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Model Reference Adaptive Controller Design for a Multi-state Reparable System

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Abstract— **This work considers the adaptive repair rate design of a multi-state reparable system modeled by coupled transport and integro-differential equations. A reparable system is one which can be restored to satisfactory operation by repair actions whenever a failure occurs. The model describes the probabilities of the system in good and failure modes. The objective is to design the adaptive repair functions so that the probability of the system in good mode can be steered to a target distribution. Rigorous analysis on the convergence of tracking error between the plant and the target states is addressed.**

I. Introduction

A reparable system is a system which can be restored to satisfactory operation by repair actions whenever a failure occurs (e.g. [1], [2]). This type of systems occur very often in product design, inventory systems, computer networking, electrical power system and complex manufacturing processes. Understanding the reliability of a reparable system is critical in reliability engineering, which relates closely to system quality and safety. Reliability is defined as the probability that the system, subsystem or component will operate successfully by a given time. Here, we consider a reparable system with possibly M modes of failure. The system is good at time zero and transitions are permitted only between the good mode 0 and the failure mode i.

In this paper, we aim at designing the adaptive repair actions so that the probability of the system in good mode can be steered to a *target* distribution. It is assumed that the target distribution is part of a reference model driven by an idealized repair rate that yields a desirable probability distribution. An adaptive control scheme developed for transport PDEs that avoid bilinear controller design, is adopted for the specific class of coupled transport and integro-differential equations to generate the adaptive laws for the repair rates. Via the appropriate selection of a Lyapunov functional for the resulting error system, the convergence of the plant probability density distribution to that of the target system is established along with the state convergence.

II. Problem formulation

The mathematical model that describes the probabilities of the system in good and failure modes is governed by a coupled system of transport and integro-differential equations. The application of Markov chain and supplementary variable techniques are used to derive the model. Specifically, the system of equations reads (see [3])

$$\frac{\mathrm{d}p_0(t)}{\mathrm{d}t} = -\big(\sum_{i=1}^{M}\lambda_i\big)p_0(t) + \sum_{j=1}^{M}\int_0^{\ell}\mu_i(x)p_i(x,t)\,\mathrm{d}x, \tag{1}$$

$$\frac{\partial p_i(x,t)}{\partial t} + \frac{\partial p_i(x,t)}{\partial x} = -\mu_i(x)p_i(x,t), \tag{2}$$

with boundary conditon

$$p_i(0,t) = \lambda_i p_0(t), \quad i = 1,2,\ldots,M, \quad t > 0, \tag{3}$$

and initial conditions

$$p_0(0) = \phi_0, \quad p_i(x,0) = \phi_i(x), \quad i = 1,2,\ldots,M, \tag{4}$$

where $p_0(t)$ stands for the probability that the device is in good mode 0 at time t; $p_i(x,t)$ stands for the probability density distribution (with respect to repair time x satisfying $0 \le x \le \ell$) that the failed device is in failure mode i at time t and has an elapsed repair time of x; $\lambda_i > 0$ is the constant *failure rate* of the device for failure mode i; and $\mu_i(x) \ge 0$ is the *repair rate* when the device is in failure state at t and has an elapsed repair time of x. Extracting the probability from the probability density, one has the probability of the device in failure mode i at time t is denoted by $\overline{p}_i(t)$ and given by

$$\overline{p}_i(t) = \int_0^{\ell} p_i(x,t)\,\mathrm{d}x.$$

It is assumed that the failure rates λ_i are constant and repair times are arbitrarily distributed. All failures are statistically independent. The repair process begins immediately when the device fails. No further failure can occur while system is down and the device functions as new after repair.

We further assume that the maximum repair time $\ell < \infty$ and the associated repair rate is bounded, i.e.,

$$0 \le \mu_i(x) \le \overline{\mu} < \infty, \quad i = 1,2,,\ldots,M. \tag{5}$$

It is clear that $\int_0^{\ell}\mu_i(x)\,dx < \infty$. Initially the sum of the probability distributions in good and failure modes is 1, i.e.,

$$\phi_0 + \sum_{i=1}^{M}\int_0^{\ell}\phi_i(x)\,\mathrm{d}x = 1, \tag{6}$$

where $\phi_0 \ge 0$, $\phi_i(x) \ge 0$, $\forall x \ge 0$, and $\phi_i \in L^1(0,l)$.

The well-posedness and asymptotic behavior of system (1)–(4) with given failure and repair rates have been well addressed using C_0-semigroup theory (see [4], [5], [6]). Optimal repair maintenance design over a finite time interval was discussed in [7], [8], which leads to a bilinear control

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problem. Also, failure rate identification using a least-squares method was discussed in [9]. In our recent work [10], we have constructed adaptive observer to estimate both the failure and repair rates. The objective here is to construct adaptive repair actions μ_i so that the probability of the system in good mode can be steered to a target distribution.

A. Well-posedness of the Model

We first recall the well-posedness of equations (1)–(4). Define the state space $X = \mathbb{R} \times \left(L^1(0,\ell)\right)^M$ with $\|\cdot\|_X = |\cdot| + \sum_{i=1}^M \|\cdot\|_{L^1(0,\ell)}$ and define the system operator $\mathcal{A}$ and its domain by

$$\mathcal{A}p = \begin{bmatrix} -\sum_{i=1}^M \lambda_i p_0 + \sum_{i=1}^M \int_0^\ell \mu_i(x) \bullet \mathrm{d}x \\ -\left(\frac{\mathrm{d}}{\mathrm{d}x} + \mu_1(x)\right)\bullet \\ \vdots \\ -\left(\frac{\mathrm{d}}{\mathrm{d}x} + \mu_M(x)\right)\bullet \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_M \end{bmatrix} \tag{7}$$

for $p = (p_0, p_1(x), \ldots, p_M(x))^T \in D(\mathcal{A})$, and

$$D(\mathcal{A}) = \Big\{ p \in X \Big| \ \frac{\mathrm{d}p_i(x)}{\mathrm{d}x} \in L^1(0,\ell), \text{ and } p_i(0) = \lambda_i p_0, \quad i = 1, 2, \ldots, M \Big\}.$$

Equations (1)–(4) can be rewritten as an abstract Cauchy problem in the non-reflexible Banach space X

$$\dot{p}(t) = \mathcal{A}p(t), \quad t > 0, \tag{8}$$

$$p(0) = (\phi_0, \phi_1, \ldots, \phi_M)^T. \tag{9}$$

It is shown in [4] that the operator $\mathcal{A}$ generates a positive C_0-semigroup of contraction. The solution to (1)–(4) is nonnegative if the initial data are nonnegative. Moreover, if

$$\int_0^l \mu_i(x)\,\mathrm{d}x = \infty, \quad i = 1, \ldots, M, \tag{10}$$

then the system is conserved in terms of the norm $\|\cdot\|_X$. In other words, the summation of the probability distributions of the system in good and failure modes is always 1 for any $t > 0$, that is,

$$p_0(t) + \sum_{i=1}^M \overline{p}(t) = \phi_0 + \sum_{i=11}^M \int_0^\ell \phi_i(x)\,\mathrm{d}x = 1, \quad \forall t > 0. \tag{11}$$

In fact, condition (10) indicates that the failed components or subsystems can be completely repaired or replaced by new ones when the repair time reaches its maximum. In this case, 0 is a simple eigenvalue of $\mathcal{A}$ and the time-dependent solution of (8)–(9) converges to its steady-state exponentially, which is the eigenfuction associated with 0 (see [4], [5], [6]). Utilizing this property, the authors in [11] have established the exact bilinear controllability of the repair actions under certain conditions on the desired distribution of the failure modes.

In the current work, we assume that $\int_0^l \mu_i\,dx < \infty$, which means that the failed components may not be fully recovered from repair, As a result,

$$p_0(t) + \sum_{i=1}^M \overline{p}(t) < 1, \quad \forall t > 0,$$

and 0 is no longer in the spectrum of $\mathcal{A}$. The solution to (8)–(9) will converge to zero exponentially.

In the rest of our discussion, we consider the state space $X = \mathbb{R} \times \left(L^2(0,\ell)\right)^M$ for addressing the convergence of the adaptive repair actions. One can show that $p_i(x,t) \in L^2(0,T;L^2(0,\ell))$ if the initial datum $\phi_i \in L^2(0,\ell)$ using the characteristic method (see (36) in [6]) and Volterra integral equation (see Remark 1 in [12]).

III. Model reference Controller Design

Viewing the repair rates $\mu_i(x)$ as the control signals in (1)–(4), we arrive at coupled bilinear control systems. Using optimal control techniques, one can solve for the optimal controls [8] but such a solution increases controller complexity and computational load. As an alternative for the bilinear control of transport PDEs, we consider adaptive control techniques presented in [13].

Central to the design of the adaptive controller is the reference model, which represents an *idealized reparable system*; this of course is the target system and the controllers μ_i must ensure that (1)–(4) follows the target system.

Assumption 1 (Idealized repair rate controllers): It is assumed that there exist unknown positive repair rates $m_i(x)$, $i = 1, \ldots, M$, each with known lower and upper bounds

$$0 < \underline{m}_i \le m_i(x) \le \overline{m}_i, \quad i = 1, \ldots, M, \tag{12}$$

such that the solution to the systems

$$\begin{aligned} \frac{\mathrm{d}q_0(t)}{\mathrm{d}t} &= -\Big(\sum_{i=1}^M \lambda_i\Big) q_0(t) + \sum_{i=1}^M \int_0^\ell m_i(x) q_i(x,t)\,\mathrm{d}x, \\ \frac{\partial q_i(x,t)}{\partial t} &+ \frac{\partial q_i(x,t)}{\partial x} = -m_i(x) q_i(x,t), \end{aligned} \tag{13}$$

with boundary condition

$$q_i(0,t) = \lambda_i q_0(t), \quad i = 1, 2, \ldots, M, \quad t > 0, \tag{14}$$

and initial conditions

$$q_0(0) = \phi_0, \quad q_i(x,0) = \phi_i(x), \quad i = 1, 2, \ldots, M \tag{15}$$

produce the solutions $(q_0(t), q_i(x,t))$ to an idealized behavior that represents the reference model (i.e., target system).

Remark 1: Please note that if the idealized controllers $m_i(x)$ were known, then applying controllers $\mu_i(x) = m_i(x)$ to the systems (1)–(2) and examining the evolution of the errors $e_0(t) = p_0(t) - q_0(t)$, $e_i(x,t) = p_i(x,t) - q_i(x,t)$, with boundary condition

$$e_i(0,t) = \lambda_i e_0(t), \quad i = 1, 2, \ldots, M, \quad t > 0, \tag{16}$$

and initial conditions

$$e_0(0) = 0, \quad e_i(x,0) = 0, \quad i = 1, 2, \ldots, M, \tag{17}$$

one can show that the errors are zero.

IV. Model reference adaptive controller design

Since the idealized controller rates $m_i(x)$ in (13), are unknown, one can replace them by their *adaptive estimates*

$\widehat{\mu}_i(x,t)$, $i = 1,\ldots,M$, in (1)–(4). The use of the reference model (13)–(15) will subsequently aid in their adaptation.

A. Stability Analysis

Application of the adaptive controllers $\widehat{\mu}_i(x,t)$ in place of the unknown controllers $\mu_i(x)$ in (1), (2), i.e., select $\mu_i(x) = \widehat{\mu}_i(x,t)$, produces the closed-loop plant

$$\begin{aligned}\frac{\mathrm{d}p_0(t)}{\mathrm{d}t} &= -\sum_{i=1}^{M}\lambda_i p_0(t) + \sum_{i=1}^{M}\int_0^\ell \widehat{\mu}_i(x,t)p_i(x,t)\,\mathrm{d}x,\\ \frac{\partial p_i(x,t)}{\partial t} + \frac{\partial p_i(x,t)}{\partial x} &= -\widehat{\mu}_i(x,t)p_i(x,t),\end{aligned} \tag{18}$$

with boundary conditions (3) and initial conditions (4).

In order to extract the update laws for the adaptive estimates $\widehat{\mu}_i(x,t)$ and to examine the performance of the adaptive controllers, one compares (18) to the model reference (13)–(15). The adaptive errors in this case are governed by

$$\begin{aligned}\frac{\mathrm{d}e_0(t)}{\mathrm{d}t} &= -\sum_{i=1}^{M}\lambda_i e_0(t) + \sum_{i=1}^{M}\int_0^\ell \widehat{\mu}_i(x,t)p_i(x,t)\,\mathrm{d}x\\ &\quad -\sum_{i=1}^{M}\int_0^\ell m_i(x)q_i(x,t)\,\mathrm{d}x\\ &= -\sum_{i=1}^{M}\lambda_i e_0(t) + \sum_{i=1}^{M}\int_0^\ell \widetilde{\mu}_j(x,t)p_i(x,t)\,\mathrm{d}x\\ &\quad +\sum_{i=1}^{M}\int_0^\ell m_i(x)e_i(x,t)\,\mathrm{d}x\end{aligned} \tag{19}$$

and

$$\begin{aligned}\frac{\partial e_i(x,t)}{\partial t} + \frac{\partial e_i(x,t)}{\partial x} &= -\widehat{\mu}_i(x,t)p_i(x,t) + m_i(x)q_i(x,t)\\ &= -\widetilde{\mu}_i(x,t)p_i(x,t) - m_i(x)e_i(x,t)\end{aligned} \tag{20}$$

where $\widetilde{\mu}_i(x,t) = \widehat{\mu}_i(x,t) - m_i(x)$, $i = 1,\ldots,M$, denote the *parameter errors*, with boundary and initial conditions given by (16), (17). The right hand side of (20) used the substitution

$$\begin{aligned}-\widehat{\mu}_i(x,t)p_i(x,t) + m_i(x)q_i(x,t) &= -\left(\widehat{\mu}_i(x,t) - m_i(x)\right)p_i(x,t)\\ &\quad - m_i(x)p_i(x,t) + m_i(x)q_i(x,t)\\ &= -\widetilde{\mu}(x,t)p_i(x,t) - m_i(x)e_i(x,t).\end{aligned}$$

Lemma 1: Consider the reparable system (1)–(3) with the repair rates assumed to satisfy the strengthened positivity (12) in Assumption 1. Moreover, assume that there exists a constant $\varepsilon > 0$ such that

$$\begin{aligned}&\textstyle\sum_{i=1}^{M}(\lambda_i - \frac{1}{2}\lambda_i^2 - \varepsilon) \ge a_1 > 0 \quad \text{and}\\ &\textstyle\sum_{i=1}^{M}(\underline{m}_i - \frac{\overline{m}_i^2\ell}{4\varepsilon}) \ge a_2 > 0,\end{aligned} \tag{21}$$

for some constants $a_1, a_2 > 0$. If the adaptive controllers are updated according to the rules

$$\begin{aligned}&\int_0^\ell \dot{\widehat{\mu}}_i(x,t)\psi_i(x)\,\mathrm{d}x =\\ &\qquad \gamma_i\int_0^\ell \left(e_i(x,t) - e_0(t)\right)p_i(x,t)\psi_i(x)\,\mathrm{d}x\end{aligned} \tag{22}$$

for $i = 1,\ldots,M$ and test functions $\psi_i(x) \in L^2(0,\ell)$, with $\gamma_i > 0$ denoting the *adaptive gains* [14], the state errors converge in the sense

$$\lim_{t\to\infty} e_0(t) = 0, \quad \lim_{t\to\infty}\int_0^\ell e_i^2(x,t)\,\mathrm{d}x = 0, \;\; i = 1,\ldots,M. \tag{23}$$

Proof: The extraction of the update laws (22) is based on Lyapunov-redesign methods. The associated Lyapunov functional is given by

$$\begin{aligned}V(t) &= \frac{1}{2}e_0^2(t) + \frac{1}{2}\sum_{i=1}^{M}\int_0^\ell e_i^2(x,t)\,\mathrm{d}x\\ &\quad + \frac{1}{2}\sum_{i=1}^{M}\int_0^\ell \frac{1}{\gamma_i}\widetilde{\mu}_i^2(x,t)\,\mathrm{d}x.\end{aligned} \tag{24}$$

When the derivative of V is evaluated along (19)–(20) with adaptive laws (22) and conditions (16), (17), one arrives at

$$\begin{aligned}\dot{V} &= -\left(\sum_{i=1}^{M}\lambda_i\right)e_0^2(t) + e_0(t)\sum_{i=1}^{M}\int_0^\ell \widetilde{\mu}_j(x,t)p_i(x,t)\,\mathrm{d}x\\ &\quad + e_0(t)\sum_{i=1}^{M}\int_0^\ell m_i(x)e_i(x,t)\,\mathrm{d}x - \sum_{i=1}^{M}\int_0^\ell e_i(x,t)e_i'(x,t)\,\mathrm{d}x\\ &\quad - \sum_{i=1}^{M}\int_0^\ell \widetilde{\mu}_i(x,t)p_i(x,t)e_i(x,t)\,\mathrm{d}x - \sum_{i=1}^{M}\int_0^\ell m_i(x)e_i^2(x,t)\,\mathrm{d}x\\ &\quad + \sum_{i=1}^{M}\int_0^\ell \widetilde{\mu}_i(x,t)\left(e_i(x,t) - e_0(t)\right)p_i(x,t)\,\mathrm{d}x\\ &= -\left(\sum_{i=1}^{M}\lambda_i\right)e_0^2(t) + e_0(t)\sum_{i=1}^{M}\int_0^\ell m_i(x)e_i(x,t)\,\mathrm{d}x\\ &\quad - \frac{1}{2}\sum_{i=1}^{M}\int_0^\ell \frac{de_i^2(x,t)}{dx}\,\mathrm{d}x - \sum_{i=1}^{M}\int_0^\ell m_i(x)e_i^2(x,t)\,\mathrm{d}x\\ &\le -\left(\sum_{i=1}^{M}\lambda_i\right)e_0^2(t) + |e_0(t)|\sum_{i=1}^{M}\|m_i\|_{L^2}\|e_i(t)\|_{L^2}\\ &\quad - \frac{1}{2}\sum_{i=1}^{M}(e_i^2(L,t) - \lambda_i^2e_0^2(t)) - \underline{m}_i\sum_{i=1}^{M}\|e_i(t)\|_{L^2}^2\\ &\le -\sum_{i=1}^{M}\left(\lambda_i - \frac{1}{2}\lambda_i^2 - \varepsilon\right)e_0^2(t) - \sum_{i=1}^{M}(\underline{m}_i - \frac{\overline{m}_i^2\ell}{4\varepsilon})\|e_i(t)\|_{L^2}^2\\ &\le -a_1|e_0(t)|^2 - a_2\sum_{i=1}^{M}\|e_i(t)\|_{L^2}^2 \le 0,\end{aligned}$$

where the cross product term is simplified via Young's inequality [15] and which results in $a_1, a_2 > 0$ by (21). The convergence (23) along with the boundedness of the adaptive estimates of the controllers follows from [16]. ■

B. Well-posedness of adaptive controller

In order to write the plant (8) in a form that provides an explicit parametrization, we consider abstracting the evolution of each probability density $p_i(x,t)$. We let H denote the state space for (2) that is equipped with an appropriate inner product $\langle\cdot,\cdot\rangle$ and associated induced norm $|\cdot|$. Considering a Gelfand triple, we also let $V = H^1(0,\ell)$ be a reflexive Banach space with norm $\|\phi\| = (|\phi|^2 + |\phi'|^2)^{1/2}$ for $\phi \in V$. We then consider the following triple $V \hookrightarrow H \hookrightarrow V^*$ with V^* denoting the dual $V^* = (H^1(0,\ell))^*$. This space represents the space of continuous conjugate linear functionals on V with the dual norm on it denoted by $\|\cdot\|_*$. As a consequence of the dense and continuous embeddings we have

$$|\phi| \leq k\|\phi|, \qquad k > 0. \tag{25}$$

A parameter space associated with the spatially varying repair rates $\mu_i(x)$ is defined as follows

$$\Theta = \left\{\theta : \theta \in L^1(0,\ell) \text{ with } \theta \text{ satisfying } 0 < \underline{\theta} \leq \theta \leq \overline{\theta}\right\} \tag{26}$$

for some constants $\underline{\theta}, \overline{\theta} > 0$.

We can now define the θ-parameterized operators associated with the PDE (2). For each $\theta \in \Theta$, define the operator $A_0(\theta) : V \to V^*$ by

$$\langle A_0(\theta)\phi, \psi\rangle = -\int_0^\ell \frac{\mathrm{d}\phi(x)}{\mathrm{d}x}\psi(x)\,\mathrm{d}x + \int_0^\ell \theta(x)\phi(x)\psi(x)\,\mathrm{d}x. \tag{27}$$

In a similar fashion to [16], this operator satisfies:

(P1) (Θ-linearity) The map $\theta \to A_0(\theta)\phi$ is affine from Θ into V^* for each test function $\phi \in V$. For each $\theta \in \Theta$ and each $\phi \in V$ we have the decomposition of $A_0(\theta)$ into a *known operator* $A_2 : V \to V^*$, and a *linearly-parameterized* operator $A_1(\theta) : V \to V^*$ as follows

$$A_0(\theta)\phi = A_1(\theta)\phi + A_2\phi. \tag{28}$$

The map $\theta \to A_1(\theta)\phi$ from Θ into V^* is linear for each $\phi \in V$.

(P2) ($V \to V^*$-boundedness) There exist bounds $\alpha_1, \alpha_2 > 0$

$$|\langle A_1(\theta)\phi, \psi\rangle| \leq \alpha_1 |\theta|_\Theta \|\phi\|\,\|\psi\|, \tag{29}$$

for $\phi, \psi \in V$ and $\theta \in \Theta$, and

$$|\langle A_2\phi, \psi\rangle| \leq \alpha_2\|\phi\|\|\psi\|, \quad \phi, \psi \in V. \tag{30}$$

We can now identify the two operators in (27): for each $\theta \in \Theta$ and $\phi, \psi \in V$ we have

$$\langle A_1(\theta)\phi, \psi\rangle = \int_0^\ell \theta(x)\phi(x)\psi(x)\,\mathrm{d}x, \tag{31}$$

and for $\phi, \psi \in V$ we have

$$\langle A_2\phi, \psi\rangle = \int_0^\ell D\phi(x)\psi(x)\,\mathrm{d}x. \tag{32}$$

To express the parameter-dependent operator $A_1(\theta)$ as an operator in Θ, we follow the work in [16]. For each $\phi \in V$, let $\Pi(\phi) : V \to \Theta \subset V^*$ be the linear mapping defined by

$$\langle \theta, \Pi(\phi)\psi\rangle_\Theta = \langle A_1(\theta)\phi, \psi\rangle, \;\; \psi \in V, \; \theta \in \Theta. \tag{33}$$

The boundedness follows from (P2) since for $\phi \in V$ we have

$$\begin{aligned} |\langle \theta, \Pi(\phi)\psi\rangle_\Theta| &= |\langle A_1(\theta)\phi, \psi\rangle| \\ &\leq \overline{\theta}|\phi|\,|\psi| \leq \overline{\theta}\|\phi\|\,\|\psi\|, \end{aligned} \tag{34}$$

and thus the mapping $\Pi(\phi) \colon V \to \Theta$ is bounded. We subsequently define $\Pi^*(\phi) \colon \Theta \to V^*$ via

$$\langle \Pi^*(\phi)\theta, \psi\rangle = \langle \theta, \Pi(\phi)\psi\rangle_\Theta = \langle A_1(\theta)\phi, \psi\rangle, \tag{35}$$

for $\psi \in V$, $\theta \in \Theta$. We can write (1)–(4) as an initial value problem

$$\partial_t p_0(t) = -\sum_{i=1}^{M}\lambda_i p_0(t) + \sum_{i=1}^{M}\langle A_1(\mu_i)p_i(t), 1\rangle, \tag{36a}$$

$$\partial_t p_i(t) = A_2 p_i(t) - A_1(\mu_i)p_i(t) + B\lambda_i p_0(t), \tag{36b}$$

$$p_0(0) = \varphi_0, \;\; p_i(x,0) = \varphi_i(x), \; i = 1,2,\ldots,M, \tag{36c}$$

where the boundary operator $B : \mathbb{R}^1 \to V^*$ is identified as

$$Bc = c\delta(x-0) \quad \text{and} \quad B^*\phi = \phi(0), \tag{37}$$

that is,

$$\langle Bc, \phi\rangle = \langle c, B^*\phi\rangle = \langle c, \phi(0)\rangle, \; \forall c \in \mathbb{R}^1, \; \forall \phi \in V, \tag{38}$$

where we used the fact that $V \hookrightarrow C[0,L]$, so $\phi(0)$ is well-defined. To view the M equations in (36b) collectively along with (36a) as the single evolution equation in (8), (9), we define the *aggregate* state space which consists of M copies of H via $\mathbb{H} = \prod_{i=1}^M H$. Similarly, we define the aggregate interpolating spaces $\mathbb{V} = \prod_{i=1}^M V$ and $\mathbb{V}^* = \prod_{i=1}^M V^*$. It is easily seen that $\mathbb{V} \hookrightarrow \mathbb{H} \hookrightarrow \mathbb{V}^*$ and the duality pairing $\langle\cdot,\cdot\rangle_{\mathbb{V}^*\times\mathbb{V}}$ is the extension by continuity of the inner product $\langle\cdot,\cdot\rangle_{\mathbb{H}}$; i.e., for an element $\boldsymbol{\Phi} = \{\phi_1, \phi_2, \ldots, \phi_M\} \in \mathbb{V}^*$ and an element $\boldsymbol{\Psi} = \{\psi_1, \psi_2, \ldots, \psi_M\} \in \mathbb{V}$ we have that $\langle \boldsymbol{\Phi}, \boldsymbol{\Psi}\rangle_{\mathbb{V}^*\times\mathbb{V}}$ reduces to $\langle \boldsymbol{\Phi}, \boldsymbol{\Psi}\rangle_{\mathbb{H}}$ if $\boldsymbol{\Phi} \in \mathbb{H}$. The aggregate state is

$$\mathbf{p}(t) = \begin{bmatrix} p_1(t) & p_2(t) & \ldots & p_M(t) \end{bmatrix}^T,$$

and the aggregate parameter set is $\boldsymbol{\Theta} = \prod_{i=1}^M \Theta \subset \mathbb{V}^*$. We define the aggregate *parameter-dependent* operator $\mathcal{A}_1(\boldsymbol{\mu}) = \mathbf{I}_M \otimes A_1(\mu_i)|_{i=1}^M$ or in matrix form

$$\mathcal{A}_1(\boldsymbol{\mu}) = \mathrm{diag}(A_1(\mu_1), A_1(\mu_2), \ldots, A_1(\mu_M)),$$

where $A_1(\mu_i)$ is given by (31), $\mathcal{A}_2 = \left(I_M \otimes A_2\right)$ with A_2 given by (32) and $\mathcal{B} = \left(I_M \otimes B\right)$, where $\boldsymbol{\mu} = \begin{pmatrix}\mu_1 & \mu_2 & \cdots & \mu_M\end{pmatrix}^T$, with $\otimes$ denoting the Kronecker product, [17].

Similarly, following (33) we define the aggregate linear mapping $\boldsymbol{\Pi}(\Phi) : \mathbb{V} \to \boldsymbol{\Theta}$ as follows

$$\begin{aligned} \langle \boldsymbol{\mu}, \boldsymbol{\Pi}(\Phi)\Psi\rangle_{\boldsymbol{\Theta}} &= [\langle \mu_1, \Pi(\phi_1)\psi_1\rangle_\Theta, \ldots, \langle \mu_M, \Pi(\phi_M)\psi_M\rangle_\Theta]^T \\ &= [\langle A_1(\mu_1)\phi_1, \psi_1\rangle, \ldots, \langle A_1(\mu_M)\phi_M, \psi_M\rangle]^T \\ &= \langle \mathcal{A}_1(\boldsymbol{\mu})\Phi, \Psi\rangle, \end{aligned}$$

yielding the aggregate version in (35) $\boldsymbol{\Pi}^*(\Phi) \colon \boldsymbol{\Theta} \to \mathbb{V}^*$ via

$$\begin{aligned} \langle \boldsymbol{\Pi}^*(\Phi)\boldsymbol{\mu}, \Psi\rangle &= [\langle \Pi^*(\phi_1)\mu_1, \psi_1\rangle, \ldots, \langle \Pi^*(\phi_M)\mu_M, \psi_M\rangle]^T \\ &= \langle \boldsymbol{\mu}, \boldsymbol{\Pi}(\Phi)\Psi\rangle_{\boldsymbol{\Theta}}. \end{aligned}$$

The equations for the M probability densities in (2) and (36b) can be written in aggregate form as

$$\dot{\mathbf{p}}(t) = -\mathcal{A}_1(\boldsymbol{\mu})\mathbf{p}(t) + \mathcal{A}_2\mathbf{p}(t) + \mathcal{B}\boldsymbol{\lambda}p_0(t),$$

or in terms of $\boldsymbol{\Pi}^*$ as

$$\dot{\mathbf{p}}(t) = -\boldsymbol{\Pi}^*(\mathbf{p}(t))\boldsymbol{\mu} + \mathcal{A}_2\mathbf{p}(t) + \mathcal{B}\boldsymbol{\lambda}p_0(t).$$

Finally, the probability $p_0(t)$ from (36a) can be written as

$$\begin{aligned} \dot{p}_0(t) &= -\left(\boldsymbol{\lambda}^T\mathbf{1}_M\right)p_0(t) + \langle \mathcal{A}_1(\boldsymbol{\mu})\mathbf{p}(t), \mathbf{1}_M\rangle \\ &= -\left(\boldsymbol{\lambda}^T\mathbf{1}_M\right)p_0(t) + \langle \boldsymbol{\Pi}^*(\mathbf{p}(t))\boldsymbol{\mu}, \mathbf{1}_M\rangle \\ &= -\left(\boldsymbol{\lambda}^T\mathbf{1}_M\right)p_0(t) + \langle \boldsymbol{\mu}, \boldsymbol{\Pi}(\mathbf{p}(t))\mathbf{1}_M\rangle_{\boldsymbol{\Theta}}, \end{aligned}$$

where

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_M \end{bmatrix}^T$$

and $\mathbf{1}_M$ is the M-dimensional column vector of 1's.

The augmented state $(p_0(t), \mathbf{p}(t))$ is written in alternate forms to facilitate the control design below

$$\begin{bmatrix} \dot{p}_0(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix} = \begin{bmatrix} -(\boldsymbol{\lambda}^T \mathbf{1}_M) p_0(t) + \langle \mathbf{\Pi}^*(\mathbf{p}(t))\boldsymbol{\mu}, \mathbf{1}_M \rangle \\ \mathcal{A}_2 \mathbf{p}(t) - \mathbf{\Pi}^*(\mathbf{p}(t))\boldsymbol{\mu} + \mathcal{B}\boldsymbol{\lambda} p_0(t) \end{bmatrix}$$

$$= \begin{bmatrix} \langle \mathbf{\Pi}^*(\mathbf{p}(t))\boldsymbol{\mu}, \mathbf{1}_M \rangle \\ -\mathbf{\Pi}^*(\mathbf{p}(t))\boldsymbol{\mu} \end{bmatrix} + \begin{bmatrix} -(\boldsymbol{\lambda}^T \mathbf{1}_M) p_0(t) \\ \mathcal{B}\boldsymbol{\lambda} p_0(t) + \mathcal{A}_2 \mathbf{p}(t) \end{bmatrix} \tag{39}$$

$p_0(0) = \varphi_0, \;\; \mathbf{p}(0) = \boldsymbol{\varphi},$

where

$$\boldsymbol{\varphi}(x) = \begin{bmatrix} \varphi_1(x) & \varphi_2(x) & \dots & \varphi_M(x) \end{bmatrix}^T \in \mathbb{H}.$$

Equation (39) is the detailed representation of the evolution equation (8), (9). The benefit of this representation is that it allows one to design the adaptive controller since it separates the dynamics into parameter-dependent and parameter-independent dynamics.

Similar to (39), one can write the reference model (13) as

$$\begin{bmatrix} \dot{q}_0(t) \\ \dot{\mathbf{q}}(t) \end{bmatrix} = \begin{bmatrix} -(\boldsymbol{\lambda}^T \mathbf{1}_M) q_0(t) + \langle \mathbf{\Pi}^*(\mathbf{q}(t))\mathbf{m}, \mathbf{1}_M \rangle \\ \mathcal{A}_2 \mathbf{q}(t) - \mathbf{\Pi}^*(\mathbf{p}(t))\mathbf{m} + \mathcal{B}\boldsymbol{\lambda} q_0(t) \end{bmatrix}$$

$$= \begin{bmatrix} \langle \mathbf{\Pi}^*(\mathbf{q}(t))\mathbf{m}, \mathbf{1}_M \rangle \\ -\mathbf{\Pi}^*(\mathbf{q}(t))\mathbf{m} \end{bmatrix} + \begin{bmatrix} -(\boldsymbol{\lambda}^T \mathbf{1}_M) q_0(t) \\ \mathcal{B}\boldsymbol{\lambda} q_0(t) + \mathcal{A}_2 \mathbf{q}(t) \end{bmatrix} \tag{40}$$

$q_0(0) = \varphi_0, \;\; \mathbf{q}(0) = \boldsymbol{\varphi}.$

Since the idealized aggregate control signal $\mathbf{m}$ is not available, then application of its adaptive estimate $\widehat{\boldsymbol{\mu}}(t)$ in (39), (40) produces the error system

$$\begin{aligned} \dot{e}_0(t) &= -(\boldsymbol{\lambda}^T \mathbf{1}_M) e_0(t) + \langle \mathbf{\Pi}^*(\mathbf{p}(t))\widetilde{\boldsymbol{\mu}}(t), \mathbf{1}_M \rangle \\ &\quad + \langle (\mathbf{\Pi}^*(\mathbf{p}(t)) - \mathbf{\Pi}^*(\mathbf{q}(t)))\,\mathbf{m}, \mathbf{1}_M \rangle \\ \dot{\mathbf{e}}(t) &= -\mathbf{\Pi}^*(\mathbf{p}(t))\widetilde{\boldsymbol{\mu}}(t) - (\mathbf{\Pi}^*(\mathbf{p}(t)) - \mathbf{\Pi}^*(\mathbf{q}(t)))\,\mathbf{m} \\ &\quad + \mathcal{A}_2 \mathbf{e}(t) + \mathcal{B}\boldsymbol{\lambda} e_0(t) \\ e_0(0) &= 0, \mathbf{e}(0) = \mathbf{0}. \end{aligned} \tag{41}$$

Using (35), we can write (41) as

$$\begin{aligned} \dot{e}_0(t) &= \langle \mathbf{\Pi}^*(\mathbf{p}(t))\widetilde{\boldsymbol{\mu}}(t), \mathbf{1}_M \rangle \\ &\quad -(\boldsymbol{\lambda}^T \mathbf{1}_M) e_0(t) + \langle \mathcal{A}_1(\mathbf{m})\mathbf{e}(t)), \mathbf{1}_M \rangle \\ \dot{\mathbf{e}}(t) &= -\mathbf{\Pi}^*(\mathbf{p}(t))\widetilde{\boldsymbol{\mu}}(t) + \mathcal{B}\boldsymbol{\lambda} e_0(t) - \mathcal{A}_1(\mathbf{m})e(t) + \mathcal{A}_2 \mathbf{e}(t) \end{aligned}$$

with initial conditions $e_0(0) = 0$, $\mathbf{e}(0) = \mathbf{0}$. The above can be combined with the adaptive laws (22) to arrive at the skew-adjoint structure of adaptive systems.

Towards that consider the above expression in matrix form

$$\begin{bmatrix} \dot{e}_0(t) \\ \dot{\mathbf{e}}(t) \end{bmatrix} = \begin{bmatrix} \langle \mathbf{\Pi}^*(\mathbf{p}(t))\widetilde{\boldsymbol{\mu}}(t), \mathbf{1}_M \rangle \\ -\mathbf{\Pi}^*(\mathbf{p}(t))\widetilde{\boldsymbol{\mu}}(t) \end{bmatrix}$$

$$+ \begin{bmatrix} -(\mathbf{1}_M^T \boldsymbol{\lambda}) & \langle \mathcal{A}_1(\mathbf{m})\cdot, \mathbf{1}_M \rangle \\ \mathcal{B}\boldsymbol{\lambda} & (-\mathcal{A}_1(\mathbf{m}) + \mathcal{A}_2) \end{bmatrix} \begin{bmatrix} e_0(t) \\ \mathbf{e}(t) \end{bmatrix}, \tag{42}$$

$e_0(0) = \varphi_0 - \varphi_0, \;\; \mathbf{p}(0) = \boldsymbol{\varphi} - \boldsymbol{\varphi}.$

then

$$\begin{aligned} \begin{bmatrix} \langle \mathbf{\Pi}^*(\mathbf{p}(t))\widetilde{\boldsymbol{\mu}}(t), \mathbf{1}_M \rangle \\ -\mathbf{\Pi}^*(\mathbf{p}(t))\widetilde{\boldsymbol{\mu}}(t) \end{bmatrix} &= \begin{bmatrix} \langle \mathbf{\Pi}^*(\mathbf{p}(t))\bullet, \mathbf{1}_M \rangle \\ -\mathbf{\Pi}^*(\mathbf{p}(t))\bullet \end{bmatrix} \widetilde{\boldsymbol{\mu}}(t) \\ &= \begin{bmatrix} 0 & \langle \mathcal{A}_1(\widetilde{\boldsymbol{\mu}}(t))\bullet, \mathbf{1}_M \rangle \\ 0 & -\mathcal{A}_1(\widetilde{\boldsymbol{\mu}}(t))\bullet \end{bmatrix} \begin{bmatrix} p_0(t) \\ \mathbf{p}(t) \end{bmatrix}. \end{aligned} \tag{43}$$

Define the extended error $E(t) \in \mathcal{H} = \mathbb{R}^1 \times \mathbb{H}$ via

$$E(t) = \begin{bmatrix} e_0(t) \\ \mathbf{e}(t) \end{bmatrix}$$

with the interpolating spaces given by $\mathcal{V} = \mathbb{R}^1 \times \mathbb{V}$ and $\mathcal{V}^* = \mathbb{R}^1 \times \mathbb{V}^*$ We define the counterpart of (35) in the spaces $\mathcal{H}, \mathcal{V}, \mathcal{V}^*$ and $\mathcal{Q}$. To extract the counterpart of the operator Π defined in (33) applied for the extended states over $\mathcal{H}$, we consider the last two expressions in (43), in weak form

$$\begin{aligned} &\langle \mathcal{A}(\boldsymbol{\mu})X, Y \rangle_{\mathcal{V}^*, \mathcal{V}} \\ &= \left\langle \begin{bmatrix} 0 & \langle \mathcal{A}_1(\boldsymbol{\mu})\bullet, \mathbf{1}_M \rangle \\ 0 & -\mathcal{A}_1(\boldsymbol{\mu})\bullet \end{bmatrix} \begin{bmatrix} x_0 \\ \mathbf{x} \end{bmatrix}, \begin{bmatrix} y_0 \\ \mathbf{y} \end{bmatrix} \right\rangle \\ &= \left\langle \begin{bmatrix} 0 & \langle \mathbf{\Pi}^*(\mathbf{x})\bullet, \mathbf{1}_M \rangle \\ 0 & -\mathbf{\Pi}^*(\mathbf{x})\bullet \end{bmatrix} \begin{bmatrix} \boldsymbol{\mu} \end{bmatrix}, \begin{bmatrix} y_0 \\ \mathbf{y} \end{bmatrix} \right\rangle \\ &= \langle \mathcal{P}^*(X)\boldsymbol{\mu}, Y \rangle_{\mathcal{V}^*, \mathcal{V}} \\ &= \left\langle \boldsymbol{\mu}, \begin{bmatrix} \langle x_0 \mathcal{B}^* \bullet, \mathbf{y} \rangle \\ -\langle \bullet, \mathbf{\Pi}(\mathbf{x})\mathbf{y} \rangle_{\mathbf{\Theta}} \end{bmatrix} \right\rangle_{\mathcal{Q}} \\ &= \left\langle \boldsymbol{\mu}, \begin{bmatrix} 0 & \langle x_0 \mathcal{B}^* \bullet, \cdot \rangle \\ 0 & -\langle \bullet, \mathbf{\Pi}(\mathbf{x})\cdot \rangle_{\mathbf{\Theta}} \end{bmatrix} \begin{bmatrix} y_0 \\ \mathbf{y} \end{bmatrix} \right\rangle_{\mathcal{Q}} \\ &= \langle \boldsymbol{\mu}, \mathcal{P}(X)Y \rangle_{\mathcal{Q}}, \end{aligned}$$

for

$$\boldsymbol{\mu} \in \mathcal{Q} = \mathbf{\Theta},$$

and

$$X = \begin{bmatrix} x_0 \\ \mathbf{x} \end{bmatrix}, Y = \begin{bmatrix} y_0 \\ \mathbf{y} \end{bmatrix} \in \mathcal{H} = \mathbb{R}^1 \times \mathbb{H},$$

where we used the identity

$$\langle \boldsymbol{\mu}, \mathbf{\Pi}(\Phi)\Psi \rangle_{\mathbf{\Theta}} = \langle \mathbf{\Pi}^*(\Phi)\boldsymbol{\mu}, \Psi \rangle = \langle \mathcal{A}_1(\boldsymbol{\mu})\Phi, \Psi \rangle.$$

Finally, we define the evolution operator $\mathbf{A}(t) : \mathcal{V} \to \mathcal{V}^*$ via

$$\mathbf{A}(t) = \begin{bmatrix} \mathcal{A}_m & \mathcal{P}^*(X(t)) \\ -\mathcal{P}(X(t)) & 0 \end{bmatrix}, \tag{44}$$

where $\mathcal{A}_m$ is the operator from (42)

$$\mathcal{A}_m = \begin{bmatrix} -(\mathbf{1}_M^T \boldsymbol{\lambda}) & \langle \mathcal{A}_1(\mathbf{m})\cdot, \mathbf{1}_M \rangle \\ \mathcal{B}\boldsymbol{\lambda} & (-\mathcal{A}_1(\mathbf{m}) + \mathcal{A}_2) \end{bmatrix},$$

and which is the operator $\mathcal{A}$ in (8), (9) evaluated at m. The associated state and parameter error is now written in the skew-adjoint form

$$\begin{bmatrix} \dot{\widetilde{X}}(t) \\ \dot{\widetilde{\mathbf{q}}}(t) \end{bmatrix} = \mathbf{A}(t) \begin{bmatrix} \widetilde{X}(t) \\ \widetilde{\mathbf{q}}(t) \end{bmatrix}, \tag{45}$$

with initial conditions $\widetilde{X}(0) = \widehat{X}(0) - X(0) = 0$ and $\widetilde{\mathbf{q}}(0) = \widehat{\mathbf{q}}(0) - \mathbf{m}$. Equation (45) is in the form presented in [18]. The evolution operator $\mathbf{A}(t)$ has the skew-adjoint structure of adaptive systems and satisfies all the conditions in [18] needed for the well-posedness of (13), (14) and (19), (20) or (13), (14) and (18) with the adaptation (22).

V. Numerical Examples

Consider (1)–(4) with $M = 1$. To approximate (2), we use the discretization scheme in [8], [12] with a uniform mesh $\{x_i = ih\}_{i=0}^N$ having a step size of $h = T/N$ in the spatial domain $[0, T]$. Denote $p_{1,i}(t) = p_1(x_i, t)$ and set $\mu_i = \mu(x_i)$. To approximate the integral, we use the rectangular rule and apply the upwind scheme [19] to discretize the spatial first-order partial derivative term in (2). This leads to the semi-discretized state equations

$$\dot{p}_0(t) = -\lambda_1 p_0(t) + h \sum_{i=1}^N \mu_i p_{1,i}(t),$$

$$\dot{p}_{1,i}(t) = -\left(\frac{p_{1,i}(t) - p_{1,i-1}(t)}{h} \right) - \mu_i p_{1,i}(t), \ 1 \le i \le N,$$

with boundary condition $p_{1,0} = \lambda_1 p_0(t)$ and initial conditions $p_0(0) = \phi_0$ and $p_{1,i}(0) = \phi_{1,i}$. Setting

$$p(t) = \begin{bmatrix} p_0(t) & p_{1,1}(t) & \dots & p_{1,N}(t) \end{bmatrix}^T$$

and

$$A_h = \begin{bmatrix} -\lambda_1 & h\mu_1 & h\mu_2 & \cdots & h\mu_N \\ \lambda_1/h & -\mu_1 - 1/h & 0 & \cdots & 0 \\ 0 & 1/h & -\mu_2 - 1/h & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & -\mu_{N-1} - 1/h & 0 \\ 0 & 0 & 0 & 1/h & -\mu_N - 1/h \end{bmatrix},$$

then (1)–(4) is viewed as an IVP of the system

$$\dot{p}(t) = A_h p(t), \quad p(0) = [1, 0, \cdots, 0]^T. \tag{46}$$

Using Trotter-Kato Theorem [20], one can show that the solution to (46) strongly converges to (8)–(9) as $N \to \infty$ [21]. The ODE system (46) can be solved by MATLAB's ODE solvers (e.g., `ode15s`).

The ODE system (46) can be recast into the finite dimensional representation of (39)

$$\dot{p}(t) = P_h^T(p(t))\boldsymbol{\mu} + C_h p(t),$$

where the $(N+1) \times N$ matrix is

$$P_h^T(p) = \begin{bmatrix} hp_{1,1} & hp_{1,2} & hp_{1,3} & \cdots & hp_{1,N} \\ -p_{1,1} & 0 & 0 & \cdots & 0 \\ 0 & -p_{1,2} & 0 & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -p_{1,N-1} & 0 \\ 0 & 0 & \cdots & 0 & -p_{1,N} \end{bmatrix},$$

the $(N+1) \times (N+1)$ matrix is

$$C_h = \begin{bmatrix} -\lambda_1 & 0 & 0 & 0 & \cdots & 0 \\ \lambda_1/h & -1/h & 0 & 0 & \cdots & 0 \\ 0 & 1/h & -1/h & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 1/h & -1/h & 0 \\ 0 & 0 & 0 & 0 & 1/h & -1/h \end{bmatrix},$$

and the N-dimensional vector is $\boldsymbol{\mu} = \begin{bmatrix} \mu_1 & \mu_2 & \dots & \mu_N \end{bmatrix}^T$. The product $P_h^T(p(t))\boldsymbol{\mu}$ above is also written as $D_h(\boldsymbol{\mu})p(t)$ where the $N \times (N+1)$ matrix $D_h(\boldsymbol{\mu})$ is

$$D_h(\theta) = \begin{bmatrix} 0 & h\mu_1 & h\mu_2 & h\mu_3 & \cdots & h\mu_N \\ 0 & -\mu_1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -\mu_2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & -\mu_{N-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & -\mu_N \end{bmatrix},$$

and which represents the matrix representation of the adjoint of the operator $\mathcal{P}^*$ in (44). In a similar way, the finite dimensional representation of the reference model is

$$\dot{q}(t) = P_h^T(q)\mathbf{m} + C_h q(t),$$

and the associated adaptive error is

$$\dot{e}(t) = P_h^T(p)\widetilde{\boldsymbol{\mu}}(t) + (D_h(\mathbf{m}) + C_h)\, e(t),$$

which produces the adaptive laws

$$\dot{\widehat{\boldsymbol{\mu}}}(t) = -\Gamma B_h(p) e(t),$$

where $\Gamma = \Gamma^T > 0$ is the adaptive gain matrix. The parameter space in this case is $\Theta = \{\boldsymbol{\mu} \in \mathbb{R}^N : \mu_1 \dots, \mu_N \ge 0\}$.

The reference model use the idealized repair rate given by $m(x) = 0.2(x/10)^4$ and a known constant rate $\lambda_1 = 0.2$. The initial conditions were selected as $\phi_0 = 0.4$ and $\phi_{1,i} = (1 - \phi_0)/(Mh)$, $i = 1, \dots, N$. Using an adaptive gain in (22) as $\gamma_i = 50$, and the initial estimates $\widehat{\mu}_i(0) = 0.1 m_i$, both the closed loop system and reference model were integrated in the time interval $[0, 20]$s.

Figure 1 depicts the evolution of the plant probability $p_0(t)$ and that of the reference model $q_0(t)$. It is observed that the state error $e_0(t) = p_0(t) - q_0(t)$ converges to zero in less that 0.4s. The plant probability density $p(x,t)$ that uses the proposed adaptive controller, and that of the reference model $q(x,t)$ are depicted in Figure 2, where it is observed that the controlled system converges to the reference model. Finally, the adaptive estimate $\widehat{\mu}(x,t)$ (also the adaptive controller) is presented in Figure 3 at the initial (dashed green line) and final (solid blue line) times as well as the idealized probability density $m(x)$ (dotted black line). It is observed

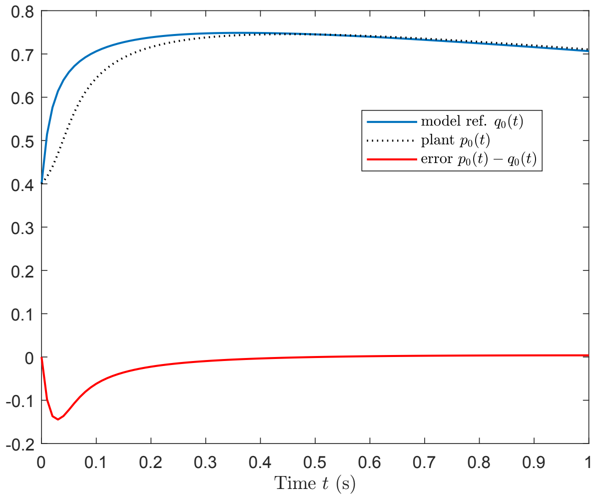

Fig. 1. Evolution of adaptively controlled $p_0(t)$ and model reference $q_0(t)$.

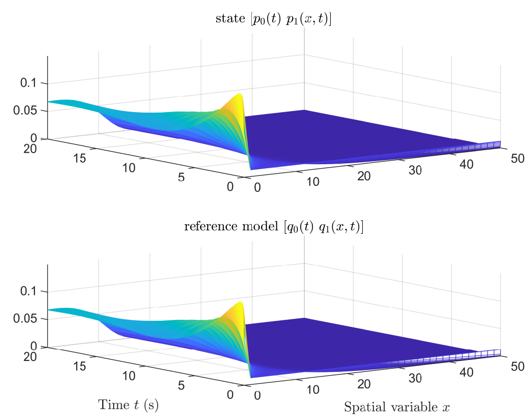

Fig. 2. Evolution of adaptively controlled $p_1(t)$ and model reference $q_1(t)$.

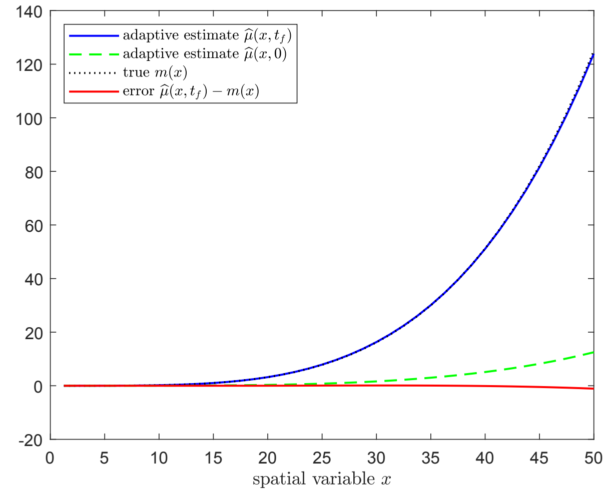

Fig. 3. Spatial evolution of parameter estimate $\widehat{\mu}(x,t)$ at the initial time and final time. The true $m(x)$ is depicted in black dashed lines and the error $\widehat{\mu}(x,t) - m(x)$ at the final time is depicted in red.

that the error at the final time is identically zero for all x.

VI. Conclusions

Following a recently proposed adaptive alternative for the bilinear control of transport integro-differential equations representing multi-rate reparable systems, we presented a model reference adaptive control scheme to ensure that the probabilities of the system in good mode can be steered to a target distribution. Assuming that there exist spatially varying repair rates representing the target distributions and generating the target transport integro-differential equations, we proposed an adaptation of the bilinear controllers and presented both the well-posedness of the resulting adaptive system and the convergence of the plant states to the model reference states in the appropriate norms. Numerical studies involving a system with a single failure model, resulting in a scalar integro-differential equation coupled to a transport PDE, demonstrated both a state and a parameter convergence.

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