A Finite Blocklength Analysis of Unequal Bit Protection for the AWGN Channel

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***Abstract*—In classical capacity analysis for point-to-point (P2P) channels, all transmitted data is equally protected. Transmitted data will be recovered if and only if transmitted at a rate below the channel's capacity. When a P2P channel is studied in the Finite Blocklength (FBL) regime the inclusion of a reliability term suggests the possibility of constructing codes that protect portions of the data differently. In this paper, we present an FBL achievable bound for the utilization of superposition (SUP) coding with successive interference cancellation (SIC) in the realization of Unequal Bit Protection (UBP) in transmission over a static, scalar additive white Gaussian noise (AWGN) channel. We also present a converse bound for the problem. Through our numerical analysis we show that in cases where data to be transmitted has known and differing reliability requirements, the use of superposition coding can increase the achievable sum rate of transmission of all classes of data protection over a uniform protection coding scheme and orthogonalization over time. However, the (SUP-SIC) achievable region and converse are not tight, and the FBL UBP capacity remains an open problem.**

I. Introduction

At times a transmitter might desire to protect from error some portions of communicated data to a different degree. This Unequal Error Protection (UEP) can take two primary forms, formalized in [1] as Unequal Bit Protection (UBP) and Unequal Message Protection (UMP). UBP is the process by which a subset of the data to be transmitted is protected to a different degree. A classic example of UBP is in the greater protection of header bits compared to payload bits. In contrast, in UMP the codebook is designed to protect specific messages. A prototypical example of UMP is the "Red Alert Problem" [2] in which a single message (the Red Alert Message) must be recovered by the receiver at a much higher reliability than other messages.

A. Prior Work

The work in [1], in addition to formalizing the distinction between UBP and UMP, also examines both problems for Discrete Memoryless Channels (DMC) from a large deviation analysis standpoint. Critically, it was shown that when the code rate matches the channel capacity it is impossible to protect even a single bit with a positive error exponent. Implementing such protection requires transmission at a rate strictly less than capacity. In contrast, when some subset of the messages is desired to have enhanced protection (UMP), the error exponent achieved by the protected set is the same as achieved when only transmitting the protected set, provided that the total codebook rate is no greater than capacity.

In [2] the UMP problem was considered for a single message for the AWGN channel, and matching bounds for the error exponent of the red alert message were found for the average power codebook. The work in [3] extends the analysis of the DMC UMP problem to an FBL analysis and finds that in a similar fashion to the error exponent analysis, different classes of messages can be transmitted at the same second-order rate as if each was being transmitted alone. For many types of channels, UMP is well understood.

The use of superposition (SUP) coding for UBP has been well-studied in the case of non-static channels. Known commonly as "The Broadcast Approach" (BCA) it was first introduced in [4] and has found applications in P2P as well as multi-user communications. For static channels, [5] considers UBP for the AWGN channel utilizing superposition coding, identical to our own problem setting. While providing an exact achievable bound, this work's reliance on the union bound as well as exact calculation of probabilistic tail values makes it difficult to compute and of less utility for larger values of n and γ. In [6] the authors consider an autoencoder (AE) based scheme for creating UBP and UMP codes including a UBP code for the static AWGN P2P channel. For this channel, the (AE) based encoders outperform the random coding utilized in [5], but no achievable bound is derived, rather the positive impact of (AE) encoding is shown over experimental results in [5] utilizing random coding.

B. Contribution

As described in the preceding, research has shown that UBP can be useful for very short blocklengths, but that asymptotically it becomes useless as rates approach capacity. A less explored question lies between the two extremes: can UBP be useful for scenarios involving intermediate blocklengths? At what combinations of blocklengths, channel conditions, and design constraints can UBP be useful?

We address this question by presenting a "normal approximation" [7], FBL achievable and converse bounds for the static AWGN UBP problem. Our achievable scheme makes use of SUP-SIC. We show that despite it being impossible for UBP to achieve a positive error exponent for a single bit when the sum rate matches capacity, sum rate gains can be achieved at FBL over both the traditional orthogonalization of transmission and

jointly encoding all data at the highest reliability. These gains are possible when there is a combination of a sufficiently large difference in required reliabilities for protection classes and when the most protected data is a sufficiently small percentage of the total data to be transferred.

We also derive a FBL converse for the UBP problem. Our achievable bound does not match the converse, indicating there is a possibility for an improved achievable scheme.

We also present a novel scenario demonstrating the potential usefulness of "medium" blocklength UBP: a node requiring the transmission of regular critical status and control information and important, but non time critical auxiliary data.

C. Notation

The capacity, in nats per channel use, of the point-to-point Gaussian channel with SNR γ is

$$\mathsf{C}(\gamma) = 1/2\ \ln(1+\gamma),\ \gamma \geq 0. \tag{1}$$

Second order results for multi-user Gaussian channels are often expressed as a function of the cross-dispersion function

$$\mathsf{V}(x,y) = \frac{x(2+y)}{2(1+x)(1+y)},\ 0 \leq x \leq y. \tag{2}$$

The point-to-point Gaussian dispersion function is

$$\mathsf{V}(x) = \mathsf{V}(x,x) = \frac{x(2+x)}{2(1+x)^2},\ x \geq 0. \tag{3}$$

The normal approximation of the second order capacity of the point-to-point Gaussian channel with SNR γ, for n channel uses and reliability ε, is denoted as

$$\kappa(n,\gamma,\varepsilon) = \mathsf{C}(\gamma) - \sqrt{\frac{\mathsf{V}(\gamma)}{n}}\mathsf{Q}^{-1}(\varepsilon),\ 0 \leq \gamma, \varepsilon \in [0,1], \tag{4}$$

which is an accurate proxy for achievable rates for values of the parameters for which $\kappa(n,\gamma,\epsilon)$ is at least comparable with $\ln(n)/n$ [8]. In (4), $\mathsf{Q}^{-1}(.)$ denotes the inverse of the function

$$\mathsf{Q}(x) = \int_x^{+\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-t^2/2}\,\mathrm{d}t,\ x \in \mathbb{R}. \tag{5}$$

II. Channel Model and Problem Formulation

We consider the memoryless, scalar, static AWGN P2P channel, where the channel between the base-station sending signal X and the receiver is modeled as $Y = hX + Z$. Here h is the constant channel state between transmitter and receiver, and Z is the Gaussian noise, assumed to be independent of the input with zero mean and variance σ^2. The input X is subject to the power constraint $\mathbb{E}[X^2] \leq P$. Given this normalization, the SNR is $\gamma := \frac{P|h|^2}{\sigma^2}$. For most memoryless point-to-point channels, it has been shown [8], [9] that $M^*(n,\varepsilon)$, defined as the maximum number of messages that can be sent within n channel uses and with an average probability of error not exceeding ϵ, behaves as

$$1/n\ \ln M^*(n,\varepsilon) = \kappa(n,\gamma,\varepsilon) + O_{\ln(n)/n}, \tag{6}$$

where the normal approximation function $\kappa(\cdot)$ was defined in (4), and where the term $\sqrt{\mathsf{V}(\gamma)/n}\mathsf{Q}^{-1}(\epsilon)$ expresses the rate penalty incurred by limiting transmission to n channel uses and allowing a probability of error no larger than $\epsilon \in (0,1)$ on a point-to-point Gaussian channel with SNR γ. We utilize O_n as a shorthand for the usual $O(n)$ notation for the asymptotic behavior as a function of n.

Definition 1. Code with bitwise unequal error protection. An $(n, M_1, M_2, P, \varepsilon_1, \varepsilon_2)$ code for the AWGN static P2P channel consists of two independent and uniformly distributed messages $m_1 \in [1 : M_1]$ and $m_2 \in [1 : M_2]$ encoded into sequences X of length n via a single encoder $f_n : [1 : M_1] \times [1 : M_2] \to \mathbb{R}^n$ and a single decoders $\phi_n : \mathbb{R}^n \to [1 : M_1] \times [1 : M_2]$ such that

$$\|f_n(m_1,m_2)\|^2 \leq nP, \forall (m_1,m_2) \in [1:M_1] \times [1:M_2], \tag{7a}$$

$$\Pr[(\widehat{m}_1) \neq (m_1), (m_1)\text{ sent}] \leq \varepsilon_1, \tag{7b}$$

$$\Pr[(\widehat{m}_2) \neq (m_2), (m_2)\text{ sent}] \leq \varepsilon_2, \tag{7c}$$

where $(\widehat{m}_1, \widehat{m}_2) = \phi_n(Y^n)$. In this definition m_1 represents the high reliability, critical message and m_2 is the auxiliary message, that is, $\varepsilon_1 < \varepsilon_2$. Writing M_1 and M_2 in their binary representation allows for a construction of M as the concatenation of the two sub-messages. Then it can be seen that this is a UBP construction.

We refer to standard P2P codes where all messages are sent with the same average reliability as homogeneous.

Definition 2 (Achievable Rate Region)**.** A non-negative rate tuple (R_1, R_2) is said to be $(n, \varepsilon_1, \varepsilon_2)$-achievable if there exists a $(n, M_{1,n}, M_{2,n}, P, \epsilon_{1,n}, \epsilon_{2,n})$ code for some n with $\epsilon_{1,n} \leq \varepsilon_1$,$\epsilon_{2,n} \leq \varepsilon_2$ and $\frac{\ln(M_{j,n})}{n} \geq R_j$ for $j \in \{1,2\}$.

The question we seek to answer is given $(n, \epsilon_1, \epsilon_2)$, what are the achievable (R_1, R_2) pairs and what are their sum rates $R := R_1 + R_2$, and conversely, which rate pairs are impossible?

III. Converse and Achievability

Heuristically, it can be reasoned that a P2P UBP code cannot operate with a rate beyond the bounds of a homogeneous code designed for operation with probability of error equal to the sum of the maximum probabilities of error in the UBP code. (The existence of such a code would imply the existence of a homogeneous code violating P2P bounds.) We provide a formal proof in Appendix A, but we state the resulting converse region for a two-tier UBP scheme as follows.

Proposition 1.

$$\mathcal{R}^{(\mathrm{CONV})}(n,\varepsilon_1,\varepsilon_2) = \bigcup \Big\{ (R_1,R_2) \in \mathbb{R}_+^2 :$$

$$R_1 + R_2 \leq \kappa(n,\gamma,\varepsilon_1+\varepsilon_2) + O_{\ln(n)/n}, \tag{8a}$$

$$R_1 \leq \kappa(n,\gamma,\varepsilon_1) + O_{\ln(n)/n}, \tag{8b}$$

$$R_2 \leq \kappa(n,\gamma,\varepsilon_2) + O_{\ln(n)/n}, \Big\}, \tag{8c}$$

One simple achievable scheme is for the transmitter to combine both messages prior to encoding and utilize a code

that is achievable under the critical reliability requirement. In the multi-user setting we have referred to this scheme in previous works as the Concatenate and Code Protocol (CCP), [10]–[12] and it is equivalent to rate splitting (RS).

Proposition 2. *CCP Achievable Rate Region: the following rates are achievable*

$$\mathcal{R}^{CCP}(n,\varepsilon_1,\varepsilon_2) = \Big\{(R_1,R_2)\in\mathbb{R}_+^2 : R_1+R_2 \le \mathsf{C}(\gamma) - \sqrt{\frac{1}{n}\mathsf{V}(\gamma)}\mathsf{Q}^{-1}(\varepsilon_1) + O_{\ln(n)/n}\Big\} \quad (9)$$

We also consider an achievable scheme based on superposition coding with rate splitting and SIC decoding.

Proposition 3. *Achievable Rate Region: the following rates are achievable*

$$\mathcal{R}^{SUP\text{-}RS}(n,\varepsilon_1,\varepsilon_2) = \bigcup_{(\alpha,\beta,\epsilon_{10},\epsilon_{11})\in[0,1]^4} \Big\{(R_1,R_2)\in\mathbb{R}_+^2 :$$

$$R_1+\beta R_2 \le \mathsf{C}\left(\frac{(1-\alpha)\gamma}{1+\alpha\gamma}\right) - \sqrt{\frac{1}{n}\mathsf{V}'(\alpha\gamma,\gamma)}\mathsf{Q}^{-1}(\epsilon_{11}) + O_{\ln(n)/n}, \quad (10a)$$

$$(1-\beta)R_2 \le \kappa(n,\alpha\gamma,\epsilon_{10}) + O_{\ln(n)/n}, \Big\}, \quad (10b)$$

where α is the power split and β the rate split. $\mathsf{V}'(\alpha\gamma,\gamma) = \mathsf{V}(\gamma)+\mathsf{V}(\alpha\gamma)-2\mathsf{V}(\alpha\gamma,\gamma)$.

The duo $(\epsilon_{10},\epsilon_{11})\in[0,1]^2$ satisfies

$$\mathsf{F}(\epsilon_{10},\epsilon_{11};r(\alpha\gamma,\gamma)) \ge 1-\varepsilon_2 \quad (10c)$$

$$1-\epsilon_{11} \ge 1-\varepsilon_1 \quad (10d)$$

where the probability of correct decoding function $\mathsf{F}(\cdot,\cdot;\cdot)$ is

$$\mathsf{F}(\epsilon_{10},\epsilon_{11};r(\alpha\gamma,\gamma)) := \Pr\left[G_2 \le \mathsf{Q}^{-1}(\epsilon_{10}), rG_2+\sqrt{1-r^2}G_3 \le \mathsf{Q}^{-1}(\epsilon_{11})\right], \quad (10e)$$

for G_2, G_3 i.i.d. standard Gaussian random variables, and the correlation coefficient $r(\alpha\gamma,\gamma)$ is defined as

$$r(\alpha\gamma,\gamma) := \frac{\mathsf{V}(\alpha\gamma,\gamma)-\mathsf{V}(\alpha\gamma,\alpha\gamma)}{\sqrt{(\mathsf{V}(\gamma,\gamma)+\mathsf{V}(\alpha\gamma,\alpha\gamma)-2\mathsf{V}(\alpha\gamma,\gamma))\mathsf{V}(\alpha\gamma,\alpha\gamma)}}. \quad (10f)$$

Due to space constraints, we do not provide the proof here, but it closely mimics that in [11, Sec. VI], specialized to a single receiver. A brief sketch of the proof is below.

Code words are chosen from successive power shells, a construction shown to minimize the dispersion versus an i.i.d construction. The decoding method is fixed as Successive-Interference-Cancellation (SIC) to extract the reliabilities for decoding each (critical, auxiliary) message. The receiver must first recover the critical message while treating the auxiliary message as noise. This must be done within the critical reliability requirement. The receiver then subtracts the codeword component of the critical message and attempts recovery of the auxiliary message. The receiver must decode both messages with a joint reliability at least as high as the auxiliary reliability requirement. The probabilities of error are first bounded through threshold decoding. A change of measure argument for the probability of an incorrect codeword exceeding the threshold is combined with an application of a multi-dimensional Berry-Esseen inequality to produce (10).

(10a) is the achievable rate for recovering the cloud center while treating the auxiliary message as noise and (10b) is the achievable rate for the satellite after interference cancellation. (10c) and (10d) constrain the region based on the required reliabilities. The relative complexity of (10c) is due to the information densities used in each step of threshold decoding not being independent.

Remark 1. $\mathcal{R}^{CCP} \subseteq \mathcal{R}^{SUP\text{-}RS}$

By inspection, $\mathcal{R}^{\text{CCP}}$ is recovered exactly by setting $\beta = 1$ and $\alpha = 0$ in $\mathcal{R}^{\text{SUP-RS}}$, thus $\mathcal{R}^{\text{SUP-RS}}$ can be no smaller than $\mathcal{R}^{\text{CCP}}$.

IV. Numerical Evaluation

A. Rate Region

In Fig. 1 we present achievable rate regions for the UBP AWGN communication channel $\mathcal{R}^{\text{SUP-RS}}$, $\mathcal{R}^{\text{CCP}}$, and Time Division Multiplexing (TDM) ($\mathcal{R}^{\text{TDM}}$) specialized from [11, Eq. 32] for a single transmitter operating orthogonally in time to realize UBP. $\mathcal{R}^{\text{CONV}}$ is also plotted. It is evident that sum rate gains can be made provided the rate demands of the critical message are not too high. In our numerical evaluations, we find that points on the boundary of $\mathcal{R}^{\text{SUP-RS}}$ β in (10) are found to be exactly either 1 or 0. This suggests that $\mathcal{R}^{\text{SUP-RS}} = \mathcal{R}^{\text{CCP}} \cup \mathcal{R}^{\text{SUP}}$.$\mathcal{R}^{\text{SUP}}$ is the evaluation of $\mathcal{R}^{\text{SUP-RS}}$ without rate splitting. It is the evaluation of (10) with β fixed to zero.

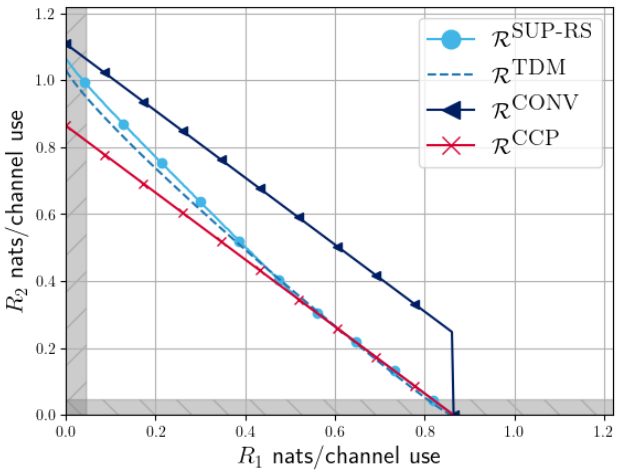
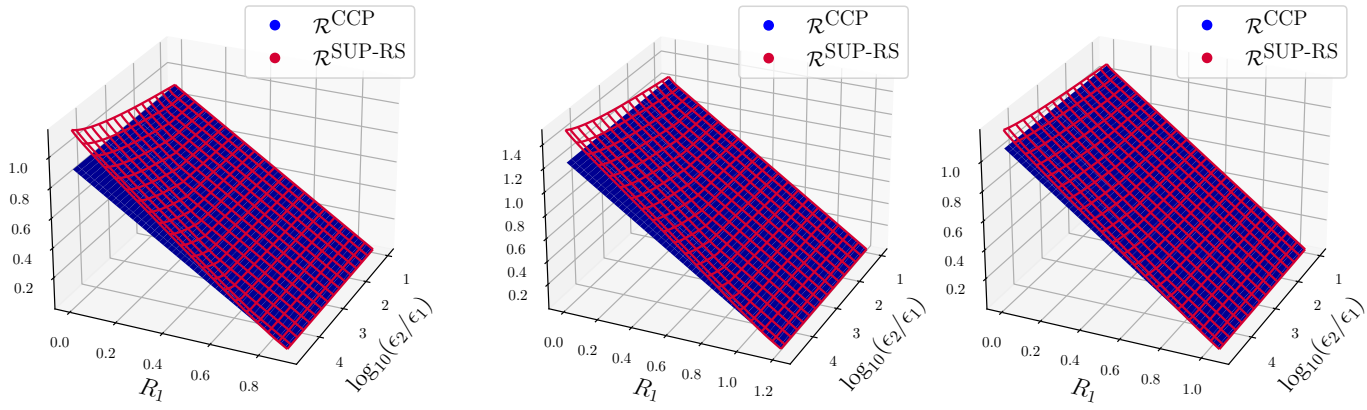


Fig. 1: $\gamma = 10, \varepsilon_1 = 10^{-6}, \varepsilon_1 = 10^{-1}, n = 100$

B. Rate region response to parameter changes

To illustrate the behavior of the rate region in response to scenario parameter changes we plot a rate region in three



(a) $\gamma = 10, n = 100$ (b) $\gamma = 20, n = 100$ (c) $\gamma = 10, n = 500$

Fig. 2: Rate Regions as a function of $\log_{10}(\epsilon_2/\epsilon_1)$, $\epsilon_1 = 10^{-6}$

dimensions where the third axis is the order of magnitude difference between the critical and auxiliary reliability. $\log_{10}(\epsilon_2/\epsilon_1)$.

In all figures we note that the regions in which $\mathcal{R}^{\text{SUP-RS}}$ are equivalent to $\mathcal{R}^{\text{CCP}}$ are similarly shaped. $\mathcal{R}^{\text{SUP-RS}}$ achieves a larger rate pair when the reliabilities ϵ_1 and ϵ_2 differ greatly, the critical rate is sufficiently small, or some combination of the two. This behavior is mostly independent of the blocklength and channel conditions. However, it is also noted that the effectiveness as measured by the relative increase for rate pairs of UBP over CCP does vary across all parameters. Increases in n and γ decrease the relative gains across all reliability pairs with increases in n limiting relative gains to a much greater degree.

C. Comparison with Previous Work

As [5] concerns a similar problem setup we compare an achievable region derived from their achievable bound, specialized to a uniform blocklength with our results. It is important to emphasize that [5] does possess two advantages: it is a complete bound and not a "normal approximation", and the work extends to heterogeneous deadlines for critical and auxiliary messages. Because it uses a known sub-optimal (in the second order) i.i.d Gaussian code construction and a union bound of the probability of error it may underestimate what is achievable. In Fig. 3 when n is small, the achievable region evaluated from [5] is outperformed by our normal approximation. Calculating the bound in [5] becomes very difficult as n or γ increases as the tail probabilities rapidly become very small.

D. An example scenario

To illustrate the potential usefulness of our SUP-SIC UBP construction, we consider a scenario arising in autonomous manufacturing. A peripheral node is connected via a static AWGN channel to a central control node. The peripheral node must regularly transmit critical high reliability data with low latency. The peripheral node is also required to upload

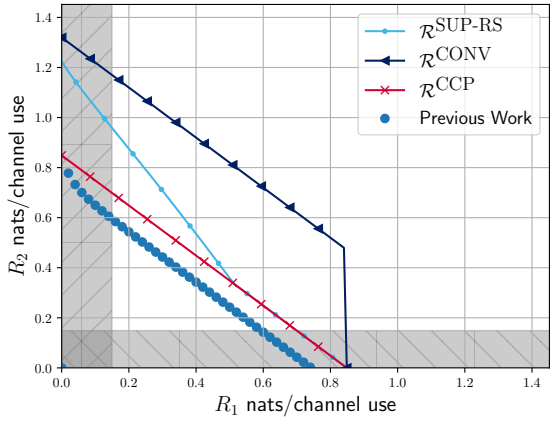


Fig. 3: Comparison with prior work [5] with $n = 20, \gamma = 20, \varepsilon_1 = 10^{-5}, \varepsilon_2 = 0.1$

less time-critical auxiliary data (e.g. diagnostic data), with asymptotically guaranteed delivery. To ensure the reliability of auxiliary data delivery, the central node confirms receipt of each transmission block in an orthogonal manner with reliability $(1 - \epsilon_{\text{CONF}})$.

In summary, the scenario consists of a peripheral node that

- Must transmit critical information at a rate R_1 and reliability $1 - \epsilon_1$ within n channel uses.
- Receives confirmation of successful transmission with reliability $1 - \epsilon_{\text{CONF}}$
- Wishes to maximize the expected rate of successful auxiliary data transmission.

In this scenario, transmission of auxiliary data is only possible if the required critical transmission rate, R_1, is below the second-order P2P capacity of the AWGN channel evaluated at the required reliability of $1-\epsilon_1$ for n channel uses. If this is the case the peripheral node may choose a scheme that allows for the auxiliary data to be simultaneously transmitted.

We explore the peripheral node schemes involving allocating message bits, channel uses, or power to accommodate auxiliary data transmission. This corresponds respectively to CCP, TDM, and the SUP-RS.

With these schemes, the peripheral node can achieve some goodput of auxiliary data where goodput is the expectation of the transmission rate over all communication blocks. The goodput of auxiliary data is given as

$$\bar{R}_2 = (1-\epsilon_2)(1-\epsilon_{\text{CONF}})R_2 \tag{11}$$

In (11) $(1-\epsilon_2)$ is the expected reliability of the transmission from the peripheral node to the central node of the auxiliary data, and $(1-\epsilon_{\text{CONF}})$ is the expected reliability of the transmission acknowledgment.

In Fig. 4 we plot the expected sum rate $R_1 + \bar{R}_2$ that the peripheral node can achieve using an allocation of message bits (CCP), channel accesses (TDM), or a combination of power and message bits (SUP-RS). SUP-RS allows the control node to transfer its auxiliary data at a faster rate as long as the critical rate demand remains small. In other cases, jointly encoding the auxiliary data with the critical data will outperform both TDM and SUP alone, in these regions rate splitting is utilized.

We have plotted two separate auxiliary reliabilities. In the absence of a transmission block level constraint on the auxiliary reliability $(1-\epsilon_2)$, ϵ_2 is a free parameter for the optimization of goodput. This optimization is constrained by the requirements of the critical transmission (rate/reliability) and the channel parameters.

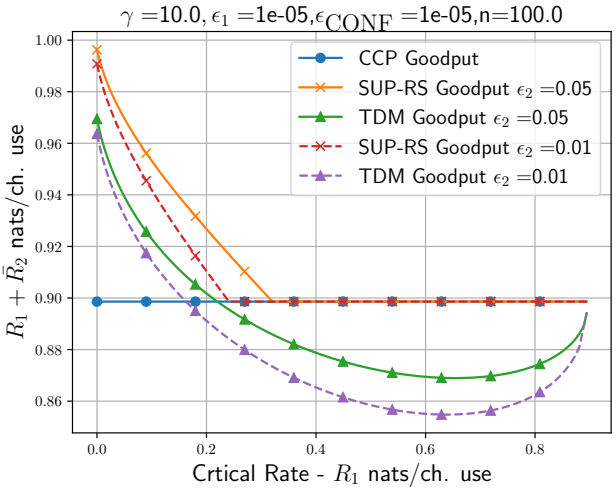


Fig. 4: Combined Rate as a function of Critical Rate R_1

V. Conclusion

UBP via SUP-RS can provide meaningful sum-rate gains provided that the data to be transmitted has sufficiently different reliability requirements and a reduction in the most reliable transmitted rate can be tolerated. The larger the difference in reliabilities the larger the gains and smaller the cost. Increasing the blocklength does not increase the cost in the form of required reduction in critical rate of UBP implementation via SUP but does reduce the scale of the rate increase benefit. A degradation in SNR surprisingly corresponds to a larger sum rate gain relative to the jointly encoding scheme. Our example scenario illustrates how UBP can be beneficial in moderate deadline scenarios. The normal approximation for the AWGN UBP problem provides a tool for evaluating UBP in many scenarios and can be readily extended to other channel models and multiple classes of bit protection.

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Appendix A
Converse

This proof follows closely the methods from [13]

A. Equal Power Constraint

The converse proof relies on all codewords having equal power. Using the same methods as [13, Sec. IV A-1]

$$\mathcal{C}_{\text{eq}}(n,\varepsilon_1,\varepsilon_2) \subseteq \mathcal{C}(n,\varepsilon_1,\varepsilon_2) \subseteq \mathcal{C}_{\text{eq}}(n+1,\varepsilon_1,\varepsilon_2), \tag{12}$$

where $\mathcal{C}$ is the UBP finite block length capacity of the AWGN P2P with a maximum codeword power constraint $\|f_n(m_1,m_2)\|^2 \leq nP, \quad \forall (m_1,m_2) \in [M_1] \times [M_2]$. Thus, through the second order $\mathcal{C}_{\text{eq}}$ and $\mathcal{C}$ are equivalent and it is sufficient to consider only equal power codewords

B. Han Converse Bound

Proposition 4. *Fix a blocklength $n \geq 1$, auxiliary output distribution $Q_Y^n(\boldsymbol{Y})$ and a constant ξ. For any $(n, |M_1|, |M_2|, P, \varepsilon_1, \varepsilon_2)$ code of fixed empirical power P, there exists a random vector $\boldsymbol{X}$ with distribution $P_{\boldsymbol{X}}$ supported on $\{\boldsymbol{x} : \|\boldsymbol{x}\|^2 = nP\}$ such that*

$$\varepsilon_2 \geq \Pr[\mathcal{A}] - e^{-n\xi} \tag{13}$$

where

$$\mathcal{A} := \left\{ \frac{1}{n} \log \frac{W^n(\boldsymbol{Y}|\boldsymbol{X})}{Q_Y^n(\boldsymbol{Y})} \leq \frac{1}{n} \log M_1 M_2 - 2\xi \right\} \tag{14}$$

The proof starts by defining error events $\mathcal{E}_1, \mathcal{E}_2$, and $\mathcal{E}$ when $f_n(m_1, m_2)$ is transmitted. $\mathcal{E}_1$ is the event $\widehat{m}_1 \neq (m_1)$, and $\mathcal{E}_2$ is the event $(\widehat{m}_2) \neq (m_2)$, and $\mathcal{E} := \mathcal{E}_1 \cup \mathcal{E}_2$. These definitions can be utilized in a basic partition.

$$\Pr[\mathcal{A}] = \Pr[\mathcal{A} \cap \mathcal{E}] + \Pr[\mathcal{A} \cap \mathcal{E}^C] \tag{15a}$$

$$\varepsilon_1 + \varepsilon_2 \geq \Pr[\mathcal{A}] - \Pr[\mathcal{A} \cap \mathcal{E}^C] \tag{15b}$$

We write an expression for the intersection of $\mathcal{A}$ and $\mathcal{E}^C$ over the channel transition $W^n(\boldsymbol{Y}|\boldsymbol{X})$ for every channel input.

$$\Pr[\mathcal{A} \cap \mathcal{E}^C] = \int\int P_{\boldsymbol{X}}(\boldsymbol{x}) W^n(\boldsymbol{y}|\boldsymbol{x}) \tag{16}$$

$$\times \mathbb{1}(\phi(W^n(\boldsymbol{y}) = (M_1, M_2)) \mathbb{1}(\mathcal{A}) \mathrm{dxdy} \tag{17}$$

Bounding $W^n(\boldsymbol{y}|\boldsymbol{x})$ via our definition of $\mathcal{A}$ give us

$$\Pr[\mathcal{A} \cap \mathcal{E}^C] \geq \int\int P_{\boldsymbol{X}}(\boldsymbol{x}) M_1 M_2 Q_{\boldsymbol{Y}}^n(\boldsymbol{y}) e^{-n\xi} \tag{18}$$

$$\times \mathbb{1}(\phi(W^n(\boldsymbol{y}) = (M_1, M_2)) \mathrm{dxdy} \tag{19}$$

$$\geq e^{-n\xi} \tag{20}$$

assuming codewords are equiprobable. So we are left with

$$\varepsilon_1 + \varepsilon_2 \geq 1 - e^{-n\xi} - \Pr\left[\frac{1}{n} \log \frac{W^n(\boldsymbol{Y}|\boldsymbol{X})}{Q_{\boldsymbol{Y}}(\boldsymbol{Y})} > \frac{1}{n} \log M_1 M_2 - 2\xi \right]_{P_{\boldsymbol{X}}(\boldsymbol{x}) W^n(\boldsymbol{y}|\boldsymbol{x})} \tag{21}$$

For readability we define

$$R_{2,n} := \frac{1}{n} \log M_2 - \xi \quad R_{1,n} := \frac{1}{n} \log M_1 - \xi \tag{23}$$

$$i(\boldsymbol{Y}; \boldsymbol{x}) := \frac{1}{n} \log \frac{\boldsymbol{W}^n(\boldsymbol{Y}|\boldsymbol{X})}{Q_{\boldsymbol{Y}}^n(\boldsymbol{Y})} \tag{24}$$

where we make use of the auxiliary distribution $Q_Y(y) \sim \mathcal{N}\left(y; 0, P + \sigma^2\right)$ and as a channel transition probability $W_{\boldsymbol{Y}|\boldsymbol{X}}(y) \sim \mathcal{N}\left(y; x, \sigma^2\right)$. we thus have sums of independent random variables of the following type, where Y_t is the channel output at time $t \in [n]$.

$$\ln \frac{W(Y_t|x_t)}{Q_Y(Y_t)} = \mathsf{C}(\gamma) + \frac{\nu_t^2 - N_t^2}{2(1+\gamma)} \gamma + \frac{\nu_t N_t}{1+\gamma}, \tag{25}$$

where we introduced the normalized quantities

$$N_t := \frac{Y_t - x_t}{\sigma} \sim \mathcal{N}(0,1), \quad \nu_t := \frac{x_t}{\sigma}. \tag{26}$$

The information density has mean $\mathbb{E}[i] = \mathsf{C}(\gamma) + \mu(\boldsymbol{x})$ with

$$\mu(\boldsymbol{x}) := \frac{\|\boldsymbol{x}\|^2/\sigma^2 - n\gamma}{n2(1+\gamma)} \tag{27a}$$

and covariance matrix $n\mathrm{Cov}[\boldsymbol{i}] = \mathsf{V}(\gamma, \gamma) + \mathcal{V}(\boldsymbol{x})$ with

$$\mathcal{V}(\boldsymbol{x}) := \frac{\|\boldsymbol{x}\|^2/\sigma^2}{n(1+\gamma)^2} - \frac{\gamma}{(1+\gamma)^2}, \tag{28a}$$

Every sequence $\boldsymbol{X}$ satisfies $\mu(\boldsymbol{x}) = 0$, and $\mathcal{V}(\boldsymbol{x}) = 0$. In (22) we complete the converse proof by applying the Berry-Esseen theorem in (22).

In the event the transmitter sends no auxiliary bit ($R_2 = 0$) or no critical bit ($R_1 = 0$) one can bound the rate as the point to point (P2P) converse of the channel operating at the corresponding reliability.

$$\Pr[i(\boldsymbol{Y}; \boldsymbol{x}|\boldsymbol{x}) - \mathsf{C}(\gamma) - \mu(\boldsymbol{x}) > R_{2,n} + R_{1,n} + \xi - \mathsf{C}(\gamma) - \mu(\boldsymbol{x})] \tag{22a}$$

$$\leq \Pr\left[\boldsymbol{Z} < -\sqrt{n}(R_{1,n} + R_{2,n} + \xi) + \sqrt{n}\mathsf{C}(\gamma) + \sqrt{n}\mu(\boldsymbol{x})\right]_{\boldsymbol{Z} \sim \mathcal{N}(0; \mathsf{V}(\gamma,\gamma) + \mathcal{V}(\boldsymbol{x}))} + \frac{B}{\sqrt{n}} \tag{22b}$$

$$\overset{\xi = \frac{\ln(n)}{2n}}{=} \Psi\left(-\sqrt{n}(R_{1,n} + R_{2,n}) + \sqrt{n}\mathsf{C}(\gamma) - \frac{\ln(n)}{2\sqrt{n}}; \mathsf{V}(\gamma,\gamma)\right) + \frac{B}{\sqrt{n}}, \tag{22c}$$

$$\varepsilon_1 + \varepsilon_2 \geq 1 - \Psi\left(-\sqrt{n}(R_{1,n} + R_{2,n} - \mathsf{C}(\gamma)) - \frac{\ln(n)}{2\sqrt{n}}; \mathsf{V}(\gamma,\gamma)\right) + \frac{B-1}{\sqrt{n}} \tag{22d}$$

$$R_{1,n} + R_{2,n} \leq \mathsf{C}(\gamma) + \frac{1}{\sqrt{n}} \mathsf{Q}_{\text{inv}}(\varepsilon_1 + \varepsilon_2; \mathsf{V}(\gamma,\gamma)) + O\left(\frac{\ln(n)}{n}\right), \tag{22e}$$