

ON THE WEAK LOCAL ARTHUR PACKETS CONJECTURE FOR SPLIT CLASSICAL GROUPS

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ABSTRACT. Recently, motivated by the theory of real local Arthur packets, making use of the wavefront sets of representations over non-Archimedean local fields F , Ciubotaru, Mason-Brown, and Okada defined the weak local Arthur packets consisting of certain unipotent representations and conjectured that they are unions of local Arthur packets. In this paper, we prove this conjecture for $\mathrm{Sp}_{2n}(F)$ and split $\mathrm{SO}_{2n+1}(F)$ with the assumption of the residue field characteristic of F being large. In particular, this implies the unitarity of these unipotent representations. We also discuss the generalization of the weak local Arthur packets beyond unipotent representations, which reveals the close connection with a conjecture of Jiang on the structure of wavefront sets for representations in local Arthur packets.

1. INTRODUCTION

Let F be a non-Archimedean field of characteristic zero. Let $G_n = \mathrm{Sp}_{2n}, \mathrm{SO}_{2n+1}, \mathrm{SO}_{2n}$ be the split classical groups, where α is a square class in F , and let $G_n = G_n(F)$. The Langlands dual groups are

$$\widehat{G}_n(\mathbb{C}) = \mathrm{SO}_{2n+1}(\mathbb{C}), \mathrm{Sp}_{2n}(\mathbb{C}), \mathrm{SO}_{2n}(\mathbb{C}),$$

respectively. Let ${}^L G_n = \widehat{G}_n(\mathbb{C}) \times W_F$ be the L -group of G_n .

In his fundamental work [Art13], Arthur introduced the local Arthur packets which are finite sets of representations of G_n , parameterized by local Arthur parameters. Local Arthur parameters are defined as a direct sum of irreducible representations

$$\psi : W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \rightarrow {}^L G_n$$

$$(1.1) \quad \psi = \bigoplus_{i=1}^r \phi_i \otimes S_{m_i} \otimes S_{n_i},$$

satisfying the following conditions:

- (1) $\phi_i(W_F)$ is bounded and consists of semi-simple elements, and $\dim(\phi_i) = k_i$;
- (2) the restrictions of ψ to the two copies of $\mathrm{SL}_2(\mathbb{C})$ are analytic, S_k is the k -dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$, and

$$\sum_{i=1}^r k_i m_i n_i = N = N_n := \begin{cases} 2n+1 & \text{when } G_n = \mathrm{Sp}_{2n}, \\ 2n & \text{when } G_n = \mathrm{SO}_{2n+1}, \mathrm{SO}_{2n}. \end{cases}$$

The first copy of $\mathrm{SL}_2(\mathbb{C})$ is called the Deligne- $\mathrm{SL}_2(\mathbb{C})$, denoted by $\mathrm{SL}_2^D(\mathbb{C})$. The second copy of $\mathrm{SL}_2(\mathbb{C})$ is called the Arthur- $\mathrm{SL}_2(\mathbb{C})$, denoted by $\mathrm{SL}_2^A(\mathbb{C})$. Let $\mathcal{O}_\psi^D$ and $\mathcal{O}_\psi^A$ be the corresponding

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nilpotent orbits via restricting ψ to the first and the second copy of $\mathrm{SL}_2(\mathbb{C})$, respectively (more precisely, see Definition 2.7). We let $\Psi(G_n)$ denote the set of local Arthur parameters of G_n . Assuming the Ramanujan conjecture, Arthur ([Art13]) showed that these local Arthur packets characterize the local components of square-integrable automorphic representations. Given a local Arthur parameter ψ as in (1.1), the local Arthur packet is denoted by Π_ψ . An irreducible admissible representation π of G_n is called of Arthur type if it lies in a local Arthur packet. As an application, Arthur proved the local Langlands correspondence for G_n .

Given a local Arthur parameter ψ as in (1.1), in a series of papers ([Mœ06a, Mœ06b, Moe09, Moe10, Moe11]), Mœglin explicitly constructed each local Arthur packet Π_ψ and showed that it is of multiplicity free. Then, Xu ([Xu17]) gave an algorithm to determine whether the representations in Mœglin's construction are nonzero, and Atobe ([Ato22a]) gave a refinement on Mœglin's construction, using the new derivatives introduced by himself and Mínguez ([AM23]), which makes it relatively easier to compute the enhanced L -parameters. Unlike local L -packets which are disjoint, local Arthur packets may have nontrivial intersections. Recently, Atobe ([Ato22b]), Hazeltine and the authors ([HLL22]) independently studied the intersection problem of local Arthur packets for symplectic and split odd special orthogonal groups, and gave different algorithms to determine when an irreducible representation is of Arthur type and what are the local Arthur packets containing it. Note that understanding the intersection of local Arthur packets is crucial towards the local non-tempered Gan-Gross-Prasad problem (see [GGP20, Conjecture 7.1, Remark 7.3]).

Let G be a connected reductive group defined over F and $G = G(F)$. Given an irreducible representation π of G , one important invariant is a set $\mathfrak{n}(\pi)$ which is defined to be all the F -rational nilpotent orbits $\mathcal{O}$ in the Lie algebra $\mathfrak{g}(F)$ of G such that the coefficient $c_{\mathcal{O}}(\pi)$ in the Harish-Chandra-Howe local expansion of the character $\Theta(\pi)$ of π is nonzero (see [HC78] and [MW87]). Let $\mathfrak{n}^m(\pi)$ be the subset of $\mathfrak{n}(\pi)$ consisting of maximal nilpotent orbits, under the closure ordering of nilpotent orbits. Let $\bar{\mathfrak{n}}(\pi)$ and $\bar{\mathfrak{n}}^m(\pi)$ be the sets of corresponding nilpotent orbits over the algebraic closure $\bar{F}$. The set $\bar{\mathfrak{n}}^m(\pi)$ is called the geometric wavefront set of π . We shall freely identify the set of nilpotent orbits of $\mathfrak{g}(\mathbb{C})$ and that of $\mathfrak{g}(\bar{F})$.

It is an interesting and long-standing question to study the structures of the sets $\mathfrak{n}(\pi)$, $\mathfrak{n}^m(\pi)$, $\bar{\mathfrak{n}}(\pi)$ and $\bar{\mathfrak{n}}^m(\pi)$. For a long time, it is expected that the geometric wavefront set is a singleton. However, recently, Tsai ([Tsa22]) constructed examples showing that the geometric wavefront set may not always be a singleton. Hence, the geometric wavefront set is still very complicated and hard to compute in general. Recently, Okada ([Oka21]) introduced new invariants, the canonical unramified wavefront sets for representations π of depth-0, denoted by $\underline{\mathrm{WF}}(\pi)$, which facilitate a lot of the computation of the geometric wavefront sets for depth-0, especially unipotent representations of p -adic reductive groups ([CMO21, CMO22, CMO23]). In particular, in [CMO22], using the canonical unramified wavefront sets $\underline{\mathrm{WF}}(\pi)$, Ciubotaru, Mason-Brown, and Okada gave a new characterization of the local Arthur packets corresponding to *basic* local Arthur parameters, i.e., those which are trivial on $W_F \times \mathrm{SL}_2^D(\mathbb{C})$, for connected reductive groups, inner to split. More precisely, they gave the following theorem.

Theorem 1.1 ([CMO22, Theorem 3.0.3]). *Let G be a connected reductive group defined and inner to split over F , and let $G = G(F)$. Assume that there is a local Arthur packets theory for G as conjectured in [Art89, Conjecture 6.1] and the residue field of F has sufficiently large characteristic. Let ψ be a basic local Arthur parameter of G and denote by λ the (real) infinitesimal parameter associated with ψ . Then the local Arthur packet corresponding to ψ can be characterized*

as

$$(1.2) \quad \Pi_\psi = \{\pi \in \Pi(G)_\lambda \mid \underline{\text{WF}}(\pi) \leq d_A(\mathcal{O}_\psi^A, 1)\}.$$

Here $\Pi(G)_\lambda$ consists of representations of G with infinitesimal parameter λ , the map d_A is the Achar's duality map defined in [Ach03, §4], and the nilpotent orbit $\mathcal{O}_\psi^A$ is defined similarly as above.

An interesting phenomenon discovered by Ciubotaru, Mason-Brown, and Okada in [CMO22] is that if replacing the canonical unramified wavefront set $\underline{\text{WF}}(\pi)$ by the geometric wavefront set $\bar{\mathbf{n}}^m(\pi)$ and replacing Achar's duality d_A by the Barbasch-Vogan duality d_{BV} , then the right hand side of (1.2) becomes much larger than Π_ψ . Here, the Barbasch-Vogan duality d_{BV} is between the nilpotent orbits of $\hat{\mathfrak{g}}(\mathbb{C})$ and $\mathfrak{g}(\mathbb{C})$ (see [Spa82, BV85, Lus84, Ach03] and §2.2 for details). Inspired by the case of real reductive groups, they conjectured that it would be a union of local Arthur packets as follows.

Conjecture 1.2 ([CMO22, Conjecture 3.1.2]). *Let G be a connected reductive group and $G = G(F)$. Assume that there is a local Arthur packets theory for G as conjectured in [Art89, Conjecture 6.1]. Let ψ be a basic local Arthur parameter of G and denote λ the (real) infinitesimal parameter associated with ψ . Then the weak local Arthur packet defined by*

$$(1.3) \quad \Pi_\psi^{\text{Weak}} := \{\pi \in \Pi(G)_\lambda \mid \bar{\mathbf{n}}^m(\pi) \leq d_{BV}(\mathcal{O}_\psi^A)\}$$

is a union of local Arthur packets.

In this paper, we prove this conjecture for $\text{Sp}_{2n}(F)$ and split $\text{SO}_{2n+1}(F)$ with the assumption of the residue field characteristic of F being large. We also generalize the definition of weak local Arthur packets to general representations of Arthur type (that is lying in some local Arthur packets) and prove the analogous results assuming Jiang's Conjecture on wavefront sets of representations in local Arthur packets as follows.

Conjecture 1.3 (Jiang's Conjecture, [Jia14]). *Let G be a connected reductive group and $G = G(F)$. Assume that there is a local Arthur packets theory for G as conjectured in [Art89, Conjecture 6.1]. Let ψ be a local Arthur parameter of G , and Π_ψ be the local Arthur packet associated with ψ . Then the followings hold.*

- (i) *For any $\pi \in \Pi_\psi$, any nilpotent orbit $\mathcal{O}$ in $\bar{\mathbf{n}}^m(\pi)$ has the property that $\mathcal{O} \leq d_{BV}(\mathcal{O}_\psi^A)$.*
- (ii) *There exists at least one member $\pi \in \Pi_\psi$ having the property that $d_{BV}(\mathcal{O}_\psi^A) \in \bar{\mathbf{n}}^m(\pi)$.*

Jiang's conjecture describes the connection between the structures of local Arthur parameters and the geometric wavefront sets of representations in local Arthur packets. It is a natural generalization of Shahidi's conjecture which says that tempered L -packets of quasi-split groups have generic members, and enhanced Shahidi's conjecture which says that a local Arthur packet of a quasi-split group has a generic member if and only if it is tempered. There has been many progresses on Jiang's conjecture, see [LS23], [HLLS23] for more details. In particular, combining the results of [CMO21, CMO22, CMO23, HLLZ22, HLLS23], Jiang's conjecture is true for all local Arthur parameters of split SO_{2n+1} and Sp_{2n} which are trivial on W_F . More precisely,

Theorem 1.4 ([HLLS23]). *Assume that the residue field of F has sufficiently large characteristic. Conjecture 1.3 holds for any local Arthur parameter ψ of split $\text{SO}_{2n+1}(F)$ and $\text{Sp}_{2n}(F)$ whose restriction to W_F is trivial.*

Let $\Lambda(G_n)$ be the set of infinitesimal parameters of G_n . For $\lambda \in \Lambda(G_n)$, we also let $\Phi(G_n)_\lambda$ (resp. $\Psi(G_n)_\lambda$) be the set of L -parameters (resp. local Arthur parameters) of G_n whose associated infinitesimal parameter is λ . Now we state our main result.

Theorem 1.5 (Theorem 4.5). *Let G_n be the split group $\mathrm{SO}_{2n+1}(F)$, $\mathrm{Sp}_{2n}(F)$ or $\mathrm{SO}_{2n}(F)$. Assume that the residue field of F has sufficiently large characteristic.*

- (a) *For any basic local Arthur parameter ψ_0 of G_n , the weak local Arthur packet $\Pi_{\psi_0}^{\mathrm{Weak}}$ is contained in a union of local Arthur packets*

$$\Pi_{\psi_0}^{\mathrm{Weak}} \subseteq \bigcup_{\psi \in (d_{BV})_{\Psi(G_n)_\lambda}^{-1}(\mathcal{O}')} \Pi_\psi,$$

where $\lambda = \lambda_{\psi_0}$, $\mathcal{O}' = d_{BV}(\mathcal{O}_{\psi_0}^A)$, and

$$(d_{BV})_{\Psi(G_n)_\lambda}^{-1}(\mathcal{O}') := \{\psi \in \Psi(G_n)_\lambda \mid d_{BV}(\mathcal{O}_\psi^A) = \mathcal{O}'\}.$$

- (b) *Moreover, assume Conjecture 1.3(i) holds for any $\psi \in \Psi(G_n)$ whose restriction to W_F is trivial. Then we have the other direction of containment*

$$\Pi_{\psi_0}^{\mathrm{Weak}} \supseteq \bigcup_{\psi \in (d_{BV})_{\Psi(G_n)_\lambda}^{-1}(\mathcal{O}')} \Pi_\psi,$$

which proves Conjecture 1.2. In particular, by Theorem 1.4, Conjecture 1.2 holds for split $\mathrm{SO}_{2n+1}(F)$ and $\mathrm{Sp}_{2n}(F)$ without the assumption of Conjecture 1.3(i).

We give two remarks. First, Part (a) of above theorem shows that weak local Arthur packets $\Pi_{\psi_0}^{\mathrm{Weak}}$ consist of unitary representations, which proves [CMO22, Conjecture 3.1.3] (see Theorem 4.7). Second, the method naturally extends to inner forms of split special orthogonal groups once Arthur's theory on the local Arthur packets being developed. Especially, in the proof, we need the results that $\Pi_{\phi_\psi} \subseteq \Pi_\psi$ and $\Pi_{\hat{\psi}} = \{\hat{\pi} \mid \pi \in \Pi_\psi\}$. Recently, we were informed that Max Gurevich and Emile Okada has an independent proof of Theorem 1.5 in [GO24].

The following proposition plays a key role in the proof of Theorem 1.5 Part (a). For any local L -parameter ϕ , we let $\mathcal{O}_\phi$ denote the corresponding nilpotent orbit via restricting ϕ to the $\mathrm{SL}_2(\mathbb{C})$. Fixing $\lambda \in \Lambda(G_n)$, there exists a unique L -parameter $\phi^0 \in \Phi(G_n)_\lambda$ such that $\mathcal{O}_{\phi^0} \geq \mathcal{O}_\phi$ for any $\phi \in \Phi(G_n)_\lambda$ (see Proposition 2.8). We call ϕ^0 the unique open L -parameter of $\Phi(G_n)_\lambda$.

Proposition 1.6 (Proposition 3.4). *Let $\lambda \in \Lambda(G_n)$. Suppose the unique open L -parameter ϕ^0 of $\Phi(G_n)_\lambda$ is tempered and let $\mathcal{O}' = d_{BV}(\mathcal{O}_{\phi^0})$. Then any L -parameter in*

$$(d_{BV})_{\Phi(G_n)_\lambda}^{-1}(\mathcal{O}') := \{\phi \in \Phi(G_n)_\lambda \mid d_{BV}(\mathcal{O}_\phi) = \mathcal{O}'\}$$

is of Arthur type.

The proof of this proposition is based on the explicit description of $d_{BV}^{-1}(\mathcal{O}') := \{\mathcal{O} \mid d_{BV}(\mathcal{O}) = \mathcal{O}'\}$ studied in [LLS24] (see Lemma 3.1). It seems that for general groups, Conjecture 1.2 does not imply Proposition 1.6 in an obvious way. Thus, Proposition 1.6 has its own interests.

If we replace the $\bar{\pi}^m(\pi)$ by $d_{BV}(\mathcal{O}_{\phi_{\bar{\pi}}})$ in the definition of weak local Arthur packet (1.3), then we can generalize Theorem 1.5 to anti-tempered local Arthur parameters, which are not necessarily basic. More precisely, we have the following. Note that in this case we do not need the characteristic assumption of the residue field of F .

Theorem 1.7 (Theorem 4.9). *Let G_n be the split group $\mathrm{SO}_{2n+1}(F)$, $\mathrm{Sp}_{2n}(F)$ or $\mathrm{SO}_{2n}(F)$. For any anti-tempered local Arthur parameter ψ_0 , we denote $\lambda := \lambda_{\psi_0}$ and $\mathcal{O}' := d_{BV}(\mathcal{O}_{\psi_0}^A)$. Consider the set of representations*

$$\Pi_{\psi_0}^{\mathrm{Weak}} := \{\pi \in \Pi(G_n)_\lambda \mid d_{BV}(\mathcal{O}_{\phi_\pi}) \leq d_{BV}(\mathcal{O}_{\psi_0}^A)\}.$$

We have an inclusion

$$(1.4) \quad \Pi_{\psi_0}^{\mathrm{Weak}} \subseteq \bigcup_{\psi \in (d_{BV})^{-1}_{\Psi(G_n)_\lambda}(\mathcal{O}')} \Pi_\psi.$$

Moreover, if $\mathcal{O}_{\phi_\pi} \geq \mathcal{O}_{\phi_\psi}$ for any $\psi \in \Psi(G_n)_\lambda$ and $\pi \in \Pi_\psi$, which has already been verified for the split groups $\mathrm{SO}_{2n+1}(F)$ and $\mathrm{Sp}_{2n}(F)$ in [HLLZ22, Theorem 1.15, Corollary 4.12(2)], then the inclusion (1.4) is an equality.

Finally, we discuss possible generalizations of the weak local Arthur packets beyond the basic local Arthur parameters, which would facilitate a better understanding of Jiang's conjecture. Let G be a connected reductive group and $G = G(F)$. We assume that there is a local Arthur packets theory for G as conjectured in [Art89, Conjecture 6.1]. In the following discussion, we let $\mathcal{O}'$ be any nilpotent orbit of $\mathfrak{g}(\overline{F})$, $\lambda \in \Lambda(G)$, and $\psi \in \Psi(G)_\lambda$. A first generalization would be

$$(1.5) \quad \Pi_\psi^{\mathrm{WF}} := \{\pi \in \Pi(G)_\lambda \mid \overline{\mathfrak{n}}^m(\pi) \leq d_{BV}(\mathcal{O}_\psi^A)\},$$

or, more generally,

$$(1.6) \quad \Pi_{\mathcal{O}', \lambda}^{\mathrm{WF}} := \{\pi \in \Pi(G)_\lambda \mid \overline{\mathfrak{n}}^m(\pi) \leq \mathcal{O}'\}.$$

However, these sets may not always be unions of local Arthur packets since they may contain representations not of Arthur type, see Example 5.1. A natural modification of (1.5) or (1.6) is to add the condition of Arthur type as follows

$$(1.7) \quad \Pi_\psi^{\mathrm{WF}, A} := \{\pi \in \Pi(G)_\lambda \text{ of Arthur type} \mid \overline{\mathfrak{n}}^m(\pi) \leq d_{BV}(\mathcal{O}_\psi^A)\},$$

or, more generally,

$$(1.8) \quad \Pi_{\mathcal{O}', \lambda}^{\mathrm{WF}, A} := \{\pi \in \Pi(G)_\lambda \text{ of Arthur type} \mid \overline{\mathfrak{n}}^m(\pi) \leq \mathcal{O}'\},$$

However, these sets still may not always be unions of local Arthur packets, see Example 5.2. Now we give the another generalization. Define

$$\Pi_{\mathcal{O}', \lambda}^{\mathrm{Weak}} := \{\pi \in \Pi(G)_\lambda \text{ of Arthur type} \mid \text{There exists a } \psi \in \Psi(\pi) \text{ such that } d_{BV}(\mathcal{O}_\psi^A) \leq \mathcal{O}'\},$$

where $\Psi(\pi) := \{\psi \in \Psi(G) \mid \pi \in \Pi_\psi\}$. Then, we have the following proposition.

Proposition 1.8 (Proposition 5.3). *Let G be a connected reductive group and $G = G(F)$. Assume that there is a local Arthur packets theory for G as conjectured in [Art89, Conjecture 6.1]. Assume Conjecture 1.3 holds for G . For any nilpotent orbit $\mathcal{O}'$ of $\mathfrak{g}(\overline{F})$ and any infinitesimal parameter λ of G , we have*

$$\Pi_{\mathcal{O}', \lambda}^{\mathrm{Weak}} = \bigcup_{\psi \in \Psi(G)_\lambda, \Pi_\psi \subseteq \Pi_{\mathcal{O}', \lambda}^{\mathrm{WF}, A}} \Pi_\psi \subseteq \Pi_{\mathcal{O}', \lambda}^{\mathrm{WF}, A},$$

where the containment can be strict.

The set $\Pi_{\mathcal{O}', \lambda}^{\mathrm{Weak}}$ can be regarded as a natural generalization of $\Pi_{\psi_0}^{\mathrm{Weak}}$ and Proposition 1.8 implies its close relation with Conjecture 1.3. Indeed, assuming Conjecture 1.3, one can see from Proposition 1.8 that $\Pi_{\mathcal{O}', \lambda}^{\mathrm{Weak}}$ is the maximal subset of $\Pi(G)_\lambda$ with the following properties.

- $\Pi_{\mathcal{O}', \lambda}^{\text{Weak}} \subseteq \{\pi \in \Pi(G)_\lambda \mid \bar{\mathfrak{n}}^m(\pi) \leq \mathcal{O}'\}$.
- $\Pi_{\mathcal{O}', \lambda}^{\text{Weak}}$ is a union of local Arthur packets.

Hence, if Conjecture 1.2 holds for a basic local Arthur parameter ψ_0 of G , then

$$\Pi_{\psi_0}^{\text{Weak}} = \Pi_{\mathcal{O}', \lambda}^{\text{Weak}},$$

where $\mathcal{O}' = d_{BV}(\mathcal{O}_{\psi_0}^A)$ and $\lambda = \lambda_{\psi_0}$.

Following is the structure of the paper. In §2, we introduce necessary preliminaries. In §3, we recall certain results from [LLS24] on the fibers of the Barbasch-Vogan duality and prove the key Proposition 1.6. In §4, we prove our main result Theorem 1.5 on the weak local Arthur packets Conjecture 1.2 and its variant Theorem 1.7. In §5, we generalize the definition of weak local Arthur packets and prove Proposition 1.8.

2. PRELIMINARIES

In this section, we recall the preliminaries and notations that will be used in this paper.

We let F be a non-Archimedean field of characteristic zero whose residue field has cardinality $q = q_F$. Let W_F denote the Weil group of F and let I_F be its inertia subgroup. The group G_n will be one of the groups $\text{Sp}_{2n}, \text{SO}_{2n+1}, \text{SO}_{2n}$ defined and split over F unless specified otherwise. We let $G_n = G_n(F)$ and let $\mathfrak{g}_n$ denote their Lie algebras. Thus, $\mathfrak{g}_n = \mathfrak{sp}_{2n}, \mathfrak{so}_{2n+1}, \mathfrak{so}_{2n}$ respectively.

2.1. Partitions and nilpotent orbits. In this subsection, we recall the basic notation for partitions and the correspondence between nilpotent orbits of $\mathfrak{g}_n(\mathbb{C})$ and partitions.

First, we denote the set of partitions of n by $\mathcal{P}(n)$. We express a partition $\underline{p} \in \mathcal{P}(n)$ in one of the following forms.

- (i) $\underline{p} = [p_1, \dots, p_N]$, such that p_i 's are non-increasing and $\sum_{i=1}^N p_i = n$. We denote the length of $\underline{p}$ by $l(\underline{p}) = |\{1 \leq i \leq N \mid p_i > 0\}|$.
- (ii) $\underline{p} = [p_1^{r_1}, \dots, p_N^{r_N}]$, such that p_i 's are decreasing and $\sum_{i=1}^N r_i p_i = n$. We assume $r_i > 0$ unless specified.

Also, we denote $|\underline{p}| = n$ if $\underline{p} \in \mathcal{P}(n)$. We let $\geq$ denote the dominance order on $\mathcal{P}(n)$. That is, if $\underline{p} = [p_1, \dots, p_r], \underline{q} = [q_1, \dots, q_s] \in \mathcal{P}(n)$, then $\underline{p} \geq \underline{q}$ if $\sum_{i=1}^k p_i \geq \sum_{i=1}^k q_i$ for any $1 \leq k \leq r$.

Next, we recall the definitions for partitions of type B, C and D .

Definition 2.1. For $\epsilon \in \{\pm 1\}$, we define

$$\mathcal{P}_\epsilon(n) = \{[p_1^{r_1}, \dots, p_N^{r_N}] \in \mathcal{P}(n) \mid r_i \text{ is even for all } p_i \text{ with } (-1)^{p_i} = \epsilon\}.$$

Then we say

- (1) $\underline{p} \in \mathcal{P}(n)$ is of type B if n is odd and $\underline{p} \in \mathcal{P}_1(n)$.
- (2) $\underline{p} \in \mathcal{P}(n)$ is of type C if n is even and $\underline{p} \in \mathcal{P}_{-1}(n)$.
- (3) $\underline{p} \in \mathcal{P}(n)$ is of type D if n is even and $\underline{p} \in \mathcal{P}_1(n)$.

We denote $\mathcal{P}_X(n)$ the set of partitions of n of type X .

Denote the set of nilpotent orbits of $\mathfrak{so}_{2n+1}(\mathbb{C})$, $\mathfrak{sp}_{2n}(\mathbb{C})$ and $\mathfrak{so}_{2n}(\mathbb{C})$ by $\mathcal{N}_B(2n+1)$, $\mathcal{N}_C(2n)$ and $\mathcal{N}_D(2n)$ respectively. Also, we denote

$$\mathcal{N}_B = \bigcup_{n \geq 0} \mathcal{N}_B(2n+1), \quad \mathcal{N}_C = \bigcup_{n \geq 0} \mathcal{N}_C(2n), \quad \mathcal{N}_D = \bigcup_{n \geq 0} \mathcal{N}_D(2n).$$

For $(X, N) \in \{(B, 2n+1), (C, 2n), (D, 2n)\}$, there is a surjection

$$\begin{aligned}\mathcal{N}_X(N) &\longrightarrow \mathcal{P}_X(N) \\ \mathcal{O} &\longmapsto \underline{p}_{\mathcal{O}}.\end{aligned}$$

The fiber of $\underline{p} = [p_1^{m_1}, \dots, p_r^{m_r}] \in \mathcal{P}_X(N)$ under this map is a singleton, which we denote by $\{\mathcal{O}_{\underline{p}}\}$, except when $X = D$ and $\underline{p}$ is “very even”, i.e., the integers p_i ’s are all even. When $\underline{p}$ is very even, the fiber consists of two nilpotent orbits, which we denote by $\mathcal{O}_{\underline{p}}^I$ and $\mathcal{O}_{\underline{p}}^{II}$.

The surjection $\mathcal{O} \mapsto \underline{p}_{\mathcal{O}}$ carries the closure ordering on $\mathcal{N}_X(N)$ to the dominance ordering on $\mathcal{P}_X(N)$ in the sense that $\mathcal{O} > \mathcal{O}'$ if and only if $\underline{p}_{\mathcal{O}} > \underline{p}_{\mathcal{O}'}$. Note that when $\underline{p}$ is very even, $\mathcal{O}_{\underline{p}}^I$ and $\mathcal{O}_{\underline{p}}^{II}$ are not comparable.

2.2. Barbasch-Vogan duality. In this subsection, following [Spa82, BV85, Lus84, Ach03], we introduce several operations on the set of partitions, and then use them to describe the definition of the Barbasch-Vogan duality on the level of partitions and nilpotent orbits.

First, we need the following operations to construct or decompose partitions.

Definition 2.2. Suppose $\underline{p} \in \mathcal{P}(n_1)$ and $\underline{q} \in \mathcal{P}(n_2)$.

- (i) Write $\underline{p} = [p_1^{r_1}, \dots, p_N^{r_N}]$ and $\underline{q} = [p_1^{s_1}, \dots, p_N^{s_N}]$, where we allow $r_i = 0$ or $s_i = 0$. Then we define

$$\underline{p} \sqcup \underline{q} = [p_1^{r_1+s_1}, \dots, p_N^{r_N+s_N}].$$

- (ii) Write $\underline{p} = [p_1, \dots, p_N]$, we define

$$\begin{aligned}\underline{p}^+ &= [p_1 + 1, p_2, \dots, p_N] \in \mathcal{P}(n_1 + 1), \\ \underline{p}^- &= [p_1, \dots, p_{N-1}, p_N - 1] \in \mathcal{P}(n_1 - 1).\end{aligned}$$

- (iii) Write $\underline{p} = [p_1, \dots, p_N] \in \mathcal{P}(n)$. We define $\underline{p}^* = [p_1^*, \dots, p_N^*] \in \mathcal{P}(n)$, the transpose (or conjugation) of $\underline{p}$, by

$$p_i^* = |\{j \mid p_j \geq i\}|.$$

Next, we recall the definition of collapse. Let n be a positive integer and let $X = B$ if n is odd and $X \in \{C, D\}$ if n is even. For any $\underline{p} \in \mathcal{P}(n)$, there exists a unique maximal partition $\underline{p}_X \in \mathcal{P}(n)$ of type X such that $\underline{p}_X \leq \underline{p}$. We call $\underline{p}_X$ the X -collapse of $\underline{p}$.

Now we recall the definition of Barbasch-Vogan duality for partitions of type X . Following the notation in [Ach03], we shall omit the parentheses between the superscript and subscript. For example, we shall write $\underline{p}_D^+ \subscript{B}^- *$ instead of $(((\underline{p}_D)^+)_B)^- *$.

- Definition 2.3.** (i) For $\underline{p} \in \mathcal{P}(2n+1)$ of type B , we define $d_{BV}(\underline{p}) := \underline{p}_C^*$, which is in $\mathcal{P}(2n)$ of type C .
(ii) For $\underline{p} \in \mathcal{P}(2n)$ of type C , we define $d_{BV}(\underline{p}) := \underline{p}_B^+$, which is in $\mathcal{P}(2n+1)$ of type B .
(iii) For $\underline{p} \in \mathcal{P}(2n)$ of type D , we define $d_{BV}(\underline{p}) := \underline{p}_D^*$, which is in $\mathcal{P}(2n)$ of type D .

Finally, we recall the definition of the Barbasch-Vogan duality on nilpotent orbits. If $\underline{p} \in \mathcal{P}_D(2n)$ is very even, then we define (see [CM93, Corollary 6.3.5])

$$(2.1) \quad d_{BV}(\mathcal{O}_{\underline{p}}^I) := \begin{cases} \mathcal{O}_{d_{BV}(\underline{p})}^I & \text{if } n \text{ is even,} \\ \mathcal{O}_{d_{BV}(\underline{p})}^{II} & \text{if } n \text{ is odd,} \end{cases} \quad d_{BV}(\mathcal{O}_{\underline{p}}^{II}) := \begin{cases} \mathcal{O}_{d_{BV}(\underline{p})}^{II} & \text{if } n \text{ is even,} \\ \mathcal{O}_{d_{BV}(\underline{p})}^I & \text{if } n \text{ is odd.} \end{cases}$$

Otherwise, we define $d_{BV}(\mathcal{O}_{\underline{p}}) := \mathcal{O}_{d_{BV}(\underline{p})}$. See [CM93, Corollary 6.3.5]. We say a nilpotent orbit or a partition is *special* if it is in the image of the Barbasch-Vogan duality map.

We remark that the formula

$$d_{BV}(\mathcal{O}_{\underline{p}}^I) = \begin{cases} \mathcal{O}_{\underline{p}}^I & \text{if } n \text{ is even,} \\ \mathcal{O}_{\underline{p}}^{II} & \text{if } n \text{ is odd.} \end{cases}$$

in the paragraph below [HLLS23, Definition 2.5] should be replaced by (2.1).

2.3. Definition of parameters. In this subsection, we recall the definition of L -parameters, local Arthur parameters and infinitesimal parameters of G_n and related notations. We allow G_n to be $\mathrm{GL}_n(F)$ in this subsection. Since G_n is split over F , we replace ${}^L G_n$ by $\widehat{G}_n(\mathbb{C})$ to simplify the notation.

Definition 2.4. An L -parameter $[\phi]$ of G_n is a $\widehat{G}_n(\mathbb{C})$ -conjugacy class of an admissible homomorphism

$$\phi : W_F \times \mathrm{SL}_2(\mathbb{C}) \rightarrow \widehat{G}_n(\mathbb{C}).$$

That is, ϕ is continuous, and

- (1) the restriction of ϕ to W_F consists of semi-simple elements;
- (2) the restriction of ϕ to $\mathrm{SL}_2(\mathbb{C})$ is a morphism of complex algebraic groups;

By abuse of notation, we do not distinguish $[\phi]$ and ϕ .

An enhanced L -parameter of G_n is a pair (ϕ, χ) , where ϕ is an L -parameter and χ is an irreducible representation of the component group $\mathcal{S}_\phi := \pi_0(\mathrm{Cent}(\mathrm{im}(\phi), \widehat{G}_n)/Z(\widehat{G}_n)^\Gamma)$.

We let $\Phi(G_n)$ denote the set of L -parameters of G_n and let $\Phi_e(G_n)$ denote the set of enhanced L -parameter of G_n .

Definition 2.5. A local Arthur parameter $[\psi]$ of G_n is a $\widehat{G}(\mathbb{C})$ -conjugacy class of a homomorphism

$$\psi : W_F \times \mathrm{SL}_2^D(\mathbb{C}) \times \mathrm{SL}_2^A(\mathbb{C}) \rightarrow \widehat{G}_n(\mathbb{C}),$$

such that

- (1) $\psi|_{W_F \times \mathrm{SL}_2(\mathbb{C})^D}$ is an L -parameter;
- (2) the restriction of ψ to $\mathrm{SL}_2^A(\mathbb{C})$ is a morphism of complex algebraic groups;
- (3) $\psi|_{W_F}$ has bounded image.

By abuse of notation, we do not distinguish $[\psi]$ and ψ . We let $\Psi(G_n)$ denote the set of local Arthur parameters of G_n .

For each $\psi \in \Psi(G_n)$, we may define another local Arthur parameter $\widehat{\psi}$ by swapping $\mathrm{SL}_2^D(\mathbb{C})$ and $\mathrm{SL}_2^A(\mathbb{C})$. Namely, the morphism $\widehat{\psi}$ is given by

$$(2.2) \quad \widehat{\psi}(w, x, y) := \psi(w, y, x).$$

Also, we may associate an L -parameter ϕ_ψ to ψ by

$$\phi_\psi(w, x) := \psi(w, x, d_w),$$

where $d_w = \mathrm{diag}(|w|^{1/2}, |w|^{-1/2}) \in \mathrm{SL}_2(\mathbb{C})$. It is proved in [Art13] that the map $\psi \mapsto \phi_\psi$ is an injection. We say an L -parameter $\phi \in \Phi(G_n)$ is of *Arthur type* if $\phi = \phi_\psi$ for some $\psi \in \Psi(G_n)$.

In [Art13], for each local Arthur parameter ψ of G_n , he constructed a finite (multi-)set Π_ψ of $\Pi(G_n)$, called the local Arthur packet of ψ . Based on the parametrization of tempered local

Arthur packets and tempered spectrum, this induces the local Langlands Correspondence (fixing a Whittaker datum)

$$\begin{aligned}\Pi(G_n) &\rightarrow \Phi_e(G_n), \\ \pi &\mapsto (\phi_\pi, \chi_\pi).\end{aligned}$$

We call ϕ_π the L -parameter of π and call the set $\Pi_\phi := \{\pi \mid \phi_\pi = \phi\}$ the L -packet of ϕ . Arthur also showed that the local Arthur packet Π_ψ must contain the associated L -packet Π_{ϕ_ψ} .

Similar to the assignment $\psi \mapsto \phi_\psi$, for each L -parameter ϕ , we may associate a morphism λ_ϕ from W_F to $\widehat{G}_n(\mathbb{C})$ by

$$\lambda_\phi(w) := \phi(w, d_w).$$

For $\psi \in \Psi(G_n)$, we shall denote $\lambda_\psi := \lambda_{\phi_\psi}$ for short. This gives an *infinitesimal parameter* of G_n in the following sense.

Definition 2.6. *An infinitesimal parameter $[\lambda]$ of G_n is a $\widehat{G}_n(\mathbb{C})$ -conjugacy class of a continuous homomorphism*

$$\lambda : W_F \rightarrow \widehat{G}_n(\mathbb{C}),$$

whose image consists of semi-simple elements. By abuse of notation, we don't distinguish $[\lambda]$ and λ . We let $\Lambda(G_n)$ denote the set of infinitesimal parameters of G_n .

It is shown in [Moe06b, Moe09] that for any $\pi \in \Pi_\psi$, we have $\lambda_{\phi_\pi} = \lambda_\psi$. For each L -parameter ϕ and each local Arthur parameter ψ of G_n , we associate nilpotent orbits $\mathcal{O}_\phi$, $\mathcal{O}_\psi^D$ and $\mathcal{O}_\psi^A$ and partitions $\underline{p}(\phi)$, $\underline{p}^D(\psi)$ and $\underline{p}^A(\psi)$ as follows.

Definition 2.7. *For $\phi \in \Phi(G_n)$ and $\psi \in \Psi(G_n)$, we define $\mathcal{O}_\phi$ (resp. $\mathcal{O}_\psi^D$, $\mathcal{O}_\psi^A$) to be the nilpotent orbit of $\widehat{\mathfrak{g}}_n(\mathbb{C})$ containing the element*

$$d(\phi|_{\mathrm{SL}_2}) \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) \quad \left(\text{resp. } d(\psi|_{\mathrm{SL}_2^D}) \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right), d(\psi|_{\mathrm{SL}_2^A}) \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) \right),$$

and define the partition $\underline{p}(\phi) := \underline{p}_{\mathcal{O}_\phi}$ (resp. $\underline{p}^D(\psi) := \underline{p}_{\mathcal{O}_\psi^D}$, $\underline{p}^A(\psi) := \underline{p}_{\mathcal{O}_\psi^A}$). Note that $\mathcal{O}_\psi^A = \mathcal{O}_\psi^D = \mathcal{O}_{\phi_\psi}$ and $\underline{p}^A(\widehat{\psi}) = \underline{p}^D(\psi) = \underline{p}(\phi_\psi)$. Here d is the differential map.

Fix a $\lambda \in \Lambda(G_n)$. There is a natural partial ordering $\geq_C$ on the set $\Phi(G_n)_\lambda$, which is induced from the closure ordering on the associated Vogan variety. See [HLLZ22, Definition 1.10] for details. In [HLLZ22, Corollary 4.12 (2)], jointly with Hazeltine and Zhang, we show that $\phi_1 \geq_C \phi_2$ implies that $\underline{p}(\phi_1) \geq \underline{p}(\phi_2)$ when $G_n = \mathrm{SO}_{2n+1}(F)$ or $\mathrm{Sp}_{2n}(F)$. Indeed, the same proof for $\mathrm{SO}_{2n}(F)$ shows that $\phi_1 >_C \phi_2$ implies that $\underline{p}(\phi_1) > \underline{p}(\phi_2)$. We recall the following.

Proposition 2.8. *Let $\lambda \in \Lambda(G_n)$. The following holds.*

- (a) *There exist unique $\phi^0, \phi_0 \in \Phi(G_n)_\lambda$ such that for any $\phi \in \Phi(G_n)_\lambda$, the inequality holds*

$$\underline{p}(\phi^0) \geq \underline{p}(\phi) \geq \underline{p}(\phi_0).$$

We call ϕ^0 (resp. ϕ_0) the open (resp. closed) L -parameter of $\Phi(G_n)_\lambda$.

- (b) *$\Phi(G_n)_\lambda$ contains an L -parameter of Arthur type if and only if ϕ^0 is tempered. Let $\psi^0 = \phi^0 \otimes S_1$ so that $\phi^0 = \phi_{\psi^0}$. Then ϕ_0 is also of Arthur type with $\phi_0 = \phi_{\widehat{\psi}^0}$.*

Proof. These statements follow from [HLLZ22, Lemma 6.2, 6.4]. □

We remark that the nilpotent orbits version of Part (a) of above proposition also holds since $\underline{p}_{\mathcal{O}} > \underline{p}_{\mathcal{O}'}$ if and only if $\mathcal{O} > \mathcal{O}'$. We also need the following adjectives on parameters.

Definition 2.9. Let $\phi \in \Phi(G_n)$, $\psi \in \Psi(G_n)$ and $\lambda \in \Lambda(G_n)$.

- (1) We say ϕ is tempered if $\phi|_{W_F}$ has bounded image. We say ψ is tempered if ϕ_ψ is tempered, or equivalently, $\psi|_{\mathrm{SL}_2^A(\mathbb{C})}$ is trivial.
- (2) We say ϕ (resp. ψ , λ) is unramified if $\phi|_{I_F}$ (resp. $\psi|_{I_F}$, $\lambda|_{I_F}$) is trivial.
- (3) We say an unramified infinitesimal parameter λ is real if the eigenvalues of $\lambda(\mathrm{Fr})$ are all real and positive, where Fr is any choice of Frobenius in W_F .

Note that any tempered L -parameter ϕ is of Arthur type since $\phi = \phi_\psi$ where $\psi = \phi \otimes S_1$, i.e., $\psi(w, x, y) := \phi(w, x)$.

Finally, we recall the definition of Deligne-Langlands-Lusztig parameters. Let G be a connected reductive group defined over F and $G = G(F)$. We assume G is inner to split for simplicity. Recall that q is the cardinality of the residue field of F .

Definition 2.10. A Deligne-Langlands-Lusztig parameter of the group G is a $\widehat{G}(\mathbb{C})$ -orbit of a triple (s, x, ρ) , where

- $s \in \widehat{G}(\mathbb{C})$ is semisimple;
- $x \in \widehat{\mathfrak{g}}(\mathbb{C})$ such that $\mathrm{Ad}(s)x = qx$;
- ρ is an irreducible representation of the component group of $\mathrm{Cent}(\{s, x\}, \widehat{G}(\mathbb{C}))$ that is trivial on the image of $Z(\widehat{G}(\mathbb{C}))$.

We denote $\Phi^{\mathrm{Lus}}(G)$ the set of Deligne-Langlands-Lusztig parameter of G .

Let $\Pi^{\mathrm{Lus}}(G) \subseteq \Pi(G)$ denote the subset of representations of unipotent cuspidal support defined in [Lus95]. The following theorem is proved by [KL87, Lus95, Lus02]. We refer the reader to [CMO23, Theorem 4.1.1] for details.

Theorem 2.11 (Deligne-Langlands-Lusztig Correspondence). *There is a bijection*

$$\begin{aligned} \Phi^{\mathrm{Lus}}(G) &\longrightarrow \Pi^{\mathrm{Lus}}(G) \\ (s, x, \rho) &\longmapsto X(s, x, \rho), \end{aligned}$$

satisfying several desiderata (see [CMO23, Theorem 4.1.1] for details).

For the groups G_n considered in this paper, the local Langlands correspondence given by Arthur's theory is compatible with the above correspondence (see [AMS21, §2.3] and [AMS22, §4]). In particular, a representation $\pi \in \Pi(G_n)$ is of unipotent cuspidal support if and only if its local L -parameter ϕ_π is unramified. In this case, let

$$s = \phi_\pi \left(\mathrm{Fr}, \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix} \right), \quad x = d(\phi_\pi|_{\mathrm{SL}_2}) \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right),$$

where Fr is any choice of Frobenius in W_F . Then $\pi = X(s, x, \rho)$ for some ρ .

Remark 2.12. Let G be a connected reductive group defined over F , inner to split. For any unramified L -parameter ϕ_π of $G = G(F)$, the elements s, x can be defined in the same way, and there is an equality

$$\mathrm{Cent}(\{s, x\}, \widehat{G}(\mathbb{C})) = \mathrm{Cent}(\mathrm{im}(\phi), \widehat{G}(\mathbb{C})).$$

Thus, if (ϕ, χ) is the enhanced L -parameter of π , then χ can also be viewed as an irreducible representation of the component group of $\mathrm{Cent}(\{s, x\}, \widehat{G}(\mathbb{C}))$ that is trivial on the image of $Z(\widehat{G}(\mathbb{C}))$. In this case, for G_n , the results in [AMS21, AMS22] also show that $\rho = \chi$ under the comparison.

2.4. Aubert-Zelevinsky involution. Let G be any connected reductive algebraic group defined over F , $G = G(F)$, and let $\mathcal{R}(G)$ be the Grothendieck group of smooth representations of finite length of G . If π is a smooth representation of finite length of G , we let $[\pi]$ denote its image in $\mathcal{R}(G)$. If P is a parabolic subgroup of G , we let Ind_P^G denote the normalized parabolic induction and let Jac_P denote the Jacquet module.

In [Aub95], Aubert showed that for any representation π of $\Pi(G)$, there exists $\varepsilon \in \{\pm 1\}$ and an irreducible representation $\widehat{\pi} \in \Pi(G)$ such that

$$[\widehat{\pi}] := \varepsilon \sum_P (-1)^{\dim(A_P)} [\text{Ind}_P^G(\text{Jac}_P(\pi))].$$

Here the sum is taken over all standard parabolic subgroups P of G and A_P is the maximal split torus of the center of the Levi subgroup of P . Moreover, the map $\pi \mapsto \widehat{\pi}$ is an involution on $\Pi(G)$. We call $\widehat{\pi}$ the Aubert-Zelevinsky involution of π .

It is expected that local Arthur packets are compatible with Aubert-Zelevinsky involution in the sense that for any $\psi \in \Psi(G)$,

$$\Pi_{\widehat{\psi}} = \{\widehat{\pi} \mid \pi \in \Pi_{\psi}\}.$$

(Recall that $\widehat{\psi}$ is defined by (2.2).) When $G = G_n$, this is discussed and proved in [Art13, §7.1] and [Xu17, §A].

3. FIBERS OF THE BARBASCH-VOGAN DUALITY

In this section, we first recall certain results from [LLS24] on the fibers of the Barbasch-Vogan duality. Then we prove a key result (Proposition 3.4 below) showing that certain L -parameters are of Arthur type, which plays an important role in the proof of the weak local Arthur packets conjecture next section.

3.1. Partitions and nilpotent orbits. Let $(X, X') \in \{(B, C), (C, B), (D, D)\}$. In this subsection, we describe the structure of the sets of partitions

$$d_{BV}^{-1}(\underline{\mathbf{p}}) := \{\underline{p} \in \mathcal{P}_X \mid d_{BV}(\underline{p}) = \underline{\mathbf{p}}\},$$

for a special partition $\underline{\mathbf{p}} \in \mathcal{P}_{X'}$. In [LLS24], jointly with Shahidi, we gave an explicit description of $d_{BV}^{-1}(\underline{\mathbf{p}})$ and related it with $d_{BV}^{-1}(\mathcal{O}')$ (see Proposition 3.3 below), which we recall now.

Recall that when we write a partition $\underline{p}$ as $[p_1, \dots, p_r]$, we require p_i 's to be non-increasing. Set $p_t = 0$ for any $t > r$ throughout this section. Given a partition $\underline{p} \in \mathcal{P}_X$, the following lemma describes a necessary condition on $\underline{q}$ such that $\underline{p} \geq \underline{q}$ and $d_{BV}(\underline{p}) = d_{BV}(\underline{q})$.

Lemma 3.1 ([LLS24, Lemma 3.5, Corollary 3.9]). *Let $X \in \{B, C, D\}$. Suppose $\underline{p} = [p_1, \dots, p_r]$, $\underline{q} = [q_1, \dots, q_s] \in \mathcal{P}_X(n)$ satisfy that $\underline{p} \geq \underline{q}$ and $d_{BV}(\underline{p}) = d_{BV}(\underline{q})$. Then there exists a sequence of pairs of positive integers $\{(x_i, y_i)\}_{i=1}^\alpha$ where*

- (a) $1 \leq x_i < y_i \leq r + 1$;
- (b) $p_{x_i} = p_{x_i+1} + 1 = \dots = p_{y_i-1} + 1 = p_{y_i} + 2$, where we set $p_{r+1} = 0$;
- (c) the sequence $(p_{x_1}, \dots, p_{x_\alpha})$ is strictly decreasing;

such that $\underline{q}$ can be obtained from $\underline{p}$ by replacing $\{p_{x_i}, p_{y_i}\}_{i=1}^\alpha$ in $\underline{p}$ with $\{p_{x_i} - 1, p_{y_i} + 1\}_{i=1}^\alpha$. In particular, for any $1 \leq t \leq r$, we have

$$(3.1) \quad 0 \leq \sum_{z=1}^t p_z - \sum_{i=1}^t q_z \leq 1.$$

We need the following refinement.

Corollary 3.2. *Let $X \in \{B, C, D\}$. Suppose $\underline{p} = [p_1, \dots, p_r], \underline{q} = [q_1, \dots, q_s] \in \mathcal{P}_X(n)$ satisfy that $\underline{p} \geq \underline{q}$ and $d_{BV}(\underline{p}) = d_{BV}(\underline{q})$. Suppose further that $\underline{p} = \bigsqcup_{j \in J} \underline{p}_j$ and $\underline{q} = \bigsqcup_{j \in J} \underline{q}_j$ such that $|\underline{p}_j| = |\underline{q}_j|$ and $\underline{p}_j \geq \underline{q}_j$ for all $j \in J$. Write $\underline{p}_j = [p_{j,1}, \dots, p_{j,r_j}]$ and $\underline{q}_j = [q_{j,1}, \dots, q_{j,s_j}]$. Then for each $j \in J$, there exists a sequence of pairs of positive integers $\{(x_{(j,k)}, y_{(j,k)})\}_{k=1}^{\alpha_j}$ where*

- (a) $1 \leq x_{(j,k)} < y_{(j,k)} \leq r_j + 1$;
- (b) $p_{j,x_{(j,k)}} = p_{j,x_{(j,k)}+1} + 1 = \dots = p_{j,y_{(j,k)}-1} + 1 = p_{j,y_{(j,k)}} + 2$, where we set $p_{j,r_j+1} = 0$;
- (c) the sequence $(p_{j,x_{(j,1)}}, \dots, p_{j,x_{(j,\alpha_j)}})$ is strictly decreasing;

such that $\underline{q}_j$ can be obtained from $\underline{p}_j$ by replacing $\{p_{j,x_{(j,k)}}, p_{j,y_{(j,k)}}\}_{k=1}^{\alpha_j}$ in $\underline{p}_j$ with $\{p_{j,x_{(j,k)}} - 1, p_{j,y_{(j,k)}} + 1\}_{k=1}^{\alpha_j}$. In particular, for any $1 \leq t \leq r_j$, we have

$$(3.2) \quad 0 \leq \sum_{z=1}^t p_{j,z} - \sum_{z=1}^t q_{j,z} \leq 1.$$

Proof. The idea of the proof is similar to [CMO23, Lemma 5.1.1]. We shall use the following two statements whose proof can be found there.

- (i) If $\underline{p}_1 \geq \underline{q}_1$ and $\underline{p}_2 \geq \underline{q}_2$, then $\underline{p}_1 \sqcup \underline{p}_2 \geq \underline{q}_1 \sqcup \underline{q}_2$.
- (ii) If $(\tau_1, \dots, \tau_\gamma)$ is a sequence of non-increasing integers, and σ is any permutation of $\{1, \dots, \gamma\}$, then for any $1 \leq t \leq \gamma$,

$$\sum_{i=1}^t \tau_i \geq \sum_{i=1}^t \tau_{\sigma(i)}.$$

It suffices to prove Parts (a), (b) and (c) for a fixed $j \in J$. By considering the decomposition

$$\underline{p} = \underline{p}_j \sqcup \left(\bigsqcup_{j' \in J \setminus \{j\}} \underline{p}_{j'} \right), \quad \underline{q} = \underline{q}_j \sqcup \left(\bigsqcup_{j' \in J \setminus \{j\}} \underline{q}_{j'} \right),$$

we may assume $|J| = 2$ and label $J = \{1, 2\}$ with $j = 1$. First, we prove the inequality (3.2).

Recall that $\underline{q} = \underline{q}_1 \sqcup \underline{q}_2$ and hence $s = s_1 + s_2$. Let $f : \{1, \dots, s_1 + s_2\} \rightarrow \{1, 2\}$ be a function such that $\underline{q}_j = \bigsqcup_{1 \leq i \leq s, f(i)=j} [q_i]$ for $j = 1, 2$. Equivalently, the function f satisfies that

$$q_i = \begin{cases} q_{1, |\{k \leq i \mid f(k)=1\}|} & \text{if } f(i) = 1, \\ q_{2, |\{k \leq i \mid f(k)=2\}|} & \text{if } f(i) = 2. \end{cases}$$

Assume $r_1 = s_1$ by adding zero to $\underline{p}_1$ if necessary, we consider a sequence of integers $\lambda = (\lambda_1, \dots, \lambda_{s_1+s_2})$ given by

$$\lambda_i := \begin{cases} p_{1, |\{k \leq i \mid f(k)=1\}|} & \text{if } f(i) = 1, \\ q_{2, |\{k \leq i \mid f(k)=2\}|} & \text{if } f(i) = 2. \end{cases}$$

Then there exists a permutation σ of $\{1, \dots, r_1 + s_2\}$ such that $\underline{p}_1 \sqcup \underline{q}_2 = [\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(r_1+s_2)}]$. Note that we require the sequence $(\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(r_1+s_2)})$ to be non-increasing. Then for any $1 \leq t \leq r$, we have

$$\sum_{z=1}^t p_z \geq \sum_{z=1}^t \lambda_{\sigma(z)} \geq \sum_{z=1}^t \lambda_z \geq \sum_{z=1}^t q_z.$$

Here the first inequality follows from $\underline{p} = \underline{p}_1 \sqcup \underline{p}_2 \geq \underline{p}_1 \sqcup \underline{q}_2$ by (i), the second inequality follows from (ii), and the last inequality follows from $\underline{p}_1 \geq \underline{q}_1$. As a consequence, for any $1 \leq t \leq r$, the inequality (3.1) in Lemma 3.1 gives

$$1 \geq \sum_{z=1}^t p_z - \sum_{z=1}^t q_z \geq \sum_{z=1}^t \lambda_z - \sum_{z=1}^t q_z \geq \sum_{z=1}^t q_z - \sum_{z=1}^t q_z = 0.$$

Since

$$\sum_{z=1}^t \lambda_z - \sum_{z=1}^t q_z = \sum_{z=1}^{|\{k \leq t \mid f(k)=1\}|} p_{1,z} - \sum_{z=1}^{|\{k \leq t \mid f(k)=1\}|} q_{1,z},$$

this proves (3.2) for the fixed $j \in J$ by varying t .

As a consequence of (3.2), for $j \in \{1, 2\}$, there exists a sequence of pairs of positive integers $\{(x_{(j,k)}, y_{(j,k)})\}_{k=1}^{\alpha_j}$ that satisfies Conditions (a) and (c) such that $\underline{q}_j$ can be obtained from $\underline{p}_j$ by replacing $\{p_{j,x_{(j,k)}}, p_{j,y_{(j,k)}}\}_{k=1}^{\alpha_j}$ in $\underline{p}_j$ with $\{p_{j,x_{(j,k)}} - 1, p_{j,y_{(j,k)}} + 1\}_{k=1}^{\alpha_j}$. Also, since

$$\{p_{1,x_{(1,k)}}\}_{k=1}^{\alpha_1} \sqcup \{p_{2,x_{(2,k)}}\}_{k=1}^{\alpha_2} = \{p_{x_i}\}_{i=1}^{\alpha}, \quad \{p_{1,y_{(1,k)}}\}_{k=1}^{\alpha_1} \sqcup \{p_{2,y_{(2,k)}}\}_{k=1}^{\alpha_2} = \{p_{y_i}\}_{i=1}^{\alpha},$$

where $\{(x_i, y_i)\}_{i=1}^{\alpha}$ is given by Lemma 3.1, Condition (b) also holds for $\{(x_{(j,k)}, y_{(j,k)})\}_{k=1}^{\alpha_j}$. This completes the proof of the corollary. $\square$

Finally, for $(X, X') \in \{(B, C), (C, B), (D, D)\}$ and a special $\mathcal{O}' \in \mathcal{N}_{X'}$, $d_{BV}^{-1}(\mathcal{O}')$ can be related with $d_{BV}^{-1}(\underline{p}_{\mathcal{O}'})$ in the following proposition.

Proposition 3.3 ([LLS24, Proposition 2.10]). *Let $(X, X') \in \{(B, C), (C, B), (D, D)\}$. For each special $\mathcal{O}' \in \mathcal{N}_{X'}$, we have the following.*

- (a) *If $\underline{\mathfrak{p}} := \underline{p}_{\mathcal{O}'}$ is not very even of type D , then any $\underline{p} \in d_{BV}^{-1}(\underline{\mathfrak{p}})$ is not very even, and*

$$d_{BV}^{-1}(\mathcal{O}') = \{\mathcal{O}_{\underline{p}} \mid \underline{p} \in d_{BV}^{-1}(\underline{\mathfrak{p}})\}.$$

- (b) *If $\underline{\mathfrak{p}} := \underline{p}_{\mathcal{O}'}$ is very even of type D , then*

$$d_{BV}^{-1}(\mathcal{O}') = \{d_{BV}(\mathcal{O}')\},$$

which is a singleton.

3.2. L -parameters. In this subsection, we prove the following proposition showing that certain L -parameters are of Arthur type.

Proposition 3.4. *Let G_n be the split group $\mathrm{SO}_{2n+1}(F)$, $\mathrm{Sp}_{2n}(F)$ or $\mathrm{SO}_{2n}(F)$ and $\lambda \in \Lambda(G_n)$. Suppose the unique open L -parameter ϕ^0 of $\Phi(G_n)_\lambda$ is tempered and denote $\mathcal{O}' := d_{BV}(\mathcal{O}_{\phi^0})$. Then any L -parameter ϕ in*

$$(d_{BV})_{\Phi(G_n)_\lambda}^{-1}(\mathcal{O}') := \{\phi \in \Phi(G_n)_\lambda \mid d_{BV}(\mathcal{O}_\phi) = \mathcal{O}'\}$$

is of Arthur type.

When $G_n = \mathrm{SO}_{2n+1}(F)$ or $\mathrm{Sp}_{2n}(F)$, the map from the nilpotent orbits of $\widehat{\mathfrak{g}}_n(\mathbb{C})$ to partitions of the corresponding type is a bijection. Therefore, we have

$$(d_{BV})_{\Phi(G_n)_\lambda}^{-1}(\mathcal{O}') = (d_{BV})_{\Phi(G_n)_\lambda}^{-1}(\underline{p}_{\mathcal{O}'}):= \{\phi \in \Phi(G_n)_\lambda \mid d_{BV}(p(\phi)) = \underline{p}_{\mathcal{O}'}\}.$$

However, the above equality fails for $\mathrm{SO}_{2n}(F)$ when $\underline{\mathfrak{p}} = \underline{p}_{\mathcal{O}'}$ is very even. In this case, we have

$$(d_{BV})_{\Phi(G_n)_\lambda}^{-1}(\underline{\mathfrak{p}}) = (d_{BV})_{\Phi(G_n)_\lambda}^{-1}(\mathcal{O}_{\underline{\mathfrak{p}}}^I) \sqcup (d_{BV})_{\Phi(G_n)_\lambda}^{-1}(\mathcal{O}_{\underline{\mathfrak{p}}}^{II}).$$

In any case, Proposition 3.4 follows from its partition version.

Proposition 3.5. *Let G_n be the split group $\mathrm{SO}_{2n+1}(F)$, $\mathrm{Sp}_{2n}(F)$ or $\mathrm{SO}_{2n}(F)$ and $\lambda \in \Lambda(G_n)$. Suppose the unique open L -parameter ϕ^0 of $\Phi(G_n)_\lambda$ is tempered and denote $\underline{\mathbf{p}} := d_{BV}(p(\phi^0))$. Then any L -parameter $\phi \in (d_{BV})_{\Phi(G_n)_\lambda}^{-1}(\underline{\mathbf{p}})$ is of Arthur type.*

Now we give a more explicit description for the partitions $\underline{p}(\phi)$, $\underline{p}^D(\psi)$ and $\underline{p}^A(\psi)$. Let $\xi : \widehat{G}_n(\mathbb{C}) \hookrightarrow \mathrm{GL}_N(\mathbb{C})$ be a standard embedding. The map

$$\begin{aligned} \Phi(G_n) &\rightarrow \Phi(\mathrm{GL}_N(F)), \\ \phi &\mapsto \xi \circ \phi \end{aligned}$$

is an injection unless $G_n = \mathrm{SO}_{2n}(F)$. For $\phi \in \Phi(\mathrm{SO}_{2n}(F))$, the set

$$\{\phi' \in \Phi(\mathrm{SO}_{2n}(F)) \mid \xi \circ \phi = \xi \circ \phi'\}$$

is either a singleton or equal to $\{\phi, \phi^c\}$ (see [GGP12, Theorem 8.1(ii)]). The L -parameter ϕ^c is called the outer conjugation of ϕ , which can be obtained from ϕ by conjugating with an element in $\mathrm{O}_{2n}(\mathbb{C}) \setminus \mathrm{SO}_{2n}(\mathbb{C})$. The above discussion works for $\psi \in \Psi(G_n)$ without change. In the following, we don't distinguish ϕ (resp. ψ) and ϕ^c (resp. ψ^c), since they give the same partition.

By identifying ϕ (resp. ψ) with $\xi \circ \phi$ (resp. $\xi \circ \psi$) we may decompose it into a direct sum of irreducible representations of $W_F \times \mathrm{SL}_2(\mathbb{C})$ (resp. $W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$) and write

$$(3.3) \quad \phi = \bigoplus_{i \in I} \rho_i \otimes S_{a_i}, \quad \psi = \bigoplus_{j \in J} \rho_j \otimes S_{a_j} \otimes S_{b_j},$$

where ρ_i 's and ρ_j 's are irreducible representations of W_F and S_a is the unique a -dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$. With this decomposition, we have

$$\underline{p}(\phi) = \bigsqcup_{i \in I} [a_i^{\dim(\rho_i)}], \quad \underline{p}^D(\psi) = \bigsqcup_{j \in J} [a_j^{\dim(\rho_j) \cdot b_j}], \quad \underline{p}^A(\psi) = \bigsqcup_{j \in J} [b_j^{\dim(\rho_j) \cdot a_j}].$$

Also, with the decomposition (3.3), we have (again composing with $\xi : \widehat{G}_n(\mathbb{C}) \hookrightarrow \mathrm{GL}_N(\mathbb{C})$)

$$\lambda_\phi = \bigoplus_{i \in I} \left(\bigoplus_{k=0}^{a_i-1} \rho_i \cdot |\cdot|^{\frac{a_i-1}{2}-k} \right), \quad \phi_\psi = \bigoplus_{j \in J} \left(\bigoplus_{k=0}^{b_j-1} \rho_j \cdot |\cdot|^{\frac{b_j-1}{2}-k} \right) \otimes S_{a_j}.$$

In particular, if ϕ is tempered and $\rho \cdot |\cdot|^x \subseteq \lambda_\phi$ where ρ is irreducible with bounded image and $x \in \mathbb{R}$, then $x \in \frac{1}{2}\mathbb{Z}$.

In the following discussion, we shall treat ϕ (resp. ψ) as a self-dual L -parameter (resp. local Arthur parameter) of some $\mathrm{GL}_N(F)$. We write $\phi \supset \phi'$ if ϕ' is equivalent to a subrepresentation of ϕ . In this case, we let $\phi - \phi'$ denote the subrepresentation of ϕ such that $\phi = \phi' + (\phi - \phi')$. We use the same notation $\lambda \supseteq \lambda'$, $\lambda - \lambda'$ for infinitesimal parameters.

The following lemma is the key observation towards Proposition 3.5.

Lemma 3.6. *Let ϕ^0 be a tempered self-dual L -parameter of $\mathrm{GL}_N(F)$ and denote $\lambda = \lambda_{\phi^0}$. Suppose $\phi \in \Phi(\mathrm{GL}_N(F))_\lambda$ is self-dual, and ϕ^0, ϕ have decompositions*

$$\phi^0 = \phi_1^0 + \phi_2^0, \quad \phi = \phi_1 + (\rho \cdot |\cdot|^z \otimes S_a) + \phi_2,$$

where

- (i) $z \in \mathbb{R}$ and ρ is irreducible with bounded image,
- (ii) $\lambda_{\phi_1^0} = \lambda_{\phi_1}$ is self-dual.

Then the followings hold.

- (a) Suppose further that any irreducible summand of ϕ_2^0 has dimension less than or equal to

$$\dim(\rho| \cdot |^z \otimes S_a) = \dim(\rho) \cdot a.$$

Then $z = 0$, and $\phi_2^0 \supseteq \rho \otimes S_a$.

- (b) Suppose further that $z \neq 0$ and any irreducible summand of ϕ_2^0 has dimension less than or equal to

$$\dim(\rho \otimes S_{a+1}) = \dim(\rho) \cdot (a + 1).$$

Then $|z| = 1/2$, and $\phi_2^0 \supseteq \rho \otimes S_{a+1}$.

Proof. Since $\lambda_{\phi^0} = \lambda = \lambda_{\phi}$ and $\lambda_{\phi_1^0} = \lambda_{\phi_1}$ by Condition (ii), we see that

$$\lambda_{\phi_2^0} = \lambda_{\rho| \cdot |^z \otimes S_a + \phi_2} \supseteq \lambda_{\rho| \cdot |^z \otimes S_a} = \bigoplus_{i=0}^{a-1} \rho| \cdot |^{z + \frac{a-1}{2} - i}.$$

This implies $z \in \frac{1}{2}\mathbb{Z}$. Since ϕ is self-dual, replacing ρ by $\rho^\vee$, the dual of ρ , if necessary, we may assume $z \geq 0$. Then $\lambda_{\phi_2^0} \supseteq \rho| \cdot |^{z + \frac{a-1}{2}}$ implies that ϕ_2^0 must contain an irreducible summand of the form $\rho \otimes S_b$ with $b \geq 2(z + \frac{a-1}{2}) + 1$ (and $b \equiv 2z + a \pmod{2}$).

For Part (a), the assumption gives

$$a \leq 2 \left(z + \frac{a-1}{2} \right) + 1 \leq b \leq a,$$

which implies $z = 0$ and $b = a$. For Part (b), the assumption implies $z \geq 1/2$ and

$$a + 1 \leq 2 \left(z + \frac{a-1}{2} \right) + 1 \leq b \leq a + 1.$$

Thus, $b = a + 1$ and $z = 1/2$. This completes the proof of the lemma. $\square$

As a corollary, we prove a special case of Proposition 3.5 using Lemma 3.1.

Corollary 3.7. *Let ϕ^0 be a tempered self-dual L -parameter of $\mathrm{GL}_N(F)$ of the form*

$$\phi^0 = \rho \otimes \left(\bigoplus_{j=1}^r S_{p_j} \right),$$

where ρ is one-dimensional, self-dual and the sequence $(p_1, \dots, p_r)$ is non-increasing. Denote $\lambda = \lambda_{\phi^0}$. Suppose $\phi \in \Phi(\mathrm{GL}_N(F))_\lambda$ is self-dual and $\underline{q} := \underline{p}(\phi)$ can be obtained from $\underline{p} = \underline{p}(\phi^0) = [p_1, \dots, p_r]$ by replacing $\{p_{x_i}, p_{y_i}\}_{i=1}^\alpha$ in $\underline{p}$ with $\{p_{x_i} - 1, p_{y_i} + 1\}_{i=1}^\alpha$, where the sequence $\{(x_i, y_i)\}_{i=1}^\alpha$ satisfies Conditions (a), (b) and (c) in Lemma 3.1 for $\underline{p}$. Then ϕ is of Arthur type. More explicitly, let $J := \{1, \dots, r\} \setminus (\{x_1, \dots, x_\alpha\} \sqcup \{y_1, \dots, y_\alpha\})$, then $\phi = \phi_\psi$, where

$$\psi = \bigoplus_{j \in J} \rho \otimes S_{p_j} \otimes S_1 + \bigoplus_{i=1}^\alpha \rho \otimes S_{p_{x_i}-1} \otimes S_2.$$

Proof. We apply induction on $\alpha = \alpha(\phi, \phi^0)$. When $\alpha = 0$, we have $\phi = \phi^0$ and the conclusion trivially holds. Now we assume $\alpha(\phi, \phi^0) = k > 0$ and the conclusion is verified for any ϕ' with $\alpha(\phi', \phi^0) < k$.

Since ϕ^0 contains $\rho \otimes (S_{p_1} + \dots + S_{p_i})$ for $1 \leq i \leq x_1 - 1$, inductively applying Lemma 3.6(a), we see that ϕ also contains $\rho \otimes (S_{p_1} + \dots + S_{p_{x_1-1}})$. Denote $\phi_1 = \phi_1^0 := \rho \otimes (S_{p_1} + \dots + S_{p_{x_1-1}})$.

Denote $a := p_{x_1} - 1$ for simplicity and let m be the multiplicity of a in $\underline{p}(\phi)$, which is greater than or equal to 2 by assumption. Write

$$\begin{aligned}\phi - \phi_1 &\supseteq \bigoplus_{j=1}^m \rho| \cdot |^{z_j} \otimes S_a, \\ \phi^0 - \phi_1^0 &= \rho \otimes S_{a+1} + (\rho \otimes S_a)^{\oplus(m-2)} + \bigoplus_{i=y_1}^r \rho \otimes S_{p_i}.\end{aligned}$$

(The $\oplus(m-2)$ of $(\rho \otimes S_a)^{\oplus(m-2)}$ means the multiplicity.) Since the multiplicity of $\rho| \cdot |^{\frac{a-1}{2}}$ in $\lambda_{\phi-\phi_1}$ is the same as that of $\lambda_{\phi^0-\phi_1^0}$, which is exactly $m-2$, we see that at least 2 of z_j 's are not zero. Now we may write

$$\phi^0 = \phi_1^0 + \phi_2^0, \quad \phi = \phi_1 + (\rho| \cdot |^z \otimes S_a) + \phi_2$$

for some $z \neq 0$. Applying Lemma 3.6(b), we obtain that $|z| = 1/2$. Since ϕ is self-dual, we may rewrite

$$\begin{aligned}\phi^0 &= \phi_1^0 + \rho \otimes S_{a+1} + \rho \otimes S_{a-1} + \tilde{\phi}_2^0, \\ \phi &= \phi_1 + \rho| \cdot |^{\frac{1}{2}} \otimes S_a + \rho| \cdot |^{-\frac{1}{2}} \otimes S_a + \left(\bigoplus_{j=1}^{m-2} \rho| \cdot |^{z_j} \otimes S_a \right) + \tilde{\phi}_2.\end{aligned}$$

Applying Lemma 3.6(a) inductively again, we see that $\{z_j\}_{j=1}^{m-2}$ are all zero. Finally, let

$$\phi' := \phi_1 + \rho \otimes S_{a+1} + \rho \otimes S_{a-1} + (\rho \otimes S_a)^{\oplus(m-2)} + \tilde{\phi}_2.$$

It is not hard to see that $\alpha(\phi', \phi^0) = \alpha(\phi, \phi^0) - 1$. Then the induction hypothesis for the pair (ϕ', ϕ^0) gives $\phi' = \phi_{\psi'}$, where we let $J' = \{1, \dots, r\} \setminus (\{x_2, \dots, x_\alpha\} \sqcup \{y_2, \dots, y_\alpha\})$, and

$$\psi' = \bigoplus_{j \in J'} \rho \otimes S_{p_j} \otimes S_1 + \bigoplus_{i=2}^{\alpha} \rho \otimes S_{p_{x_i}-1} \otimes S_2.$$

Comparing ϕ and ϕ' , we get the desired conclusion. This completes the proof of the corollary. $\square$

Now we prove Proposition 3.5.

Proof. We may write

$$\phi^0 = \bigoplus_{i \in I_{nsd}} \bigoplus_{j \in J_i} (\rho_i + \rho_i^\vee) \otimes S_{a_j} + \bigoplus_{i \in I_{sd}} \bigoplus_{j \in J_i} \rho_i \otimes S_{a_j},$$

where $\rho_i \not\cong \rho_i^\vee$ for $i \in I_{nsd}$ and $\rho_i \cong \rho_i^\vee$ for $i \in I_{sd}$, and $\rho_{i_1} \not\cong \rho_{i_2} \not\cong \rho_{i_1}^\vee$ for $i_1 \neq i_2 \in I_{nsd}$ and $\rho_{i_1} \not\cong \rho_{i_2}$ for $i_1 \neq i_2 \in I_{sd}$. Let $I = I_{nsd} \sqcup I_{sd}$ and for $i \in I$, denote

$$\phi_i^0 := \begin{cases} \bigoplus_{j \in J_i} (\rho_i + \rho_i^\vee) \otimes S_{a_j} & \text{if } i \in I_{nsd}, \\ \bigoplus_{j \in J_i} \rho_i \otimes S_{a_j} & \text{if } i \in I_{sd}, \end{cases}$$

which we regard as a self-dual tempered L -parameter of $\mathrm{GL}_{n_i}(F)$ that factors through $\mathrm{Sp}_{n_i}(\mathbb{C})$ if $G_n = \mathrm{SO}_{2n+1}(F)$, and factors through $\mathrm{O}_{n_i}(\mathbb{C})$ if $G_n = \mathrm{Sp}_{2n}(F)$ or $\mathrm{SO}_{2n}(F)$.

Suppose $\phi \in (d_{BV})_{\Phi(G_n)_\lambda}^{-1}(\underline{\mathfrak{p}})$, where $\underline{\mathfrak{p}} = d_{BV}(p(\phi^0))$. Then we have a decomposition

$$\phi = \bigoplus_{i \in I} \phi_i,$$

where $\lambda_{\phi_i} = \lambda_{\phi_i^0}$. Write

$$\underline{p}_i := \underline{p}(\phi_i^0) = [p_{i,1}, \dots, p_{i,r_i}], \quad \underline{q}_i = \underline{p}(\phi_i) = [q_{i,1}, \dots, q_{i,s_i}].$$

For each $i \in I$, we have $|\underline{p}_i| = |\underline{q}_i|$ and $\underline{p}_i \geq \underline{q}_i$ by Proposition 2.8(a) since ϕ_i^0 is a tempered L -parameter of $\mathrm{GL}_{n_i}(F)$. Therefore, Corollary 3.2 implies that for any $i \in I$ and $1 \leq t \leq r_i$,

$$(3.4) \quad 0 \leq \varepsilon(\underline{p}_i, \underline{q}_i, t) := \sum_{j=1}^t p_{i,j} - \sum_{j=1}^t q_{i,j} \leq 1.$$

For $i \in I_{nsd}$, we may write $\underline{p}_i = \underline{p}_i' \sqcup \underline{p}_i'$ and $\underline{q}_i = \underline{q}_i' \sqcup \underline{q}_i'$. Therefore, for $1 \leq t \leq r_i/2$, if we define $\varepsilon(\underline{p}_i', \underline{q}_i', t)$ similarly, then

$$\varepsilon(\underline{p}_i, \underline{q}_i, 2t) = 2 \cdot \varepsilon(\underline{p}_i', \underline{q}_i', t),$$

and hence (3.4) implies $\varepsilon(\underline{p}_i', \underline{q}_i', t) = 0$. We conclude that $\underline{p}_i' = \underline{q}_i'$ and hence $\underline{p}_i = \underline{q}_i$. We take $\psi_i = \phi_i^0 \otimes S_1$ in this case.

For $i \in I_{sd}$ such that $\dim(\rho) > 1$, we have

$$\underline{p}_i = \underbrace{\underline{p}_i' \sqcup \dots \sqcup \underline{p}_i'}_{\dim(\rho) \text{ copies}}, \quad \underline{q}_i = \underbrace{\underline{q}_i' \sqcup \dots \sqcup \underline{q}_i'}_{\dim(\rho) \text{ copies}},$$

and hence the same argument shows that $\underline{p}_i = \underline{q}_i$. We take $\psi_i = \phi_i^0 \otimes S_1$ in this case.

Finally, for $i \in I_{sd}$ such that $\dim(\rho) = 1$, Corollary 3.2 implies that the pair (ϕ_i^0, ϕ_i) satisfies the assumption of Corollary 3.7, and hence there is a ψ_i such that $\phi_i = \phi_{\psi_i}$.

In conclusion, we have constructed $\psi = \bigoplus_{i \in I} \psi_i$ so that $\phi = \phi_{\psi}$. This completes the proof of the proposition. $\square$

The proof above also gives the following corollary, which will be used in future work.

Corollary 3.8. *Suppose $\lambda \in \Lambda(G_n)$ has a decomposition*

$$\lambda = \bigoplus_{i \in I} \rho_i | \cdot |^{x_i},$$

where ρ_i 's are irreducible representations of W_F with bounded image, and $x_i \in \mathbb{R}$. Suppose further that ρ_i 's are either non-self-dual or $\dim(\rho_i) > 1$. Then for any $\phi, \phi' \in \Phi(G_n)_{\lambda}$ such that $\underline{p}(\phi) \geq \underline{p}(\phi')$, we have $d_{BV}(\underline{p}(\phi)) = d_{BV}(\underline{p}(\phi'))$ if and only if $\phi = \phi'$.

We end this subsection by demonstrating an example that the set $(d_{BV})_{\Phi(G_n)_{\lambda_{\phi}}}^{-1}(d_{BV}(\underline{p}(\phi)))$ may contain an L -parameter not of Arthur type when ϕ is of Arthur type but not tempered.

Example 3.9. *Let $G_n = \mathrm{SO}_{11}(F)$. Consider*

$$\begin{aligned} \psi &= 1 \otimes S_2 \otimes S_3 + 1 \otimes S_4 \otimes S_1, \\ \phi_{\psi} &= 1 \otimes S_4 + 1 \otimes S_2 + | \cdot |^1 \otimes S_2 + | \cdot |^{-1} \otimes S_2. \end{aligned}$$

We have $\underline{p}^D(\psi) = \underline{p}(\phi_{\psi}) = [4, 2^3]$, and we set $\underline{\mathbf{p}} = d_{BV}([4, 2^3]) = [5, 3, 1^3]$. Then $(d_{BV})_{\Phi(G_n)_{\lambda}}^{-1}(\underline{\mathbf{p}}) = \{\phi_{\psi}, \phi_1, \phi_2\}$, where

$$\begin{aligned} \phi_1 &= 1 \otimes S_4 + 1 \otimes S_2 + 1 \otimes S_2 + | \cdot |^{3/2} \otimes S_1 + | \cdot |^{-3/2} \otimes S_1, \\ \phi_2 &= 1 \otimes S_4 + | \cdot |^1 \otimes S_2 + | \cdot |^{-1} \otimes S_2 + | \cdot |^{1/2} \otimes S_1 + | \cdot |^{-1/2} \otimes S_1, \end{aligned}$$

are both not of Arthur type.

4. UNIPOTENT REPRESENTATIONS WITH REAL INFINITESIMAL CHARACTER

In this section, we apply the results in Section 3.2 to prove the conjecture for weak local Arthur packets of basic local Arthur parameters for $G_n = \mathrm{Sp}_{2n}(F)$ and split $\mathrm{SO}_{2n+1}(F)$. We assume the residue field of F has sufficiently large characteristic throughout the section.

First, we recall the definition of *weak local Arthur packets* for *basic local Arthur parameters*.

Definition 4.1. *Let G be a connected reductive group and $G = G(F)$. Assume that there is a local Arthur packets theory for G as conjectured in [Art89, Conjecture 6.1]. We say $\psi \in \Psi(G)$ is basic if $\psi|_{W_F \times \mathrm{SL}_2^D(\mathbb{C})}$ is trivial. For each basic local Arthur parameter ψ_0 of G , we define the weak local Arthur packet associated with ψ_0 by*

$$(4.1) \quad \Pi_{\psi_0}^{\mathrm{Weak}} := \{\pi \in \Pi(G)_{\lambda_{\psi_0}} \mid \bar{\mathfrak{n}}^m(\pi) \leq d_{BV}(\mathcal{O}_{\psi_0}^A)\}.$$

Here, we recall that $\bar{\mathfrak{n}}^m(\pi)$ is the wavefront set of π consists of nilpotent orbits over $\bar{F}$.

We say an unramified infinitesimal parameter λ is real if after composing with an embedding ${}^L G \rightarrow {}^L \mathrm{GL}_N$, it decomposes as

$$\lambda = \bigoplus_{i \in I} |\cdot|^{x_i},$$

where x_i are all real numbers. For $\psi \in \Psi(G_n)$, it is not hard to see that λ_ψ is unramified (resp. real) if and only if $\psi|_{I_F}$ (resp. $\psi|_{W_F}$) is trivial. In particular, if ψ is basic, then λ_ψ is real. Now we recall the following result from [CMO23].

Theorem 4.2 ([CMO23, Theorem 1.4.1]). *Let G be a connected reductive algebraic group defined over F , inner to split. Assume that the residue field of F has sufficiently large characteristic. Suppose π is a unipotent representation of $G(F)$ with real infinitesimal parameter. Then the geometric wavefront set of π is a singleton, and*

$$(4.2) \quad \bar{\mathfrak{n}}^m(\pi) = \{d_{BV}(\mathcal{O}_{\phi_{\bar{\pi}}})\},$$

where $\mathcal{O}_{\phi_{\bar{\pi}}}$ is a nilpotent orbit of $\widehat{\mathfrak{g}}(\mathbb{C})$ associated to the L -parameter $\phi_{\bar{\pi}}$ (see Definition 2.7).

Remark 4.3. By [CMO23, Corollary 6.0.5], we may replace the $\leq$ in (4.1) by $=$. Moreover, applying Theorem 4.2, we have

$$\begin{aligned} \Pi_{\psi_0}^{\mathrm{Weak}} &= \{\pi \in \Pi(G_n)_\lambda \mid \bar{\mathfrak{n}}^m(\pi) = d_{BV}(\mathcal{O}_{\psi_0}^A)\} \\ &= \{\pi \in \Pi(G_n)_\lambda \mid d_{BV}(\mathcal{O}_{\phi_{\bar{\pi}}}) = d_{BV}(\mathcal{O}_{\phi^0})\} \\ &= \{\pi \in \Pi(G_n)_\lambda \mid \phi_{\bar{\pi}} \in (d_{BV})_{\Phi(G_n)_\lambda}^{-1}(d_{BV}(\mathcal{O}_{\phi^0}))\}, \end{aligned}$$

where $\phi^0 = \phi_{\widehat{\psi_0}}$, the unique tempered L -parameter of $\Phi(G)_{\lambda_{\psi_0}}$.

Let us recall the statement of the Conjecture for weak local Arthur packets.

Conjecture 4.4 ([CMO22, Conjecture 3.1.2]). *Let ψ be a basic local Arthur parameter of G and denote λ the (real) infinitesimal parameter associated with ψ . Then Π_ψ^{Weak} is a union of local Arthur packets.*

With the results in Section 3.2, we prove Conjecture 4.4 for the split groups $G_n = \mathrm{SO}_{2n+1}(F)$, $\mathrm{Sp}_{2n}(F)$ or $\mathrm{SO}_{2n}(F)$ assuming Conjecture 1.3(i) holds for all ψ whose restriction to W_F is trivial.

Theorem 4.5. *Let G_n be the split group $\mathrm{SO}_{2n+1}(F)$, $\mathrm{Sp}_{2n}(F)$ or $\mathrm{SO}_{2n}(F)$. Assume that the residue field of F has sufficiently large characteristic.*

- (a) For any basic local Arthur parameter ψ_0 of G_n , the weak local Arthur packet $\Pi_{\psi_0}^{\text{Weak}}$ is contained in a union of local Arthur packets

$$(4.3) \quad \Pi_{\psi_0}^{\text{Weak}} \subseteq \bigcup_{\psi \in (d_{BV})_{\Psi(G_n)_\lambda}^{-1}(\mathcal{O}')} \Pi_\psi,$$

where $\lambda = \lambda_{\psi_0}$, $\mathcal{O}' = d_{BV}(\mathcal{O}_{\psi_0}^A)$, and

$$(d_{BV})_{\Psi(G_n)_\lambda}^{-1}(\mathcal{O}') := \{\psi \in \Psi(G_n)_\lambda \mid d_{BV}(\mathcal{O}_\psi^A) = \mathcal{O}'\}.$$

- (b) Moreover, assume that Conjecture 1.3(i) holds for all $\psi \in \Psi(G_n)$ whose restriction to W_F is trivial. Then we have the other direction of containment

$$(4.4) \quad \Pi_{\psi_0}^{\text{Weak}} \supseteq \bigcup_{\psi \in (d_{BV})_{\Psi(G_n)_\lambda}^{-1}(\mathcal{O}')} \Pi_\psi,$$

which proves Conjecture 4.4. In particular, by Theorem 1.4, Conjecture 4.4 holds for split $\text{SO}_{2n+1}(F)$ and $\text{Sp}_{2n}(F)$ without the assumption of Conjecture 1.3(i).

Proof. Note that

$$\mathcal{O}' = d_{BV}(\mathcal{O}_{\psi_0}^A) = d_{BV}(\mathcal{O}_{\widehat{\psi}_0}^D) = d_{BV}(\mathcal{O}_{\phi^0}),$$

where $\phi^0 = \phi_{\widehat{\psi}_0}$ is the unique tempered L -parameter in $\Phi(G_n)_\lambda$. For Part (a), suppose $\pi \in \Pi_{\psi_0}^{\text{Weak}}$. Then Theorem 4.2 gives

$$\mathcal{O}' = \bar{\mathfrak{n}}^m(\pi) = d_{BV}(\mathcal{O}_{\phi_\pi}).$$

Namely, $\phi_\pi \in (d_{BV})_{\Phi(G_n)_\lambda}^{-1}(d_{BV}(\mathcal{O}_{\phi^0}))$. Therefore, Proposition 3.4 implies that the L -parameter ϕ_π is of Arthur type. Say $\phi_\pi = \phi_{\widehat{\psi}}$. Then $\psi \in (d_{BV})_{\Psi(G_n)_\lambda}^{-1}(\mathcal{O}')$. Also, $\widehat{\pi} \in \Pi_{\phi_\pi} = \Pi_{\phi_{\widehat{\psi}}} \subseteq \Pi_{\widehat{\psi}}$, and hence $\pi \in \Pi_\psi$. This verifies (4.3).

For Part (b), suppose $\pi \in \Pi_\psi$ where $\psi \in (d_{BV})_{\Psi(G_n)_\lambda}^{-1}(\mathcal{O}')$. Conjecture 1.3(i) implies

$$\bar{\mathfrak{n}}^m(\pi) \leq d_{BV}(\mathcal{O}_\psi^A) = \mathcal{O}' = d_{BV}(\mathcal{O}_{\psi_0}^A).$$

Therefore, π is in the weak local Arthur packet $\Pi_{\psi_0}^{\text{Weak}}$. This proves (4.4) and completes the proof of the theorem. $\square$

- Remark 4.6.**
1. The same proof works for inner forms of split groups $\text{SO}_{2n+1}(F)$ or $\text{SO}_{2n}(F)$ once Arthur's theory on the local Arthur packets is developed. Especially, we need the results that $\Pi_{\phi_\psi} \subseteq \Pi_\psi$ and $\Pi_{\widehat{\psi}} = \{\widehat{\pi} \mid \pi \in \Pi_\psi\}$ in the proof.
 2. If one can verify an analogue of Proposition 3.4 for any connected reductive algebraic group G , inner to split, then Theorem 4.5 also holds for G by similar arguments. However, Conjecture 4.4 do not imply Proposition 3.4 in an obvious way. Therefore, Proposition 3.4 for G_n has its own interests.
 3. Part (a) of above theorem (together with Arthur's theory) implies that any representation in a weak local Arthur packet is unitary. This proves [CMO22, Conjecture 3.1.3] for the split group $\text{SO}_{2n+1}(F)$, $\text{Sp}_{2n}(F)$ or $\text{SO}_{2n}(F)$ without assumptions. We record it below using their notation.

Theorem 4.7. Assume that the residue field of F has sufficiently large characteristic. Let $(q^{\frac{1}{2}h^\vee}, x, \rho)$ be a Deligne-Langlands-Lusztig parameter of the split groups $\text{SO}_{2n+1}(F)$, $\text{Sp}_{2n}(F)$ or $\text{SO}_{2n}(F)$ such that x belongs to the special piece of $\mathbb{O}^\vee$. Then the irreducible representation $X(q^{\frac{1}{2}h^\vee}, x, \rho)$ is unitary.

Finally, we give a remark of a characterization of anti-tempered local Arthur packets, which directly comes from the proof of [CMO22, Theorem 3.0.3].

Remark 4.8. For any basic local Arthur parameter ψ_0 of G , it is stated in [CMO22, Theorem 3.0.3] that

$$\Pi_{\psi_0} = \{\pi \in \Pi(G)_\lambda \mid {}^K\text{WF}(\pi) = d_A(\mathcal{O}_{\psi_0}^A, 1)\},$$

where ${}^K\text{WF}(\pi)$ is the canonical unramified wavefront set of π , d_A is the duality defined in [Ach03]. See [CMO22] for details of these notations. By [CMO22, Theorem 2.6.2(1)], we may rewrite it as

$$(4.5) \quad \Pi_{\psi_0} = \{\pi \in \Pi(G)_\lambda \mid d_A(\mathcal{O}_{\phi_{\hat{\pi}}}, 1) = d_A(\mathcal{O}_{\psi_0}^A, 1)\}.$$

The right hand side of (4.5) makes sense for any local Arthur parameter ψ_0 , which is not necessarily basic. Indeed, the same proof of [CMO22, Theorem 3.0.3] implies that (4.5) holds for any anti-tempered local Arthur parameter ψ_0 of G . This gives a characterization of anti-tempered local Arthur packets.

Similarly, if we replace the $\bar{\pi}^m(\pi)$ by $d_{BV}(\mathcal{O}_{\phi_{\hat{\pi}}})$ in the definition of weak local Arthur packet (4.1), then we can generalize Theorem 4.5 to any anti-tempered local Arthur parameter, which is not necessarily basic. More precisely, we have the following theorem. Note that in this case we do not need the characteristic assumption of the residue field of F since we don't need to make use of Theorem 4.2.

Theorem 4.9. Let G_n be the split group $\text{SO}_{2n+1}(F)$, $\text{Sp}_{2n}(F)$ or $\text{SO}_{2n}(F)$. For any anti-tempered local Arthur parameter ψ_0 , we denote $\lambda := \lambda_{\psi_0}$ and $\mathcal{O}' := d_{BV}(\mathcal{O}_{\psi_0}^A)$. Consider the set of representations

$$\Pi_{\psi_0}^{\text{Weak}} := \{\pi \in \Pi(G_n)_\lambda \mid d_{BV}(\mathcal{O}_{\phi_{\hat{\pi}}}) \leq d_{BV}(\mathcal{O}_{\psi_0}^A)\}.$$

We have an inclusion

$$(4.6) \quad \Pi_{\psi_0}^{\text{Weak}} \subseteq \bigcup_{\psi \in (d_{BV})_{\Psi(G_n)_\lambda}^{-1}(\mathcal{O}')} \Pi_\psi.$$

Moreover, if $\mathcal{O}_{\phi_{\hat{\pi}}} \geq \mathcal{O}_{\phi_\psi}$ for any $\psi \in \Psi(G_n)_\lambda$ and $\pi \in \Pi_\psi$, which has already been verified for the split groups $\text{SO}_{2n+1}(F)$ and $\text{Sp}_{2n}(F)$ in [HLLZ22, Theorem 1.15, Corollary 4.12(2)], then the inclusion (1.4) is an equality.

Proof. The proof of the inclusion (4.6) is exactly the same as the proof of Theorem 4.5(a) (without using Theorem 4.2), which we omit. Conversely, suppose π is in the right hand side of (4.6), i.e., there exists a local Arthur parameter ψ such that $\pi \in \Pi_\psi$ and $d_{BV}(\mathcal{O}_{\psi}^A) = d_{BV}(\mathcal{O}_{\psi_0}^A)$. Then $\hat{\pi} \in \Pi_{\hat{\psi}}$ and the assumption gives $\mathcal{O}_{\phi_{\hat{\pi}}} \geq \mathcal{O}_{\phi_{\hat{\psi}}}$. Taking Barbasch-Vogan duality, we obtain

$$d_{BV}(\mathcal{O}_{\phi_{\hat{\pi}}}) \leq d_{BV}(\mathcal{O}_{\phi_{\hat{\psi}}}) = d_{BV}(\mathcal{O}_{\psi_0}^A),$$

which implies that $\pi \in \Pi_{\psi_0}^{\text{Weak}}$. This completes the proof of the theorem. $\square$

5. GENERALIZATIONS OF WEAK LOCAL ARTHUR PACKETS AND EXAMPLES

In this section, we discuss possible generalizations on the definition of weak local Arthur packets such that Conjecture 1.2 or (4.3) has a chance to be true, towards a better understanding of Jiang's conjecture.

Throughout the section, we let $G = G(F)$ be the F -point of a connected reductive algebraic group G defined over F , and assume there is a local Arthur packets theory for G as conjectured in

[Art89, Conjecture 6.1]. Let λ be any infinitesimal parameter of G . Without loss of generality, we may assume that $\lambda = \lambda_\psi$ for some local Arthur parameter ψ , otherwise there is no representation of Arthur type in $\Pi(G)_\lambda$.

A first naive generalization is to consider the set

$$(5.1) \quad \Pi_{\mathcal{O}', \lambda}^{\text{WF}} := \{\pi \in \Pi(G)_\lambda \mid \bar{\mathfrak{n}}^m(\pi) \leq \mathcal{O}'\},$$

for any nilpotent orbit $\mathcal{O}'$ of $\mathfrak{g}(\overline{F})$. However, $\Pi_{\mathcal{O}', \lambda}^{\text{WF}}$ is not always a union of local Arthur packets since it may contain representations not of Arthur type, as shown in the following example.

Example 5.1. Take ϕ^0 to be a tempered unramified L -parameter of $\text{SO}_{2n+1}(F)$ or $\text{Sp}_{2n}(F)$ such that $\lambda := \lambda_{\phi^0}$ is real, and that there exists a representation $\pi_{\text{bad}} \in \Pi(G_n)_\lambda$ that is not of Arthur type (see [HLL22, Example 7.10] for the existence of such π_{bad}). For any $\pi \in \Pi(G_n)_\lambda$, we have

$$\bar{\mathfrak{n}}^m(\pi) = \{d_{BV}(\mathcal{O}_{\phi_\pi})\}$$

by Theorem 4.2. Let ϕ_0 be the closed L -parameter in $\Phi(G_n)_\lambda$. Then since $\mathcal{O}_{\phi_\pi} \geq \mathcal{O}_{\phi_0}$, we conclude that

$$\bar{\mathfrak{n}}^m(\pi) \leq d_{BV}(\mathcal{O}_{\phi_0}).$$

Therefore, taking $\mathcal{O}' = d_{BV}(\mathcal{O}_{\phi_0})$, we have

$$\Pi_{\mathcal{O}', \lambda}^{\text{WF}} = \Pi(G_n)_\lambda,$$

which contains π_{bad} not of Arthur type.

A natural modification of (5.1) is to add the condition of Arthur type.

$$(5.2) \quad \Pi_{\mathcal{O}', \lambda}^{\text{WF}, \text{A}} := \{\pi \in \Pi(G)_\lambda \text{ of Arthur type} \mid \bar{\mathfrak{n}}^m(\pi) \leq \mathcal{O}'\}.$$

Then assuming Conjecture 1.3(i), we have

$$(5.3) \quad \Pi_{\mathcal{O}', \lambda}^{\text{WF}, \text{A}} \supseteq \bigcup_{\psi \in \Psi(G_n)_\lambda, d_{BV}(\mathcal{O}_\psi^A) \leq \mathcal{O}'} \Pi_\psi.$$

Moreover, assuming Conjecture 1.3(ii), the right hand side of (5.3) is exactly the union of all local Arthur packets contained in $\Pi_{\mathcal{O}', \lambda}^{\text{WF}, \text{A}}$, see the proof of Proposition 5.3 below for details. However, the containment (5.3) can be strict even if we assume $\mathcal{O}' = d_{BV}(\mathcal{O}_\psi^A)$ for some local Arthur parameter ψ in $\Psi(G)_\lambda$, as shown in the following example.

Example 5.2. In this example, we adopt the notation in [AM23] for enhanced L -parameter, which is called L -data there. Consider the following unipotent representations of $\text{SO}_{11}(F)$ with real infinitesimal parameter λ ,

$$\begin{aligned} \pi &= L(\Delta[-1/2, -3/2]; \pi((1/2)^-, (3/2)^-)), \\ \hat{\pi} &= L(\Delta[-3/2, -3/2]; \pi((1/2)^-, (1/2)^-, (3/2)^+)). \end{aligned}$$

The L -parameter associated with $\hat{\pi}$ is

$$\phi_{\hat{\pi}} = |\cdot|^{-3/2} \otimes S_1 + |\cdot|^{3/2} \otimes S_1 + 1 \otimes S_2 + 1 \otimes S_2 + 1 \otimes S_4,$$

which is not of Arthur type. We have (identifying nilpotent orbits of $\text{SO}_{11}(\mathbb{C})$ as partitions)

$$\bar{\mathfrak{n}}^m(\pi) = \{d_{BV}(\mathcal{O}_{\phi_\pi})\} = \{d_{BV}([4, 2, 2, 1, 1])\} = \{[5, 3, 1^3]\}.$$

On the other hand, the representation π is of Arthur type, and

$$\Psi(\pi) := \{\psi \mid \pi \in \Pi_\psi\} = \{\psi_1, \psi_2\},$$

where

$$\begin{aligned}\psi_1 &= 1 \otimes S_2 \otimes S_3 + 1 \otimes S_4 \otimes S_1, \\ \psi_2 &= 1 \otimes S_1 \otimes S_2 + 1 \otimes S_1 \otimes S_4 + 1 \otimes S_4 \otimes S_1.\end{aligned}$$

One can compute that

$$d_{BV}(\mathcal{O}_{\psi_1}^A) = [7, 2^2], \quad d_{BV}(\mathcal{O}_{\psi_2}^A) = [7, 1^4].$$

By Conjecture 1.3(ii), for $i = 1, 2$, there exists at least one representation $\pi_i \in \Pi_{\psi_i}$ with $\bar{\mathbf{n}}^m(\pi_i) = d_{BV}(\mathcal{O}_{\psi_i}^A)$. Indeed, by the explicit formula in Theorem 4.2, we can take

$$\begin{aligned}\pi_1 &= L(\Delta[-1/2, -3/2]; \pi((1/2)^+, (3/2)^+), \\ \pi_2 &= L(\Delta[-3/2, -3/2], \Delta[-1/2, -1/2]; \pi((1/2)^+, (3/2)^+).\end{aligned}$$

Therefore, any local Arthur packet Π_ψ that contains π is not contained in the set

$$\Pi_{[5,3,1^3],\lambda}^{WF,A} = \{\pi' \in \Pi(\mathrm{SO}_{11}(F))_\lambda \mid \bar{\mathbf{n}}^m(\pi') \leq [5, 3, 1^3], \text{ and } \pi' \text{ is of Arthur type.}\}.$$

However, π is in $\Pi_{[5,3,1^3],\lambda}^{WF,A}$. Therefore, $\Pi_{[5,3,1^3],\lambda}^{WF,A}$ can not be written as a union of local Arthur packets. We remark that $[5, 3, 1^3] = d_{BV}(\mathcal{O}_\psi^A)$ where $\psi = 1 \otimes S_3 \otimes S_2 + 1 \otimes S_1 \otimes S_4 \in \Psi(\mathrm{SO}_{11}(F))_\lambda$.

The failure of the containment (5.3) in the above example comes from Conjecture 1.3(ii) and the fact that there does not exist a ψ in

$$\Psi(\pi) := \{\psi \in \Psi(G) \mid \pi \in \Pi_\psi\}$$

such that $d_{BV}(\mathcal{O}_\psi^A) \leq \mathcal{O}'$. This suggests the final modification for the definition of weak local Arthur packets. For any nilpotent orbit $\mathcal{O}'$ of $\mathfrak{g}(\bar{F})$ and any infinitesimal parameter λ of G , we define the weak local Arthur packet as follows

(5.4)

$$\Pi_{\mathcal{O}',\lambda}^{\mathrm{Weak}} := \{\pi \in \Pi(G)_\lambda \text{ of Arthur type} \mid \text{There exists a } \psi \in \Psi(\pi) \text{ such that } d_{BV}(\mathcal{O}_\psi^A) \leq \mathcal{O}'\}.$$

We summarize these phenomena in the following proposition.

Proposition 5.3. *Let G be a connected reductive group and $G = G(F)$. Assume that there is a local Arthur packets theory for G as conjectured in [Art89, Conjecture 6.1]. Assume Conjecture 1.3 holds for the group $G = G(F)$. For any nilpotent orbit $\mathcal{O}'$ of $\mathfrak{g}(\bar{F})$ and any infinitesimal parameter λ of G , we have*

$$\Pi_{\mathcal{O}',\lambda}^{\mathrm{Weak}} = \bigcup_{\psi \in \Psi(G)_\lambda, \Pi_\psi \subseteq \Pi_{\mathcal{O}',\lambda}^{WF,A}} \Pi_\psi \subseteq \Pi_{\mathcal{O}',\lambda}^{WF,A},$$

where the containment can be strict.

Proof. Suppose $\pi \in \Pi_{\mathcal{O}',\lambda}^{\mathrm{Weak}}$. Then there exists a ψ such that $d_{BV}(\mathcal{O}_\psi^A) \leq \mathcal{O}'$ and $\pi \in \Pi_\psi$. By Conjecture 1.3(i), for any $\pi' \in \Pi_\psi$, we have

$$\bar{\mathbf{n}}^m(\pi') \leq d_{BV}(\mathcal{O}_\psi^A) \leq \mathcal{O}',$$

and hence $\Pi_\psi \subseteq \Pi_{\mathcal{O}',\lambda}^{WF,A}$.

Conversely, suppose $\pi \in \Pi_{\psi'}$ where $\Pi_{\psi'} \subseteq \Pi_{\mathcal{O}',\lambda}^{WF,A}$. By Conjecture 1.3(ii), there exists a $\pi' \in \Pi_{\psi'}$ such that $\bar{\mathbf{n}}^m(\pi') = \{d_{BV}(\mathcal{O}_{\psi'}^A)\}$, and hence $d_{BV}(\mathcal{O}_{\psi'}^A) \leq \mathcal{O}'$. This completes the proof of the proposition. $\square$

Assuming Conjecture 1.3, one can see from Proposition 5.3 that $\Pi_{\mathcal{O}',\lambda}^{\text{Weak}}$ is the maximal subset of $\Pi(G)_\lambda$ with the following properties.

- $\Pi_{\mathcal{O}',\lambda}^{\text{Weak}} \subseteq \{\pi \in \Pi(G)_\lambda \mid \bar{\pi}^m(\pi) \leq \mathcal{O}'\}$.
- $\Pi_{\mathcal{O}',\lambda}^{\text{Weak}}$ is a union of local Arthur packets.

Hence, if Conjecture 4.4 holds for a basic local Arthur parameter ψ_0 of G , then

$$\Pi_{\psi_0}^{\text{Weak}} = \Pi_{\mathcal{O}',\lambda}^{\text{Weak}},$$

where $\mathcal{O}' = d_{BV}(\mathcal{O}_{\psi_0}^A)$ and $\lambda = \lambda_{\psi_0}$. Therefore, the set $\Pi_{\mathcal{O}',\lambda}^{\text{Weak}}$ can be regarded as a natural generalization of $\Pi_{\psi_0}^{\text{Weak}}$ and reveals the close connection with Conjecture 1.3.

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