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RESEARCH ARTICLE

Splines on Cayley graphs of the symmetric group

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Abstract

A spline is an assignment of polynomials to the vertices of a graph whose edges are labeled by ideals, where the difference of two polynomials labeling adjacent vertices must belong to the corresponding ideal. The set of splines forms a ring. We consider spline rings where the underlying graph is the Cayley graph of a symmetric group generated by a collection of transpositions. These rings generalize the GKM construction for equivariant cohomology rings of flag, regular semisimple Hessenberg and permutohedral varieties. These cohomology rings carry two actions of the symmetric group S_n whose graded characters are both of general interest in algebraic combinatorics. In this paper, we generalize the graded S_n-representations from the cohomologies of the above varieties to splines on Cayley graphs of S_n and then (1) give explicit module and ring generators for whenever the S_n-generating set is minimal, (2) give a combinatorial characterization of when graded pieces of one S_n-representation is trivial, and (3) compute the first degree piece of both graded characters for all generating sets.

Contents

1. Introduction

Let $\mathcal{G}$ be a graph with edges labeled by ideals in $\mathbb{C}[t_\bullet] := \mathbb{C}[t_1, \ldots, t_n]$. A *spline* on $\mathcal{G}$ is an assignment of polynomials to vertices such that the difference of two polynomials labeling adjacent vertices must be in the corresponding ideal. The *Cayley graph* for a group G and generating set $S \subseteq G$ has vertex set G and edge set $\{(g, gs) \mid g \in G, s \in S\}$. When the group G is a symmetric group S_n and the generating set S consists of inversions, there is a natural edge labeling for the corresponding Cayley graph. This labeled Cayley graph, and thereby the splines on it, are entirely determined by the data of the inversion graph $\Gamma = ([n], S)$. This paper determines algebraic structures of splines on Cayley graphs of symmetric groups using the combinatorial data of the inversion graph Γ.

To discuss the results below, we begin with some notation. Let Γ be a connected simple graph with vertex set $[n] := \{1, \ldots, n\}$, and identify the edges in its edge set $E(\Gamma)$ with transpositions in S_n. This paper studies how properties of Γ determine the algebraic structure of splines on the Cayley graph $\mathcal{G}_\Gamma$ of S_n with generating set $E(\Gamma)$ and edge label $(w, w(i,j)) \mapsto \langle t_{w(i)} - t_{w(j)} \rangle$. Formally, the *ring of splines* is defined as

$$\mathcal{M}_\Gamma := \left\{ \bar{\rho} \in \prod_{w \in S_n} \mathbb{C}[t_\bullet] \;\middle|\; \bar{\rho}(w) - \bar{\rho}(w(i,j)) \in \langle t_{w(i)} - t_{w(j)} \rangle \text{ when } (i,j) \in E(\Gamma) \right\},$$

with (graded) S_n-module structure $w \cdot \bar{\rho}(v) = w\bar{\rho}(w^{-1}v)$ and (graded) $\mathbb{C}[t_\bullet]$-module structure given by multiplication.

This definition of the ring of splines generalizes the case where $\mathcal{G}_\Gamma$ is the moment graph of a geometric object called a regular semisimple Hessenberg variety and the ring of splines is isomorphic to the equivariant cohomology of that variety [13, 16, 28]. We call this the *geometric case*, and in this case, the corresponding graph Γ is a *Hessenberg graph*, commonly characterized in algebraic combinatorics as being the indifference graph of a $3+1$- and $2+2$-free poset. The more general setting considered in this paper allows one to spot patterns in rich algebraic structure that would otherwise be restricted for geometric reasons. For example, in the geometric case, $\mathcal{M}_\Gamma$ is always a free module over the polynomial ring, whereas for general Γ, it is not.

The S_n-module structure on $\mathcal{M}_\Gamma$ was first defined in the geometric case as the *dot action* on equivariant cohomology by Tymoczko in [28]. There are two natural S_n-equivariant quotients, L_Γ and R_Γ, of $\mathcal{M}_\Gamma$ that are in fact graded $\mathbb{C}$-vector spaces. The graded S_n-module structure of $\mathcal{M}_\Gamma$ induces graded S_n-representations on the quotients, admitting (via the Frobenius character map **ch**) two different graded symmetric functions:

$$\mathbf{ch}(\mathrm{L}_\Gamma) := \bigoplus_i \mathbf{ch}(\mathrm{L}_\Gamma)_i \quad \text{and} \quad \mathbf{ch}(\mathrm{R}_\Gamma) := \bigoplus_i \mathbf{ch}(\mathrm{R}_\Gamma)_i.$$

These are manifestly Schur-positive symmetric function invariants of any simple graph.

The graded symmetric functions $\mathbf{ch}(\mathrm{L}_\Gamma)$ and $\mathbf{ch}(\mathrm{R}_\Gamma)$ are historically of interest to algebraic combinatorists because of their connections to chromatic symmetric functions [8, 19, 25] and LLT polynomials [3, 5, 19] in the geometric case. The two bases of symmetric functions we consider here are *Schur functions* $\{s_\lambda\}$ and *homogeneous symmetric functions* $\{h_\lambda\}$. In the geometric case, two major open problems seek (1) a homogeneous basis expansion of $\mathbf{ch}(\mathrm{L}_\Gamma)$ ([1, 8, 12, 14, 18, 20, 25, 26], and many others), and (2) a Schur basis expansion of $\mathbf{ch}(\mathrm{R}_\Gamma)$ ([2, 7, 19, 21, 22, 23] and many others). Again, our object of study is more general, and because of this, we can identify patterns otherwise masked by geometric structure. For example, the Stanley–Stembridge conjecture [27] claims that the homogeneous basis expansion of $\mathbf{ch}(\mathrm{L}_\Gamma)$ has only nonnegative integer coefficients (h-positivity) in the geometric case. We observe below that this is not the case for general Γ, but h-positivity seems to occur whenever $\mathcal{M}_\Gamma$ is a free module over $\mathbb{C}[t_\bullet]$.

This paper begins with several fundamental properties of $\mathcal{M}_\Gamma$. First, we establish the algebraic structure of $\mathcal{M}_\Gamma$ as an invariant of the graph Γ.

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Lemma 1.1. *An isomorphism of graphs $\Gamma \cong \Gamma'$ induces a ring isomorphism of splines $\mathcal{M}_\Gamma \cong \mathcal{M}_{\Gamma'}$ and equality of graded symmetric functions:* $\mathbf{ch}(\mathrm{L}_\Gamma) = \mathbf{ch}(\mathrm{L}_{\Gamma'})$ *and* $\mathbf{ch}(\mathrm{R}_\Gamma) = \mathbf{ch}(\mathrm{R}_{\Gamma'})$.

In particular, Lemma 1.1 shows that the graded symmetric functions $\mathbf{ch}(\mathrm{L}_\Gamma)$ and $\mathbf{ch}(\mathrm{R}_\Gamma)$ are (Schur-positive) invariants of unlabeled simple graphs. Lemma 1.1 is proved via Propositions 2.18 and 2.20 below.

Then when Γ is a tree, we determine explicit ring and module generators of $\mathcal{M}_\Gamma$ called *coset splines* (Definition 3.4).

Theorem 1.2. *If Γ is a tree, then the set of coset splines is a $\mathbb{C}[t_\bullet]$-module generating set of $\mathcal{M}_\Gamma$, and the set of linear and constant coset splines is a ring generating set of $\mathcal{M}_\Gamma$.*

Since they generate, one can compute $\mathcal{M}_\Gamma$ explicitly with coset splines using a computer algebra system. Theorem 1.2 is Theorem 3.7 and Corollary 3.8 below.

We use Theorem 1.2 to show that $\mathcal{M}_\Gamma$ is *not* always a free $\mathbb{C}[t_\bullet]$-module, and $\mathbf{ch}(\mathrm{L}_\Gamma)$ is *not* always h-positive (see Appendix A). One example is if $\Gamma = ([4], \{(1,4), (2,4), (3,4)\})$, then $\mathcal{M}_\Gamma$ is not a free module and $\mathbf{ch}(\mathrm{L}_\Gamma)_2$ is not h-positive. This also confirms that $\mathcal{M}_\Gamma$ is not always the equivariant cohomology of an (equivariantly formal) algebraic variety as in [16], since in that case, $\mathcal{M}_\Gamma$ is a free $\mathbb{C}[t_\bullet]$-module.

Our next main results, Theorems 1.3 and 1.4 below, explicitly compute certain graded pieces of the symmetric functions $\mathbf{ch}(\mathrm{L}_\Gamma)$ and $\mathbf{ch}(\mathrm{R}_\Gamma)$. Specifically, we determine when graded pieces of $\mathbf{ch}(\mathrm{L}_\Gamma)$ and $\mathbf{ch}(\mathrm{R}_\Gamma)$ are equal to $\mathbf{ch}(\mathrm{L}_{K_n})$ and $\mathbf{ch}(\mathrm{R}_{K_n})$ where K_n is the complete graph ($\Gamma = K_n$ is a very special geometric case), and we compute $\mathbf{ch}(\mathrm{L}_\Gamma)_1$ and $\mathbf{ch}(\mathrm{R}_\Gamma)_1$ for all connected graphs Γ.

For a variety of reasons, for example by formulae in [25] or by some geometric observations, in the geometric case, it is straightforward to tell from a Hessenberg graph H whether the symmetric function $\mathbf{ch}(\mathrm{L}_H)_d$ corresponds to a trivial representation. We achieve an analogous result for arbitrary graphs. The k-connectivity (Definition 2.2) of a graph is a combinatorial invariant that measures how many vertices can be removed from a graph before it might become disconnected.

Theorem 1.3. *Let Γ be a connected simple graph. The following are equivalent:*

1) *The graph Γ is k-connected.*
2) *For all $d < k$, the symmetric function $\mathbf{ch}(\mathrm{L}_\Gamma)_d$ corresponds to a trivial representation.*
3) *For all $d < k$, the d-th graded piece of $\mathcal{M}_\Gamma$ is isomorphic to the d-th graded piece of $\mathcal{M}_{K_n}$, where K_n is the complete graph on n vertices.*

Geometrically, the d-th graded piece of $\mathcal{M}_{K_n}$ is isomorphic to the $2d$-th equivariant cohomology of the full flag variety and is thus spanned by equivariant Schubert classes whose spline formula is given in [6]. Theorem 1.3 is a consequence of Theorem 4.2 below.

When Γ is a Hessenberg graph, the first graded piece of $\mathbf{ch}(\mathrm{L}_\Gamma)$ has been computed in a variety of ways. The Schur expansion is computed by counting P-tableaux [25]. Expansions in the homogeneous basis have been computed with P-tableaux [11], geometrically [10], as well as with splines [4]. Our methods here most directly generalize those in [4].

Theorem 1.4. *D Let Γ be any connected simple graph. The first-degree pieces of the graded symmetric functions $\mathbf{ch}(\mathrm{L}_\Gamma)$ and $\mathbf{ch}(\mathrm{R}_\Gamma)$ can be computed in both the Schur and homogeneous bases of symmetric functions from the data of (1) cut edges of Γ and (2) cut vertices of Γ and the number of connected components those vertices separate.*

Formally, there exist a subset E_1 of cut edges, a subset E_2 of 2-connected subgraphs, a nonnegative integer $k \in \mathbb{N}$, and a function $e \mapsto \lambda_e$ from E_1 to the set of partitions of n, such that

$$\mathbf{ch}(\mathrm{L}_\Gamma)_1 = \sum_{e \in E_1} h_{\lambda_e} + (|E_2| - 1) h_{n-1,1} + k h_n$$

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and

$$\mathbf{ch}(\mathrm{R}_\Gamma)_1 = \sum_{e \in E_1} \left(h_{\lambda_e} - s_n\right) + |E_2| s_{n-1,1}.$$

Theorem 1.4 is Theorem 9.2 and Corollary 9.3 below. The subsets E_1 and E_2 are defined using a combinatorial construction from the block-cut tree (Definition 6.2) of Γ in Section 8.

The paper is structured as follows. Section 2 constructs $\mathcal{M}_\Gamma$ and proves some of the fundamental algebraic properties, including the isomorphism of Lemma 1.1. Section 3 builds tools for computing spline conditions from paths in Γ and $\mathcal{G}_\Gamma$. It also contains the construction of coset splines for trees and the proof that coset splines generate $\mathcal{M}_\Gamma$, Theorem 1.2 above. Section 4 leverages the tools in Section 3 to prove our result on k-connectedness, Theorem 1.3 above. Sections 5, 6, 7, 8 and 9 are all to compute the representations in Theorem 1.4 above. Section 5 defines a set of linear splines on the graph $\mathcal{G}_\Gamma$ and proves some linear relations within that set. Section 6 reduces the computation to a subclass of graphs Γ that will be used in all of the remaining sections. Section 7 proves that the set of splines from Section 5 is in fact a $\mathbb{C}$-spanning set for linear splines, and Section 8 computes the $\mathbb{C}$-dimension of this space. Finally, Section 9 computes the first graded piece of $\mathbf{ch}(\mathrm{L}_\Gamma)$ and $\mathbf{ch}(\mathrm{R}_\Gamma)$, Theorem 1.4. Appendix A contains a table of $\mathbf{ch}(\mathrm{L}_\Gamma)$ and $\mathbf{ch}(\mathrm{R}_\Gamma)$ for graphs with 3 or 4 vertices and a table of the rank-generating functions for graphs of size 5 (which gives the graded dimension of the representations).

2. Background

There is a natural action of the symmetric group S_n on the polynomial ring $\mathbb{C}[t_\bullet]$ by

$$w f(t_1, \ldots, t_n) \mapsto f\left(t_{w(1)}, \ldots, t_{w(n)}\right). \tag{2.1}$$

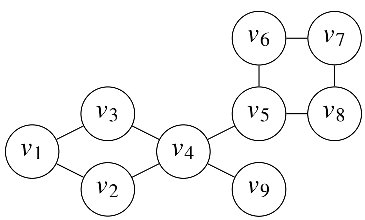
We use both one-line and cycle notation for elements of S_n. We denote a permutation's cycle notation with parentheses and commas, and its one-line notation without, so that $(1, 2, 3) = 231$.

2.1. Graphs: simple and Cayley

This subsection establishes the basic definitions, results and notation from graph theory needed below. A *graph* is a tuple $\Gamma = (V, E)$ where V is the set of vertices and $E \subset V \times V$ is the set of edges. Graphs here are understood to be undirected and simple (i.e., finite, loopless and without multiple edges). We will always take Γ to be connected and may remind the reader of this assumption where particularly important. Write $E(\Gamma)$ for the edge set of a graph Γ and $V(\Gamma)$ for the vertex set. Inclusion $v \in \Gamma$ means $v \in V(\Gamma)$.

If the vertex set V has some natural linear order (in particular, when $V = [n]$), then an edge between vertices $i < j$ will always be written with the lower vertex first (i, j), unless explicitly stated otherwise. Note that these edges are undirected, so an edge (i, j) is the same as an edge (j, i).

We denote graphs pictorially with circles as vertices and lines as edges between them; for example, we would display a particular graph Γ on 9 vertices as



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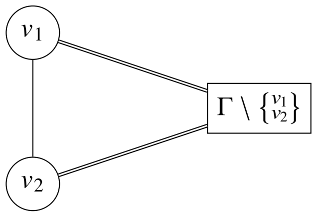
The *induced subgraph* of Γ with vertex set $V \setminus A$ is $\Gamma \setminus A := (V \setminus A, E')$, where $E' = E \cap (V \setminus A \times V \setminus A)$. We write $\Gamma - v$ for $\Gamma \setminus \{v\}$. When collapsing a subgraph in drawing, we reference the subgraph in a square to distinguish that there are multiple vertices being referenced, and double lines connecting to acknowledge the possibility of multiple edges. For example, we may display Γ above as



if the structure within $\Gamma \setminus \{v_1, v_2\}$ is not needed.

Definition 2.1. For a graph Γ, a set $A \subset V(\Gamma)$ is a *cut set* if $\Gamma \setminus A$ is disconnected. Similarly, $v \in V(\Gamma)$ is a *cut vertex* of Γ (denoted $v \vdash \Gamma$) if $\Gamma - v$ is disconnected.

An edge $e \in E(\Gamma)$ is a *cut edge* if the graph $(V(\Gamma), E(\Gamma) \setminus \{e\})$ is disconnected.

A *path* in Γ from vertex v_0 to vertex v_ℓ of length ℓ is a sequence of vertices $(v_0, v_1, \ldots, v_\ell)$, where $(v_k, v_{k+1}) \in E(\Gamma)$ for $k = 0, \ldots, \ell - 1$. Define the *distance* $d(v, w)$ between v and w as the minimum length over all paths from v to w, and let $d(v, w) := \infty$ if no such path exists.

Definition 2.2. A graph $\Gamma = (V, E)$ is *k-connected* if $\Gamma \setminus A$ is connected for all $A \subset V$ such that $|A| \leq k - 1$.

In other words, a graph is k-connected if there exists no cut set A where $|A| < k$. The following is an equivalent characterization used in §4.

Theorem 2.3 (Menger's Theorem). *A graph Γ is k-connected if and only if for every pair of vertices $i, j \in \Gamma$, there exist at least k vertex-disjoint paths from i to j.*

An *R-labeled graph* is a tuple (V, E, L), where (V, E) is a graph and L is a function $L\colon E \to R$ for some set R. A *Cayley graph* of a group G and a set of generators S is the graph $\big(G, \{(g, h) \mid g^{-1}h \in S\}\big)$. Cayley graphs are usually directed graphs, but all generators considered here will be involutions, and so the Cayley graphs will be undirected simple graphs. Note that $g^{-1}h \in S$ if and only if $h = gs$ for $s \in S$, so edges in a Cayley graph correspond to right multiplication by generators.

This paper concerns graphs Γ on vertex set $[n]$ and labeled Cayley graphs of the symmetric group with generators being some set of transpositions. The edge labels are principle ideals in $\mathbb{C}[t_\bullet]$.

Definition 2.4. Let Γ be a graph on $[n]$. Identify each edge $(i, j) \in E(\Gamma)$ with the transposition $(i, j) \in S_n$. The *labeled Cayley graph associated to* Γ is $\mathcal{G}_\Gamma := (\mathcal{V}, \mathcal{E}, \mathcal{L})$, where

- $\mathcal{V} = S_n$,
- $\mathcal{E} = \{(w, v) \mid w^{-1}v \in E(\Gamma)\}$, and
- $\mathcal{L}(w, v) = \langle t_i - t_j \rangle$, where $(i, j) = wv^{-1}$.

Note $w^{-1}v$ is conjugate to wv^{-1}, so if $w = v(i, j)$, then $\mathcal{L}(w, v) = \langle t_{w(i)} - t_{w(j)} \rangle = \langle t_{v(i)} - t_{v(j)} \rangle$. Note also that $\mathcal{L}$ is defined whenever wv^{-1} is a transposition.

Example 2.5. Let $\Gamma = ([3], \{(1, 2), (2, 3)\})$. Then $\mathcal{G}_\Gamma$ has vertex set S_3, edges $\{(w, v) \mid w^{-1}v \in \{(1, 2), (2, 3)\}$, and labels of the form $\langle t_i - t_j \rangle$, where $i, j \in [3]$. Below is $\mathcal{G}_\Gamma$, with labeling ideals denoted by generators.

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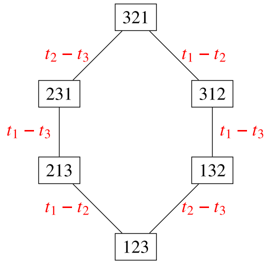


Consider the edge $(132, 312)$. These permutations have their first and second positions swapped, corresponding to right multiplication by $(1,2) \in E(\Gamma)$. The edge is labeled $\langle t_1 - t_3 \rangle$ because these permutations have the *entries* 1 and 3 swapped, corresponding to left multiplication by $(1,3)$.

The Γ-*length* of a permutation $w \in S_n$ is

$$\ell_\Gamma(w) := \min\{\ell \mid w = s_1 \cdots s_\ell, \ \ \{s_1, \ldots, s_\ell\} \subseteq E(\Gamma)\}. \tag{2.2}$$

This is also the value of $d(e, w)$ in $\mathcal{G}_\Gamma$. When Γ is the path graph, Γ-length is the traditional length function on permutations.

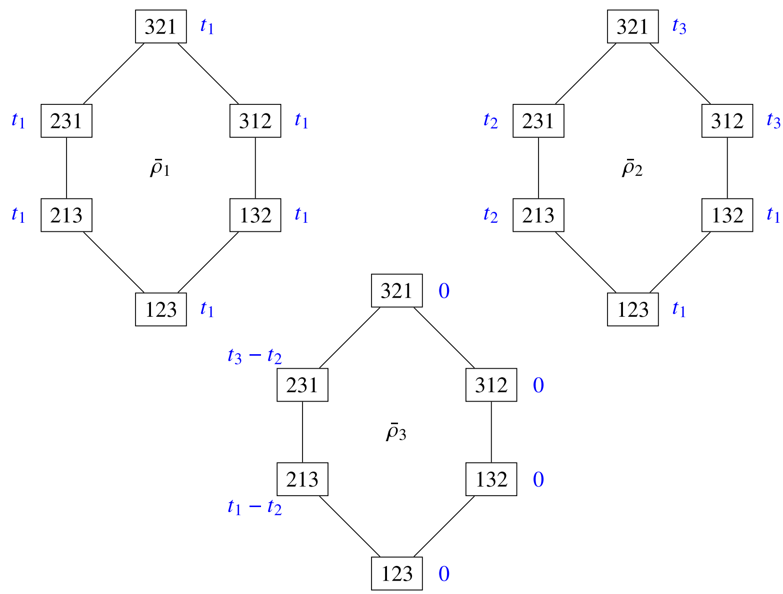
2.2. Splines

This section introduces the ring of splines on a labeled Cayley graph. The lemmas in this subsection are well known and straightforward, but we include proofs for completeness.

Definition 2.6. Let Γ be a graph on $[n]$. A *spline* on $\mathcal{G}_\Gamma$ is a function $\bar{\rho} \colon S_n \to \mathbb{C}[t_\bullet]$ such that $\bar{\rho}(w) - \bar{\rho}(v) \in \mathcal{L}(w, v)$ whenever $(w, v) \in E(\mathcal{G}_\Gamma)$. The *support* of the spline $\bar{\rho}$ is the set $\operatorname{supp}(\bar{\rho}) := \{w \mid \bar{\rho}(w) \neq 0\}$.

To distinguish from polynomials, we always denote a spline with a bar.

Example 2.7. Again, consider $\Gamma = ([3], \{(1,2), (2,3)\})$. Drawn below (omitting edge-labels) are three examples of splines on $\mathcal{G}_\Gamma$.



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So $\bar{\rho}_1(w) = t_1$ for all $w \in S_3$, $\bar{\rho}_2(w) = t_{w(1)}$ for all $w \in S_3$, and $\bar{\rho}_3(w) = \begin{cases} t_1 - t_2 & \text{if } w = 213 \\ t_3 - t_2 & \text{if } w = 231 \\ 0 & \text{otherwise.} \end{cases}$

The set of splines is closed under addition, as well as multiplication.

Lemma 2.8. *Let Γ be a graph on $[n]$. If $\bar{\rho}$ and $\bar{\sigma}$ are splines on $\mathcal{G}_\Gamma$, then so is $\bar{\sigma}\bar{\rho}$, the spline constructed via pointwise multiplication.*

Proof. Let $(w, v) \in E(\mathcal{G}_\Gamma)$. By assumption, $\bar{\rho}(w) - \bar{\rho}(v) \in \mathcal{L}(w, v)$ and $\bar{\sigma}(w) - \bar{\sigma}(v) \in \mathcal{L}(w, v)$. We have

$$\begin{aligned} \bar{\rho}(w)\bar{\sigma}(w) - \bar{\rho}(v)\bar{\sigma}(v) &= \bar{\rho}(w)\bar{\sigma}(w) - \bar{\sigma}(v)\bar{\rho}(w) + \bar{\sigma}(v)\bar{\rho}(w) - \bar{\rho}(v)\bar{\sigma}(v) \\ &= \bar{\rho}(w)(\bar{\sigma}(w) - \bar{\sigma}(v)) + \bar{\sigma}(v)(\bar{\rho}(w) - \bar{\rho}(v)), \end{aligned}$$

and the sum is clearly in $\mathcal{L}(w, v)$. □

Definition 2.9. The *ring of splines* on $\mathcal{G}_\Gamma$ is the subring

$$\mathcal{M}_\Gamma := \left\{ \bar{\rho} \in \prod_{w \in S_n} \mathbb{C}[t_\bullet] \;\middle|\; \bar{\rho}(w) - \bar{\rho}(v) \in \mathcal{L}(w, v) \text{ for all } (w, v) \in E(\mathcal{G}_\Gamma) \right\}$$

of $\prod_{w \in S_n} \mathbb{C}[t_\bullet]$ with pointwise addition and multiplication.

Lemma 2.10. *The ring $\mathcal{M}_\Gamma$ is graded by degree, so $\mathcal{M}_\Gamma = \bigoplus_{i \geq 0} \mathcal{M}_\Gamma^i$.*

Proof. Let $\bar{\rho}$ be a spline in $\mathcal{M}_\Gamma$ and let $\bar{\rho}_k(w)$ be the k-th graded piece of the polynomial $\bar{\rho}(w)$. We aim to show that $\bar{\rho}_k$ is a spline as well. For each $(w, v) \in E(\mathcal{G}_\Gamma)$, the ideal $\mathcal{L}(w, v)$ is a homogeneous ideal. Thus, $\bar{\rho}(w) - \bar{\rho}(v) \in \mathcal{L}(w, v)$, and it follows that $\bar{\rho}_k(w) - \bar{\rho}_k(v) \in \mathcal{L}(w, v)$, so $\bar{\rho}_k$ is a spline. For two homogeneous splines $\bar{\rho}$ and $\bar{\sigma}$ of degrees p and q, respectively, the product $\bar{\rho}\bar{\sigma}$ is homogeneous of degree $p + q$ on its support. □

We now construct two sets of splines and the *identity spline*, each are elements of $\mathcal{M}_\Gamma$ for all Γ. Let

$$\begin{aligned} &\bar{\mathbb{1}} : S_n \to \mathbb{C}[t_\bullet] \text{ be } \bar{\mathbb{1}}(w) := 1 && \text{for all } w \in S_n, \\ &\bar{t}_i : S_n \to \mathbb{C}[t_\bullet] \text{ be } \bar{t}_i(w) := t_i && \text{for all } w \in S_n, i \in \{1, \ldots, n\}, \textit{and} \\ &\bar{x}_i : S_n \to \mathbb{C}[t_\bullet] \text{ be } \bar{x}_i(w) := t_{w(i)} && \text{for all } w \in S_n, i \in \{1, \ldots, n\}. \end{aligned}$$

The ring $\mathcal{M}_\Gamma$ is an infinite-dimensional $\mathbb{C}$-vector space in the natural way and can also be viewed as a finitely generated graded $\mathbb{C}[t_\bullet]$-module in two ways via the following module actions:

$$f(t_1, \ldots, t_n).\bar{\rho} = f(\bar{t}_1, \ldots, \bar{t}_n)\bar{\rho} \tag{2.3}$$

and

$$f(t_1, \ldots, t_n).\bar{\rho} = f(\bar{x}_1, \ldots, \bar{x}_n)\bar{\rho}, \tag{2.4}$$

where the right-hand side of both (2.3) and (2.4) work by substituting splines for variables in to the polynomial f then multiplying as in the ring structure of $\mathcal{M}_\Gamma$. For both actions, the constant $f(0, \ldots, 0)$ is naturally mapped to $f(0, \ldots, 0)\bar{\mathbb{1}}$. Since $\mathcal{M}_\Gamma$ is a $\mathbb{C}[t_\bullet]$-submodule of $\prod_{w \in S_n} \mathbb{C}[t_\bullet]$ for either module action, it is finitely generated. We call the module action (2.3) the *left action* and the module action (2.4) the *right action* of $\mathbb{C}[t_\bullet]$ on $\mathcal{M}_\Gamma$. Given any $\omega \in S_n$, both actions may be twisted by sending $f \mapsto \omega f$ first in the polynomial ring. Both the left and right actions are naturally compatible with the grading on $\mathcal{M}_\Gamma$.

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Example 2.11. Let $\bar{\rho} \in \mathcal{M}_\Gamma$ and let $f(t_\bullet) = t_1^3 + t_2^2 + t_3$. Let $\omega = (1, 2, 3) \in S_n$. The left action of f on $\bar{\rho}$ evaluated at any $v \in S_n$ is

$$f(t_\bullet).\bar{\rho}(v) = \left[((\bar{t}_1)^3 + (\bar{t}_2)^2 + \bar{t}_3)\bar{\rho}\right](v) = (t_1^3 + t_2^2 + t_3)\bar{\rho}(v),$$

the right action of f on $\bar{\rho}$ evaluated at any $v \in S_n$ is

$$f(t_\bullet).\bar{\rho}(v) = \left[((\bar{x}_1)^3 + (\bar{x}_2)^2 + \bar{x}_3)\bar{\rho}\right](v) = (t_{v(1)}^3 + t_{v(2)}^2 + t_{v(3)})\bar{\rho}(v),$$

the ω-twisted left action of f on $\bar{\rho}$ evaluated at any $v \in S_n$ is

$$f(t_\bullet).\bar{\rho}(v) = \left[((\bar{t}_{\omega(1)})^3 + (\bar{t}_{\omega(2)})^2 + \bar{t}_{\omega(3)})\bar{\rho}\right](v) = (t_2^3 + t_3^2 + t_1)\bar{\rho}(v),$$

and the ω-twisted right action of f on $\bar{\rho}$ evaluated at any $v \in S_n$ is

$$f(t_\bullet).\bar{\rho}(v) = \left[((\bar{x}_{\omega(1)})^3 + (\bar{x}_{\omega(2)})^2 + \bar{x}_{\omega(3)})\bar{\rho}\right](v) = (t_{v(2)}^3 + t_{v(3)}^2 + t_{v(1)})\bar{\rho}(v).$$

The ring of splines has a S_n-module structure, originally defined for Hessenberg graphs in [28, 29].

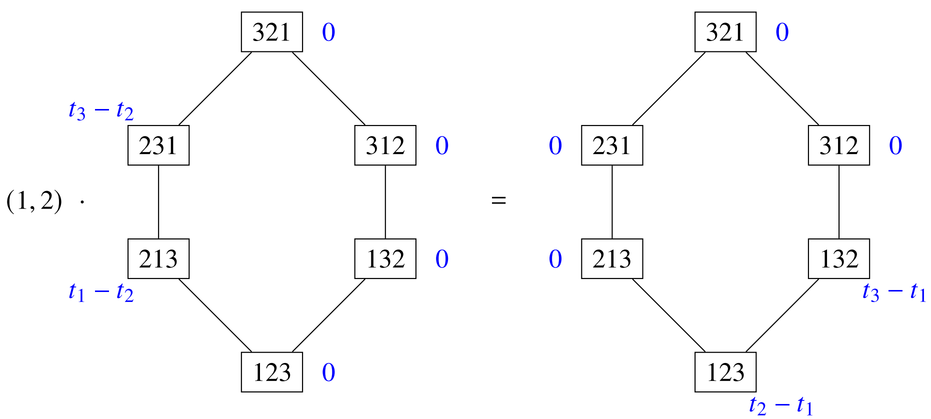
Definition 2.12. Let $\bar{\rho} \in \mathcal{M}_\Gamma$. The *dot action* of S_n on $\mathcal{M}_\Gamma$ is given by

$$w \cdot \bar{\rho}(v) := w\bar{\rho}(w^{-1}v)$$

for $w, v \in S_n$. Any $\omega \in S_n$ may twist the dot action by first sending $v \rightarrow \omega v \omega^{-1}$ (conjugating by ω). Since conjugation is an inner automorphism of S_n, the standard and ω-twisted S_n-module structures on $\mathcal{M}_\Gamma$ are isomorphic.

Using our standard for visualizing splines, the dot action by w moves polynomials around $\mathcal{G}_\Gamma$ by sending the polynomial at v to wv (for all $v \in S_n$) and then acts on every polynomial by w as in Equation (2.1).

Example 2.13. The dot action of the transposition $(1, 2)$ on the spline $\bar{\rho}_3$ from Example 2.7 is computed below.



Computed below is the $\omega = (1, 2, 3)$-twisted action of the transposition $(1, 2)$ on the spline $\bar{\rho}_3$ from Example 2.7. Note this is the same as the untwisted action of $(1, 2, 3)(1, 2)(1, 2, 3)^{-1} = (2, 3)$.

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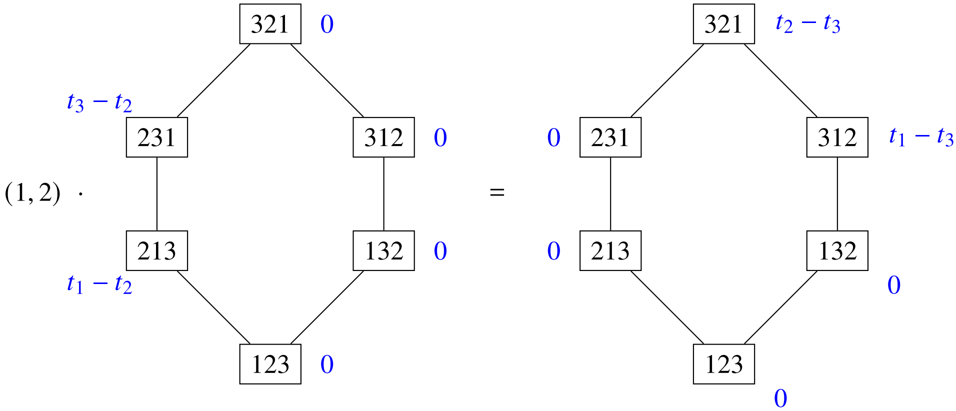

Remark 2.14. The dot action is well defined. We have for all $(v_1, v_2) \in E(\mathcal{G}_\Gamma)$ that

$$\begin{aligned} w \cdot \bar{\rho}(v_1) - w \cdot \bar{\rho}(v_2) &= w\bar{\rho}(w^{-1}v_1) - w\bar{\rho}(w^{-1}v_2) \\ &= w(\bar{\rho}(w^{-1}v_1) - \bar{\rho}(w^{-1}v_2)) \\ &\in w\mathcal{L}(w^{-1}v_1, w^{-1}v_2). \end{aligned}$$

If $v_1 v_2^{-1} = (i, j)$, then $w^{-1}v_1v_2^{-1}w = (w^{-1}(i), w^{-1}(j))$. So

$$w\mathcal{L}(w^{-1}v_1, w^{-1}v_2) = \left\langle w(t_{w^{-1}(i)} - t_{w^{-1}(j)})\right\rangle = \left\langle t_i - t_j \right\rangle = \mathcal{L}(v_1, v_2).$$

Thus, $w \cdot \bar{\rho}(v_1) - w \cdot \bar{\rho}(v_2) \in \mathcal{L}(v_1, v_2)$, and $w \cdot \bar{\rho} \in \mathcal{M}_\Gamma$.

Finally, consider the quotients

$$\mathrm{L}_\Gamma := \mathcal{M}_\Gamma \big/ \langle \bar{t}_1, \ldots, \bar{t}_n \rangle \tag{2.5}$$

and

$$\mathrm{R}_\Gamma := \mathcal{M}_\Gamma \big/ \langle \bar{x}_1, \ldots, \bar{x}_n \rangle . \tag{2.6}$$

Call L_Γ and R_Γ the left and right quotients of $\mathcal{M}_\Gamma$, respectively. As $\mathbb{C}[t_\bullet]$-modules for the left and right action, both quotients are $\mathcal{M}_\Gamma / I\mathcal{M}_\Gamma$, where I is the 'irrelevant ideal' $\langle t_1, ..., t_n \rangle$ of $\mathbb{C}[t_1, \ldots, t_n]$. Thus, L_Γ and R_Γ each inherit the structure of a finite-dimensional graded $\mathbb{C}$-vector space from the left- and right-module structure of $\mathcal{M}_\Gamma$, respectively. Any homogeneous module-generating set over $\mathbb{C}[t_\bullet]$ projects to a spanning set over $\mathbb{C}$ in the quotient.

The ideals $\langle \bar{t}_1, \ldots, \bar{t}_n \rangle$ and $\langle \bar{x}_1, \ldots, \bar{x}_n \rangle$ are homogeneous and S_n-equivariant, and so the graded S_n-module structure on $\mathcal{M}_\Gamma$ projects to graded S_n-representations on both L_Γ and R_Γ. *Symmetric functions* are formal power series in $\{x_1, x_2, ...\}$ invariant under permuting the variables. The Frobenius character map gives an isomorphism from the algebra of representations of symmetric groups to the algebra of symmetric functions. The two bases of symmetric functions we consider are *Schur functions* $\{s_\lambda\}$, which correspond to irreducible representations, and *homogeneous symmetric functions* $\{h_\lambda\}$, which correspond to induced representations of trivial representations on Young subgroups to symmetric groups. Both Schur and homogeneous symmetric functions are indexed by integer partitions. Denote the Frobenius character of these (q-graded) S_n-representations as $\mathbf{ch}(\mathrm{L}_\Gamma)$ and $\mathbf{ch}(\mathrm{R}_\Gamma)$, respectively. Since both $\mathbf{ch}(\mathrm{L}_\Gamma)$ and $\mathbf{ch}(\mathrm{R}_\Gamma)$ correspond to graded representations, and all representations are sums of irreducible representations, both $\mathbf{ch}(\mathrm{L}_\Gamma)$ and $\mathbf{ch}(\mathrm{R}_\Gamma)$ are manifestly Schur-positive graded symmetric functions.

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Example 2.15. Again, consider $\Gamma = ([3], \{(1,2),(2,3)\})$. Then

$$\mathbf{ch}(\mathrm{L}_\Gamma) = s_3 + (s_{2,1} + 2s_3)q + s_3q^2 = h_3 + (h_{2,1} + h_3)q + h_3q^2$$

and

$$\mathbf{ch}(\mathrm{R}_\Gamma) = s_3 + 2s_{2,1}q + s_3q^2.$$

The following Lemma 2.16 is useful for computer calculations.

Lemma 2.16. *Let Γ and Γ' be two graphs on $[n]$, and $\Gamma \cup \Gamma' := ([n], E(\Gamma) \cup E(\Gamma'))$. Then*

$$\mathcal{M}_{\Gamma\cup\Gamma'} = \mathcal{M}_\Gamma \cap \mathcal{M}_{\Gamma'}$$

Proof. This easily follows from the set-theoretic definition

$$\mathcal{M}_\Gamma = \left\{ \bar{\rho} \in \prod_{w\in S_n} \mathbb{C}[t_\bullet] \;\middle|\; \bar{\rho}(w) - \bar{\rho}(v) \in \mathcal{L}(w,v) \text{ for all } (w,v) \in E(\mathcal{G}_\Gamma) \right\}. \qquad \square$$

2.3. Isomorphisms

It is natural to expect that if two graphs Γ and Γ' on $[n]$ are isomorphic, that the resulting algebraic structures on $\mathcal{M}_\Gamma$ and $\mathcal{M}_{\Gamma'}$ should also have meaningful isomorphisms between them. This section shows that an isomorphism $\Gamma \to \Gamma'$ induces a labeled-graph isomorphism $\mathcal{G}_\Gamma \to \mathcal{G}_{\Gamma'}$, a ring isomorphism $\mathcal{M}_\Gamma \to \mathcal{M}_{\Gamma'}$, a collection of different $\mathbb{C}[t_\bullet]$-module isomorphisms $\mathcal{M}_\Gamma \to \mathcal{M}_\Gamma$, and an S_n-module isomorphism $\mathcal{M}_\Gamma \to \mathcal{M}_\Gamma$ that leads to equalities $\mathbf{ch}(\mathrm{L}_\Gamma) = \mathbf{ch}(\mathrm{L}_{\Gamma'})$ and $\mathbf{ch}(\mathrm{R}_\Gamma) = \mathbf{ch}(\mathrm{R}_{\Gamma'})$.

Throughout this subsection, let Γ and Γ' be graphs on $[n]$ and say that $\omega\colon \Gamma \to \Gamma'$ is a graph isomorphism. Then ω is also naturally an element of S_n, viewed as a bijection from $[n]$ to itself. Let ω denote both the graph isomorphism and associated permutation.

Our first construction is an isomorphism between the corresponding labeled Cayley graphs. The following Lemma 2.17 states that $\mathcal{G}_\Gamma$ and $\mathcal{G}_{\Gamma'}$ are related as graphs by conjugation, and the associated labels are related via the action on ideals induced by the action on polynomials in Equation (2.1).

Lemma 2.17. *Let $\omega\colon \Gamma \to \Gamma'$ be a graph isomorphism. Then $v \mapsto \omega v \omega^{-1}$ is a graph isomorphism $\mathcal{G}_\Gamma \to \mathcal{G}_{\Gamma'}$. Additionally, if $\mathcal{L}$ is the label on $\mathcal{G}_\Gamma$, $\mathcal{L}'$ the label on $\mathcal{G}_{\Gamma'}$, and $(v_1, v_2) \in E(\mathcal{G}_\Gamma)$, then $\mathcal{L}'(\omega v_1 \omega^{-1}, \omega v_2 \omega^{-1}) = \omega\mathcal{L}(v_1, v_2)$.*

Proof. Conjugation is a group automorphism of S_n. Say $(v_1, v_2) \in E(\Gamma)$ and in particular that $v_1^{-1}v_2 = (i,j) \in E(\Gamma)$. Then

$$\begin{aligned}\left(\omega v_1 \omega^{-1}\right)^{-1}\left(\omega v_2 \omega^{-1}\right) &= \omega v_1^{-1} v_2 \omega^{-1}\\ &= \omega(i,j)\omega^{-1}\\ &= (\omega(i), \omega(j)) \in E(\Gamma').\end{aligned}$$

Thus, conjugation by ω defines a graph isomorphism $\mathcal{G}_\Gamma \to \mathcal{G}_{\Gamma'}$. For the labels on $\mathcal{G}_\Gamma$ and $\mathcal{G}_{\Gamma'}$, the computation above also shows that if $v_1 v_2^{-1} = (p,q)$, then $\left(\omega v_1 \omega^{-1}\right)\left(\omega v_2 \omega^{-1}\right)^{-1} = (\omega(p), \omega(q))$. It follows that $\left(\omega v_1 \omega^{-1}, \omega v_2 \omega^{-1}\right) \in E(\mathcal{G}_{\Gamma'})$ is labeled $\left\langle t_{\omega(p)} - t_{\omega(q)} \right\rangle = \omega\left\langle t_p - t_q \right\rangle$. The claim follows. $\square$

Define $\Omega\colon \mathcal{M}_\Gamma \to \mathcal{M}_{\Gamma'}$ by $\Omega(\bar{\rho})(v) := \omega\bar{\rho}\left(\omega^{-1} v \omega\right)$. The following Proposition 2.18 proves Ω is a ring isomorphism and is actually a consequence of Lemma 2.17 and a more general Proposition of Gilbert, Tymoczko and Viel [15, Prop 2.7]. We include the proof here for completeness.

Proposition 2.18. *The map $\Omega\colon \mathcal{M}_\Gamma \to \mathcal{M}_{\Gamma'}$ is a ring isomorphism.*

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Proof. Let $\bar{\rho} \in \mathcal{M}_\Gamma$. First, we show that $\Omega(\bar{\rho}) \in \mathcal{M}_{\Gamma'}$. Let $(v_1, v_2) \in E(\mathcal{G}_{\Gamma'})$. By Lemma 2.17, there is an edge $(\omega^{-1}v_1\omega, \omega^{-1}v_2\omega) \in E(\mathcal{G}_\Gamma)$, and so $\bar{\rho}(\omega^{-1}v_1\omega) - \bar{\rho}(\omega^{-1}v_2\omega) \in \mathcal{L}(\omega^{-1}v_1\omega, \omega^{-1}v_2\omega)$. Now we have

$$\begin{aligned}\Omega(\bar{\rho})(v_1) - \Omega(\bar{\rho})(v_2) &= \omega\bar{\rho}\Big(\omega^{-1}v_1\omega\Big) - \omega\bar{\rho}\Big(\omega^{-1}v_2\omega\Big)\\ &= \omega\Big(\bar{\rho}\Big(\omega^{-1}v_1\omega\Big) - \bar{\rho}\Big(\omega^{-1}v_2\omega\Big)\Big)\\ &\in \omega\mathcal{L}\Big(\omega^{-1}v_1\omega, \omega^{-1}v_2\omega\Big) = \mathcal{L}'(v_1, v_2).\end{aligned}$$

Thus, $\Omega(\bar{\rho}) \in \mathcal{M}_{\Gamma'}$. It is easy to verify that this map is a ring homomorphism, and the inverse from $\mathcal{M}_{\Gamma'}$ to $\mathcal{M}_\Gamma$ is constructed in the same manner with the map $\omega^{-1} : \Gamma' \to \Gamma$. □

The following lemma gives three instances in which Ω is also a module isomorphism between $\mathcal{M}_\Gamma$ and $\mathcal{M}_{\Gamma'}$.

Lemma 2.19. *The ring isomorphism Ω is a module isomorphism from $\mathcal{M}_\Gamma$ to $\mathcal{M}_{\Gamma'}$ with respect to the following actions:*

1. *the left $\mathbb{C}[t_\bullet]$-action on $\mathcal{M}_\Gamma$ to the ω-twisted left $\mathbb{C}[t_\bullet]$-action on $\mathcal{M}_{\Gamma'}$,*
2. *the right $\mathbb{C}[t_\bullet]$-action on $\mathcal{M}_\Gamma$ to the ω-twisted right $\mathbb{C}[t_\bullet]$-action on $\mathcal{M}_{\Gamma'}$, and*
3. *the dot action of S_n on $\mathcal{M}_\Gamma$ to the ω-twisted dot action of S_n on $\mathcal{M}_{\Gamma'}$.*

Proof. Both $\mathbb{C}[t_\bullet]$-module statements follow from two straightforward computations,

$$\Omega(\bar{t}_i)(v) = \bar{t}_{\omega(i)}(v) \quad \text{and} \quad \Omega(\bar{x}_i)(v) = \bar{x}_{\omega(i)}(v).$$

Say for the left action, if $f \in \mathbb{C}[t_\bullet]$ and $\bar{\rho} \in \mathcal{M}_\Gamma$, then $\Omega(f(t_1, \ldots, t_n).\bar{\rho}) = f(\bar{t}_{\omega(1)}, \ldots, \bar{t}_{\omega(n)})\Omega(\bar{\rho})$, precisely the twisted action. The same holds for the right $\mathbb{C}[t_\bullet]$-action to the ω-twisted right $\mathbb{C}[t_\bullet]$-action. It is easy to show that ring isomorphism Ω^{-1} is the inverse for Ω as a $\mathbb{C}[t_\bullet]$-module morphism for both pairs of actions, and so Ω is a $\mathbb{C}[t_\bullet]$-module isomorphism as in (1) and (2).

Given the dot action on $\mathcal{M}_\Gamma$, the induced action of $u \in S_n$ on $\bar{\rho} \in \mathcal{M}_{\Gamma'}$ is

$$(u, \bar{\rho}) := \Omega\Big(u \cdot \Omega^{-1}(\bar{\rho})\Big).$$

To check that the action is compatible with multiplication of elements $v, u \in S_n$, compute

$$\begin{aligned}(v, (u, \bar{\rho})) &= \Big(v, \Omega\Big(u \cdot \Omega^{-1}(\bar{\rho})\Big)\Big)\\ &= \Omega\Big(v \cdot \Omega^{-1}\Omega\Big(u \cdot \Omega^{-1}(\bar{\rho})\Big)\Big)\\ &= \Omega\Big(v \cdot \Big(u \cdot \Omega^{-1}(\bar{\rho})\Big)\Big)\\ &= \Omega\Big(vu \cdot \Omega^{-1}(\bar{\rho})\Big)\\ &= (vu, \bar{\rho}).\end{aligned}$$

Now compute for $u, v \in S_n$ that

$$\begin{aligned}(u, \bar{\rho})(v) &= \Omega\Big(u \cdot \Omega^{-1}(\bar{\rho})\Big)(v)\\ &= \omega\Big(u \cdot \Omega^{-1}(\bar{\rho})\Big)(\omega^{-1}v\omega)\\ &= \omega u\Big(\Omega^{-1}(\bar{\rho})\Big)(u^{-1}\omega^{-1}v\omega)\end{aligned}$$

$$= \omega u \omega^{-1} \bar{\rho}(\omega u^{-1} \omega^{-1} v)$$
$$= \omega u \omega^{-1} \cdot \bar{\rho}(v).$$

This is precisely the ω-twisted dot action of u on $\mathcal{M}_{\Gamma'}$. Again, by computing with Ω^{-1}, it follows that Ω is an S_n-module isomorphism. □

Note that any generating set for the left or right $\mathbb{C}[t_\bullet]$-module structures on $\mathcal{M}_\Gamma$ must necessarily be generators for the ω-twisted versions as well. As such, when searching for generators, we may choose any graph isomorphic to Γ for explicit calculations.

Proposition 2.20. *If Γ and Γ' are isomorphic, then* $\mathbf{ch}(\mathrm{L}_\Gamma) = \mathbf{ch}(\mathrm{L}_{\Gamma'})$ *and* $\mathbf{ch}(\mathrm{R}_\Gamma) = \mathbf{ch}(\mathrm{R}_{\Gamma'})$.

Proof. Let ω be an isomorphism from Γ to Γ'. The S_n-isomorphism Ω from Lemma 2.19 (3) preserves the ideal $\langle \bar{t}_1, \ldots, \bar{t}_n \rangle$ from $\mathcal{M}_\Gamma$ to $\mathcal{M}_{\Gamma'}$. Thus, we have a S_n-module isomorphism from L_Γ to the ω-twisted $\mathrm{L}_{\Gamma'}$. Twisting by ω is an inner automorphism of S_n, so the ω-twisted $\mathrm{L}_{\Gamma'}$ is in turn isomorphic to the untwisted $\mathrm{L}_{\Gamma'}$ as an S_n-representation. The exact same argument holds for R_Γ and $\mathrm{R}_{\Gamma'}$. Isomorphic representations have identical traces (i.e., equal characters), and the equalities follow. □

By Proposition 2.20, we may consider any graph isomorphic to Γ when calculating $\mathbf{ch}(\mathrm{L}_\Gamma)$ and $\mathbf{ch}(\mathrm{R}_\Gamma)$.

Corollary 2.21. *The graded symmetric functions* $\mathbf{ch}(\mathrm{L}_\Gamma)$ *and* $\mathbf{ch}(\mathrm{R}_\Gamma)$ *are invariants of simple graphs.*

3. Module structure of $\mathcal{M}_\Gamma$

This section establishes some algebraic properties of $\mathcal{M}_\Gamma$ as a module over the polynomial ring $\mathbb{C}[t_\bullet]$. It begins with two results: one that establishes the size of a minimal homogeneous $\mathbb{C}[t_\bullet]$-module generating set as an invariant of Γ and a second that proves the module generated by constant and linear splines is a free module over $\mathbb{C}[t_\bullet]$. This section continues with subsection 3.1, which establishes an algebraic relation that must be satisfied by elements of $\mathcal{M}_\Gamma$. This section ends with subsection 3.2, which gives an explicit and combinatorially meaningful generating set of $\mathcal{M}_\Gamma$ as a $\mathbb{C}[t_\bullet]$-module when Γ is a tree (proving Theorem 1.2).

Before continuing, we will briefly describe what is already known in the geometric case. If Γ is a Hessenberg graph, then

- $\mathcal{M}_\Gamma$ is a free $\mathbb{C}[t_\bullet]$-module with a combinatorial formula for its rank-generating function [13], and furthermore,
- $\mathcal{M}_\Gamma$ has explicit upper-triangular generators are achieved from a Białynicki-Birula decomposition of the corresponding variety [9, 13].

In the geometric case, the rank-generating function is equivalent (substituting $q \mapsto q^2$) to the Poincaré polynomial of the corresponding variety. If Γ is not in the geometric case, then $\mathcal{M}_\Gamma$ is not always a free module. We now prove that the number of generators in each degree of a homogeneous generating set is still an invariant of Γ. We compute the minimal number of linear generators for $\mathcal{M}_\Gamma$ in Section 8.

A generating set F of a finitely generated $\mathbb{C}[t_\bullet]$-module M is minimal if there exists a collection of polynomials $\{c_f \mid f \in F\} \subset \mathbb{C}[t_\bullet]$ such that $\sum_{f \in F} c_f . f = 0$. Then $c_f \notin \mathbb{C} \setminus \{0\}$ for all $f \in F$ (i.e., no c_f is a unit). In other words, no proper subset of F generates M. If M is graded, then a set F is homogeneous if every element $f \in F$ is homogeneous.

The following lemma is known, essentially as a corollary to the graded Nakayama lemma, and holds in greater generality (i.e., for other graded rings over a field). We include a proof for completeness.

Lemma 3.1. *Let M be a finitely generated $\mathbb{N}$-graded module over $\mathbb{C}[t_\bullet]$. Then every minimal homogeneous generating set has the same number of elements of each degree.*

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Proof. Let $I = \langle t_\bullet \rangle$ be the irrelevant ideal. As $\mathbb{C}[t_\bullet]/_I \cong \mathbb{C}$, the quotient $M/_{IM}$ is a graded $\mathbb{C}$-module. In particular, $M/_{IM}$ is a graded $\mathbb{C}$-vector space of dimension $(d_0, \dots, d_n)$, and we will prove that any homogeneous minimal generating set for M projects to a graded basis in $M/_{IM}$. Let $F \coloneqq \left\{ f_k^i \mid 0 \le i \le N,\ k \in [K_i],\ \deg(f_k^i) = i \right\}$ be a minimal homogeneous generating set of M with K_i elements of degree i. It is easy to reason that $n = N$ since an element in M of degree greater than N is in $IF = IM$ (so $n \le N$), and if $f_1^N \in IM$, then F is not minimal (so $n \ge N$). In fact, if $f \in F$ and $f \in IM$, then F is not minimal (f would be in the $\mathbb{C}[t_\bullet]$-span of lower degree elements of F), and moreover, since we are assuming that F is minimal, we know that the image of $f \in F$ in $M/_{IM}$ is nonzero. We will show that $K_i = d_i$ for all $i = 1, \dots, N$. Let $\pi \colon M \to M/_{IM}$ be the quotient map.

We know $\pi(F)$ is a homogeneous spanning set for the graded vector space $M/_{IM}$. Since $M/_{IM}$ is a graded vector space, we may prove linear independence degree-by-degree. Say for $c_1, \dots, c_{K_i} \in \mathbb{C}$ that $\sum_{k=1}^{K_i} c_k \pi(f_k^i) = 0$. We will show that $c_1 = \cdots = c_{K_i} = 0$. It follows that $\pi\left(\sum_{k=1}^{K_i} c_k f_k^i\right) = 0$, and so $\sum_{k=1}^{K_i} c_k f_k^i \in IM$. So there exists some finite set P that indexes two subsets $\{r_p \mid p \in P\} \subset I$ and $\{h_p \mid P \in P\} \subset M$ such that $\sum_{k=1}^{K_i} c_k f_k^i = \sum_{p\in P} r_p.h_p$. Since $\sum_{k=1}^{K_i} c_k f_k^i$ is homogeneous of degree i, it suffices to consider only the i-th graded piece of each element $r_p h_p$.

Say $i = 0$. Since each $r_p \in I$ has no degree 0 component, neither does $r_p h_p$, so $\sum_{k=1}^{K_0} c_k f_k^0 = 0$. Since F is a minimal generating set for M, it follows that $c_1 = \cdots = c_{k_0} = 0$.

Now say $i > 0$. Each h_p is degree at most $i-1$, so $\sum_{p\in P} r_p.h_p \in \mathbb{C}[t_\bullet]\{f_q^j \mid 0 \le j < i,\ q \in [K_j]\}$. So

$$\sum_{k=1}^{K_i} c_k f_k^i = \sum_{p\in P} r_p h_p = \sum_{\substack{0\le j<i \\ 1\le q\le K_j}} c_{j,q}(t_\bullet) f_q^j.$$

This is a relation in M of elements from F and thus cannot have any nonzero constant coefficients, so $c_1 = \cdots = c_{k_i} = 0$. Thus, $\{\pi(f_k^i) \mid k \in [K_i]\}$ is a basis of the i-th graded piece of the vector space $M/_{IM}$, and so $K_i = d_i$ is independent of the choice of F. □

The proof of Lemma 3.1 also ensures that a minimal graded generating set of $\mathcal{M}_\Gamma$ with respect to either the left or right module structure projects to a basis of L_Γ or R_Γ, respectively.

Lemma 3.2 below shows that the first graded piece of the $\mathbb{C}[t_\bullet]$-module is free. Note Lemma 3.2 is independent of the polynomial action chosen (e.g., left, right, and twisted alternatives).

Lemma 3.2. *The $\mathbb{C}[t_\bullet]$-submodule $\mathcal{M}_\Gamma^{\le 1}$ generated by the constant and linear splines on $\mathcal{G}_\Gamma$ is a free module.*

Proof. Let $(e, w_2, w_3, \dots, w_{n!})$ be a linear order on S_n, where $\ell_\Gamma(v) < \ell_\Gamma(w)$ implies that $v < w$. Since Γ is connected, if $w \ne e$, there exists $(i, j) \in E(\Gamma)$ such that $w(i, j) < w$.

Let $F \coloneqq \{\bar{\mathbb{1}}, \bar{f}_1, \dots, \bar{f}_k\}$ be a minimal generating set of $\mathcal{M}_\Gamma^{\le 1}$, where each $\bar{f}_1, \dots, \bar{f}_k$ is a linear spline. Then $\{\bar{\mathbb{1}}, \bar{f}_1 - \bar{f}_1(e)\bar{\mathbb{1}}, \dots, \bar{f}_n - \bar{f}_n(e)\bar{\mathbb{1}}\}$ is a homogeneous generating set of the same size and is therefore minimal by Lemma 3.1. This new generating set has the property that the $\bar{\mathbb{1}}$ is the unique spline whose minimal element is e, so assume that $\bar{f}(e) = 0$ for all $f \in F \setminus \{\bar{\mathbb{1}}\}$.

Let $F_v \coloneqq \{f \in F \mid \min(\mathrm{supp}(f)) = v\}$. So $F_e = \{\bar{\mathbb{1}}\}$. We iteratively construct a minimal generating set such that $|F_v| \in \{0, 1\}$ for all $v \in S_n$. Say that $|F_v| \in \{0, 1\}$ for all $v < w$, and $|F_w| \ge 2$. Let $F_w = \{\bar{g}_1, \dots, \bar{g}_r\}$. Since the linear order on S_n is an extension of Γ-length, there exists $(a, b) \in E(\Gamma)$ such that $w(a, b) < w$, and so $\bar{g}(w(a, b)) = 0$ for all $\bar{g} \in F_w$. Thus, there exist $c_1, \dots, c_r \in \mathbb{C}^*$ such that $\bar{g}_i(w) = c_i(t_{w(a)} - w_{w(b)})$. For $j = 2, \dots, r$, the spline $\bar{g}_j - \frac{c_j}{c_1}\bar{g}_1$ is supported strictly above w. Let

$$F' = (F \setminus F_w) \cup \left\{ \bar{g}_1, \bar{g}_2 - \frac{c_2}{c_1}\bar{g}_1, \dots, \bar{g}_r - \frac{c_r}{c_1}\bar{g}_1 \right\}$$

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be still a minimal generating set, and $|F'_v| \in \{0,1\}$ for all $v \leq w$. Iterate this process, letting $F = F'$. Eventually, $|F_v| \in \{0,1\}$ for all $v \in S_n$. In particular, this F is upper triangular with respect to our total order (the minimal element in the support of each spline is unique to that spline), and so F generates a free $\mathbb{C}[t_\bullet]$-module. □

We note that this submodule is precisely where this paper proves h-positivity in Theorem 9.2 and Corollary 9.3.

3.1. Implied conditions on splines

This subsection gives algebraic conditions that an element $\bar{\rho} \in \mathcal{M}_\Gamma$ must satisfy that are not explicitly in the definition. Specifically, given $w, v \in S_n$, we want to infer conditions on $\bar{\rho}(w) - \bar{\rho}(v)$ when (w, v) is not necessarily an edge in $\mathcal{G}_\Gamma$. Let $w, v \in S_n$ and $(w = v_0, v_1, \ldots, v_m = v)$ be a path from w to v in $\mathcal{G}_\Gamma$. Say for each edge (v_{k-1}, v_k) that $v_k v_{k-1}^{-1} = (i_k, j_k)$, so that $\mathcal{L}(v_{k-1}, v_k) = \langle t_{i_k} - t_{j_k} \rangle$ for each $k = 1, \ldots, m$. Then

$$\bar{\rho}(w) - \bar{\rho}(v) = \sum_{k=1}^{m} (\bar{\rho}(v_{k-1}) - \bar{\rho}(v_k)) \in \langle t_{i_k} - t_{j_k} \mid k \in [m] \rangle. \tag{3.1}$$

Define

$$I_B^w := \langle t_{w(i)} - t_{w(j)} \mid (i, j) \in B \subseteq E(T) \rangle. \tag{3.2}$$

Lemma 3.3 below is particularly useful when $\bar{\rho} \in \mathcal{M}_\Gamma$ satisfies $\bar{\rho}(v) = 0$ for some $v \in S_n$. Recall that we identify $B \subset E(\Gamma)$ with a subset of transpositions, and write $w\langle B \rangle$ for the left coset at w of the reflection subgroup generated by the transpositions in B.

Lemma 3.3. *Let T be a spanning tree of Γ. Let $v \in w\langle B \rangle$, where $B \subseteq E(T)$. If $\bar{\rho} \in \mathcal{M}_\Gamma$, then $\bar{\rho}(w) - \bar{\rho}(v) \in I_B^w$.*

Proof. Let $w^{-1}v = b_1 \cdots b_m$, where $b_1, \ldots, b_m \in B$. Let $(w = v_0, v_1, \ldots, v_m = v)$ be the path from w to v, where $v_k^{-1} v_{k-1} = b_k \in B$ for all $k \in [m]$. Say that $v_k v_{k-1}^{-1} = (i_k, j_k)$, so that $\mathcal{L}(v_{k-1}, v_k) = t_{i_k} - t_{j_k}$. By Equation (3.1),

$$\bar{\rho}(w) - \bar{\rho}(v) \in \langle t_{i_k} - t_{j_k} \mid k \in [m] \rangle.$$

For each $k \in [m]$, since $v_k v_{k-1}^{-1} = (i_k, j_k)$, we have that $b_k = v_k^{-1} v_{k-1} = (v_k^{-1}(i_k), v_k^{-1}(j_k))$. Each edge $b_k \in B$, so the integers $v_k^{-1}(i_k) = (wb_1 \cdots b_k)^{-1}(i_k)$ and $v_k^{-1}(j_k) = (wb_1 \cdots b_k)^{-1}(j_k)$ must be in the same connected component of $([n], B)$.

Since $(wb_1 \cdots b_k)^{-1} = (b_k \cdots b_1)w^{-1}$, it follows that $w^{-1}(i_k)$ and $w^{-1}(j_k)$ are vertices in the same connected component of $([n], B)$ for all $k \in [m]$. If $(q_0, \ldots, q_\ell)$ is a path in $([n], B)$ from $q_0 = w^{-1}(i_k)$ to $q_\ell == w^{-1}(j_k)$, then $t_{q_0} - t_{q_\ell} = \sum_{r=1}^{\ell} t_{q_{r-1}} - t_{q_r}$, and thus, $t_{w^{-1}(i_k)} - t_{w^{-1}(j_k)} \in I_B^e$. It follows that $t_{i_k} - t_{j_k} \in I_B^w$ for all $k \in [m]$, and so $\bar{\rho}(w) - \bar{\rho}(v) \in I_B^w$. □

A monomial ideal in $\mathbb{C}[t_\bullet]$ is an ideal I generated by monomials. Monomial ideals are particularly nice when computing intersections; if $I_1 = \langle m_1, \ldots, m_k \rangle$ and $I_2 = \langle n_1, \ldots, n_\ell \rangle$ are both monomial ideals, then $I_1 \cap I_2 = \langle \operatorname{lcm}(m_i, n_j) \mid i \in [k],\ j \in [\ell] \rangle$.

Let T be a spanning tree of Γ, where $E(T) = \{(a_1, b_1), \ldots, (a_{n-1}, b_{n-1})\}$. Ideals of the form $\langle t_{a_i} - t_{b_i} \mid (a_i, b_i) \in B \subset E(T) \rangle$ can be considered monomial ideals, via the graded automorphism

$$\mathbb{C}[t_1, \ldots, t_n] \cong \mathbb{C}[t_{a_1} - t_{b_1}, \ldots, t_{a_{n-1}} - t_{b_{n-1}}, t_n]$$

defined by $t_i \mapsto \begin{cases} t_{a_i} - t_{b_i} & \text{if } i \in [n-1] \\ t_n & \text{if } i = n \end{cases}$. Since $t_i \mapsto t_{w(i)}$ is also a graded automorphism of $\mathbb{C}[t_\bullet]$, the ideals I_B^w from Equation 3.2 can also be considered as monomial ideals (taking care to **fix** T and

$w \in S_n$). We will fix T and w and then treat ideals of the form I_B^w as monomial ideals to compute intersections in the proofs of Theorem 3.7 and Lemma 4.1 in the following two sections.

3.2. *Coset splines and trees*

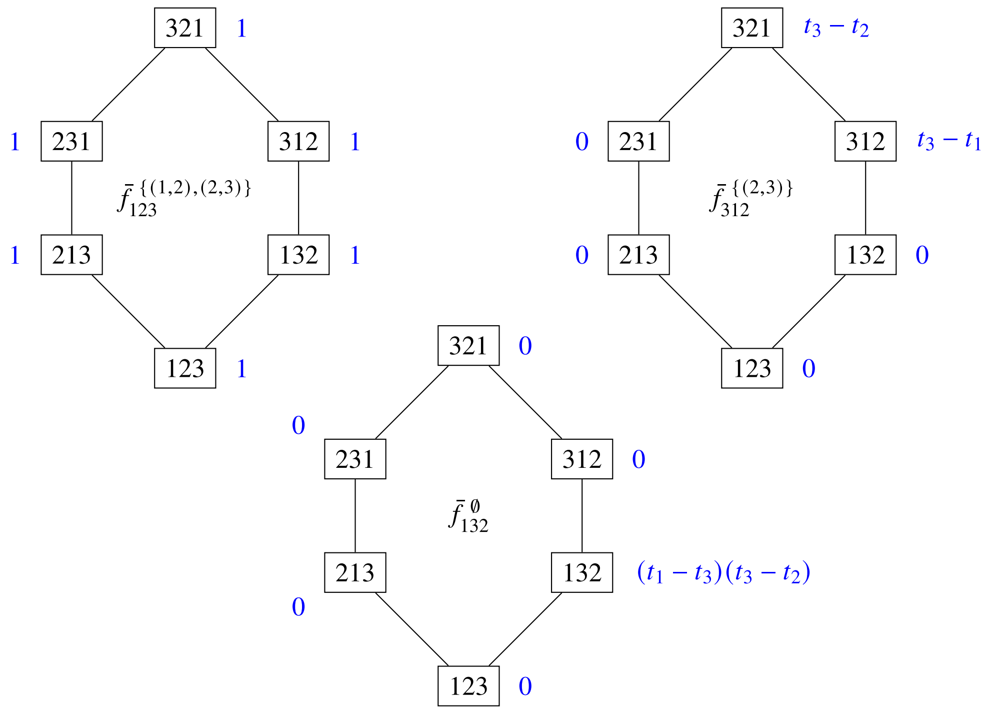
This subsection establishes a set of splines called *coset splines* (Definition 3.4) that generate $\mathcal{M}_\Gamma$ as a module over the polynomial ring when Γ is a tree. This subsection also identifies a subset of those coset splines that generate $\mathcal{M}_\Gamma$ as a ring when Γ is a tree.

Definition 3.4. Let Γ be a tree, $E := E(\Gamma)$ and $B \subseteq E$. The *coset spline at the identity* $\bar{f}_e^B : S_n \to \mathbb{C}[t_\bullet]$ is

$$\bar{f}_e^B(w) := \begin{cases} \prod_{(i,j)\in E\setminus B} \left(t_{w(i)} - t_{w(j)}\right) & w \in \langle B\rangle \\ 0 & \text{otherwise} \end{cases}$$

The *coset spline at* w is $\bar{f}_w^B := w \cdot \bar{f}_e^B$. We adopt the conventions that a product over the empty set $\emptyset$ is 1 (so $f_w^\emptyset = \mathbb{1}$) and that the subgroup generated by the empty set is the identity (so $\langle\emptyset\rangle = \{e\}$).

Example 3.5. Again, consider $\Gamma = ([3], \{(1,2), (2,3)\})$. Drawn below are three examples of coset splines on $\mathcal{G}_\Gamma$.



Lemma 3.6. *When Γ is a tree, coset splines are elements of $\mathcal{M}_\Gamma$. Additionally, if $w, v \in S_n$ are in the same coset of $\langle B\rangle$, then $\bar{f}_w^B = \bar{f}_v^B$.*

Proof. It suffices to show $\bar{f}_e^B$ is a spline. Let $w \in \langle B\rangle$ and $v \in S_n$, where $vw^{-1} = (i,j) \in E$.

If $(i,j) \in E \setminus B$, then $v \notin \langle B\rangle$, and so $\bar{f}_e^B(v) = 0$. Thus, $\bar{f}_e^B(w) - \bar{f}_w^B(v) = t_{w(i)} - t_{w(j)} \in \mathcal{L}(w,v)$, as desired.

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If $(i,j) \in B$, then

$$\begin{aligned}
\bar{f}_e^B(w) - \bar{f}_e^B(v) &= \prod_{(r_1,s_1)\in E\setminus B} \left(t_{w(r_1)} - t_{w(s_1)}\right) - \prod_{(r_2,s_2)\in E\setminus B} \left(t_{w(i,j)(r_2)} - t_{w(i,j)(s_2)}\right) \\
&= w\left(\prod_{(r_1,s_1)\in E\setminus B} \left(t_{r_1} - t_{s_1}\right) - (i,j)\left(\prod_{(r_2,s_2)\in E\setminus B} \left(t_{r_2} - t_{s_2}\right)\right)\right) \\
&= w\left(\sum_{0\le p,q} g_{pq}(t_\bullet) t_i^p t_j^q - (i,j)\left(\sum_{0\le r,s} g_{rs}(t_\bullet) t_i^r t_j^s\right)\right) \\
&= w\left(\sum_{0\le p,q} g_{pq}(t_\bullet)(t_i^p t_j^q - t_i^q t_j^p)\right).
\end{aligned}$$

As $t_i^p t_j^q - t_i^q t_j^p \in \langle t_i - t_j \rangle$, it follows that $\bar{f}_e^B(w) - \bar{f}_e^B(v) \in \langle t_{w(i)} - t_{w(j)} \rangle = \mathcal{L}(w,v)$. Thus, $\bar{f}_e^B$ is a spline.

Now we prove that coset splines are uniquely determined by the coset. For all $u \in \langle B \rangle$, we have

$$\begin{aligned}
u \cdot \bar{f}_e^B(w) &= \begin{cases} u\left(\prod_{(i,j)\in E-B} \left(t_{u^{-1}w(i)} - t_{u^{-1}w(j)}\right)\right) & u^{-1}w \in \langle B \rangle \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \prod_{(i,j)\in E-B} \left(t_{w(i)} - t_{w(j)}\right) & w \in \langle B \rangle \\ 0 & \text{otherwise} \end{cases} \\
&= \bar{f}_e^B(w).
\end{aligned}$$

If $w\langle B \rangle = v\langle B \rangle$, then $w = vu$, for some $u \in \langle B \rangle$, and so

$$\bar{f}_w^B = w \cdot \bar{f}_e^B = (vu) \cdot \bar{f}_e^B = v \cdot (u \cdot \bar{f}_e^B) = v \cdot \bar{f}_e^B = \bar{f}_v^B. \qquad \square$$

Note that the following Theorem 3.7 is independent of the left or right module structure.

Theorem 3.7. *Let Γ be a tree. The set of coset splines $\{\bar{f}_w^B \mid w \in S_n,\ B \subseteq E(\Gamma)\}$ is a $\mathbb{C}[t_\bullet]$-generating set of $\mathcal{M}_\Gamma$.*

Proof. Let $\bar{\rho} \in \mathcal{M}_\Gamma$, we will show that $\bar{\rho} \in \mathbb{C}[t_\bullet]\{\bar{f}_B^w \mid w \in S_n, B \subseteq E(\Gamma)\}$ by induction on containment of the support $\operatorname{supp}(\bar{\rho})$. If $\bar{\rho} \equiv 0$, this is clearly in the span of the coset splines, and the base case $\operatorname{supp}(\bar{\rho}) = \emptyset$ is done. Otherwise, $\operatorname{supp}(\bar{\rho}) \neq \emptyset$, and we assume all splines $\bar{\kappa}$ where $\operatorname{supp}(\bar{\kappa}) \subsetneq \operatorname{supp}(\bar{\rho})$ are in $\mathbb{C}[t_\bullet]\{\bar{f}_B^w \mid w \in S_n, B \subseteq E(\Gamma)\}$. Replacing $\bar{\rho}$ by $\bar{\rho} - \bar{\rho}(e)\bar{\mathbb{1}}$ if necessary, we assume $\bar{\rho}(e) = 0$. This also handles the case where $\operatorname{supp}(\bar{\rho}) = S_n$.

Fix $w \in S_n$ such that $\bar{\rho}(w) \neq 0$ and w is adjacent in $\mathcal{G}_\Gamma$ to some $w' \in S_n$ where $\bar{\rho}(w') = 0$. Define

$$\mathcal{B}_w := \{B \mid B \subset E(\Gamma),\ \ \exists v \in w\langle B \rangle \text{ such that } \bar{\rho}(v) = 0\}.$$

Since $\bar{\rho}(w') = 0$, this set is nonempty. Each element in $\mathcal{B}_w$ is a generating set for a reflection subgroup whose left coset at w contains an element not in $\operatorname{supp}(\bar{\rho})$. Note if $B \subset B'$ and $B \in \mathcal{B}_w$, then $B' \in \mathcal{B}_w$.

By Lemma 3.3,

$$\bar{\rho}(w) \in \bigcap_{B\in\mathcal{B}_w} I_B^w = \bigcap_{B\in\mathcal{B}_w} \left\langle t_{w(i)} - t_{w(j)} \mid (i,j) \in B \subset E(\Gamma) \right\rangle =: \mathcal{I}_\rho^w. \tag{3.3}$$

Following the logic of Subsection 3.1 (i.e., treating $\{t_{w(i)} - t_{w(j)} \mid (i,j) \in E(\Gamma)\}$ as variables), $\mathcal{I}_\rho^w$ is a monomial ideal generated by the monomials that are contained within *every* element of the intersection.

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A monomial $\mathfrak{m} = \prod_{(i,j)\in E} (t_{w(i)} - t_{w(j)})^{\alpha_{ij}}$ is contained within the ideal $\mathcal{I}_\rho^w$ if and only if for every $B \in \mathcal{B}_w$, there is at least one $(i, j) \in B$ such that $\alpha_{ij} > 0$. For generators of $\mathcal{I}_\rho^w$, it suffices to consider only those monomials such that $\alpha_{ij} \in \{0, 1\}$ for all $(i, j) \in E(\Gamma)$. Since $\alpha_{ij} \in \{0, 1\}$, the monomials that generate $\mathcal{I}_\rho^w$ are a subset of $\{\bar{f}_w^B(w) \mid B \subseteq E(\Gamma)\}$. In particular, we have the equality $\langle \bar{f}_w^D(w) \mid D \subseteq E(\Gamma),\ \bar{f}_w^D(w) \in \mathcal{I}_\rho^w \rangle = \mathcal{I}_\rho^w$.

Consider the coset splines $\{\bar{f}_w^D \mid \bar{f}_w^D(w) \in \mathcal{I}_\rho^w\}$. By definition, for any $D \subset E(\Gamma)$,

$$\bar{f}_w^D(w) = \prod_{(i,j)\in E(\Gamma)\setminus D} t_{w(i)} - t_{w(j)}.$$

Now $\bar{f}_w^D(w) \in \mathcal{I}_\rho^w$ if and only if $(E(\Gamma) \setminus D) \cap B \neq \emptyset$ for all $B \in \mathcal{B}_w$. Thus, $\bar{f}_w^D(w) \in \mathcal{I}_\rho^w$ if and only if $B \not\subset D$ for all $B \in \mathcal{B}_w$. Since $\mathcal{B}_w$ is closed under supersets, $\bar{f}_w^D(w) \in \mathcal{I}_\rho^w$ if and only if $D \notin \mathcal{B}_w$. Thus,

$$\bar{\rho}(w) \in \mathcal{I}_\rho^w = \langle \bar{f}_w^D(w) \mid \text{ for all } v \in w\langle D\rangle,\ \bar{\rho}(v) \neq 0 \rangle.$$

Let $\bar{f} \in \mathbb{C}[t_\bullet]\{\bar{f}_w^D \mid \text{ for all } v \in w\langle D\rangle,\ \bar{\rho}(v) \neq 0\}$ such that $\bar{\rho}(w) = \bar{f}(w)$ (a different $\bar{f}$ may be chosen for the left and right module structure, but either way, such a $\bar{f}$ exists since it is only required to agree with $\bar{\rho}$ at w). Since $\operatorname{supp}(\bar{f}) \subseteq \operatorname{supp}(\bar{\rho})$ and $\bar{f}(w) = \bar{\rho}(w) \neq 0$, it follows that $\operatorname{supp}(\bar{\rho}) \supsetneq \operatorname{supp}(\bar{\rho} - \bar{f})$. Thus, $\bar{\rho} - \bar{f} \in \mathbb{C}[t_\bullet]\{\bar{f}_B^w \mid w \in S_n, B \subseteq E(\Gamma)\}$. Since $\bar{f}$ is also a sum of coset splines, $\bar{\rho} \in \mathbb{C}[t_\bullet]\{\bar{f}_B^w \mid w \in S_n, B \subseteq E(\Gamma)\}$. □

The collection of all coset splines is not a minimal generating set. One might significantly decrease the size of this set by fixing the linear order on S_n in the proof of Lemma 3.2, and only considering the largest (by support) coset splines supported 'above' a permutation. There is no guarantee that these generators are minimal for all degrees, but it is easy to reason that this collection is minimal for the module $\mathcal{M}_\Gamma^{\leq 2}$ generated by the constant, linear and quadratic splines.

We also achieve a generating set for $\mathcal{M}_\Gamma$ as a ring in Corollary 3.8 below.

Corollary 3.8. *Let Γ be a tree. The constant and linear coset splines along with either $\{\bar{t}_i \mid i \in [n]\}$ or $\{\bar{x}_i \mid i \in [n]\}$ generate $\mathcal{M}_\Gamma$ as a ring.*

Proof. It follows immediately from the definition that

$$f_w^B = \prod_{s\in E(\Gamma)\setminus B} f_w^{E(\Gamma)\setminus\{s\}}.$$

So every coset spline except $\bar{\mathbb{1}}$ is a product of linear coset splines, which generate $\mathcal{M}_\Gamma$ together with either $\{\bar{t}_i \mid i \in [n]\}$ or $\{\bar{x}_i \mid i \in [n]\}$ by Theorem 3.7. □

We can leverage Theorem 3.7 to compute $\mathcal{M}_\Gamma$ for all graphs Γ. Any graph Γ can be expressed as the union of spanning trees $\Gamma = T_1 \cup \cdots \cup T_k$. Lemma 2.16 says that $\mathcal{M}_\Gamma = \bigcap_{i=1}^k \mathcal{M}_{T_i}$, and Theorem 3.7 gives explicit generators for each $\mathcal{M}_{T_i}$. This is most useful in computer calculations, where the task of constructing modules from generators and intersecting them can be completed by a computer algebra system.

4. Connectedness and $\mathcal{M}_\Gamma^k$

This section proves an equivalence between the k-connectivity of Γ and which graded pieces of the representation $\mathbf{ch}(\mathrm{L}_\Gamma)$ are trivial.

Lemma 4.1 below infers new conditions on $\mathcal{M}_\Gamma$ from collections of vertex-disjoint paths in Γ.

Lemma 4.1. *Say that there exist k vertex-disjoint paths from i to j in Γ. Let $\Gamma' = ([n], E(\Gamma) \cup \{(i, j)\})$. Then $\mathcal{M}_\Gamma^{k-1} = \mathcal{M}_{\Gamma'}^{k-1}$.*

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Proof. We will show both directions of containment. Clearly, $\mathcal{M}_\Gamma^{k-1} \supseteq \mathcal{M}_{\Gamma'}^{k-1}$.

Say that $\bar{\rho} \in \mathcal{M}_\Gamma^{k-1}$, and let $w, v \in S_n$ such that $w^{-1}v = (i, j)$. Let $p_r = (i, s_{r,1}, \ldots, s_{r,\ell_r}, j)$ for $r = 1, \ldots, k$ be the k vertex-disjoint paths from i to j in Γ. By Lemma 3.3,

$$\begin{aligned}\bar{\rho}(w) - \bar{\rho}(v) &\in \left\langle t_{w(i)} - t_{w(s_{r,1})}, t_{w(s_{r,1})} - t_{w(s_{r,2})}, \ldots, t_{w(s_{r,\ell_r})} - t_{w(j)} \right\rangle \\ &= \left\langle t_{w(i)} - t_{w(j)}, t_{w(i)} - t_{w(s_{r,1})}, \ldots, t_{w(s_{r,\ell_r-1})} - t_{w(s_{r,\ell_r})} \right\rangle\end{aligned}$$

for all $r = 1, \ldots, k$. Since the paths $p_1, \ldots, p_k$ are vertex independent, the set of edges

$$A = \{(i, j)\} \cup \bigcup_{r=1}^{k} \{(i, s_{r,1}), (s_{r,1}, s_{r,2}), \ldots, (s_{r,\ell_r-1}, s_{r,\ell_r})\}$$

contains no cycles and thus forms a tree. In particular, we may consider $\{t_a - t_b \mid (a, b) \in A\}$ as monomials in $\mathbb{C}[t_\bullet]$. Let $s_{r,0} := i$ when it is convenient for indexing. It remains to compute

$$\begin{aligned}\bar{\rho}(w) &\in \bigcap_{r=1}^{k} \left\langle t_{w(i)} - t_{w(j)}, t_{w(s_{r,0})} - t_{w(s_{r,1})}, \ldots, t_{w(s_{r,\ell_r-1})} - t_{w(s_{r,\ell_r})} \right\rangle \\ &= \left\langle t_{w(i)} - t_{w(j)}, \prod_{r=1}^{k} t_{w(s_{r,m_r-1})} - t_{w(s_{r,m_r})} \,\middle|\, 0 < m_r \leq \ell_r \right\rangle.\end{aligned}$$

Each generator of this ideal is a homogeneous polynomial, one of degree 1 and all others of degree k. Since $\bar{\rho}(w) - \bar{\rho}(v)$ is degree $k-1$, it follows that $\bar{\rho}(w) - \bar{\rho}(v) \in \langle t_{w(i)} - t_{w(j)} \rangle$.

Since v, w were arbitrary such that $w^{-1}v = (i, j)$, we know that $\bar{\rho} \in \mathcal{M}_{\Gamma'}^{k-1}$. Since $\bar{\rho}$ was arbitrary, $\mathcal{M}_\Gamma^{k-1} \subseteq \mathcal{M}_{\Gamma'}^{k-1}$. □

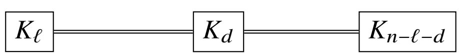
Theorem 4.2. *Let $\Gamma = ([n], E)$ be a connected graph on n vertices. The following are equivalent:*

1. *Γ is k-connected.*
2. *$\mathcal{M}_\Gamma^d = \mathcal{M}_{K_n}^d$ for all $d < k$, where K_n is the complete graph.*
3. *$\mathbf{ch}(\mathrm{L}_\Gamma)_d$ is trivial for all $d < k$.*

Proof. (1) ⇒ (2). If Γ is k-connected, by Menger's theorem, every $(i, j) \in [n] \times [n]$ has k vertex-disjoint paths connecting them in Γ. By Lemma 4.1, $\mathcal{M}_\Gamma^{k-1} = \mathcal{M}_{K_n}^{k-1}$.

(2) ⇒ (3). The ring $\mathcal{M}_{K_n}$ corresponds to the equivariant cohomology of the full flag variety, where the dot action is known to be trivial [28].

(3) ⇒ (1). Assume that Γ is *not* k-connected. Let d be the integer such that Γ is d-connected but not $(d+1)$-connected (so $0 < d < k$). We will show that $\mathbf{ch}(\mathrm{L}_\Gamma)_d$ is not trivial. Then Γ has a cut set of size d, and so Γ is (isomorphic to) a sub-graph of the graph $H = ([n], \{(i, j) \mid 1 \leq i < j < \ell + d \text{ or } \ell < i < j \leq n\})$ drawn below:



(the center K_d is the cut set). The graph H is also d-connected. By (1) ⇒ (2), the graded pieces $\mathcal{M}_\Gamma^p = \mathcal{M}_H^p = \mathcal{M}_{K_n}^p$ for all $0 \leq p < d$. Since Γ is an edge-subgraph of H, it follows directly from the definitions that $\mathcal{M}_\Gamma \supseteq \mathcal{M}_H$.

If $I = \langle t_1, \ldots, t_n \rangle$, then for any graded $\mathbb{C}[t_\bullet]$-module $M := \oplus_{p \geq 0} M^p$, the following equality is by definition

$$\left(M \big/ IM \right)^p = M^p \big/ IM \cap M^p.$$

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Since multiplication by elements in I must increase degree, the d-th degree component of $I\mathcal{M}_\Gamma$ and the d-th degree component of $I\mathcal{M}_H$ are equal. In particular,

$$I\mathcal{M}_\Gamma \cap \mathcal{M}_\Gamma^d = I\left(\mathcal{M}_\Gamma^{\leq d-1}\right) \cap \mathcal{M}_\Gamma^d = I\left(\mathcal{M}_H^{\leq d-1}\right) \cap \mathcal{M}_\Gamma^d = I\mathcal{M}_H \cap \mathcal{M}_\Gamma^d.$$

It follows that for the quotients

$$\left(\mathcal{M}_\Gamma/ I\mathcal{M}_\Gamma\right)^d = \mathcal{M}_\Gamma^d / I\mathcal{M}_\Gamma \cap \mathcal{M}_\Gamma^d = \mathcal{M}_\Gamma^d / I\mathcal{M}_H \cap \mathcal{M}_\Gamma^d = \left(\mathcal{M}_\Gamma / I\mathcal{M}_H\right)^d,$$

we get containment in the vector spaces

$$(\mathrm{L}_\Gamma)_d = \left(\mathcal{M}_\Gamma/ I\mathcal{M}_\Gamma\right)^d = \left(\mathcal{M}_\Gamma/ I\mathcal{M}_H\right)^d \supseteq \left(\mathcal{M}_H/ I\mathcal{M}_H\right)^d = (\mathrm{L}_H)_d.$$

In particular, the representation with character $\mathbf{ch}(\mathrm{L}_H)_d$ is a sub-representation of the representation with character $\mathbf{ch}(\mathrm{L}_\Gamma)_d$. The graph H is in fact a Hessenberg graph, and it is easy to compute with P-tableaux from [25] that the d-th graded piece of $\mathbf{ch}(\mathrm{L}_H)$ is non-trivial, so the d-th graded piece of $\mathbf{ch}(\mathrm{L}_\Gamma)$ contains a nontrivial sub-representation and is thus nontrivial. □

Remark 4.3. The graph H in the proof of Theorem 4.2 is the Hessenberg graph associated to the vector

$$h = (\overbrace{\ell + d, \dots, \ell + d}^{\ell \text{ times}}, n, \dots, n).$$

The 3 + 1– and 2 + 2–free poset P on $[n]$ for which H is the indifference graph has relations $\{i <_P j \mid i \in [\ell],\ j \in \{d + \ell + 1, \dots, n\}\}$.

The following corollary is a consequence of Theorem 4.2.

Corollary 4.4. *If Γ is k-connected, then $\mathbf{ch}(\mathrm{R}_\Gamma)_d$ is equal to $\mathbf{ch}\left(\mathrm{R}_{K_n}\right)_d$, which is the d-th degree piece of the graded regular representation.*

5. Generators for linear splines

The remaining sections are devoted to computing the first degree piece of the graded symmetric functions $\mathbf{ch}(\mathrm{L}_\Gamma)$ and $\mathbf{ch}(\mathrm{R}_\Gamma)$ for all connected graphs Γ. We show that the first degree piece of $\mathcal{M}_\Gamma$ is computable from the data of cut vertices and cut edges, in particular the block-cut tree of Γ (Definition 6.2).

This section defines a set $\mathcal{F}_\Gamma$ that we will eventually show is a $\mathbb{C}$-spanning set for $\mathcal{M}_\Gamma^1$. Subsection 5.1 proves several $\mathbb{C}$-linear relations within the set $\mathcal{F}_\Gamma$ that will turn out to be sufficient for reducing to a basis.

First, we will introduce (in fact, reintroduce) a collection of linear splines that depend on cut edges in Γ. Let $s = (i, j)$ be a cut edge of Γ, and let G_s be one of the two connected components of $([n], E(\Gamma) \setminus \{s\})$. We are free to choose either component; see Remark 5.1 below. For each subset $A \subset [n]$ such that $|A| = |G_s|$, we define $\bar{f}_A^s \colon S_n \to \mathbb{C}[t_\bullet]$ by

$$\bar{f}_A^s(w) := \begin{cases} t_{w(i)} - t_{w(j)} & \text{if } w^{-1}(A) = V(G_s) \\ 0 & \text{otherwise,} \end{cases}$$

for all $w \in S_n$. We associate to Γ the collection

$$\mathcal{C}_\Gamma := \left\{\bar{f}_A^s \mid s \text{ is a cut edge of } \Gamma,\ A \subset [n],\ |A| = |G_s|\right\}.$$

Note that these splines $\bar{f}_A^s$ are actually the linear coset splines from Definition 3.4. We make the change in notation for several reasons, one being that the subset A uniquely determines the coset whose support is

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$\bar{f}_A^s$ (as opposed to many elements w defining the same $\bar{f}_w^B$). Since G_s is one of two connected components in the graph $([n], E(\Gamma) \setminus \{s\})$, it follows that $v \in w\langle E(\Gamma) \setminus \{s\}\rangle$ if and only if $w(V(G_s)) = v(V(G_s))$. In particular, we have equality $\bar{f}_A^s = \bar{f}_w^B$ precisely when $B = E(\Gamma) \setminus \{s\}$ and $w^{-1}(A) = V(G_s)$.

Remark 5.1. When defining $\bar{f}_A^s$, we chose G_s to be one of the two connected components in the graph $([n], E(\Gamma) \setminus \{s\})$. This choice does not affect the set of splines in $\mathcal{C}_\Gamma$. More precisely, if H is the other connected component in $([n], E(\Gamma) \setminus \{s\})$, then $|H| = n - |G_s|$, and we have that

$$\bar{f}_A^s(w) = \begin{cases} t_{w(i)} - t_{w(j)} & \text{if } w^{-1}(A) = V(G_s) \\ 0 & \text{otherwise} \end{cases} = \begin{cases} t_{w(i)} - t_{w(j)} & \text{if } w^{-1}(A^c) = V(H) \\ 0 & \text{otherwise.} \end{cases}$$

In particular, for a fixed cut edge s, the set of linear coset splines $\{\bar{f}_A^s\}$ associated to that cut edge is unaffected by the choice of G_s.

Now we will introduce a (truly new) collection of linear splines that depend on cut vertices j and the connected components of $\Gamma - j$, as well as an integer k. Let $j \vdash \Gamma$ be a cut vertex, G be a connected component of $\Gamma - j$, and $k \in [n]$. We define $\bar{y}_{G,k}^j : S_n \to \mathbb{C}[t_\bullet]$ by

$$\bar{y}_{G,k}^j(w) := \begin{cases} t_k - t_{w(j)} & \text{if } w^{-1}(k) \in G \\ 0 & \text{otherwise} \end{cases}$$

for all $w \in S_n$. We associate to Γ the collection

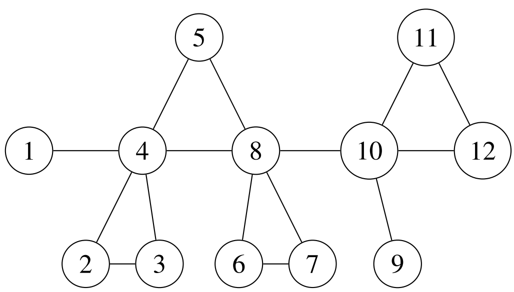
$$\mathcal{Y}_\Gamma := \left\{ \bar{y}_{G,k}^j \middle| j \vdash \Gamma,\ G \text{ a connected component of } \Gamma - j,\ k \in [n] \right\}.$$

Finally, recall the splines $\mathcal{T}_n := \{\bar{t}_i \mid i \in [n]\}$ and $\mathcal{X}_n := \{\bar{x}_i \mid i \in [n]\}$ from Subsection 2.2. Now we define

$$\mathcal{F}_\Gamma := \mathcal{T}_n \cup \mathcal{X}_n \cup \mathcal{C}_\Gamma \cup \mathcal{Y}_\Gamma. \tag{5.1}$$

We will eventually show that $\mathcal{F}_\Gamma$ is a $\mathbb{C}$-spanning set of $\mathcal{M}_\Gamma^1$.

Example 5.2. Let Γ be the graph drawn below.



Since Γ has three cut edges $(1,4)$, $(8,10)$, and $(9,10)$, we have that

$$\mathcal{C}_\Gamma = \left\{ \bar{f}_A^{(1,4)} \mid A \subset [12],\ |A| = 1 \right\} \cup \left\{ \bar{f}_A^{(8,10)} \mid A \subset [12],\ |A| = 8 \right\} \cup \left\{ \bar{f}_A^{(9,10)} \mid A \subset [12],\ |A| = 1 \right\}.$$

One such element $\bar{f}^{(9,10)}_{\{6\}} \in \mathcal{C}_\Gamma$ takes the form

$$\bar{f}^{(9,10)}_{\{6\}}(w) := \begin{cases} t_{w(9)} - t_{w(10)} & w^{-1}(\{6\}) = \{9\} \\ 0 & \text{otherwise,} \end{cases}$$

and is supported on the coset $\{w \in S_n \mid w(9) = 6\}$.

Since Γ has three cut vertices 4, 8, and 10, we have that

$$\mathcal{Y}_\Gamma = \left\{ \bar{y}^4_{G,k} \middle| V(G) \in \left\{ \begin{matrix} \{1\}, \\ \{2,3\}, \\ \{5,\ldots,12\} \end{matrix} \right\},\ k \in [12] \right\} \cup \left\{ \bar{y}^8_{G,k} \middle| V(G) \in \left\{ \begin{matrix} \{1,..,,5\}, \\ \{6,7\}, \\ \{9,\ldots,12\} \end{matrix} \right\},\ k \in [12] \right\}$$
$$\cup \left\{ \bar{y}^{10}_{G,k} \middle| V(G) \in \left\{ \begin{matrix} \{1,\ldots,8\}, \\ \{9\}, \\ \{11,12\} \end{matrix} \right\},\ k \in [12] \right\}.$$

One such element $\bar{y}^8_{G,3} \in \mathcal{Y}_\Gamma$ takes the form

$$\bar{y}^8_{\{6,7\},3}(w) := \begin{cases} t_3 - t_{w(8)} & w^{-1}(3) \in \{6,7\} \\ 0 & \text{otherwise} \end{cases}$$

and is supported on the set $\{w \in S_n \mid w(6) = 3 \text{ or } w(7) = 3\}$.

Now Lemma 5.3 below shows that $\mathcal{F}_\Gamma$ is in fact a subset of $\mathcal{M}^1_\Gamma$.

Lemma 5.3. *Let Γ be a graph on $[n]$. The four sets $\mathcal{T}_n$, $\mathcal{X}_n$, $\mathcal{C}_\Gamma$ and $\mathcal{Y}_\Gamma$ are subsets of $\mathcal{M}_\Gamma$.*

Proof. We already know that $\bar{t}_i$ and $\bar{x}_i$ are elements of $\mathcal{M}_\Gamma$ for all $i \in [n]$, so $\mathcal{T}_n$ and $\mathcal{X}_n$ are subsets.

Now we show that each element of $\mathcal{C}_\Gamma$ is a well-defined spline. Recall that these are coset splines, and so are well defined for trees. If s is a cut edge of Γ, then every spanning tree T of Γ must have s as an edge. Fix $A \subset [n]$, where $|A| = |G_s|$, and for all spanning trees T, choose T_s to be the connected component where $V(T_s) = V(G_s)$. It follows that $\bar{f}^s_A \in \mathcal{M}_T$ for every spanning tree T, and so $\bar{f}^s_A \in \mathcal{M}_\Gamma$ by Lemma 2.16.

Finally, we show that every element $\bar{y}^j_{G,k} \in \mathcal{Y}_\Gamma$ is a linear spline on $\mathcal{G}_\Gamma$. We will verify this from the definition, edge by edge. Let $(w,v) \in E(\mathcal{G}_\Gamma)$, where $w = v(p,q)$. We prove that $\bar{y}^j_{G,k}(w) - \bar{y}^j_{G,k}(v) \in \mathcal{L}(w,v) = \langle t_{v(p)} - t_{v(q)} \rangle$ in three cases, depending on the values of $w^{-1}(k)$ and $v^{-1}(k)$.

Case 1: $w^{-1}(k), v^{-1}(k) \notin G$. Then by definition both $\bar{y}^j_{G,k}(w) = 0$ and $\bar{y}^j_{G,k}(v) = 0$, so the difference is clearly in $\mathcal{L}(w,v)$.

Case 2: $w^{-1}(k), v^{-1}(k) \in G$. So $\bar{y}^j_{G,k}$ contains both w and v in its support. We compute from the definition that

$$\begin{aligned} \bar{y}^j_{G,k}(w) - \bar{y}^j_{G,k}(v) &= t_k - t_{w(j)} - t_k + t_{v(j)} \\ &= t_{v(j)} - t_{w(j)} = \begin{cases} \pm(t_{v(p)} - t_{v(q)}) & j \in \{p,q\} \\ 0 & j \notin \{p,q\}. \end{cases} \end{aligned}$$

In either case, this difference is in the ideal $\mathcal{L}(w,v)$.

Case 3: $w^{-1}(k) \in G$, $v^{-1}(k) \notin G$. In particular, w is in the support of $\bar{y}^j_{G,k}$ whereas v is not. Since $w^{-1}(k) = (p,q)v^{-1}(k)$, we know that one of either p or q is in G and the other is not. Without loss of generality, say $p \in G$ and $q \notin G$. In particular, $w^{-1}(k) = p$ and $v^{-1}(k) = q$. Since $(p,q) \in E(\Gamma)$ and the only element in $[n] \setminus V(G)$ that elements of G are connected to is the vertex j, it follows that

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$v^{-1}(k) = q = j$. So $v(q) = k$ and $w(j) = v(p,q)(j) = v(p)$. Compute that

$$\bar{y}^j_{G,k}(w) - \bar{y}^j_{G,k}(v) = t_k - t_{w(j)} = t_{v(q)} - t_{v(p,q)(j)} = t_{v(q)} - t_{v(p)} \in \mathcal{L}(w,v).$$

Thus, $\bar{y}^j_{G,k}$ is an element of $\mathcal{M}_\Gamma$. □

The splines in $\mathcal{F}_\Gamma$ are defined from graph properties that are intrinsic to the isomorphism class of Γ. Lemma 5.4 below makes this precise.

Lemma 5.4. *Let $\omega\colon \Gamma \to \Gamma'$ be a graph isomorphism and Ω be as in Subsection 2.3. Then $\mathcal{F}_{\Gamma'} = \Omega(\mathcal{F}_\Gamma)$.*

Proof. It follows directly from the definitions that $\Omega(\mathcal{X}_n) = \mathcal{X}_n$ and $\Omega(\mathcal{T}_n) = \mathcal{T}_n$.

The image of the coset spline $\bar{f}^{(i,j)}_A \in \mathcal{M}_\Gamma$ can be computed to be the coset spline $\bar{f}^{(w(i),w(j))}_{w^{-1}(A)} \in \mathcal{M}_{\Gamma'}$, where we consistently choose the connected component $\omega(G_s)$. From this, it is straightforward from the definitions to verify that $\mathcal{C}_{\Gamma'} = \Omega(\mathcal{C}_\Gamma)$

Similarly, it is easy to verify that $\bar{y}^j_{G,k} \mapsto \bar{y}^{w(j)}_{w(G),w(k)}$, and so $\mathcal{Y}_{\Gamma'} = \Omega(\mathcal{Y}_\Gamma)$. □

By Lemma 5.4, it suffices to prove that $\mathcal{F}_\Gamma$ spans $\mathcal{M}^1_\Gamma$ for any particular graph in isomorphism class of Γ.

5.1. Some relations

This set $\mathcal{F}_\Gamma$ is not a $\mathbb{C}$-basis of $\mathcal{M}^1_\Gamma$. Indeed, the following Lemmas 5.5, 5.6 and 5.7 give relations between elements of $\mathcal{F}_\Gamma$.

The first set of relations in the Lemma 5.5 are relatively straightforward.

Lemma 5.5. *For $\bar{t}_i \in \mathcal{T}_n$, $\bar{x}_i \in \mathcal{X}_n$, $\bar{f}^s_A \in \mathcal{F}_\Gamma$ and $\bar{y}^j_{G,k} \in \mathcal{Y}_\Gamma$, the following relations hold:*

(1) $\sum_{r=1}^{n} \bar{x}_r = \sum_{r=1}^{n} \bar{t}_r$,

(2) *if (i,j) is a cut edge and $G_{(i,j)}$ is the component containing the vertex i, then $\sum_A \bar{f}^{(i,j)}_A = \bar{x}_i - \bar{x}_j$, where the sum is over all $A \subset [n]$ such that $|A| = \left|G_{(i,j)}\right|$,*

(3) *if $j \vdash \Gamma$, then $\sum_{k=1}^{n} \bar{y}^j_{G,k} = \left(\sum_{r\in G} \bar{x}_r\right) - |G|\bar{x}_j$ for any connected component G of $\Gamma - j$, and*

(4) *if $j \vdash \Gamma$ and $k \in [n]$ is fixed, $\sum_G \bar{y}^i_{G,k} = \bar{t}_k - \bar{x}_j$, where the sum is over all connected components G of $\Gamma - j$.*

Proof. Relation (1) is easy, as is relation (2) once it is noted that the support of each $\bar{f}^{(i,j)}_A$ for a fixed (i,j) is disjoint. Fix $w \in S_n$.

For relation (3), compute that

$$\sum_{k=1}^{n} \bar{y}^j_{G,k}(w) = \sum_{\substack{k\in[n]\\ w^{-1}(k)\in G}} t_k - t_{w(j)} = \left(\sum_{\substack{k\in[n]\\ w^{-1}(k)\in G}} t_k\right) - |G|t_{w(j)}$$
$$= \left(\sum_{r\in G} t_{w(r)}\right) - |G|t_{w(j)} = \left(\left(\sum_{r\in G} \bar{x}_r\right) - |G|\bar{x}_j\right)(w).$$

For relation (4), note that either $w^{-1}(k) = j$ or $w^{-1}(k) \in G$ for one and only one connected component G of $\Gamma - j$. As such, if $w^{-1}(k) = j$, in which case w is not in the support of any of the $\bar{y}^j_{G,k}$ and likewise $\bar{t}_k(w) - \bar{x}_j(w) = 0$. Otherwise, $w^{-1}(k) \in G$ for some particular connected component G and $\bar{y}^j_{G,k}(w) = t_k - t_{w(j)}$. This is precisely $\bar{t}_k(w) - \bar{x}_j(w)$, and we have (4). □

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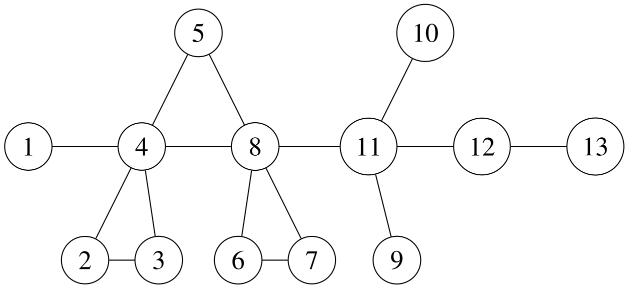
Lemma 5.6 below shows that if j is a cut vertex and a connected component G of $\Gamma - j$ is connected to j by a cut edge (i, j), then the spline $\bar{y}^{j}_{G,k}$ can be written as a sum of other splines from $\mathcal{F}_\Gamma$. In particular, it will allow us to remove from $\mathcal{F}_\Gamma$ the splines in $\mathcal{Y}_\Gamma$ that correspond to components connected by cut edges.

Lemma 5.6. *Let Γ be a graph, j a cut vertex and (i, j) a cut edge, choosing $G_{(i,j)}$ to be the component containing the vertex i. Let C be the connected component of the forest with vertex set $V(G_{(i,j)}) \cup \{j\}$ and edge set $\{s \mid s$ is a cut edge of $G_{(i,j)}\} \cup \{(i, j)\}$ that contains the vertex j. Then for all $k \in [n]$,*

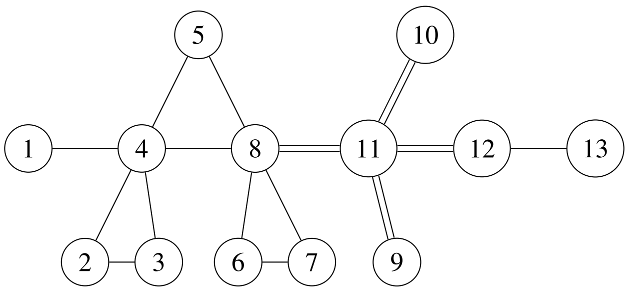
$$\sum_{\substack{v\in C-j\\ v\vdash\Gamma}} \sum_{\substack{G\\ G\cap C=\emptyset}} \bar{y}^{v}_{G,k} + \sum_{a\in E(C)} \sum_{\substack{A\ni k\\ |A|=|G_a|}} \bar{f}^{a}_{A} = \bar{y}^{j}_{G_{(i,j)},k},$$

where G in the first double-sum is a connected component of $\Gamma - v$, where G_a is the connected component that is a subset of $G_{(i,j)}$, and where A in the second double-sum is a subset of $[n]$.

Since Lemma 5.6 is rather technical, we will walk through an example before seeing the full proof. Let Γ be the graph on 13 vertices below:



where $(11, 12)$ is cut edge and $G_{(11,12)}$ is the component that contains 11. The forest of cut edges with vertex set $V(G_{(11,12)}) \cup \{12\}$ has edge set $\{(1, 4), (8, 11), (9, 11), (10, 11), (12, 11)\}$. If we mark in Γ the component of this forest that contains the vertex 11 with double lines, we get the following graph:



In essence, Lemma 5.6 says that for all $k \in [13]$, the spline $\bar{y}^{12}_{G_{(11,12)},k}$ can be written as a sum of some splines in $\mathcal{C}_\Gamma$ associated to the cut edges $(8, 11)$, $(9, 11)$ and $(10, 11)$ (since they are a part of that marked tree) and some splines in $\mathcal{Y}_\Gamma$ associated to cut vertex 8, since $\Gamma - 8$ has components that 'hang off of' that marked tree.

For each of the cut edges $a \in \{(8, 11), (9, 11), (10, 11)\}$, we must choose G_a to be the component contained within $G_{(11,12)}$, so $V(G_{(9,11)}) = \{1, \ldots, 11\}$, $V(G_{(9,11)}) = \{9\}$, and $V(G_{(10,11)}) = \{10\}$.

More formally, Lemma 5.6 states that for all $k \in [13]$, the following equality holds (we will denote the specific subgraph by their vertex set):

$$\bar{y}^{8}_{[5],k} + \bar{y}^{8}_{\{6,7\},k} + \sum_{\substack{|A|=8\\ k\in A}} \bar{f}^{(8,11)}_{A} + \sum_{\substack{|A|=1\\ k\in A}} \bar{f}^{(9,11)}_{A} + \sum_{\substack{|A|=1\\ k\in A}} \bar{f}^{(10,11)}_{A} + \sum_{\substack{|A|=11\\ k\in A}} \bar{f}^{(11,12)}_{A} = \bar{y}^{12}_{[11],k}.$$

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The proof proceeds in cases by the value of $w^{-1}(k)$. For example, say we wished to evaluate both sides of the above expression at a $w \in S_{13}$ such that $w^{-1}(k) = 3$. This makes it easier to determine which splines in the sum above on the left are supported at w. In particular,

1. $\bar{y}^{8}_{[5],k}(w) = t_k - t_{w(8)}$ since $3 \in [5]$,
2. $\bar{y}^{8}_{\{6,7\},k}(w) = 0$ since $3 \notin \{6,7\}$,
3. $\sum_{\substack{|A|=8\\ k\in A}} \bar{f}^{(8,11)}_A(w) = t_{w(8)} - t_{w(11)}$ since the set $A = w(V(G_{(8,11)})$ contains $w(3) = k$,
4. $\sum_{\substack{|A|=1\\ k\in A}} \bar{f}^{(9,11)}_A(w) = 0$ and $\sum_{\substack{|A|=1\\ k\in A}} \bar{f}^{(10,11)}_A(w) = 0$ since $k \notin w(\{9\})$ and $k \notin w(\{10\})$, and finally,
5. $\sum_{\substack{|A|=11\\ k\in A}} \bar{f}^{(11,12)}_A(w) = t_{w(11)} - t_{w(12)}$ since the set $A = w(V(G_{(11,12)})$ contains $w(3) = k$.

If we add these all up, the sum telescopes and the evaluation is

$$(t_k - t_{w(8)}) + (t_{w(8)} - t_{w(11))} + (t_{w(11)} - t_{w(12)}) = t_k - t_{w(12)},$$

which is precisely $\bar{y}^{12}_{[11]}(w)$. The essence of the proof, which we will now provide, is that the splines in the sum with support at a particular $w \in S_n$ can be determined from any simple path from $w^{-1}(k)$ to j and that the sum always telescopes as it did in the example above.

Proof of Lemma 5.6. Let $w \in S_n$. We will show that both sides of the claimed equality are equal when evaluated at w. Let $k \in [n]$, and we will proceed in cases based off of the value $w^{-1}(k)$.

First, say $w^{-1}(k) \notin V(G_{(i,j)})$. All paths that begin in $G_{(i,j)}$ and leave must contain the edge (i,j) and thus visit the vertex j. In particular, for any $v \in C$ where $c \vdash \Gamma$, the connected component of $\Gamma - v$ that contains $w^{-1}(k)$ also contains $j \in C$. So $w^{-1}(k) \notin G$ for any G in the first double-sum, and so

$$\sum_{\substack{v\in C-j\\ v\vdash\Gamma}} \sum_{\substack{G\\ G\cap C=\emptyset}} \bar{y}^{v}_{G,k}(w) = 0.$$

If $a \in E(C)$, then G_a was chosen to be contained within $G_{(i,j)}$, so $w^{-1}(k) \notin G_a$ as well. In particular, if $k \in A$, then $w^{-1}(A) \neq G_a$, and so

$$\sum_{a\in E(C)} \sum_{\substack{A\ni k\\ |A|=|G_a|}} \bar{f}^{a}_{A}(w) = 0.$$

It is direct from the definition that $\bar{y}^{v}_{G_{(i,j)},k}(w) = 0$, and so the claim holds when $w^{-1}(k) \notin G_{(i,j)}$.

Now assume that $w^{-1}(k) \in G_{(i,j)}$. Let $P = (p_0, \ldots, p_\ell, p_{\ell+1})$ be a simple path from $p_0 := w^{-1}(k)$ to $p_{\ell+1} := j$. Note that p_ℓ must be the vertex i. This path P may start outside of C, but must eventually enter the tree C. Say that $m \in \{0, \ldots, \ell\}$ is the lowest index such that $p_m \in C$. Since $i \in C$ and $p_\ell = i$, this integer m does exist. Simple paths from vertex to vertex within trees are unique, so there is a unique simple path from p_m to j in C. This path is $P^C := (p_m, \ldots, p_{\ell+1})$.

First, we will determine the value of $\bar{f}^{a}_{A}(w)$ for $a \in E(C)$. Say the edge $a \in C$ is not an edge in the path P^C. Then the vertices $w^{-1}(k)$ and j are in the same connected component of the graph $([n], E(\Gamma) - a)$. In particular, $w^{-1}(k) \notin G_a$, so if $k \in A$, then $w^{-1}(A) \neq V(G_a)$. It follows that for all A such that $k \in A$, if $a \notin P^C$, then $\bar{f}^{a}_{A}(w) = 0$.

However, if $a \in P^C$, then we may let $A := w(V(G_a))$, and then $k \in A$. So for each $a \in P^C$, there exists a single spline $\bar{f}^{a}_{A}$ in the sum that is supported at w. In particular, if $a = (p,q)$, then $\bar{f}^{q}_{A}(w) = t_{w(p)} - t_{w(q)}$.

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At this point, we have that

$$\sum_{\substack{v \in C-j \\ v \vdash \Gamma}} \sum_{\substack{G \\ G \cap C = \emptyset}} \bar{y}^v_{G,k}(w) + \sum_{a \in E(C)} \sum_{\substack{A \ni k \\ |A| = |G_a|}} \bar{f}^a_A = \sum_{\substack{v \in C-j \\ v \vdash \Gamma}} \sum_{\substack{G \\ G \cap C = \emptyset}} \bar{y}^v_{G,k}(w) + \sum_{(p,q) \in P^C} t_{w(p)} - t_{w(q)}.$$

Now we will have two cases: if $w^{-1}(k) \in C$ and if $w^{-1}(k) \notin C$.

Case 1: $w^{-1}(k) \in C$. So $m = 0$. Then for all $v \in G_{(i,j)}$, if a connected component G of $\Gamma - v$ contains $w^{-1}(k)$, then so does $G \cap C$. In particular,

$$\sum_{\substack{v \in C-j \\ v \vdash \Gamma}} \sum_{\substack{G \\ G \cap C = \emptyset}} \bar{y}^v_{G,k}(w) = 0.$$

Now we compute

$$\begin{aligned} \sum_{(r,v) \in P^C} t_{w(v)} - t_{w(r)} &= \sum_{i=0}^{\ell} t_{w(p_i)} - t_{w(p_{i+1})} \\ &= t_{w(p_0)} - t_{w(p_{\ell+1})} \\ &= t_k - t_{w(j)}. \end{aligned}$$

This is precisely $\bar{y}^j_i(w)$, and so the equality holds if $w^{-1}(k) \in C$.

Case 2: $w^{-1}(k) \notin C$. Then $m \neq 0$, and consider the vertex p_{m-1}. Since $p_m \in C$ and $\Gamma - p_m$ separates $w^{-1}(k)$ from j, the vertex p_m is a cut vertex of Γ. If $v' \in C$ is any vertex other than p_m, then p_m and $w^{-1}(k)$ are in the same connected component of $\Gamma - v'$ (connected via the path $(p_0, \ldots, p_m)$). In particular, any connected component of $\Gamma - v'$ that contains $w^{-1}(k)$ intersects nontrivially with C. So for $v = p_m \in C - j$, there is precisely one component G of $\Gamma - v$ that contains $w^{-1}(k)$. If this component G intersected nontrivially with C, then (p_{m-1}, p_m) would have to be an edge in C, but (p_{m-1}, p_m) was explicitly assumed not to be a cut edge. In particular, w is supported on one and only one spline in the first double-sum (the one where $v = p_m$ and $G \ni p_{m-1}$)), and so

$$\sum_{\substack{v \in C-j \\ v \vdash \Gamma}} \sum_{\substack{G \\ G \cap C = \emptyset}} \bar{y}^v_{G,k}(w) = t_k - t_{w(p_m)}.$$

It follows that

$$\begin{aligned} \sum_{\substack{v \in C-j \\ v \vdash \Gamma}} \sum_{\substack{G \\ G \cap C = \emptyset}} \bar{y}^v_{G,k}(w) + \sum_{(p,q) \in P^C} t_{w(p)} - t_{w(q)} &= t_k - t_{w(p_m)} + \sum_{r=m}^{\ell} t_{w(p_r)} - t_{w(p_{r+1})} \\ &= t_k - t_{w(p_m)} + t_{w(p_m)} - t_{w(p_{\ell+1})} \\ &= t_k - t_{w(j)}. \end{aligned}$$

So in either case, the sum evaluates to $\bar{y}^j_{G_{(i,j)},k}(w)$. □

Now Lemma 5.6 two very important consequences. First, as mentioned, it will allow us to disregard those splines $\bar{y}^j_{G,k}$ where G is connected to j via a cut edge. Second, observe we only required j to be a cut vertex so that $\bar{y}^j_{G,k}$ is defined. We may, however, remove this restriction and 'force through' the argument as follows. If (p,q) is a cut edge and q is not a cut vertex, then q is a leaf in Γ. Let $G_{(p,q)}$ be the connected component containing p (and thereby all of $[n] \setminus \{q\}$). We might abuse notation and let for all $w \in S_n$,

$$\bar{y}^q_{G_{(p,q)},k}(w) = \begin{cases} t_k - t_{w(q)} & \text{if } w^{-1}(k) \in [n] \setminus \{q\} \\ 0 & \text{otherwise} \end{cases}$$
$$= \bar{t}_k(w) - \bar{x}_q(w),$$

and get another relation from Lemma 5.6. A consequence of this is Lemma 5.7 below.

Lemma 5.7. *Let (i, j) be a leaf edge in Γ with j the leaf vertex, and $G_{(i,j)}$ the connected component of $([n], E(\Gamma) \setminus \{s\})$ that contains i. Then for all $A \subset [n]$ such that $|A| = \left|G_{(i,j)}\right| = n-1$, we have that*

$$\bar{f}_A^{(i,j)} \in \mathbb{C}\left\{\bar{\rho} \in \mathcal{F}_\Gamma \middle| \bar{\rho} \neq \bar{f}_B^{(i,j)} \text{ for any } B \subset [n]\right\}.$$

Proof. Let C be the connected component of the forest with vertex set $V(G_{(i,j)}) \cup \{j\} = [n]$ and edge set $\{s \mid s \text{ is a cut edge of } \Gamma)\}$ that contains the vertex j. By Lemma 5.6 and the discussion above, for all $k \in [n]$,

$$\sum_{\substack{v \in C - j \\ v \vdash \Gamma}} \sum_{\substack{G \\ G \cap C = \emptyset}} \bar{y}^v_{G,k} + \sum_{a \in E(C)} \sum_{\substack{A \ni k \\ |A| = |G_a|}} \bar{f}^a_A = \bar{t}_k - \bar{x}_j,$$

where G in the first double-sum is a connected component of $\Gamma - v$, G_a is the connected component of $([n], E(\Gamma) \setminus \{(i, j)\})$ that is a subgraph of $G_{(i,j)}$ (i.e., does not contain j), and A in the second double-sum is a subset of $[n]$. In particular,

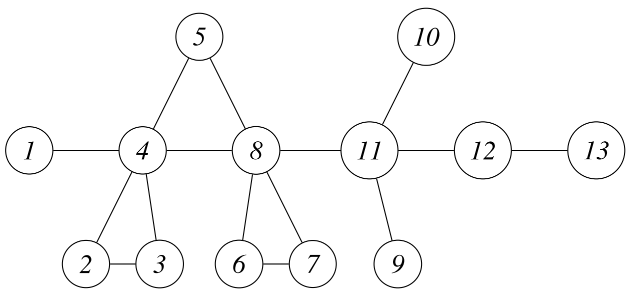
$$\sum_{\substack{|A|=n-1 \\ k \in A}} \bar{f}_A^{(i,j)} = \bar{t}_k - \bar{x}_n - \sum_{\substack{v \in C - j \\ v \vdash \Gamma}} \sum_{\substack{G \\ G \cap C = \emptyset}} \bar{y}^v_{G,k} - \sum_{\substack{a \in E(C) \\ a \neq (i,j)}} \sum_{\substack{A \ni k \\ |A| = |G_a|}} \bar{f}^a_A.$$

Now the right-hand side is in $\mathbb{C}\left\{\bar{\rho} \in \mathcal{F}_\Gamma \middle| \bar{\rho} \neq \bar{f}_B^{(i,j)} \text{ for any } B \subset [n]\right\}$. Let $\bar{\sigma}_k := \sum_{\substack{|A|=n-1 \\ k \in A}} \bar{f}_A^{(i,j)}$, so the above relation says $\bar{\sigma}_k \in C\left\{\bar{\rho} \in \mathcal{F}_\Gamma \middle| \bar{\rho} \neq \bar{f}_B^{(i,j)} \text{ for any } B \subset [n]\right\}$. For each $p \in [n]$, we have that

$$\bar{f}^{(i,j)}_{[n]\setminus\{p\}} = \left(\frac{1}{n-1} \sum_{k \in [n]} \bar{\sigma}_k\right) - \bar{\sigma}_p.$$

Thus, $\bar{f}^{(i,j)}_{[n]\setminus\{p\}} \in \mathbb{C}\left\{\bar{\rho} \in \mathcal{F}_\Gamma \middle| \bar{\rho} \neq \bar{f}_B^{(i,j)} \text{ for any } B \subset [n]\right\}$, and as all subsets of size $|A| = n - 1$ take the form $A = [n] \setminus \{p\}$, the claim follows. □

Example 5.8. If Γ is the graph below where $(12, 13)$ is a leaf,



then Lemma 5.7 states that $\mathbb{C}\mathcal{F}_\Gamma$ is identical to $\mathbb{C}\left\{\bar{\rho} \in \mathcal{F}_\Gamma \middle| \bar{\rho} \neq \bar{f}_B^{(12,13)} \text{ for any } B \subset [n]\right\}$.

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Since $(1,4)$ is also a leaf, we have that $\mathbb{C}\mathcal{F}_\Gamma$ is equal to $\mathbb{C}\left\{\bar{\rho} \in \mathcal{F}_\Gamma \middle| \bar{\rho} \neq \bar{f}_B^{(1,4)} \text{ for any } B \subset [n]\right\}$ as well. Note we cannot remove the coset splines for $(1,4)$ and $(12,13)$ from $\mathcal{F}_\Gamma$ at the same time and maintain the $\mathbb{C}$-span; we have to pick a particular leaf to remove and stick with it.

6. Natural labels and cliqued graphs

In this section, we reduce the computation for arbitrary Γ in two ways. First, we show that Γ may be replaced by a cliqued graph (defined below) without altering $\mathcal{M}_\Gamma^1$. Second, we replace Γ with a particular representative of the isomorphism class we call naturally labeled. Subsection 6.2 gives three technical lemmas on splines that hold for these constructions.

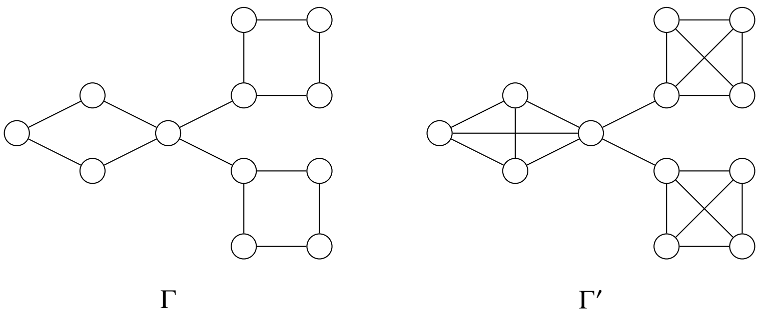
First, if Γ does not have a cut vertex, then it is 2-connected. Thus, $\mathbf{ch}(\mathrm{L}_\Gamma)_1$ is trivial and $\mathbf{ch}(\mathrm{R}_\Gamma)_1$ is the first degree piece of the graded regular representation by Theorem 4.2 and Corollary 4.4. So we may assume that Γ has a cut vertex; in particular, we may assume that Γ has at least three vertices.

A *clique* is a subgraph isomorphic to a complete graph. Let Γ by any (connected) graph on $[n]$. Call Γ *cliqued* if two vertices are connected by an edge in Γ whenever there exists two vertex-disjoint paths between them. Define

$$\Gamma' = ([n], E(\Gamma) \cup \{(i,j) \mid \text{ exists two vertex-disjoint paths from } i \text{ to } j \text{ in } \Gamma\}).$$

Now Γ' is cliqued, and we call Γ' the *cliqued version* of Γ. By Lemma 4.1, the first degree pieces of $\mathcal{M}_\Gamma$ and $\mathcal{M}_{\Gamma'}$ are equal. Therefore, it suffices to consider cliqued graphs Γ when proving results on the structure of $\mathcal{M}_\Gamma^1$.

Example 6.1. Below is an example of a graph Γ and the cliqued graph Γ' such that $\mathcal{M}_\Gamma^1 = \mathcal{M}_{\Gamma'}^1$:



A *2-connected component* of a graph Γ is a subgraph of Γ that is 2-connected. A *block* is a maximal 2-connected component. In a cliqued graph, every block is a clique, and the process of cliquing a graph simply converts every block to a clique.

Definition 6.2. Let Γ be a graph. The *block-cut tree* of Γ is the tree with vertex set

$$\{v \mid v \vdash \Gamma\} \cup \{B \mid B \text{ is a block in } \Gamma\}$$

consisting of cut vertices and blocks in Γ and edge set $\{(v,B) \mid v \in B\}$.

Example 6.3. The graph on the left below is Γ from Example 6.3 with the blocks and cut vertices labeled. The graph on the right below is the associated the block-cut tree. Note that the block-cut tree for the cliqued version Γ' of Γ in Example 6.3 would be the same.

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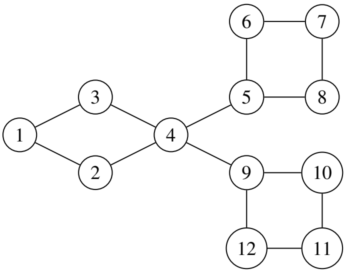
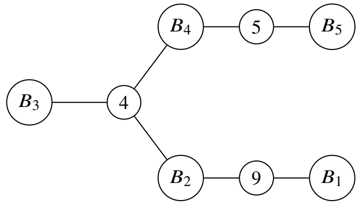




where

- B_1 is the induced subgraph on vertices $\{9, 10, 11, 12\}$
- B_2 is the induced subgraph on vertices $\{4, 9\}$
- B_3 is the induced subgraph on vertices $\{1, 2, 3, 4\}$
- B_4 is the induced subgraph on vertices $\{4, 5\}$
- B_5 is the induced subgraph on vertices $\{5, 6, 7, 8\}$

It is easy to reason that the block-cut tree is indeed a tree by arguing that it cannot contain cycles. In a block-cut tree of a graph Γ, every leaf is a block in Γ and not a cut vertex, and paths in the block-cut tree alternate between cut-vertices and blocks. Since the block-cut tree ignores the internal structure of a 2-connected component, the block-cut tree of a graph Γ and the block-cut tree of its cliqued version Γ' are isomorphic as graphs.

We now use the block-cut tree to construct a particular representative of the isomorphism class of Γ. We will construct a bijection $\phi\colon [n] \to [n]$ to relabel the vertices of Γ.

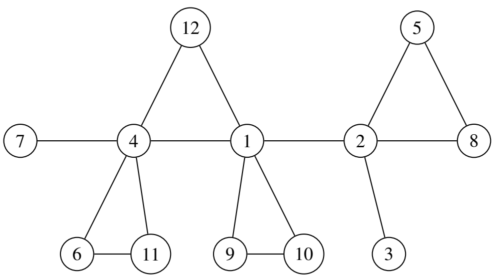
Choose a cut vertex $v \vdash \Gamma$ such that v is adjacent to at most 1 block that is not a leaf in the block-cut tree of Γ. One may obtain such a vertex v by (1) removing all leaves from the block-cut tree of Γ (which must be blocks) and then (2) choosing a leaf from the tree that remains (which must be cut vertices). The vertices 5 and 9 satisfy this condition in Example 6.3.

Let B be the largest block that is also a leaf adjacent to v in the block-cut tree of Γ. Since B is a leaf in the block-cut tree, there exists only a single cut vertex in B – namely, $v \in B$. The following is the algorithm that produces $\phi\colon [n] \to [n]$.

1. Choose $i \in B$ so that $i \neq v$. Define $\phi(i) := n$. Note n is adjacent to at most one cut vertex of Γ and is not itself a cut vertex of Γ.
2. Define ϕ on the remaining vertices in B as follows. Define ϕ on $B \setminus \{i, v\}$ so that $d(j, i) < d(k, i)$ implies $\phi(j) > \phi(k)$ for all $j, k \in B \setminus \{i, v\}$. Let $\phi(v) := n - |B| + 1$
3. Define ϕ on the remaining vertices of Γ as follows. Let ϕ be any bijection satisfying if $j, k \in \Gamma \setminus V(B)$, then $d(j, i) < d(k, i)$ implies $\phi(j) > \phi(k)$.

Now that we have a bijection $\phi\colon [n] \to [n]$, define a new graph $\Gamma'' := ([n], \{(\phi(j), \phi(k)) \mid (j, k) \in E(\Gamma)\})$. This graph is clearly isomorphic to Γ. We call the graph Γ'' constructed in this manner *naturally labeled*.

Example 6.4. This example will construct a naturally labeled graph from the not-naturally labeled graph with 12 vertices drawn below:



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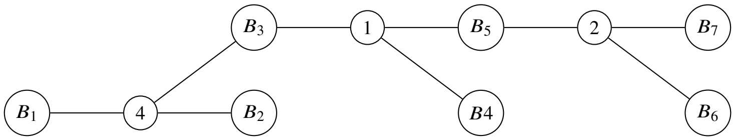
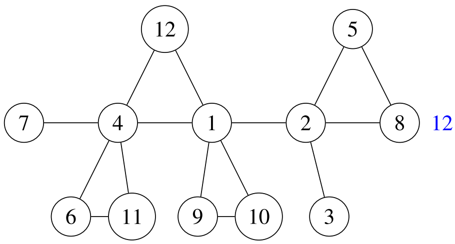
This has block-cut tree



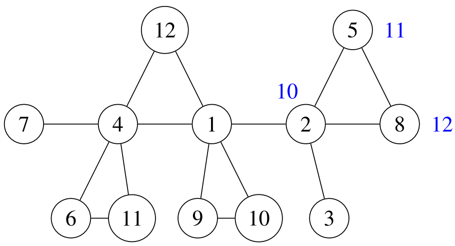
with blocks

- B_1 on $\{4, 7\}$
- B_2 on $\{4, 6, 11\}$
- B_3 on $\{1, 4, 12\}$
- B_4 on $\{1, 9, 10\}$
- B_5 on $\{1, 2\}$
- B_6 on $\{2, 3\}$
- B_7 on $\{2, 5, 8\}$

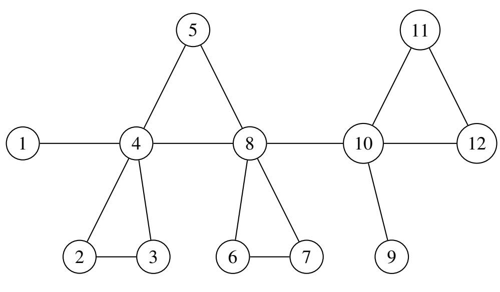
To choose a vertex i such that $\phi(i) = 12$, first we identify an appropriate cut vertex v. There are two cut vertices adjacent to only one non-leaf vertex in the block-cut tree, 2 and 4. Let $v = 2$. The vertex 2 is adjacent to blocks B_6 and B_7 in the block-cut tree. Since B_7 is bigger than B_6, we know that either of $i = 5$ or $i = 8$ will work. We choose $i = 8$ so $\phi(8) = 12$, which concludes step (1).



Now the block B_7 has three vertices, which leaves no choice for defining ϕ on the remainder of B_7. So $\phi(5) = 11$ and $\phi(4) = 10$. This concludes step (2).



Finally, we define the rest of ϕ based on distance from the vertex $i = 8$ and replace the old graph with the naturally labeled one. One possible natural label is the following:



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We note how many choices were made along the way. In particular, the isomorphism class of a graph Γ may have many different naturally labeled members.

We can use a natural label to more efficiently identify cut edges and connected components of a graph. The following definitions formalize this.

Definition 6.5. Let Γ be a naturally labeled graph. If $j \in \Gamma$ is a cut vertex, then $i < j$ is *j-dominant* if i is the maximal value vertex in a connected component of $\Gamma - j$ that does not contain the vertex n. We call such an (i, j) a *dominant pair*. A dominant pair (i, j) is *strongly dominant* if (i, j) is a cut edge of Γ, denoted $(i, j) \gg \Gamma$. Otherwise, (i, j) is *weakly dominant*, denoted $(i, j) > \Gamma$.

There are four things to note about dominant pairs.

1. Even if $(n-1, n)$ is a cut edge, $(n-1, n)$ is never a strongly dominant pair since n is not a cut vertex by the definition of a natural label. However, every other cut edge in a naturally labeled graph Γ is a strongly dominant pair, since the higher-labeled vertex in the cut edge must be a cut vertex.
2. If $j \vdash \Gamma$, then all vertices larger than j must be concentrated in one connected component of $\Gamma - j$. If k is in a connected component of $\Gamma - j$ that does not contain the vertex n, then any path from k to n must pass through j, and so $d(k, n) > d(j, n)$, and thus, $k < j$ by the definition of a natural label. In particular, the only connected component of $\Gamma - j$ whose maximal vertex is greater than j is the one that contains n, so each connected component of $\Gamma - j$ that does not contain n contains precisely one j-dominant vertex.
3. Since a natural label is constructed by distance from n, for each cut vertex j, the maximal-labeled vertices in a connected component of $\Gamma - j$ that does not contain n must be adjacent to j. In particular, dominant pairs are also edges.
4. The lower vertex in the dominant pair uniquely determines the pair. In particular, if for contradiction we assume (i, j) and (i, k) are both dominant pairs, then i and k are in the same connected component of $\Gamma - j$, and $i < k$, and so (i, j) is not a dominant pair.

Definition 6.6. Let Γ be naturally labeled and fix $j \in \Gamma$. Let $\mathfrak{C}_\Gamma(j) := \{i \mid (i, j) > \Gamma \text{ or } (i, j) \gg \Gamma\}$, and $\mathfrak{c}_\Gamma(j) := |\mathfrak{C}_\Gamma(j)|$. If j is not a cut vertex, then $\mathfrak{C}_\Gamma(j) = \emptyset$ and $\mathfrak{c}_\Gamma(j) = 0$. If j is a cut vertex of Γ, then the *cut decomposition* of $\Gamma - j$ is

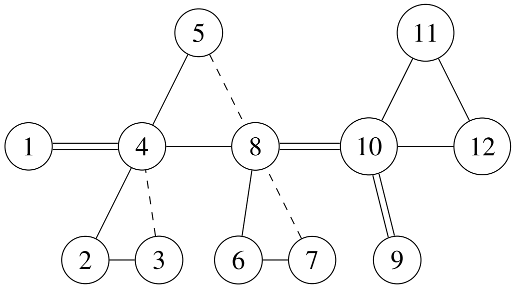
$$\Gamma - j := \Gamma_0^j \cup \bigcup_{i \in \mathfrak{C}_\Gamma(j)} \Gamma_i^j,$$

where Γ_i^j the connected component of $\Gamma - j$ such that $i \in \Gamma_i^j$ and Γ_0^j denotes the single connected component of $\Gamma - j$ where $n \in \Gamma_0^j$.

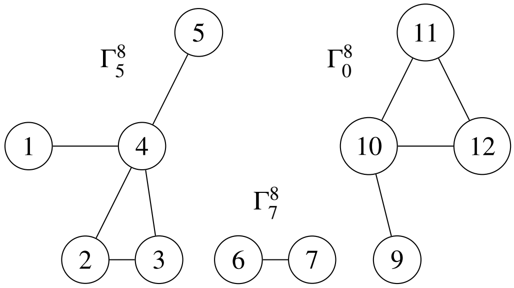
So if $j \vdash \Gamma$ is a cut vertex of Γ and $k \in [n]$ such that $k > j$, then $k \in \Gamma_0^j$. When Γ is obvious from context, we write $\mathfrak{C}(j)$ and $\mathfrak{c}(j)$ without the subscripts.

Remark 6.7. If (i, j) is a cut edge and j is a cut vertex, then Γ_i^j as in the cut decomposition of $\Gamma - j$ is one of the two connected components of $([n], E(\Gamma) \setminus \{(i, j)\})$. In particular, Γ_i^j is one of the two valid choices for G_s when defining $\bar{f}_A^{(i,j)}$ at the beginning of Section 5. From now on, even if Γ is not naturally labeled and even if the cut edge is $(i, j) = (n-1, n)$, we will choose G_s to be the connected component of $(V(\Gamma), E(\Gamma) - (i, j))$ that contains $i < j$, so that the notation always agrees with Definition 6.6.

Example 6.8. The following is the cliqued and naturally labeled graph Γ from Example 6.4. We have labeled the strongly dominant pairs in double lines and the weakly dominant pairs in dashed lines:



The cut vertices are $\{4, 8, 10\}$, and $\mathfrak{c}_\Gamma(10) = \mathfrak{c}_\Gamma(8) = \mathfrak{c}_\Gamma(4) = 2$. The graph $\Gamma - 8$ is displayed below, with the cut decomposition labeled:



Lemma 6.9 below summarizes some properties of a cliqued and naturally labeled graph.

Lemma 6.9. *If Γ is cliqued and naturally labeled, then*

(A) *if $i, j \in \mathrm{N}(n)$ are both adjacent to the vertex n in Γ, then $(i, j) \in E(\Gamma)$,*
(B) *if $n-1 \vdash \Gamma$ is a cut vertex of Γ, then at most one of the connected components Γ_i^{n-1} for $i \in \mathfrak{C}_\Gamma(n-1)$ in the cut decomposition of $\Gamma - (n-1)$ is not a single vertex,*
(C) *If $r, k \in \Gamma_i^j \cap \mathrm{N}(j)$ are vertices both adjacent to $j \vdash \Gamma$ and in the same connected component of $\Gamma - j$, then $(r, k) \in E(\Gamma)$.*

Proof. Let B_0 be the block in Γ that contains n.

(1) If $i, j \in \mathrm{N}(n)$, then $i, j \in B_0$. Since Γ is cliqued, B_0 must be a clique and so $(i, j) \in E(\Gamma)$.

(2) The vertex $n - 1$ is a cut vertex of Γ if and only if n is a leaf, and B_0 is size 2. Since Γ is naturally labeled, $n - 1$ is adjacent to at most one block that is not a leaf in the block-cut tree of Γ, and B_0 is of maximal size among those leaves in the block-cut tree. Thus, the blocks adjacent to $n - 1$ in the block-cut tree of Γ are either not a leaf in the block-cut tree (of which there can only be one) or a leaf in the block-cut tree and size no greater than 2.

(3) There exists a path (r, j, k) in Γ, and another path from r to k in Γ_i^j, which does not contain the vertex j. Since Γ is cliqued and we know there are two vertex-disjoint paths from r to k in Γ, it follows that $(r, k) \in E(\Gamma)$. □

We use Lemma 6.9 to categorize cliqued and naturally labeled graphs in to three types, based on the structure of Γ near the vertex n.

Lemma 6.10. *If Γ is cliqued and naturally labeled, then it falls in to one of the following three types:*

(A) *The edge $(n - 1, n)$ is a cut edge of Γ, and at most one component of $\Gamma - (n - 1)$ is not an isolated vertex.*
(B) *The vertex $n - 2$ is a cut vertex, and the vertices $\{n - 2, n - 1, n\}$ form a block in Γ.*
(C) *None of the vertices $\{n, n - 1, n - 2\}$ are cut vertices, and $\{n - 2, n - 1, n\}$ form a clique in Γ.*

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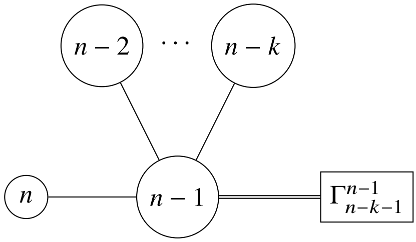
Proof. Categorize Γ by the size of the neighborhood $\mathrm{N}(n)$. For every Γ, exactly one of the following is true: $|\mathrm{N}(n)| = 1$, $|\mathrm{N}(n)| = 2$ or $|\mathrm{N}(n)| > 2$. Note that, since every block is a clique and n is not a cut vertex, the block B containing n has vertices $N(n) \cup \{n\}$.

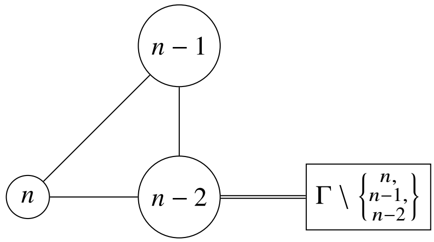
If $|\mathrm{N}(n)| = 1$, then $n-1$ is a cut vertex, and Γ is type A. The rest of the claim for type A is Lemma 6.9(2).

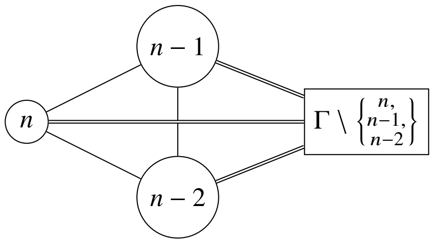
Suppose $|\mathrm{N}(n)| = 2$. Since Γ is naturally labeled, $n-2$ must be a cut vertex. We also have $(n-2, n-1) \in E(\Gamma)$ by Lemma 6.9(1).

Finally, suppose $|\mathrm{N}(n)| > 2$. Since Γ is naturally labeled, none of $\{n-2, n-1, n\}$ is a cut vertex. Once again, $(n-2, n-1) \in E(\Gamma)$ by Lemma 6.9(1). □

Visually, Lemma 6.10 says that if Γ is cliqued and naturally labeled, then Γ can be represented diagrammatically in one of the following three ways:

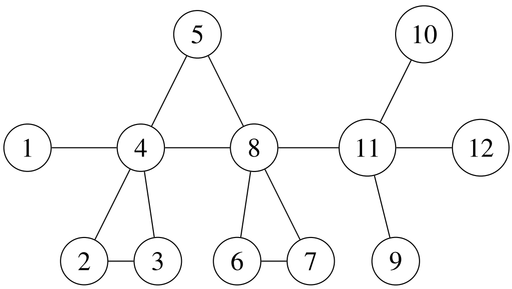
A:


B:


C:


In the diagram for type A above, $\mathfrak{c}_\Gamma(n-1) = k$. We remark that in type B, the induced subgraph $\Gamma \setminus \{n-2, n-2, n\}$ may be disconnected (such as it would be for the graph in Example 6.8). However, in type C, the vertex $n-3$ must be in the same block as n, $n-1$ and $n-2$, so the induced subgraph $\Gamma \setminus \{n-2, n-1, n\}$ is actually connected.

Example 6.11. The graph Γ from Example 6.8 is type B. The following is a naturally labeled type A graph:



A different natural label on the same graph, such as the one below, can have a different classification. The following naturally labeled graph is in the same isomorphism class as the previous, but is type B:

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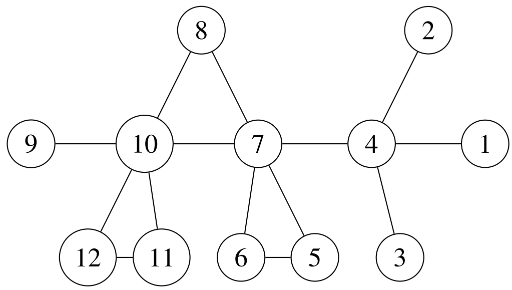

6.1. The spanning set revisited

A natural labeling provides a more convenient indexing for the splines in $\mathcal{Y}_\Gamma$ and $\mathcal{C}_\Gamma$, using the cut decomposition from Definition 6.6. In particular, we write

$$\bar{y}^j_{i,k} \coloneqq \bar{y}^j_{\Gamma^j_i,k}$$

so that

$$\mathcal{Y}_\Gamma = \left\{ \bar{y}^j_{i,k} \middle| j \vdash \Gamma,\ i \in \mathfrak{C}_\Gamma(j) \cup \{0\},\ k \in [n] \right\}.$$

Additionally, while the notation for individual splines $\bar{f}^s_A \in \mathcal{C}_\Gamma$ does not change, we note that, as stated in Remark 6.7, we choose G_s for $s = (i < j)$ to be equal to Γ^j_i.

We collect the important naturally-labeled versions of Lemmas 5.6, 5.5 and 5.7 below in Proposition 6.12.

Proposition 6.12. *Let Γ be naturally labeled, and let*

$$\mathcal{B}_\Gamma \coloneqq \{\bar{t}_i \mid i \in [n]\} \cup \{\bar{x}_i \mid i \in [n]\} \cup \left\{ \bar{f}^s_A \middle| \begin{array}{c} s = (i,j) \gg \Gamma, \\ |A| = \left|\Gamma^j_i\right| \end{array} \right\} \cup \left\{ \bar{y}^j_{i,k} \middle| \begin{array}{c} (i,j) > \Gamma, \\ k \in [n] \end{array} \right\}.$$

Then the following hold:

1. *If $(i,j) \gg \Gamma$, then $\bar{y}^j_{i,k} \in \mathbb{C}\mathcal{B}_\Gamma$ for all $k \in [n]$.*
2. *If $j \vdash \Gamma$, then $\bar{y}^j_{0,k} \in \mathbb{C}\mathcal{B}_\Gamma$ for all $k \in [n]$.*
3. *If $(n-1,n)$ is a cut edge of Γ, then $\bar{f}^{(n-1,n)}_A \in \mathbb{C}\mathcal{B}_\Gamma$ for all $A \subset [n]$ where $|A| = n-1$.*

In particular, $\mathbb{C}\mathcal{B}_\Gamma = \mathbb{C}\mathcal{F}_\Gamma$.

Proof. The first relation (1) is exactly Lemma 5.6.

The second relation (2) follows from (1) together with Lemma 5.5(4).

The third relation (3) is Lemma 5.7, applied to the cut edge $(n-1,n)$. □

6.2. Technical lemmas

This subsection contains three lemmas that are used within the proof of Theorem 7.2.

Let $S^i_n \coloneqq \{w \in S_n \mid w(i) = n\}$ be a left coset of S_{n-1} in S_n. The first Lemma 6.13 establishes what values a linear spline $\bar{\rho}$ may take on S^{n-1}_n if $\bar{\rho}$ is not supported on $S^n_n = S_{n-1}$.

Lemma 6.13. *Let Γ be naturally labeled, and $\bar{\rho} \in \mathcal{M}^1_\Gamma$, where $\bar{\rho} \equiv 0$ on S^n_n. If $w, v \in S^{n-1}_n$, then $\bar{\rho}(w) = c_w\left(t_n - t_{w(n)}\right)$ and $\bar{\rho}(v) = c_v\left(t_n - t_{v(n)}\right)$ for some $c_w, c_v \in \mathbb{C}$. Furthermore, if $w(n) = v(n)$, or Γ is type B/C, then $c_w = c_v$.*

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Proof. First, since Γ is naturally labeled, it follows that $(n-1, n) \in E(\Gamma)$. We will show the first part of the claim for $w \in S_n^{n-1}$, and the same will hold for $v \in S_n^{n-1}$. If $w \in S_n^{n-1}$, then $w(n-1,n) \in S_n^n$, and $\mathcal{L}(w, w(n-1,n)) = \langle t_{w(n-1)} - t_{w(n)} \rangle = \langle t_n - t_{w(n)} \rangle$. Since $\bar{\rho}$ is linear and $\bar{\rho}(w(n-1,n)) = 0$, the first part of the claim follows.

Now we prove the second part of the claim. First, we assume $w(n) = v(n)$. Two permutations $w, v \in S_n^{n-1}$ have the property $w(n) = v(n)$ if and only if $v \in wS_{n-2}$. The claim will follow if $c_w = c_v$ for $v = w(r,s)$ whenever $(r,s) \in S_{n-2}$. Let $\{r,s\} \subset [n-2]$. As Γ is naturally labeled and so n is not a cut vertex of Γ, there exists a simple path $(r_0, r_1, \ldots, r_m)$ in Γ from $r = r_0$ to $s = r_m$ that does not contain n, and by Lemma 3.3,

$$\bar{\rho}(w) - \bar{\rho}(w(r,s)) \in \left\langle t_{w(r_i)} - t_{w(r_{i-1})} \mid i \in [m] \right\rangle.$$

In particular,

$$(c_w - c_{w(r,s)})t_n - (c_w - c_{w(r,s)})t_{w(n)} \in \left\langle t_{w(r_i)} - t_{w(r_{i-1})} \mid i \in [m] \right\rangle.$$

The monomial $t_{w(n)}$ does not appear in $\{t_{w(r_i)} - t_{w(r_{i-1})} \mid i \in [m]\}$, and thus, $c_w = c_{w(r,s)}$. Since the transpositions (r,s) generate S_{n-2}, the claim follows.

Now we prove the claim if Γ is type B/C. If Γ is type B or type C, then $\Gamma - (n-1)$ is connected. It suffices to prove $c_w = c_{w(r,s)}$ for $n-1 \notin \{r,s\}$. Since $\Gamma - (n-1)$ is connected, there exists a path $(r_0, \ldots, r_m)$ from $r = r_0$ to $s = r_m$ in Γ that does not visit the vertex $n-1$. So

$$(c_w - c_{w(r,s)})t_n - c_w t_{w(n)} - c_{w(r,s)} t_{w(r,s)(n)} \in \left\langle t_{w(r_i)} - t_{w(r_{i-1})} \mid i \in [m] \right\rangle.$$

Now $t_n = t_{w(n-1)}$ never appears in $\left\{ t_{w(r_i)} - t_{w(r_{i-1})} \mid i \in [m] \right\}$, and thus, $c_w = c_v$. □

The second two lemmas assume that (i,j) is a dominant pair in Γ and establish what values a linear spline $\bar{\rho}$ may take on S_n^i if $\operatorname{supp}(\bar{\rho}) \cap S_n^j = \emptyset$. Lemma 6.14 below assumes (i,j) is strongly dominant and relates $\bar{\rho}(w)$ to $\bar{\rho}(v)$ if w and v are in the same coset S_n^i and they are also in the same coset of the reflection subgroup generated by the transpositions $E(\Gamma) \setminus \{(i,j)\}$.

Lemma 6.14. *Let Γ be cliqued and naturally labeled, and say $(i,j) \gg \Gamma$. Let $\bar{\rho} \in \mathcal{M}_\Gamma^1$, where $\bar{\rho} \equiv 0$ on S_n^j. If $w, v \in S_n^i$, then $\bar{\rho}(w) = c_w\left(t_n - t_{w(j)}\right)$ and $\bar{\rho}(v) = c_v\left(t_n - t_{v(j)}\right)$ for some $c_w, c_v \in \mathbb{C}$. Furthermore, if $v \in w\langle E(\Gamma) \setminus (i,j)\rangle$, then $c_w = c_v$.*

Proof. First, since Γ is naturally labeled, it follows that $(i,j) \in E(\Gamma)$. We will show the first part of the claim for $w \in S_n^i$, and the same will hold for $v \in S_n^i$. If $w \in S_n^i$, then $w(i,j) \in S_n^j$, and $\mathcal{L}(w, w(i,j)) = \langle t_{w(i)} - t_{w(j)} \rangle = \langle t_n - t_{w(j)} \rangle$. Since $\bar{\rho}$ is linear and $\bar{\rho}(w(i,j)) = 0$, the first part of the claim follows from the definition of a spline on $\mathcal{G}_\Gamma$.

The reflection subgroup $\langle E(\Gamma) \setminus \{(i,j)\}\rangle$ is also generated by the transpositions $\{(r,s) \mid \{r,s\} \subset V(\Gamma_i^j)\} \cup \{(p,q) \mid \{p,q\} \subset [n] \setminus V(\Gamma_i^j)\}$, as this set contains $E(\Gamma) \setminus \{i,j\}$. We will show that, for those generating transpositions, $c_w = c_{w(r,s)}$ and $c_w = c_{w(p,q)}$.

First, we show $c_w = c_{w(r,s)}$ for $\{r,s\} \subset V(\Gamma_i^j)$. Let $(r_0, \ldots, r_m)$ be a path in Γ_i^j from $r = r_0$ to $r_m = s$. Then by Lemma 3.3,

$$\bar{\rho}(w) - \bar{\rho}(w(r,s)) \in \left\langle t_{w(r_k)} - t_{w(r_{k-1})} \mid k \in [m] \right\rangle.$$

In particular,

$$(c_w - c_{w(r,s)})t_n - (c_w - c_{w(r,s)})t_{w(j)} \in \left\langle t_{w(r_k)} - t_{w(r_{k-1})} \mid k \in [m] \right\rangle.$$

Since j is not in the path $(r_0, \ldots, r_m)$, the monomial $t_{w(j)}$ does not appear in $\{t_{w(r_k)} - t_{w(r_{k-1})} \mid k \in [m]\}$, and thus, $c_w = c_{w(r,s)}$.

Now let $\{p,q\} \subset [n] \setminus V(\Gamma_i^j)$. Since (i,j) is a cut edge, the induced subgraph of Γ with vertex set $[n] \setminus V(\Gamma_i^j)$ is connected. Let $(p_0, ..., p_m)$ be a path from $p = p_0$ to $q = p_m$ in Γ that does *not* contain i. By Lemma 3.3,

$$\bar{\rho}(w) - \bar{\rho}(w(p,q)) \in \left\langle t_{w(p_k)} - t_{w(p_{k-1})} \mid k \in [m] \right\rangle.$$

In particular,

$$(c_w - c_{w(p,q)})t_n - c_w t_{w(j)} + c_{w(p,q)} t_{w(p,q)(j)} \in \left\langle t_{w(p_k)} - t_{w(p_{k-1})} \mid k \in [m] \right\rangle.$$

Since i is not in the path $(p_0, ..., p_m)$, the monomial $t_n = t_{w(i)}$ does not appear in $\{t_{w(p_k)} - t_{w(p_{k-1})} \mid k \in [m]\}$, and thus, $c_w = c_{w(p,q)}$.

Since the reflection subgroup $\langle E(\Gamma) \setminus (i,j) \rangle$ is generated by $\{(r,s) \mid \{r,s\} \subset \Gamma_i^j\} \cup \{(p,q) \mid \{p,q\} \in [n] \setminus \Gamma_i^j\}$, the claim follows. □

Lemma 6.15 below assumes that (i,j) is weakly dominant and then relates $\bar{\rho}(w)$ and $\bar{\rho}(v)$ if w and v are in the same coset S_n^i.

Lemma 6.15. *Let Γ be cliqued and naturally labeled, and say $(i,j) > \Gamma$. Let $\bar{\rho} \in \mathcal{M}_\Gamma^1$, where $\bar{\rho} \equiv 0$ on S_n^j. If $w, v \in S_n^i$, then $\bar{\rho}(w) = c_w\left(t_n - t_{w(j)}\right)$ and $\bar{\rho}(v) = c_v\left(t_n - t_{v(j)}\right)$ for some $c_w, c_v \in \mathbb{C}$. Furthermore, $c_w = c_v$.*

Proof. The proof of the first part of this claim is identical to the first part of the proof of Lemma 6.14.

Since (i,j) is not a cut edge and Γ is naturally labeled, there exists $(k,j) \in E(\Gamma)$ with $k < i < j$ and $k \in \Gamma_i^j$. By Lemma 6.9(3), $(i,k) \in E(\Gamma)$. If $u \in S_n^k$, then $\bar{\rho}(u) = c_u\left(t_n - t_{u(j)}\right)$ for the same reason that $\bar{\rho}(w)$ and $\bar{\rho}(v)$ take this form. We will prove the slightly stronger claim that $c_w = c_v$ for all $w, v \in S_n^i \sqcup S_n^k$. We proceed for now assuming that the induced subgraph of $\mathcal{G}_\Gamma$ with vertex set $S_n^i \sqcup S_n^k$ is connected, and we will verify that this assumption holds afterwards.

If $S_n^i \sqcup S_n^k$ is connected, it will suffice to check if $c_w = c_v$ for adjacent elements $w, v \in S_n^i \sqcup S_n^k$. We check edges in two cases: those within S_n^i (resp. S_n^k), and the edges $(w, w(i,k))$ between S_n^i and S_n^k.

If $(w, w(p,q))$ is an edge in $\mathcal{G}_\Gamma$ between elements of S_n^i, then $i \notin \{p,q\}$ and

$$\bar{\rho}(w) - \bar{\rho}(w(p,q)) = (c_w - c_{w(p,q)})t_n - c_w t_{w(j)} + t_{w(p,q)(j)} \in \left\langle t_{w(q)} - t_{w(p)} \right\rangle.$$

Since $n \notin \{w(p), w(q)\}$, it follows $c_w = c_{w(p,q)}$. The same logic holds for edges in $\mathcal{G}_\Gamma$ between two elements of S_n^k.

For edges $(w, w(i,k))$ in $\mathcal{G}_\Gamma$ between $w \in S_n^i$ and $w(i,k) \in S_n^k$, compute

$$\bar{\rho}(w) - \bar{\rho}(w(i,k)) = c_w t_n - c_{w(i,k)} t_{w(k)} - (c_w - c_{w(i,k)})t_{w(j)} \in \left\langle t_{w(i)} - t_{w(k)} \right\rangle.$$

Since $w(j) \notin \{w(i), w(k)\}$, it follows that $c_w = c_{w(i,k)}$.

If $S_n^i \sqcup S_n^k$ is connected, and equality holds on every edge, it follows that $c_w = c_v$ for all $w, v \in S_n^i$.

Now we will prove that the induced subgraph of $\mathcal{G}_\Gamma$ with vertex set $S_n^i \sqcup S_n^k$ is connected. Since Γ is cliqued, $(i,k) \in E(\Gamma)$. In particular, if $w \in S_n^i$, then w is connected in $\mathcal{G}_\Gamma$ directly to $w(i,k)$ in S_n^k.

Let $\{r,s\} \subset [n] \setminus \{i\}$. We prove in three cases that for all $r, s \neq i$, the permutation $w \in S_n^i$ is connected to $w(r,s) \in S_n^i$ within the induced subgraph of $\mathcal{G}_\Gamma$ with vertex set $S_n^i \sqcup S_n^k$. It will follow by symmetry (replace i with k) and that the induced subgraph of $\mathcal{G}_\Gamma$ with vertex set $S_n^i \sqcup S_n^k$ is connected. The three cases are (i) there exists a simple path from r to s in Γ that does not visit the vertex i, (ii) simple paths from r to s in Γ must visit i, but need not visit k, and (iii) simple paths from r to s must visit the vertex i and the vertex k.

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(i) If there exists a path $(p_0, \ldots, p_\ell)$ in $\Gamma - i$ from $r = p_0$ to $s = p_\ell$, then w is connected to $w(r,s)$ via only elements in S_n^i since

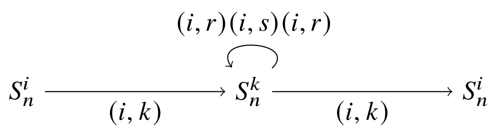
$$w(r,s) = (p_0, p_1) \cdots (p_{\ell-1}, p_\ell) \cdots (p_0, p_1).$$

We will use this path computation implicitly in (ii) and (iii).

(ii) If there exists a path $(r, \ldots, i, \ldots, s)$ from r to s in Γ containing i but not k, then r, i and s are in the same connected component in $\Gamma - k$. Consider

$$w(r,s) = w(i,k)(i,r)(i,s)(i,r)(i,k).$$

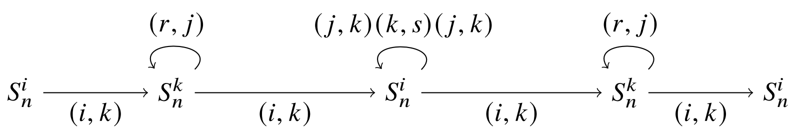
This sequence of transpositions gives a path in $S_n^i \sqcup S_n^k$ from w to $w(r,s)$. Below is a diagram that shows how each right multiplication moves between S_n^i and S_n^k:



(iii) If both i and k must be in a simple path from r to s, it suffices to assume this path takes the form $(r, \ldots, i, k, \ldots, s)$. In particular, the first piece $(r, \ldots, i)$ is a path in $\Gamma - k$ and the second piece $(k, \ldots, s)$ is a path in $\Gamma - i$. Consider

$$w(r,s) = w(i,k)(r,j)(i,k)(j,k)(k,s)(j,k)(i,k)(r,j)(i,k).$$

This sequence of transpositions gives a path from w to $w(r,s)$ in $S_n^i \sqcup S_n^k$. Below is a diagram detailing how each right multiplication moves between S_n^i and S_n^k:



So subgraph with vertices $S_n^i \sqcup S_n^k$ is connected; our earlier assumption is verified and we have the claim. ◻

7. Proof of the linear spanning theorem

This section shows that the collection $\mathcal{F}_\Gamma$ from Equation 5.1 (below Lemma 5.3) is a $\mathbb{C}$-spanning set of $\mathcal{M}_\Gamma^1$. In other words, we prove $\mathbb{C}\mathcal{F}_\Gamma = \mathcal{M}_\Gamma^1$. First, we require a lemma on the compatibility of these splines on S_n with splines on S_{n-1}.

Lemma 7.1. *Let* Γ *on* $[n]$ *be cliqued and naturally labeled. Let* $\mathcal{F}_\Gamma^{(n)} := \{\bar{\rho}|_{S_{n-1}} \mid \bar{\rho} \in \mathcal{F}_\Gamma,\ \bar{\rho}(w) \in \mathbb{C}[t_1, \ldots, t_{n-1}]$ *for all* $w \in S_{n-1}\}$. *Then*

$$\mathbb{C}\mathcal{F}_\Gamma^{(n)} = \mathbb{C}\mathcal{F}_{\Gamma - n}.$$

Proof. First, note that the Cayley graph $\mathcal{G}_{\Gamma-n}$ is equal to the induced subgraph of $\mathcal{G}_\Gamma$ with vertex set S_{n-1}. In particular, each element of $\mathcal{F}_\Gamma^{(n)}$ is in fact a spline in $\mathcal{M}_{\Gamma-n}$

By Lemma 5.5(4), for each cut vertex j in Γ and G the connected component of $\Gamma - j$ that contains n, we may remove the splines $\{\bar{y}_{G,k}^j|_{S_{n-1}} \mid k \in [n]\}$ from $\mathcal{F}_\Gamma^{(n)}$ without changing the $\mathbb{C}$-span. Similarly,

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by Lemma 5.5(4), for each cut vertex $j \neq n-1$ of $\Gamma - n$ and connected component G of $(\Gamma - n) - j$ that contains $n-1$, we may remove the splines $\{\bar{y}^j_{G,k} \mid k \in [n]\}$ from $\mathcal{F}_{\Gamma-n}$.

Let $\bar{\rho} \in \mathcal{F}_\Gamma$ such that $\bar{\rho}|_{S_{n-1}}$ is a nonzero element of $\mathcal{F}_\Gamma^{(n)}$. This means that $\bar{\rho} \neq \bar{t}_n$ and $\bar{\rho} \neq \bar{x}_n$. Additionally, by the definitions, if $\bar{\rho} = \bar{f}^s_A$, we must have $n \notin A$ (otherwise, $\bar{\rho} \equiv 0$) and $s \neq (n-1, n)$, and if $\bar{\rho} = \bar{y}^j_{i,k}$, then $k \neq n$. So we have a combinatorial description for the elements of $\mathcal{F}_\Gamma^{(n)}$. In particular, as collections of functions from S_{n-1} to $\mathbb{C}[t_\bullet]$, we wish to show that the following two sets have the same $\mathbb{C}$-span:

$$\mathcal{F}_\Gamma^{(n)} = \mathcal{T}_{n-1} \cup \mathcal{X}_{n-1} \cup \left\{ \bar{f}^s_A \left| \begin{matrix} s \text{ cut edge of } \Gamma, \\ |A| = |G_s| \\ s \neq (n-1, n), \\ n \notin A \end{matrix} \right. \right\} \cup \left\{ \bar{y}^j_{G,k} \left| \begin{matrix} j \vdash \Gamma, \\ n \notin G, \\ k \in [n-1] \end{matrix} \right. \right\}$$

and

$$\mathcal{F}_{\Gamma-n} = \mathcal{T}_{n-1} \cup \mathcal{X}_{n-1} \cup \left\{ \bar{f}^s_A \left| \begin{matrix} s \text{ cut edge of } \Gamma - n, \\ |A| = |G_s| \end{matrix} \right. \right\} \cup \left\{ \bar{y}^j_{G,k} \left| \begin{matrix} j \vdash \Gamma - n, \\ n-1 \notin G, \\ k \in [n-1] \end{matrix} \right. \right\},$$

where for the cut edges $s = (i < j)$ of either Γ or $\Gamma - n$, the component G_s is the connected component of the graph with edge s removed that contains i.

The equalities for $\bar{t}_i \in \mathcal{T}_{n-1}$ and $\bar{x}_i \in \mathcal{X}_{n-1}$ are obvious, so we focus on the latter two subsets. The remainder of the proof is in three cases: whether the cliqued and naturally labeled graph Γ is type A, B or C. Each argument amounts to matching the cut vertices and cut edges of Γ to those in $\Gamma - n$ (and vice versa). Each match gives pairs of splines in the third and fourth subsets above that are in fact equal to each other. Then we ensure that wherever these graph objects do not align, the 'unmatched' splines in each set are contained within the other's $\mathbb{C}$-span.

Type A: First, we compare the cut edges of Γ and $\Gamma - n$ and ensure that each spline in the third subsets of both $\mathcal{F}_\Gamma^{(n)}$ and $\mathcal{F}_{\Gamma-n}$ are contained within the span of the other set. If Γ is type A, then every cut edge of $\Gamma - n$ is also a cut edge of Γ. Every cut edge of Γ that is not $(n-1, n)$ is also a cut edge of $\Gamma - n$. Finally, if $s = (i < j) \neq (n-1, n)$ is a cut edge, then the connected component of $([n]E(\Gamma) \setminus \{s\})$ that contains i and the connected component of $([n-1], E(\Gamma - n) \setminus \{s\})$ that contains i are equal, since these are the components with lower-valued vertices and thus are unaffected by removing n. So the third subsets in $\mathcal{F}_\Gamma^{(n)}$ and $\mathcal{F}_{\Gamma-n}$ above are actually equal.

Second, we compare the cut vertices and associated connected components of Γ and $\Gamma - n$ and ensure that each spline in the fourth subsets of both $\mathcal{F}_\Gamma^{(n)}$ and $\mathcal{F}_{\Gamma-n}$ are contained within the span of the other set. There are two cases: $\mathfrak{c}_\Gamma(n-1) = 1$ and $\mathfrak{c}_\Gamma(n-1) > 1$. If $\mathfrak{c}_\Gamma(n-1) > 1$, every cut vertex in Γ is a cut vertex in $\Gamma - n$, and vice versa. If $j \vdash \Gamma - n$ where $j \neq n-1$, if G is the component of $\Gamma - j$ that contains n then $G - n$ is the component of $(\Gamma - n) - j$ that contains $n-1$. If $j = n-1$, then $\mathcal{F}_{\Gamma-n}$ contains every $\bar{y}^{n-1}_{G,k}$ and $\mathcal{F}_\Gamma^{(n)}$ contains every $\bar{y}^{n-1}_{G,k}$ such that $n \notin G$. Either way, these two collections of splines are identical, so the fourth subsets in $\mathcal{F}_\Gamma^{(n)}$ and $\mathcal{F}_{\Gamma-n}$ are in fact equal.

If $\mathfrak{c}_\Gamma(n-1) = 1$, then $n-1$ is not a cut vertex of $\Gamma - n$, so $\mathcal{F}_{\Gamma-n}$ does not contain the spline $\bar{y}^{n-1}_{G,k} \in \mathcal{F}_\Gamma^{(n)}$ where $V(G) = [n-2]$. However, in this case, for all $w \in S_{n-1}$, we compute

$$\begin{aligned} \bar{y}^{n-1}_{G,k}(w) &= \begin{cases} t_k - t_{w(n-1)} & w^{-1}(k) \in [n-2] \\ 0 & w^{-1}(k) = n-1. \end{cases} \\ &= \bar{t}_k(w) - \bar{x}_{n-1}(w) \in \mathbb{C}\mathcal{F}_{\Gamma-n}. \end{aligned}$$

Every other cut vertex $j \vdash \Gamma - n$ and connected component G of $(\Gamma - n) - j$ (that does not contain $n-1$) is also a cut vertex of Γ and connected component of $\Gamma - j$ (that does not contain n), so each spline of the form $\bar{y}^j_{G,k}$ in $\mathcal{F}_{\Gamma-n}$ has a direct counterpart in $\mathcal{F}_\Gamma^{(n)}$. Thus $\mathbb{C}\mathcal{F}_\Gamma^{(n)} = \mathbb{C}\mathcal{F}_{\Gamma-n}$, and the claim holds in type A.

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Type B: First, we compare the cut vertices and associated connected components to match elements in the fourth subsets. If Γ is type B, then every cut vertex in $\Gamma - n$ is also a cut vertex of Γ, and for all $j \vdash \Gamma$, the connected component of $\Gamma - j$ that contains n also contains $n-1$, so the fourth subsets in $\mathcal{F}_\Gamma^{(n)}$ and $\mathcal{F}_{\Gamma-n}$ are equal.

Now compare the cut edges to match elements in the third subsets. Every cut edge in Γ is a cut edge in $\Gamma - n$; however, the edge $(n-2, n-1)$ is a cut edge in $\Gamma - n$ but not in Γ. For this cut edge, we let $G_{(n-2,n-1)}$ be the subgraph with vertex set $[n-2]$. So $\mathcal{F}_{\Gamma-n}$ has a subset $\left\{\bar{f}^{(n-2,n-1)}_{[n-1]\backslash k} \mid k \in [n-1]\right\}$ of splines that is not a subset of $\mathcal{F}_\Gamma^{(n)}$. We will show that these splines are contained within the span $\mathbb{C}\mathcal{F}_\Gamma^{(n)}$. By Lemma 5.7 applied to the leaf $(n-2, n-1)$ in $\Gamma - n$, each spline in $\left\{\bar{f}^{(n-2,n-1)}_{[n-1]\backslash k} \in\mid k \in [n-1]\right\} \subset \mathcal{F}_{\Gamma-n}$ is a linear combination of the remaining splines in $\mathcal{F}_{\Gamma-n}$. Since the fourth subsets are equal and every *other* cut edge of $\Gamma - n$ is a cut edge of Γ, each of these remaining splines is also in $\mathcal{F}_\Gamma^{(n)}$. So the one subset $\left\{\bar{f}^{(n-2,n-1)}_{[n-1]\backslash k} \mid k \in [n-1]\right\}$ of unmatched splines in $\mathcal{F}_{\Gamma-n}$ is contained within the span $\mathbb{C}\mathcal{F}_\Gamma^{(n)}$, and so $\mathbb{C}\mathcal{F}_\Gamma^{(n)} = \mathbb{C}\mathcal{F}_{\Gamma-n}$.

Type C: If Γ is type C, then every cut vertex or edge in $\Gamma - n$ is also a cut vertex or edge in Γ, and vice versa. Additionally, for any $j \vdash \Gamma$, the connected component of $\Gamma - j$ containing n also contains $n-1$. So all indexing data is the same, and so $\mathcal{F}_\Gamma^{(n)} = \mathcal{F}_{\Gamma-n}$. Thus, the claim holds in Type C. □

The proof of Theorem 7.2 below assumes a natural label, so we will use the indexing conventions for $\mathcal{F}_\Gamma$ described in Subsection 6.1. In particular, we will heavily use the weakly dominant $(i, j) > \Gamma$ and strongly dominant $(i, j) \gg \Gamma$ pairs in Definition 6.5. Now we are able to prove that $\mathbb{C}\mathcal{F}_\Gamma = \mathcal{M}_\Gamma^1$ and compute a recursive dimension formula.

Theorem 7.2. *Let Γ be a connected graph. The splines $\mathcal{F}_\Gamma$ from Equation (5.1) form a $\mathbb{C}$-spanning set of $\mathcal{M}_\Gamma^1$. Furthermore, if Γ is cliqued and naturally labeled, then*

$$\dim_{\mathbb{C}}(\mathcal{M}_\Gamma^1) = 1 + \dim_{\mathbb{C}}(\mathcal{M}_{\Gamma-n}^1) + \begin{cases} \binom{n-1}{1} & \text{if } \Gamma \text{ is type A} \\ 1 & \text{if } \Gamma \text{ is type B/C} \end{cases}$$
$$+ \sum_{(i,j)\gg\Gamma} \binom{n-1}{\left|\Gamma_i^j\right| - 1} + |\{(i,j) > \Gamma\}|.$$

Proof. It suffices to assume for both parts of the claim that Γ is cliqued and naturally labeled. Recall the decomposition $S_n = S_n^1 \sqcup \cdots \sqcup S_n^n$, where $S_n^i := \{w \in S_n \mid w(i) = n\}$. Note $S_n^n = S_{n-1}$. Let $\bar{\rho} \in \mathcal{M}_\Gamma^1$. We will prove that $\bar{\rho} \in \mathbb{C}\mathcal{F}_\Gamma$ and proceed by induction on n. The base case is $n = 3$, where $\mathcal{M}_\Gamma = \mathbb{C}\mathcal{F}_\Gamma$ is easily verified by hand (there are only two connected graphs on three vertices) and either way follows from [4].

In each of the three steps to the proof given below, we use elements of $\mathcal{F}_\Gamma$ to replace $\bar{\rho}$ with a spline supported on a strictly smaller subset of S_n. To track $\dim_{\mathbb{C}}(\mathcal{M}_\Gamma^1)$, we will create a set $\mathcal{B}$ of linearly independent elements of $\mathcal{M}_\Gamma^1$.

(**Step 1:** S_n^n) This step applies the induction assumption to replace $\bar{\rho}$ with a spline supported on $S_n^1 \sqcup \cdots \sqcup S_n^{n-1}$. If $w, v \in S_n^n = S_{n-1}$ with $w^{-1}v = (i, j) \in E(\Gamma - n)$, then $\bar{\rho}(w) - \bar{\rho}(v) = c\big(t_{w(i)} - t_{w(j)}\big)$, where $c \in \mathbb{C}$. Since $w(i) \neq n$ and $w(j) \neq n$, the coefficient $[t_n]\bar{\rho}(w)$ of t_n in $\bar{\rho}(w)$ must be equal to $[t_n]\bar{\rho}(v)$. Since $\Gamma - n$ is connected, the induced subgraph of $\mathcal{G}_\Gamma$ with vertex set S_n^n is connected. Moreover, the coefficient of t_n is the same for all $\bar{\rho}(u)$, where $u \in S_n^n$. Let $c_n := [t_n]\bar{\rho}(u)$ for $u \in S_n^n$. Then $[t_n](\bar{\rho} - c_n\bar{t}_n)(u) = 0$ for all $u \in S_{n-1}$.

So we replace $\bar{\rho}$ with $\bar{\rho} - c_n\bar{t}_n$, and now $\bar{\rho}(u) \in \mathbb{C}[t_1, \ldots, t_{n-1}]$ when $u \in S_n^n$. Let $\mathcal{B} := \{\bar{t}_n\}$. We will add linearly independent elements to $\mathcal{B}$ throughout the proof and keep track of $|\mathcal{B}|$.

By Lemma 7.1 and the induction hypothesis, $\mathcal{M}_{\Gamma-n}^1 = \mathbb{C}\mathcal{F}_{\Gamma-n} = \mathbb{C}\mathcal{F}_\Gamma^{(n)}$. So we may assume that $\bar{\rho}|_{S_{n-1}} \equiv 0$. Add to $\mathcal{B}$ the $\dim_{\mathbb{C}}(\mathcal{M}_{\Gamma-n}^1)$-many splines required. Note these splines are independent once restricted to $S_n^n = S_{n-1}$, so any nontrivial linear combination will have elements of S_n^n in its support. At this point, $|\mathcal{B}| = \dim_{\mathbb{C}}(\mathcal{M}^1_{\Gamma-n}) + 1$, and $\bar{\rho} \equiv 0$ on S_n^n.

(**Step 2:** S_n^{n-1}) Next, we use elements of $\mathcal{F}_\Gamma$ to replace $\bar{\rho}$ with a spline that evaluates to 0 on $S_n^{n-1} \sqcup S_n^n$. The process is slightly different for graphs of type A and types B/C.

Since Γ is cliqued and naturally labeled, $(n-1, n) \in E(\Gamma)$. Thus, for all $w \in S_n^{n-1}$, there is an edge $(w, w(n-1,n)) \in E(\mathcal{G}_\Gamma)$ where $w(n-1,n) \in S_n^n$. Each of these edges are labeled $\langle t_{w(n-1)} - t_{w(n)} \rangle = \langle t_n - t_{w(n)} \rangle$. Since $\bar{\rho} \equiv 0$ on S_n^n, we have that $\bar{\rho}(w) = c_w (t_n - t_{w(n)})$ for some $c_w \in \mathbb{C}$ for all $w \in S_n^{n-1}$.

Type A: By Lemma 6.13, if $w, v \in S_n^{n-1}$ and $w(n) = v(n)$, then $c_w = c_v$. Let $c_k := c_w$ when $w(n) = k$. If $w(n-1) = n$ and $w(n) = k < n$, then by definition, $\bar{f}^{(n-1,n)}_{[n]\setminus\{k\}}(w) = t_n - t_{w(n)}$. However, if $w(n) = n$, then $\bar{f}^{(n-1,n)}_{[n]\setminus\{k\}}(w) = 0$ whenever $k \neq n$. It follows that

$$\bar{\rho} - \sum_{i=1}^{n-1} c_k \bar{f}^{(n-1,n)}_{[n]\setminus\{k\}} \equiv 0$$

on $S_n^{n-1} \sqcup S_n^n$. Add the $n-1$ coset splines $\bar{f}^{(n-1,n)}_{[n]\setminus\{k\}}$ used above to $\mathcal{B}$, which are linearly independent from the splines already in $\mathcal{B}$ since they are not supported on S_n^n and have disjoint support on S_n^{n-1}. In type A, at this point, $|\mathcal{B}| = 1 + \dim_{\mathbb{C}}(\mathcal{M}^1_{\Gamma-n}) + \binom{n-1}{1}$, and $\mathcal{B}$ is linearly independent, even if we restrict the splines in $\mathcal{B}$ to $S_n^{n-1} \sqcup S_n^n$.

Type B/C: We proceed in the same format as type A. By Lemma 6.13, if $w, v \in S_n^{n-1}$, then $c_w = c_v$. Write $c := c_w$ for $w \in S_n^{n-1}$. Then

$$\bar{\rho} - c(\bar{t}_n - \bar{x}_n) \equiv 0$$

on $S_n^{n-1} \sqcup S_n^n$. Add the single linearly independent spline $\bar{t}_n - \bar{x}_n$ to $\mathcal{B}$. In type B/C at this point, $|\mathcal{B}| = \dim_{\mathbb{C}}(\mathcal{M}^1_{\Gamma-n}) + 2$, and $\mathcal{B}$ is linearly independent, even if we restrict each spline to $S_n^{n-1} \sqcup S_n^n$.

(**Step 3:** S_n^i) Now given a spline $\bar{\rho}$ such that $\bar{\rho} \equiv 0$ on $S_n^{i+1} \sqcup \cdots S_n^n$, we show how to replace it with a spline that vanishes on $S_n^i \sqcup \cdots S_n^n$. This step is repeated until $\bar{\rho}$ vanishes on all of S_n. Assume that $\bar{\rho} \equiv 0$ on $S_n^{i+1} \sqcup \cdots \sqcup S_n^n$. Additionally, we assume that the splines in $\mathcal{B}$ are linearly independent; moreover, the set remains linearly independent once each spline is restricted to $S_n^{i+1} \sqcup \cdots \sqcup S_n^n$. In particular, any nontrivial linear combination of splines in $\mathcal{B}$ is nonzero on $S_n^{i+1} \sqcup \cdots \sqcup S_n^n$ (and therefore S_n). The remainder of the proof is type A/B/C-independent but still requires three cases.

First, a formulation of $\bar{\rho}(w)$ for $w \in S_n^i$ will be used in each case. Since Γ is naturally labeled and $i \neq n$, there exists $j \in [n]$ such that $i < j$ and $(i, j) \in E(\Gamma)$, so $(w, w(i,j)) \in E(\mathcal{G}_\Gamma)$. If $w \in S_n^i$, then $w(i,j) \in S_n^j$, so

$$\bar{\rho}(w) = c_w (t_{w(i)} - t_{w(j)}) = c_w (t_n - t_{w(j)})$$

for some $c_w \in \mathbb{C}$.

Case 1: If i is *not* j-dominant for any $j \in [n]$, since Γ is naturally labeled, there exist (at least) two vertices j, k where $i < j < k$ and $\{(i,j), (i,k)\} \subset E(\Gamma)$. It follows that

$$\bar{\rho}(w) = c_w (t_n - t_{w(j)}) = c'_w (t_n - t_{w(k)}).$$

This is not possible for $c_w, c'_w \in \mathbb{C}$ unless $c_w = c'_w = 0$, and so $\bar{\rho}(w) = 0$. In short, $\bar{\rho} \equiv 0$ on S_n^i, and we do not need any splines from $\mathcal{F}_\Gamma$ to achieve this.

Case 2: If there exists $j \in [n]$ where $(i, j) \gg \Gamma$, then i is the maximal vertex in its connected component of $\Gamma - j$. Thus, the vertex j is the *only* element in the neighborhood $N(i)$ of the vertex i that is greater than i (so i is not k-dominant for any $k \neq j$), and (i, j) is a cut edge of Γ. By Lemma 6.14, if $v \in w\langle E(\Gamma) \setminus (i, j) \rangle$, then $c_v = c_w$.

Recall that $v \in w\langle E(\Gamma) \setminus (i,j)\rangle$ if and only if $w(V(\Gamma_i^j)) = v(V(\Gamma_i^j))$. Let

$$\mathcal{A} := \left\{A \subset [n] \middle| |A| = \left|\Gamma_i^j\right|,\ n \in A\right\},$$

and write $c_A := c_w$ if $w(V(\Gamma_i^j)) = A$. If $w(i) = n$, then $w^{-1}(V(\Gamma_i^j)) = A \in \mathcal{A}$, and we compute $\bar{f}_A^{(i,j)}(w) = t_n - t_{w(j)}$. Also, since Γ is naturally labeled, if $k > i$, then $k \notin \Gamma_i^j$. In particular, if $k > i$ and $w(k) = n$, then $w^{-1}(V(\Gamma_i^j)) \notin \mathcal{A}$, and so $\bar{f}_A^{(i,j)}(w) = 0$ for all $w \in S_n^{i+1} \sqcup \cdots \sqcup s_n^n$. It follows that

$$\bar{\rho} - \sum_{A \in \mathcal{A}} c_A \bar{f}_A^{(i,j)} \equiv 0$$

on $S_n^i \sqcup \cdots \sqcup S_n^n$. We replace $\bar{\rho}$ with this spline. The coset splines $\bar{f}_A^{(i,j)}$ for $A \in \mathcal{A}$ have disjoint support among themselves and are only supported on S_n^r for $r \leq i$, so $\{\bar{f}_A^{(i,j)} \mid A \in \mathcal{A}\} \cup \mathcal{B}$ is linearly independent, even when each spline is restricted to $S_n^i \sqcup \cdots \sqcup S_n^n$. Each time we use Case 2 (i.e., for each $(i,j) \gg \Gamma$), we add $|\mathcal{A}| = \binom{n-1}{\left|\Gamma_i^j\right| - 1}$-many splines to $\mathcal{B}$.

Case 3: If there exists $j \in [n]$ where $(i,j) > \Gamma$, the vertex j is the *only* element in the neighborhood $N(i)$ of the vertex i that is greater than i (so i is not k-dominant for any $k \neq j$), but (i,j) is not a cut edge. By Lemma 6.15, if $w, v \in S_n^i$, then $c_w = c_v =: c$. Finally, we confirm that

$$\bar{\rho} - c\,\bar{y}_{i,n}^j \equiv 0$$

on $S_n^i \sqcup \cdots \sqcup S_n^n$. We replace $\bar{\rho}$ with this spline. The single spline $\bar{y}_{i,n}^j$ is supported on S_n^r for $r \leq i$, and so $\{\bar{y}_{i,n}^j\} \cup \mathcal{B}$ is linearly independent, even when restricted to $S_n^i \sqcup \cdots \sqcup S_n^n$. Each time Case 3 is used (i.e., for each $(i,j) > \Gamma$), we add 1 spline to $\mathcal{B}$.

When $i = 1$ is reached, we have used $\mathcal{F}_\Gamma$ to replace $\bar{\rho}$ with a spline $\bar{\rho} \equiv 0$ on all of S_n. Thus, $\bar{\rho} \in \mathbb{C}\mathcal{B}$, and the set $\mathcal{B} \subseteq \mathcal{F}_\Gamma$ is linearly independent (the restriction is now to the whole symmetric group $S_n^1 \sqcup \cdots \sqcup S_n^n = S_n$), and $|\mathcal{B}| = \dim_{\mathbb{C}}(\mathcal{M}_\Gamma^1)$ is as claimed. □

8. The linear dimension formula

This section constructs a combinatorial invariant of simple graphs that is also the $\mathbb{C}$-dimension of the associated linear splines.

Let Γ be a connected graph on at least three vertices. First, if $j \vdash \Gamma$ is a cut vertex, let $\mathfrak{c}_\Gamma(j) + 1$ be the number of connected components in $\Gamma - j$. This is a straightforward expansion of the definition we gave for $\mathfrak{c}_\Gamma(j)$ from naturally labeled graphs to all graphs. When Γ is fixed, we may drop the subscript and write $\mathfrak{c}(j)$.

Definition 8.1. Recall the construction of a block-cut tree in Definition 6.2. In this tree, every leaf is a block of Γ. Let LB_Γ be the set of blocks in Γ that are leaves in the block-cut tree. We call elements of LB_Γ *leaf blocks* of Γ. Let IB_Γ be the set of blocks in Γ that are not leaves in the block-cut tree. We call elements of IB_Γ *internal blocks* of Γ. Note when Γ is 2-connected, $\mathrm{LB}_\Gamma = \emptyset$ and $\mathrm{IB}_\Gamma = \{\Gamma\}$; in particular, when the block-cut tree of Γ is a single vertex (i.e., when Γ is 2-connected), we consider Γ to be an internal block.

If a block B in Γ is size $|B| = 2$, that block must consist of two vertices in Γ connected by an edge. Since blocks are maximal 2-connected subgraphs, this edge must be a cut edge. In particular, blocks B in IB_Γ of size $|B| = 2$ are in bijection with cut edges of Γ that are not leaf edges. Let $\mathrm{IC}_\Gamma := \{(i,j) \in E(\Gamma) \mid V(B) = \{i,j\}$ for some $B \in \mathrm{IB}_\Gamma\}$. The elements of IC_Γ are *internal cut edges* of Γ.

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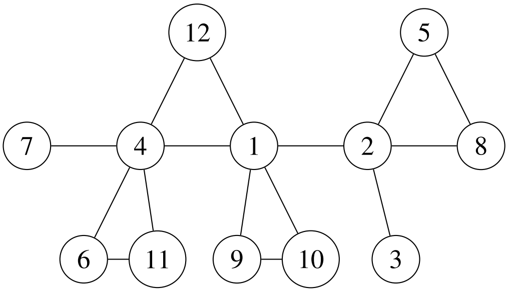
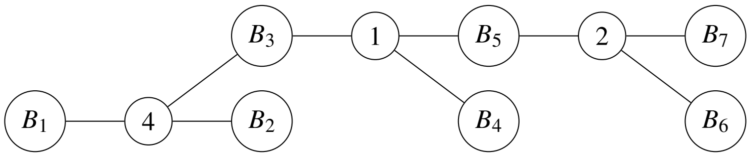
Example 8.2. Consider the graph



with block-cut tree



This is the construction from the beginning of Example 6.4. The leaf blocks are

$$\mathrm{LB}_\Gamma = \{B_1, B_2, B_4, B_6, B_7\}.$$

The internal blocks are

$$\mathrm{IB}_\Gamma = \{B_3, B_5\}.$$

Within those internal blocks, $|B_5| = 2$ and B_5 corresponds to the cut edge $(1, 2)$. So

$$\mathrm{IC}_\Gamma = \{(1, 2)\},$$

and $(1, 2)$ is the only cut edge in Γ that is not a leaf edge.

For a connected graph Γ on n vertices, we define

$$D_\Gamma \coloneqq 2n - 1 - \sum_{j \vdash \Gamma} \mathfrak{c}_\Gamma(j) + n(|\mathrm{LB}_\Gamma| + |\{B \in \mathrm{IB}_\Gamma \mid |B| > 2\}| - 1) + \sum_{s \in \mathrm{IC}_\Gamma} \binom{n}{|G_s|}, \tag{8.1}$$

where G_s is defined as in Section 5 (i.e., G_s is one of the two connected components of the graph $([n], E(\Gamma) \setminus \{s\})$). This formula is unaffected by a choice of component, as the sizes of the two connected components in $([n], E(\Gamma) \setminus \{s\})$ sum to n and $\binom{n}{|G_s|} = \binom{n}{n - |G_s|}$.

Remark 8.3. The invariant D_Γ might be more concisely written as

$$D_\Gamma = n - 1 - \sum_{j \vdash \Gamma} \mathfrak{c}_\Gamma(j) + n(|\mathrm{LB}_\Gamma| + |\{B \in \mathrm{IB}_\Gamma \mid |B| > 2\}|) + \sum_{s \in \mathrm{IC}_\Gamma} \binom{n}{|G_s|},$$

but the format in Equation 8.1 is more conducive to the proofs that follow.

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Example 8.4. Consider the graph Γ from Example 8.2 above. The three cut vertices are 4, 1 and 2. Each of those cut vertices separate Γ in to three connected components, so $\mathfrak{c}_\Gamma(4) = \mathfrak{c}_\Gamma(1) = \mathfrak{c}_\Gamma(2) = 2$. The block B_3 is the only block in IB_Γ with more than two vertices, so $|\{B \in \mathrm{IB}_\Gamma \mid |B| > 2\}| = 1$. The only internal cut edge is $(1, 2)$, and this cut edge separates Γ in to a component of size 8 and a component of size 4. We choose the component with vertex set $\{1, 4, 6, 7, 9, 10, 11, 12\}$, but note that $\binom{12}{8} = \binom{12}{12-8} = \binom{12}{4}$. We compute that

$$D_\Gamma = 2 \cdot 12 - 1 - (2 + 2 + 2) + 12(5 + 1 - 1) + \binom{12}{8} = 572.$$

Since D_Γ is defined using only the block-cut tree, cut edges and cut vertices of Γ, it is an invariant of the isomorphism class of Γ. We note that if Γ is 2-connected, it follows that $D_\Gamma = 2n - 1$.

Lemma 8.5 below gives a formulation of D_Γ specific to naturally labeled graphs. Its proof constructs important bijections that will be used later in the proofs of Proposition 8.9 and Corollary 9.3.

Lemma 8.5. *Let Γ be a naturally labeled graph. Then*

$$D_\Gamma = 2n - 1 + \sum_{(i,j) > \Gamma} n + \sum_{(i,j) \gg \Gamma} \binom{n}{\left|\Gamma_i^j\right|} - \sum_{j \vdash \Gamma} \mathfrak{c}_\Gamma(j).$$

Proof. We compare the right-hand side of the formula above with (8.1). The clear cancellation between the two sides of the claimed equality is $2n - 1 - \sum_{j \vdash \Gamma} \mathfrak{c}_\Gamma(j)$. Thus, the claim will follow if

$$n(|\mathrm{LB}_\Gamma| + |\{B \in \mathrm{IB}_\Gamma \mid |B| > 2\}| - 1) + \sum_{s \in \mathrm{IC}_\Gamma} \binom{n}{|G_s|} = \sum_{(i,j) > \Gamma} n + \sum_{(i,j) \gg \Gamma} \binom{n}{\left|\Gamma_i^j\right|}. \tag{*}$$

Since Γ is naturally labeled, the block containing n is a leaf in the block-cut tree. So $|\mathrm{LB}_\Gamma| - 1$ can be interpreted as the number of leaf blocks that do not contain n.

In the remainder of the proof, we pair blocks in LB_Γ and IB_Γ that contribute on the left side of the claimed equality (*) with dominant pairs (i, j) that contribute to the right side, in order to identify cancellations. The pairings that we will prove and use are as follows:

1. Leaf blocks $B \in \mathrm{LB}_\Gamma$ with $|B| = 2$ and $n \notin B$ contribute n to the left side of (*) and are in bijection with strongly dominant pairs (i, j) where $\left|\Gamma_i^j\right| = 1$, which contribute $\binom{n}{1} = n$ to the right side.
2. The set of internal cut edges IC_Γ is equal to the set of strongly dominant pairs (i, j) such that $\left|\Gamma_i^j\right| > 1$, and they both contribute $\binom{n}{|G_s|} = \binom{n}{\left|\Gamma_i^j\right|}$.
3. Leaf blocks and internal blocks of size at least three (i.e., all blocks of size at least 3) that do *not* contain the vertex n contribute n to the left side of the claimed equality and are in bijection with weakly dominant pairs (i, j), each of which contributes n to the right side.

This list also serves as an outline of the proof that follows.

(1) Let B be a leaf block of size 2 with vertex set $V(B) = \{i, j\}$ where $i < j$. Then the single edge (i, j) within B is a cut edge of Γ, and since B is a leaf block, that cut edge (i, j) must separate a single vertex. If this block does not contain n (so $j \neq n$), then since Γ is naturally labeled, the cut edge must be a dominant pair and that separated vertex must be i. So $(i, j) \gg \Gamma$ and $V(\Gamma_i^j) = \{i\}$, and $\left|\Gamma_i^j\right| = 1$. However, if $(i, j) \gg \Gamma$ and $\left|\Gamma_i^j\right| = 1$, then i must be a leaf, and the subgraph $(\{i, j\}, \{(i, j)\})$ is a block

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in Γ that does not contain n. In particular, we have shown

$$\{B \in \mathrm{LB}_\Gamma \mid |B| = 2,\ n \notin B\} = \{B \in \mathrm{LB}_\Gamma \mid V(B) = \{i, j\},\ (i, j) \gg \Gamma\}.$$

So the leaf blocks B in Γ of size 2 are in natural bijection with the dominant pairs $(i, j) \gg \Gamma$ where $\left|\Gamma_i^j\right| = 1$. Formally, $|\{B \in \mathrm{LB}_\Gamma \mid |B| = 2,\ n \notin B\}| = \left|\left\{(i, j) \gg \Gamma \mid \left|\Gamma_i^j\right| = 1\right\}\right|$. Thus,

$$\begin{aligned}
|\mathrm{LB}_\Gamma| + |\{B \in \mathrm{IB}_\Gamma \mid |B| > 2\}| - 1 &= |\{B \in \mathrm{LB}_\Gamma \mid |B| = 2\}| + |\{B \in \mathrm{LB}_\Gamma \mid |B| > 2\}| \\
&\qquad + |\{B \in \mathrm{IB}_\Gamma \mid |B| > 2\}| - 1 \\
&= \left|\left\{(i, j) \gg \Gamma \mid \left|\Gamma_i^j\right| = 1\right\}\right| + |\{B \in \mathrm{LB}_\Gamma \mid |B| > 2,\ n \notin B\}| \\
&\qquad + |\{B \in \mathrm{IB}_\Gamma \mid |B| > 2\}| \\
&= \left|\left\{(i, j) \gg \Gamma \mid \left|\Gamma_i^j\right| = 1\right\}\right| \\
&\qquad + |\{B \mid B \text{ a block in } \Gamma,\ |B| > 2,\ n \notin B\}|.
\end{aligned}$$

We cancel $n\left|\left\{(i, j) \gg \Gamma \mid \left|\Gamma_i^j\right| = 1\right\}\right|$ from the left side and $\sum_{\substack{(i,j)\gg\Gamma \\ \left|\Gamma_i^j\right|=1}} \binom{n}{1}$ from the right side of (*), and it remains to prove that

$$n|\{B \mid B \text{ a block in } \Gamma,\ |B| > 2,\ n \notin B\}| + \sum_{s\in \mathrm{IC}_\Gamma} \binom{n}{|G_s|} = \sum_{(i,j)>\Gamma} n + \sum_{\substack{(i,j)\gg\Gamma \\ \left|\Gamma_i^j\right|>1}} \binom{n}{\left|\Gamma_i^j\right|}. \tag{**}$$

(2) Now if $s = (i, j) \in \mathrm{IC}_\Gamma$, then s is a cut edge, and since Γ is naturally labeled, $n \notin \{i, j\}$. So if $(i, j) \in \mathrm{IC}_\Gamma$, then $(i, j) \gg \Gamma$ and $\left|\Gamma_i^j\right| > 1$; otherwise, i is a leaf and the block B where $V(B) = \{i, j\}$ is not an internal block. However, if $(i, j) \gg \Gamma$ and $\left|\Gamma_i^j\right| > 1$, then i cannot be a leaf, (i, j) must be a cut edge and the block B where $V(B) = \{i, j\}$ is not a leaf in the block-cut tree. So $\mathrm{IC}_\Gamma = \left\{(i, j) \gg \Gamma \mid \left|\Gamma_i^j\right| > 1\right\}$. In particular,

$$\sum_{s\in \mathrm{IC}_\Gamma} \binom{n}{|G_s|} = \sum_{\substack{(i,j)\gg\Gamma \\ \left|\Gamma_i^j\right|>1}} \binom{n}{\left|\Gamma_i^j\right|}.$$

After cancelling this value from both sides of (**), it remains to show that

$$n|\{B \mid B \text{ a block in } \Gamma,\ |B| > 2,\ n \notin B\}| = \sum_{(i,j)>\Gamma} n. \tag{***}$$

(3) We argue that

$$|\{B \mid B \text{ a block in } \Gamma,\ |B| > 2,\ n \notin B\}| = |\{(i, j) > \Gamma\}|.$$

The bijection is as follows. If B is a block in Γ that does not contain n, then there is a unique path from B to the block B_0 that contains n in the block-cut tree of Γ. The first edge in that path is from B to a cut vertex of Γ that is contained within B. Let j be this cut vertex, and let i be the maximal vertex in B that is *not* equal to j. Note that B is not connected to n in $\Gamma - j$.

Since Γ is naturally labeled, (i, j) is an edge and i must also be the maximal vertex in its connected component of $\Gamma - j$ (every path from n to that component first passes through j, and i is the largest

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vertex in that component that is adjacent to j). Since $|B| > 2$, there is at least one other element of B in the neighborhood of j, so (i, j) is not a cut edge. In particular, $(i, j) > \Gamma$.

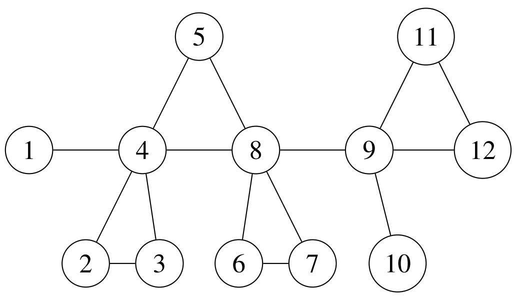
However, if $(i, j) > \Gamma$, let B be the block containing i. This block cannot contain n by the definition of a natural label and must be of size at least 2 since (i, j) is not a cut edge.

So we have a bijection, proving as a consequence (***), and the claim follows. □

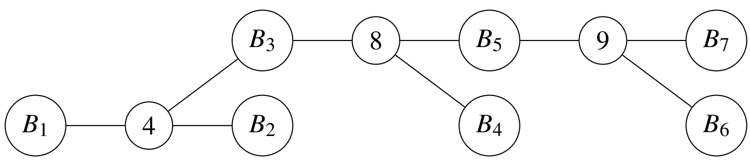
The bijection between certain dominant pairs and blocks that we constructed in the proof of Lemma 8.5 above will be used again to compute a recursive formula for D_Γ and provide a label-independent formula for $\mathbf{ch}(\mathrm{L}_\Gamma)_1$ and $\mathbf{ch}(\mathrm{R}_\Gamma)_1$ in Corollary 9.3 below. Visually, when Γ is naturally labeled, we have the following correspondences between blocks in Γ and dominant pairs.

Blocks B	$\left\{\begin{matrix} \lvert B\rvert > 2 \\ n \notin B \end{matrix}\right\}$	$\left\{\begin{matrix} B \in \mathrm{LB}_\Gamma \\ \lvert B\rvert = 2,\ n \notin B \end{matrix}\right\}$	$\left\{\begin{matrix} B \in \mathrm{IB}_\Gamma \\ \lvert B\rvert = 2 \end{matrix}\right\}$
	$\updownarrow$	$\updownarrow$	$\updownarrow$
Dominant pairs(i, j)	$\{(i, j) > \Gamma\}$	$\left\{\begin{matrix} (i, j) \gg \Gamma \\ \left\lvert \Gamma_i^j \right\rvert = 1 \end{matrix}\right\}$	$\left\{\begin{matrix} (i, j) \gg \Gamma \\ \left\lvert \Gamma_i^j \right\rvert > 1 \end{matrix}\right\}$

Example 8.6. Consider the naturally labeled graph Γ drawn below:



with block-cut tree



The blocks of size at least 2 that do not contain n correspond to weakly dominant pairs in the following manner:

Blocks	Pairs
B_2 on $\{2, 3, 4\}$	$(3, 4) > \Gamma$
B_3 on $\{4, 5, 8\}$	$(5, 8) > \Gamma$
B_4 on $\{6, 7, 8\}$	$(7, 8) > \Gamma$

The leaf blocks of size 2 that do not contain n correspond to strongly dominant pairs (i, j) where $\left|\Gamma_i^j\right| = 1$ in the following manner:

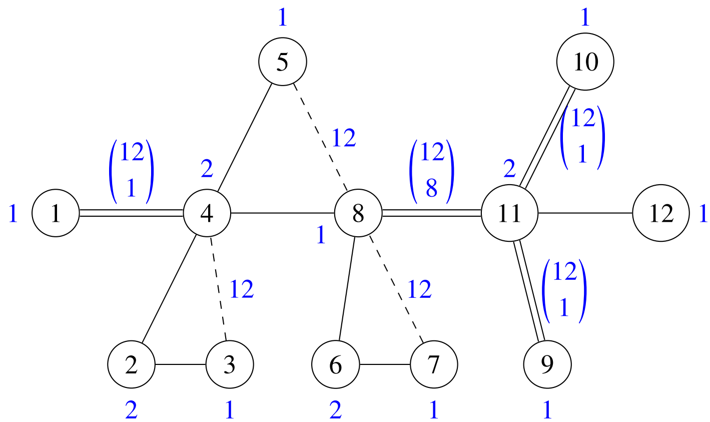
Blocks	Pairs
B_1 on $\{1,4\}$	$(1,4) > \Gamma$
B_6 on $\{9,10\}$	$(9,10) > \Gamma$

Finally, the internal block B_5 on $\{8,9\}$ (so of size 2) corresponds to the dominant pair $(8,9) \gg \Gamma$.

Example 8.7. When Γ is naturally labeled, each sum appearing in the formula in Lemma 8.5 has the following combinatorial interpretation:

- n : Every vertex of Γ contributes 1 to the sum,
- $n - 1 - \sum_{j \vdash \Gamma} \mathfrak{c}_\Gamma(j)$: Every vertex that is *not* the lower vertex in a dominant pair or the vertex n contributes an additional 1 to the sum (recall that the lower vertices in a dominant pair determine the pair uniquely).
- $\sum_{j \vdash \Gamma} \sum_{(i,j) > \Gamma} n$: Every weakly dominant pair contributes n to the sum.
- $\sum_{j \vdash \Gamma} \sum_{(i,j) \gg \Gamma} \binom{n}{\left|\Gamma_i^j\right|}$: Every strongly dominant pair (i, j) contributes $\binom{n}{\left|\Gamma_i^j\right|}$ to the sum.

Below, we have drawn Γ from Example 6.11, with the associated values for D_Γ in blue. In this picture, dashed lines indicate weak dominance and double lines indicate strong dominance.



Remark 8.8. It is not obvious that the formula in Lemma 8.5 is the same for different naturally labeled graphs in the isomorphism class. However, it is clear from the definition in (8.1) that D_Γ is an invariant, so they must be equal.

The following Proposition 8.9 gives an inductive formula for D_Γ when Γ is cliqued and naturally labeled that matches the inductive formula for $\dim_{\mathbb{C}}(\mathcal{M}_\Gamma^1)$ from Theorem 7.2.

Proposition 8.9. *If Γ is a cliqued and naturally labeled graph on at least 4 vertices, then D_Γ can be computed recursively as follows:*

$$D_\Gamma = 1 + D_{\Gamma - n} + \begin{cases} \binom{n-1}{1} & \text{if } \Gamma \text{ is type A} \\ 1 & \text{if } \Gamma \text{ is type B/C} \end{cases}$$
$$+ \sum_{(i,j) \gg \Gamma} \binom{n-1}{\left|\Gamma_i^j\right| - 1} + |\{(i,j) > \Gamma\}|.$$

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Before proving Proposition 8.9, we prove several computational lemmas and construct a finer categorization of type A graphs. The key idea is to concretely describe the block-cut tree of $\Gamma - n$ in terms of the block-cut tree of Γ.

If B is a block in Γ that does not contain n, then B is still 2-connected in $\Gamma - n$. Additionally, if i and k where $n \notin \{i, k\}$ are vertices in Γ such that i and k are in different connected components of $\Gamma - j$, then i and k are also in different connected components of $(\Gamma - n) - j$. In particular, If B is a block in Γ that does not contain n, then B is a block in $\Gamma - n$.

Lemma 8.10 below describes the simplest case, when Γ is either type B or type C

Lemma 8.10. *Let Γ be a cliqued and naturally labeled graph on at least 4 vertices. If Γ is of type B or C, then the following equalities hold:*

- $|\mathrm{LB}_\Gamma| = |\mathrm{LB}_{\Gamma-n}|$,
- $\mathrm{IB}_\Gamma = \mathrm{IB}_{\Gamma-n}$, *and*
- $\mathrm{IC}_\Gamma = \mathrm{IC}_{\Gamma-n}$.

Proof. Let B_0 be the block in Γ that contains n. Since Γ is type B or C, $|B_0| > 2$. Since Γ is cliqued, the subgraph $B_0 - n$ has at least 2 vertices and is also a clique, and in particular, $B_0 - n$ is a block in $\Gamma - n$. Every other block or cut vertex in Γ is a block or cut vertex in $\Gamma - n$. Thus, the block-cut tree of Γ is isomorphic to the block-cut tree of $\Gamma - n$.

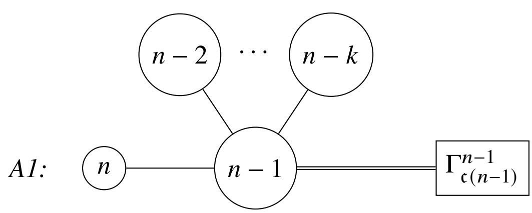
Since the block-cut trees are isomorphic they have the same number of leaves, and so $|\mathrm{LB}_\Gamma| = |\mathrm{LB}_{\Gamma-n}|$. Since the only block that changes is B_0 and all non-leaf blocks in Γ are non-leaf blocks in $\Gamma - n$, it follows that $\mathrm{IB}_\Gamma = \mathrm{IB}_{\Gamma-n}$. This equality implies $\mathrm{IC}_\Gamma = \mathrm{IC}_{\Gamma-n}$ by definition. □

We now assume that Γ is type A. We will give a similar computation for type-A graphs, but there are several cases to consider. Since $(n-1, n)$ is a cut edge, it corresponds to a block B_0 containing n in Γ that has no natural counterpart in the block-cut tree of $\Gamma - n$. Not only that, but $n - 1$ may not be a cut vertex in $\Gamma - n$.

Addressing this requires further decomposition of type A graphs, which we denote A1, A2, A3, A4 and A5. We will define them carefully below, but the consequences in each case in terms of moving from the block-cut tree of Γ to that of $\Gamma - n$ are essentially as follows (recall B_0 is the block in Γ containing n):

(A1) The block-cut tree of $\Gamma - n$ is simply that of Γ with the block B_0 removed.
(A2) The block-cut tree of $\Gamma - n$ is the block-cut tree of Γ with the block B_0 and the cut vertex $n - 1$ removed, and an internal block B' of size 2 for Γ becomes a leaf block for $\Gamma - n$.
(A3) The block-cut tree of $\Gamma - n$ is the block-cut tree of Γ with the block B_0 and the cut vertex $n - 1$ removed, but every other internal (resp. leaf) block of Γ remains an internal (resp. leaf) block of $\Gamma - n$.
(A4) The block-cut tree of $\Gamma - n$ is the block-cut tree of Γ with the block B_0 and the cut vertex $n - 1$ removed, and an internal block B' of size greater than 2 for Γ becomes a leaf block for $\Gamma - n$.
(A5) The graph $\Gamma - n$ is 2-connected.

First, a type A graph Γ is type A1 if $\mathfrak{c}_\Gamma(n-1) > 1$. Graphically, Γ looks like



Note that in type A1, $\Gamma - n$ has the same cut vertices as Γ, so the block-cut tree of $\Gamma - n$ is the block-cut tree of Γ with B_0 removed.

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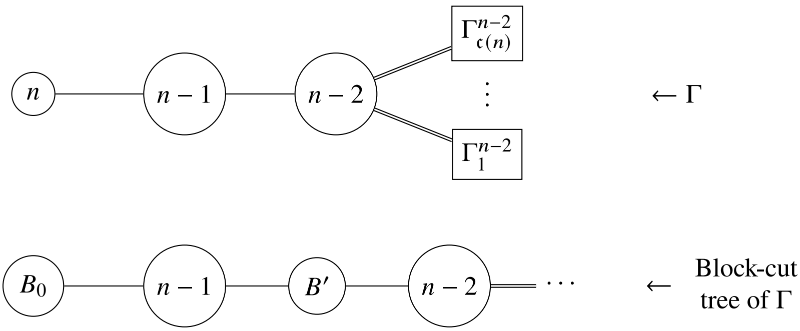
If Γ is type A but not type A1, then $n-1$ is a cut vertex of Γ, but it is not a cut vertex of $\Gamma - n$. So the block-cut tree of $\Gamma - n$ is the block-cut tree of Γ with both the block B_0 and the cut vertex $n-1$ removed. In particular, $n-1$ is contained within precisely two blocks in Γ: one that contains n (so B_0) and one that contains $n-2$. Let B' be the block in Γ that contains $n-1$ and $n-2$.

We say Γ is type A2 if $|B'| = 2$ (i.e., $V(B') = \{n-2, n-1\}$). Graphically, Γ and its block-cut tree look like

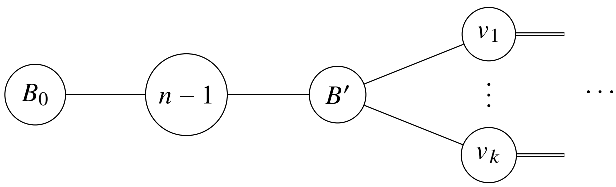
A2:



Note that the B' where $V(B') = \{n-2, n-1\}$ is associated to an internal cut edge in Γ and is a leaf block in $\Gamma - n$.

The graph Γ is type A3 if $|B'| > 2$ and B' contains more than 2 cut vertices of Γ. So B' is adjacent to more than two vertices in the block-cut tree of Γ. Graphically, the block-cut tree of Γ looks like

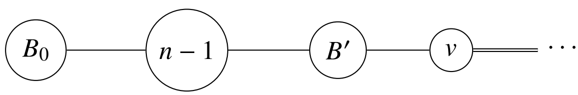
A3:



where $k > 1$. Note that B' is an internal block in both Γ and $\Gamma - n$.

The graph Γ is type A4 if $|B'| > 2$, and B' contains precisely 2 cut vertices of Γ, $n-1$ and some other cut vertex v of Γ. In particular, B' is adjacent to precisely two vertices in the block-cut tree of Γ. Graphically, the block-cut tree of Γ looks like

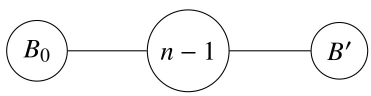
A4:



Note that B' is an internal block of size at least 3 in Γ and a leaf block in $\Gamma - n$.

Finally, the graph Γ is type A5 if $n-1$ is the only cut vertex in B'. In particular, B_0 and B' are the only two blocks in Γ, so the block-cut tree of Γ is

A5:



The computations for type A5 are generally easy, as $\Gamma - n$ is 2-connected.

Lemmas 8.11, 8.12 and 8.13 below use this finer categorization of type A graphs to explicitly compute the relationship between sums in the formulas of D_Γ and $D_{\Gamma-n}$. For the proofs of Lemmas 8.12, 8.13 and 8.11 below, B_0 is the block in Γ that contains n and (if Γ type A2-A5) B' is the block in Γ that contains $n-1$ but not n (so B' is still a block in $\Gamma - n$).

Lemma 8.11. *Let Γ be a cliqued and naturally labeled graph on at least 4 vertices. Then the number of leaf blocks in $\Gamma - n$ is*

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$$|\mathrm{LB}_{\Gamma-n}| = \begin{cases} |\mathrm{LB}_\Gamma| & \textit{if } \Gamma \textit{ is type A2 or A4} \\ |\mathrm{LB}_\Gamma| - 1 & \textit{if } \Gamma \textit{ is type A1 or A3} \\ |\mathrm{LB}_\Gamma| - 2 & \textit{if } \Gamma \textit{ is type A5.} \end{cases}$$

Proof. If Γ is type A2 or A4, then $B' \in \mathrm{LB}_{\Gamma-n}$ is a leaf block of $\Gamma - n$, but $B' \in \mathrm{IB}_\Gamma$ is an internal block of Γ. So $\mathrm{LB}_{\Gamma-n} = (\mathrm{LB}_\Gamma \setminus \{B_0\}) \cup \{B'\}$. Less formally, we lose a block B_0 and gain a block B', maintaining the same size.

If Γ is type $A1$ or $A3$, every leaf block in $\Gamma - n$ is a leaf block in Γ, but we still lose B_0. So the size decrements by 1.

If Γ is type A5, then Γ has two leaf blocks (B_0 and B'), whereas $\Gamma - n$ is a clique, with zero leaf blocks. We directly compute $|\mathrm{LB}_\Gamma| = 2$ and $|\mathrm{LB}_{\Gamma-n}| = 0$. □

Lemma 8.12. *Let* Γ *be a cliqued and naturally labeled graph on at least* 4 *vertices. Then*

$$|\{B \in \mathrm{IB}_{\Gamma-n} \mid |B| > 2\}| = \begin{cases} |\{B \in \mathrm{IB}_\Gamma \mid |B| > 2\}| & \textit{if } \Gamma \textit{ is type A1, A2, or A3} \\ |\{B \in \mathrm{IB}_\Gamma \mid |B| > 2\}| - 1 & \textit{if } \Gamma \textit{ is type A4} \\ |\{B \in \mathrm{IB}_\Gamma \mid |B| > 2\}| + 1 & \textit{if } \Gamma \textit{ is type A5.} \end{cases}$$

Proof. If Γ is type A1, then every internal block of Γ is an internal block of Γ' and vice versa. If Γ is type A2, then B' where $V(B') = \{n-2, n-1\}$ is an internal block for Γ but a leaf block for $\Gamma - n$. However, B' is not counted above since $|B'| = 2$. Every other internal block of Γ is an internal block of $\Gamma - n$ and vice versa. If Γ is type A3, then B' is still an internal block in $\Gamma - n$ because it is adjacent to more than 2 cut vertices in the block-cut tree of $\Gamma - n$. Every other internal block is also the same, and so the equality follows.

If Γ is type A4, then B' is an internal block of Γ but a leaf block in $\Gamma - n$. Every other internal block in $\Gamma - n$ is an internal block in Γ, so $|\{B \in \mathrm{IB}_{\Gamma-n} \mid |B| > 2\}| = |\{B \in \mathrm{IB}_\Gamma \mid |B| > 2\}| - 1$.

If Γ is type A5, then $\mathrm{IB}_\Gamma = \emptyset$. However, $\mathrm{IB}_{\Gamma-n} = \{B'\}$. Since Γ has at least 4 vertices, we know that $|B'| > 2$, and the claim follows. □

Lemma 8.13. *Let* Γ *be a cliqued and naturally labeled graph on at least* 4 *vertices. Then the internal cut edges in* $\Gamma - n$ *are*

$$\mathrm{IC}_{\Gamma-n} = \begin{cases} \mathrm{IC}_\Gamma & \textit{if } \Gamma \textit{ is type A1, A3, A4, or A5} \\ \mathrm{IC}_\Gamma \setminus \{(n-2, n-3)\} & \textit{if } \Gamma \textit{ is type A2.} \end{cases}$$

Proof. The set IC_Γ is the set of internal cut edges in Γ. The two types where B' is an internal block in Γ but a leaf block in $\Gamma - n$ are types A2, A3 and A4. In types A3 and A4, the block B' is assumed to have size $|B| > 2$ and so does not correspond to an element of IC_Γ. If Γ is type A2, B' contributes to IC_Γ but not $\mathrm{IC}_{\Gamma-n}$, and that contribution is precisely the cut edge $(n-2, n-1)$. □

Now we are ready to prove the recursive formula for D_Γ.

Proof of Proposition 8.9. Consider the sum $\sum_{j \vdash \Gamma - n} \mathfrak{c}_{\Gamma-n}(j)$. If Γ is type B or C, then every cut vertex of Γ is a cut vertex of $\Gamma - n$ and vice versa. For each such cut vertex $j \vdash \Gamma$ (and $j \vdash \Gamma - n$), since Γ is type B/C, no connected component of $\Gamma - j$ consists of only the vertex n, so $\mathfrak{c}_\Gamma(j) = \mathfrak{c}_{\Gamma-n}(j)$.

If Γ is type A1, then every cut vertex of Γ is a cut vertex of $\Gamma - n$ but $\mathfrak{c}_\Gamma(n-1) = \mathfrak{c}_{\Gamma-n}(n-1) + 1$. If Γ is type A2, A3, A4 or A5, then $n-1$ is not a cut vertex of $\Gamma - n$ but $\mathfrak{c}_\Gamma(n-1) = 1$. So we set $\mathfrak{c}_{\Gamma-n}(n-1) := 0$, and the same relationship as in type A1 applies. Now for every other $j \neq n-1$, the vertex $j \vdash \Gamma$ if and only if $j \vdash \Gamma - n$, and then $\mathfrak{c}_\Gamma(j) = \mathfrak{c}_{\Gamma-n}(j)$. So we compute

$$\sum_{j \vdash \Gamma - n} \mathfrak{c}_{\Gamma-n}(j) = \sum_{j \vdash \Gamma} \mathfrak{c}_\Gamma(j) - \begin{cases} 1 & \text{if } \Gamma \text{ type A} \\ 0 & \text{if } \Gamma \text{ type B/C.} \end{cases}$$

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Next, we compare the contributions to D_Γ and $D_{\Gamma-n}$ by leaf blocks and internal blocks of size greater than 2. By Lemmas 8.10, 8.12 and 8.13, it follows that

$$|LB_{\Gamma-n}| + |\{B \in \mathrm{IB}_{\Gamma-n} \mid |B| > 2\}| = |LB_\Gamma| + |\{B \in \mathrm{IB}_\Gamma \mid |B| > 2\}| - \begin{cases} 1 & \text{if } \Gamma \text{ type A but not A2,} \\ 0 & \text{if } \Gamma \text{ type B, C, or A2.} \end{cases}$$

Since Γ is naturally labeled, if $s \in \mathrm{IC}_{\Gamma-n}$ (so $s \neq (n, n-1)$), then the connected component of $([n-1], E(\Gamma - n) \setminus s)$ that does not contain the vertex $n-1$ is equal to the connected component of $([n], E(\Gamma) \setminus \{s\})$ that does not contain the vertex n. It is important for proving the recursion that for each internal cut edge $s \in \mathrm{IC}_{\Gamma-n} \subset \mathrm{IC}_\Gamma$, we always pick the component G_s to be equal in Γ and $\Gamma - n$ (i.e., always choose $n \notin G_s \subset \Gamma$), so we adopt this convention. If Γ is type A2 (i.e., $\mathrm{IC}_\Gamma \neq \mathrm{IC}_{\Gamma-n}$), we remove the 'over-counting' from the internal cut edge $(n-2, n-1) \in \mathrm{IC}_\Gamma$ below and get that

$$\sum_{s \in \mathrm{IC}_{\Gamma-n}} \binom{n-1}{|G_s|} = \sum_{s \in \mathrm{IC}_\Gamma} \binom{n-1}{|G_s|} - \begin{cases} \binom{n-1}{n-2} & \Gamma \text{ is type A2} \\ 0 & \text{otherwise.} \end{cases}$$

Now we compute

$$\begin{aligned} D_{\Gamma-n} &= 2n - 3 - \sum_{j \vdash \Gamma - n} \mathfrak{c}_{\Gamma-n}(j) + (n-1)(|\mathrm{LB}_{\Gamma-n}| + |\{B \in \mathrm{IB}_{\Gamma-n} \mid |B| > 2\}| - 1) \\ &\quad + \sum_{s \in \mathrm{IC}_{\Gamma-n}} \binom{n-1}{|G_s|} \\ &= 2n - 3 - \sum_{j \vdash \Gamma} \mathfrak{c}_\Gamma(j) + \begin{cases} 1 & \text{if } \Gamma \text{ type A} \\ 0 & \text{if } \Gamma \text{ type B/C.} \end{cases} \\ &\quad + (n-1)(|LB_\Gamma| + |\{B \in \mathrm{IB}_\Gamma \mid |B| > 2\}| - 1) - \begin{cases} n-1 & \text{if } \Gamma \text{ type A but not A2} \\ 0 & \text{if } \Gamma \text{ type B, C, or A2.} \end{cases} \\ &\quad + \sum_{s \in \mathrm{IC}_\Gamma} \binom{n-1}{|G_s|} - \begin{cases} \binom{n-1}{n-2} & \Gamma \text{ is type A2} \\ 0 & \text{otherwise.} \end{cases} \\ &= 2n - 2 - \sum_{j \vdash \Gamma} \mathfrak{c}_\Gamma(j) - \begin{cases} n-1 & \text{if } \Gamma \text{ type A} \\ 1 & \text{if } \Gamma \text{ type B/C.} \end{cases} \\ &\quad + (n-1)(|LB_\Gamma| + |\{B \in \mathrm{IB}_\Gamma \mid |B| > 2\}| - 1) + \sum_{s \in \mathrm{IC}_\Gamma} \binom{n-1}{|G_s|} \end{aligned}$$

Now we have $D_{\Gamma-n}$ in a form that closely resembles that of D_Γ. Recall Pascal's identity $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. For each $s \in \mathrm{IC}_\Gamma$, we have that

$$\binom{n}{|G_s|} - \binom{n-1}{|G_s|} = \binom{n-1}{|G_s| - 1}.$$

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In particular,

$$\sum_{s\in \mathrm{IC}_\Gamma}\binom{n}{|G_s|} - \sum_{s\in \mathrm{IC}_\Gamma}\binom{n-1}{|G_s|} = \sum_{s\in \mathrm{IC}_\Gamma}\binom{n-1}{|G_s|-1}.$$

So we compute that the difference $D_\Gamma - D_{\Gamma-n}$ is equal to

$$1 + (|\mathrm{LB}_\Gamma| + |\{B \in \mathrm{IB}_\Gamma \mid |B| > 2\}| - 1) + \sum_{s\in \mathrm{IC}_\Gamma}\binom{n-1}{|G_s|-1} + \begin{cases} n-1 & \text{if } \Gamma \text{ type A} \\ 1 & \text{if } \Gamma \text{ type B/C.}\end{cases}$$

The remainder of the proof is essentially the same as the proof for Lemma 8.5. In particular,

1. Leaf blocks of size 2 that do not contain the vertex n are in bijection with strongly dominant pairs (i, j) where $\left|\Gamma_i^j\right| = 1$,
2. The set of internal cut edges IC_Γ is equal to the set of strongly dominant pairs (i, j), where $\left|\Gamma_i^j\right| > 1$, and
3. Leaf blocks and internal blocks of size at least three (i.e., all blocks of size at least 3) that do *not* contain the vertex n are in bijection with weakly dominant pairs (i, j).

We note that $\binom{n-1}{1-1} = 1$ and get that

$$(|\mathrm{LB}_\Gamma| + |\{B \in \mathrm{IB}_\Gamma \mid |B| > 2\}| - 1) + \sum_{s\in \mathrm{IC}_\Gamma}\binom{n-1}{|G_s|-1} = \sum_{(i,j)\gg\Gamma}\binom{n-1}{\left|\Gamma_i^j\right|-1} + |\{(i,j) > \Gamma\}|.$$

Thus, $D_\Gamma - D_{\Gamma-n}$ has the claimed form. □

Corollary 8.14. *Let Γ be a connected graph on $[n]$ where $n \geq 3$, and D_Γ as defined in Equation (8.1). Then $\dim_{\mathbb{C}}(\mathcal{M}^1_\Gamma) = D_\Gamma$. Moreover, $\dim_{\mathbb{C}}(\mathrm{L}_\Gamma)_1 = \dim_{\mathbb{C}}(\mathrm{R}_\Gamma)_1 = D_\Gamma - n$.*

Proof. It suffices to assume that Γ is cliqued and naturally labeled. In this case, by Theorem 7.2 and Proposition 8.9, both $\dim_{\mathbb{C}}(\mathcal{M}^1_\Gamma)$ and D_Γ follow the same recursion. Thus, it suffices to show equality on all connected graphs on 3 vertices. Quick computation confirms that

$$\dim_{\mathbb{C}}(\mathcal{M}^1_{K_3}) = 5 = D_{K_3} \text{ and } \dim_{\mathbb{C}}(\mathcal{M}^1_{P_3}) = 7 = D_{P_3},$$

and so the two statistics must be equal. The 'moreover' part follows from the fact that $\mathbb{C}\{\bar{t}_1, \ldots, \bar{t}_n\}$ and $\mathbb{C}\{\bar{x}_1, \ldots, \bar{x}_n\}$ are the (n-dimensional) linear subspace of $\mathcal{M}^1_\Gamma$ quotiented to obtain L_Γ and R_Γ, respectively. □

We now have a closed combinatorial formula for the $\mathbb{C}$-dimension of the first graded piece of $\mathcal{M}^1_\Gamma$. In particular, we have the dimension of the representations corresponding to $\mathbf{ch}(\mathrm{L}_\Gamma)_1$ and $\mathbf{ch}(\mathrm{R}_\Gamma)_1$. We are also quite close to constructing $\mathbb{C}$-bases of $\mathcal{M}^1_\Gamma$, as the formulae for D_Γ in Lemma 8.5 and the natural one for $|\mathcal{B}_\Gamma|$ are very similar.

9. The left and right linear representations

This subsection computes $\mathbf{ch}(\mathrm{L}_\Gamma)_1$ and $\mathbf{ch}(\mathrm{R}_\Gamma)_1$ for all connected Γ, proving Theorem 1.4 from the introduction. We prove this assuming that Γ is naturally labeled and get the label-independent formula as a corollary. The computation is direct and achieved by computing the dot action on two subsets $\mathcal{LS}_\Gamma$ and $\mathcal{RS}_\Gamma$ of $\mathcal{B}_\Gamma$ that project to bases of $(\mathrm{L}_\Gamma)_1$ and $(\mathrm{R}_\Gamma)_1$, respectively.

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Say Γ is a naturally labeled connected graph. By Proposition 6.12 and Theorem 7.2, if

$$\mathcal{B}_\Gamma := \{\bar{t}_i \mid i \in [n]\} \cup \{\bar{x}_i \mid i \in [n]\} \cup \left\{ \bar{f}^s_A \middle| \begin{matrix} s = (i,j) \gg \Gamma, \\ |A| = \left|\Gamma^j_i\right| \end{matrix} \right\} \cup \left\{ \bar{y}^j_{i,k} \middle| \begin{matrix} (i,j) > \Gamma, \\ k \in [n] \end{matrix} \right\},$$

then $\mathcal{M}^1_\Gamma = \mathbb{C}\mathcal{B}_\Gamma$. For the first graded piece of L_Γ and R_Γ, we will remove elements from $\mathcal{B}_\Gamma$ using the relations in Lemma 5.5, prove that the image of what remains is a basis by dimension, and then compute the representations on $(\mathrm{L}_\Gamma)_1$ and $(\mathrm{R}_\Gamma)_1$ using Lemma 9.1 below.

Lemma 9.1. *Let $w \in S_n$. Then*

$$w \cdot \bar{f}^{(i,j)}_A = \bar{f}^{(i,j)}_{w(A)} \text{ and } w \cdot \bar{y}^j_{r,k} = \bar{y}^j_{r,w(k)}.$$

The proof of Lemma 9.1 is direct from the definitions.

Now we will define two subsets of $\mathcal{B}_\Gamma$ – one for $(\mathrm{L}_\Gamma)_1$ and one for $(\mathrm{R}_\Gamma)_1$ – that project to $\mathbb{C}$-bases in these quotients.

For the linear piece of the left quotient $(\mathrm{L}_\Gamma)_1$, we first remove from $\mathcal{B}_\Gamma$ the splines $\{\bar{t}_1, \ldots, \bar{t}_n\}$. We may also discard

(1) The single spline $\bar{x}_n$ by Lemma 5.5(1),
(2) The splines $\{\bar{x}_i \mid (i,j) \gg \Gamma\}$ by Lemma 5.5(2), and
(3) The splines $\{\bar{x}_i \mid (i,j) > \Gamma\}$ by Lemma 5.5(3).

Note the set of splines in (2) and (3) is size $|\{\bar{x}_i \mid (i,j) \gg \Gamma\} \cup \{\bar{x}_i \mid (i,j) > \Gamma\}| = \sum_{j \vdash \Gamma} \mathfrak{c}(j)$. So the image of

$$\mathcal{LS}_\Gamma := \{\bar{x}_r \in \mathcal{X}_{n-1} \mid r \text{ is not } s\text{-dominant } \forall s \in [n]\} \cup \left\{\bar{f}^s_A \mid s \gg \Gamma\right\} \cup \left\{\bar{y}^j_{r,k} | (r,j) > \Gamma, \ \ k \in [n]\right\}$$

in $(\mathrm{L}_\Gamma)_1$ is a spanning set. Note that the size of $\{\bar{x}_r \in \mathcal{X}_{n-1} \mid r$ is not s-dominant $\forall s \in [n]\}$ is $n - 1 - \sum_{j \vdash \Gamma} \mathfrak{c}(j)$, the size of $\left\{\bar{f}^s_A \mid s \gg \Gamma\right\}$ is $\sum_{(i,j) \gg \Gamma} \binom{n}{\left|\Gamma^j_i\right|}$, and the size of $\left\{\bar{y}^j_{r,k} | (r,j) > \Gamma, \ \ k \in [n]\right\}$ is $\sum_{(i,j) > \Gamma} n$. Thus, the size of $\mathcal{LS}_\Gamma$ is precisely the dimension $D_\Gamma - n$ of $(\mathrm{L}_\Gamma)_1$, as computed in Lemma 8.5. So $\mathcal{LS}_\Gamma$ projects to a basis of $(\mathrm{L}_\Gamma)_1$. In fact, $\mathcal{LS}_\Gamma$ is a permutation basis for the dot action representation, from which it is easy to compute the dot action representation (we will state and prove this in Theorem 9.2).

For the linear piece of the right quotient $(\mathrm{R}_\Gamma)_1$, we first remove from $\mathcal{B}_\Gamma$ the splines $\{\bar{x}_1, \ldots, \bar{x}_n\}$. Let $m_{ij} := \left|\Gamma^j_i\right|$, and let $\left\{A_p \mid p \in \left[\binom{n}{m_{ij}}\right]\right\}$ be an enumeration of the $\binom{n}{m_{ij}}$-many subsets A associated to a strongly dominant pair $(i,j) \gg \Gamma$. By Lemma 5.5, the following three relations hold in R_Γ:

$$\sum_{r=1}^n \bar{t}_r \sim 0, \quad \sum_{A \subset [n]} \bar{f}^{(i,j)}_A \sim 0 \text{ and } \sum_{k=1}^n \bar{y}^j_{i,k} \sim 0.$$

The natural subset of $\mathcal{B}_\Gamma$ whose image spans $(\mathrm{R}_\Gamma)_1$ is therefore

$$\mathcal{RS}_\Gamma := \{\bar{t}_r - \bar{t}_{r+1} \mid r \in [n-1]\} \cup \left\{ \bar{f}^{(i,j)}_{A_p} - \bar{f}^{(i,j)}_{A_{p+1}} \middle| \begin{matrix} p \in \left[\binom{n}{m_{ij}} - 1\right], \\ (i,j) \gg \Gamma \end{matrix} \right\} \cup \left\{ \bar{y}^j_{r,k} - \bar{y}^j_{r,k+1} \middle| \begin{matrix} (r,j) > \Gamma, \\ k \in [n-1] \end{matrix} \right\}.$$

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The first subset is size $n-1$. The second subset is size $\sum_{(i,j)\gg\Gamma}\left(\binom{n}{\left|\Gamma_i^j\right|}-1\right)$, and the third subset is size $\sum_{(i,j)>\Gamma}(n-1)$. The number of (strong or weak) dominant pairs is

$$|(i,j)\in E(\Gamma)\mid (i,j)\gg\Gamma \text{ or } (i,j)>\Gamma\}|=\sum_{j\vdash\Gamma}\mathfrak{c}(j),$$

so

$$\sum_{(i,j)\gg\Gamma}\left(\binom{n}{\left|\Gamma_i^j\right|}-1\right)+\sum_{(i,j)>\Gamma}(n-1)=\sum_{(i,j)\gg\Gamma}\binom{n}{\left|\Gamma_i^j\right|}+\sum_{(i,j)>\Gamma}n-\sum_{j\vdash\Gamma}\mathfrak{c}(j).$$

So the image of $\mathcal{RS}_\Gamma$ is a basis for $(\mathrm{R}_\Gamma)_1$.

Theorem 9.2. *Let Γ be a naturally labeled graph. If $(i,j)\gg\Gamma$, define the partition $\lambda_{ij}:=\left(n-\left|\Gamma_i^j\right|,\left|\Gamma_i^j\right|\right)$ (reordered if necessary). Then*

$$\mathbf{ch}(\mathrm{L}_\Gamma)_1=\sum_{(i,j)\gg\Gamma}h_{\lambda_{ij}}+\sum_{(i,j)>\Gamma}h_{n-1,1}+\left(n-1-\sum_{j\vdash\Gamma}\mathfrak{c}(j)\right)h_n,$$

and

$$\mathbf{ch}(\mathrm{R}_\Gamma)_1=s_{n-1,1}+\sum_{(i,j)\gg\Gamma}\left(h_{\lambda_{ij}}-s_n\right)+\sum_{(i,j)>\Gamma}s_{n-1,1}.$$

Proof. Since $\mathbb{C}\mathcal{LS}_\Gamma$ and $\mathbb{C}\mathcal{RS}_\Gamma$ are S_n-invariant vector spaces, the dot action on each is a representation. Since the projection of these spaces to $(\mathrm{L}_\Gamma)_1$ and $(\mathrm{R}_\Gamma)_1$ are in fact isomorphisms, the symmetric functions $\mathbf{ch}(\mathrm{L}_\Gamma)_1$ and $\mathbf{ch}(\mathrm{R}_\Gamma)_1$ are the characters of the dot action representation on $\mathbb{C}\mathcal{LS}_\Gamma$ and $\mathbb{C}\mathcal{RS}_\Gamma$, respectively. Each of the identified subsets in bases $\mathcal{LS}_\Gamma$ and $\mathcal{RS}_\Gamma$ span S_n-invariant subspaces of $\mathbb{C}\mathcal{LS}_\Gamma$ and $\mathbb{C}\mathcal{RS}_\Gamma$, respectively.

First, we will compute each part of $\mathbf{ch}(\mathrm{L}_\Gamma)_1$. The dot action fixes each $\bar{x}_i$, and so the character of the dot action representation on $\mathbb{C}\{\bar{x}_r\in\mathcal{X}_{n-1}\mid r \text{ is not } s\text{-dominant } \forall s\in[n]\}$ is $\left(n-1-\sum_{j\vdash\Gamma}\mathfrak{c}(j)\right)h_n$. By Lemma 9.1, the character of the dot action representation restricted to $\mathbb{C}\left\{\bar{f}_A^s\mid s\gg\Gamma\right\}$ is $\sum_{(i,j)\gg\Gamma}h_{\lambda_{ij}}$. By Lemma 9.1 as well, the character of the dot action representation on $\mathbb{C}\left\{\bar{y}_{r,k}^j|(r,j)>\Gamma,\ \ k\in[n]\right\}$ is $\sum_{(i,j)>\Gamma}h_{n-1,1}$.

Now we will compute each part of $\mathbf{ch}(\mathrm{R}_\Gamma)_1$. Each of the following computations use the same principle argument. If K is an integer, and the set $\{e_i\mid i\in[K]\}$ is a permutation basis of some permutation representation of S_n with character h_λ, then the vector $\sum_{i=1}^{K}e_i$ is invariant under that representation. Furthermore, the character of the representation on the orthogonal subspace spanned by $\{e_{i+1}-e_i\mid i\in[K-1]\}$ is $h_\lambda-s_n$.

The character of the dot action representation on $\mathbb{C}\{t_i\mid i\in[n]\}$ is $h_{n-1,n}$, and so the dot action representation on $\mathbb{C}\{\bar{t}_r-\bar{t}_{r+1}\mid r\in[n-1]\}$ is $h_{n-1,n}-s_n=s_{n-1,1}$. The character of the dot action representation on $\mathbb{C}\left\{\bar{f}_{A_p}^{(i,j)}\middle|p\in\left[\binom{n}{m_i}\right],\ (i,j)\gg\Gamma\right\}$ is $\sum_{(i,j)\gg\Gamma}h_{\lambda_{ij}}$, and so the character of the dot action representation on $\mathbb{C}\left\{\bar{f}_{A_p}^{(i,j)}-\bar{f}_{A_{p+1}}^{(i,j)}\middle|p\in\left[\binom{n}{m_i}-1\right],\ (i,j)\gg\Gamma\right\}$ is $\sum_{(i,j)\gg\Gamma}h_{\lambda_{ij}}-s_n$. The character of the dot action representation on $\mathbb{C}\left\{\bar{y}_{r,k}^j\middle|(r,j)>\Gamma,\ k\in[n]\right\}$ is $\sum_{(i,j)>\Gamma}h_{n-1,1}$, and so the character of the dot action representation on $\mathbb{C}\left\{\bar{y}_{r,k}^j-\bar{y}_{r,k+1}^j\middle|(r,j)>\Gamma,\ k\in[n-1]\right\}$ is $\sum_{(i,j)>\Gamma}(h_{n-1,1}-s_n)=\sum_{(i,j)>\Gamma}s_{n-1,1}$. □

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The Schur-expansion of $h_{\lambda_{ij}}$ is easy to compute since λ_{ij} is only a two-part partition. In particular, if K is the larger of $\left|\Gamma_i^j\right|$ and $n-\left|\Gamma_i^j\right|$, then $h_{\lambda_{ij}} - s_n = \sum_{m=0}^{n-K-1} s_{K+m,n-K-m}$.

The following corollary gives the label-independent description, from the statistics on block-cut trees described in Section 8.

Corollary 9.3. *Let* Γ *be a connected simple graph, and let* LB_Γ*,* IB_Γ *and* IC_Γ *be the leaf blocks, internal blocks and internal cut edges of* Γ *as defined in the beginning of Section 8. For* $(i,j) \in \mathrm{IC}_\Gamma$*, let* λ_{ij} *be the partition* $\left(n - \left|G_{(i,j)}\right|, \left|G_{(i,j)}\right|\right)$ *(reordered if necessary), where* $G_{(i,j)}$ *is a connected component of the graph* $([n], E(\Gamma) \setminus \{(i,j)\})$*. Then*

$$\mathbf{ch}(\mathrm{L}_\Gamma)_1 = \sum_{(i,j)\in \mathrm{IC}_\Gamma} h_{\lambda_{ij}} + (|\mathrm{LB}_\Gamma| + |\{B \in \mathrm{IB}_\Gamma \mid |B| > 2\}| - 1) h_{n-1,1} + \left(n - 1 - \sum_{j \vdash \Gamma} \mathfrak{c}(j)\right) h_n,$$

and

$$\mathbf{ch}(\mathrm{R}_\Gamma)_1 = \sum_{(i,j)\in \mathrm{IC}_\Gamma} \left(h_{\lambda_{ij}} - s_n\right) + (|\mathrm{LB}_\Gamma| + |\{B \in \mathrm{IB}_\Gamma \mid |B| > 2\}|) s_{n-1,1}.$$

Proof. This follows directly from Theorem 9.2 and the bijections/equalities described in the proof of Lemma 8.5 (and also the proof of Proposition 8.9). □

We note that in the statement of Theorem 1.4 in the introduction, the sets are $E_1 = \mathrm{IC}_\Gamma$ and $E_2 = \mathrm{LB}_\Gamma \cup \{B \in \mathrm{IB}_\Gamma \mid |B| > 2\}$, and the integer $k = n - 1 - \sum_{j \vdash \Gamma} \mathfrak{c}(j)$.

Example 9.4. Let Γ be the graph from Example 8.6. We may compute the representations with either Theorem 9.2 or Corollary 9.3. Then

$$\mathbf{ch}(\mathrm{L}_\Gamma)_1 = 6h_{12} + 4h_{11,1} + h_{8,4}$$

and

$$\mathbf{ch}(\mathrm{R}_\Gamma)_1 = 4s_{11,1} + h_{8,4} - s_{12} = 4s_{11,1} + (s_{8,4} + s_{9,3} + s_{10,2} + s_{11,1}).$$

We note that, by this formula, the symmetric function $\mathbf{ch}(\mathrm{L}_\Gamma)_1$ is h-positive for all graphs Γ. So Theorem 9.2 and Corollary 9.3 prove an extension of the linear part of the graded Stanley–Stembridge conjecture from Hessenberg graphs to all connected graphs.

A. Tables of polynomials

Without geometric methods, it is quite difficult to compute these representations. One can, however, compute the dimension more easily using [17, 24]. For example, despite our current inability to compute the representation, we do know that $\dim(\mathrm{L}_{C_4})_3 = 9 = \dim(\mathrm{R}_{C_4})_3$. We also note that $\mathcal{M}_{C_4}^{\leq 3}$ is *not* a free module, but $\mathcal{M}_{C_4}^{\leq 2}$ is free (and $\mathbf{ch}\left(\mathrm{L}_{C_4}\right)$ is, in degree ≤ 2, h-positive).

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Table 1. *The polynomials* $\sum_{i\geq 0}\mathbf{ch}(\mathrm{L}_\Gamma)_i q^i$ *in the homogeneous basis and* $\sum_{i\geq 0}\mathbf{ch}(\mathrm{R}_\Gamma)_i q^i$ *in the Schur basis for all graphs on 3 and 4 vertices, excluding* $\mathbf{ch}(\mathrm{L}_{C_4})_3$ *and* $\mathbf{ch}(\mathrm{R}_{C_4})_3$.

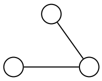
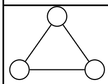
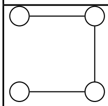
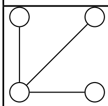
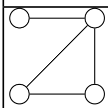
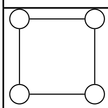
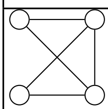
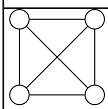
Γ	$\sum_{i\geq 0}\mathbf{ch}(\mathrm{L}_\Gamma)_i q^i$	$\sum_{i\geq 0}\mathbf{ch}(\mathrm{R}_\Gamma)_i q^i$
	$h_3 + (h_3 + h_{2,1})q + (h_3)q^2$	$s_3 + (2s_{2,1})q + (s_{1,1,1})q^2$
	$h_3(1 + 2q + 2q^2 + q^3)$	$s_3 + (s_{2,1})q + (s_{2,1})q^2 + (s_{1,1,1})q^3$
	$h_4 + (h_{2,2} + h_{3,1} + h_4)q + (h_{2,2} + h_{3,1} + h_4)q^2 + (h_4)q^3$	$s_4 + (s_{2,2} + 3s_{3,1})q + (3s_{2,1,1} + s_{2,2})q^2 + (s_{1,1,1,1})q^3$
	$h_4 + (h_4 + 2h_{3,1})q + (h_4 + 2h_{3,1} - h_{2,2} + h_{2,1,1})q^2 + (h_4)q^3$	$s_4 + (3s_{3,1})q + (3s_{2,2} + 3s_{2,1,1})q^2 + (s_{1,1,1,1})q^3$
	$h_4 + (h_{3,1} + 2h_4)q + (2h_{3,1} + 2h_4)q^2 + (h_{3,1} + 2h_4)q^3 + (h_4)q^4$	$s_4 + (2s_{3,1})q + (s_{2,1,1} + 2s_{2,2} + s_{3,1})q^2 + (2s_{2,1,1})q^4 + (s_{1,1,1,1})q^4$
	$h_4 + (3h_4)q + (h_{2,2} + h_{3,1} + 3h_4)q^2 + \mathbf{ch}(\mathrm{L}_\Gamma)_3 q^3 + (h_4)q^4$	$s_4 + (s_{3,1})q + (2s_{2,2} + 3s_{3,1})q^2 + \mathbf{ch}(\mathrm{R}_\Gamma)_3 q^3 + (s_{1,1,1,1})q^4$
	$h_4 + (3h_4)q + (h_{3,1} + 4h_4)q^2 + (h_{3,1} + 4h_4)q^3 + (3h_4)q^4 + (h_4)q^5$	$s_4 + (s_{3,1})q + (s_{2,2} + 2s_{3,1})q^2 + (2s_{2,1,1} + s_{2,2})q^3 + (s_{2,1,1})q^4 + (s_{1,1,1,1})q^5$
	$h_4(1 + 3q + 5q^2 + 6q^3 + 5q^4 + 3q^5 + q^6)$	$s_4 + (s_{3,1})q + (s_{2,2} + s_{3,1})q^2 + (s_{2,1,1} + s_{3,1})q^3 + (s_{2,1,1} + s_{2,2})q^4 + (s_{2,1,1})q^5 + (s_{1,1,1,1})q^6$

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Table 2. *The rank-generating functions and total rank for all isomorphism classes of graphs on 5 vertices. Geometric cases are marked with †.*

$E(\Gamma)$	$\sum \dim(\mathrm{L}_\Gamma)_i q^i = \sum \dim(\mathrm{R}_\Gamma)_i q^i$	Total
(1, 5), (2, 5), (3, 5), (4, 5)	$1 + 16q + 66q^2 + 56q^3 + q^4$	140
(1, 4), (1, 5), (2, 5), (3, 5)	$1 + 21q + 71q^2 + 31q^3 + q^4$	125
(1, 4), (1, 5), (2, 5), (3, 5), (4, 5)	$1 + 12q + 42q^2 + 52q^3 + 22q^4 + q^5$	130
(1, 4), (1, 5), (2, 4), (2, 5), (3, 5)	$1 + 8q + 38q^2 + 68q^3 + 13q^4 + q^5$	129
(1, 4), (1, 5), (2, 4), (3, 5), (4, 5)†	$1 + 12q^1 + 47q^2 + 47q^3 + 12q^4 + q^5$	120
(1, 4), (1, 5), (2, 4), (2, 5), (3, 5), (4, 5)	$1 + 8q + 24q^2 + 49q^3 + 34q^4 + 8q^5 + q^6$	125
(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)	$1 + 4q + 17q^2 + 47q^3 + 62q^4 + 6q^5 + q^6$	138
(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)	$1 + 4q + 17q^2 + 33q^3 + 43q^4 + 27q^5 + 4q^6 + q^7$	130
(1, 3), (1, 5), (2, 4), (2, 5)†	$1 + 26q^1 + 66q^2 + 26q^3 + 1q^4$	120
(1, 3), (1, 5), (2, 4), (2, 5), (3, 5)†	$1 + 17q^1 + 42q^2 + 42q^3 + 17q^4 + 1q^5$	120
(1, 3), (1, 5), (2, 4), (2, 5), (3, 5), (4, 5)†	$1 + 8q^1 + 29q^2 + 44q^3 + 29q^4 + 8q^5 + 1q^6$	120
(1, 3), (1, 4), (2, 4), (2, 5), (3, 5)	$1 + 4q + 49q^2 + 69q^3 + 14q^4 + q^5$	138
(1, 3), (1, 4), (1, 5), (2, 4), (2, 5), (3, 5)	$1 + 4q + 26q^2 + 51q^3 + 36q^4 + 5q^5 + q^6$	124
(1, 3), (1, 4), (1, 5), (2, 4), (2, 5), (3, 5), (4, 5)†	$1 + 4q^1 + 17q^2 + 38q^3 + 38q^4 + 17q^5 + 4q^6 + q^7$	120
(1, 3), (1, 4), (1, 5), (2, 5), (3, 4), (3, 5)†	$1 + 8q^1 + 29q^2 + 44q^3 + 29q^4 + 8q^5 + q^6$	120
(1, 3), (1, 4), (1, 5), (2, 5), (3, 4), (3, 5), (4, 5)†	$1 + 8q^1 + 20q^2 + 31q^3 + 31q^4 + 20q^5 + 8q^6 + q^7$	120
(1, 3), (1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)†	$1 + 4q^1 + 13q^2 + 26q^3 + 32q^4 + 26q^5 + 13q^6 + 4q^7 + q^8$	120
(1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 5)	$1 + 4q + 13q^2 + 35q^3 + 45q^4 + 24q^5 + 5q^6 + q^7$	128
(1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 5), (4, 5)	$1 + 4q + 9q^2 + 23q^3 + 39q^4 + 33q^5 + 10q^6 + 4q^7 + q^8$	124
(1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)†	$1 + 4q^1 + 9q^2 + 19q^3 + 27q^4 + 27q^5 + 19q^6 + 9q^7 + 4q^8 + q^9$	120
(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)†	$1 + 4q^1 + 9q^2 + 15q^3 + 20q^4 + 22q^5 + 20q^6 + 15q^7 + 9q^8 + 4q^9 + q^{10}$	120

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