \int (m'(x-x_0)-m'(\alpha-\alpha_0))\Psi(\alpha,\beta)\,d\alpha \\ &= -\int_0^1 \int m'(d(k,\alpha))\partial_\alpha\left[\frac{\operatorname{Re} W(\alpha,\beta)-\operatorname{Re} W(\alpha_0,0)}{1+\operatorname{Re} W_\alpha}\Psi(\alpha,\beta)\right] d\alpha dk. \end{aligned}$$

Here m' is a bump function with unit integral, so taking absolute values we get

$$|D(\beta,t)| \lesssim \sup_{|\alpha-\alpha_0|<|\beta|} |W_\alpha||\Psi| + |\operatorname{Re} W(\alpha,\beta)-\operatorname{Re} W(\alpha_0,0)|(|\Psi_\alpha|+|W_{\alpha\alpha}||\Psi|).$$

In this context we have

$$|W_{\alpha\alpha}| \lesssim \epsilon|\beta|^{-1},$$

so the conclusion follows from (6.11).

□

6.4. **Bounds for $\eta = \operatorname{Im} W$.** Here we have the straightforward equivalence

$$\|\eta\|_{LE^0} \approx \|\operatorname{Im} W\|_{LE^0} \tag{6.12}$$

as η and $\operatorname{Im} W$ are one and the same function up to a biLipschitz change of coordinates. Our first aim will be to understand the bounds for the low frequencies of $\operatorname{Im} W$ on the top:

Lemma 6.5. *For each dyadic frequency $\lambda < 1$ we have*

$$\|\operatorname{Im} W_{\le\lambda}\|_{L^2_t L^\infty_{loc}(B_\lambda)} \lesssim \|\operatorname{Im} W\|_{LE^0}. \tag{6.13}$$

Proof. Since $LE^0 = L^2_t L^2_{loc}(B_1)$, this bound is a direct application of Lemma 6.3 (b). □

On the other hand, for nonlinear estimates, we also need bounds in depth, precisely over the regions $A_\lambda(x_0)$. There we have

Lemma 6.6. *For each dyadic frequency $\lambda < 1$ we have*

$$\|\operatorname{Im} W\|_{L^2_t L^\infty_{loc}(A_\lambda)} + \lambda^{-1}\|W_\alpha\|_{L^2_t L^\infty_{loc}(A_\lambda)} \lesssim \|\operatorname{Im} W\|_{LE^0}. \tag{6.14}$$

Proof. We start by recalling that $\operatorname{Im} W$ is harmonic in the strip with Dirichlet boundary condition on the bottom. Then $\operatorname{Im} W(\alpha, \beta)$ is given by

$$\operatorname{Im} W(\beta) = P_D(\beta, D) \operatorname{Im} W(0),$$

where the symbol $p_D(\xi, \beta)$ of the multiplier $P_D(\beta)$ is

$$p_D(\xi, \beta) = \frac{\sinh(\xi(\beta + h))}{\sinh(\xi h)}.$$

For $|\beta| \approx \lambda^{-1}$ these symbols satisfy uniformly the condition (6.4) with $\lambda = \lambda_1$. Then the kernel bound (6.8) also holds uniformly, and the conclusion of Lemma 6.3 applies also uniformly. This yields the bound for the first term on the left. The bound for the second term on the left is similar, by applying the same argument to the operators $\lambda^{-1}\partial_\alpha P_D(\beta)$ and $\lambda^{-1}\partial_\beta P_D(\beta)$ uniformly in $|\beta| \approx \lambda^{-1}$.

Alternatively, we note that one can obtain the bound for W_α or equivalently for $\nabla \operatorname{Im} W$ by elliptic regularity. We have already obtained estimates for $\operatorname{Im} W$ in the region $A_\lambda(x_0)$, which has size λ^{-1}, and so using the elliptic regularity we can estimate the derivatives of a harmonic function in a domain in terms of the solution on a larger domain:

$$\|\nabla_{\alpha,\beta} \operatorname{Im} W(\alpha, \beta)\|_{L^\infty(A_\lambda(x_0))} \lesssim \lambda \|\theta(\alpha, \beta)\|_{L^\infty(cA_\lambda(x_0))}, \quad c > 1.$$

□

Also connected to $\theta = \operatorname{Im} W$, we need to estimate the difference $\theta - H_N(\eta)$. Here we are comparing two harmonic functions with same Dirichlet data on the top, but with homogeneous Dirichlet vs. Neumann boundary condition the bottom. The regions over which we compare the difference are of size h:

$$\mathbf{B}_h(x_0) := \{(\alpha,\, \beta) \,:\, \beta \in [-h, 0],\ |\alpha - \alpha_0| \lesssim h\}.$$

We have

Lemma 6.7. *For the difference $\theta - H_N(\eta)$ have*

$$\|h^j \nabla^j (\theta - H_N(\eta))\|_{L^2_t L^\infty_{loc}(\mathbf{B}_h)} \lesssim \|\operatorname{Im} W\|_{LE^0}, \qquad j = 0, 1, 2. \tag{6.15}$$

Proof. We first compute

$$(\theta - H_N(\eta))(\beta) = C(\beta, D)\eta,$$

where

$$C(\beta, \xi) := \frac{\sinh((h + \beta)\xi)}{\sinh(h\xi)} - \frac{\cosh((h + \beta)\xi)}{\cosh(h\xi)} = \frac{2\sinh(\beta\xi)}{\sinh(2h\xi)}$$

has size 1 for $|\xi| < h^{-1}$ and decays exponentially for larger ξ. Thus these kernels satisfy uniformly the condition (6.4) with $\lambda_2 = 1/h$.

Hence the bound for $\operatorname{Im} W - H_N(\operatorname{Im} W)$ follows by Lemma 6.3 (b) with $\lambda_1 = 1$ and $\lambda_2 = 1/h$.

We now turn our attention to the $j = 1$ case, namely the map

$$\eta \to \nabla(\theta - H_N(\eta))(\alpha, \beta).$$

Differentiating the previous symbol in either α or β yields another factor of ξ, namely leads to the symbols

$$\frac{\xi \cdot \sinh(\xi\beta)}{\sinh(2h\xi)}, \qquad \frac{\xi \cdot \cosh(\xi\beta)}{\sinh(2h\xi)}.$$

Both are bump functions on the h^{-1} scale, but now their size is improved to h^{-1}. Thus both operators equal h^{-1} times an averaging operator on the h scale. Hence Lemma 6.3(b) again applies, but yields another h^{-1} factor. The same argument applies as well for the second order derivatives of $\theta - H_N(\eta)$.

□

Now we are already able to estimate the easiest of the error terms:

Proof of the Err_4 bound. We estimate the difference between the two integrals I_1 and I_1^{hol} using Proposition 6.4 with $\Psi = \theta(\theta - H_N(\eta))_\beta$. We also need to account for the difference

$$\theta(\theta - H_N(\eta))_y - \theta(\theta - H_N(\eta))_\beta,$$

which, by chain rule, is readily estimated by

$$|\theta(\theta - H_N(\eta))_y - \theta(\theta - H_N(\eta))_\beta| \lesssim |\theta||\nabla_{\alpha,\beta}(\theta - H_N(\eta))||W_\alpha|.$$

Combining this with Proposition 6.4 and using $|W_\alpha| < \epsilon$, we obtain

$$|I_1 - I_1^{hol}| \lesssim \epsilon \int_0^T \iint_{A_1(\alpha_0)} |\theta||\nabla(\theta - H_N(\eta))|\, d\beta d\alpha dt + \epsilon \int_0^T \int_{-h}^{-1} \sup_{|\alpha-\alpha_0|<|\beta|} |\theta||\nabla(\theta - H_N(\eta))| + |\beta||\partial_\alpha(\theta(\theta - H_N(\eta))_\beta)|\, d\beta dt.$$

It remains to bound the two integrals by $\|\eta\|_{LE}^2$, both of which are straightforward in view of Lemma 6.6 and Lemma 6.7. □

6.5. **Estimates for Y.** Here we prove a local energy bound for the auxiliary holomorphic function

$$Y = \frac{W_\alpha}{1 + W_\alpha}.$$

Lemma 6.8. *a) For $\lambda > 1$ we have*

$$\|Y_\lambda\|_{L_t^2 L_{loc}^2} \lesssim \lambda M. \tag{6.16}$$

b) For $\lambda \leq 1$ we have

$$\|Y_\lambda\|_{L_t^2 L_\alpha^\infty (B_\lambda(x_0))} \lesssim \lambda M. \tag{6.17}$$

We note that both estimates follow directly from Lemma 6.6 if Y is replaced by W_α. However to switch to Y one would seem to need some Moser type inequalities, which unfortunately do not work in negative Sobolev spaces. The key observation is that in both of these estimates it is critical that W_α is holomorphic, and Y is an analytic function of W_α.

Proof. We will bound Y on the top using bounds for its holomorphic extension. Based on the bounds for W in (6.12) and (6.14), this satisfies estimates as follows:

A. If $-1 < \beta < 0$ then on each unit strip S we have

$$\|\partial_\alpha^j Y(\cdot, \beta)\|_{L_t^2 L_\alpha^2(B_1(x_0)} \lesssim |\beta|^{-1-j} M. \tag{6.18}$$

B. If $-h < \beta < -1$ then on each $|\beta|$ strip we have

$$\|\partial_\alpha^j Y(\cdot, -\beta)\|_{L_t^2 L_\alpha^\infty(B_\beta(x_0))} \lesssim |\beta|^{-1-j} M. \tag{6.19}$$

We use the following representation of Y on the top,

$$\begin{aligned} Y(\alpha, 0) &= Y(\alpha, -h) + i \int_{-h}^0 Y_\alpha(\alpha, \beta)\, d\beta \\ &= Y(\alpha, -h) + ihY_\alpha(\alpha, -h) + \int_{-h}^0 \beta Y_{\alpha\alpha}(\alpha, \beta)\, d\beta. \end{aligned}$$

The function $Y(\alpha, -h)$ is at frequency $1/h$, and obeys the bounds (6.19) therefore the first two terms above easily satisfy the bounds in the lemma.

It remains to consider the integral term, where we treat the integrand differently depending on β and on λ.

Case I: $\lambda > 1$. Here we are only interested in unit strips, and use L^2 bounds. Depending on β, we differentiate as follows:

Case I.a: Small β, $-\lambda^{-1} < \beta < 0$. There we use (6.18) to estimate

$$\|P_\lambda \partial_\alpha^2 Y(\cdot, \beta)\|_{L_t^2 L_{loc}^2} \lesssim \lambda^2 \|Y(\cdot, \beta)\|_{L_t^2 L_{loc}^2} \lesssim |\beta|^{-1} \lambda^2 M,$$

which suffices for the β integration.

Case I.b: Large β, $-h < \beta < \lambda^{-1}$. There depending on the size of β we use either (6.18) or (6.19) to estimate

$$\|P_\lambda \partial_\alpha^2 Y(\cdot, \beta)\|_{L_t^2 L_{loc}^2} \lesssim |\beta|^{-3} M,$$

which again suffices for the β integration.

Case II: $\lambda > 1$. Here we are only interested in strips of width λ^{-1}, and use L^∞ bounds. Depending on β, we differentiate as follows:

Case II.a: Very small β, $-1 < \beta < 0$. Here we cover the λ^{-1} strip with unit strips, use Hölder's inequality, then Bernstein's inequality to get

$$\|P_\lambda \partial_\alpha^2 Y(\cdot,\beta)\|_{L_t^2 L^\infty_{loc}(B_\lambda)} \lesssim \lambda^{\frac{5}{2}} \|Y(\cdot,\beta)\|_{L_t^2 L^2_{loc}(B_\lambda)} \lesssim \lambda^2 \|Y(\cdot,\beta)\|_{L_t^2 L^2_{loc}} \lesssim |\beta|^{-1}\lambda^2 M,$$

which is enough.

Case II.b: Small β, $-\lambda^{-1} < \beta < -1$. Here we cover the λ^{-1} strip with $|\beta|^{-1}$ strips, use Hölder inequality, then Bernstein's inequality to get

$$\begin{aligned}\|P_\lambda \partial_\alpha^2 Y(\cdot,\beta)\|_{L_t^2 L^\infty_{loc}(B_\lambda)} &\lesssim \lambda^{\frac{5}{2}} \|Y(\cdot,\beta)\|_{L_t^2 L^2_{loc}(B_\lambda)} \lesssim \lambda^2 \beta^{-\frac{1}{2}} \|Y(\cdot,\beta)\|_{L_t^2 L^2_{loc}(B_\beta)} \\ &\lesssim \lambda^2 \|Y(\cdot,\beta)\|_{L_t^2 L^\infty_{loc}(B_\beta)} \lesssim |\beta|^{-1}\lambda^2 M,\end{aligned}$$

which is enough.

Case II.c: Large β, $-h < \beta < \lambda^{-1}$. There we use (6.19) to estimate

$$\|P_\lambda \partial_\alpha^2 Y(\cdot,\beta)\|_{L_t^2 L^2_{loc}(B_\lambda)} \lesssim |\beta|^{-3} M,$$

which again suffices for the β integration.

□

6.6. **Bounds for $\partial\phi = R$.** This is not as easy as for $\eta = \operatorname{Im} W$, because the strips in the Eulerian and holomorphic setting do not agree, and can in effect be quite different. Nevertheless, we will still prove

Proposition 6.9. *Assume (1.7) holds. Then we have*

$$\|\psi\|_{LE^{-\frac{1}{2}}} \approx \|R\|_{LE^{-\frac{1}{2}}} \qquad (mod\ \epsilon \|\operatorname{Im} W\|_{LE^0}). \tag{6.20}$$

Here the equivalence should be interpreted as the double inequality

$$\|\psi\|_{LE^{-\frac{1}{2}}} \lesssim \|R\|_{LE^{-\frac{1}{2}}} + \epsilon \|\operatorname{Im} W\|_{LE^0}, \qquad \|R\|_{LE^{-\frac{1}{2}}} \lesssim \|\psi\|_{LE^{-\frac{1}{2}}} + \epsilon \|\operatorname{Im} W\|_{LE^0}.$$

Proof. We recall that $|\nabla\phi|^2 = |R|^2$, so all we need is to transfer the L^2 local bound from unit strips in the Eulerian setting to unit strips in the holomorphic setting.

To switch from one strip to another we will critically use the bound in Proposition 6.4, which uses the fact that in depth the distance between the two strips is smaller than $\epsilon|\beta|$. Because of this, we start with a preliminary result, which is more easily proved:

Lemma 6.10. *For each dyadic $\lambda < 1$ we have*

$$\|R\|_{L_t^2 L^\infty_{loc}(A_\lambda)} + \lambda^{-2} \|\nabla R\|_{L_t^2 L^\infty_{loc}(A_\lambda)} \lesssim \lambda \|R\|^2_{LE^{-\frac{1}{2}}}. \tag{6.21}$$

Proof. From the definition of the local energy functional associated to R we know that we have L^2 control over R inside every vertical strip of width 1. However, initially we do not have any information on the top or on the bottom of the strip. As a consequence we first prove the

desired bound in a region $A_\lambda(x_0)$ that avoids the case $|\beta| \approx h$. In order to use the control we have on R we split the region $2A_\lambda(x_0)$ in strips of width 1 and then add the λ^{-1} bounds on strips to obtain

$$\|R\|^2_{L^2(2A_\lambda(x_0))} \leq \lambda^{-1} \|R\|_{LE^{-\frac{1}{2}}}.$$

Then the bound in the lemma follows by elliptic regularity.

Finally, if $|\beta| \approx h$ we use the homogeneous boundary conditions Dirichlet or Neumann for $\operatorname{Re} R$ and $\operatorname{Im} R$ to separately mirror them in a symmetric domain below the bottom via reflection principle, and then proceed as above.

□

We now return to the proof of the Proposition 6.9. For this we need to compare the integrals

$$\mathcal{I}_E = \int_0^T \iint m_x(x-x_0)|\nabla\phi|^2 \, dxdydt, \qquad \mathcal{I}_H = \int_0^T \iint m_x(\alpha-\alpha_0)|R|^2 \, dxdydt,$$

and show that

$$|\mathcal{I}_E - \mathcal{I}_H| \lesssim \epsilon(\|R\|^2_{LE^{-\frac{1}{2}}} + \|W\|^2_{LE^0}).$$

Since $|\nabla\phi|^2 = R^2$, we can apply directly Proposition 6.4. This yields

$$\begin{aligned} |\mathcal{I}_E - \mathcal{I}_H| \lesssim \epsilon \int_0^T \iint_{A_1(x_0)} |R|^2 \, d\beta d\alpha dt \\ + \int_0^T \int_{-h}^1 (c_\beta + \sup_{|\alpha-\alpha_0|<\beta} |W_\alpha|) \sup_{|\alpha-\alpha_0|<\beta} |R|^2 + |RR_\alpha| \, d\beta dt. \end{aligned}$$

The first integral is directly estimated by $\|R\|^2_{LE^{-\frac{1}{2}}}$. For the contribution of c_β we use the dyadic summability of c_β along with Lemma 6.10. Hence we are left with

$$\int_0^T \int_{-h}^1 \sup_{|\alpha-\alpha_0|<\beta} |W_\alpha| \sup_{|\alpha-\alpha_0|<\beta} |R|^2 + |RR_\alpha| \, d\beta dt.$$

To bound this last integral we switch roles and use the local energy norm for W_α via Lemma 6.5 and for R via Lemma 6.10, while for R_α we use the control norm and Bernstein's inequality to get the bound $|R_\alpha| \lesssim |\beta|^{\frac{1}{2}} c_\beta$. This yields the fixed β bound

$$c_\beta |\beta|^{-1} \|R\|_{LE^{-\frac{1}{2}}} \|\operatorname{Im} W\|_{LE^0}.$$

Finally we integrate with respect to $\beta \in [-h, -1]$ to obtain

$$\epsilon \|R\|_{LE^{-\frac{1}{2}}} \|\operatorname{Im} W\|_{LE^0}.$$

□

The local energy norm for R measures the function inside the entire strip. However, we also need to estimate it on the top:

Lemma 6.11. *For $R \in LE^{-\frac{1}{2}}$ we have the following high frequency bound on the top:*

$$\|R\|_{L^2_t H^{-\frac{1}{2}}_{loc}} \lesssim \|R\|_{LE^{-\frac{1}{2}}}, \tag{6.22}$$

respectively the low frequency bound

$$\|R_\lambda\|_{L^2_t L^\infty(B_\lambda)} \lesssim \lambda^{\frac{1}{2}} \|R\|_{LE^{-\frac{1}{2}}}, \qquad \lambda \leq 1. \tag{6.23}$$

Proof. The first part follows from the trace theorem, as $R \in L^2_t L^2_{loc}(A_1)$ is a harmonic function. The second part is more delicate, but we can use the same argument as in Lemma 6.8. Precisely, we write

$$R(\alpha, 0) = R(\alpha, -h) + \int_{-h}^{0} iR_\alpha(\alpha, \beta)\, d\beta.$$

For the first term we can use Lemma 6.10. For the second term we split the integral into

$$R_1 = \int_{-h}^{-\lambda^{-1}} iR_\alpha(\alpha, \beta)\, d\beta, \qquad R_2 = \int_{-\lambda^{-1}}^{0} iR_\alpha(\alpha, \beta)\, d\beta.$$

For R_1 we use the gradient bound in Lemma 6.10, to compute

$$\|R_1\|_{L^2_t L^\infty_{loc}(B_\lambda)} \lesssim \int_{-h}^{-\lambda^{-1}} |\beta|^{-\frac{3}{2}}\, d\beta \lesssim \lambda^{\frac{1}{2}} M,$$

and the spectral projector P_λ is harmless.

For R_2 on the other hand we use the spectral projector for Bernstein's inequality in Lemma 6.3, and then to eliminate the derivative

$$\begin{aligned}
\|P_\lambda R_2\|_{L^2_t L^\infty_{loc}(B_\lambda)} &\lesssim \lambda^{\frac{1}{2}} \|P_\lambda R_2\|_{L^2_t L^2_{loc}(B_\lambda)} \\
&\lesssim \lambda^{\frac{3}{2}} \left\| \int_{-\lambda^{-1}}^{0} R(\alpha, \beta)\, d\beta \right\|_{L^2_t L^2_\alpha(B_\lambda)} \\
&\lesssim \lambda \|R\|_{L^2_t L^2_\alpha(\mathbf{B}_\lambda)},
\end{aligned}$$

where at the last step we have used Hölder's inequality in β. To estimate R over a square $\mathbf{B}_\lambda(x_0)$ of width λ^{-1}, we cover the square with λ^{-1} strips $\mathbf{S}(x_0 + j)$ with $|j| \lesssim \lambda^{-1}$, and then use Holder's inequality again to get

$$\|P_\lambda R_2\|_{L^2_t L^\infty_{loc}(B_\lambda)} \lesssim \lambda^{\frac{1}{2}} \|R\|_{LE^{-\frac{1}{2}}}.$$

□

6.7. **Bilinear estimates for $|\nabla\phi|^2 = |R|^2$ and its harmonic extension.** Here we will prove the following bound:

Lemma 6.12. *a) The function $|\nabla\phi|^2 = |R|^2$ restricted to the top satisfies the following estimate:*

$$\||R|^2\|_{LE^0} \lesssim \epsilon M. \tag{6.24}$$

b) Its low frequency part satisfies

$$\|P_\lambda |R|^2\|_{L^2_t L^\infty_{loc}(B_\lambda(x_0))} \lesssim c_\lambda M. \tag{6.25}$$

c) In addition, for each $\lambda < 1$ there is a decomposition

$$P_{<\lambda}|R|^2 = G^1_\lambda + G^2_\lambda,$$

where

$$\sup_\lambda \|G^1_\lambda\|_{L^1 L^\infty(B_\lambda(x_0))} \lesssim \lambda M^2, \tag{6.26}$$

while

$$\|G^2_\lambda\|_{L^2 L^\infty(B_\lambda(x_0))} \lesssim c_\lambda M. \tag{6.27}$$

Proof. a) We restate this as a bound for R,

$$\||R|^2\|_{LE^0} \lesssim \|R\|_{LE^{-\frac{1}{2}}} \|R\|_{\ell^1 L^\infty_t H^1_h}, \tag{6.28}$$

where the ℓ^1 summability is measured using the control frequency envelope $\{c_\lambda\}$. For this we use a Littlewood-Paley decomposition

$$R\bar{R} = \sum_\lambda (R_{<\lambda}\bar{R}_\lambda R_\lambda \bar{R}_{<\lambda}) + \sum_\lambda R_\lambda \bar{R}_\lambda,$$

and analyze each component separately. We discuss two cases: first when $\lambda \geq 1$ and the second is when $\lambda < 1$. For now we discuss the first case, i.e., $\lambda \geq 1$. To bound R we will use either the control norm X, or the local energy norm $LE^{-\frac{1}{2}}$. Correspondingly, we have the following bounds for the dyadic pieces

$$\|R_{<\lambda}\|_{L^2_t L^2_{loc}} \lesssim \lambda^{\frac{1}{2}} \|R_{<\lambda}\|_{L^2_t H^{-\frac{1}{2}}_{h,loc}} \lesssim \lambda^{\frac{1}{2}} M,$$

respectively

$$\|R_\lambda\|_{L^\infty L^2} \leq \lambda^{-1} \|R_\lambda\|_{L^\infty H^1_h} \lesssim \lambda^{-1} c_\lambda.$$

We begin with the low-high frequency term where we compute using Bernstein's inequality in Lemma 6.2

$$\|R_\lambda \bar{R}_{<\lambda}\|_{L^2_t L^2_{loc}} \lesssim \lambda^{\frac{1}{2}} \|R_\lambda \bar{R}_{<\lambda}\|_{L^2_t L^1_{loc}} \lesssim \lambda^{\frac{1}{2}} \|R_\lambda\|_{L^\infty_t L^2} \|\bar{R}_{<\lambda}\|_{L^2_t L^2_{loc}} \lesssim c_\lambda M.$$

Here we can sum up with respect to dyadic λ as needed.

For $\sum_\lambda R_\lambda \bar{R}_\lambda$ we perform a similar analysis, and consider the product's output at frequency ν, where $\nu \lesssim \lambda$. Here, ν can be ≥ 1 or < 1. We assume first that $\nu \geq 1$, and return to the other case later in the proof. From Bernstein's inequality in Lemma 6.3

$$\|P_\nu \left(R_\lambda \bar{R}_\lambda\right)\|_{L^2_t L^2_{loc}} \lesssim \nu^{\frac{1}{2}} \|P_\nu \left(R_\lambda \bar{R}_\lambda\right)\|_{L^2_t L^1_{loc}},$$

and further, by Cauchy's inequality, we get

$$\|P_\nu \left(R_\lambda \bar{R}_\lambda\right)\|_{L^2_t L^2_{loc}} \lesssim \nu^{\frac{1}{2}} \|R_\lambda\|_{L^2_t L^2_{loc}} \|\bar{R}_\lambda\|_{L^2_t L^2} \lesssim \left(\frac{\nu}{\lambda}\right)^{\frac{1}{2}} \|R_\lambda\|_{L^2_t H^{-\frac{1}{2}}_{h,loc}} \|\bar{R}_\lambda\|_{L^\infty_t H^1_h} \lesssim \left(\frac{\nu}{\lambda}\right)^{\frac{1}{2}} c_\lambda M.$$

The λ summation is again straightforward.

Therefore (6.28) holds for the high frequency (≥ 1) part of the output. The remaining case in (6.28) corresponds to low frequency output and will follow from the proof of part (b) below.

b) The goal here is to prove the following estimate

$$\sum_{\lambda<1} \|P_\lambda |R|^2\|_{L^2_t L^\infty_{loc}(B_\lambda)} \lesssim \|R\|_{LE^{-\frac{1}{2}}} \|R\|_{\ell^1 L^\infty_t H^1_h}, \tag{6.29}$$

which in particular suffices to finish the proof of part a) of the proposition. Again we use the control frequency envelope $\{c_\lambda\}$ to measure the ℓ^1 summation in the second factor on the right, and will show that

$$\|P_\lambda |R|^2\|_{L^2_t L^\infty_{loc}(B_\lambda)} \lesssim c_\lambda M. \tag{6.30}$$

We need to consider the expressions $P_\lambda\left(R_\nu \bar{R}_\mu\right)$, where by Littlewood-Paley trichotomy, we have several cases to discuss:

i.) Case $\nu \approx \mu$, $\mu > \lambda$ and $\mu > 1$.

In this case both input frequencies are comparable and larger than 1 but the output frequency is $\lambda < 1$. We use Bernstein's inequality and Hölder's inequality in both space and time to obtain

$$\|P_\lambda\left(R_\mu \bar{R}_\mu\right)\|_{L^2_t L^\infty_{loc}(B_\lambda)} \lesssim \lambda \|R_\mu \bar{R}_\mu\|_{L^2_t L^1_{loc}(B_\lambda)} \lesssim \lambda \|R_\mu\|_{L^2_t L^2_{loc}(B_\lambda)} \|\bar{R}_\mu\|_{L^\infty_t L^2}.$$

Since the input frequencies are higher than 1, we estimate the first factor using Lemma (6.12) adapted for the dyadic pieces, together with the fact that in an interval of size λ^{-1} we have about λ^{-1} size 1 subintervals. For the second factor we use the control envelope c_λ. This yields

$$\begin{aligned}
\lambda \|R_\mu\|_{L^2_t L^2_{loc}(B_\lambda)} \|\bar{R}_\mu\|_{L^\infty_t L^2(B_\lambda)} &\lesssim \lambda \mu^{\frac{1}{2}} \lambda^{-\frac{1}{2}} \|R_\mu\|_{L^2_t H^{-\frac{1}{2}}_{loc}} \|R_\mu\|_{L^\infty_t L^2_\alpha} \\
&\lesssim \lambda^{\frac{1}{2}} \mu^{-\frac{1}{2}} \|R\|_{LE^{-\frac{1}{2}}} \|R_\mu\|_{L^\infty_t H^1_\alpha} \\
&\lesssim \lambda^{\frac{1}{2}} \mu^{-\frac{1}{2}} c_\mu M.
\end{aligned}$$

Now the μ summation is straightforward due to the off-diagonal decay.

ii.) Case $\nu \approx \mu$, $\mu > \lambda$ and $\mu < 1$.

This case is a harder one because we deal with different scale localizations. More explicitly the input frequencies are on the scale μ^{-1} which is less than the output frequency which lives on the scale λ^{-1}. Thus, we first use Bernstein's inequality in Lemma 6.3, followed by Hölder's inequality in both space and time:

$$\|P_\lambda\left(R_\mu \bar{R}_\mu\right)\|_{L^2_t L^\infty_{loc}(B_\lambda)} \lesssim \lambda \|R_\mu \bar{R}_\mu\|_{L^2_t L^1_{loc}(B_\lambda)} \lesssim \lambda \|R_\mu\|_{L^\infty_t L^2_{loc}(B_\lambda)} \|\bar{R}_\mu\|_{L^2_t L^2_{loc}(B_\lambda)}, \tag{6.31}$$

and then we use the control envelope c_μ to arrive to

$$\begin{aligned}\lambda\|R_\mu\|_{L_t^\infty L^2_{loc}(B_\lambda)}\|\bar{R}_\mu\|_{L_t^2 L^2_{loc}(B_\lambda)} &\lesssim \lambda\mu^{-1}\|R_\mu\|_{L_t^\infty H^1_{h,loc}(B_\lambda)}\|\bar{R}_\mu\|_{L_t^2 L^2_{loc}(B_\lambda)}\\ &\lesssim \lambda\mu^{-1}c_\mu\|\bar{R}_\mu\|_{L_t^2 L^2_{loc}(B_\lambda)}.\end{aligned} \tag{6.32}$$

In the second term on the right we switch from λ^{-1} width strips to μ^{-1} wide strips using Holder's inequality, followed by Hölder's inequality again and then Lemma (6.10) to obtain

$$\|\bar{R}_\mu\|^2_{L_t^2 L^2_{loc}(B_\lambda)} \lesssim \left(\frac{\mu}{\lambda}\right)^{\frac{1}{2}}\|\bar{R}_\mu\|^2_{L_t^2 L^2_{loc}(B_\mu)} \lesssim \frac{\mu}{\lambda^{\frac{1}{2}}}\|\bar{R}_\mu\|^2_{L_t^2 L^\infty_{loc}(B_\mu)} \lesssim \left(\frac{\mu}{\lambda}\right)^{\frac{1}{2}} M. \tag{6.33}$$

Using this in (6.32) we have proved that

$$\|P_\lambda\left(R_\mu \bar{R}_\mu\right)\|_{L_t^2 L^\infty_{loc}(B_\lambda)} \lesssim \lambda^{\frac{1}{2}}\mu^{-\frac{1}{2}}c_\mu M. \tag{6.34}$$

The μ summation is again straightforward.

iii.) Case $\nu < \lambda$ and $\mu \approx \lambda$.

Here we observe that we can drop the projection P_λ, and then we can use Lemma 6.10 for the first factor and Bernstein's inequality for the second one

$$\|R_\nu \bar{R}_\mu\|_{L_t^2 L^\infty_{loc}(B_\lambda)} \lesssim \|R_\nu\|_{L_t^2 L^\infty_{loc}(B_\lambda)}\|R_\mu\|_{L_t^\infty L^2_\alpha} \lesssim \nu^{\frac{1}{2}}\mu^{-\frac{1}{2}}c_\mu M. \tag{6.35}$$

We do have off-diagonal decay since $\nu < \mu$, and summing over such ν yields a bound of $c_\lambda M$ as desired.

c) We observe that we only need to place low-low interactions in G^1 and high-high interactions in G^2. In this context by low-low we mean that both input frequencies are smaller than λ, and then their output is also smaller than λ, and by high-high interaction we refer to larger than λ input frequencies that give rise to a smaller than λ output frequency.

We begin with the input frequencies μ and ν both smaller than λ, and by Hölder's inequality in time we get that

$$\|R_\mu R_\nu\|_{L_t^1 L^\infty_{loc}(B_\lambda)} \lesssim \|R_\mu\|_{L_t^2 L^\infty_{loc}(B_\lambda)}\|R_\nu\|_{L_t^2 L^\infty_{loc}(B_\lambda)}.$$

Since both μ and ν are smaller than λ we can apply Lemma 6.10 and get

$$\|R_\mu\|_{L_t^2 L^\infty_{loc}(B_\lambda)}\|R_\nu\|_{L_t^2 L^\infty_{loc}(B_\lambda)} \lesssim \mu^{\frac{1}{2}}\nu^{\frac{1}{2}}\|R\|^2_{LE^{-\frac{1}{2}}}.$$

Summing over both $\mu, \nu < \lambda$ we get that indeed

$$\sum_\lambda \|R_\mu R_\nu\|_{L_t^1 L^\infty_{loc}(B_\lambda)} \lesssim \lambda\|R\|^2_{LE^{-\frac{1}{2}}},$$

which finishes the proof of (6.26).

For the high-high case the analysis in part (i) and (ii) applies together with the summation over λ and μ. The bound for G_2 follows.

□

Using the $|R|^2$ bound, we are able to estimate two more of the error terms:

The estimate for Err_1 in Proposition 5.2. We recall that

$$Err_1 := \int_0^T \int \sigma m_x \eta \mathcal{N}(\eta)\psi \, dxdt.$$

Since $\mathcal{N}(\eta)\psi = |\nabla\phi|^2$ on the top, this is a direct consequence of Lemma 6.12 (a).

□

Proof of the Err_2 estimate. We recall that the expression for Err_2 is given by

$$Err_2 := \int_0^T \iint m_x(x - x_0)\theta_y H_N(|\nabla\phi|^2) \, dxdydt.$$

We first recast it in holomorphic coordinates,

$$Err_2 = \int_0^T \iint m_x(x - x_0)(\operatorname{Re} W_\alpha + |W_\alpha|^2) H_N(|R|^2) \, d\alpha d\beta dt.$$

To estimate it we will combine the bounds in Lemma 6.6 with those in Lemma 6.12. We exploit these bound in two steps.

1. High frequency bounds. Here we consider the contributions where at least one of the W_α and $H_N(|R|^2)$ factors is at high frequency (≥ 1). In this case the corresponding harmonic extension decays exponentially in β on the unit scale, therefore the bound for the corresponding part of Err_2 is localized both in α and in β on the unit scale. On this scale, by elliptic regularity, we have local bounds

$$W_\alpha \in L^2_t(L^2_\beta H^{-\frac{1}{2}}_\alpha)_{loc}, \qquad H_N(|R|^2) \in L^2_t(L^2_\beta H^{\frac{1}{2}}_\alpha)_{loc}$$

in terms of the LE^0 norms for θ and $|R|^2$ on the top. These are dual spaces. Furthermore, the remaining W_α factors are harmless since from the X bound we have

$$W_\alpha \in L^\infty_t L^\infty_\beta(\ell^1 H^{\frac{1}{2}}_\alpha).$$

2. Low frequency bounds. Here we use the decomposition in part (c) of the last lemma, where λ is matched to the depth $|\beta| \approx \lambda^{-1}$.

For G^1_λ we combine (6.26) with the trivial L^∞ bound for W_α derived fom the X norm, where the latter comes with ℓ^1 summability.

For G^1_λ instead we combine (6.26) with the bound (6.6) for W_α. □

6.8. **Bilinear estimates for $\operatorname{Im} W \cdot \operatorname{Re} W_\alpha$ and its harmonic extension.** This expression appears in the normal form correction part of the proof of our nonlinear Morawetz inequality. Here we will prove the following bound:

Lemma 6.13. *a) The function $\operatorname{Im} W \cdot \operatorname{Re} W_\alpha$ restricted to the top satisfies the following high frequency estimate:*

$$\|\operatorname{Im} W \cdot \operatorname{Re} W_\alpha\|_{LE^0} \lesssim \epsilon \|\operatorname{Im} W\|_{LE^0}. \tag{6.36}$$

b) Its low frequencies satisfy the additional bound

$$\|P_\lambda(\operatorname{Im} W \cdot \operatorname{Re} W_\alpha)\|_{L^2_t L^\infty_{loc}(B_\lambda(x_0))} \lesssim c_\lambda \|\operatorname{Im} W\|_{LE^0}. \tag{6.37}$$

c) In addition, for each $\lambda < 1$ there is a decomposition

$$P_{<\lambda}(\operatorname{Im} W \cdot \operatorname{Re} W_\alpha) = G^1_\lambda + G^2_\lambda,$$

where

$$\sup_\lambda \|G^1_\lambda\|_{L^1_t L^\infty_{loc}(B_\lambda(x_0))} \lesssim \lambda M^2, \tag{6.38}$$

while

$$\|G^2_\lambda\|_{L^2_t L^\infty_{loc}(B_\lambda(x_0))} \lesssim c_\lambda M. \tag{6.39}$$

Proof. a) Here we use the fact that W_α is bounded in L^∞

$$\begin{aligned} \|\operatorname{Im} W \cdot \operatorname{Re} W_\alpha\|_{LE^0} &= \|\operatorname{Im} W \cdot \operatorname{Re} W_\alpha\|_{L^2_t L^2_{loc}(B_\lambda(x_0))} \\ &\lesssim \|\operatorname{Im} W\|_{L^2_t L^2_{loc}(B_\lambda(x_0))} \|\operatorname{Re} W_\alpha\|^\infty_L \\ &\lesssim \epsilon \|\operatorname{Im} W\|_{LE^0}. \end{aligned}$$

b) The proof is exactly as in Lemma (6.12) with the corresponding adjustments that come from the fact that $\operatorname{Im} W$ and $\operatorname{Re} W_\alpha$ are differently balanced in comparison with R: one is $1/2$ derivative less than R and one is $1/2$ derivative above R, respectively.

The only slight technical difference that arises, is when one considers the case of low-high interactions, with the high frequency on $\operatorname{Re} W_\alpha$. In this case, instead of looking separately at the norms

$$\|\operatorname{Im} W_\nu \cdot \operatorname{Re} W_{\mu,\alpha}\|_{L^2_t L^\infty_{loc}(B_\lambda(x_0))}, \qquad \nu < \mu \lesssim 1,$$

and then sum over ν with $\nu < \mu$, we group terms and analyze directly

$$\|\operatorname{Im} W_{<\mu} \cdot \operatorname{Re} W_{\mu,\alpha}\|_{L^2_t L^\infty_{loc}(B_\lambda(x_0))}.$$

By doing so we avoid the potentially troublesome ν summation.

Thus, we proceed as follows

$$\|(\operatorname{Im} W)_{<\mu} \cdot (\operatorname{Re} W_\alpha)_\mu\|_{L^2_t L^\infty_{loc}(B_\mu)} \lesssim \|(\operatorname{Im} W)_{<\mu}\|_{L^2_t L^\infty_{loc}(B_\mu)} \|(\operatorname{Re} W_\alpha)_\mu\|_{L^\infty} \lesssim c_\mu M,$$

where for the first factor we have used Lemma 6.5, while the dyadic bound for $\operatorname{Re} W_\alpha$ follows from Proposition 3.3. This suffices for both parts (b) and (c) of the lemma.

□

Proof of the Err_5^2 estimate. Here we consider the bound for the second term in Err_5, namely

$$Err_5^2 := g \int_0^T \iint \operatorname{Im}\left(\frac{1}{1+W_\alpha}\mathcal{T}W_\alpha\right) H_D(\operatorname{Im} W \operatorname{Re} W_\alpha)\, d\alpha d\beta dt.$$

The same proof as for Err_2 applies, using Lemma 6.6, which now is combined with Lemma 6.13 instead of Lemma 6.12. □

Proof of the Err_5^3 estimate. Here we consider the third term in Err_5, namely

$$Err_5^3 = \int_0^T \iint \frac{1}{1+W_\alpha} \operatorname{Im} P[|R|^2]_\alpha\, H_D(\operatorname{Im} W \cdot \operatorname{Re} W_\alpha)\, d\alpha d\beta dt.$$

This is again the same proof as for Err_2, using Lemma 6.12 and Lemma 6.13. □

7. Bounds involving F

The aim of this section is to prove the error estimates involving F. These are all tied to the normal form correction we use to deal with the unbounded error term Err_3. We recall that

$$F = P\left[\frac{2i \operatorname{Im} Q_\alpha}{J}\right] = R + P\left[2i \operatorname{Im}\left(R\bar{Y}\right)\right] := R + F^{[2]},$$

where we have separated the linear part F and the quadratic and higher order part $F^{[2]}$. The imaginary part of $F^{[2]}$ is explicit on the top:

$$\operatorname{Im} F^{[2]} = \operatorname{Im}\left(R\bar{Y}\right).$$

Thus in the fluid domain we can write

$$\operatorname{Im} F^{[2]} = H_D\left(\operatorname{Im}\left(R\bar{Y}\right)\right).$$

In Eulerian coordinates, the expression $H_D\left(\operatorname{Im}\left(R\bar{Y}\right)\right)$ arises as the nonlinear component of θ_t, see (2.6). Indeed, in holomorphic coordinates, we compute on the top

$$\nabla\phi\nabla\psi = \operatorname{Im}\left(R\frac{\bar{W}_\alpha}{1+\bar{W}_\alpha}\right) = \operatorname{Im}\left(F - \frac{Q_\alpha}{1+W_\alpha}\right) = \operatorname{Im}(F-R).$$

Understanding $\operatorname{Re} F^{[2]}$, on the other hand, is a slightly more delicate matter, since a-priori it is only determined modulo constants. In our

setting, the constant in $\operatorname{Re} F$ is determined by

$$\begin{aligned} \operatorname{Re} F^{[2]}(\alpha_0, 0) &= \left(\frac{\operatorname{Im} F \operatorname{Im} W_\alpha}{1 + \operatorname{Re} W_\alpha} - \operatorname{Re} R \right)(\alpha_0, 0) \\ &= \left(\frac{\operatorname{Im}(R\bar{Y}) \operatorname{Im} W_\alpha}{J(1 + \operatorname{Re} W_\alpha)} - \operatorname{Re} R \right)(\alpha_0, 0). \end{aligned} \tag{7.1}$$

We will not use the full expression in the sequel, but merely the bound

$$|\operatorname{Re} F^{[2]}(\alpha_0, 0)| \lesssim |R(\alpha_0, 0)|. \tag{7.2}$$

In what follows we will first establish direct bounds for $\operatorname{Im} F^{[2]}$, which has a bilinear structure as described above. The real part will satisfy similar bounds except at very low frequencies $\leq 1/h$.

7.1. **Bilinear estimate for $\operatorname{Im} F^{[2]} = H_D(\nabla\phi\nabla\psi)$.** For this expression we will prove the following bounds, which are needed in order to switch from Err_3 to Err_3^{hol} and prove Proposition 5.3:

Lemma 7.1. *a) The function $F^{[2]}$ restricted to the top satisfies the following high frequency estimate with $\lambda \geq 1$.*

$$\|P_\lambda F^{[2]}\|_{LE} \lesssim \lambda^{\frac{1}{2}} \epsilon M, \qquad \lambda \geq 1. \tag{7.3}$$

b) It also satisfies the low frequency bound

$$\|P_\lambda F^{[2]}\|_{L^2_t L^\infty_{loc}(B_\lambda)} \lesssim \lambda^{\frac{1}{2}} \epsilon M, \qquad 1/h < \lambda < 1. \tag{7.4}$$

c) Finally, at very low frequencies we have:

$$\|P_{<1/h} \operatorname{Im} F^{[2]}\|_{L^2_t L^\infty_{loc}(B_{1/h})} \lesssim h^{-\frac{1}{2}} \epsilon M. \tag{7.5}$$

We also list some straightforward consequences of the above Lemma:

Corollary 7.2. *The low frequency part of $\operatorname{Im} F^{[2]}$ satisfies on the top*

$$\|P_{\leq 1} \operatorname{Im} F^{[2]}\|_{LE} \lesssim \epsilon M. \tag{7.6}$$

Its harmonic extension satisfies the bound

$$\|H_D(\nabla\theta\nabla\phi)\|_{L^2_t L^\infty(A_\lambda)} \lesssim \lambda^{\frac{1}{2}} \epsilon M. \tag{7.7}$$

The estimates in part a) are not entirely satisfactory because the ℓ^1 summation with respect to λ is missing for $\lambda > 1$. Similarly the ℓ^1 summation with respect to $\lambda < 1$ is missing in part (b). To compensate for that, we complement the above result as follows:

Lemma 7.3. *a) The function $F_h^{[2]} = F_{\geq 1}^{[2]}$ restricted to the top admits the following high frequency decomposition*

$$F_h^{[2]} = F_h^{[2],1} + F_h^{[2],1},$$

where the dyadic pieces of $F_h^{[2],1}$ satisfy

$$\|F_\lambda^{[2],1}\|_{L^2_t H^{-\frac{1}{2}}_{loc}} \lesssim M c_\lambda, \tag{7.8}$$

while the dyadic pieces of $F_h^{[2],2}$ *satisfy*

$$\|F_\lambda^{[2],2}\|_{L_t^\infty H^1} \lesssim \epsilon c_\lambda. \tag{7.9}$$

As a consequence of the previous lemma and interpolation (or by a similar direct proof), we have

Corollary 7.4. *The function* $F_h^{[2],2}$ *in the last lemma also satisfies the interpolated bounds*

$$\|F_h^{[2],2}\|_{L_t^p H_{loc}^s} \lesssim \epsilon^{2-\frac{2}{p}} M^{\frac{2}{p}}, \tag{7.10}$$

where

$$2 < p < \infty, \qquad \frac{1}{p} = \frac{1-s}{3}.$$

Similarly, to account for the lack of summability in the low frequency bound (7.4), we have the following:

Lemma 7.5. *We can decompose* $F_l^{[2]} := F_{[1/h,1]}^{[2]}$ *into*

$$F_l^{[2]} = F_l^{[2],1} + F_l^{[2],2},$$

where the dyadic pieces of $F_l^{[2],1}$ *satisfy the dyadic bounds*

$$\lambda^{-\frac{1}{2}} \|F_\lambda^{[2],1}\|_{L_t^2 L_{loc}^\infty(B_\lambda)} \lesssim c_\lambda M, \tag{7.11}$$

while the dyadic pieces of $F_l^{[2]}$ *satisfy the weaker bound*

$$\sup_\lambda \lambda^{-\frac{1}{2}} \|F_\lambda^{[2],1}\|_{L_t^2 L_{loc}^\infty(B_\lambda)} \lesssim \epsilon M, \tag{7.12}$$

as well as the uniform bound

$$\lambda^{\frac{1}{2}} \|F_\lambda^{[2],2}\|_{L^\infty} \lesssim c_\lambda \epsilon. \tag{7.13}$$

The bounds in Lemma 7.1 will be used in order to estimate trilinear terms. For quartic terms on one hand we have more flexibility, and Lemmas 7.3, 7.5 are more useful.

We now successively prove the above lemmas.

Proof of Lemmas 7.1, 7.3, 7.5. Here we use the dyadic local energy bounds for R in Lemma 6.10 as well as the local energy bounds for Y in Lemma 6.8. On the other hand, in terms of the control norm, we have the bounds:

$$\|R_\lambda\|_{L^2} \lesssim \lambda^{-1} c_\lambda, \qquad \|Y_\lambda\|_{L^2} \lesssim \lambda^{-\frac{1}{2}} c_\lambda.$$

As usual we consider the Littlewood-Paley decomposition of $F_\lambda^{[2]}$,

$$F_\lambda^{[2]} = \mathbf{P}[\mathrm{Im}(R_\lambda \bar{Y}_{<\lambda}) + \mathrm{Im}(R_{<\lambda} \bar{Y}_\lambda)] + \mathbf{P} \sum_{\mu \geq \lambda} P_\lambda \,\mathrm{Im}(R_\mu \bar{Y}_\mu).$$

The first two terms in this decomposition, namely the high-low and the low-high interaction, are estimated in the same manner as in Lemmas 6.12,6.13 to obtain the high frequency bounds

$$\|\mathbf{P}\,\mathrm{Im}(R_\lambda \bar{Y}_{<\lambda})\|_{L^2_t L^2_{loc}} + \|\mathbf{P}\,\mathrm{Im}(R_{<\lambda}\bar{Y}_\lambda)]\|_{L^2_t L^2_{loc}} \lesssim c_\lambda M \lambda^{\frac{1}{2}},$$

respectively the low frequency bounds

$$\|\mathbf{P}\,\mathrm{Im}(R_\lambda \bar{Y}_{<\lambda})\|_{L^2_t L^\infty_{loc}(B_\lambda)} + \|\mathbf{P}\,\mathrm{Im}(R_{<\lambda}\bar{Y}_\lambda)]\|_{L^2_t L^\infty_{loc}(B_\lambda)} \lesssim c_\lambda M \lambda^{\frac{1}{2}}.$$

These both suffice for Lemma 7.1, and show that these contributions can be placed in $F^{[2],1}_h$ for Lemma 7.3, respectively in $F^{[2],1}_l$ for Lemma 7.5.

Thus it remains to consider the case of high-high interactions, $P_\lambda\,\mathrm{Im}(R_\mu \bar{Y}_\mu)$. Here we separate the analysis into low and high frequencies.

A. High frequencies $1 \le \lambda \le \mu$. Here we estimate again as in Lemmas 6.12,6.13,

$$\|P_\lambda\,\mathrm{Im}(R_\mu \bar{Y}_\mu)\|_{L^2_t L^2_{loc}} \lesssim \lambda^{\frac{1}{2}} \|\,\mathrm{Im}(R_\mu \bar{Y}_\mu)\|_{L^2_t L^1_{loc}} \lesssim \lambda^{\frac{1}{2}} \|R_\mu\|_{L^2_t L^2_{loc}} \|\bar{Y}_\mu\|_{L^2_t L^2} \lesssim \lambda^{\frac{1}{2}} c_\mu M.$$

This suffices for the μ summation, which yields the conclusion of Lemma 7.1(a), but yields no λ summation due to a lack of off-diagonal decay. Because of this, for Lemma 7.3 we place this term in $F^{[2],2}_h$ and estimate it by

$$\|P_\lambda\,\mathrm{Im}(R_\mu \bar{Y}_\mu)\|_{L^\infty_t L^2} \lesssim \lambda^{\frac{1}{2}} \|\,\mathrm{Im}(R_\mu \bar{Y}_\mu)\|_{L^\infty_t L^1} \lesssim \lambda^{\frac{1}{2}} \|R_\mu\|_{L^\infty L^2} \|Y_\mu\|_{L^\infty_t L^2} \lesssim \lambda^{\frac{1}{2}} \mu^{-\frac{3}{2}} c_\mu^2,$$

where we have off-diagonal decay,

$$\sum_{\mu > \lambda} \|P_\lambda\,\mathrm{Im}(R_\mu \bar{Y}_\mu)\|_{L^\infty_t L^2} \lesssim \lambda^{-1} c_\lambda,$$

as desired.

B. Low frequencies $1 \le \lambda \le \mu$. Here we should also consider two cases, $\mu \le 1$ and $\mu > 1$. The latter case is similar but simpler, so it is omitted. Assuming $\lambda \le \mu \le 1$ we compute

$$\begin{aligned}\|P_\lambda\,\mathrm{Im}(R_\mu \bar{Y}_\mu)\|_{L^2_t L^\infty_{loc}(B_\lambda)} &\lesssim \lambda \|\,\mathrm{Im}(R_\mu \bar{Y}_\mu)\|_{L^2_t L^1_{loc}(B_\lambda)} \lesssim \lambda \|R_\mu\|_{L^2_t L^2_{loc}(B_\lambda)} \|\bar{Y}_\mu\|_{L^\infty_t L^2} \\ &\lesssim \mu^{\frac{1}{2}} \lambda^{\frac{1}{2}} \|R_\mu\|_{L^2_t L^2_{loc}(B_\mu)} \|\bar{Y}_\mu\|_{L^\infty_t L^2} \lesssim \lambda^{\frac{1}{2}} c_\mu M.\end{aligned}$$

This is again good enough for Lemma 7.1(c), but there is no λ summation. Hence, for Lemma 7.5 we place these contributions in $F^{[2],2}_\lambda$, and estimate them exactly as in the high frequency case. □

The bound for $Err_3 - Err_3^{hol}$: proof of Proposition 5.3. The expression Err_3 is given by

$$Err_3 = \int_0^T \iint m_x(x - x_0)\,\mathrm{Im}\, R\, H_D(\nabla\theta \nabla\phi)\, dxdydt,$$

while its holomorphic counterpart is

$$Err_3^{hol} = \int_0^T \iint m_\alpha(\alpha - \alpha_0)\,\mathrm{Im}\, R\, H_D(\nabla\theta \nabla\phi)\, dxdydt.$$

In their difference we obtain errors due to (i) Jacobian terms and (ii) the switch between Eulerian vertical strips and the vertical strips in holomorphic coordinates. We estimate the difference using Proposition 6.4, which yields

$$|Err_3 - Err_3^{hol}| \lesssim D_1 + D_2,$$

where

$$D_1 := \int_0^T \iint_{A_1} (|W_\alpha| + |\operatorname{Re} W(-\operatorname{Re} W(\alpha_0, 0)|)|R||H_D(\nabla\theta\nabla\phi)|\, d\alpha d\beta dt.$$

$$D_2 := \int_0^T \int_{-h}^{-1} (c_\beta + \sup_{|\alpha-\alpha_0|<|\beta|} |W_\alpha|) \sup_{|\alpha-\alpha_0|<|\beta|} |R||H_D(\nabla\theta\nabla\phi)| + |\beta||\partial_\alpha[RH_D(\nabla\theta\nabla\phi)]|\, d\beta dt.$$

A. The estimate for D_1. Here we use the decomposition in Lemma 7.3. The harmonic extension of (the high frequencies of) $F_h^{[2],1}$ belongs to $L_t^2 L_{loc}^2$ by elliptic regularity, which is combined with the similar bound for R, and suffices. To deal with $F_h^{[2],2}$ we imbalance a bit the scales using Corollary 7.4. Working with $s \in (-\frac{1}{2}, \frac{1}{2})$ we obtain that its harmonic extension satisfies

$$\|H_N(F_h^{[2],2})\|_{L_t^p L_{loc}^q} \lesssim M^{\frac{2}{p}} \epsilon^{2-\frac{2}{p}}, \qquad 2 < q < \infty, \qquad \frac{3}{p} - \frac{2}{q} = \frac{1}{2}.$$

Consider first the W_α term, which we also imbalance, interpolating in a similar manner between the energy and the local energy bound. This yields

$$\|W_\alpha\|_{L_t^{p_1} L_{loc}^{q_1}} \lesssim M^{\frac{1}{p_1}} \epsilon^{1-\frac{1}{p_1}}, \qquad 2 \le q_1 \le \infty, \qquad \frac{3}{p_1} - \frac{2}{q_1} = 0.$$

We choose exponents appropriately so that

$$\frac{1}{p} + \frac{1}{p_1} = \frac{1}{1} + \frac{1}{q_1} = \frac{1}{2}.$$

Then we multiply, combining with the L^2 local energy bound for R and using Hölder's inequality.

Next we consider the $|\operatorname{Re} W(\alpha, \beta) - \operatorname{Re} W(\alpha_0, 0)|$ term. To argue as for the previous difference we simply estimate it by the Fundamental Theorem of Calculus

$$\|\operatorname{Re} W(\alpha, \beta) - \operatorname{Re} W(\alpha_0, 0)\|_{L_t^{p_1} L_{loc}^{q_1}} \lesssim \|W_\alpha\|_{L_t^{p_1} L_{loc}^{q_1}},$$

and conclude in the same manner.

B. The estimate for D_2. Consider a dyadic frequency $\lambda < 1$. Then in the corresponding regions A_λ we have by (7.4)

$$\|H_D(\nabla\theta\nabla\phi)\|_{L_t^2 L^\infty(A_\lambda)} \lesssim \lambda^{-1} \|\nabla H_D(\nabla\theta\nabla\phi)\|_{L_t^2 L^\infty(A_\lambda)} \lesssim \epsilon M \lambda^{\frac{1}{2}}.$$

Combined with the bound for R in Lemma 6.10 this suffices for the c_β term. It remains to consider the W_α term. For that it suffices to match the above bound with a corresponding bound for RW_α,

$$\| \sup_{A_\lambda} |R| \sup_{A_\lambda} |W_\alpha| \|_{L^2_t} \lesssim c_\lambda M \lambda^{\frac{1}{2}}. \tag{7.14}$$

It remains to prove (7.14). Harmlessly neglecting the exponentially decaying tails at higher frequencies, we write in A_λ

$$R = \sum_{\mu \leq \lambda} R_\mu, \quad W_\alpha = \sum_{\nu \leq \lambda} W_{\nu,\alpha}.$$

For $\mu < \nu$ we write

$$\| \sup_{A_\lambda} |R_\mu| \sup_{A_\lambda} |W_{\nu,\alpha}| \|_{L^2_t} \lesssim \|R_\mu\|_{L^2_t L^\infty_{loc}(B_\lambda)} \|W_{\nu,\alpha}\|_{L^\infty_t L^\infty_\alpha} \lesssim \mu^{\frac{1}{2}} c_\nu M,$$

while for $\nu < \mu$ we have

$$\| \sup_{A_\lambda} |R_\mu| \sup_{A_\lambda} |W_{\nu,\alpha}| \|_{L^2_t} \lesssim \|R_\mu\|_{L^\infty_t L^\infty_\alpha} \|W_{\nu,\alpha}\|_{L^2_t L^\infty_{loc}(B_\lambda)} \lesssim \mu^{-\frac{1}{2}} \nu c_\nu M,$$

and (7.14) follows in both cases after μ, ν summation.

The proof of Proposition 5.3 is concluded.

□

7.2. **Estimates involving F.** There are three error terms which involve the full F, namely Err_5, Err_6 and Err_7. In this section we will estimate these terms. We need to deal with F in the following combinations:

(1) The harmonic function

$$G_1 := \mathrm{Im}(FR_\alpha). \tag{7.15}$$

(2) The harmonic function

$$G_2 := \mathrm{Im}(F^{[2]} W_\alpha). \tag{7.16}$$

(3) The harmonic extension

$$G_3 := H_D(\mathrm{Im}\, W \,\mathrm{Re}\, F(1 + W_\alpha)). \tag{7.17}$$

We will state our main bounds directly in terms of these expressions, rather than in terms of F. This is because the bounds for G_1, G_2 and G_3 are better viewed as trilinear bounds, rather than more directly as iterated bilinear bounds. We begin with G_1 and G_2, where the results are easier to state:

Proposition 7.6. *a) High frequency bounds. The functions G_1, and G_2 have the following regularity in the fluid domain:*

$$\|G_1\|_{L^2_t L^2_\beta H^{-\frac{1}{2}}_\alpha (A^1(x_0))} \lesssim \epsilon M, \tag{7.18}$$

$$\|G_2\|_{L^2_t L^2(A_1(x_0))} \lesssim \epsilon^2 M. \tag{7.19}$$

.

b) Low frequency bounds:

$$\|G_1\|_{L^2_t L^\infty(A_\lambda(x_0))} \lesssim \lambda c_\lambda M, \tag{7.20}$$

$$\||G_2\|_{L^2_t L^\infty_\alpha(A_\lambda(x_0))} \lesssim \lambda^{\frac{1}{2}} c_\lambda M. \tag{7.21}$$

We postpone the proof of the proposition, in order to complete the proof of the Err_5^1 and Err_6 bounds.

Proof of the bound for Err_5^1. We recall that

$$Err_5^1 := \int_0^T \iint m_\alpha \operatorname{Im}(FR_\alpha) H_D \left(\operatorname{Im} W \operatorname{Re} W_\alpha\right) d\alpha d\beta dt.$$

At high frequency this is estimated combining (7.18) and (6.36). At low frequency instead we combine (7.20) and (6.37). □

Proof of the bound for Err_6. We recall that

$$Err_6 = \int_0^T \iint m_\alpha \operatorname{Im} R \operatorname{Im}((F-R)W_\alpha)\, d\alpha d\beta dt.$$

This we can estimate using Lemma 6.10 for R, and (7.19), (7.21) for the second factor. □

Next we consider the bounds for G_3, which are summarized in the following:

Proposition 7.7. *For each $1/h < \mu < 1$, the function G_3 admits a decomposition*

$$G_3 = G_3^{\mu,1} + G_3^{\mu,2},$$

where the two components satisfy estimates as follows:

a) High frequency bounds.

$$\|\partial_\alpha G_3^{\mu,1}\|_{L^2_t L^2(A_1(\alpha_0))} \lesssim \mu^{-\frac{1}{2}} c_\mu M, \tag{7.22}$$

$$\sup_\mu \mu^{-\frac{1}{2}} \|\partial_\alpha G_3^{\mu,2}\|_{L^1 L^2(A_1(\alpha_0))} \lesssim M^2. \tag{7.23}$$

b) Low frequency bounds:

$$\mu^{\frac{1}{2}} \|\partial_\alpha G_3^{\mu,1}\|_{L^2_t L^\infty(A_\mu(\alpha_0))} \lesssim c_\mu M, \tag{7.24}$$

respectively

$$\sup_\mu \mu^{-\frac{1}{2}} \|\partial_\alpha G_3^{\mu,2}\|_{L^1_t L^\infty(A_\mu(\alpha_0))} \lesssim M^2. \tag{7.25}$$

We now use this Proposition to estimate the remaining error:

Proof of the bound for Err_7. We recall that

$$Err_7 := \int_0^T \iint m_\alpha \operatorname{Im} R\partial_\alpha G_3 \, d\alpha d\beta dt.$$

We first estimate the bound for $R_{\geq 1}$, using the decomposition above with $\mu = 1$. The $G_3^{1,1}$ contribution is easy to bound using the local energy for R. The $G_3^{1,2}$ contribution is also easy to bound using the uniform H_h^1 control norm for R.

Then we estimate the contribution of R_μ, where we use the above decomposition associated exactly to the frequency μ. Precisely, we match the $G_3^{\mu,1}$ bound with the local energy estimate for R_μ, while on the other hand we match the $G_3^{\mu,2}$ bound with the uniform bound for R using the control norm. □

The remainder of the section is devoted to the proof of Propositions 7.6,7.7. In estimating the contributions of F we will separately consider three regimes:

I. High frequencies: ≥ 1. Here the real and imaginary part of F satisfy similar estimates.

II. Low frequencies: $\in [h^{-1}, 1]$. Again the real and imaginary part of F satisfy similar estimates. We also include here the very low frequencies of $\operatorname{Im} F$ and R.

III. Very low frequencies: $\leq h^{-1}$ for $\operatorname{Re}(F-R)$. It is here that the difference between $\operatorname{Re} F$ and $\operatorname{Im} F$ comes into play, along with the assignment of constants as discussed in the beginning of the section.

I. The high frequencies of F. For the R_h component of F_h we simply use Lemma 6.10. For the high frequencies $F_h^{[2]}$ of $F^{[2]}$ we instead rely on Lemmas 7.1, 7.3. Only in a few cases we need to backtrack further and use the structure of $F_h^{[2],1}$ and $F_h^{[2],2}$.

I.a. The contribution of F_h to G_1. We use the bilinear Littlewood-Paley expansion

$$P_\lambda(F_h R_\alpha) = F_{h,\lambda} R_{<\lambda,\alpha} + F_{h,<\lambda} R_{\lambda,\alpha} + \sum_{\mu \geq \lambda} P_\lambda(F_\mu R_{\mu,\alpha}). \tag{7.26}$$

For F_h we use the expansion

$$F_h = R_h + F_h^{[2],1} + F_h^{[2],2},$$

where the last two terms are as in Lemma 7.3. We successively consider the three terms in (7.26).

For the first term in (7.26) it is easy to bound the output of the $F_\lambda^{[2],2}$ component. Indeed, using the local energy norm for R and Lemma 6.11, we have

$$\|R_\alpha\|_{L_t^2 H_{loc}^{-\frac{3}{2}}} \lesssim M,$$

which can be in turn easily combined with (7.9).

The output of the $R_\lambda + F_\lambda^{[2],1}$ component is more difficult to estimate. We recall that $F_\lambda^{[2],1}$ arises from unbalanced frequency interactions, so we expand it as

$$F_\lambda^{[2],1} = \mathbf{P}[\mathrm{Im}(R_\lambda(\bar{Y}_{<\lambda} + 2)) + \mathrm{Im}(R_{<\lambda}\bar{Y}_\lambda)].$$

Multiplying this by $R_{<\lambda,\alpha}$ we obtain a trilinear form, for which we need to balance the three input frequencies. There are two terms to consider, and we only consider the worst one,

$$\mathbf{P}(Y_\lambda \bar{R}_{<\lambda}) R_{<\lambda,\alpha}.$$

This is estimated using Hölder's inequality and Bernstein's inequality as follows:

$$\begin{aligned}
\|\mathbf{P}(Y_\lambda \bar{R}_{<\lambda}) R_{<\lambda,\alpha}\|_{L_t^2 H_{loc}^{-1}} &\lesssim \lambda^{-1} \|Y_\lambda \bar{R}_{<\lambda} R_{<\lambda,\alpha}\|_{L_t^2 L_{loc}^2} \\
&\lesssim \lambda^{-1} \|Y_\lambda\|_{L_t^\infty L^2} \|\bar{R}_{<\lambda}\|_{L_t^2 L^\infty} \|R_{<\lambda,\alpha}\|_{L_t^\infty L^\infty} \\
&\lesssim \lambda^{-1} \|Y_\lambda\|_{L_t^\infty L^2} \lambda^{\frac{1}{2}} \|\bar{R}_{<\lambda}\|_{L_t^2 L^2} \lambda^{\frac{1}{2}} \|R_{<\lambda,\alpha}\|_{L_t^\infty L^2} \\
&\lesssim \|Y_\lambda\|_{L_t^\infty \dot{H}^{\frac{1}{2}}} \|R\|_{L_t^2 H_{loc}^{-\frac{1}{2}}} \|R_\alpha\|_{L_t^\infty L^2} \\
&\lesssim c_\lambda M \epsilon.
\end{aligned}$$

Here the three factors on the right are estimated using the uniform control norm, local energy, respectively the uniform control norm.

For the second term in (7.26) it is easy to bound the output of the R and the $F^{[2],1}$ component, using the uniform control norm for R.

We are left with the $F^{[2],2}$ component, which arises from balanced interactions of R and Y. Thus we obtain again a trilinear form. Precisely, we need to bound in $L_t^\infty L_{loc}^2$ the expression

$$\sum_{\mu_1 \lesssim \mu, \lambda} P_{\mu_1}(R_\mu \bar{Y}_\mu) R_{\lambda,\alpha} = \sum_{\mu < \lambda} (R_\mu \bar{Y}_\mu) R_{\lambda,\alpha} + \sum_{\mu \geq \lambda} P_{<\lambda}(R_\mu \bar{Y}_\mu) R_{\lambda,\alpha}.$$

Here for the first sum where $\mu < \lambda$ we use Hölder's inequality followed by Bernstein's inequality and arrive at the bound

$$\big\|(R_\mu \bar{Y}_\mu) R_{\lambda,\alpha}\big\|_{L_t^2 H_{loc}^{-1}} \lesssim \mu \|R\|_{L_t^2 H_{loc}^{-\frac{1}{2}}} \|Y\|_{L_t^\infty L^\infty} \|R_{\lambda,\alpha}\|_{L^\infty L^2} \lesssim \mu c_\lambda \epsilon M, \tag{7.27}$$

which suffices after μ summation. Here we used the local energy for R and the uniform control norm for the remaining factors.

On the other hand for the second sum which corresponds to the range $\mu \geq \lambda$ we estimate

$$\left\|(R_\mu \bar{Y}_\mu) R_{\lambda,\alpha}\right\|_{L^2_t H^{-1}_{loc}} \lesssim \lambda^2 \mu^{-1} \|R_\mu\|_{L^\infty_t H^1} \|Y\|_{L^\infty_t L^\infty} \|R_\alpha\|_{L^2_t H^{-\frac{3}{2}}_{loc}} \lesssim \mu^{-1} \lambda^2 c_\mu \epsilon M, \tag{7.28}$$

which again suffices. This corresponds to using local energy for R_α and the uniform control norm for the remaining factors.

Finally, the third term in (7.26) is negligible since we are multiplying two holomorphic functions, so the output at low frequency is exponentially small.

I.b. The contribution of F_h to G_2. This is given by $F_h^{[2]} W_\alpha$. We use again the Littlewood-Paley trichotomy,

$$P_\lambda(F_h^{[2]} W_\alpha) = F_{h,\lambda}^{[2]} W_{<\lambda,\alpha} + F_{h,<\lambda}^{[2]} W_{\lambda,\alpha} + \sum_{\mu>\lambda} P_\lambda(F_{h,\mu}^{[2]} W_{\mu,\alpha}). \tag{7.29}$$

The first term is easy for the $F_h^{[2],2}$ component, where we use the local energy norm for $\operatorname{Im} W$.

For the $F_h^{[2],1}$ component we again expand $F_h^{[2],1}$ as a bilinear form in R and Y which contains only high-low interactions, obtaining a trilinear form. As before we have two contributions, of which we describe the worst, namely

$$\sum_{\lambda>1} \mathbf{P}(\bar{R}_{<\lambda} Y_\lambda) W_{<\lambda,\alpha}.$$

This is estimated by

$$\left\|\mathbf{P}((\bar{R}_{<\lambda} Y_\lambda) W_{<\lambda,\alpha}\right\|_{L^2_t H^{-\frac{1}{2}}_{loc}} \lesssim \|R\|_{L^2_t H^{-\frac{1}{2}}_{loc}} \|Y_\lambda\|_{L^\infty_t L^\infty_\alpha} \|W_\alpha\|_{L^\infty_t L^\infty_\alpha} \lesssim c_\lambda \epsilon M. \tag{7.30}$$

Consider now the second term in (7.29). The bound is easy for the $F_h^{[2],1}$ component. For the $F_h^{[2],2}$ component we need to consider the sum

$$\sum_{\mu_1<\mu,\lambda} P_{\mu_1}(R_\mu \bar{Y}_\mu) W_{\alpha,\lambda} = \sum_{\mu<\lambda} (R_\mu \bar{Y}_\mu) W_{\alpha,\lambda} + \sum_{\lambda\leq\mu} P_{<\lambda}(R_\mu \bar{Y}_\mu) W_{\alpha,\lambda}.$$

Again we estimate the two terms differently. For the first sum where $\mu < \lambda$ we compute by Hölder and Bernstein's inequalities at fixed time

$$\left\| \sum_{1\leq\mu<\lambda} R_\mu \bar{Y}_\mu W_{\alpha,\lambda} \right\|_{L^2_t H^{-\frac{1}{2}}_{loc}} \lesssim \lambda^{-\frac{1}{2}} \mu^{\frac{1}{2}} \|R\|_{L^2_t \dot{H}^{-\frac{1}{2}}_{loc}} \|Y\|_{L^\infty_t L^\infty_\alpha} \|W_{\lambda,\alpha}\|_{L^\infty_t H^{\frac{1}{2}}_h} \lesssim \lambda^{-\frac{1}{2}} \mu^{\frac{1}{2}} c_\lambda \epsilon M,$$

using the local energy bound for R and the uniform control norm for the other two factors. Similarly, for the second sum, where $\mu \gtrsim \lambda$, we have the fixed time bound

$$\left\|P_{<\lambda}(R_\mu \bar{Y}_\mu) W_{\alpha,\lambda}\right\|_{L^2_t H^{-\frac{1}{2}}_{loc}} \lesssim \lambda \mu^{-1} \|R_\mu\|_{L^\infty_t H^1_h} \|Y\|_{L^\infty_t L^\infty_\alpha} \|W\|_{L^2_t L^2_{loc}} \lesssim \lambda \mu^{-1} c_\mu \epsilon M,$$

which again suffices.

Finally, the third term in (7.29) is negligible as the two factors are holomorphic.

I.c. The contribution of F_h to G_3. Here we will estimate directly the product $F_h \operatorname{Im} W$, as $F_h W_\alpha$ is easily seen to satisfy the same bounds as F_h. We use again the Littlewood-Paley trichotomy,

$$P_\lambda(F_h W) = F_{h,\lambda} \operatorname{Im} W_{<\lambda} + F_{h,<\lambda} W_\lambda + \sum_{\mu>\lambda} P_\lambda(F_{h,\mu} W_\mu). \tag{7.31}$$

The contribution of R to the first term is easy to bound in $L^2_t H^{-\frac{1}{2}}_{loc}$, and so can be included in $G_3^{1,1}$. Consider now the expression

$$\sum_{\lambda \geq 1} F^{[2]}_{h,\lambda} \operatorname{Im} W_{<\lambda}.$$

Here it is easy to estimate the contribution of $F_h^{[2],2}$, using the local energy bound for $\operatorname{Im} W$. Hence we consider the contribution of $F_h^{[2],1}$, which contains the high-low interactions of R and Y in $F^{[2]}$. We expand this as a trilinear form, obtaining two terms depending on whether R or Y is at high frequency. The better term is

$$R_\lambda Y_{<\lambda} W_{<\lambda},$$

where the second factor is harmless so this is no different than the corresponding contribution of R.

The worst term is

$$Y_\lambda \bar{R}_{<\lambda} \operatorname{Im} W_{<\lambda}.$$

To bound it we consider several cases depending on the frequencies of R and $\operatorname{Im} W$:

(i) Both frequencies ≥ 1. Then we have the fixed time estimate

$$\begin{aligned} \| Y_\lambda \bar{R}_{[1,\lambda)} \operatorname{Im} W_{[1,\lambda)} \|_{L^2_t H^{\frac{1}{2}}_{loc}} &\lesssim \| Y_\lambda \|_{L^\infty_t H^{\frac{1}{2}}_h} (\| R \|_{L^\infty_t \dot{H}^1_h} \| \operatorname{Im} W \|_{L^2_t L^2_{loc}} + \| R \|_{L^2_t H^{-\frac{1}{2}}_{loc}} \| \operatorname{Im} W \|_{L^\infty_t H^{\frac{3}{2}}_\alpha}) \\ &\lesssim c_\lambda \epsilon M, \end{aligned}$$

where we balance norms depending on which of the frequencies of R and W is larger. This contribution is included in $G_3^{1,1}$.

(ii) One frequency ≥ 1, and one ≤ 1. Here the same argument as above applies, where we bound the low frequency factor in $L^2_t L^\infty_{loc}$. Again here we use $G_3^{1,1}$.

(iii) Both frequencies ≤ 1. This is the more difficult term, where we need the parameter μ and the $G_3^{\mu,2}$ component. This is where we differentiate depending on the frequency of R. If the frequency of R is less than μ then we use the local energy bound for R, and add that

contribution to $G_3^{\mu,2}$, If the frequency of R is larger than μ then we use the energy bound for R, and add that contribution to $G_3^{\mu,1}$.

The low-high case, i.e. the second term in (7.31), is similar to the like one for G_1, and goes into $G_3^{1,1}$.

Unlike in the case of G_1 or G_2, here the high-high to low case is also nontrivial. We consider it next. For the two components of $F_h^{[2]}$ we estimate for $1 < \lambda \lesssim \mu$

$$\|P_\lambda(F_{h,\mu}^{[2],1} \operatorname{Im} W_\mu)\|_{L_t^2 H_{loc}^{\frac12}} \lesssim \lambda^{\frac12}\mu^{-\frac12}\|F_{h,\mu}^{[2],1}\|_{L_t^2 H_{loc}^{-\frac12}}\|\operatorname{Im} W_\mu\|_{L_t^\infty H_h^{\frac32}} \lesssim \lambda^{\frac12}\mu^{-\frac12}c_\mu M,$$

respectively

$$\|P_\lambda(F_{h,\mu}^{[2],2} \operatorname{Im} W_\mu)\|_{L_t^2 H_{loc}^{\frac12}} \lesssim \lambda^{\frac12}\mu^{-\frac12}\|F_{h,\mu}^{[2],2}\|_{L_t^\infty H_h^1}\|\operatorname{Im} W\|_{L_t^2 L_{loc}^2} \lesssim \lambda^{\frac12}\mu^{-\frac12}c_\mu M,$$

both of which suffice after μ summation. Both of these components go into $G_3^{1,1}$.

The same estimates also apply for $\lambda = 1$ when P_1 is replaced by $P_{\le 1}$. This addresses the low frequency bounds ≤ 1 in $G_3^{1,1}$.

II. The low frequencies of F. Here we consider the low frequencies

$$F_l := F_{[h^{-1},1]}.$$

Our main tool will be the decomposition for $F_l^{[2]}$ provided by Lemma 7.5. One consequence of Lemma 7.5 is the bound

$$\|F_l\|_{L_t^2 L_{loc}^\infty} \lesssim M, \tag{7.32}$$

which will be used to handle with the contribution of F_l to the high frequencies of G_1, G_2 and G_3.

II.a. The contribution of F_l to G_1. The contribution of R is easy to estimate using Lemma 6.10. The high frequencies ≥ 1 are in turn directly estimated using (7.32).

Here it remains to bound the expression

$$\sum_{\lambda_1,\lambda_2 \le 1} F_{l,\lambda_1}^{[2]} R_{\lambda_2,\alpha}$$

in $L_t^2 L_{loc}^\infty(A_\lambda)$. Restricting to A_λ limits the frequencies λ_1, λ_2 to $[1/h, \lambda]$ with only exponentially decaying tails at higher frequencies. We consider two cases:

If $\lambda_1 \le \lambda_2$ then we can use (7.11) and (7.12), and combine this with the uniform control bound for R and Bernstein's inequality,

$$\|F_{l,\lambda_1}^{[2]} R_{\lambda_2,\alpha}\|_{L_t^2 L_{loc}^\infty(A_\lambda)} \lesssim \|F_{l,\lambda_1}^{[2]}\|_{L_t^2 L_{loc}^\infty(A_\lambda)}\|R_{\lambda_2,\alpha}\|_{L_t^\infty L_\alpha^\infty} \lesssim \lambda_1^{\frac12}\lambda_2^{\frac12}c_{\lambda_2}M,$$

which suffices after λ_1, λ_2 summation.

If $\lambda_1 > \lambda_2$ we can still estimate the contribution of $F_l^{[2],2}$ using (7.13) combined with the pointwise bound for $R_{\lambda_2,\alpha}$ derived from local energy in Lemma 6.10,

$$\|F_{l,\lambda_1}^{[2]} R_{\lambda_2,\alpha}\|_{L_t^2 L_{loc}^\infty(A_\lambda)} \lesssim \|F_{l,\lambda_1}^{[2]}\|_{L_t^\infty L_\alpha^\infty} \|R_{\lambda_2,\alpha}\|_{L_t^2 L_{loc}^\infty(A_\lambda)} \lesssim \lambda_1^{-\frac{1}{2}} \lambda_2^{\frac{3}{2}} c_{\lambda_1} M.$$

This leaves us only with the contribution of $F_l^{[2],1}$, which we expand to a trilinear expression, arriving at an expression of the form

$$\sum_{\lambda_2,\lambda_3 \le \lambda_1 \le \lambda} R_{\lambda_3} Y_{\lambda_1} R_{\lambda_2,\alpha} + R_{\lambda_1} Y_{\lambda_3} R_{\lambda_2,\alpha}.$$

Here we apply the local energy bound for the factor with the lowest frequency $\lambda_{\min}$, and use the uniform control norm for the two highest frequencies. Estimating as above this yields a bound

$$\|R_{\lambda_3} Y_{\lambda_1} R_{\lambda_2,\alpha} + R_{\lambda_1} Y_{\lambda_3} R_{\lambda_2,\alpha}\|_{L_t^2 L_{loc}^\infty(A_\lambda)} \lesssim \lambda_{min}^{\frac{1}{2}} \lambda_1^{\frac{1}{2}} c_{\lambda_1} M,$$

where we have off-diagonal decay for the summation.

II.b. The contribution of F_l to G_2. The contribution of R is easy to estimate using Lemma 6.10. The high frequencies ≥ 1 are in turn directly estimated using (7.32).

It remains to estimate the low frequency contribution of $F_l^{[2]}$, for which we consider the decomposition in Lemma 7.5. This time the contribution of $F_l^{[2],1}$ is easy to bound, using the pointwise estimate for W_α,

$$\|F_{l,<\lambda}^{[2],1} W_{<\lambda,\alpha}\|_{L_t^2 L^\infty(A_\lambda)} \lesssim \lambda^{\frac{1}{2}} c_\lambda M.$$

This leaves us with the contribution of $F_l^{[2],2}$, i.e., with terms of the form

$$\sum_{\lambda_1,\lambda_2<\lambda} F_{l,\lambda_1}^{[2],2} W_{\lambda_2,\alpha}.$$

Here we consider two cases.

a) If $\lambda_1 < \lambda_2$ then we use (7.12) for the first factor combined with the pointwise bound derived from the control norm for the second, which yields

$$\|F_{l,\lambda_1}^{[2],2} W_{\lambda_2,\alpha}\|_{L_t^2 L_\alpha^\infty(A_\lambda)} \lesssim \lambda_1^{\frac{1}{2}} \epsilon c_{\lambda_2} M,$$

with off-diagonal decay which insures the summation with respect to $\lambda_1, \lambda_2 < \lambda$.

b) If $\lambda_1 \geq \lambda_2$ then we use (7.13) for the first factor combined with local energy for the second, which yields

$$\|F_{\lambda_1}^{[2],2} W_{\lambda_2,\alpha}\|_{L_t^2 L_\alpha^\infty(A_\lambda)} \lesssim \lambda_1^{-\frac{1}{2}} \lambda_2 \epsilon c_{\lambda_1} M.$$

This again suffices.

II.c. The contribution of F_l to G_3. For all terms except a single one, it suffices to use only $G_3^{1,1}$. We consider first the contribution of R, which is

$$H_D(\operatorname{Im} W \cdot \operatorname{Re} R_l).$$

Here we use the Littlewood-Paley trichotomy, combining a local energy bound for the low frequency factor with the uniform control norm bound for the high frequency factor. The estimates follow from Hölder's and Bernstein's inequalities. We briefly describe the estimates:

a) In the high-low case $\lambda > \mu$ we have

$$\| \operatorname{Im} W_\lambda \operatorname{Re} R_\mu\|_{L^2_t L^\infty_{loc}(B_\lambda)} \lesssim \| \operatorname{Im} W_\lambda\|_{L^\infty_t L^\infty_\alpha(B_\lambda)} \| \operatorname{Re} R_\mu\|_{L^2_t L^\infty_{loc}(B_\lambda)} \lesssim \mu^{\frac{1}{2}} \lambda^{-1} c_\lambda M.$$

b) In the low-high case we have

$$\| \operatorname{Im} W_{<\lambda} \operatorname{Re} R_\lambda\|_{L^2_t L^\infty_{loc}(B_\lambda)} \lesssim \|W_{<\lambda}\|_{L^2_t L^\infty_\alpha(B_\lambda)} \| \operatorname{Re} R_\lambda\|_{L^\infty_t L^\infty_\alpha(B_\lambda)} \lesssim \lambda^{-\frac{1}{2}} c_\lambda M.$$

c) In the high-high case we have

$$\begin{aligned} \|P_\lambda(\operatorname{Im} W_\mu \operatorname{Re} R_\mu)\|_{L^2_t L^\infty_{loc}(B_\lambda)} &\lesssim \lambda \| \operatorname{Im} W_\mu \operatorname{Re} R_\mu\|_{L^2_t L^1_{loc}(B_\lambda)} \\ &\lesssim \lambda \| \operatorname{Im} W_\mu\|_{L^\infty_t L^2_\alpha} \| \operatorname{Re} R_\mu\|_{L^2_t L^2_{loc}(B_\lambda)} \\ &\lesssim \lambda^{\frac{1}{2}} \mu^{-1} c_\mu M. \end{aligned}$$

Next we consider the contribution of $F_l^{[2]}$ using the Littlewood-Paley trichotomy:

a) The expression $W_{<\lambda} F^{[2]}_{l,\lambda}$. This is the most delicate case. We can easily dispense with $F^{[2],2}_{l,\lambda}$ via (7.13) combined with the local energy bound for W in Lemma 6.6. It remains to consider the contribution of $F^{[2],1}_{l,\lambda}$, which, we recall, is produced from unbalanced interactions of R and Y. Then we are left with trilinear expressions of two types.

a.1) The trilinear form

$$W_{<\lambda} R_\lambda \bar{Y}_{<\lambda}.$$

Here we use local energy for W and the uniform control norm for the remaining two factors; this is identical to case (b) before since Y is bounded.

a.2) The trilinear form

$$W_{<\lambda} R_{<\lambda} \bar{Y}_\lambda,$$

where using one local energy bound does not seem to suffice. It is only here that the decomposition $G_3^{\mu,1} + G_3^{\mu,2}$ is needed. We consider three cases depending on how the two low frequencies λ_1, λ_2 compare with μ.

a.2.i) $\lambda_1, \lambda_2 < \mu$. Then we group terms as

$$R_{<\mu} W_{<\mu} \bar{Y}_\lambda,$$

and use the pointwise bound derived from local energy for the first two factors to obtain

$$\|R_{<\mu} W_{<\mu} \bar{Y}_\lambda\|_{L^1_t L^\infty_{loc}(B_\lambda)} \lesssim \mu^{-1} M^2 c_\lambda.$$

This term is placed in $G_3^{\mu,2}$.

a.2.ii) $\lambda_1 > \mu$. Then also $\lambda > \mu$. We group terms as

$$R_{\lambda_1} W_{<\lambda} \bar{Y}_\lambda.$$

Then we use the local energy bound for W and the control norm for R to get a bound of

$$\|R_{\lambda_1} W_{<\lambda} \bar{Y}_\lambda\|_{L^2_t L^\infty_{loc}(B_\lambda)} \lesssim \lambda_1^{-\frac{1}{2}} c_\lambda c_{\lambda_1} M,$$

where we use the summation for both λ_1 and λ. This term is placed in $G_3^{\mu,1}$.

a.2.iii) $\lambda_1 < \mu < \lambda_2$. Now we switch roles and use local energy for R and the control norm for W. The estimate is similar to the previous case but better. This term is also placed in $G_3^{\mu,1}$.

b) The expression $W_\lambda F^{[2]}_{l,<\lambda}$. This is easier, using the control norm for W_λ and local energy (7.11) (7.12) for $F^{[2]}$.

c) The expression $P_\lambda(W_\nu F^{[2]}_{l,\nu})$ is similar to the above, using either local energy or the control norm for W corresponding to the two components of $F^{[2]}$. This term is also placed in $G_3^{1,1}$.

III. The very low frequencies of F. Here we consider the very low frequencies

$$F_{vl} = \mathrm{Re}(F - R)_{<\frac{1}{h}}.$$

We freely omit the imaginary part of F, as well as R, which fit within the purview of the analysis in the low frequency case.

The size of F_{vl} depends on the choice of the constants, but its derivative does not, so we estimate that first:

Lemma 7.8. *The function ∇F_{vl} satisfies the bound*

$$\|\nabla F_{vl}\|_{L^2_t L^\infty_{loc}(A_{1/h})} \lesssim h^{-\frac{3}{2}} M\epsilon. \tag{7.33}$$

This is an immediate consequence of Lemma 7.5 and the proof is omitted. This estimate allows us to estimate the contribution of $F_{vl} - F_{vl}(\alpha_0)$ to Proposition 7.6. By direct integration, this function satisfies

$$|F_{vl}(z) - F_{vl}(\alpha_0)| \lesssim g(t)|z - \alpha_0|, \qquad \|g\|_{L^2_t} \lesssim h^{-\frac{3}{2}} M\epsilon. \tag{7.34}$$

We now consider its effect on Propositions 7.6, 7.7.

III.a. The contribution of F_{vl} to G_1. This is easily estimated combining the pointwise estimate (7.34) with the pointwise bound for R_α derived from the control norm.

III.b. The contribution of F_{vl} to G_2. This is also straightforward using the pointwise estimate (7.34) together with the pointwise bound for W_α derived from the control norm.

III.c. The contribution of F_{vl} to G_3. Here we need to consider the expression

$$G_{3,vl} = (F_{vl}(\alpha) - F_{vl}(\alpha_0)) \operatorname{Im} W(1 + W_\alpha).$$

To estimate G_3 at frequency λ we use Bernstein's inequality to bound $\operatorname{Im} W$ and W_α in L^∞ in terms of the control norm. To do this we take into account the fact that the lowest frequency must be at least h^{-1}, the highest frequency must be at least λ, as well as the fact that from (7.34) we get a factor of $\lambda^{-1}h^{-\frac{3}{2}}$ in the region $B_\lambda(x_0)$. The worst case scenario is when $\nu < \lambda$ and we estimate in $A_\lambda(\alpha_0)$

$$|P_\lambda[(F_{vl}(\alpha) - F_{vl}(\alpha_0)) \operatorname{Im} W_\nu W_{\lambda,\alpha})]| \lesssim f(t)\lambda^{-1}h^{-\frac{3}{2}}\mu^{-1} \lesssim f(t)\lambda^{-\frac{1}{2}},$$

with trivial ℓ^1 summation. Thus this contribution is directly placed in G_3^1.

Here μ is limited below by $1/h$ because we use the inhomogeneous norms in X.

IV. The constant in F. We denote the constant by $c(t)$ which we will simply estimate via (7.2), which we recall here:

$$|c(t)| \lesssim |R(\alpha_0)|.$$

To evaluate the contribution of c we will use the following

Lemma 7.9. *For each $\lambda > h^{-1}$ we have a decomposition*

$$c = c_\lambda^1 + c_\lambda^2,$$

where

$$\|c_\lambda^1\|_{L_t^2} \lesssim \lambda^{\frac{1}{2}} M, \tag{7.35}$$

respectively

$$\|c_\lambda^2\|_{L_t^\infty} \lesssim \lambda^{-\frac{1}{2}} c_\lambda, \tag{7.36}$$

with additional ℓ^1 summability at low frequency in the last bound.

Proof. This corresponds to the decomposition

$$R = R_{\le\lambda} + R_{\ge\lambda},$$

where for the first term we use the local energy bounds and for the second the X bound. □

We now evaluate the effect of c in Propositions 7.6, 7.7.

IV.ab. The contribution of F_{vl} to G_1, G_2. For G_1 and G_2 we use the above decomposition to estimate FR_λ, respectively $FW_{\lambda,\alpha}$. For the c^1_λ term we use the control norm for its co-factor, and for the c^1_λ term we use local energy for its co-factor.

IV.c. The contribution of F_{vl} to G_1, G_2. The same idea as above applies the $c \operatorname{Im} W$ component of G_3. Finally, for the term

$$c \operatorname{Im} W \cdot W_\alpha$$

we apply a similar argument, but splitting c depending on the lowest of the two frequencies.

Appendix A. Nonlinear computations

In this appendix, we prove another Morawetz's inequality which holds under a very mild smallness assumption on the free surface elevation η (and without restriction on ψ). The proof is entirely different. It exploits the positivity of the pressure to deduce through a virial type argument a control of the kinetic energy. As a result, we obtain a bound of the local energy, which is a quadratic quantity, in terms of the momentum density I_1, which contains a linear term. However, by so doing, we loose the uniformity in the depth h as well as the control of the low-frequency component of the velocity potential.

Theorem A.1. *Let $g \in (0, +\infty)$. Let $s > 5/2$ and T be an arbitrary positive real number. Consider any solution $(\eta, \psi) \in C^0([0,T]; H^s(\mathbb{R}) \times H^s(\mathbb{R}))$ of the water-wave system (1.4). Given $\varepsilon > 0$ and $r > 1/2$, set*

$$m(x) = \int_0^x \frac{d\sigma}{(1+\varepsilon^2\sigma^2)^r}.$$

Assume that

$$(i) \qquad \inf_{(t,x)\in[0,T]\times\mathbb{R}} \eta(t,x) \geq -\frac{h}{2},$$

$$(ii) \qquad \sup_{(t,x)\in[0,T]\times\mathbb{R}} |\eta_x(t,x)| \leq \frac{1}{3},$$

$$(iii) \qquad \varepsilon r\Big(h + \|\eta\|_{L^\infty}\Big) \leq \frac{1}{42}.$$

Then there holds

$$\int_0^T \left\{ \int m_x(x)\eta^2(t,x)\,dx + \iint_{\Omega(t)} m_x(x)\,|\nabla_{x,y}\phi(t,x,y)|^2\,dydx \right\} dt \leq 14 \int_{\mathbb{R}} m(x) I_1(t,x)\,dx \Big|_0^T + 2 \int m(x) I_2(t,x)\,dx \Big|_0^T, \tag{A.1}$$

where

$$I_1(t,x) = \int_{-h}^{\eta(t,x)} \phi_x(t,y)\,dy, \quad I_2(t,x) = \eta(t,x)\psi_x(t,x),$$

and where we used the notation $\int f(t,x)\,dx\,\Big|_0^T = \int f(T,x)\,dx - \int f(0,x)\,dx$.

Proof. The proof is in two different steps. We first estimate the local kinetic energy by using the momentum density I_1 and the positivity of the pressure. Then we estimate the local potential energy by using the momentum density of I_2.

Step 1: kinetic energy. We begin by proving that

$$\int_0^T \iint_{\Omega(t)} m_x(x)\,|\nabla_{x,y}\phi(t,x,y)|^2\,dydxdt \le 7\int_{\mathbb{R}} m(x)I_1(t,x)\,dx\,\Big|_0^T. \tag{A.2}$$

To do so we use the local conservation law $\partial_t I_1 + \partial_x S_1 = 0$ where recall that

$$S_1(t,x) := -\int_{-h}^{\eta(t,x)} (\partial_t\phi + gy)\,dy + \frac{1}{2}\int_{-h}^{\eta(t,x)} \left(\phi_x^2 - \phi_y^2\right) dy.$$

By multiplying the equation $\partial_t I_1 + \partial_x S_1 = 0$ by $m = m(x)$ and integrating by parts, one obtains that

$$\iint_{Q_T} S_1(t,x) m_x\,dxdt = \int mI_1\,dx\,\Big|_0^T,$$

where $Q_T = [0,T]\times\mathbb{R}$. We will prove a stronger result than (A.2). Namely, we will prove that

$$\begin{aligned}\iint_{Q_T} S_1(t,x)m_x\,dxdt \ge \frac{1}{4}\int_0^T \iint_{\Omega(t)} m_x\,|\nabla_{x,y}\phi|^2\,dydxdt \\ + \int_0^T \iint_{\Omega(t)} m_x P\,dydxdt \\ + \frac{h}{2}\int_0^T \int m_x\phi_x^2(x,-h)\,dxdt.\end{aligned} \tag{A.3}$$

This will imply (A.2) since the third term in the right-hand side of (A.3) is obviously positive and since the second one also since $P \ge 0$ (this classical result follows from the maximum principle, the fact that P is sub-harmonic and the boundary condition on the bottom; see Lannes [28]).

To obtain (A.3), we start from

$$\partial_t\phi + \frac{1}{2}|\nabla_{x,y}\phi|^2 + P + gy = 0,$$

which allows us to write S_1 under the form

$$S_1(t,x) := \frac{1}{2}\int_{-h}^{\eta(t,x)} \Big(|\nabla_{x,y}\phi|^2 + P \Big)\, dy + \frac{1}{2}\int_{-h}^{\eta(t,x)} (\phi_x^2 - \phi_y^2)\, dy.$$

Then, to obtain (A.3), the key point is to prove that

$$\iint_{\Omega(t)} (\phi_x^2 - \phi_y^2)\, dydx$$

can be written as the sum of a positive term and a remainder term. This will be deduced from the following identity.

Notation A.2. From now on we use the shorthand notations

$$\iint f\, dx\, dt = \iint_{Q_T} f(t,x)\, dxdt, \quad \iiint f\, dy\, dx\, dt = \int_0^T \iint_{\Omega(t)} f(t,x,y)\, dydxdt.$$

Lemma A.3. *For any function $w = w(x)$ we have*

$$\iint w(\phi_x^2 - \phi_y^2)\, dydx = \int w(h+\eta)\phi_x^2(x,-h)\, dx - 2\iint w\eta_x\phi_x\phi_y\, dydx + 2\iint w_x(y-\eta)\phi_x\phi_y\, dydx. \quad \text{(A.4)}$$

Proof. This identity is proved in [2] when $w = 1$. The time variable is seen as a parameter and we skip it. Set

$$u(x,y) = -w(x)(y-\eta(x))\phi_y(x,y)^2.$$

Then $u(x,\eta(x)) = 0$ and $u(x,-h) = 0$ so $\int_{-h}^{\eta(x)} \partial_y u\, dy = 0$. On the other hand

$$\partial_y u = -2w(y-\eta)\phi_y\phi_{yy} - w\phi_y^2,$$

so integrating on $y \in [-h,\eta(x)]$ and then on x we obtain, remembering that $\phi_{yy} = -\phi_{xx}$,

$$0 = \iint u_y = -\iint w\phi_y^2 + 2\iint w(y-\eta)\phi_y\phi_{xx}.$$

Since $\phi_y = 0$ on $y = -h$, by integrating by parts we infer that

$$0 = -\iint w\phi_y^2 - \iint w(y-\eta)\partial_y\phi_x^2 + 2\iint w\eta_x\phi_x\phi_y - 2\iint w_x(y-\eta)\phi_x\phi_y.$$

Thus

$$0 = -\iint w\phi_y^2 - \iint \partial_y\big(w(y-\eta)\phi_x^2\big) + \iint w\phi_x^2 + 2\iint w\eta_x\phi_x\phi_y - 2\iint w_x(y-\eta)\phi_x\phi_y.$$

Since

$$\int_{\mathbb{R}}\int_{-h}^{\eta} \partial_y\big(w(y-\eta)\phi_x^2\big)\, dy\, dx = \int_{\mathbb{R}} w(h+\eta)\phi_x^2(x,-h)\, dx,$$

this proves the desired result. □

Set

$$\Sigma := \frac{1}{2}\iiint m_x \left|\nabla_{x,y}\phi\right|^2 \, dy\, dx\, dt + \frac{1}{2}\iiint m_x\big(\phi_x^2 - \phi_y^2\big)\, dydxdt.$$

It follows from the previous lemma that $\Sigma = \Sigma_1 + \Sigma_2$ with

$$\Sigma_1 = \iiint \Big(\frac{m_x}{2} - m_x\eta_x + m_{xx}(y-\eta)\Big)\left|\nabla_{x,y}\phi\right|^2 \, dydxdt,$$
$$\Sigma_2 = \iint m_x(h+\eta)\phi_x^2(x,-h)\, dxdt.$$

Now we assume that $\eta \geq -h/2$. Then

$$\Sigma_2 \geq \frac{h}{2}\iint m_x\phi_x^2(x,-h)\, dxdt.$$

Now recall that by definition,

$$m(x) = \int_0^x \frac{d\sigma}{(1+\varepsilon^2\sigma^2)^r},$$

with $r > 1/2$ and where ε has to be chosen. Then

$$m_{xx}(x) = -r\frac{2\varepsilon^2 x}{(1+\varepsilon^2x^2)^{r+1}} = \varepsilon C(\varepsilon,x)m_x(x) \quad \text{with} \quad C(\varepsilon,x) = -2r\frac{\varepsilon x}{1+\varepsilon^2x^2}.$$

Since $|C(\varepsilon,x)| \leq r$, we obtain that $|m_{xx}(x)| \leq \varepsilon r m_x(x)$. As a result,

$$|m_{xx}(y-\eta)| \leq \varepsilon r(h + \|\eta\|_{L^\infty})m_x.$$

Since, on the other hand, one has $|\eta_x| \leq \frac{1}{3}$ by assumption, we conclude that

$$\frac{m_x}{2} - m_x\eta_x + m_{xx}(y-\eta) \geq \Big(\frac{1}{6} - \varepsilon r(h + \|\eta\|_{L^\infty})\Big)m_x.$$

Then, assuming that

$$\varepsilon r\Big(h + \|\eta\|_{L^\infty}\Big) \leq \frac{1}{42},$$

we conclude that

$$\Sigma_1 \geq \frac{1}{7}\iiint m_x \left|\nabla_{x,y}\phi\right|^2 \, dy\, dx\, dt,$$

which completes the proof of (A.3) and hence the proof of (A.2).

Step 2: estimate of the potential energy. In light of (A.2), to prove Theorem A.1, it is sufficient to prove the following estimate about the potential energy:

$$\int_0^T \int g m_x(x)\eta^2(t,x)\,dx\,dt \le \iint_{\Omega(t)} m_x(x)\,|\nabla_{x,y}\phi(t,x,y)|^2\,dydxdt + 2\int m(x)I_2(t,x)\,dx\Big|_0^T, \tag{A.5}$$

where recall that $I_2(t,x) = \eta(t,x)\psi_x(t,x)$.

We now work with the density momentum I_2 and the associated flux force S_2. Recall that

$$S_2 = -\eta\psi_t - \frac{g}{2}\eta^2 + \frac{1}{2}\int_{-h}^{\eta}(\phi_x^2 - \phi_y^2)\,dy,$$

Again, it follows from the local conservation law $\partial_t I_2 + \partial_x S_2 = 0$ that, for any weight $m = m(x)$ and any time T, one has

$$\iint_{Q_T} S_2(t,x)m_x\,dxdt = \int_{\mathbb{R}} m(x)I_2(T,x)\,dx - \int_{\mathbb{R}} m(x)I_2(0,x)\,dx, \tag{A.6}$$

where $Q_T = [0,T]\times\mathbb{R}$.

Let us introduce a notation. Set

$$N(\eta)\psi = \frac{1}{2}\psi_x^2 - \frac{1}{2}\frac{(G(\eta)\psi + \eta_x\psi_x)^2}{1+\eta_x^2},$$

so that the Bernouilli equation reads

$$\partial_t\psi + g\eta + N(\eta)\psi = 0.$$

We begin by reporting the expression for $\partial_t\psi$ given by (1.4) to obtain

$$S_2 = \frac{g}{2}\eta^2 + \eta N(\eta)\psi + \frac{1}{2}\int_{-h}^{\eta(t,x)}(\phi_x^2 - \phi_y^2)\,dy.$$

Let us recall a lemma from [2] which allows to handle the integral involving $N(\eta)\psi$.

Lemma A.4. *For any function $\mu = \mu(x)$ there holds*

$$\int_{\mathbb{R}} \mu N(\eta)\psi\,dx = -\iint_{\Omega} \mu_x\phi_x\phi_y\,dydx + \frac{1}{2}\int \mu\phi_x^2|_{y=-h}\,dx. \tag{A.7}$$

Proof. One can check that

$$N(\eta)\psi = \mathcal{N}\,\big|_{y=\eta}$$

with

$$\mathcal{N} = \frac{1}{2}\phi_x^2 - \frac{1}{2}\phi_y^2 + \eta_x\phi_x\phi_y.$$

The proof then relies on the following identity

$$\partial_y\big(\phi_y^2-\phi_x^2\big)+2\partial_x\big(\phi_x\phi_y\big)=2\phi_y\Delta_{x,y}\phi,$$

which implies that, since ϕ is harmonic and $\partial_y\mu=0$,

$$\partial_y\big(\mu\phi_y^2-\mu\phi_x^2\big)+2\partial_x\big(\mu\phi_x\phi_y\big)=2\mu_x\phi_x\phi_y.$$

We deduce that the vector field $X\colon\Omega\to\mathbb{R}^2$ defined by $X=(-\mu\phi_x\phi_y;\frac{\mu}{2}\phi_x^2-\frac{\mu}{2}\phi_y^2)$ satisfies $\operatorname{div}_{x,y}(X)=-\mu_x\phi_x\phi_y$. Since $\nabla_{x,y}\phi$ belongs to $C^1(\overline{\Omega})$ and since one has the boundary conditions

$$\phi_y|_{y=-h}=0,$$

an application of the divergence theorem gives that

$$\begin{aligned}-\iint_\Omega\mu_x\phi_x\phi_y\,dydx&=\iint_\Omega\operatorname{div}_{x,y}X\,dydx\\&=\int_{\partial\Omega}X\cdot n\,d\sigma=\int\mu\mathcal{N}\,\big|_{y=\eta}\,dx-\frac12\int\mu\phi_x^2|_{y=-h}\,dx.\end{aligned}$$

This completes the proof. □

By combining this result with Lemma A.3, we conclude that

$$\begin{aligned}\iint m_xS_2\,dxdt&=\iint\frac{g}{2}m_x\eta^2\,dxdt\\&\quad+\iint m_x\left(\frac{h}{2}+\eta\right)\phi_x^2|_{y=-h}\,dxdt\\&\quad+\iiint\left(m_{xx}y-2\eta m_{xx}-2m_x\eta_x\right)\phi_x\phi_y\,dydxdt.\end{aligned}\tag{A.8}$$

Now, by assumptions, one has

$$\frac{h}{2}+\eta\geq0,\quad|m_{xx}|\leq\varepsilon r\,|m_x|\,,\quad\sup|\eta_x|\leq\frac13.$$

Consequently,

$$\begin{aligned}|m_{xx}y-2\eta m_{xx}-2m_x\eta_x|&\leq\left(\varepsilon r(h+\|\eta\|_{L^\infty})+2\varepsilon r\,\|\eta\|_{L^\infty}+\frac23\right)|m_x|\\&\leq\left(3\varepsilon r(h+\|\eta\|_{L^\infty})+\frac23\right)|m_x|\\&\leq\left(\frac{3}{42}+\frac23\right)|m_x|\leq|m_x|\,.\end{aligned}$$

So, (A.8) implies that

$$\iint\frac{g}{2}m_x\eta^2\,dxdt\leq\iint m_xS_2\,dxdt+\frac12\iiint m_x\,|\nabla_{x,y}\phi|^2\,dydxdt.$$

The desired result (A.5) then follows from (A.2) and (A.6).

This completes the proof of Theorem A.1. □

References

[1] Albert Ai. Low regularity solutions for gravity water waves. arXiv:1712.07821.

[2] Thomas Alazard. Boundary observability of gravity water waves. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, to appear.

[3] Thomas Alazard. Stabilization of gravity water waves. *Journal Math. Pures Appl.*, 114:51–84, 2018.

[4] Thomas Alazard, Nicolas Burq and Claude Zuily. On the Cauchy problem for gravity water waves. *Invent. Math.*, 198, 71-163, 2014.

[5] Thomas Alazard, Nicolas Burq and Claude Zuily. Strichartz estimates and the Cauchy problem for the gravity water waves equations. *Memoirs of the AMS*, to appear.

[6] Thomas Alazard and Jean-Marc Delort. Sobolev estimates for two dimensional gravity water waves Astérisque, 374 (2015), viii+241 pages.

[7] Thomas Alazard and Jean-Marc Delort. Global solutions and asymptotic behavior for two dimensional gravity water waves. *Ann. Sci. Norm. Supér.*, 48, 1149-1238, 2015.

[8] Thomas Alazard and Guy Métivier. Paralinearization of the Dirichlet to Neumann operator, and regularity of three-dimensional water waves. *Comm. Partial Differential Equations*, 34(10-12):1632–1704, 2009.

[9] Serge Alinhac. Paracomposition et opérateurs paradifférentiels. *Comm. Partial Differential Equations*, 11(1):87–121, 1986.

[10] Serge Alinhac. Existence d'ondes de raréfaction pour des systèmes quasi-linéaires hyperboliques multidimensionnels. *Comm. Partial Differential Equations*, 14(2):173–230, 1989.

[11] Hajer Bahouri, Jean-Yves Chemin, and Raphaël Danchin. *Fourier analysis and nonlinear partial differential equations*, volume 343 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2011.

[12] T Brooke Benjamin. Impulse, flow force and variational principles. *IMA Journal of applied Mathematics*, 32(1-3):3–68, 1984.

[13] T. Brooke Benjamin and Peter J. Olver. Hamiltonian structure, symmetries and conservation laws for water waves. *J. Fluid Mech.*, 125:137–185, 1982.

[14] Jean-Michel Bony. Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires. *Ann. Sci. École Norm. Sup. (4)*, 14(2):209–246, 1981.

[15] Angel Castro, Diego Córdoba, Charles Fefferman, Francisco Gancedo, Javier Gómez-Serrano. Finite time singularities for the free boundary incompressible Euler equations. *Ann. of Math.* (2) 178:1061–1134, 2013.

[16] Demetrios Christodoulou and Hans Lindblad. On the motion of the free surface of a liquid. *Comm. Pure Appl. Math.*, 53(12):1536–1602, 2000.

[17] James Colliander, Markus Keel, Gigliola Staffilani, Hideo Takaoka, and Terence C. Tao. Global well-posedness and scattering for the energy-critical nonlin- ear Schrödinger equation in R^3, *Ann. of Math.* (2) 167 (2008), no. 3, 767Đ865.

[18] Peter Constantin and Jean-Claude Saut Local smoothing properties of dispersive equations. *J. Amer. Math. Soc.* 1 (1988), no. 2, 413–439.

[19] Daniel Coutand and Steve Shkoller. Well-posedness of the free-surface incompressible Euler equations with or without surface tension. *J. Amer. Math. Soc.*, 20(3):829–930 (electronic), 2007.

[20] Daniel Coutand and Steve Shkoller. On the finite-time splash and splat singularities for the 3-D free-surface Euler equations. *Comm. Math. Phys.* 325: 143–183, 2014.

[21] A. Dyachenko, E. Kuznetsov, M. Spector, and V. Zakharov. Analytical description of the free surface dynamics of an ideal fluid (canonical formalism and conformal mapping). *Physics Letters A*, 221(1Ð2):73 – 79, 1996.

[22] Pierre Germain, Nader Masmoudi and Jalal Shatah. Global solutions for the gravity water waves equation in dimension 3. *Annals of Mathematics* (2) 175, no. 2, 691-754, 2012.

[23] Lars Hörmander. *Lectures on linear hyperbolic differential equations*, *Mathématiques & Applications (Berlin) [Mathematics & Applications]*, volume 26. Springer-Verlag, Berlin, 1997.

[24] Benjamin Harrop-Griffiths, Mihaela Ifrim, Daniel Tataru. Finite depth gravity water waves in holomorphic coordinates. *Ann. PDE* 3, no. 1, 2017.

[25] John Hunter, Mihaela Ifrim, Daniel Tataru. Two dimensional water waves in holomorphic coordinates. *Comm. Math. Phys.*, 346, no. 2, 483–552, 2016.

[26] Mihaela Ifrim, Daniel Tataru. Two dimensional water waves in holomorphic coordinates II: global. *Bulletin de la Société Mathématique de France*, 144(2), 369-394, 2016.

[27] Alexandru Ionescu and Fabio Pusateri. Global solutions for the gravity water waves system in 2*d*. *Invent. Math.*, 199 (3), 653-804, 2015.

[28] David Lannes. Well-posedness of the water-waves equations. *J. Amer. Math. Soc.*, 18(3):605–654 (electronic), 2005.

[29] David Lannes. *Water waves: mathematical analysis and asymptotics*, volume 188 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2013.

[30] Hans Lindblad. Well-posedness for the motion of an incompressible liquid with free surface boundary. *Ann. of Math. (2)*, 162(1):109–194, 2005.

[31] Michael S Longuet-Higgins. On integrals and invariants for inviscid, irrotational flow under gravity. *Journal of Fluid Mechanics*, 134:155–159, 1983.

[32] Jeremy Marzuola, Jason Metcalfe, and Daniel Tataru. Strichartz estimates and local smoothing estimates for asympototically flat Schrödinger equations. *J. Funct. Anal.*, 255(6):1497–1553, 2008.

[33] Jason Metcalfe and Daniel Tataru. Decay estimates for variable coefficient wave equations in exterior domains. In *Advances in phase space analysis of partial differential equations*, volume 78 of *Progr. Nonlinear Differential Equations Appl.*, pages 201–216. Birkhäuser Boston Inc., Boston, MA, 2009.

[34] Guy Métivier. *Para-differential calculus and applications to the Cauchy problem for nonlinear systems*, volume 5 of *Centro di Ricerca Matematica Ennio De Giorgi (CRM) Series*. Edizioni della Normale, Pisa, 2008.

[35] Cathleen S Morawetz. Time decay for the nonlinear klein-gordon equation. In *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, volume 306, pages 291–296. The Royal Society, 1968.

[36] V. I. Nalimov. The Cauchy-Poisson problem. *Dinamika Splošn. Sredy*, (Vyp. 18 Dinamika Zidkost. so Svobod. Granicami):104–210, 254, 1974.

[37] Tohru Ozawa and Keith Rogers. Sharp Morawetz estimates. *Journal d Analyse Mathématique*, 121(1) 2013.

[38] Jalal Shatah and Chongchun Zeng. Geometry and a priori estimates for free boundary problems of the Euler equation. *Comm. Pure Appl. Math.*, 61(5):698–744, 2008.

[39] Nathan Totz and Sijue Wu. A Rigorous Justification of the Modulation Approximation to the 2D Full Water Wave Problem. *Comm. Math. Phys.*, 310:817–883, 2012.
[40] Luis Vega. Schrödinger equations: pointwise convergence to the initial data. *Proc. Amer. Math. Soc.* 102 (1988), no. 4, 874–878.
[41] Sijue Wu. Well-posedness in Sobolev spaces of the full water wave problem in 2-D. *Invent. Math.*, 130(1):39–72, 1997.
[42] Sijue Wu. Well-posedness in Sobolev spaces of the full water wave problem in 3-D. *J. Amer. Math. Soc.*, 12(2):445–495, 1999.
[43] Sijue Wu. Almost global wellposedness of the $2 - D$ full water wave problem. *Invent. Math.*, 177: 45–135, 2009.
[44] Sijue Wu. Global wellposedness of the 3-D full water wave problem. *Invent. Math.*, 184:125–200, 2011.
[45] Xuecheng Wang. On 3D water waves system above a flat bottom. arXiv:1508.06223.
[46] Vladimir E. Zakharov. Stability of periodic waves of finite amplitude on the surface of a deep fluid. *Journal of Applied Mechanics and Technical Physics*, 9(2):190–194, 1968.

Thomas Alazard
CNRS and CMLA, École Normale Supérieure de Paris-Saclay, Cachan, France

Mihaela Ifrim
Department of Mathematics, University of Wisconsin–Madison, USA

Daniel Tataru
Department of Mathematics, University of California, Berkeley, USA