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Polarization purity and cross-channel intensity correlations

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We consider the question of monitoring polarization purity, that is, measuring deviations from orthogonality δ_τ and δ_ϵ of an ostensibly orthogonal polarization basis with a reference channel of ellipticity ϵ and tilt τ. A simple result was recently derived for a phase-sensitive receiver observing unpolarized radiation [IEEE Trans. Geosci. Remote Sens. 62, 2003610 (2024)]: with $\rho^{(1)}$ denoting the Pearson complex correlation coefficient between channel *complex fields*, it states that $\mp\cos(2\epsilon)\delta_\tau \pm i\delta_\epsilon \approx \rho^{(1)}$ when $\delta_{\tau,\epsilon} \ll 1$. However, phase-sensitive (in-phase and quadrature) data are seldom available at optical frequencies. To that end, here we generalize the result by deriving a new equation for the polarization "alignment" error: $\cos^2(2\epsilon)\delta_\tau^2 + {\delta_\epsilon}^2 \approx \rho^{(2)}$, where $\rho^{(2)}$ is the intensity cross-correlation coefficient. Only the measurement of the *(real) intensity* cross-correlation coefficient is needed when observing unpolarized light. For the special case of a linearly polarized basis, the tilt error is simply $\delta_\tau \approx \sqrt{\rho^{(2)}}$, and for the circular basis case, with ellipticity deviation δ_ϵ from circular helicity $\pi/4$ (the reference channel of opposite helicity), $\delta_\epsilon \approx \sqrt{\rho^{(2)}}$. These results provide simple means to gauge the quality of polarimeters and depolarizers.

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1. INTRODUCTION

In a recent article [1], the following question was posed in the context of dual-polarized antennae: how close to orthogonal are the two polarization basis antennae (channels) of a radar detector? To that end, we showed that in unpolarized Gaussian radiation, the electric fields detected by the two antennas are statistically independent if the antennas are truly orthogonal, but any deviation from orthogonality induces cross-channel correlations. Such correlations can be used to quantify departures from the orthogonality of the channels. Here, we generalize the result to intensity measurements only. This generalization supplies a simple means to evaluate the purity of a polarimetric optical device by exposing it to thermal radiation and measuring the cross-channel correlations. In particular, one can test channel orthogonality of snapshot division of aperture Stokes polarimeters such as in [2,3] as well as test whether incident radiation is unpolarized [4] by measuring correlations between carefully calibrated orthogonal channels. In the latter case, correlations might be due to non-Gaussian statistics rather than misalignment. One can also gauge the performance of polarization scramblers and rough surface depolarizers [5–7] by evaluating the output in terms of uniformity on the Poincaré sphere (see Fig. 1) [8,9]. Similarly, one can also test polarization beam splitters by monitoring the orthogonality of the outputs.

2. BACKGROUND

For the reader's convenience, we briefly summarize the main result of [1]. Let u_1, u_2 be the normalized Jones vectors of two channels: e.g., $u_1 = (|E_{1,x}|e^{i\phi_{1,x}}, |E_{1,y}|e^{i\phi_{1,y}})^T$ describes a polarization state determined by the relative electric field amplitudes $|E_{1,x}|$, $|E_{1,y}|$ and phases $\phi_{1,x}$, $\phi_{1,y}$ in channel 1, and similarly for channel 2, $u_2 = (|E_{2,x}|e^{i\phi_{2,x}}, |E_{2,y}|e^{i\phi_{2,y}})^T$. Then, let $E(t) = [E_x(t), E_y(t)]^T$ be the incident electric field at the time t, with joint complex circular Gaussian statistics appropriate for unpolarized natural radiation [10]. With † denoting the complex conjugate transpose, the fields detected in each channel are given by the projections $E_i(t) = u_i^\dagger E(t)$, and the complex cross-correlation coefficient between the zero mean electric fields $E_1(t)$, $E_2(t)$ received by the two channels is shown to be related to the inner product of channel Jones vectors as

$$\rho^{(1)} = \frac{\langle E_1(t)E_2^*(t)\rangle_t}{\sqrt{\langle |E_1|^2\rangle_t \langle |E_2|^2\rangle_t}} = u_1^\dagger u_2, \tag{1}$$

where angular brackets denote the time average and a superscript (1) indicates first-order coherence (fields) as typically used in quantum optics [11,12]. The stationary ergodic random process is assumed so that $\rho^{(1)}$ is not a function of time, and ensemble interpretation can be used [10,13].

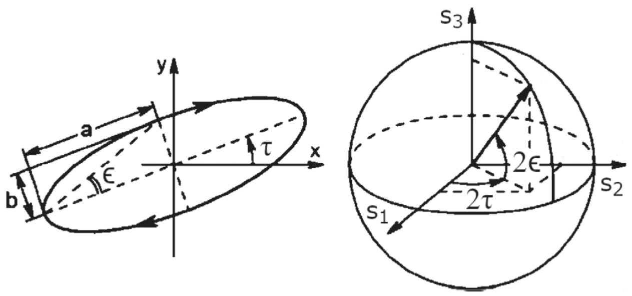

Fig. 1. (Left) polarization ellipse, with the ellipticity angle ϵ and the tilt angle τ, and (right) the mapping to the Poincaré sphere. The tilt ranges from $-\pi/2$ to $\pi/2$ and the ellipticity from $-\pi/4$ to $\pi/4$. Note that the sign of the ellipticity tracks the handedness of the polarization ellipse. The axes in the right panel are normalized Stokes parameters $s_1 = \cos(2\epsilon)\cos(2\tau)$, $s_2 = \cos(2\epsilon)\sin(2\tau)$, and $s_3 = \sin(2\epsilon)$.

The complex inner product on the right-hand side of Eq. (1) can be measured and, for unpolarized incident radiation, reflects the degree of orthogonality between the channels [1], e.g., zero for orthogonal channels and unity for parallel channels. A description of how channel fields and their correlation are measured using antennas in the microwave regime is included in [4]. The result (1) was anticipated in [1,14] by noticing that the correlation coefficient has all the properties of an inner product in a space of random functions [15]. For $|\rho^{(1)}| \ll 1$, Eq. (1) yields [1,16]

$$\mp \cos(2\epsilon)\delta_\tau \pm i\delta_\epsilon \approx u_1^\dagger u_2 = \rho^{(1)}, \quad \textbf{(2)}$$

where δ_τ, δ_ϵ are the tilt and ellipticity (see Fig. 1) angle deviations from orthogonality, and ϵ is the ellipticity of the reference channel [1]. Note that for $\epsilon = \pm\pi/4$ (right or left circularly polarized states), the error comes from the ellipticity only. If $\epsilon = 0$ (linear polarization), both the tilt and ellipticity errors come into play, although they do so separately in the real and imaginary parts of $\rho^{(1)}$, respectively. The lower/upper sign is for the case where the tilt of the reference channel τ is in the first/fourth quadrant of the axis system in Fig. 1.

In summary, the deviation from a perfectly orthogonal polarization basis is related to an experimentally measurable correlation coefficient. However, in contrast to the microwave domain, in optics heterodyne polarimetry, data are seldom available and only intensities are typically measured (interferometry is not considered because of the emphasis on practical polarimetry). Therefore, the purpose of this note is to generalize Eq. (1) to intensity measurements only.

3. INTENSITY CROSS-CORRELATION AND POLARIZATION PURITY

As just remarked, cross-channel correlations of the electric field that enter Eq. (2) require phase measurements, but such heterodyne measurements of the electric field are seldom available at optical frequencies. Therefore, we look to quantify polarization purity by measuring intensities only, e.g., using a photomultiplier tube.

As above, let the reference channel have ellipticity ϵ and tilt τ and denote ellipticity and tilt deviations of the other channel from orthogonality by δ_ϵ, δ_τ, respectively. Also, let $\rho^{(2)}$ be the intensity [second-order and hence, superscript (2)], cross-channel correlation coefficient. Our main finding is then that for incident unpolarized radiation:

$$\cos^2(2\epsilon)\delta_\tau^2 + \delta_\epsilon{}^2 \approx \rho^{(2)}. \quad \textbf{(3)}$$

The essential ingredient for deriving Eq. (3) is the relation

$$\rho^{(2)} = |\rho^{(1)}|^2 = |u_1^\dagger u_2|^2, \quad \textbf{(4)}$$

derived in the following section. Note that Eq. (4) holds for any polarized channels, with Jones vectors u_1, u_2 not necessarily nearly orthogonal. Symbolically, Eq. (4) may have a superficial resemblance to results in coherence and quantum optics, but here it is derived from the Jones formalism of polarimetry and, to the best of our knowledge, is new. Note that incoherence between any two orthogonal basis states plays an essential role and holds for thermal unpolarized radiation.

To gain intuition, let us consider a couple of special cases. For a linear polarization basis, $\epsilon = 0$ and $\delta_\epsilon = 0$, and one obtains from Eq. (4) that the tilt error δ_τ simply equals the square root of the intensity cross-correlation coefficient $\rho^{(2)}$:

$$\delta_\tau \approx \sqrt{\rho^{(2)}}, \quad \textbf{(5)}$$

whereas for a circular basis, $\epsilon = \pi/4$, and the result reduces to a simple square root form again:

$$\delta_\epsilon \approx \sqrt{\rho^{(2)}}. \quad \textbf{(6)}$$

These results give information about *relative* errors only, i.e., departures from orthogonality between the two channels rather than their absolute polarization states.

4. DERIVATION OF EQ. (4) FOR UNPOLARIZED INCIDENT RADIATION

To derive Eq. (4), consider thermal unpolarized radiation and the definition of the intensity cross-correlation coefficient:

$$\rho^{(2)} = \left\langle \left(\frac{I_1(t) - \langle I_1(t)\rangle_t}{\sigma_{I_1}} \right) \left(\frac{I_2(t) - \langle I_2(t)\rangle_t}{\sigma_{I_2}} \right) \right\rangle_t. \quad \textbf{(7)}$$

Here, $I_1(t)$, $I_2(t)$ are the intensities in channels 1 and 2 at the time t and are the stationary and ergodic random processes, with standard deviations σ_{I_1}, σ_{I_2}, $\sigma_{I_i}^2 = \langle I_i(t)^2\rangle_t - \langle I_i(t)\rangle_t^2$, $i = 1, 2$. Angular brackets with a subscript t denote the time. An alternate form of Eq. (7) is

$$\rho^{(2)} = \frac{\langle I_1(t) I_2(t)\rangle_t - \langle I_1(t)\rangle_t \langle I_2(t)\rangle_t}{\sigma_{I_1}\sigma_{I_2}}. \quad \textbf{(8)}$$

To link with the incident field, we return to $I_i(t) = |E_i(t)|^2$, where $E_i(t)$ is the complex electric field amplitude in the ith channel at the time t. These detected (channel) fields are projections of the incident field on the basis of Jones vectors: $E_i(t) = u_i^\dagger E(t)$, where $E(t) = [E_x(t), E_y(t)]^T$ is the incident electric field Jones vector at the time t. In what follows, it will be useful to write the inner product as $\sum_\alpha u_{i,\alpha}^* E_\alpha(t) = u_{i,\alpha}^* E_\alpha(t)$, where we sum over repeated indices, and $\alpha \in \{x, y\}$ (Greek indices indicate Cartesian components of the incident field

(x, y), while Latin is reserved for the channel, (1,2).). We next consider the intensity correlator:

$$\langle I_1(t) I_2(t)\rangle_t = \left\langle E_1(t)^* E_1(t) E_2(t)^* E_2(t)\right\rangle_t$$
$$= \left\langle u_{1,\alpha} E_\alpha^*(t) u_{1,\beta}^* E_\beta(t) u_{2,\gamma} E_\gamma^*(t) u_{2,\delta}^* E_\delta(t)\right\rangle_t$$
$$= u_{1,\alpha} u_{1,\beta}^* u_{2,\gamma} u_{2,\delta}^* \left\langle E_\alpha^*(t) E_\beta(t) E_\gamma^*(t) E_\delta(t)\right\rangle_t. \quad (9)$$

We evaluate the four-field time average with the help of the Isserlis–Wick theorem [17]. This is possible since $E_x(t)$, $E_y(t)$ are multivariate circular Gaussian-distributed incident fields corresponding to unpolarized thermal (chaotic) radiation [10]. That is, their real and imaginary parts are joint real Gaussian zero-mean random processes. Defining $v(t) = [E_x'(t), E_x''(t), E_y'(t), E_y''(t)]^T$, where single primes denote real parts and double primes denote imaginary parts, and the probability distribution of $v(t)$ is of the form [10]

$$p(v(t)) = \frac{1}{(2\pi)^2 \det(C)^{1/2}} \exp\left(-\frac{1}{2} v(t)^T C^{-1} v(t)\right), \quad (10)$$

where C is the covariance matrix $C = E[v(t) v(t)^T]$, with E here denoting the expected value. Polarization states for such a random process are uniformly distributed on the surface of the Poincaré sphere (panel b of Fig. 1) as discussed in e.g., [1,9]. The four complex fields (at the time t) on the right side of Eq. (9) are linear functions of these joint real Gaussian fields (e.g., $E_x = E_x' + i E_x''$). Therefore, letting f denote a linear function, the expectation value is of the form $\langle f_1 f_2 f_3 f_4\rangle_e$.

We now make use of the remarkable property of Gaussian variates that all higher-order correlations are expressible in terms of second-order correlations between pairs of variables. This Gaussian moment theorem is also discussed in [17] as Theorem 2.9, (Wick's formula) $\langle f_1 f_2 f_3 f_4\rangle_e = \langle f_1 f_2\rangle_e \langle f_3 f_4\rangle_e + \langle f_1 f_3\rangle_e \langle f_2 f_4\rangle_e + \langle f_1 f_4\rangle_e \langle f_2 f_3\rangle_e$. Applying this theorem to our case and using ergodicity yields

$$\langle E_\alpha^*(t) E_\beta(t) E_\gamma^*(t) E_\delta(t)\rangle_t = \langle E_\alpha^*(t) E_\beta(t) E_\gamma^*(t) E_\delta(t)\rangle_e$$
$$= \langle E_\alpha^*(t) E_\beta(t)\rangle_e \langle E_\gamma^*(t) E_\delta(t)\rangle_e$$
$$+ \langle E_\alpha^*(t) E_\delta(t)\rangle_e \langle E_\beta(t) E_\gamma^*(t)\rangle_e$$
$$+ \langle E_\alpha^*(t) E_\gamma^*(t)\rangle_e \langle E_\beta(t) E_\delta(t)\rangle_e, \quad (11)$$

where the subscript e indicates the ensemble average. Next, we make use of statistical independence of the zero-mean processes $E_x(t)$, $E_y(t)$ and circularity $\langle E_x(t)^2\rangle_e = \langle E_x^*(t)^2\rangle_e = 0$ to see that the third term vanishes identically for all values of the subscripts. Back-filling into Eq. (9), and distributing the Jones vectors, we obtain the following equation for the correlator:

$$\langle I_1(t) I_2(t)\rangle_t = \langle I_1(t)\rangle_e \langle I_2(t)\rangle_e + \left|\langle E_1(t)^* E_2(t)\rangle_e\right|^2. \quad (12)$$

Thus,

$$\langle I_1(t) I_2(t)\rangle_t - \langle I_1(t)\rangle_e \langle I_2(t)\rangle_e = \left|\langle E_1(t)^* E_2(t)\rangle_e\right|^2. \quad (13)$$

To evaluate the standard deviations, we recall that

$$\sigma_{I_1}^2 = \langle I_1(t)^2\rangle_t - \langle I_1(t)\rangle_e^2, \quad (14)$$

$$\sigma_{I_2}^2 = \langle I_2(t)^2\rangle_t - \langle I_2(t)\rangle_e^2. \quad (15)$$

These can be found simply by letting $1 \to 2$ or $2 \to 1$ in Eq. (13). The result is

$$\sigma_{I_1}^2 = \langle |E_1(t)|^2\rangle_e^2, \quad (16)$$

$$\sigma_{I_2}^2 = \langle |E_2(t)|^2\rangle_e^2. \quad (17)$$

We now arrive at

$$\rho^{(2)} = \frac{\langle I_1(t) I_2(t)\rangle_t - \langle I_1(t)\rangle_e \langle I_2(t)\rangle_e}{\sigma_{I_1}\sigma_{I_2}}$$
$$= \frac{\left|\langle E_1(t)^* E_2(t)\rangle_e\right|^2}{\langle |E_1(t)|^2\rangle_e \langle |E_2(t)|^2\rangle_e}. \quad (18)$$

But by Eq. (1), this is exactly $|\rho^{(1)}|^2$, thus proving that $|\rho^{(1)}|^2 = |u_1^\dagger u_2|^2 = \rho^{(2)}$ as claimed. This permits one to glean information about the departures of the polarization basis from orthogonality, without knowing the channel polarization states per se, solely from the measured cross-channel intensity correlation coefficient $\rho^{(2)}$ as given by the main result (3).

5. ACCOUNTING FOR TIME AVERAGING OF A PHOTODETECTOR

So far, the fundamental random variable $I_i(t)$ has tacitly been regarded as the "instantaneous" intensity in the channel i at a time instant t, proportional to the square of the electric field amplitude. But what are the time scales relevant to such intensity measurements? For example, photomultipliers have rise times on the order of 10^{-11} s [18], and the time scale on which intensity might change could be approximated with the coherence time of the radiation at hand, which for thermal light is of order $t_{\mathrm{coh}} \sim 10^{-8}$ s [13]. Thus, the instantaneous intensity is indeed measurable. In general, though, for a slower detector, one is measuring, to take the simplest case, a moving average [19]. In this section, we examine this practical case to ask whether our results continue to hold. To that end, we define a time-averaged intensity (over a period T) and consider cross-channel correlations. Let $I_1(t)$, $I_2(t)$ be the instantaneous intensities in the two channels at time t. Then, let us define the measured average intensities:

$$i_1(t_0) = \frac{1}{T}\int_{t_0-\frac{T}{2}}^{t_0+\frac{T}{2}} I_1(t)\mathrm{d}t, \quad (19)$$

$$i_2(t_0) = \frac{1}{T}\int_{t_0-\frac{T}{2}}^{t_0+\frac{T}{2}} I_2(t)\mathrm{d}t. \quad (20)$$

Now, $i_1(t_0)$ and $i_2(t_0)$ are the fundamental random processes of interest, and we wish to find their correlation coefficient. Putting the pieces together, we can find the correlator of the two measured intensities i_1 and i_2:

$$\langle i_1(t_0) i_2(t_0)\rangle_t = \frac{1}{T^2}\int_{t_0-\frac{T}{2}}^{t_0+\frac{T}{2}}\int_{t_0-\frac{T}{2}}^{t_0+\frac{T}{2}} u_{1,\alpha}^* u_{1,\beta} u_{2,\gamma}^* u_{2,\delta} \times \langle E_\alpha(t) E_\beta^*(t) E_\gamma(t') E_\delta^*(t')\rangle_e \mathrm{d}t\mathrm{d}t'. \quad (21)$$

We next turn to the four-field expectation value, paying special attention to the difference of times in the arguments (t and t'):

$$\langle E_\alpha(t) E_\beta^*(t) E_\gamma(t') E_\delta^*(t')\rangle_e = \langle E_\alpha(t) E_\beta^*(t)\rangle_e \langle E_\gamma(t') E_\delta^*(t')\rangle_e + \langle E_\alpha(t) E_\delta^*(t')\rangle_e \langle E_\beta^*(t) E_\gamma(t')\rangle_e + \langle E_\alpha(t) E_\gamma(t')\rangle_e \langle E_\beta^*(t) E_\delta^*(t')\rangle_e. \quad (22)$$

We again observe that the third term is zero for chaotic light because $E_x(t)$, $E_y(t)$ are zero-mean statistically independent processes with $\langle E_x(t)E_x(t')\rangle_e = \langle E_y(t)E_y(t')\rangle_e = 0$.

Our argument for the four-field decomposition is as follows. Fields $(E_x(t), E_y(t), E_x(t'), E_y(t'))$ are joint complex Gaussian random variables as the incident radiation is unpolarized and of thermal origin. Thus, the real and imaginary parts of these four fields are eight real jointly Gaussian-distributed random variables, with an appropriate covariance matrix. The four fields are linear functions of the eight real jointly Gaussian-random variables. Therefore, the conditions of the Gaussian moment theorem, e.g., [17] are met, and the decomposition follows. Inserting Eq. (22) back into (21), we get the correlator:

$$\langle i_1(t_0) i_2(t_0)\rangle_t = \frac{1}{T^2}\int_{t_0-\frac{T}{2}}^{t_0+\frac{T}{2}}\int_{t_0-\frac{T}{2}}^{t_0+\frac{T}{2}} (\langle I_1(t)\rangle_e \langle I_2(t')\rangle_e + u_{1,\alpha}^* u_{1,\beta} u_{2,\gamma}^* u_{2,\delta}\delta_{\alpha\delta}\delta_{\beta\gamma} \times R_\alpha(t'-t)^* R_\beta(t'-t))\mathrm{d}t\mathrm{d}t', \quad (23)$$

where the autocorrelation $R_\alpha(\tau) = \langle E_\alpha^*(t) E_\alpha(t+\tau)\rangle_e$ of incident radiation contains coherence time information, e.g., $\sim 10^{-8}$ s characteristic of thermal light [13]. The delta functions $\delta_{\alpha\beta}$ appear because $E_x(t)$, $E_y(t)$ are statistically independent (δ-correlated in time) and zero-mean. Thus,

$$\langle i_1(t_0) i_2(t_0)\rangle_t - \langle i_1(t_0)\rangle_t \langle i_2(t_0)\rangle_t = u_{1,\alpha}^* u_{2,\alpha} u_{1,\beta} u_{2,\beta}^* \frac{1}{T^2}\int_{t_0-\frac{T}{2}}^{t_0+\frac{T}{2}}\int_{t_0-\frac{T}{2}}^{t_0+\frac{T}{2}} R_\alpha(t'-t) \times R_\beta(t'-t)^* \mathrm{d}t\mathrm{d}t'. \quad (24)$$

At this point, we make use of $R_\alpha(\tau) = R(\tau)$, true for statistically identical processes $E_x(t)$, $E_y(t)$:

$$\langle i_1(t_0) i_2(t_0)\rangle_t - \langle i_1(t_0)\rangle_t \langle i_2(t_0)\rangle_t = |u_1^\dagger u_2|^2 \frac{1}{T^2}\int_{t_0-\frac{T}{2}}^{t_0+\frac{T}{2}}\int_{t_0-\frac{T}{2}}^{t_0+\frac{T}{2}} |R(t'-t)|^2 \mathrm{d}t\mathrm{d}t'. \quad (25)$$

Normalizing by standard deviations, which are found by letting $2 \to 1$ in Eq. (25), one again obtains that

$$\rho^{(2)} = \frac{\langle i_1(t_0) i_2(t_0)\rangle_t - \langle i_1(t_0)\rangle_t \langle i_2(t_0)\rangle_t}{\sigma_{i_1}\sigma_{i_2}} = |u_1^\dagger u_2|^2, \quad (26)$$

thus generalizing earlier results to the more practical case of measured time-averaged intensities. Even if intensity is measured with a slow photodetector, the relationship between measured intensities and channel Jones vector inner product still holds.

6. CONFIRMATION VIA MONTE CARLO SIMULATION

We test Eq. (5) through an easy-to-reproduce numerical simulation. Recall that Eq. (5) holds for thermal unpolarized radiation in continuous time. Since such processes cannot be directly simulated on a computer, we test Eq. (5) with discrete time series and equally spaced samples of thermal radiation, separated by many coherence times t_{coh}, to ensure statistical independence.

We used MATLAB to simulate statistically independent and identically distributed (IID) complex circular Gaussian random variables [10]. This generates a time series of electric field samples $E_x(t_n)$ and $E_y(t_n)$, where n is an integer ranging from 1 to N. Next, we find the field projections on two Jones vectors, representing, say, linear orthogonal arms of a beam splitter.

We chose to simulate the Jones vectors as $u_1 = (1, 0)^T$ and $u_2 = (\cos(90° - \delta_\tau), \sin(90° - \delta_\tau))^T$ so that these are nearly 90° apart, but not quite, with an error $\delta_\tau = 5°$. The intensity cross-correlation coefficient between the channel field time series is then calculated.

For a generic function f, the time average is of the form, $\langle f\rangle_t = \lim_{T\to\infty}\frac{1}{T}\int_0^T f(t)\mathrm{d}t$, approximated with the discrete form $\langle f\rangle_t \approx \frac{1}{N}\sum_{n=1}^N f(t_n)$. Then the cross-channel correlation coefficient is calculated as $\rho^{(2)} = \frac{\langle I_1 I_2\rangle_t - \langle I_1\rangle_t\langle I_2\rangle_t}{\sigma_1\sigma_2}$, $\sigma_i^2 = \langle I_i^2\rangle_t - \langle I_i\rangle_t^2$. Because the time series are of finite length ($N = 10^4$), the correlation coefficient is not precisely δ_τ^2, valid only in the limit $N \to \infty$. The *sample* correlation coefficient, denoted by $\tilde{\rho}^{(2)}$, is itself a random variable. By repeating the simulation many times, as might be done in experiments, the distribution of $\tilde{\rho}^{(2)}$ is found and shown in Fig. 2(a). This distribution is centered on δ_τ^2 (see Fig. 2) with a standard deviation observed to be $\approx 1/\sqrt{N} = 0.01$. Insofar as simulations of $\tilde{\rho}^{(2)}$ involve finite time duration as do experimental measurements of $\rho^{(2)}$, Fig. 2 mimics laboratory experiments. We also simulated the ellipticity error δ_ϵ in a circular basis configuration and confirmed Eq. (6) as shown in Fig. 2(b).

7. CONCLUDING REMARKS

Although the simplest results hold when $E_x(t)$, $E_y(t)$ are independent and identical (circular) Gaussian distributions, corresponding to isotropic fields such as direct sunlight, it is also possible to address the case of components with different variances. To that end, it is important to observe that deriving the result $|\rho^{(1)}|^2 = \rho^{(2)}$ relies on the assumption of the multivariate Gaussian distribution of $E_{x,y}$, but *not on the "circularity,"* i.e., the Gaussian random fields in the x and y directions need not be identically distributed.

To get simple results in the unequal variance case, we let one of the polarization basis channels lie along the x (or y) direction. Then, as shown in [1], $\rho^{(1)} = \frac{\sigma_{E_1}}{\sigma_{E_2}} u_1^\dagger u_2$, where subscript 1 refers to the x direction. Using the result $|\rho^{(1)}|^2 = \rho^{(2)}$ then

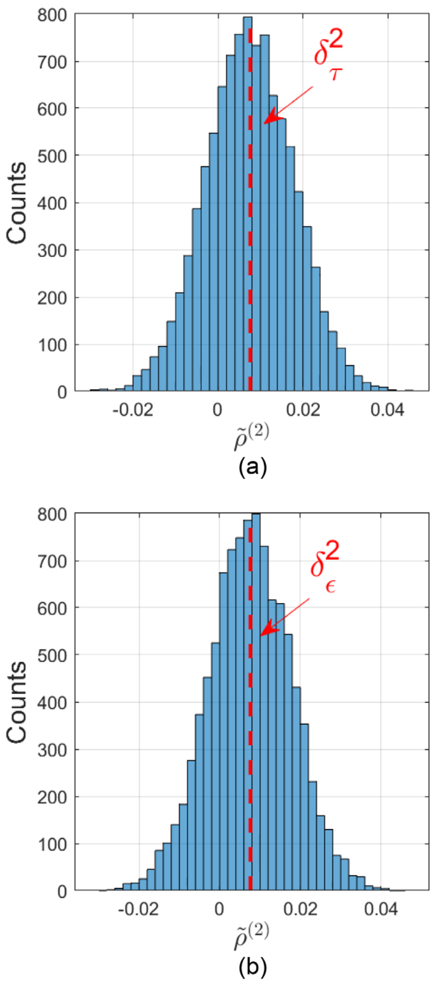

Fig. 2. Intensity correlation histogram over $n = 10^4$ independent trials (realizations) of the sample cross-correlation coefficient $\tilde{\rho}^{(2)}$, each value for a time-series of length $N = 10^4$. The simulated built-in errors (deviation from orthogonality) are: (a) $\delta_\tau = 5° = 0.0873$ rad from perpendicular linear; (b) $\delta_\epsilon = 5° = 0.0873$ rad from orthogonal circular. The histogram means ≈ 0.0075 are very close to $\delta_\tau^2 = 0.0076$, shown by a dashed red line. The standard deviation is $\approx 0.01 = 1/\sqrt{N}$.

yields $\rho^{(2)} = \frac{\langle I_1 \rangle}{\langle I_2 \rangle} |u_1^\dagger u_2|^2$. For the practically important case of a linearly polarized basis, with error δ_τ from $\pi/2$, these results become $\delta_\tau \approx \rho^{(1)} \frac{\sigma_{E_2}}{\sigma_{E_1}}$ and $\delta_\tau \approx \sqrt{\rho^{(2)} \frac{\langle I_2 \rangle}{\langle I_1 \rangle}}$.

In summary, our results are useful for checking the orthogonality of polarization channels in a polarimetric device operating at optical frequencies. All that is needed is a measurement of the channel intensity cross-correlation while observing randomly polarized light, and polarization purity (departure from orthogonality) of the instrument can be instantly ascertained, possibly in real time. In addition to the applications mentioned in the introduction, this could also be used to check cross-channel isolation (purity) in dual-polarized lidar [7].

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