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RESEARCH ARTICLE

A raising operator formula for Macdonald polynomials

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Abstract
We give an explicit raising operator formula for the modified Macdonald polynomials $\tilde{H}_\mu(X;q,t)$, which follows from our recent formula for ∇ on an LLT polynomial and the Haglund-Haiman-Loehr formula expressing modified Macdonald polynomials as sums of LLT polynomials. Our method just as easily yields a formula for a family of symmetric functions $\tilde{H}^{1,n}(X;q,t)$ that we call $1,n$-Macdonald polynomials, which reduce to a scalar multiple of $\tilde{H}_\mu(X;q,t)$ when $n=1$. We conjecture that the coefficients of $1,n$-Macdonald polynomials in terms of Schur functions belong to $\mathbb{N}[q,t]$, generalizing Macdonald positivity.

1. Introduction

Tracing back to work of Young, raising operator formulas have been used as a powerful tool in classical symmetric function theory through modern Schubert calculus – see, for example, [1, 8, 10, 12, 13, 16, 18, 19, 27, 33, 34, 38, 39, 41]. Their applications in symmetric function theory include formulas for fundamental bases such as Schur functions and Schur Q-functions, as well as bases in the more contemporary framework involving a parameter t, such as the modified Hall-Littlewood polynomials [20, 31], given by the raising operator formula

$$H_\mu(X;t) = \mathrm{pol}_X\, \boldsymbol{\sigma}\Big(\frac{\mathbf{z}^\mu}{\prod_{\alpha_{ij}\in R_+}(1-t\,z_i/z_j)}\Big). \tag{1}$$

Many research directions have emerged from modifications to classical raising operator formulas. For example, other important families of symmetric functions, including the parabolic Hall-Littlewood polynomials of Shimozono-Weyman [36], and k-Schur functions [8, 15], can be defined by generalizing the formula (1).

The raising operator methodology lies at the foundation of Macdonald's development in [29] of classical symmetric function theory and its one-parameter generalizations, including Hall-Littlewood polynomials. However, for the two-parameter generalization to Macdonald polynomials, no raising operator cornerstone similar to (1) has previously been known, forcing the theory of Macdonald polynomials to be developed in a more indirect way.

Here, we establish a raising operator formula for the modified Macdonald polynomials $\tilde{H}_\mu(X;q,t)$, which reduces at $q=0$ to a previously known formula for Hall-Littlewood polynomials. Using notation defined in §3.1, our formula reads

$$\tilde{H}_\mu(X;q,t) = \omega\,\mathrm{pol}_X\,\boldsymbol{\sigma}\left(\frac{z_1\cdots z_l \prod_{\alpha_{ij}\in R_{\mu^*}\setminus\widehat{R}_{\mu^*}}\left(1-q^{\mathrm{arm}(\mu^*[i])+1}t^{-\mathrm{leg}(\mu^*[i])}\frac{z_i}{z_j}\right)\prod_{\alpha_{ij}\in\widehat{R}_{\mu^*}}\left(1-qt\,\frac{z_i}{z_j}\right)}{\prod_{\alpha_{ij}\in R_+}\left(1-q\,\frac{z_i}{z_j}\right)\prod_{\alpha_{ij}\in R_{\mu^*}}\left(1-t\,\frac{z_i}{z_j}\right)}\right). \quad (2)$$

The proof begins with the Haglund-Haiman-Loehr formula [21] for $\tilde{H}_\mu(X;q,t)$ as a weighted sum of LLT polynomials. We then apply the operator ∇, which has $\tilde{H}_\mu(X;q,t)$ as an eigenfunction, and use the formula for ∇ on an LLT polynomial established in our recent work [7].

A consequence of (2) is the emergence of an intriguing new family of higher Macdonald polynomials $\tilde{H}^{1,n}_\mu(X;q,t)$, given by a formula similar to (2) with $(z_1\cdots z_l)^n$ in place of $z_1\cdots z_l$ – see Theorem 5.2.1. We conjecture (Conjecture 5.2.2) that the coefficients of the resulting polynomials in terms of Schur functions belong to $\mathbb{N}[q,t]$, generalizing Macdonald positivity. As we will see, this conjecture can be formulated for all n simultaneously as the statement that the expression on the right-hand side of (2), before applying $\omega\,\mathrm{pol}_X$, has coefficients in $\mathbb{N}[q,t]$ when regarded as an infinite series of GL_l characters.

In §6, we also derive a new raising operator formula for the integral form Macdonald polynomials $J_\mu(X;q,t)$.

Other raising operator formulas for Macdonald polynomials have previously appeared in the literature. Lassalle-Schlosser [28] inverted the Pieri formula for Macdonald polynomials $Q_\mu(X;q,t)$ (which differ from $J_\mu(X;q,t)$ by a scalar factor) to obtain a formula for $Q_\mu(X;q,t)$ that can be interpreted as a raising operator formula. Shiraishi [37] conjectured a similar raising operator formula for $Q_\mu(X;q,t)$, later proven by Noumi and Shiraishi in their work [32] on the bispectral problem of the Macdonald-Ruijnesaars q-difference operators. However, these formulas are quite different and more intricate than ours.

2. Background

2.1. Partitions and symmetric functions

The (French style) diagram of a partition $\mu = (\mu_1 \geq \cdots \geq \mu_k > 0)$ is the set $\{(i,j)\in\mathbb{Z}^2_+ : 1\leq j\leq k,\ 1\leq i\leq \mu_j\}$. We identify (i,j) with the lattice square or *box* whose northeast corner has coordinates $(x,y)=(i,j)$ and refer to this box as being in *column* i and row j. We set $|\mu| = \mu_1+\cdots+\mu_k$ and let $\ell(\mu)=k$ be the number of nonzero parts of μ. We write μ^* for the transpose of μ. The *arm* and *leg* of a box $b\in\mu$ are the number of boxes in μ strictly east of b and strictly north of b, respectively.

Let $\Lambda=\Lambda(X)$ be the algebra of symmetric functions in infinitely many variables $X = x_1, x_2, \ldots$, with coefficients in the field $\mathbf{k}=\mathbb{Q}(q,t)$. We follow Macdonald's notation [29] for the graded bases of Λ, and for the automorphism $\omega\colon \Lambda\to\Lambda$ given on Schur functions by $\omega s_\lambda = s_{\lambda^*}$. We also work with series and symmetric functions in finitely many variables $\mathbf{z}=z_1,\ldots,z_l$. If $f(X)\in\Lambda$ is a formal symmetric function, then $f(\mathbf{z})$ or $f(z_1,\ldots,z_l)$ denotes its specialization with $X = z_1,\ldots,z_l,0,0,\ldots$.

Given a symmetric function $f\in\Lambda$ and any expression A involving indeterminates, the plethystic evaluation $f[A]$ is defined by writing f as a polynomial in the power-sums p_k and evaluating with $p_k\mapsto p_k[A]$, where $p_k[A]$ is the result of substituting a^k for every indeterminate a occurring in A. The variables q,t from our ground field $\mathbf{k}$ count as indeterminates.

By convention, the name of an alphabet $X = x_1, x_2, \ldots$ stands for $x_1+x_2+\cdots$ inside a plethystic evaluation. Then $f[X] = f[x_1+x_2+\cdots] = f(x_1,x_2,\ldots) = f(X)$. For example, the evaluation $f[X/(1-t^{-1})]$ is the image of $f(X)$ under the $\mathbf{k}$-algebra automorphism of Λ that sends p_k to $p_k/(1-t^{-k})$.

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The *modified Macdonald polynomials* $\tilde{H}_\mu = \tilde{H}_\mu(X;q,t)$ of [17] are defined in terms of the Macdonald polynomials $Q_\mu(X;q,t)$ [29, VI (4.12)] or their integral forms $J_\mu(X;q,t)$ [29, VI (8.3)] by

$$\tilde{H}_\mu(X;q,t) = t^{\mathsf{n}(\mu)} J_\mu\Big[\frac{X}{1-t^{-1}};q,t^{-1}\Big] = t^{\mathsf{n}(\mu)}\Big(\prod_{b\in\mu}(1-q^{\mathrm{arm}(b)+1}t^{-\mathrm{leg}(b)})\Big)Q_\mu\Big[\frac{X}{1-t^{-1}};q,t^{-1}\Big], \tag{3}$$

where

$$\mathsf{n}(\mu) = \sum_i (i-1)\mu_i\,. \tag{4}$$

The $\tilde{H}_\mu(X;q,t)$ also have a direct combinatorial description [21], which we will recall in Theorem 4.2.2.

When $q=0$, the modified Macdonald polynomials reduce to the *modified Hall-Littlewood polynomials*

$$\tilde{H}_\mu(X;0,t) = t^{\mathsf{n}(\mu)} Q_\mu\Big[\frac{X}{1-t^{-1}};t^{-1}\Big], \tag{5}$$

where the Hall-Littlewood polynomials $Q_\mu(X;t)$ are as defined in [29, III (2.11)]. At $t=1$ and $t=\infty$, the $\tilde{H}_\mu(X;0,t)$ specialize to the complete homogeneous symmetric functions $\tilde{H}_\mu(X;0,1) = h_\mu(X)$ and Schur functions $t^{-\mathsf{n}(\mu)}\tilde{H}_\mu(X;0,t)|_{t=\infty} = s_\mu(X)$. We will also work with the following variant of the modified Hall-Littlewood polynomials:

$$H_\mu(X;t) \overset{def}{=} t^{\mathsf{n}(\mu)}\tilde{H}_\mu(X;0,t^{-1}) = Q_\mu[X/(1-t)]. \tag{6}$$

2.2. Weyl symmetrization and related operators

The *Weyl symmetrization operator* $\boldsymbol{\sigma}$ for GL_l is defined by

$$\boldsymbol{\sigma}(f(z_1,\dots,z_l)) = \sum_{w\in\mathcal{S}_l} w\Big(\frac{f(z_1,\dots,z_l)}{\prod_{i<j}(1-z_j/z_i)}\Big) = \sum_{w\in\mathcal{S}_l} w\Big(\frac{f(z_1,\dots,z_l)}{\prod_{\alpha_{ij}\in R_+}(1-z_j/z_i)}\Big), \tag{7}$$

where $f\in\mathbf{k}[z_1^{\pm1},\dots,z_l^{\pm1}]$ is a Laurent polynomial, $\mathcal{S}_l$ acts by permuting the variables $z_1,\dots,z_l$, and $R_+ = R_+(\mathrm{GL}_l) = \{\alpha_{ij} : 1\le i<j\le l\}$ denotes the set of positive roots for GL_l, with $\alpha_{ij} = \epsilon_i - \epsilon_j \in \mathbb{Z}^l$.

When $\mathbf{z}^\nu = z_1^{\nu_1}\cdots z_l^{\nu_l}$ for a dominant weight ν (a weight $\nu\in\mathbb{Z}^l$ is dominant if $\nu_1\ge\cdots\ge\nu_l$), $\boldsymbol{\sigma}(\mathbf{z}^\nu) = \chi_\nu$ is an irreducible GL_l character. For an arbitrary weight $\gamma\in\mathbb{Z}^l$, either $\boldsymbol{\sigma}(\mathbf{z}^\gamma) = \pm\chi_\nu$ for a suitable dominant weight ν, or $\boldsymbol{\sigma}(\mathbf{z}^\gamma) = 0$. We extend $\boldsymbol{\sigma}$ to an operator on formal $\mathbf{k}$-linear combinations $\sum_{\gamma\in\mathbb{Z}^l} c_\gamma \mathbf{z}^\gamma$ by applying it term by term, giving an infinite formal linear combination of irreducible GL_l characters $\sum_\nu a_\nu\chi_\nu = \sum_{\gamma\in\mathbb{Z}^l} c_\gamma\boldsymbol{\sigma}(\mathbf{z}^\gamma)$. This makes sense because for each dominant weight ν, the set of monomials $\mathbf{z}^\gamma$ such that $\boldsymbol{\sigma}(\mathbf{z}^\gamma) = \pm\chi_\nu$ is finite.

Recall that the *polynomial characters* of GL_l are the irreducible characters χ_ν for which ν is a partition, that is, $\nu_l\ge 0$. Given any formal $\mathbf{k}$-linear combination $\sum_\nu a_\nu\chi_\nu$ of irreducible GL_l characters, we define its *polynomial truncation* by

$$\mathrm{pol}_X\Big(\sum_\nu a_\nu\chi_\nu\Big) = \sum_{\nu_l\ge 0} a_\nu s_\nu(X). \tag{8}$$

In principle, the right-hand side is an infinite formal sum of symmetric functions, but, for instance, if $\sum a_\nu\chi_\nu$ is homogeneous of degree d, then the right-hand side is an ordinary symmetric function, homogeneous of degree d.

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We define a related operator $\mathbf{h}_X$ on Laurent polynomials $f(\mathbf{z})$ by

$$\mathbf{h}_X(f(\mathbf{z})) = \mathrm{pol}_X\, \boldsymbol{\sigma}\left(\frac{f(\mathbf{z})}{\prod_{\alpha_{ij}\in R_+}(1 - z_i/z_j)}\right), \tag{9}$$

where the factors $(1 - z_i/z_j)^{-1}$ are expanded as geometric series in z_i/z_j before applying $\boldsymbol{\sigma}$. When f is a monomial, it is well known [39] that

$$\mathbf{h}_X(\mathbf{z}^\gamma) = h_\gamma(X), \tag{10}$$

where for any integer vector $\gamma \in \mathbb{Z}^l$, we define $h_\gamma = h_{\gamma_1} \cdots h_{\gamma_l}$ to be the product of complete homogeneous symmetric functions, with h_d for $d \leq 0$ interpreted as $h_0 = 1$, or $h_d = 0$ for $d < 0$.

We again extend the definition to formal linear combinations of monomials, so that $\mathbf{h}_X(\sum_{\gamma\in\mathbb{Z}^l} c_\gamma \mathbf{z}^\gamma) = \sum_{\gamma\in\mathbb{Z}^l} c_\gamma h_\gamma(X)$. With this interpretation, (9) still remains valid when f is a power series in z_i/z_j for $i < j$. As with pol_X, in principle, $\mathbf{h}_X(\sum_\gamma c_\gamma \mathbf{z}^\gamma)$ is an infinite formal sum of symmetric functions, but for instance, if $\sum_\gamma c_\gamma \mathbf{z}^\gamma$ is homogeneous of degree d, then $\mathbf{h}_X(\sum_\gamma c_\gamma \mathbf{z}^\gamma)$ is an ordinary symmetric function, homogeneous of degree d.

Remark 2.2.1. Below we will write other formulas involving $\boldsymbol{\sigma}$ applied to an expression with denominator factors resembling those in (9). Our convention is always to expand denominator factors of the form $(1 - c\, z_i/z_j)$ for $c \in \mathbf{k}$ and $i < j$ as geometric series $(1 - c z_i/z_j)^{-1} = 1 + c z_i/z_j + \cdots$ before applying $\boldsymbol{\sigma}$.

2.3. Raising operator formulas for modified Hall-Littlewood polynomials

To set the stage for our raising operator formula for modified Macdonald polynomials, we review two different raising operator formulas for the modified Hall-Littlewood polynomials. Both formulas naturally reflect the geometry of the flag variety G/B; one realizes $\tilde{H}_\mu(X; 0, t)$ as the graded Euler characteristic of the cotangent bundle of G/B twisted by a line bundle of weight $-\mu$, while the other is the graded Euler characteristic of the cotangent bundle of G/P_μ, where P_μ is the parabolic subgroup whose block sizes are the parts of μ. See [11] and [36] for details.

The first raising operator formula for $\tilde{H}_\mu(X; 0, t)$ is the one mentioned in the introduction, which we reproduce here (see [29, III (6.3)] or [31, (4.28) and §2]):

$$t^{\mathrm{n}(\mu)} \tilde{H}_\mu(X; 0, t^{-1}) = H_\mu(X; t) = \mathrm{pol}_X\, \boldsymbol{\sigma}\left(\frac{\mathbf{z}^\mu}{\prod_{\alpha_{ij}\in R_+}(1 - t\, z_i/z_j)}\right), \tag{11}$$

where the denominator factors are expanded as geometric series in accordance with Remark 2.2.1.

A second raising operator formula follows from the work of Weyman and Shimozono-Weyman (see [40, Theorem 6.10] and [36, §2.3 (2) and (2.3)–(2.5)]). In this formula, the input partition μ appears in the set of roots, instead of in the weight $\mathbf{z}^\mu$, as it does in formula (11). Given a partition μ of l, consider the set partition of $\{1, \ldots, l\}$ into intervals of lengths $\mu_{\ell(\mu)}, \ldots, \mu_1$, and let B_μ denote the set of roots α_{ij} such that $i < j$ appear in distinct blocks of this partition. Then

$$\tilde{H}_\mu(X; 0, t) = \omega\, \mathrm{pol}_X\, \boldsymbol{\sigma}\left(\frac{z_1 \cdots z_l}{\prod_{\alpha_{ij}\in B_\mu}(1 - t\, z_i/z_j)}\right). \tag{12}$$

Here, we chose to take the parts of μ in reverse order for compatibility with the formula (21) given later, but the order does not actually matter in (12).

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Remark 2.3.1. Formulas such as (11) and (12) are traditionally written using an informal notation – as in [29], [31, (4.28)], or [38, §2] – in which formula (11), for example, would be written as

$$t^{\mathrm{n}(\mu)}\tilde{H}_\mu(X;0,t^{-1}) = \frac{1}{\prod_{\alpha_{ij}\in R_+}\left(1-t\mathbf{R}_{ij}\right)}\cdot s_\mu\,, \tag{13}$$

with raising 'operators' $\mathbf{R}_{ij}$ which act on the subscript of a Schur function s_γ by $\mathbf{R}_{ij}\gamma = \gamma+\epsilon_i-\epsilon_j$. Here, Schur functions indexed by non-partition weights are defined by $s_\gamma(X) = \mathrm{pol}_X\,\boldsymbol{\sigma}(\mathbf{z}^\gamma)$, which is equal to 0 or to ± 1 times a Schur function of partition weight. Note that all the raising operators must be applied before converting Schur functions of non-partition weights to ones indexed by partition weights. The $\mathbf{R}_{ij}$ are not true operators (e.g., $\mathbf{R}_{23}s_{(1,1,1)} = \mathrm{pol}_X\,\boldsymbol{\sigma}(z_1z_2^2) = 0$ but $\mathbf{R}_{12}\mathbf{R}_{23}s_{(1,1,1)} = s_{(2,1,0)} \neq 0$), so we think of (13) as a convenient but informal version of (11).

Remark 2.3.2. Raising operators as used here should not be confused with the creation or vertex operators of Bernstein (see, for example, [29, p. 96]) and Jing [25] for Schur functions and Hall-Littlewood polynomials.

3. Raising operator formula for modified Macdonald polynomials

3.1. The formula

We in fact give many different raising operator formulas for $\tilde{H}_\mu(X;q,t)$, one for each rearrangement β of the parts of μ^*.

For $\beta = (\beta_1,\dots,\beta_k)\in\mathbb{Z}_+^k$, we define the *column diagram* of β to be the set

$$\boldsymbol{\beta} = \left\{(i,j)\in\mathbb{Z}_+^2 : 1\le i\le k,\ 1\le j\le\beta_i\right\}. \tag{14}$$

We identify (i,j) with the box whose northeast corner has coordinates $(x,y) = (i,j)$; we say that this box is in column i and row j. The *reading order* on $\boldsymbol{\beta}$ is the total order $\prec$ on the boxes of $\boldsymbol{\beta}$ given by $(i,j)\prec(i',j')$ if $j>j'$, or $j=j'$ and $i<i'$. We let $\boldsymbol{\beta}[1],\dots,\boldsymbol{\beta}[l]$ denote the boxes of $\boldsymbol{\beta}$ listed in increasing reading order, which is the list of boxes of $\boldsymbol{\beta}$ read by rows from left to right starting from the top row, as shown in Example 3.1.5. For a box $b=(i,j)$, $\mathrm{south}(b) = (i,j-1)$ denotes the box immediately south of b. Define subsets of $R_+ = R_+(\mathrm{GL}_l)$ by

$$R_\beta = \left\{\alpha_{ij}\in R_+ : \mathrm{south}(\boldsymbol{\beta}[i])\in\boldsymbol{\beta},\ \mathrm{south}(\boldsymbol{\beta}[i])\preceq\boldsymbol{\beta}[j]\right\}, \tag{15}$$

$$\widehat{R}_\beta = \left\{\alpha_{ij}\in R_+ : \mathrm{south}(\boldsymbol{\beta}[i])\in\boldsymbol{\beta},\ \mathrm{south}(\boldsymbol{\beta}[i])\prec\boldsymbol{\beta}[j]\right\}. \tag{16}$$

For $\beta\in\mathbb{Z}_+^k$ and a box $b=(i,j)\in\boldsymbol{\beta}$ with $j>1$, define the *arm* and *leg* of b by

$$\mathrm{leg}(b) = \beta_i - j = (\text{number of boxes strictly north of } b), \tag{17}$$

$$\mathrm{arm}(b) = \left|\left\{i'\in\{1,\dots,i-1\} : j-1\le\beta_{i'}<\beta_i\right\}\sqcup\left\{i'\in\{i+1,\dots,k\} : j\le\beta_{i'}\le\beta_i\right\}\right|. \tag{18}$$

In words, $\mathrm{arm}(b)$ is the number of boxes strictly east of b in columns of height $\beta_{i'}\le\beta_i$ or strictly west of $\mathrm{south}(b)$ in columns of height $\beta_{i'}<\beta_i$.

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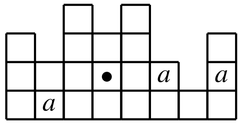
Example 3.1.1. For $\beta = (3,2,4,3,4,2,1,3)$, the column diagram of β is displayed below, along with the arm of box $\bullet = (4,2)$ where the a's mark the boxes contributing to $\operatorname{arm}(\bullet)$.

$\boldsymbol{\beta}$ =



$$\operatorname{arm}(\bullet) = 3 \tag{19}$$

Remark 3.1.2. When β is weakly decreasing (i.e., β is a partition), the column diagram $\boldsymbol{\beta}$ is the diagram of the transpose of the partition β, and the arms and legs of the boxes of $\boldsymbol{\beta}$ agree with the usual notions for partition diagrams mentioned in §2.1.

Definition 3.1.3. For $\beta \in \mathbb{Z}_+^k$, define the *Macdonald series* by

$$\mathbf{H}_\beta(\mathbf{z}; q, t) = \boldsymbol{\sigma}\left(\frac{\prod_{\alpha_{ij} \in R_\beta \setminus \widehat{R}_\beta} \left(1 - q^{\operatorname{arm}(\boldsymbol{\beta}[i])+1} t^{-\operatorname{leg}(\boldsymbol{\beta}[i])} z_i/z_j\right) \prod_{\alpha_{ij} \in \widehat{R}_\beta} \left(1 - q\,t\, z_i/z_j\right)}{\prod_{\alpha_{ij} \in R_+} \left(1 - q\, z_i/z_j\right) \prod_{\alpha_{ij} \in R_\beta} \left(1 - t\, z_i/z_j\right)} \right), \tag{20}$$

which we interpret as an infinite formal linear combination of irreducible GL_l characters by expanding the denominator factors as geometric series, in accordance with Remark 2.2.1.

We have the following raising operator formulas for the modified Macdonald polynomials $\tilde{H}_\mu(X; q, t)$. The proof will be given in §4.

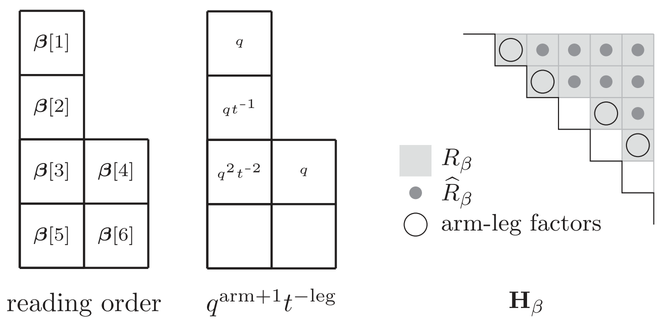
Theorem 3.1.4. *For any partition μ of l and any rearrangement β of μ^*,*

$$\tilde{H}_\mu(X; q, t) = \omega \operatorname{pol}_X\left(z_1 \cdots z_l\, \mathbf{H}_\beta(\mathbf{z}; q, t)\right), \tag{21}$$

where pol_X *is as defined in (8).*

The case of Theorem 3.1.4 when $\beta = \mu^*$ is the formula (2) previewed in the introduction.

Example 3.1.5. (i) For $\beta = (4,2)$, $l = 6$, we visualize the data for the series $\mathbf{H}_\beta$ below, with the subsets of roots R_β and $\widehat{R}_\beta$ drawn in an $l \times l$ grid, labeled by matrix-style coordinates and specified by the legend, and with large circles marking the presence of the factors involving arm and leg, which are $(1 - q\,z_1/z_2)$, $(1 - qt^{-1}z_2/z_3)$, $(1 - q^2t^{-2}z_3/z_5)$, $(1 - q\,z_4/z_6)$.



(ii) For $\beta = (1,4,2,4)$, $l = 11$, we visualize the data for $\mathbf{H}_\beta$ with the same conventions; the factors involving arm and leg, marked by the large circles, are $(1 - q^2 z_1/z_3)$, $(1 - qz_2/z_4)$, $(1 - q^2t^{-1}z_3/z_5)$, $(1 - q^2t^{-1}z_4/z_7)$, $(1 - q^4t^{-2}z_5/z_9)$, $(1 - q^2 z_6/z_{10})$, $(1 - q^3t^{-2}z_7/z_{11})$.

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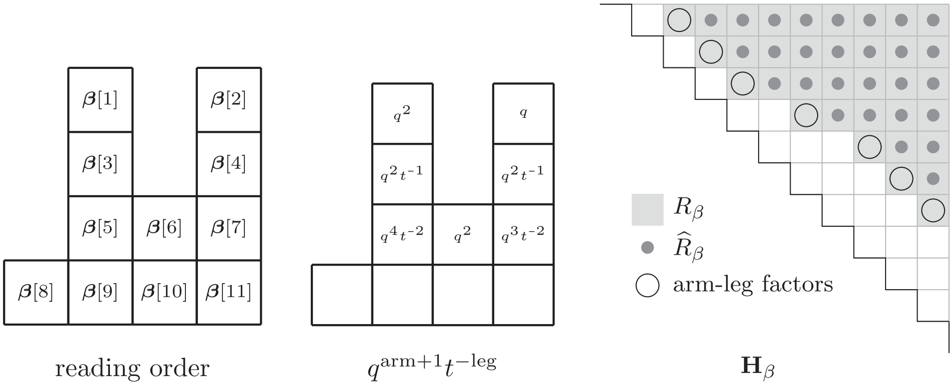

3.2. *Specializations*

It is instructive to see how the well-known specializations of Macdonald polynomials, $\tilde{H}_\mu(X;1,1) = e_1^{|\mu|}(X)$ and the Hall-Littlewood specialization $\tilde{H}_\mu(X;0,t)$, can be recovered from formula (21). This can be done for general β, but it is a little simpler when β is a partition. Accordingly, for this subsection, we now consider only the case $\beta = \mu^*$.

First, we consider the specialization $q = t = 1$. After specializing, the arm-leg and $(1 - q\,t\,z_i/z_j)$ numerator factors in the definition (20) of $\mathbf{H}_\beta$ cancel with the $(1 - t\,z_i/z_j)$ factors in the denominator. Hence,

$$\mathbf{H}_\beta(\mathbf{z};1,1) = \boldsymbol{\sigma}\Big(\frac{1}{\prod_{\alpha_{ij}\in R_+}(1 - z_i/z_j)}\Big). \tag{22}$$

Then, using (9) and (10), we recover the specialization

$$\tilde{H}_\mu(X;1,1) = \omega\,\mathrm{pol}_X(z_1\cdots z_l\,\mathbf{H}_\beta(\mathbf{z};1,1)) = e_{1^l} = e_1^l. \tag{23}$$

Now consider the specialization $q = 0$. After specializing (20), all numerator factors and the $(1 - qz_i/z_j)$ denominator factors reduce to 1. This gives

$$z_1\cdots z_l\,\mathbf{H}_\beta(\mathbf{z};0,t) = \boldsymbol{\sigma}\Big(\frac{z_1\cdots z_l}{\prod_{\alpha_{ij}\in R_\beta}(1 - t\,z_i/z_j)}\Big). \tag{24}$$

Let B_μ be the set of roots defined before (12), which can also be described as the set of positive roots above a block diagonal matrix with block sizes $\mu_{\ell(\mu)},\ldots,\mu_1$. Now R_β is contained in B_μ and $B_\mu \setminus R_\beta$ consists of triangular regions of roots between each pair of consecutive blocks. We reach the Hall-Littlewood raising operator formula using the identity

$$\boldsymbol{\sigma}\Big(\frac{z_1\cdots z_l}{\prod_{\alpha_{ij}\in B_\mu}(1 - t\,z_i/z_j)}\Big) = \boldsymbol{\sigma}\Big(\frac{z_1\cdots z_l}{\prod_{\alpha_{ij}\in R_\beta}(1 - t\,z_i/z_j)}\Big). \tag{25}$$

This is proven by removing these triangular regions from B_μ one root at a time (starting with the bottommost region), and using the following simplified version of [9, Lemma 8.9] to show that the corresponding functions remain the same at each step.

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Lemma 3.2.1. *Let $k \in \{1, \dots, l-1\}$ and suppose that $B \subseteq R_+(\mathrm{GL}_l)$ is a set of roots such that $\prod_{\alpha_{ij} \in B}(1 - tz_i/z_j)$ is fixed by the simple reflection s_k. If B contains a root $\alpha = \alpha_{k+1,j}$ for some $j > k+1$, then*

$$\boldsymbol{\sigma}\Big(\frac{z_1 \cdots z_l}{\prod_{\alpha_{ij} \in B}(1 - t z_i/z_j)}\Big) = \boldsymbol{\sigma}\Big(\frac{z_1 \cdots z_l}{\prod_{\alpha_{ij} \in B \setminus \alpha}(1 - t z_i/z_j)}\Big). \tag{26}$$

Combining (24) and (25) now shows that the raising operator formula (21) for modified Macdonald polynomials reduces at $q = 0$ to the raising operator formula (12) for Hall-Littlewood polynomials.

4. Proof of Theorem 3.1.4

We prove Theorem 3.1.4 using two main ingredients: the Haglund-Haiman-Loehr formulas [21, 22] and our recent formula for ∇ on an LLT polynomial [7]. We explain these two ingredients after a recap of LLT polynomials.

4.1. LLT polynomials

We recall the definition and basic properties of LLT polynomials [26], using the 'attacking inversions' formulation from [23].

A *skew diagram* is a difference $\nu = \lambda/\mu$ of partition diagrams $\mu \subseteq \lambda$. The *content* of a box $b = (i, j)$ in row j, column i of a (skew) diagram is $c(b) = i - j$.

Let $\boldsymbol{\nu} = (\nu_{(1)}, \dots, \nu_{(k)})$ be a tuple of skew diagrams. We consider the set of boxes in $\boldsymbol{\nu}$ to be the disjoint union of the sets of boxes in the $\nu_{(i)}$, and define the *adjusted content* of a box $a \in \nu_{(i)}$ to be $\tilde{c}(a) = c(a) + i\epsilon$, where ϵ is a fixed positive number such that $k\epsilon < 1$.

A *diagonal* in $\boldsymbol{\nu}$ is the set of boxes of a fixed adjusted content – that is, a diagonal of fixed content in one of the $\nu_{(i)}$.

The *reading order* on $\boldsymbol{\nu}$ is the total ordering $<$ on the boxes of $\boldsymbol{\nu}$ such that $a < b \Rightarrow \tilde{c}(a) \le \tilde{c}(b)$ and boxes on each diagonal increase from southwest to northeast. See Example 4.2.3. An *attacking pair* is an ordered pair of boxes (a, b) in $\boldsymbol{\nu}$ such that $a < b$ in reading order and $0 < \tilde{c}(b) - \tilde{c}(a) < 1$.

A *semistandard tableau* on the tuple $\boldsymbol{\nu}$ is a map $T\colon \boldsymbol{\nu} \to \mathbb{Z}_+$ which restricts to a semistandard Young tableau on each component $\nu_{(i)}$. The set of these is denoted $\mathrm{SSYT}(\boldsymbol{\nu})$. An *attacking inversion* in T is an attacking pair (a, b) such that $T(a) > T(b)$. The number of attacking inversions in T is denoted $\mathrm{inv}(T)$.

Definition 4.1.1. The *LLT polynomial* indexed by a tuple of skew diagrams $\boldsymbol{\nu}$ is the generating function

$$\mathcal{G}_{\boldsymbol{\nu}}(X; q) = \sum_{T \in \mathrm{SSYT}(\boldsymbol{\nu})} q^{\mathrm{inv}(T)} \mathbf{x}^T, \tag{27}$$

where $\mathbf{x}^T = \prod_{a \in \boldsymbol{\nu}} x_{T(a)}$. By [23, 26], $\mathcal{G}_{\boldsymbol{\nu}}(X; q)$ is known to be symmetric.

4.2. The Haglund-Haiman-Loehr formula

Haglund-Haiman-Loehr [21] gave a formula for the modified Macdonald polynomials $H_\mu(X; q, t)$ as a positive sum of LLT polynomials, and this was generalized in [22] to give many different expressions for $H_\mu(X; q, t)$ as a positive sum of LLT polynomials, one for each rearrangement β of μ^*. We now recall this formula.

A *ribbon* is a connected skew shape containing no 2×2 block of boxes.

For $\beta \in \mathbb{Z}_+^k$, let $V_\beta = \{(\boldsymbol{\beta}[i], \boldsymbol{\beta}[j]) : \boldsymbol{\beta}[j] = \mathrm{south}(\boldsymbol{\beta}[i])\}$ be the set of ordered pairs of boxes that form vertical dominoes in $\boldsymbol{\beta}$.

Definition 4.2.1. For each $S \subseteq V_\beta$, define $\boldsymbol{\nu}(S) = (\nu_{(1)}, \dots, \nu_{(k)})$ to be the k-tuple of ribbons where the i-th ribbon $\nu_{(i)}$ is determined by

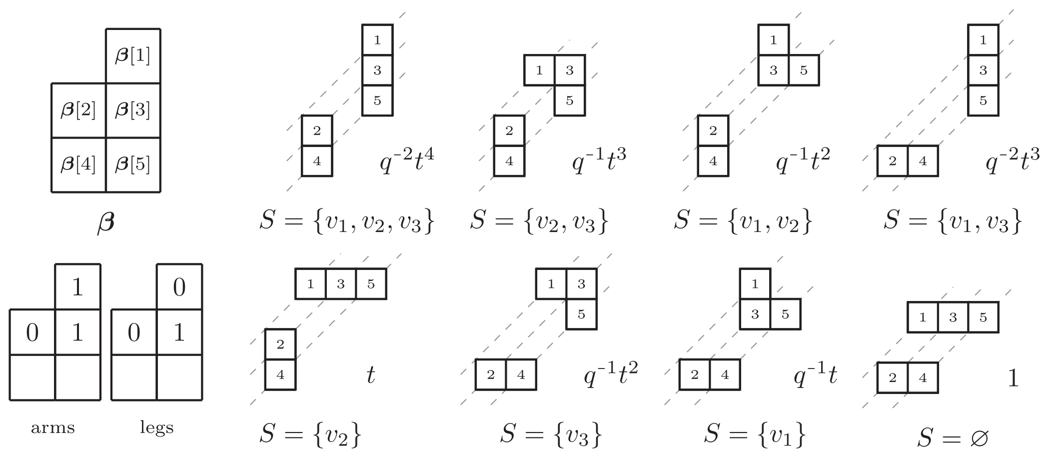
(i) $\nu_{(i)}$ has β_i boxes, of contents $-1, -2, \dots, -\beta_i$, and
(ii) the boxes of contents $-j$ and $-j+1$ in $\nu_{(i)}$ form a vertical domino if and only if the domino $((i,j),(i,j-1))$ in V_β belongs to S.

Theorem 4.2.2 [21, 22]. *Let μ be a partition and let β be any rearrangement of μ^*. Then*

$$\tilde{H}_\mu(X;q,t) = \sum_{S \subseteq V_\beta} \Bigg(\prod_{(\beta[i],\beta[j]) \in S} q^{-\operatorname{arm}(\beta[i])} \, t^{\operatorname{leg}(\beta[i])+1} \Bigg) \mathcal{G}_{\boldsymbol{\nu}(S)}(X;q). \tag{28}$$

Theorem 4.2.2 in the case that β is a partition is immediate from [21, Theorem 2.2, equation (23), and Proposition 3.4], while the generalization to any composition β is addressed in [22, Theorem 5.1.1]. Note that our conventions for diagrams, arms, and legs are the same as those in [22] except that we have reversed the order of the columns (which makes our notation consistent with that in [21]); we have also used the symmetry property $H_\mu(X;q,t) = H_{\mu^*}(X;t,q)$ to translate from the exact version stated in [22, Theorem 5.1.1].

Example 4.2.3. For $\beta = (2,3)$, the 8 summands appearing on the right-hand side of (28) are illustrated by drawing $\boldsymbol{\nu}(S)$ with boxes labeled in reading order and with the corresponding coefficient $\prod_{(\beta[i],\beta[j]) \in S} q^{-\operatorname{arm}(\beta[i])} t^{\operatorname{leg}(\beta[i])+1}$ beside it; the vertical dominoes of $\boldsymbol{\beta}$ are denoted $v_1 = (\boldsymbol{\beta}[1], \boldsymbol{\beta}[3])$, $v_2 = (\boldsymbol{\beta}[2], \boldsymbol{\beta}[4])$, $v_3 = (\boldsymbol{\beta}[3], \boldsymbol{\beta}[5])$. The arm and leg statistics for $\boldsymbol{\beta}$ are shown on the left.



4.3. *A formula for ∇ on any LLT polynomial*

The operator ∇, introduced in [2], is the linear operator on symmetric functions which acts diagonally on the basis of modified Macdonald polynomials $\tilde{H}_\mu(X;q,t)$ by $\nabla \tilde{H}_\mu = q^{\mathrm{n}(\mu^*)} t^{\mathrm{n}(\mu)} \tilde{H}_\mu$.

In [7], we give a raising operator formula for ∇ on any LLT polynomial. The formula takes a simpler form in the case that the LLT polynomial is indexed by a tuple of ribbons. We state the result for the tuple $\boldsymbol{\nu}(S)$ in Definition 4.2.1, making use of the notation $V_\beta, R_\beta, \widehat{R}_\beta$ defined in §4.2, (15), and (16). Also let A_β denote the number of attacking pairs in $\boldsymbol{\nu}(S)$, which depends only on β and not on $S \subseteq V_\beta$.

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Theorem 4.3.1 [7]. *For $\beta \in \mathbb{Z}_+^k$ and $S \subseteq V_\beta$, consider the tuple of ribbons $\boldsymbol{\nu}(S)$. We have the following formula for the operator ∇ applied to the LLT polynomial $\mathcal{G}_{\boldsymbol{\nu}(S)}(X;q)$:*

$$\nabla \mathcal{G}_{\boldsymbol{\nu}(S)}(X;q) = \omega\, \mathrm{pol}_X\, \boldsymbol{\sigma}\left((-qt)^{|V_\beta \setminus S|} q^{A_\beta} \frac{z_1 \cdots z_l \prod_{(\beta[i],\beta[j]) \in V_\beta \setminus S} z_i/z_j \prod_{\alpha_{ij} \in \widehat{R}_\beta} (1 - qt\, z_i/z_j)}{\prod_{\alpha_{ij} \in R_+} (1 - q\, z_i/z_j) \prod_{\alpha_{ij} \in R_\beta} (1 - t\, z_i/z_j)} \right), \tag{29}$$

where $l = |\beta|$ and $R_+ = R_+(\mathrm{GL}_l)$.

Proof. Our raising operator formula [7, Corollary 9.4.1] for ∇ on any LLT polynomial reduces to (29) using the following facts which hold when the LLT polynomial is indexed by a tuple of ribbons.

(1) The magic number $p(\boldsymbol{\nu}(S))$ of $\boldsymbol{\nu}(S)$ defined in [7, §8.2] is equal to the number of boxes of $\boldsymbol{\nu}(S)$ which are not the first box in a row, which is the same as $|V_\beta \setminus S|$.

(2) The weight λ in [7, Corollary 9.4.1], defined in [7, Definition 8.1.2], is obtained as follows in the case that the tuple of skew shapes consists of ribbons: fill the boxes of each row of $\boldsymbol{\nu}(S)$ with $1, 0, \ldots, 0, -1$ or just 0 for a row of length 1, and then read this filling by increasing reading order. It is then easily seen that $\mathbf{z}^\lambda = \prod_{(\beta[i],\beta[j]) \in V_\beta \setminus S} z_i/z_j$.

(3) Under the bijection $f : \beta \to \boldsymbol{\nu}(S)$ which takes the i-th box $\boldsymbol{\beta}[i]$ of $\boldsymbol{\beta}$ in reading order to the i-th box of $\boldsymbol{\nu}(S)$ in reading order, the set $\{(f(\boldsymbol{\beta}[i]), f(\boldsymbol{\beta}[j])) : \alpha_{ij} \in R_\beta\}$ is exactly the set of non-attacking pairs in $\boldsymbol{\nu}(S)$. Thus, R_β agrees with the set of roots denoted R_t in [7, Definition 8.1.2 and Remark 8.1.3 (i)]. □

4.4. Proof of Theorem 3.1.4

Applying ∇ to (28) and substituting (29) into this yields

$$\nabla \tilde{H}_\mu(X;q,t) = \omega\, \mathrm{pol}_X\, \boldsymbol{\sigma}\left(\frac{z_1 \cdots z_l \cdot \Upsilon \cdot \prod_{\alpha_{ij} \in \widehat{R}_\beta} (1 - qt\, z_i/z_j)}{\prod_{\alpha_{ij} \in R_+} (1 - q\, z_i/z_j) \prod_{\alpha_{ij} \in R_\beta} (1 - t\, z_i/z_j)} \right), \tag{30}$$

where

$$\Upsilon = \sum_{S \subseteq V_\beta} (-qt)^{|V_\beta \setminus S|} q^{A_\beta} \prod_{(\beta[i],\beta[j]) \in S} q^{-\mathrm{arm}(\beta[i])}\, t^{\mathrm{leg}(\beta[i])+1} \prod_{(\beta[i],\beta[j]) \in V_\beta \setminus S} z_i/z_j. \tag{31}$$

Defining $d = A_\beta - \mathsf{n}(\mu^*) - \sum_{(\beta[i],\beta[j]) \in V_\beta} \mathrm{arm}(\boldsymbol{\beta}[i])$ and using $\mathsf{n}(\mu) = \sum_{(\beta[i],\beta[j]) \in V_\beta} (\mathrm{leg}(\boldsymbol{\beta}[i]) + 1)$, the quantity Υ can be simplified as follows:

$$\begin{aligned}
\Upsilon &= q^{\mathsf{n}(\mu^*)+d}\, t^{\mathsf{n}(\mu)} \sum_{S \subseteq V_\beta} \Bigg((-qt)^{|V_\beta \setminus S|} \prod_{(\beta[i],\beta[j]) \in V_\beta} q^{\mathrm{arm}(\beta[i])}\, t^{-\mathrm{leg}(\beta[i])-1} \\
&\qquad\qquad \times \prod_{(\beta[i],\beta[j]) \in S} q^{-\mathrm{arm}(\beta[i])}\, t^{\mathrm{leg}(\beta[i])+1} \prod_{(\beta[i],\beta[j]) \in V_\beta \setminus S} z_i/z_j \Bigg) \\
&= q^{\mathsf{n}(\mu^*)+d}\, t^{\mathsf{n}(\mu)} \sum_{S \subseteq V_\beta} \prod_{(\beta[i],\beta[j]) \in V_\beta \setminus S} \left(-q^{\mathrm{arm}(\beta[i])+1}\, t^{-\mathrm{leg}(\beta[i])} z_i/z_j \right) \\
&= q^{\mathsf{n}(\mu^*)+d}\, t^{\mathsf{n}(\mu)} \prod_{(\beta[i],\beta[j]) \in V_\beta} \left(1 - q^{\mathrm{arm}(\beta[i])+1}\, t^{-\mathrm{leg}(\beta[i])} z_i/z_j \right).
\end{aligned}$$

Thus, plugging this back in for Υ in (30) and recalling the definition of $\mathbf{H}_\beta(\mathbf{z};q,t)$ (Definition 3.1.3), we have

$$\nabla \tilde{H}_\mu(X;q,t) = q^{\mathsf{n}(\mu^*)+d}\, t^{\mathsf{n}(\mu)} \omega\, \mathrm{pol}_X \left(z_1 \cdots z_l\, \mathbf{H}_\beta(\mathbf{z};q,t) \right).$$

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By the definition of ∇, this implies $\tilde{H}_\mu(X;q,t) = q^d \omega\, \mathrm{pol}_X\big(z_1\cdots z_l\, \mathbf{H}_\beta(\mathbf{z};q,t)\big)$. It remains to show that $d=0$. This follows from the fact that the coefficient of $s_{1^l}(X) = \mathrm{pol}_X(z_1\cdots z_l)$ in the Schur expansion of both $\omega\tilde{H}_\mu(X;q,t)$ and $\mathrm{pol}_X\big(z_1\cdots z_l\, \mathbf{H}_\beta(\mathbf{z};q,t)\big)$ is 1; the former is well known, while the latter can be seen directly by expanding the series $z_1\cdots z_l\, \mathbf{H}_\beta(\mathbf{z};q,t)$ to see that it is equal to $\boldsymbol{\sigma}(z_1\cdots z_l) = \chi_{1^l}$ plus terms of the form $a_\nu\chi_\nu$ for $\nu > 1^l$ in dominance order.

5. The m,n-Macdonald polynomials

For every pair of coprime positive integers (m,n), the action of the Burban-Schiffmann elliptic Hall algebra $\mathcal{E}$ on $\Lambda(X)$ gives rise to a family of symmetric functions that we call m,n-Macdonald polynomials. The subfamily of $1,n$-Macdonald polynomials is closely connected with the Macdonald series $\mathbf{H}_\beta$ from Definition 3.1.3. In this section, we will construct raising operator formulas for all m,n-Macdonald polynomials, reducing to Theorem 3.1.4 in the case $m=n=1$.

To define m,n-Macdonald polynomials we need to recall some facts about the algebra $\mathcal{E}$, defined by Burban and Schiffmann [14] in terms of Hall algebras of coherent sheaves on elliptic curves. For each pair of coprime integers (m,n), the algebra $\mathcal{E}$ contains a distinguished subalgebra $\Lambda(X^{m,n})$ isomorphic to the algebra of symmetric functions over $\mathbf{k}$; these subalgebras generate $\mathcal{E}$, subject to relations given in [14]. There is also an action of $\mathcal{E}$ on the space of symmetric functions $\Lambda(X)$, constructed by Schiffmann and Vasserot [35]. Our notation here is the same as in [4, 7, 5, 6] – in particular, we use the version of the action of $\mathcal{E}$ on $\Lambda(X)$ given by [6, Proposition 3.3.1]. The translation between our notation and that of [14, 35] can be found in [6, §§3.2–3.3]; the defining relations of $\mathcal{E}$ written in our notation are given in [5, §3.2].

Definition 5.1.1. Set $M=(1-q)(1-t)\in\mathbf{k}$. For coprime positive integers m,n, define the m,n-*Macdonald polynomial* $\tilde{H}^{m,n}_\mu = \tilde{H}^{m,n}_\mu(X;q,t)$ by

$$\tilde{H}^{m,n}_\mu = \tilde{H}_\mu[-MX^{m,n}]\cdot 1, \tag{32}$$

the symmetric function obtained by acting on $1\in\Lambda(X)$ with (a plethystic transformation of) a modified Macdonald polynomial in the distinguished subalgebra $\Lambda(X^{m,n})\subseteq\mathcal{E}$.

Remark 5.1.2. (i) For context, we note that $e_k[-MX^{m,n}]\cdot 1$ defines the symmetric function side of the (km,kn)-shuffle theorem of [3, 30].

(ii) $\tilde{H}^{m,n}_\mu$ is a homogeneous symmetric function of degree $n|\mu|$, as follows from the definition of the action of the Schiffmann algebra on symmetric functions [6, Proposition 3.3.1].

5.2. The $1,n$-Macdonald polynomials

By [7, Proposition 4.5.1], $\tilde{H}^{m,1}_\mu = \tilde{H}_\mu[-MX^{m,1}]\cdot 1 = \nabla^m\tilde{H}_\mu = q^{m\,\mathrm{n}(\mu^*)}t^{m\,\mathrm{n}(\mu)}\tilde{H}_\mu$, so this case is familiar. The $1,n$-Macdonald polynomials, however, carry essentially the same data as the Macdonald series $\mathbf{H}_\beta$ by the following theorem, which will be proven in §5.3 as part of a more general result (see Remark 5.3.4 (ii)).

Theorem 5.2.1. *For any partition μ of l and any rearrangement β of μ^*,*

$$\tilde{H}^{1,n}_\mu(X;q,t) = \omega\,\mathrm{pol}_X\Big(q^{\mathrm{n}(\mu^*)}t^{\mathrm{n}(\mu)}\,(z_1\cdots z_l)^n\,\mathbf{H}_\beta(\mathbf{z};q,t)\Big). \tag{33}$$

Hence, also $\mathbf{H}_\beta(\mathbf{z};q,t) = q^{-\mathrm{n}(\mu^*)}t^{-\mathrm{n}(\mu)}\lim_{n\to\infty}(z_1\cdots z_l)^{-n}(\omega\tilde{H}^{1,n}_\mu)(z_1,\dots,z_l)$.

This also shows that $\mathbf{H}_\beta(\mathbf{z};q,t)$ depends only on the partition rearrangement μ^* of β.

The Macdonald polynomial $\tilde{H}_\mu(X;q,t)$ is known [24] to be *Schur positive* (i.e., the coefficients $\tilde{K}_{\lambda,\mu}(q,t)$ in its Schur expansion are in $\mathbb{N}[q,t]$). Based on extensive computations, we were led to the following positivity conjecture for the $1,n$-Macdonald polynomials, which generalizes the positivity theorem for Macdonald polynomials.

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Conjecture 5.2.2. *The* $1,n$*-Macdonald polynomials* $\tilde{H}^{1,n}_\mu(X;q,t)$ *are Schur positive.*

Equivalently (by Theorem 5.2.1) for any partition μ *of* l *and rearrangement* β *of* μ^**, the series* $\mathbf{H}_\beta(\mathbf{z};q,t)$ *is a positive sum of irreducible* GL_l *characters; that is, the coefficients in*

$$\mathbf{H}_\beta(\mathbf{z};q,t) = \sum_\nu \mathbf{K}_{\nu,\mu}(q,t)\,\chi_\nu \tag{34}$$

are polynomials $\mathbf{K}_{\nu,\mu}(q,t) \in \mathbb{N}[q,t]$ *with nonnegative integer coefficients.*

Remark 5.2.3. For $m \neq 1$ and $n \neq 1$, the m,n-Macdonald polynomials are typically not Schur positive.

Example 5.2.4. The Schur expansions of the $1,n$-Macdonald polynomials $\tilde{H}^{1,n}_\mu(X;q,t)$ for $n=2$ and $|\mu| = 2,3$, written as in (33), are

$$\begin{aligned}
\tilde{H}^{1,2}_{2} &= \omega\, q\big(q^2 s_4 + q\, s_{31} + s_{22}\big)\\
\tilde{H}^{1,2}_{11} &= \omega\, t\big(t^2 s_4 + t\, s_{31} + s_{22}\big)\\
\tilde{H}^{1,2}_{3} &= \omega\, q^3\big(q^6 s_6 + (q^5+q^4)s_{51} + (q^4+q^3+q^2)s_{42} + q^3 s_{33} + q^3 s_{411} + (q^2+q)s_{321} + s_{222}\big)\\
\tilde{H}^{1,2}_{21} &= \omega\, qt\big(q^2t^2 s_6 + (q^2t+qt^2)s_{51} + (q^2+qt+t^2)s_{42} + qt\, s_{33} + qt\, s_{411} + (q+t)s_{321} + s_{222}\big)\\
\tilde{H}^{1,2}_{111} &= \omega\, t^3\big(t^6 s_6 + (t^5+t^4)s_{51} + (t^4+t^3+t^2)s_{42} + t^3 s_{33} + t^3 s_{411} + (t^2+t)s_{321} + s_{222}\big).
\end{aligned}$$

Proposition 5.2.5. *The* m,n*-Macdonald polynomials satisfy the same* q,t *symmetry property as ordinary modified Macdonald polynomials:* $\tilde{H}^{m,n}_\mu(X;q,t) = \tilde{H}^{m,n}_{\mu^*}(X;t,q)$*.*

Proof. This follows from the symmetry property for Macdonald polynomials, $\tilde{H}_\mu(X;q,t) = \tilde{H}_{\mu^*}(X;t,q)$, and the fact that for any symmetric function f with coefficients in $\mathbb{Q}$, $f[-MX^{m,n}]\cdot 1$ is symmetric in q and t, which in turn follows from the description of the action of the Schiffmann algebra on symmetric functions in [6, Proposition 3.3.1]. □

We obtain several specializations of the $1,n$-Macdonald polynomials easily from the raising operator formula in Theorem 5.2.1.

Proposition 5.2.6. *Let* μ *be a partition of* l*. The* $q=t=1$*,* $q=1$ *and* $q=0$ *specializations of the* $1,n$*-Macdonald polynomials are given by*

$$\tilde{H}^{1,n}_\mu(X;1,1) = e_{(n^l)}(X), \tag{35}$$

$$\big(q^{-\mathsf{n}(\mu^*)}t^{-\mathsf{n}(\mu)}\tilde{H}^{1,n}_\mu(X;q,t)\big)\big|_{q=0} = \omega\, \mathrm{pol}_X\, \boldsymbol{\sigma}\Big(\frac{(z_1\cdots z_l)^n}{\prod_{\alpha_{ij}\in B_\mu}(1-t\,z_i/z_j)}\Big), \tag{36}$$

$$\tilde{H}^{1,n}_\mu(X;1,t) = t^{\mathsf{n}(\mu)}\prod_{r=1}^{\mu_1}\omega H_{(n^{\nu_r})}(X;t), \tag{37}$$

where $\nu = \mu^*$*,* $H_\mu(X;t) = t^{\mathsf{n}(\mu)}\tilde{H}_\mu(X;0,t^{-1})$ *is as in (6), and* B_μ *is as defined before (12).*

Remark 5.2.7. Like the familiar right-hand sides of (35) and (37), the right-hand side of (36) is also well studied. Its Schur expansion coefficients are instances of the generalized Kostka polynomials introduced by Shimozono-Weyman [36] corresponding to the sequence of rectangle shapes $(n^{\mu_\ell}),\dots,(n^{\mu_2}),(n^{\mu_1})$ for $\ell = \ell(\mu)$.

Proof. Throughout this proof, we only need the case $\beta = \mu^* = \nu$ of Theorem 5.2.1.

By the same argument as in §3.2, the $q=t=1$ specialization of $\mathrm{pol}_X((z_1\cdots z_l)^n\mathbf{H}_\nu)$ is the complete homogeneous symmetric function $h_{(n^l)}$. Hence, (35) follows from Theorem 5.2.1. Similarly, (36) follows from Theorem 5.2.1 and from noting that $\mathrm{pol}_X((z_1\cdots z_l)^n\mathbf{H}_\nu(\mathbf{z};0,t))$ can be simplified just as $\mathrm{pol}_X(z_1\cdots z_l\,\mathbf{H}_\nu(\mathbf{z};0,t))$ was in §3.2.

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We now prove (37). By Theorem 5.2.1 and (9),

$$q^{-\mathsf{n}(\mu^*)}t^{-\mathsf{n}(\mu)}\omega\tilde{H}^{1,n}_{\mu}(X;q,t) = \mathbf{h}_X\Big((z_1\cdots z_l)^n\prod_{\alpha_{ij}\in R_+}(1-z_i/z_j)\widehat{\mathbf{H}}_\nu\Big), \tag{38}$$

where $\widehat{\mathbf{H}}_\nu(\mathbf{z};q,t)$ is the expression inside the $\boldsymbol{\sigma}(\cdot)$ on the right side of (20), with β equal to ν (so that $\mathbf{H}_\nu = \boldsymbol{\sigma}(\widehat{\mathbf{H}}_\nu)$). Let $C_1,\dots,C_{\mu_1}$ denote the columns of $\boldsymbol{\nu}$ and note that $R_\nu\setminus\widehat{R}_\nu$ is equal to $\bigsqcup_{r=1}^{\mu_1}\{\alpha_{ij} : \boldsymbol{\nu}[j] = \mathrm{south}(\boldsymbol{\nu}[i]),\ \boldsymbol{\nu}[i]\in C_r\}$. Hence, setting $q=1$ in (38) yields

$$t^{-\mathsf{n}(\mu)}\omega\tilde{H}^{1,n}_{\mu}(X;1,t) = \mathbf{h}_X\left(\prod_{r=1}^{\mu_1}\frac{\big(\prod_{\boldsymbol{\nu}[i]\in C_r} z_i^n\big)\prod_{\alpha_{ij}\in R_\nu\setminus\widehat{R}_\nu,\,\boldsymbol{\nu}[i]\in C_r}\big(1-t^{-\mathrm{leg}(\boldsymbol{\nu}[i])}z_i/z_j\big)}{\prod_{\alpha_{ij}\in R_\nu\setminus\widehat{R}_\nu,\,\boldsymbol{\nu}[i]\in C_r}\big(1-tz_i/z_j\big)}\right) \tag{39}$$

$$= \prod_{r=1}^{\mu_1}\mathbf{h}_X\left(\frac{\big(\prod_{\boldsymbol{\nu}[i]\in C_r} z_i^n\big)\prod_{\alpha_{ij}\in R_\nu\setminus\widehat{R}_\nu,\,\boldsymbol{\nu}[i]\in C_r}\big(1-t^{-\mathrm{leg}(\boldsymbol{\nu}[i])}z_i/z_j\big)}{\prod_{\alpha_{ij}\in R_\nu\setminus\widehat{R}_\nu\,\boldsymbol{\nu}[i]\in C_r}\big(1-tz_i/z_j\big)}\right), \tag{40}$$

where the second equality follows from (10) and the fact that $h_\gamma = h_\delta$ for $\gamma,\delta\in\mathbb{Z}^l$ which are rearrangements of each other.

The factor $\mathbf{h}_X(\cdot)$ in (40) for a given index r is equal to $t^{-\mathsf{n}(1^{\nu_r})}\omega\tilde{H}^{1,n}_{(1^{\nu_r})}(X;1,t)$, by the computation we have just done, but with the partition 1^{ν_r} in place of μ. It follows that

$$\tilde{H}^{1,n}_{\mu}(X;1,t) = \prod_{r=1}^{\mu_1}\tilde{H}^{1,n}_{(1^{\nu_r})}(X;1,t). \tag{41}$$

Finally, we will show that each $\tilde{H}^{1,n}_{(1^{\nu_r})}(X;1,t)$ is essentially a Hall-Littlewood polynomial. Using the particularly simple form of the series $\mathbf{H}_\nu$ when $\mu=(d)$ is a single row, Theorem 5.2.1 and (11) yield $q^{-\binom{d}{2}}\omega\tilde{H}^{1,n}_{(d)}(X;q,t) = H_{(n^d)}(X;q)$. By Proposition 5.2.5, $\tilde{H}^{1,n}_{\mu}(X;q,t) = \tilde{H}^{1,n}_{\mu^*}(X;t,q)$, and thus,

$$t^{-\binom{d}{2}}\omega\tilde{H}^{1,n}_{(1^d)}(X;q,t) = H_{(n^d)}(X;t). \tag{42}$$

Formula (37) now follows from (41) and (42). □

5.3. *A raising operator formula for the* m,n*-Macdonald polynomials*

We now give a raising operator formula for $\tilde{H}^{m,n}_\mu$ which generalizes the raising operator formula for $\tilde{H}_\mu$ in Theorem 3.1.4, and is derived in a similar way. Specifically, we combine the Haglund-Haiman-Loehr formula (Theorem 4.2.2) with an m,n version of Theorem 4.3.1. This latter result requires some notation involving the dilation of a column diagram.

For $\beta\in\mathbb{Z}_+^k$, let $m\beta = (m\beta_1,\dots,m\beta_k)$. We think of the column diagram $\boldsymbol{m\beta}$ of $m\beta$ as the result of dilating the column diagram $\boldsymbol{\beta}$ of β vertically by a factor of m, so that each box of $\boldsymbol{\beta}$ gives rise to m boxes of $\boldsymbol{m\beta}$. To be more precise, define the map of boxes

$$\tau\colon \boldsymbol{m\beta}\to\boldsymbol{\beta},\quad (i,j)\mapsto (i,\lfloor (j-1)/m\rfloor + 1). \tag{43}$$

Thus, in the dilation process, each box b of $\boldsymbol{\beta}$ gives rise to a set $\tau^{-1}(b)$ of m boxes of $\boldsymbol{m\beta}$, called a *dilated box*, which forms a contiguous subset of a column of $\boldsymbol{m\beta}$.

As in §3.1, we write $\boldsymbol{\beta}[1],\dots,\boldsymbol{\beta}[d]$ for the boxes of $\boldsymbol{\beta}$ listed in increasing reading order and $\boldsymbol{m\beta}[1],\dots,\boldsymbol{m\beta}[l]$ for the boxes of $\boldsymbol{m\beta}$ in increasing reading order, where $d=|\beta|$ and $l=dm$. Define a map $\mathsf{s}\colon V_\beta\to V_{m\beta}$ which takes a vertical domino $(\boldsymbol{\beta}[i],\boldsymbol{\beta}[j])$ of $\boldsymbol{\beta}$ to the vertical domino

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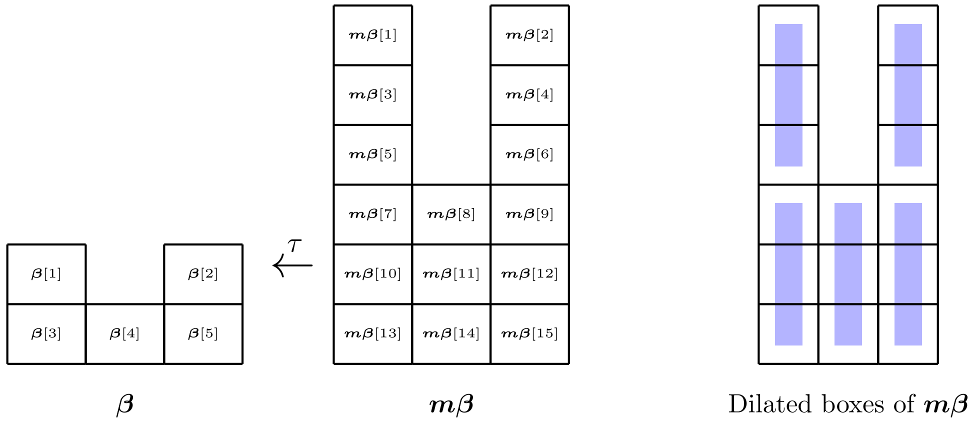
$(\boldsymbol{m\beta}[i'], \boldsymbol{m\beta}[j'])$ of $\boldsymbol{m\beta}$ such that $\tau(\boldsymbol{m\beta}[i']) = \boldsymbol{\beta}[i]$ and $\tau(\boldsymbol{m\beta}[j']) = \boldsymbol{\beta}[j]$. Note that $\mathsf{s}(V_\beta)$ is the set of vertical dominoes of the dilated diagram $\boldsymbol{m\beta}$ which straddle two dilated boxes.

Example 5.3.1. For $m = 3$ and $\beta = (2, 1, 2)$, the dilated boxes of $\boldsymbol{m\beta}$ are shown below along with the labeling of the boxes of $\boldsymbol{\beta}$ and $\boldsymbol{m\beta}$ by reading order. To clarify the definitions of τ and s, note that $\tau^{-1}(\boldsymbol{\beta}[3]) = \{\boldsymbol{m\beta}[7], \boldsymbol{m\beta}[10], \boldsymbol{m\beta}[13]\}$ and $\mathsf{s}(V_\beta) = \{(\boldsymbol{m\beta}[5], \boldsymbol{m\beta}[7]), (\boldsymbol{m\beta}[6], \boldsymbol{m\beta}[9])\}$.



Given $(m, n) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, we define the sequence of m integers as in [7, (9.5)]

$$\mathbf{b}(m,n)_i = \lceil in/m \rceil - \lceil (i-1)n/m \rceil \qquad (i = 1, \dots, m). \tag{44}$$

We then define, for any $\beta \in \mathbb{Z}_+^k$ of size $d = |\beta|$, a weight $\mathbf{b}(m, n, \beta) \in \mathbb{Z}^{dm}$ as follows: fill each dilated box of $\boldsymbol{m\beta}$ with the sequence $\mathbf{b}(m, n)$ from north to south, and then read this filling by the reading order on $\boldsymbol{m\beta}$. Equivalently, $\mathbf{b}(m, n, \beta)_r = \mathbf{b}(m, n)_a$, where a is the integer $a \in \{1, \dots, m\}$ such that $a \equiv -j+1 \pmod m$ for j the row index of the r-th box of $\boldsymbol{m\beta}$ in reading order (i.e., $\boldsymbol{m\beta}[r] = (i, j)$ for some i).

Theorem 5.3.2 [7]. *Let m, n be coprime positive integers, let $\beta \in \mathbb{Z}_+^k$, and set $d = |\beta|$, $l = dm$. For $S \subseteq V_\beta$, let $\boldsymbol{\nu}(S)$ be the k-tuple of ribbons in Definition 4.2.1. The action of the LLT polynomial $\mathcal{G}_{\boldsymbol{\nu}(S)}[-MX^{m,n}] \in \Lambda(X^{m,n})$ on 1 is given by*

$$\mathcal{G}_{\boldsymbol{\nu}(S)}[-MX^{m,n}] \cdot 1 = \omega \operatorname{pol}_X \boldsymbol{\sigma}\Bigg((-qt)^{|V_\beta \setminus S|} q^{A_\beta + (m-1)\mathrm{n}(\beta^+)} \mathbf{z}^{\mathbf{b}(m,n,\beta)} \\ \times \frac{\prod_{(\boldsymbol{m\beta}[i], \boldsymbol{m\beta}[j]) \in \mathsf{s}(V_\beta \setminus S)} z_i/z_j \prod_{\alpha_{ij} \in \widehat{R}_{m\beta}} (1 - q\,t\,z_i/z_j)}{\prod_{\alpha_{ij} \in R_+(\mathrm{GL}_l)} (1 - q\,z_i/z_j) \prod_{\alpha_{ij} \in R_{m\beta}} (1 - t\,z_i/z_j)} \Bigg), \tag{45}$$

where A_β is the number of attacking pairs of $\boldsymbol{\nu}(S)$ as in §4.3, β^+ is the partition rearrangement of β, and $R_{m\beta}$, $\widehat{R}_{m\beta} \subseteq R_+(\mathrm{GL}_l)$ are as in (15, 16).

Proof. This is obtained by combining [7, Theorem 9.3.1] and [4, Proposition 2.3.2] and specializing to the case that the LLT polynomial is indexed by the tuple of ribbons $\boldsymbol{\nu}(S)$. The notation here and that in [7, Theorem 9.3.1] are matched using the discussion in the proof of Theorem 4.3.1 and the following: the weight $\mathbf{b}(m, n, \beta)$ is the same as $\tilde{\mathbf{b}}$ defined in [7, Definition 9.2.1], and the weight λ defined there satisfies $\mathbf{z}^\lambda = \mathbf{z}^{\mathbf{b}(m,n,\beta)} \prod_{(\boldsymbol{m\beta}[i], \boldsymbol{m\beta}[j]) \in \mathsf{s}(V_\beta \setminus S)} z_i/z_j$. □

We now give our raising operator formula for the m, n-Macdonald polynomials $\tilde{H}^{m,n}_\mu(X; q, t)$.

Theorem 5.3.3. *Let m, n be coprime positive integers. Given $\beta \in \mathbb{Z}_+^k$ and setting $d = |\beta|$ and $l = dm$, define the m, n-Macdonald series by*

$$\mathbf{H}_\beta^{m,n}(\mathbf{z}; q, t) = \boldsymbol{\sigma}\left(\frac{\mathbf{z}^{\mathbf{b}(m,n,\beta)} \prod_{(\boldsymbol{m\beta}[i], \boldsymbol{m\beta}[j]) \in \mathsf{s}(V_\beta)} \left(1 - q^{\operatorname{arm}(\tau(\boldsymbol{m\beta}[i]))+1} t^{-\operatorname{leg}(\tau(\boldsymbol{m\beta}[i]))} \frac{z_i}{z_j}\right) \prod_{\alpha_{ij} \in \widehat{R}_{m\beta}} \left(1 - qt\, \frac{z_i}{z_j}\right)}{\prod_{\alpha_{ij} \in R_+(\mathrm{GL}_l)} \left(1 - q\, \frac{z_i}{z_j}\right) \prod_{\alpha_{ij} \in R_{m\beta}} \left(1 - t\, \frac{z_i}{z_j}\right)} \right), \tag{46}$$

which we regard as an infinite formal linear combination of irreducible GL_l characters using the convention of Remark 2.2.1. Then, for any partition μ and any rearrangement β of μ^*,

$$\tilde{H}_\mu^{m,n}(X; q, t) = \omega \operatorname{pol}_X \left(q^{m\,\mathsf{n}(\mu^*)} t^{\mathsf{n}(\mu)} \mathbf{H}_\beta^{m,n}(\mathbf{z}; q, t) \right). \tag{47}$$

Remark 5.3.4. (i) In (46), the indices of z_i correspond to the boxes of the dilated diagram $\boldsymbol{m\beta}$, while the arm and leg are taken with respect to the original diagram $\boldsymbol{\beta}$.

(ii) Since $\mathbf{b}(1, n, \beta) = n^l$, $\mathbf{H}_\beta^{1,n} = (z_1 \cdots z_l)^n \mathbf{H}_\beta$, where $\mathbf{H}_\beta$ is the Macdonald series from Definition 3.1.3. Hence, Theorem 5.3.3 proves Theorem 5.2.1 by setting $m = 1$ with n arbitrary. It also specializes to Theorem 3.1.4 when $(m, n) = (1, 1)$.

(iii) Expanding on (ii), the series $\mathbf{H}_\beta^{m,n}$ simultaneously encodes the m, n-Macdonald polynomials $\{H_\mu^{m,n'} : n' \in (n + m\mathbb{Z}) \cap \mathbb{Z}_+\}$, in the following sense:

$$\tilde{H}_\mu^{m,n+am} = \omega \operatorname{pol}_X \left(q^{m\,\mathsf{n}(\mu^*)} t^{\mathsf{n}(\mu)} (z_1 \cdots z_l)^a \mathbf{H}_\beta^{m,n} \right). \tag{48}$$

This follows from Theorem 5.3.3 using that $\mathbf{b}(m, n + am, \beta) = \mathbf{b}(m, n, \beta) + a^l$, which in turn holds by $\mathbf{b}(m, n + am) = \mathbf{b}(m, n) + a^m$.

Theorem 5.3.3 and Remark 5.3.4 (iii) have the following corollary.

Corollary 5.3.5. *The m, n-Macdonald series $\mathbf{H}_\beta^{m,n}$ depends only on the multiset of parts of β and not on their order: $\mathbf{H}_\beta^{m,n} = \mathbf{H}_\gamma^{m,n}$ for any rearrangement γ of β.*

Proof of Theorem 5.3.3. By a plethystic transformation and change of variables we can replace X in (28) with $-MX^{m,n}$. Letting both sides act on $1 \in \Lambda(X)$ and then substituting in (45) yields

$$\tilde{H}_\mu[-MX^{m,n}] \cdot 1 = \omega \operatorname{pol}_X \boldsymbol{\sigma}\left(\frac{\mathbf{z}^{\mathbf{b}(m,n,\beta)} \cdot \Upsilon \cdot \prod_{\alpha_{ij} \in \widehat{R}_{m\beta}} (1 - qt\, z_i/z_j)}{\prod_{\alpha_{ij} \in R_+} (1 - q\, z_i/z_j) \prod_{\alpha_{ij} \in R_{m\beta}} (1 - t\, z_i/z_j)} \right), \tag{49}$$

where

$$\Upsilon = \sum_{S \subseteq V_\beta} (-qt)^{|V_\beta \setminus S|} q^{A_\beta + (m-1)\mathsf{n}(\mu^*)} \prod_{(\beta[i], \beta[j]) \in S} q^{-\operatorname{arm}(\beta[i])} t^{\operatorname{leg}(\beta[i])+1} \prod_{(\boldsymbol{m\beta}[i], \boldsymbol{m\beta}[j]) \in \mathsf{s}(V_\beta \setminus S)} z_i/z_j. \tag{50}$$

The proof of Theorem 3.1.4 establishes that $A_\beta = \mathsf{n}(\mu^*) + \sum_{(\beta[i], \beta[j]) \in V_\beta} \operatorname{arm}(\boldsymbol{\beta}[i])$, as can also be checked combinatorially. Using this, Υ simplifies essentially the same way it did in that proof:

$$\Upsilon = q^{m\,\mathsf{n}(\mu^*)} t^{\mathsf{n}(\mu)} \prod_{(\boldsymbol{m\beta}[i], \boldsymbol{m\beta}[j]) \in \mathsf{s}(V_\beta)} \left(1 - q^{\operatorname{arm}(\tau(\boldsymbol{m\beta}[i]))+1} t^{-\operatorname{leg}(\tau(\boldsymbol{m\beta}[i]))} z_i/z_j\right). \tag{51}$$

Plugging this back into (49) completes the proof. □

We obtain yet other expressions for the modified Macdonald polynomials from Theorem 5.3.3.

Corollary 5.3.6. *Let μ be a partition of d and β a rearrangement of μ^*. The modified Macdonald polynomial $\tilde{H}_\mu(X;q,t)$ can be expressed in terms of the $m,1$-Macdonald series for any m as follows:*

$$\tilde{H}_\mu(X;q,t) = \omega \operatorname{pol}_X \big(t^{(-m+1)\mathsf{n}(\mu)}\, \mathbf{H}^{m,1}_\beta(\mathbf{z};q,t)\big). \tag{52}$$

Proof. By [7, Proposition 4.5.1], $\tilde{H}_\mu[-MX^{m,1}]\cdot 1 = \nabla^m \tilde{H}_\mu$. Substitute this into the left side of (47). □

6. A raising operator formula for the Macdonald polynomials $Q_\mu(X;q,t)$ and $J_\mu(X;q,t)$

Our formulas for the modified Macdonald polynomials $\tilde{H}_\mu(X;q,t)$ can be converted into formulas for the integral form Macdonald polynomials $J_\mu(X;q,t)$ and for the $Q_\mu(X;q,t)$ which differ from the J_μ by a scalar factor. Recall that $\tilde{H}_\mu(X;q,t) = t^{\mathsf{n}(\mu)} J_\mu[\frac{X}{1-t^{-1}};q,t^{-1}]$ as in (3), hence $\tilde{H}_\mu[X(1-t^{-1});q,t] = t^{\mathsf{n}(\mu)} J_\mu(X;q,t^{-1})$, or equivalently,

$$J_\mu(X;q,t) = t^{\mathsf{n}(\mu)} \tilde{H}_\mu[X(1-t);q,t^{-1}]. \tag{53}$$

Let $\mathbf{e}'_X : \mathbf{k}[z_1^{\pm 1},\dots,z_l^{\pm 1}] \to \Lambda$ denote the linear operator determined by

$$\mathbf{e}'_X(\mathbf{z}^\gamma) = e_\gamma[X(1-t)] = \prod_{i=1}^{l} \sum_{j=0}^{\gamma_i} (-t)^j e_{\gamma_i - j}(X) h_j(X), \tag{54}$$

where for $\gamma \in \mathbb{Z}^l$, we define $e_\gamma = e_{\gamma_1}\cdots e_{\gamma_l}$ to be the product of elementary symmetric functions, with e_d for $d \le 0$ interpreted as $e_0 = 1$, or $e_d = 0$ for $d < 0$. We extend $\mathbf{e}'_X$ to an operator on formal linear combinations of monomials just as we did for $\boldsymbol{\sigma}$ and $\mathbf{h}_X$ in §2.2. The operators $\mathbf{e}'_X$ and $\mathbf{h}_X$ are related as follows: for a formal linear combination $g = \sum_{\gamma\in\mathbb{Z}^l} c_\gamma \mathbf{z}^\gamma$ such that $\mathbf{h}_X(g)$ is a symmetric function and not just an infinite formal sum of symmetric functions, $\mathbf{e}'_X(g) = \theta\circ\omega\circ\mathbf{h}_X(g)$, where $\theta\colon \Lambda\to\Lambda$ is the automorphism sending $f(X)$ to $f[X(1-t)]$.

Theorem 6.1.1. *For any partition μ of l and any rearrangement β of μ^*, the integral form Macdonald polynomial $J_\mu(X;q,t)$ is given by*

$$J_\mu(X;q,t) = t^{\mathsf{n}(\mu)} \mathbf{e}'_X\Big(z_1\cdots z_l \prod_{\alpha_{ij}\in R_+} (1 - z_i/z_j)\, \widehat{\mathbf{H}}_\beta(\mathbf{z};q,t^{-1})\Big), \tag{55}$$

where $\widehat{\mathbf{H}}_\beta(\mathbf{z};q,t)$ is the expression inside the $\boldsymbol{\sigma}(\cdot)$ on the right side of (20) (so that $\mathbf{H}_\beta = \boldsymbol{\sigma}(\widehat{\mathbf{H}}_\beta)$). Alternatively, using informal notation similar to Remark 2.3.1,

$$\begin{aligned} J_\mu(X;q,t) = t^{\mathsf{n}(\mu)} &\left(\frac{\prod_{\alpha_{ij}\in R_+}(1-\mathsf{R}_{ij}) \prod_{\alpha_{ij}\in \widehat{R}_\beta} \big(1 - q t^{-1}\,\mathsf{R}_{ij}\big)}{\prod_{\alpha_{ij}\in R_+} \big(1 - q\,\mathsf{R}_{ij}\big) \prod_{\alpha_{ij}\in R_\beta} \big(1 - t^{-1}\,\mathsf{R}_{ij}\big)} \right) \\ &\times \prod_{\alpha_{ij}\in R_\beta\setminus \widehat{R}_\beta} \big(1 - q^{\operatorname{arm}(\beta[i])+1} t^{\operatorname{leg}(\beta[i])} \mathsf{R}_{ij}\big)\cdot e_{1^l}[X(1-t)], \end{aligned} \tag{56}$$

where R_{ij} acts on subscripts of $e_\gamma[X(1-t)]$ by $\mathsf{R}_{ij}\gamma = \gamma + \epsilon_i - \epsilon_j$.

Proof. By Theorem 3.1.4 and (9),

$$\tilde{H}_\mu(X;q,t) = \omega \mathbf{h}_X\big(z_1\cdots z_l \prod_{\alpha_{ij}\in R_+} (1 - z_i/z_j)\, \widehat{\mathbf{H}}_\beta\big). \tag{57}$$

The result then follows from (53) and the definition of $\mathbf{e}'_X$. □

We also record the consequence of Corollary 5.3.6 for the integral form Macdonald polynomials.

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Corollary 6.1.2. *Let μ be a partition of d and β a rearrangement of μ^*. Then for any $m \in \mathbb{Z}_+$,*

$$J_\mu(X;q,t) = t^{m\,\mathsf{n}(\mu)}\mathbf{e}'_X\Big(\prod_{\alpha_{ij}\in R_+(\mathrm{GL}_l)}(1-z_i/z_j)\,\widehat{\mathbf{H}}^{m,1}_\beta(\mathbf{z};q,t^{-1})\Big), \tag{58}$$

where $l = dm$ and $\widehat{\mathbf{H}}^{m,1}_\beta(\mathbf{z};q,t)$ is the expression inside $\boldsymbol{\sigma}(\cdot)$ on the right side of (46), with n set to 1.

Competing interest. The authors have no competing interest to declare.

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