Effective Field Theory of Conformal Boundaries

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We introduce an effective field theory (EFT) for conformal impurity by considering a pair of transversely displaced impurities and integrating out modes with mass inversely proportional to the separation distance. This EFT captures the universal signature of the impurity seen by a heavy local operator. We focus on the case of conformal boundaries and derive universal formulas from this EFT for the boundary structure constants at high energy. We point out that the more familiar thermal EFT for conformal field theory is a special case of this EFT with distinguished conformal boundaries. We also derive, for general conformal impurities, nonpositivity and convexitylike constraints on the Casimir energy which determines the leading EFT coefficient.

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Introduction and summary—Matter generally contains impurities that create discontinuities and irregularities in an otherwise homogeneous material. One might assume that their effects are suppressed. However, even isolated impurities can drastically modify the physical behavior of the host at the macroscopic scale, especially when the underlying system is gapless. The archetypal example is the Kondo effect, where magnetic impurities in a metallic host produce an anomalous dip in the resistivity [1]. The theoretical challenge is to explain and extract features of such many-body phenomena with impurities that typically involve strong interactions.

Conformal field theory (CFT) is a powerful nonperturbative framework to describe many-body systems at and near criticality, where exponents and correlation functions are completely determined by an algebra of primary local operators $\mathcal{O}_i(x)$ of scaling dimension Δ_i and structure constants C_{ijk}. This operator algebra is further subject to consistency conditions from conformal symmetry, unitarity and associativity of the operator-product expansion (OPE). This opens the way to systematically constrain specific CFTs by the bootstrap method (see [2]). Moreover, by exploiting these consistency conditions, we can also deduce universal properties of the operator algebra. A famous example is the Cardy formula for $d=2$ CFTs, which gives a precise prediction for the density $\rho(\Delta)$ of high energy states in terms of the central charge c [3]. The Cardy formula was originally derived from the modular invariance of the $d=2$ CFT but can be equivalently obtained from a thermal effective field theory in one dimension lower [4–7]. This is because these high energy states dominate the behavior of the CFT at high temperature, which can be characterized alternatively by an effective field theory with local action $S_{\mathrm{eff}}(\beta)$ from reduction on the Euclidean time circle $\mathbb{S}^1_\beta$ where β is the inverse temperature [8–10],

$$Z(\beta) \equiv \int d\Delta \rho(\Delta) e^{-\beta\Delta} = e^{-S_{\mathrm{EFT}}(\beta)}. \tag{1}$$

This second perspective generalizes the Cardy formula immediately to CFTs in general dimensions [4–7], where the entropy $\log \rho(\Delta)$ at high energy is controlled by the thermal effective action $S_{\mathrm{eff}}(\beta)$ which admits a derivative expansion

$$S_{\mathrm{EFT}}(\beta) = \frac{1}{\beta^{d-1}} \int d^{d-1}x \sqrt{g}(-f + \cdots), \tag{2}$$

where the leading Wilson coefficient f is the negative of thermal free energy density, which is proportional to c in $d=2$.

In this Letter we introduce a generalization of the thermal effective theory that will capture the universal behavior of new CFT data in the presence of impurities. Impurities in critical systems are generally described by conformal defects at long distance. In particular, the Kondo effect is explained by viewing the impurity as a nontrivial boundary condition for the $d=2$ free fermion CFT modeling the electron gas upon an s-wave reduction [11–14]. The nontrivial conformal boundary that emerges at long distances captures the physical signatures of the Kondo impurity. In general, conformal defects are characterized by the one-point function of bulk local operators, which no longer vanish due to the defect insertion and introduce new structure constants for the combined system, known as defect CFT (DCFT) [15–17]. For example, the one-point function of a scalar primary operator $\mathcal{O}_i$ of dimension Δ_i in the presence of a flat conformal boundary $\mathcal{B}$ takes the following form with constant coefficient C_i:

$$\langle \mathcal{O}_i(x) \rangle_{\mathcal{B}} = \frac{C_i}{2^{\Delta_i} |x_\perp|^{\Delta_i}}, \tag{3}$$

where $x_\perp$ is the perpendicular distance to $\mathcal{B}$ and spinning primaries have vanishing one-point functions. Defect data (e.g., C_i above) are subject to further bootstrap-type constraints from the consistency of defect observables, such as the associativity of two-point function of bulk local operators which may factor through either a bulk or a defect OPE channel [16,17]. The main goal of this Letter is to derive universal behavior of such defect data by exploiting an effective field theory (EFT) approach, which we refer to as the defect EFT (DEFT), that generalizes Cardy formula for CFT data in the absence of defects. Our approach is applicable to defects of general codimensions. We focus on the case of conformal boundaries in this Letter to illustrate the main ideas. One of the main results is a Cardy-like formula for the behavior of the structure constant C_i in (3) for operators of high dimensions.

The natural setup where the DEFT arises involves a pair of parallel planar defects $\mathcal{D}_1(0)$ and $\mathcal{D}_2(\delta)$ in flat spacetime with transverse separation δ. This setup introduces a scale to the otherwise gapless system and by integrating out modes of mass $\propto (1/\delta)$, which are supported along the common longitudinal directions of the defects, we arrive at the EFT that captures the fusion limit [i.e., small δ expansion of the defect two-point function $\langle \mathcal{D}_1(0)\mathcal{D}_2(\delta)\rangle$]. The same consideration carries over when the defect world volume Σ is curved as long as the curvature scale is large compared to δ. For a pair of conformal boundaries on Σ (and Σ_δ related by a transverse displacement), the defect effective action takes the same form as in (2) [19],

$$S_{\mathrm{BEFT}}(\delta) = \frac{1}{\delta^{d-1}} \int d^{d-1}x \sqrt{g}(-\mathcal{E} + \cdots), \tag{4}$$

where the leading Wilson coefficient $\mathcal{E}$ is the negative of the Casimir energy density and the subleading Wilson coefficients multiply extrinsic and intrinsic curvature invariants, whose influence on the DCFT data can be found in [18]. The similarity between (4) and (2) is not a coincidence. In fact, the thermal EFT can be thought of as a special case of the boundary EFT (BEFT) introduced here. This is achieved by folding CFT $\mathcal{T}$ on the thermal geometry $\mathbb{S}^1_\beta \times \mathbb{S}^{d-1}$, which produces the tensor product CFT $\mathcal{T} \otimes \bar{\mathcal{T}}$, where $\bar{\mathcal{T}}$ is defined with the orientation reversed, on an interval of length $(\beta/2)$ with boundary condition $\mathcal{B}$ (and its orientation reversal) from folding the trivial interface. Therefore, the thermal EFT is equivalent to the BEFT with $\delta = (\beta/2)$ with these distinguished boundary conditions. The inclusion of angular momentum chemical potentials in the thermal EFT can be similarly accounted for in this BEFT by replacing $\mathcal{B}$ at one end of the interval by the boundary condition that arises from the fusion of the rotation symmetry defect of $\mathcal{T}$ with $\mathcal{B}$, which amounts to a twisted identification for spinning states of $\mathcal{T}$ on $\mathbb{S}^{d-1}$.

We also derive general constraints on the Wilson coefficient $\mathcal{E}$ for general conformal defects of dimension p from unitarity (reflection positivity). In particular, we prove that $\mathcal{E}$ is non-negative when the pair of parallel p-dimensional defects $\mathcal{D}_1$, $\mathcal{D}_2$ are related by orientation reversal and, more generally, $\mathcal{E}$ satisfies a convexitylike property (see the Appendix).

To see how the DEFT encodes universal data of the DCFT, it is convenient to consider the setup where the defects wrap concentric spheres, which are conformally equivalent to disjoint spheres of the same radius. We refer to the two conformal frames as the annulus frame and the bulk-OPE frame, respectively. By tuning the dimensionless modulus in the defect two-point function, it is clear that the annulus frame is naturally described by the DEFT, while the bulk-OPE frame involves decomposition of the individual spherical defects into bulk local operators at their centers and then summing over the two-point functions. It is via the second frame that the DCFT data such as (3) naturally enter and consistency connect their asymptotic behavior at large Δ to the Wilson coefficients in the DEFT.

In the rest of the Letter, we will provide more details on the points summarized above. We point out that our analysis for conformal boundaries in $d = 2$ CFTs is equivalent to studying the Cardy condition (also known as the closed-open duality) for the annulus (cylinder) partition function [15,20]. The DEFT approach advocated here is generalization to general d. In particular, important technical ingredients that enter in the $d = 2$ Cardy condition such as Ishibashi states and annulus blocks have generalizations at higher d, some of which are already available from [21,22], which we make use of here for the boundary case, and a more complete analysis is included in [18]. Similar to the case of thermal EFT analyzed in [6,7], we expect the DEFT to give access to more general defect observables than just the defect structure constants [as in (3)], such as a generalization of the $d = 2$ results in [23,24] to higher dimensions.

Finally, a major triumph of the Cardy formula is its prediction for the microscopic entropy of black hole microstates, which can be understood via the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence [25,26]. More generally, the statistics of heavy operators and OPE data in CFT with anti–de Sitter (AdS) dual is captured by the on-shell actions of nontrivial saddle points such as black hole and wormholes in the gravitational path integral [5,27–31]. It would be interesting to compare our findings to such bulk geometries. For top-down holographic constructions where the defects are realized by probe brane-antibrane pairs in string theory, it

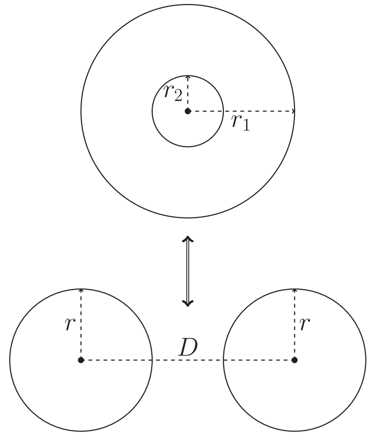

FIG. 1. Annulus and bulk-OPE conformal frames for a pair of conformal boundaries.

would also be interesting to use our CFT results to learn about open-string tachyon condensation and the emission of closed-string radiation [32,33].

Ishibashi states and annulus two-point function in general dimension—We start by setting up the defect two-point function for a pair of conformal boundaries $\mathcal{B}$ and its orientation reversal $\bar{\mathcal{B}}$. In the annulus frame, the conformal boundaries wrap concentric spheres of radius r_1 and r_2, respectively, with $r_1 > r_2$. In the bulk-OPE frame, the conformal boundaries carve out two balls of size r separated by distance D between their centers. The two frames are related by a conformal transformation (see Fig. 1 and, e.g., [34]) with

$$r = \frac{4r_1 r_2}{r_1 - r_2}, \qquad D = \frac{4\sqrt{r_1 r_2}(r_1 + r_2)}{r_1 - r_2}. \tag{5}$$

We introduce the dimensionless modulus β [35],

$$e^{\frac{\beta}{2}} = \frac{r_1}{r_2}, \qquad \cosh\frac{\beta}{4} = \frac{D}{2r}. \tag{6}$$

The defect two-point function in this case is the annulus partition function for the boundary state $|\mathcal{B}\rangle$,

$$Z_{\mathcal{B}\bar{\mathcal{B}}}(\beta) \equiv \langle \mathcal{B}| e^{-\frac{\beta}{2}H} |\mathcal{B}\rangle. \tag{7}$$

where H is the radial Hamiltonian. From (6), the fusion limit corresponds to $\beta \ll 1$, while the bulk-OPE limit is $\beta \gg 1$. As discussed in the Introduction, the former is described by the DEFT (4) with $\delta = (\beta/2)$, and the latter is a sum over two-point functions of scalar operators that appear in the decomposition of the spherical boundaries.

The boundary state $|\mathcal{B}\rangle$ on the unit sphere $\mathbb{S}^{d-1}$ can be decomposed as

$$|\mathcal{B}\rangle = \sum_{\phi} C_\phi |\phi\rangle\rangle, \tag{8}$$

into Ishibashi states $|\phi\rangle\rangle$ which are in one-to-one correspondence with scalar primary operators ϕ and automatically preserve the residual conformal symmetry $\mathfrak{so}(d,1)$. The coefficient C_ϕ is the boundary structure constant that determines the one-point function of ϕ as in (3) and the 2^{Δ_ϕ} is a consequence of the conformal transformation from the planar defect to the spherical defect. The Ishibashi states in general d were first introduced in [21] and we give an alternative derivation in Supplemental Material, Sec. I [36]. The result is the following combination of ϕ and its scalar descendants:

$$|\phi\rangle\rangle = \sum_{n=0}^{\infty} \kappa_n^\phi \Box^n \phi(0)|0\rangle, \qquad \kappa_n^\phi = \frac{2^{-2n}}{n!\left(\Delta_\phi + 1 - \frac{d}{2}\right)_n}, \tag{9}$$

where $(a)_n \equiv \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol.

Although the Ishibashi states are not normalizable, they have well-defined and diagonal matrix elements under radial evolution, giving rise to the annulus blocks,

$$\begin{aligned}\chi^{\text{Annulus}}_{\Delta_\phi}(q) &\equiv \langle\langle \phi | e^{-\frac{\beta}{2}H} |\phi\rangle\rangle \\ &= \frac{q^{\frac{\Delta_\phi}{2}}}{(1-q)^{d-1}} {}_2F_1\left(1-d+\Delta_\phi, 1-\frac{d}{2}, 1-\frac{d}{2}+\Delta_\phi, q\right),\end{aligned} \tag{10}$$

where $q \equiv e^{-\beta}$ and the last line follows from (9). It is easy to check that the annulus blocks are eigenfunctions for the differential equation

$$\frac{4q^{\frac{d}{2}+1}}{(1-q)^d}\frac{d}{dq}\left(\frac{(1-q)^d}{q^{\frac{d}{2}-1}}\frac{d}{dq}\right)\chi^{\text{Annulus}}_{\Delta_\phi} = \Delta_\phi(\Delta_\phi - d)\chi^{\text{Annulus}}_{\Delta_\phi}, \tag{11}$$

which agrees with the conformal Casimir equation of [22].

Putting the above together, we obtain the general decomposition of the annulus partition function into annulus blocks labeled by scalar primaries ϕ,

$$Z_{\mathcal{B}\bar{\mathcal{B}}}(\beta) = \sum_\phi C_\phi^2 \chi^{\text{Annulus}}_{\Delta_\phi}(e^{-\beta}). \tag{12}$$

Comparison to thermal blocks by folding: CFT on the thermal geometry $\mathbb{S}^1_\beta \times \mathbb{S}^{d-1}$ is related to that on the annulus by folding. The thermal partition function

$$Z(\beta) = \int_0^\infty d\Delta \sum_{\vec{J}} \rho^{\text{prim}}(\Delta, \vec{J}) \chi_{\Delta,\vec{J}}(q), \tag{13}$$

decomposes into thermal blocks weighted by the density $\rho^{\text{prim}}(\Delta, \vec{J})$ of conformal primaries with dimension Δ and spin $\vec{J}$. Each thermal block is a conformal character at zero angular chemical potential [41],

$$\chi_{\Delta,\bar{J}}(q)=\frac{q^{\Delta}}{(1-q)^{d}}\dim_{\bar{J}}, \tag{14}$$

where $\dim_{\bar{J}}$ is replaced by the SO(d) character and the denominators are modified accordingly when angular chemical potentials are turned on.

Consistency upon folding and unitarity requires the thermal blocks to decompose into annulus blocks (10) with non-negative coefficients a_n,

$$\chi_{\Delta,\bar{J}}(q)=\dim_{\bar{J}}\sum_{n=0}^{\infty}a_n\chi^{\mathrm{Annulus}}_{2\Delta+2n}(q). \tag{15}$$

Said differently, each thermal block $\chi_{\Delta,\bar{J}}$ for CFT $\mathcal{T}$ corresponds to an *extended* annulus block of dimension 2Δ for the CFT $\mathcal{T}\otimes\bar{\mathcal{T}}$ due to the higher spin symmetries from the extra conserved stress-energy tensor. The explicit coefficients (see Supplemental Material, Sec. II [36]) are

$$a_n=\frac{1}{n!}\frac{\left(\frac{d}{2}\right)_n(2\Delta+n+1-d)_n}{\left(2\Delta+n-\frac{d}{2}\right)_n}, \tag{16}$$

which will be needed to translate our general universal formula for CFT with defects to the special case of CFT without defects.

Asymptotic boundary structure constants—We now derive a universal formula for the behavior of boundary structure constants C_ϕ by analyzing the fusion limit of the annulus partition function (12). We define

$$B(\Delta)\equiv\sum_{\phi}C_{\phi}^{2}\delta(\Delta_\phi-\Delta), \tag{17}$$

and consider $\beta\ll 1$ in (12). The annulus block simplifies in this limit to $\chi^{\mathrm{Annulus}}_{\Delta_\phi}\to e^{-(\beta/2)\Delta_\phi}\beta^{-(d/2)}$ (see Supplemental Material, Sec. II [36] for details) and we obtain

$$\frac{1}{\beta^{\frac{d}{2}}}\int_0^\infty d\Delta B(\Delta)e^{-\frac{\beta}{2}\Delta}\xrightarrow{\beta\ll 1}e^{\frac{2^{d-1}}{\beta^{d-1}}S_{d-1}\mathcal{E}}, \tag{18}$$

where the rhs is the leading contribution from the DEFT (4) and $S_{d-1}\equiv 2\pi^{(d/2)}/\Gamma(d/2)$. Note that this immediately implies that the boundary Casimir energy between $\mathcal{B}$ and $\bar{\mathcal{B}}$ is nonpositive (i.e., $\mathcal{E}\geq 0$) due to the positivity and divergent behavior on the lhs. See the Appendix for an alternative argument that holds for general p-dimensional conformal defects [42].

Performing the inverse Laplace transform by saddle point approximation and taking into account the one-loop contributions, we obtain the following universal formula for the weighted squared boundary structure constants at high energy,

$$B(\Delta)\sim\frac{2^{\frac{d-1}{2}}}{\sqrt{d\pi}}\frac{[(d-1)\mathcal{E}S_{d-1}]^{\frac{d+1}{2d}}}{\Delta^{\frac{2d+1}{2d}}}e^{d\left(\frac{\Delta}{d-1}\right)^{\frac{d-1}{d}}(\mathcal{E}S_{d-1})^{\frac{1}{d}}}. \tag{19}$$

As in the case of other Cardy-type formulas, the above holds only after averaging over a microcanonical energy window that is small compared to Δ. These statements can be made more rigorous, including the leading corrections and precise size of the energy window, by using Tauberian theorems (see [44–49]).

For the special annulus setup related by folding the CFT $\mathcal{T}$ on the thermal geometry, we have the following relations between the BEFT quantities (in the doubled theory) and those in the thermal EFT:

$$\mathcal{E}=\frac{f_{\mathrm{thermal}}}{2^{d-1}},\qquad \Delta=2\Delta_{\mathrm{thermal}}, \tag{20}$$

where the first equation comes from the reduction $\beta\to\beta/2$ from folding, and the second equation comes from the identification between scalar operators in the doubled theory $\mathcal{T}\otimes\bar{\mathcal{T}}$ and general operators in $\mathcal{T}$ [see (15)]. From these relations we see immediately that the exponential behavior in (19) matches the expected asymptotic density of primary operators in $\mathcal{T}$. The one-loop factors also match after taking into account the relation between the annulus and the thermal blocks (see Supplemental Material, Sec. II [36] for intermediate steps).

To further isolate the asymptotic behavior of the structure constants C_ϕ at high energy, we introduce the density of scalar primaries

$$\rho^{\mathrm{prim}}(\Delta)\equiv\sum_{\phi}\delta(\Delta_\phi-\Delta), \tag{21}$$

which can be derived (see Supplemental Material, Sec. III [36]) following the general discussion in [6],

$$\rho^{\mathrm{prim}}(\Delta)\sim\alpha_d\frac{f^{\frac{(d+1)(d+2)}{4d}}}{\Delta^{\frac{d^3+5d+2}{4d}}}e^{d\left(\frac{\Delta}{d-1}\right)^{\frac{d-1}{d}}(fS_{d-1})^{\frac{1}{d}}}, \tag{22}$$

where α_d is a positive constant independent of f and Δ. Therefore, the boundary one-point function of a typical heavy primary operator is

$$C_{\phi}^{\mathrm{typical}}\equiv\sqrt{\frac{B(\Delta)}{\rho^{\mathrm{prim}}(\Delta)}}\propto\Delta^{\frac{d^2+1}{8}}e^{\frac{d}{2}\left(\frac{\Delta}{d-1}\right)^{\frac{d-1}{d}}S_{d-1}^{\frac{1}{d}}(\mathcal{E}^{\frac{1}{d}}-f^{\frac{1}{d}})}, \tag{23}$$

where a constant Δ independent prefactor has been dropped as it gives subleading contribution to $\log C_{\phi}^{\mathrm{typical}}$ in the limit $\Delta\gg 1$. Note that this constant also receives contribution from the subleading Wilson coefficient suppressed by δ^{d-1} in the effective action (4) for odd d. Note that the asymptotic behavior in (23) crucially depends on the magnitude of $\mathcal{E}$ versus f. It would be interesting to see if there exists a universal upper bound on $\mathcal{E}$ by f.

Examples—Here we provide simple examples to illustrate the universal formulas presented in the last section for the boundary structure constants.

$d=2$ CFTs: In our convention, the thermal free energy of a $d=2$ CFT of central charge c is

$$f=\frac{\pi c}{6}. \tag{24}$$

For a simple conformal boundary state $|\mathcal{B}\rangle$ of the $d=2$ CFT, the boundary Casimir energy is universal (i.e., $|\mathcal{B}\rangle$ independent) and given by [50]

$$\mathcal{E}=\frac{\pi c}{24}=\frac{f}{4}. \tag{25}$$

This is because a strip with the corresponding boundary condition $\mathcal{B}$ and its orientation reversal on the two sides is related to a half-space with a single boundary condition $\mathcal{B}$ by a conformal transformation. The ground state on the strip is the identity operator on the boundary after this conformal transformation and the Casimir energy is determined in terms of the bulk central charge c via the usual Schwarzian derivative.

Therefore, we have

$$B(\Delta)\sim\frac{\pi c^{\frac{3}{4}}}{2^{\frac{3}{2}}3^{\frac{3}{4}}\Delta^{\frac{5}{4}}}e^{\pi\sqrt{\frac{c\Delta}{3}}}, \tag{26}$$

and correspondingly,

$$C_{\phi}^{\text{typical}}\propto\Delta^{\frac{5}{8}}e^{-\frac{\pi}{2}\sqrt{\frac{c\Delta}{3}}}, \tag{27}$$

which implies that the boundary structure constants decay exponentially for typical heavy operators in the bulk.

Note that for $d=2$, the annulus block in (10) coincides with the chiral thermal character for a scalar quasiprimary, for which the above formulas apply. It is straightforward to generalize to the case of Virasoro primaries, which amounts to studying the usual Cardy condition on the cylinder in the limit of small length,

$$\int_0^\infty d\Delta B_{\text{Vir}}(\Delta)e^{-\frac{\beta}{2}\Delta}\overset{\beta\ll 1}{\longrightarrow}\sqrt{\frac{2\pi}{\beta}}e^{\frac{\pi^2(c-1)}{6\beta}}, \tag{28}$$

where we have used that, the Virasoro character behaves as

$$\chi_{\Delta}^{\text{Vir}}(\beta)=\frac{q^{\frac{\Delta}{2}-\frac{c-1}{24}}}{\eta(q)}\overset{\beta\ll 1}{\longrightarrow}e^{-\frac{\beta}{2}(\Delta-\frac{c-1}{12})}\sqrt{\frac{\beta}{2\pi}}e^{\frac{\pi^2}{6\beta}}. \tag{29}$$

Consequently, from inverse Laplace transform as before, we find

$$B_{\text{Vir}}(\Delta)\sim\frac{1}{2\sqrt{\Delta}}e^{\pi\sqrt{\frac{(c-1)\Delta}{3}}}, \tag{30}$$

and the boundary structure constant for a typical heavy Virasoro primary behaves as

$$C_{\phi\text{Vir}}^{\text{typical}}\equiv\sqrt{\frac{B_{\text{Vir}}(\Delta)}{\rho_{\text{Vir}}^{\text{prim}}(\Delta)}}\propto\Delta^{\frac{1}{4}}e^{-\frac{\pi}{2}\sqrt{\frac{(c-1)\Delta}{3}}}, \tag{31}$$

where we have used that the density of scalar Virasoro primary states is (see, e.g., [51])

$$\rho_{\text{Vir}}^{\text{prim}}(\Delta)\sim\frac{1}{2\Delta}e^{2\pi\sqrt{\frac{(c-1)\Delta}{3}}}. \tag{32}$$

Again, we find that the boundary structure constant for heavy Virasoro primaries decay exponentially.

Boundaries for free scalars and free fermions: The relation (25) between the Casimir energy $-\mathcal{E}$ and the thermal free energy f does not hold for CFTs of dimension $d>2$. Nonetheless, our universal formulas for the boundary structure constants still apply given the BEFT.

Here we discuss free theories with conformal boundary conditions. For concreteness, we focus on the theory of a real scalar and a Dirac fermion in $d=3$, though the discussion generalizes straightforwardly to such free theories in higher d. We will construct their boundary states $|\mathcal{B}\rangle$ similar to what was done for $d=2$ in [52] and verify (18) by comparing with the corresponding boundary Casimir energy.

We start with the theory of a free real scalar Φ in $d=3$ on the Euclidean cylinder $\mathbb{R}\times\mathbb{S}^2$ where τ labels the Euclidean time coordinate. The Φ field has the following decomposition into eigenfunctions of the Laplacian on $\mathbb{S}^2$ with coordinates (θ,φ):

$$\Phi(\theta,\varphi,\tau)=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{Y_{\ell,m}(\theta,\varphi)}{\sqrt{2\omega_\ell}}\left(a_{\ell,m}e^{\omega_\ell\tau}+a_{\ell,m}^{\dagger}e^{-\omega_\ell\tau}\right), \tag{33}$$

where $Y_{\ell,m}$ are the standard real spherical harmonics, the frequency is fixed by the equation of motion to be $\omega_\ell\equiv\ell+\frac{1}{2}$, and the creation and annihilation operators obey the commutation relation $[a_{\ell,m},a_{\ell',m'}^{\dagger}]=\delta_{\ell,\ell'}\delta_{m,m'}$.

The normal ordered Hamiltonian of the free scalar on $\mathbb{S}^2$ is given by

$$H=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\omega_\ell\left(a_{\ell,m}^{\dagger}a_{\ell,m}+\frac{1}{2}\right). \tag{34}$$

The thermal partition function follows immediately,

$$Z_{\Phi}(\beta)\equiv\operatorname{tr}e^{-\beta H}=\prod_{\ell=0}^{\infty}\frac{1}{\left(1-e^{-\beta(\ell+\frac{1}{2})}\right)^{2\ell+1}}, \tag{35}$$

where the contribution from zero modes in (34) vanishes by ζ function regularization. The above expression, after taking the logarithm, can be rewritten as

$$\log Z_\Phi(\beta) = -\sum_{\ell=0}^{\infty}(2\ell+1)\log\left(1-e^{-\beta\left(\ell+\frac{1}{2}\right)}\right) = \sum_{m=1}^{\infty}\frac{\sinh(m\beta)}{4m\sinh^3\left(m\frac{\beta}{2}\right)}, \quad (36)$$

where the second line comes from expanding the log and summing over ℓ first. In general dimensions, this becomes [6,53,54]

$$\log Z_\Phi(\beta) = \sum_{m=1}^{\infty}\frac{\sinh(m\beta)}{2^{d-1}m\sinh^d\left(m\frac{\beta}{2}\right)} = \frac{2\zeta(d)}{\beta^{d-1}} - \frac{(d-4)\zeta(d-2)}{12\beta^{d-3}} + \mathcal{O}(\beta^4), \quad (37)$$

where we have included the high temperature expansion in the second line and the leading term determines the negative thermal free energy density $f_\Phi = [2\zeta(d)/S_{d-1}]$ [55].

Familiar conformal boundary conditions of a free scalar are Dirichlet and Neumann boundary conditions and we denote the corresponding boundary states as $|\mathcal{B}_D\rangle$ and $|\mathcal{B}_N\rangle$, respectively. The Dirichlet boundary condition $\Phi(\theta,\varphi,\tau=0)|\mathcal{B}_D\rangle=0$ demands that $(a_{\ell,m}+a^\dagger_{\ell,m})|\mathcal{B}_D\rangle=0$ for all ℓ, m; similarly the Neumann boundary condition $\partial_\tau\Phi(\theta,\varphi,\tau=0)|\mathcal{B}_N\rangle=0$ is solved by $(a_{\ell,m}-a^\dagger_{\ell,m})|\mathcal{B}_N\rangle=0$. The boundary states on $\mathbb{S}^2$ at $\tau=0$ are then fixed,

$$|\mathcal{B}_D\rangle = g_D\left(\prod_{\ell=0}^{\infty}\prod_{m=-\ell}^{\ell}e^{-\frac{1}{2}a^\dagger_{\ell,m}a^\dagger_{\ell,m}}\right)|0\rangle,$$
$$|\mathcal{B}_N\rangle = g_N\left(\prod_{\ell=0}^{\infty}\prod_{m=-\ell}^{\ell}e^{\frac{1}{2}a^\dagger_{\ell,m}a^\dagger_{\ell,m}}\right)|0\rangle, \quad (38)$$

up to overall constants g_D, g_N, which coincide with the boundary g function for even d and are scheme dependent for odd d. These expressions are direct generalizations for the boundary states of the noncompact free boson in two dimensions [52].

The cylinder partition function with two such Dirichlet boundary states separated by a distance $\beta/2$ is

$$Z_{\mathcal{B}_D\mathcal{B}_D}(\beta) = \langle\mathcal{B}_D|e^{-\frac{\beta}{2}H}|\mathcal{B}_D\rangle = g_D^2\sqrt{Z_\Phi(\beta)}, \quad (39)$$

where we have used (38) explicitly and compared to (35) in the last equality. Similarly, for a pair of Neumann boundary states, the cylinder partition function reads

$$Z_{\mathcal{B}_N\mathcal{B}_N}(\beta) = \langle\mathcal{B}_N|e^{-\frac{\beta}{2}H}|\mathcal{B}_N\rangle = g_N^2\sqrt{Z_\Phi(\beta)}, \quad (40)$$

and the mixed partition function vanishes $Z_{\mathcal{B}_D\mathcal{B}_N}(\beta)=0$. These are consistent with folding the free scalar theory on $\mathbb{S}^1_\beta\times\mathbb{S}^2$, which implies

$$Z_\Phi(\beta) = Z_{\mathcal{B}_D\mathcal{B}_D}(\beta)Z_{\mathcal{B}_N\mathcal{B}_N}(\beta), \quad (41)$$

as long as $g_N g_D = 1$. The boundary state for the doubled theory from folding is a direct sum of Neumann and Dirichlet boundary states for two real scalars and the cylinder partition function factorizes accordingly as in (41).

From the above, we deduce that the corresponding boundary Casimir energies are related to the thermal free energy of the $d=3$ real scalar by [recall $\delta=(\beta/2)$ in (4)]

$$\mathcal{E}_D = \mathcal{E}_N = \frac{f_\Phi}{8}. \quad (42)$$

It is straightforward to repeat this analysis for general d and obtain

$$\mathcal{E}_D = \mathcal{E}_N = \frac{f_\Phi}{2^d} = \frac{\zeta(d)}{2^{d-1}S_{d-1}}, \quad (43)$$

in agreement with previous results (see, for instance, [19,56]).

We now discuss boundary states for the theory of a Dirac fermion Ψ in $d=3$. Since it is conceptually similar to the free scalar case described above, we will outline the main results here and relegate the details to Supplemental Material, Sec. IV [36].

We denote the two sets of creation and annihilation operators from the decomposition of fermion field $\Psi(\theta,\varphi,\tau)$ as $b^\dagger_{j,m}, b_{j,m}$ and $c^\dagger_{j,m}, c_{j,m}$ with positive half-integer j and half-integer m satisfying $|m|\le j$. The two boundary states $|\mathcal{B}_\pm\rangle$ corresponding to the conformal boundary conditions $\sigma^3\Psi=\pm\Psi$ at $\tau=0$ are

$$|\mathcal{B}_\pm\rangle = g_\pm\prod_{j=\frac{1}{2}}^{\infty}\prod_{m=-j}^{j}\left(1\mp ic^\dagger_{j,m}b^\dagger_{j,m}\right)|0\rangle, \quad (44)$$

with constant coefficients $g_\pm$.

The thermal partition function of the free fermion is

$$Z_\Psi(\beta) = \mathrm{tr}\, e^{-\beta H} = \prod_{j=\frac{1}{2}}^{\infty}\left(1+e^{-\beta\left(j+\frac{1}{2}\right)}\right)^{4j+2}, \quad (45)$$

whose logarithm can be rewritten as

$$\log Z_\Psi(\beta) = \sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m\sinh^2\frac{m\beta}{2}} = \frac{3\zeta(3)}{\beta^2} - \frac{\log 2}{3} + \mathcal{O}(\beta^4), \quad (46)$$

and we have included its high temperature expansion in the second line. The leading contribution determines the thermal free energy which is negative of $f_\Psi = [3\zeta(3)/4\pi]$ in $d=3$. Once again the cylinder partition functions are related to the thermal partition function by

$$Z_{\mathcal{B}_+\mathcal{B}_+}(\beta) = g_+^2\sqrt{Z_\Psi(\beta)}, \quad Z_{\mathcal{B}_-\mathcal{B}_-}(\beta) = g_-^2\sqrt{Z_\Psi(\beta)}, \quad (47)$$

while $Z_{\mathcal{B}_+\mathcal{B}_-}(\beta) = 0$. Consequently, we conclude as before that

$$\mathcal{E}_+ = \mathcal{E}_- = \frac{f_\Psi}{8} = \frac{3\zeta(3)}{32\pi}. \quad (48)$$

It is easy to see that these relations generalize for arbitrary d to

$$\mathcal{E}_+ = \mathcal{E}_- = \frac{f_\Psi}{2^d} = 2^{\lfloor\frac{d}{2}\rfloor+1}\left(1 - 2^{1-d}\right)\frac{\zeta(d)}{2^d S_{d-1}}, \quad (49)$$

where $\lfloor d/2 \rfloor$ is the greatest integer less than or equal to $(d/2)$, and we have used $f_\Psi = 2^{\lfloor d/2\rfloor+1}\left(1 - 2^{1-d}\right)[\zeta(d)/S_{d-1}]$ [53]. These results for the boundary Casimir energy are in agreement with previous results in [19,57].

Note added—Recently, we also became aware that the similar result is derived independently in [58].

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[1] J. Kondo, Prog. Theor. Phys. **32**, 37 (1964).
[2] D. Poland, S. Rychkov, and A. Vichi, Rev. Mod. Phys. **91**, 015002 (2019).
[3] J. L. Cardy, Nucl. Phys. **B270**, 186 (1986).
[4] E. Shaghoulian, Phys. Rev. D **93**, 126005 (2016).
[5] E. Shaghoulian, Phys. Rev. D **94**, 104044 (2016).
[6] N. Benjamin, J. Lee, H. Ooguri, and D. Simmons-Duffin, J. High Energy Phys. 03 (2024) 115.
[7] N. Benjamin, J. Lee, S. Pal, D. Simmons-Duffin, and Y. Xu, arXiv:2405.17562.
[8] S. Bhattacharyya, S. Lahiri, R. Loganayagam, and S. Minwalla, J. High Energy Phys. 09 (2008) 054.
[9] K. Jensen, M. Kaminski, P. Kovtun, R. Meyer, A. Ritz, and A. Yarom, Phys. Rev. Lett. **109**, 101601 (2012).
[10] N. Banerjee, J. Bhattacharya, S. Bhattacharyya, S. Jain, S. Minwalla, and T. Sharma, J. High Energy Phys. 09 (2012) 046.
[11] I. Affleck, Nucl. Phys. **B336**, 517 (1990).
[12] I. Affleck and A. W. Ludwig, Nucl. Phys. **B360**, 641 (1991).
[13] I. Affleck and A. W. Ludwig, Nucl. Phys. **B352**, 849 (1991).
[14] I. Affleck, Acta Phys. Pol. B **26**, 1869 (1995); arXiv:cond-mat/9512099.
[15] J. L. Cardy, Nucl. Phys. **B324**, 581 (1989).
[16] J. L. Cardy and D. C. Lewellen, Phys. Lett. B **259**, 274 (1991).
[17] M. Billò, V. Gonçalves, E. Lauria, and M. Meineri, J. High Energy Phys. 04 (2016) 091.
[18] O. Diatlyk, H. Khanchandani, F. K. Popov, and Y. Wang (to be published).
[19] O. Diatlyk, H. Khanchandani, F. K. Popov, and Y. Wang, J. High Energy Phys. 09 (2024) 006.
[20] J. L. Cardy, arXiv:hep-th/0411189.
[21] Y. Nakayama and H. Ooguri, J. High Energy Phys. 10 (2015) 114.
[22] A. Gadde, J. High Energy Phys. 01 (2020) 038.
[23] Y. Kusuki, J. High Energy Phys. 03 (2022) 161.
[24] T. Numasawa and I. Tsiares, J. High Energy Phys. 08 (2022) 156.
[25] A. Strominger, J. High Energy Phys. 02 (1998) 009.
[26] T. Hartman, C. A. Keller, and B. Stoica, J. High Energy Phys. 09 (2014) 118.
[27] P. Kraus and A. Maloney, J. High Energy Phys. 05 (2017) 160.
[28] J. Chandra, S. Collier, T. Hartman, and A. Maloney, J. High Energy Phys. 12 (2022) 069.
[29] Y. Kusuki, Phys. Rev. D **106**, 066020 (2022).
[30] J. Chandra and T. Hartman, J. High Energy Phys. 10 (2023) 030.
[31] J. Chandra, T. Hartman, and V. Meruliya, arXiv:2404.15183.
[32] A. Sen, Phys. Rev. D **68**, 106003 (2003).
[33] A. Sen, Int. J. Mod. Phys. A **20**, 5513 (2005).
[34] E. Eisenriegler and U. Ritschel, Phys. Rev. B **51**, 13717 (1995).
[35] The normalization is chosen such that β is the standard inverse temperature for the CFT on the thermal geometry before folding into the annulus geometry (conformally equivalent to the cylinder) as mentioned in the Introduction.
[36] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.133.261601 for details, which includes Refs. [37–39].
[37] A. A. Abrikosov, Jr., arXiv:hep-th/0212134.
[38] R. Camporesi and A. Higuchi, J. Geom. Phys. **20**, 1 (1996).
[39] N. Ishibashi, Mod. Phys. Lett. A **04**, 251 (1989).
[40] J. S. Rosenthal, Ann. Probab. **22**, 398 (1994).
[41] F. A. Dolan, J. Math. Phys. (N.Y.) **47**, 062303 (2006).
[42] This nonpositivity property of the Casimir energy also generalizes to conformal defects of higher dimensions following the same reasoning here [18]. This property for line defects is also derived independently in [43].
[43] P. Kravchuk, A. Radcliffe, and R. Sinha, arXiv:2406.04561.
[44] D. Das, S. Datta, and S. Pal, J. High Energy Phys. 11 (2017) 183.
[45] J. Qiao and S. Rychkov, J. High Energy Phys. 12 (2017) 119.
[46] B. Mukhametzhanov and A. Zhiboedov, J. High Energy Phys. 10 (2019) 261.
[47] S. Pal and Z. Sun, J. High Energy Phys. 01 (2020) 135.

[48] B. Mukhametzhanov and S. Pal, SciPost Phys. **8**, 088 (2020).
[49] D. Das, Y. Kusuki, and S. Pal, J. High Energy Phys. 04 (2021) 288.
[50] J. L. Cardy and I. Peschel, Nucl. Phys. **B300**, 377 (1988).
[51] N. Benjamin, H. Ooguri, S.-H. Shao, and Y. Wang, Phys. Rev. D **100**, 066029 (2019).
[52] M. R. Gaberdiel, Fortschr. Phys. **50**, 783 (2002).
[53] P. Chang and J. S. Dowker, Nucl. Phys. **B395**, 407 (1993).
[54] L. Iliesiu, M. Koloğlu, R. Mahajan, E. Perlmutter, and D. Simmons-Duffin, J. High Energy Phys. 10 (2018) 070.
[55] Note that the thermal partition function is divergent at $d=3$, as can be seen from the order β^2 term in (37). Relatedly, the free energy $\log Z_\Phi(\beta)$ contains a $\log\beta$ term in its high temperature expansion. These arise from the gapless mode of the scalar compactified on S^1 (see also [6]). In any case, the thermal free energy f_Φ still captures the leading growth of the density of heavy states.
[56] H. W. Diehl and F. M. Schmidt, New J. Phys. **13**, 123025 (2011).
[57] R. D. M. De Paola, R. B. Rodrigues, and N. F. Svaiter, Mod. Phys. Lett. A **14**, 2353 (1999).
[58] G. Cuomo, Y.-C. He, and Z. Komargodski, J. High Energy Phys. 11 (2024) 061.
[59] C. P. Bachas, J. Phys. A **40**, 9089 (2007).

End Matter

Appendix: General bounds on Casimir energy and one-point function—In this appendix, taking inspiration from [59] and using unitarity (reflection positivity), we derive general constraints on the Casimir energy $-\mathcal{E}$ [defined in a similar way as in (4)] for a pair of parallel p-dimensional conformal defects $\mathcal{D}_1$ and $\mathcal{D}_2$ (see, e.g., [19] for a general discussion). In addition, we also derive a strict upper bound for the one-point function coefficients of bulk local operators in the presence of a boundary.

Let us start by considering two parallel conformal defects $\mathcal{D}_i$ and $\bar{\mathcal{D}}_j$ of dimension p in flat space separated by a transverse distance z. Then as discussed in previous sections (see also [19]), the extensive piece in the defect volume V is controlled by the Casimir energy $-\mathcal{E}_{ij}$ as below,

$$\lim_{V/z^p\to\infty}\frac{1}{V}\log\langle\mathcal{D}_i(0)\bar{\mathcal{D}}_j(z)\rangle=\frac{\mathcal{E}_{ij}}{z^p}. \tag{A1}$$

In the following we will work in this limit.

Let us consider a plane parallel to the defects and located at a distance z_1 from the first defect and z_2 from the second defect (see Fig. 2). Then, taking z to be the Euclidean time direction, the defect correlator computes the overlap of the (time-evolved) defect states; we thus have the following Cauchy-Schwartz inequality:

$$|\langle\mathcal{D}_i(z_1)|\mathcal{D}_j(z_2)\rangle|^2\le\langle\mathcal{D}_i(z_1)|\mathcal{D}_i(z_1)\rangle\langle\mathcal{D}_j(z_2)|\mathcal{D}_j(z_2)\rangle. \tag{A2}$$

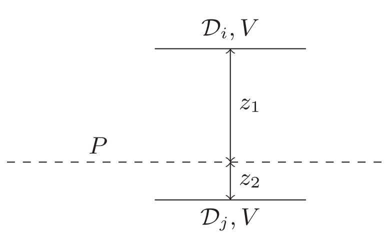

FIG. 2. Configuration of two conformal defects of dimension p parallel to a plane P, the first defect is separated from plane P by a distance z_1 and the second is separated by a distance z_2.

Writing out explicitly these overlaps, we get the following inequality for $\mathcal{E}_{ij}$ (see also [59]):

$$\frac{2\mathcal{E}_{ij}}{(z_1+z_2)^p}\le\frac{\mathcal{E}_{ii}}{(2z_1)^p}+\frac{\mathcal{E}_{jj}}{(2z_2)^p}, \tag{A3}$$

after taking the limit $V\to\infty$ and keeping $z_{1,2}$ fixed.

In the special case where the defects are related by orientation reversal (i.e., $i=j$), the above becomes

$$\frac{2\mathcal{E}_{ii}}{(z_1+z_2)^p}\le\frac{\mathcal{E}_{ii}}{(2z_1)^p}+\frac{\mathcal{E}_{ii}}{(2z_2)^p}. \tag{A4}$$

Using the convexity of the function $(1/z^p)$ we conclude that this inequality could be satisfied only if $-\mathcal{E}_{ii}\le 0$ leading to the nonpositivity of the Casimir energy as stated around (18).

Now we derive an upper bound for the general $\mathcal{E}_{ij}$ in the form of a convexitylike constraint. For that we rescale $z_1+z_2=1$ and set $z_1=t$, $z_2=1-t$, leading to

$$\mathcal{E}_{ij}\le\frac{\mathcal{E}_{ii}}{2^{p+1}t^p}+\frac{\mathcal{E}_{jj}}{2^{p+1}(1-t)^p},\quad \forall\ t\in[0,1]. \tag{A5}$$

After minimizing the rhs of the above equation, we obtain the following inequality:

$$\min_{t\in[0,1]}\left(\frac{\mathcal{E}_{ii}}{2^{p+1}t^p}+\frac{\mathcal{E}_{jj}}{2^{p+1}(1-t)^p}\right)=\left(\frac{1}{2}\mathcal{E}_{ii}^{\frac{1}{p+1}}+\frac{1}{2}\mathcal{E}_{jj}^{\frac{1}{p+1}}\right)^{p+1}\ge\mathcal{E}_{ij}, \tag{A6}$$

where the strongest bound is achieved at

$$t=t_*\equiv\frac{\mathcal{E}_{ii}^{\frac{1}{p+1}}}{\mathcal{E}_{ii}^{\frac{1}{p+1}}+\mathcal{E}_{jj}^{\frac{1}{p+1}}}. \tag{A7}$$

Note that this inequality is saturated only when $\mathcal{D}_i=\mathcal{D}_j$.

This method can also be used to derive a general bound for the one-point function of bulk primary operators in the presence of a conformal defect. For illustration, here we restrict to the case of a conformal boundary. We consider a cylinder geometry $\mathbb{R}_\tau \times \mathbb{S}^{d-1}$ and a boundary state $\mathcal{B}$ that is located at time $\tau = -\tau_1$ and a primary scalar state (corresponding to operator ϕ) at time $\tau = \tau_2$. We have the following Cauchy-Schwartz inequality as before:

$$|\langle \mathcal{B}(-\tau_1)|\phi(\tau_2)\rangle|^2 \leq \langle \mathcal{B}(-\tau_1)|\mathcal{B}(\tau_1)\rangle\langle \phi(-\tau_2)|\phi(\tau_2)\rangle, \quad \text{(A8)}$$

or more explicitly, the following inequality on the one-point function coefficient C_ϕ:

$$\frac{1}{2^{2\Delta_\phi}} \frac{C_\phi^2}{\sinh^{2\Delta_\phi}(\tau_1 + \tau_2)} \leq e^{E(2\tau_1)} \frac{1}{2^{2\Delta_\phi}\sinh^{2\Delta_\phi}(\tau_2)} \Rightarrow C_\phi^2 \leq \min_{\tau_{1,2}} \left[e^{E(2\tau_1)} \left(\frac{\sinh(\tau_1 + \tau_2)}{\sinh(\tau_2)} \right)^{2\Delta_\phi} \right], \quad \text{(A9)}$$

where we have introduced $E(\tau_1 + \tau_2) \equiv \log\langle \mathcal{B}(-\tau_1)|\mathcal{B}(\tau_2)\rangle$. Note that the rhs as a function of τ_2 is monotonically decreasing for all τ_1. Therefore, we reduce the minimization problem to

$$2\log[C_\phi] \leq -\max_{\tau_1}[(-\Delta_\phi)(2\tau_1) - E(2\tau_1)] = -\tilde{E}(-\Delta_\phi), \quad \text{(A10)}$$

where $\tilde{E}(\cdot)$ denotes the Legendre transform of the function $E(\cdot)$. Note that this inequality is never saturated. Indeed, if we had an equality, from Cauchy inequality we must conclude that $|\mathcal{B}\rangle \propto |\phi\rangle$, which is impossible. Thus, this inequality must be strict,

$$2\log[C_\phi] < -\tilde{E}(-\Delta_\phi). \quad \text{(A11)}$$