A NEW PROOF OF THE DESCRIPTION OF THE CONVEX HULL OF SPACE CURVES WITH TOTALLY POSITIVE TORSION

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Abstract. We give new proofs of the description convex hulls of space curves $\gamma : [a, b] \mapsto \mathbb{R}^d$ having totally positive torsion. These are curves such that all the leading principal minors of $d \times d$ matrix $(\gamma', \gamma'', \dots, \gamma^{(d)})$ are positive. In particular, we recover parametric representation of the boundary of the convex hull, different formulas for its surface area, and the volume of the convex hull , and the solution to a general moment problem corresponding to γ.

1. Introduction and a ~~summary~~ Summary of ~~main results~~the Main Results

Convex hull of a set $K \subset \mathbb{R}^d$ is defined as

$$\mathrm{conv}(K) = \left\{ \sum_{j=1}^{m} \lambda_j x_j,\ x_j \in K,\ \sum_{j=1}^{m} \lambda_j = 1,\ \lambda_j \geq 0,\ j = 1, \dots, m \text{ for all } m \geq 1 \right\}.$$

Describing the convex hull of a given set K is a basic problem in mathematics. By imposing additional geometric structures on K, one may hope to give a *simpler* description of $\mathrm{conv}(K)$. Perhaps a good starting point is when K is a space curve, which is the topic of our paper.

Let $[a, b]$ be an interval in $\mathbb{R}$, and let $\gamma_1(t), \dots, \gamma_{n+1}(t)$ be real valued functions on $[a, b]$. We start with two main questions, which are ultimately related to each other.

Question 1. *Describe the boundary of the convex hull of* $\gamma([a, b])$*, where*

$$\gamma(t) = (\gamma_1(t), \dots, \gamma_{n+1}(t)), \quad t \in [a, b].$$

The next question, known as the *general moment problem* [16, 14, 15], is a certain probabilistic reformulation of Question 1.

Question 2. *Find*

$$M^{\sup}(x_1, \dots, x_n) \stackrel{\text{def}}{=} \sup \{\mathbb{E}\gamma_{n+1}(Y) : \mathbb{E}\gamma_1(Y) = x_1, \dots, \mathbb{E}\gamma_n(Y) = x_n\}, \tag{1.1}$$

$$M^{\inf}(x_1, \dots, x_n) \stackrel{\text{def}}{=} \inf \{\mathbb{E}\gamma_{n+1}(Y) : \mathbb{E}\gamma_1(Y) = x_1, \dots, \mathbb{E}\gamma_n(Y) = x_n\}, \tag{1.2}$$

where supremum or infimum is taken over all random variables Y with values in $[a, b]$ such that $\gamma_j(Y)$ are measurable for all j, $1 \leq j \leq n + 1$.

The answers to both of these questions are given in terms of *lower and upper principal representations* in two remarkable monographs [15, 14] (see also a brief survey [8]) under ~~the~~ assumption (A1), which says that the sequences $(1, \gamma_1(t), \dots, \gamma_n(t))$ and $(1, \gamma_1(t), \dots, \gamma_{n+1}(t))$ are T_+-systems on $[a, b]$, we refer the reader to Subsection 1.1.3 for more details.

In this paper we give a new self-contained geometric approach to both of these questions for a subclass of (A1), curves with so called *totally positive torsion*.

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Definition. *A curve* $\gamma \in C^{n+1}((a,b),\mathbb{R}^{n+1}) \cap C([a,b],\mathbb{R}^{n+1})$ *is said to have totally positive torsion if all the leading principal minors of the matrix*

$$(\gamma'(t), \gamma''(t), \dots, \gamma^{(n+1)}(t)) \tag{1.3}$$

are positive for all $t \in (a,b)$.

Perhaps an instructive example to keep in mind is $\gamma(t) = (t, t^2, \dots, t^n, \gamma_{n+1}(t))$, where the total positivity of the torsion on (a,b) is the same as $\gamma_{n+1}^{(n+1)}(t) > 0$ on (a,b).

In fact the only property that will be needed from the principal minors of ~~the~~ matrix (1.3) is that they are ~~non-vanishing~~nonvanishing. Indeed, we can consider an invertible linear image of γ, namely a new curve $t \mapsto (\varepsilon_1\gamma_1(t), \dots, \varepsilon_{n+1}\gamma_{n+1}(t))$ with an appropriate choice of signs $\varepsilon_j = \pm 1$ and reduce the study of the convex hulls to the curves with totally positive torsion (an invertible linear transformation T maps convex hull of a set K to the convex hull of the image $T(K)$).

In Section 1.1 we provide an overview of the literature on results related to Questions 1 and 2. Section 2 is devoted to the statements of the main results of the paper, and Section 3 contains the proofs. Here we give a short summary of the theorems that we recover in this paper. The results we state hold in $\mathbb{R}^{n+1}$ for all $n \geq 1$, and all space curves $\gamma : [a,b] \to \mathbb{R}^{n+1}$ with totally positive torsion. Set $\bar{\gamma}(t) \stackrel{\text{def}}{=} (\gamma_1(t), \dots, \gamma_n(t))$, and let us denote by $\mathrm{conv}(\gamma([a,b]))$ the convex hull of the image of $[a,b]$ under the map γ.

Summary of the results:

1. Boundary of the convex hull of $\gamma([a,b])$ will be given in a parametric form.
2. Explicit diffeomorphism will be constructed between the interior of simplicies and the interior of the convex hull of $\gamma([a,b])$.
3. Formulas for the surface area of the boundary of the convex hull of $\gamma([a,b])$ will be obtained, Corollary 2.8, and different formulas for the volume of the convex hull will be presented, Corollary 2.7.
4. Any single affine hyperplane intersects the space curve $\gamma : [a,b] \to \mathbb{R}^{n+1}$ in at most $n+1$ points. Minimal number k points required to represent any point $x \in \mathrm{conv}(\gamma([a,b]))$ as a convex combination of k points of $\gamma([a,b])$ is at most $\lfloor \frac{n+3}{2} \rfloor$. Moreover, $k = \lfloor \frac{n+3}{2} \rfloor$ for any interior point of $\mathrm{conv}(\gamma([a,b]))$.
5. Parametric representations will be given for functions $M^{\sup}$ and $M^{\inf}$. The obtained parametric forms change depending on whether n is even or odd.

   (i) If n is even then

$$M^{\sup}\left(\lambda_0\bar{\gamma}(b) + \sum_{j=1}^{\frac{n}{2}} \lambda_j \bar{\gamma}(x_j)\right) = \lambda_0\gamma_{n+1}(b) + \sum_{j=1}^{\frac{n}{2}} \lambda_j \gamma_{n+1}(x_j),$$

$$M^{\inf}\left(\lambda_0\bar{\gamma}(a) + \sum_{j=1}^{\frac{n}{2}} \lambda_j \bar{\gamma}(x_j)\right) = \lambda_0\gamma_{n+1}(a) + \sum_{j=1}^{\frac{n}{2}} \lambda_j \gamma_{n+1}(y_j),$$

   for all $\lambda_0, \lambda_j \in [0,1], x_j \in [a,b]$, $j = 1, \dots, \frac{n}{2}$ with $\sum_{0 \leq k \leq \frac{n}{2}} \lambda_k = 1$.

   (ii) If n is odd then

$$M^{\sup}\left(\lambda_0\bar{\gamma}(a) + \lambda_1\bar{\gamma}(b) + \sum_{j=2}^{\frac{n+1}{2}} \lambda_j \bar{\gamma}(x_j)\right) = \lambda_0\gamma_{n+1}(a) + \lambda_1\gamma_{n+1}(b) + \sum_{j=2}^{\frac{n+1}{2}} \lambda_j \gamma_{n+1}(x_j),$$

$$M^{\inf}\left(\sum_{j=1}^{\frac{n+1}{2}} \beta_j \bar{\gamma}(x_j)\right) = \sum_{j=1}^{\frac{n+1}{2}} \beta_j \gamma_{n+1}(x_j),$$

for all $\lambda_0, \lambda_j, \beta_j \in [0,1], x_j \in [a,b]$, $j = 1, \dots, \frac{n+1}{2}$ with $\sum_{0 \le j \le \frac{n+1}{2}} \lambda_j = \sum_{1 \le j \le \frac{n+1}{2}} \beta_j = 1$.

(6) Explicit random variables Y ~~will be constructed~~ , which attain supremum and infimum correspondingly in (1.1) and (1.2), will be constructed for each given $x = (x_1, \dots, x_n)$ from the domain of definition of $M^{\sup}$ and $M^{\inf}$.

We will also see that

$$\partial \operatorname{conv}(\gamma([a,b])) = \{(x, M^{\sup}(x)), x \in \operatorname{conv}(\bar{\gamma}([a,b]))\} \cup \{(x, M^{\inf}(x)), x \in \operatorname{conv}(\bar{\gamma}([a,b]))\},$$

~~i.e.~~that is, the *upper hull* of $\operatorname{conv}(\gamma([a,b]))$ coincides with the graph of $M^{\sup}$ ~~,~~ and the lower hull with the graph of $M^{\inf}$. Besides this summary, we also recover several results previously known to Karlin–Sharpley [13] for *moment curves* using our techniques (see Corollary 2.4). In Proposition 2.2, we also show that the results obtained in this paper are sensitive to the assumption on a curve having totally positive torsion.

1.1. What ~~is known~~ Is Known about Questions 1 and 2?

In what follows we set $x \stackrel{\text{def}}{=} (x_1, \dots, x_n) \in \mathbb{R}^n$ ~~,~~ and $\mathbb{E}\bar{\gamma}(Y) \stackrel{\text{def}}{=} (\mathbb{E}\gamma_1(Y), \dots, \mathbb{E}\gamma_n(Y))$. We remark that both $M^{\sup}$ and $M^{\inf}$ depend on $n \ge 1$, $x \in \mathbb{R}^n$, $[a,b] \subset \mathbb{R}$, and γ. We shall remind the basic fact that the convex hull of a compact set is compact. For simplicity we shall use the symbol M for $M^{\sup}(x)$.

There are series of results describing M for some particular γ. A common goal is to have a parametric representation for it. However, as soon as n is large, it becomes difficult to find parametric representation for M in such generality.

1.1.1. *Convex ~~envelopes~~ Envelopes and Carathéodory ~~number~~Number.*

Under some mild assumptions on γ, say γ is continuous on $[a,b]$ is sufficient (see [16, 20]), M is defined on $\operatorname{conv}(\bar{\gamma}([a,b]))$. Moreover, for any $x \in \operatorname{conv}(\bar{\gamma}([a,b]))$, $M(x)$ is the solution of the *dual problem*

$$M(x) = \inf_{d_0 \in \mathbb{R}, d \in \mathbb{R}^n} \{d_0 + \langle d, x \rangle \text{ such that } d_0 + \langle d, \bar{\gamma}(t) \rangle \ge \gamma_{n+1}(t) \text{ for all } t \in [a,b]\}, \tag{1.4}$$

where $\langle a, b \rangle$ denotes the dot product in $\mathbb{R}^n$. Thus M is the minimal concave function defined on $\operatorname{conv}(\bar{\gamma}([a,b]))$ with the obstacle condition $M(\bar{\gamma}(t)) \ge \gamma_{n+1}(t)$ for all $t \in [a,b]$. So the graph $(x, M(x))$, $x \in \operatorname{conv}(\bar{\gamma}([a,b]))$ belongs to the boundary of $\operatorname{conv}(\gamma([a,b]))$. Carathéodory's theorem says that $(x, M(x))$ is convex combination of at most $n+2$ points from $\gamma([a,b])$. However, due to the fact $(x, M(x)) \in \partial \operatorname{conv}(\gamma([a,b]))$, one can see that $n+1$ points suffice by considering any affine hyperplane H supporting $\operatorname{conv}(\gamma([a,b]))$ at $(x, M(x))$. Since $\gamma([a,b])$ lies on one side of H, it follows that the points, whose convex combination is $(x, M(x))$, must lie in H, and we can apply Carathéodory's theorem to $H \cap \gamma([a,b])$ in $n+1$ dimensional space H. This leads ~~us~~ to another representation

$$M(x) = \sup_{\sum_{j=1}^{n+1} c_j \bar{\gamma}(t_j) = x} \left\{ \sum_{j=1}^{n+1} c_j \gamma_{n+1}(t_j) \ : \ \sum_{j=1}^{n+1} c_j = 1, \ c_\ell \ge 0, \ t_\ell \in [a,b], \ 1 \le \ell \le n+1 \right\}. \tag{1.5}$$

Probabilistic way of looking at (1.5) is that the supremum and infimum in (1.1) and (1.2) ~~is~~ are attained on random variables Y whose density is the sum of delta masses on at most $n+1$ points in $[a,b]$, ~~i.e.~~that is, $\sum_{j=1}^{n+1} c_j \delta_{t_j}$, with $t_j \in [a,b]$ for all $j = 1, \dots, n+1$.

A direction of research focuses on understanding for which curves γ ~~,~~ the number $n+1$ appearing in $\sum_{j=1}^{n+1} c_j \delta_{t_j}$ can be made smaller. As we just described, this is related to the following question: ~~*given*~~ *Given a curve $\gamma : [a,b] \to \mathbb{R}^{n+1}$ ~~,~~ and a point $y \in \partial \operatorname{conv}(\gamma([a,b]))$, find the smallest number of points $b(y)$ on $\gamma([a,b])$ whose convex combination coincides with x.* The integer $b(y)$ is called Carathéodory number for y, and it is defined for all $y \in \operatorname{conv}(\gamma([a,b]))$. Carathéodory number $b(\gamma)$ of a set $\gamma([a,b])$ is defined as

$$b(\gamma) \stackrel{\text{def}}{=} \sup_{x \in \operatorname{conv}(\gamma([a,b]))} b(x). \tag{1.6}$$

By Carathéodory's theorem $b(\gamma) \le n+2$ for curves in $\mathbb{R}^{n+1}$. For certain curves γ, the number $b(\gamma)$ can be strictly smaller than $n+2$. Fenchel's theorem [5, 7] asserts that if the compact set $\gamma([a,b])$ cannot be separated by a hyperplane into two ~~non-empty disjoint sets~~nonempty disjoint sets, then $b(\gamma) \le n+1$. In particular, for continuous curves γ over closed intervals $[a,b]$, the Carathéodory's number is at most $n+1$ giving one more justification of (1.5) for continuous maps γ. See [2] where Carathéodory number and an extension of Fenchel's theorem ~~is~~ are studied for certain type of sets in $\mathbb{R}^{n+1}$.

1.1.2. *A Convex Optimization Approach.* Another direction of research reduces (1.4) to what is called *positive semidefinite optimization problem* under the assumption

$$\gamma(t) = (t, t^2, \dots, t^n, \mathbb{1}_I(t)),$$

where I is an interval in $\mathbb{R}$. Finding upper or lower bounds on $\mathbb{E}\mathbb{1}_I(Y) = \mathbb{P}(Y \in I)$ given the first n moments of Y is of important interests ~~as~~ because it would refine the classical Chebyshev and Markov inequalities. To give a feeling how the corresponding positive semidefinite optimization problem looks like, we cite Theorem 11 in [3]: the tight upper bound on $\mathbb{P}(Y \ge 1)$ over all nonnegative random variables Y given the first n moments $\mathbb{E}Y^j = x_j$, $1 \le j \le n$, coincides with

$$M^{\sup}(x) = \min_{d_0,\dots,d_n \in \mathbb{R}} \quad d_0 + \sum_{j=1}^{n} d_j x_j.$$

Subject to

$$\begin{aligned}
&0 = \sum_{i,j\,:\,i+j=2\ell-1} t_{ij}, && \ell = 1,\dots,n,\\
&(d_0 - 1) + \sum_{j=\ell}^{n} d_j \binom{j}{\ell} = t_{00}, && \\
&\sum_{j=\ell}^{n} d_j \binom{j}{\ell} = \sum_{i,j\,:\,i+j=2\ell} t_{ij}, && \ell = 1,\dots,n,\\
&0 = \sum_{i,j\,:\,i+j=2\ell-1} z_{ij}, && \ell = 1,\dots,n,\\
&\sum_{j=0}^{\ell} d_j \binom{n-j}{\ell-j} = \sum_{i,j\,:\,i+j=2\ell} z_{ij} && \ell = 0,\dots,n,\\
&T, Z \ge 0, &&
\end{aligned}$$

where $T, Z \ge 0$ means that the matrices $T = \{t_{ij}\}_{i,j=0}^{n}$, $Z = \{z_{ij}\}_{i,j=0}^{n}$ are positive semidefinite.

The advantage of having such a semidefinite optimization problem is that it can be solved in a *polynomial time*. However, it is not clear to us how practical are these results if one wants to verify bounds $M(x) \le R(x)$ for a given function R and all x in $\mathrm{conv}(\overline{\gamma}([0,1]))$. In [3] the authors provide explicit formulas for the tight upper bound on $\mathbb{P}(Y > \lambda)$ for $n = 3$ over all nonnegative random variables with given first ~~3~~ three moments.

1.1.3. *Tchebysheff ~~systems~~Systems, ~~convex curves~~Convex Curves, and Markov ~~moment problem~~Moment Problem.* The system of continuous functions $(\gamma_0(t), \dots, \gamma_n(t))$ ~~,~~ on an interval $[a,b]$ is called Tchebysheff system (or T-system) if any nontrivial linear combination $\sum_{j=0}^{n} a_j \gamma_j(t)$ has at ~~moat~~ most n roots on $[a,b]$. ~~As~~ Since the monographs [15, 14] deal with general Markov moment problem with arbitrary Borel measures, and in this paper we consider only probability measures, in what follows we will be assuming that $\gamma_0(t) = 1$ to make the presentation consistent with [15, 14]. Under such an assumption the corresponding curve $t \mapsto (\gamma_1(t), \dots, \gamma_n(t))$ is called *convex curve*.

The sequence $(\gamma_0(t), \ldots, \gamma_n(t))$ is called T_+-system if

$$\det(\{\gamma_i(t_j)\}_{i,j=0}^{n}) > 0 \tag{1.7}$$

on the simplex $\Sigma = \{a \leq t_0 < \ldots, < t_n \leq b\}$. Notice that any T-system can be made into T_+-system just by flipping the sign in front of γ_n if necessary. If $(\gamma_0(t), \ldots, \gamma_k(t))$ is a T_+-system on $[a, b]$ for any $k = 0, \ldots, n$, then the sequence $(\gamma_0(t), \ldots, \gamma_n(t))$ is called M_+-system on $[a, b]$. Checking the positivity of ~~the~~ determinant (1.7) seems a bit unpractical ~~as~~ because one needs to verify the inequality on the simplex Σ. The following proposition gives a simple sufficient criteria for the system to be M_+ system.

Theorem 1.1 (Chapter VIII, [14]). *Let $\gamma_0(t), \ldots, \gamma_n(t)$ be in $C([a, b]) \cap C^n((a, b))$. Then, for the sequence $(\gamma_0(t), \ldots, \gamma_n(t))$ to be M_+-system on $[a, b]$, it is necessary*[1] *that $\det(\{\gamma_i^{(j)}(t)\}_{i,j=0}^{k}) \geq 0$ on (a, b) for all $k = 0, \ldots, n$, and it is sufficient that $\det(\{\gamma_i^{(j)}(t)\}_{i,j=0}^{k}) > 0$ on (a, b) for all $k = 1, \ldots, n$.*

We say that $(\gamma_1(t), \ldots, \gamma_{n+1}(t))$ satisfies $(A1)$ condition if $\gamma_1(t), \ldots, \gamma_{n+1}(t)$ are in $C([a, b]) \cap C^{n+1}((a, b))$ such that

$$(1, \gamma_1(t), \ldots, \gamma_n(t)) \quad \text{and} \quad (1, \gamma_1(t), \ldots, \gamma_{n+1}(t)) \quad \text{are} \quad T_+ - \text{systems on} \quad [a, b] \quad (A1)$$

Clearly, if $\gamma(t) = (\gamma_1(t), \ldots, \gamma_{n+1}(t))$ has totally positive torsion on (a, b) then the condition $(A1)$ holds by Theorem 1.1. On the other hand, if the sequence $(\gamma_0(t), \ldots, \gamma_{n+1})$ satisfies only ~~the~~ assumption (A1) then the probability distribution of a random variable X achieving supremum or infimum in Question 2 is given in terms of *upper and lower principal representations*, see Chapter III and IV in [15], and also Proposition 2 in a brief survey [8]. In particular, Carathéodory number is at most $\lfloor \frac{n+3}{2} \rfloor$ for the curves $t \mapsto (\gamma_1(t), \ldots, \gamma_{n+1}(t))$ in $\mathbb{R}^{n+1}$ satisfying ~~the~~ assumption (A1).

A typical example of the convex curve is the moment curve

$$\gamma(t) = (t, \ldots, t^{n+1}) \in \mathbb{R}^{n+1}.$$

Assume $[a, b] = [0, 1]$. In [13] the authors show that if $x = (x_1, \ldots, x_n)$ belongs to the interior of $\mathrm{conv}(\bar{\gamma}([0, 1]))$ then $M^{\sup}(x)$ and $M^{\inf}(x)$ are the unique solutions x_{n+1} of the linear equations

$$\det(K_{n+1}) = 0 \quad \text{and} \quad \det(S_{n+1}) = 0, \tag{1.8}$$

correspondingly, where K_k, S_k are defined as

$$S_{2k} = \begin{pmatrix} 1 & x_1 & \ldots & x_k \\ \vdots & & & \\ x_k & x_{k+1} & \ldots & x_{2k} \end{pmatrix}, \quad S_{2k+1} = \begin{pmatrix} x_1 & x_2 & \ldots & x_{k+1} \\ \vdots & & & \\ x_{k+1} & x_{k+2} & \ldots & x_{2k+1} \end{pmatrix}, \tag{1.9}$$

and

$$K_{2k} = \begin{pmatrix} x_1 - x_2 & x_2 - x_3 & \ldots & x_k - x_{k+1} \\ \vdots & & & \\ x_k - x_{k+1} & x_{k+1} - x_{k+2} & \ldots & x_{2k-1} - x_{2k} \end{pmatrix}, \tag{1.10}$$

$$K_{2k+1} = \begin{pmatrix} 1 - x_1 & x_1 - x_2 & \ldots & x_k - x_{k+1} \\ \vdots & & & \\ x_k - x_{k+1} & x_{k+1} - x_{k+2} & \ldots & x_{2k} - x_{2k+1} \end{pmatrix}.$$

A point $(x_1, \ldots, x_{n+1})$ belongs to $\mathrm{conv}(\gamma([0, 1]))$ if and only if the matrices K_{n+1} and S_{n+1} are positive semidefinite, see Theorem 16.1a, and Theorem 16.1b in [13], see also "truncated moment problem", Chapter 10 in [21]; Chapter IV, Section 2 in [1]. Also the point $(x_1, \ldots, x_{n+1})$ belongs to the interior of $\mathrm{conv}(\gamma([0, 1]))$ if and only if the matrices K_{n+1} and S_{n+1} are positive definite.

[1] Here $\gamma_j^{(0)}(t) = \gamma_j(t)$

An important contribution of [13] is that the authors give complete description of $\partial \operatorname{conv}(\gamma([0,1]))$, which allowed them to obtain a geometric point of view on the classical orthogonal polynomials. For example, knowing the width in x_{n+1} direction of the set $\operatorname{conv}(\gamma([0,1]))$, one can recover the classical fact that among all polynomials of degree $n+1$ on $[0,1]$ with the leading coefficient 1 the ~~Tchebyshev~~ Chebyshev polynomials minimize the maximum of the absolute value on $[0,1]$ (Theorem 25.2 in [13]).

~~Karlin–Sharpley~~ Karlin and Sharpley did announce an intend to settle the case when $[a,b]$ is replaced by $[-1,1]$, $\mathbb{R}^+$, or $\mathbb{R}$. After looking into a literature, to the best of our knowledge, the corresponding results appeared in the monograph of ~~Karlin–Studden~~Karlin and Studden [14].

In [22] Schoenberg obtained a formula for the volume of a smooth closed[2] convex curve $\nu : [0,2\pi] \mapsto \mathbb{R}^n$ in an even-dimensional Euclidean space:

$$\operatorname{Vol}(\operatorname{conv}(\nu([0,2\pi]))) = \pm\frac{1}{n!(n/2)!}\int_{[0,2\pi]^{\frac{n}{2}}} \det(\nu(t_1),\ldots,\nu(t_{n/2}),\nu'(t_1),\ldots,\nu'(t_{n/2}))dt_1\ldots dt_{n/2},$$

and as a corollary, using Fourier series, he derived an isoperimetric inequality

$$(\operatorname{length}(\nu))^n \geq (\pi n)^{n/2}(n/2)!n!\operatorname{Vol}(\operatorname{conv}(\nu([0,2\pi]))),$$

where $\operatorname{length}(\nu)$ denotes the Euclidean length of ν, and $\operatorname{Vol}(\cdot)$ denotes the Euclidean volume. The volumes of the convex hull of $\gamma([a,b])$, such that $\gamma(0)=0$ and the sequence $(1,\gamma_1(t),\ldots,\gamma_n(t))$ forms the T-system, were obtained both in odd and even dimensions in [15, 14], see for example ~~,~~ Theorem 6.1, Ch. IV in [14].

1.1.4. *Other ~~results~~ Results for ~~systems different~~ Systems Different from T-system.* In [24, 25] Sedykh describes possible *singularities* of the boundary of convex hulls of a curve in $\mathbb{R}^3$. In [18], using tools from algebraic geometry, namely, *De Jonquières' formula*, the authors compute a number of *complex tritangent planes* of the *algebraic boundary* of the convex hull of an algebraic space curve in $\mathbb{R}^3$ in terms of its genus and degree of the curve. Moreover, in [18] the authors also find an algebraic elimination method for computing *tritangent planes* and *edge surfaces* of the boundary of the convex hulls of algebraic space curves in $\mathbb{R}^3$. *Algebraic boundary* of the convex hull of an algebraic variety was studied [19], where the authors extended several results from [18] to higher dimensions. In [6], using topological results, it is shown that the number of tritangent planes to a smooth *generic* curve in $\mathbb{R}^3$ with nonvanishing torsion is even.

Convex hulls of space curves have appeared implicitly or explicitly in other works in relation to problems not directly related to them. We do not intend to provide the full list of references, however, let us mention some of the examples. Finding sharp constants in such classical estimates as John–Nirenberg inequality is related to finding convex hulls *in ~~non-convex~~ nonconvex domains* of certain space curves. In particular, in [12, 11], an algorithm is presented, which finds the convex hull of a space curve $\gamma(t)=(t,t^2,f(t))$ defined on $\mathbb{R}$, under the assumption that $f'''(t)$ changes sign finitely many times (notice that the sign of f''' coincides with the sign of the torsion of $\gamma(t)$). ~~As~~ Since the number of sign changes of f''' increase the "complexity" of computing, the convex hull of $\gamma(t)$ increases too. The method obtained in [12, 11] is illustrated on a particular example in [26] for the family of space curves $\gamma_\alpha(t)=(t,t^2,g_\alpha(t))$, where $g_\alpha(t)$ is a parametric family of functions defined for all $\alpha>0$ as follows:

$$g_\alpha(t)=\begin{cases}-\cos(t), & |t|\leq\alpha\\ \frac{1}{2}(t^2-\alpha^2)\cos\alpha+(\sin\alpha-\alpha\cos\alpha)(|t|-\alpha)-\cos\alpha, & |t|\geq\alpha.\end{cases}$$

Notice that the quadratic part for $|t|\geq\alpha$ is chosen in such a way that $g_\alpha\in C^2(\mathbb{R})$. Clearly, $g_\alpha'''(t)=-\sin(t)$ for $|t|\leq\alpha$ ~~,~~ and $g_\alpha'''(t)=0$ for $|t|\geq\alpha$. We see that as α increases the number of sign changes of $g_\alpha'''(t)$ increases too. In [26] the upper boundary of the convex hull of the space

[2] Here closed curve means $\nu(0)=\nu(2\pi)$

curve $\gamma_\alpha(t)$, $t \in \mathbb{R}$, is found in the ~~non-convex parametric domain~~nonconvex parametric domain[3]. $\Omega_\varepsilon = \{(x, y) \in \mathbb{R}^2 \,:\, x^2 \le y \le x^2 + \varepsilon^2\}$. In the limiting case $\varepsilon \to \infty$, one recovers the upper boundary of the convex hull of the space curve $\gamma_\alpha(t)$.

In sharpening the triangle inequality in L^p spaces, for each $p \in \mathbb{R} \setminus \{0\}$, the paper [9] finds the boundary of the convex hull of a space curve $\gamma(t) = (t, \sqrt{1-t^2}, ((1-t)^{1/p} + (1+t)^{1/p})^p), t \in [-1, 1]$. In [10] the boundary of the convex hull of a closed space curve is described, which is the union of the following three curves:

$$\left(\frac{1}{t^p + (1-t)^p + 1}, \frac{t^p}{t^p + (1-t)^p + 1}, \frac{(1+t)^p}{t^p + (1-t)^p + 1}\right), \quad t \in [0, 1];$$
$$\left(\frac{(1-t)^p}{t^p + (1-t)^p + 1}, \frac{1}{t^p + (1-t)^p + 1}, \frac{(2-t)^p}{t^p + (1-t)^p + 1}\right), \quad t \in [0, 1];$$
$$\left(\frac{t^p}{t^p + (1-t)^p + 1}, \frac{(1-t)^p}{t^p + (1-t)^p + 1}, \frac{|1-2t|^p}{t^p + (1-t)^p + 1}\right), \quad t \in [0, 1].$$

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2. Statements of ~~main results~~the Main Results

For any $v = (v_1, \ldots, v_d) \in \mathbb{R}^d$, we set $\overline{v} = (v_1, \ldots, v_{d-1})$ to be the projection onto the first $d-1$ coordinates, and we set $v^z = v_d$ to be the projection onto the last coordinate. For any $a < b$, define the following sets:

$$\Delta_c^k := \{(r_1, \ldots, r_k) \in \mathbb{R}^k \,:\, r_j \ge 0, j = 1, \ldots, k,\, r_1 + \ldots + r_k \le 1\},$$
$$\Delta_*^k := \{(y_1, \ldots, y_k) \in \mathbb{R}^k \,:\, a \le y_1 \le y_2 \le \ldots \le y_k \le b\}.$$

Let $n \ge 1$. If $n = 2\ell$, we define

$$U_n : \Delta_c^\ell \times \Delta_*^\ell \ni (\lambda_1, \ldots, \lambda_\ell, x_1, \ldots, x_\ell) \mapsto \sum_{j=1}^{\ell} \lambda_j \gamma(x_j) + (1 - \sum_{j=1}^{\ell} \lambda_j)\gamma(b);$$

$$L_n : \Delta_c^\ell \times \Delta_*^\ell \ni (\lambda_1, \ldots, \lambda_\ell, x_1, \ldots, x_\ell) \mapsto (1 - \sum_{j=1}^{\ell} \lambda_j)\gamma(a) + \sum_{j=1}^{\ell} \lambda_j \gamma(x_j),$$

and if $n = 2\ell - 1$, we define

$$U_n : \Delta_c^\ell \times \Delta_*^{\ell-1} \ni (\beta_1, \ldots, \beta_\ell, x_2, \ldots, x_\ell) \mapsto (1 - \sum_{j=1}^{\ell} \beta_j)\gamma(a) + \sum_{j=2}^{\ell} \beta_j \gamma(x_j) + \beta_1 \gamma(b);$$

$$L_n : \Delta_c^{\ell-1} \times \Delta_*^\ell \ni (\beta_2, \ldots, \beta_\ell, x_1, \ldots, x_\ell) \mapsto (1 - \sum_{j=2}^{\ell} \beta_j)\gamma(x_1) + \sum_{j=2}^{\ell} \beta_j \gamma(x_j).$$

If $n = 1$, we set $U_1 : [0, 1] =: \Delta_c^1 \times \Delta_*^0 \mapsto (1 - \beta_1)\gamma(a) + \beta_1 \gamma(b)$ ~~,~~ and $L_1 : [a, b] =: \Delta_c^0 \times \Delta_*^1 \mapsto \gamma(x_1)$. We will see that the maps U_n and L_n parameterize the upper and lower envelopes, respectively. The letters U and L are chosen as the first letters of the words *Upper* and *Lower*.

[3]By the convex hull of γ_α in Ω_ε we mean all possible convex combinations of those points on γ_α such that the projection of the resulting convex hull of these points onto $\mathbb{R}^2$ lies inside Ω_ε

Together with maps U_n and L_n we define functions $B^{\sup}$ (and $B^{\inf}$) on the image of $\overline{U}$ (or $\overline{L}$) such that

$$B^{\sup}(\overline{U}_n) = U_n^z, \tag{2.1}$$

$$B^{\inf}(\overline{L}_n) = L_n^z. \tag{2.2}$$

We remark that at this moment $B^{\sup}$ (and $B^{\inf}$) is not *well defined*, ~~i.e.~~that is, it could be that there are points s_1, s_2, $s_1 \neq s_2$ such that $\overline{U}_n(s_1) = \overline{U}_n(s_2)$ and at the same time $U_n^z(s_1) \neq U_n^z(s_2)$. However, we will see that the next theorem, in particular, claims that both functions $B^{\sup}, B^{\inf}$ are well defined.

Theorem 2.1. *Let $\gamma : [a,b] \mapsto \mathbb{R}^{n+1}$ be in $C([a,b]) \cap C^{n+1}((a,b))$ with totally positive torsion.*
If $n = 2\ell$, $\ell \geq 1$, we have

$$\overline{U}_{2\ell}(\partial\,(\Delta_c^{\ell} \times \Delta_*^{\ell})) = \overline{L}_{2\ell}(\partial\,(\Delta_c^{\ell} \times \Delta_*^{\ell})) = \partial\,\mathrm{conv}(\overline{\gamma}([a,b])), \tag{2.3}$$

$$\overline{U}_{2\ell} : \mathrm{int}(\Delta_c^{\ell} \times \Delta_*^{\ell}) \mapsto \mathrm{int}(\mathrm{conv}(\overline{\gamma}([a,b]))) \quad \textit{is diffeomorphism}, \tag{2.4}$$

$$\overline{L}_{2\ell} : \mathrm{int}(\Delta_c^{\ell} \times \Delta_*^{\ell}) \mapsto \mathrm{int}(\mathrm{conv}(\overline{\gamma}([a,b]))) \quad \textit{is diffeomorphism.} \tag{2.5}$$

If $n = 2\ell - 1$, we have

$$\overline{U}_{2\ell-1}(\partial\,(\Delta_c^{\ell} \times \Delta_*^{\ell-1})) = \overline{L}_{2\ell-1}(\partial\,(\Delta_c^{\ell-1} \times \Delta_*^{\ell})) = \partial\,\mathrm{conv}(\overline{\gamma}([a,b])), \tag{2.6}$$

$$\overline{U}_{2\ell-1} : \mathrm{int}(\Delta_c^{\ell} \times \Delta_*^{\ell-1}) \mapsto \mathrm{int}(\mathrm{conv}(\overline{\gamma}([a,b]))) \quad \textit{is diffeomorphism}, \tag{2.7}$$

$$\overline{L}_{2\ell-1} : \mathrm{int}(\Delta_c^{\ell-1} \times \Delta_*^{\ell}) \mapsto \mathrm{int}(\mathrm{conv}(\overline{\gamma}([a,b]))) \quad \textit{is diffeomorphism.} \tag{2.8}$$

For all $n \geq 1$,

$$B^{\sup}, B^{\inf} \quad \textit{are well defined}, \quad B^{\sup}, B^{\inf} \in C(\mathrm{conv}(\overline{\gamma}([a,b]))) \cap C^1(\mathrm{int}(\mathrm{conv}(\overline{\gamma}([a,b])))). \tag{2.9}$$

Next, for all $n \geq 1$~~we have~~, we have[4]

$$B^{\sup} \quad \textit{is minimal concave on} \quad \mathrm{conv}(\overline{\gamma}([a,b])) \quad \textit{with} \quad B^{\sup}(\overline{\gamma}) = \gamma_{n+1}; \tag{2.10}$$

$$B^{\inf} \quad \textit{is maximal convex on} \quad \mathrm{conv}(\overline{\gamma}([a,b])) \quad \textit{with} \quad B^{\inf}(\overline{\gamma}) = \gamma_{n+1}. \tag{2.11}$$

Moreover,

$$B^{\inf}(y) = B^{\sup}(y) \quad \textit{if and only if} \quad y \in \partial\,\mathrm{conv}(\overline{\gamma}([a,b])), \tag{2.12}$$

$$\partial\,\mathrm{conv}(\gamma([a,b])) = \{(x, B^{\sup}(x)), x \in \mathrm{conv}(\bar{\gamma}([a,b]))\} \cup \{(x, B^{\inf}(x)), x \in \mathrm{conv}(\bar{\gamma}([a,b]))\}. \tag{2.13}$$

The statement of Theorem 2.1 may seem a bit technical~~,~~; however, we think that the intuition behind the construction of the convex hulls is natural. We refer the reader to schematic pictures in ~~Fig.~~Figure 1 for better understanding of the claims made in the theorem. In ~~Fig.~~Figure 2 the domain $\mathrm{conv}(\overline{\gamma}([a,b]))$ of $B^{\sup}$ in $\mathbb{R}^3$ is foliated by triangles where $B^{\sup}$ is linear on each such triangle.

Perhaps it may seem that the total positivity of the torsion, ~~i.e.~~that is, the fact that the leading principal minors of $(\gamma', \ldots, \gamma^{(n+1)})$ have positive signs on (a,b), is a redundant assumption for Theorem 2.1 to hold true. However, the next proposition shows that the total positivity is a sensitive assumption.

Proposition 2.2. *There exists a curve $\gamma : [-1,1] \to \mathbb{R}^{2+1}$ in $C^{\infty}([-1,1])$ such that the leading principal minors of $(\gamma', \gamma'', \gamma''')$ are positive on $[-1,1]$ except the 2×2 and 3×3 principal minors vanish at $t = 0$, and the map $B^{\sup}$ defined by (2.1) is not concave on* $\mathrm{conv}(\overline{\gamma}([-1,1]))$.

[4]When $n = 1$, the equality $B^{\sup}(\overline{\gamma}) = \gamma_2$ should be replaced by $B^{\sup}(\overline{\gamma}) \geq \gamma_2$.

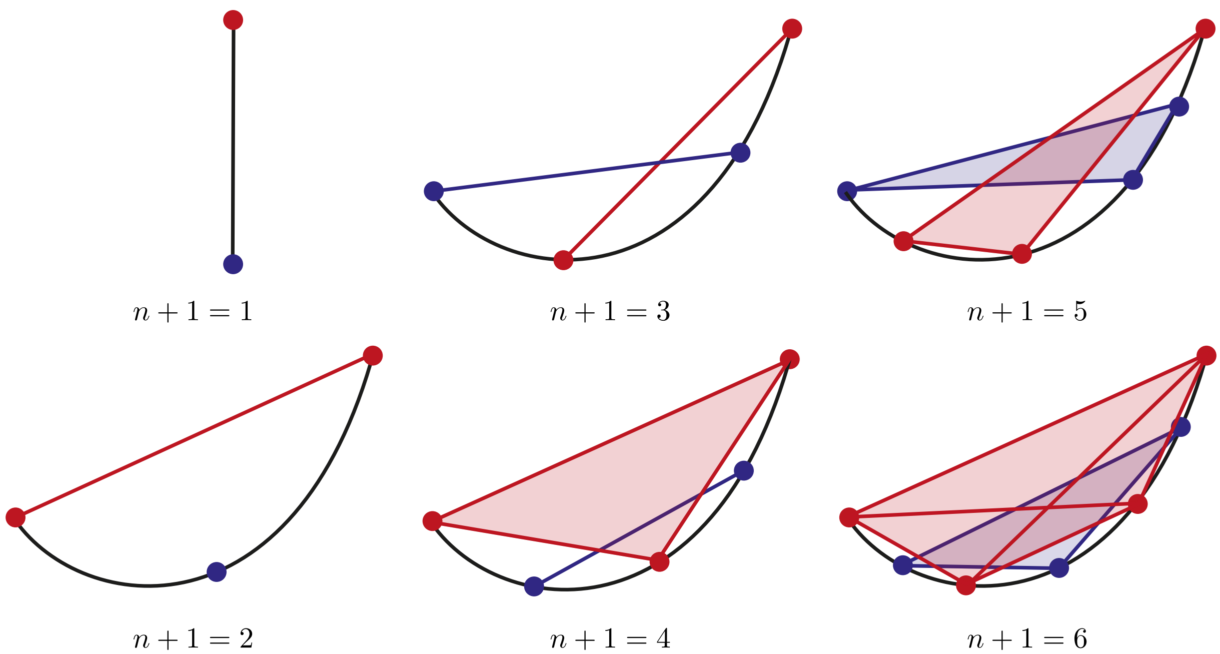

FIGURE 1. These schematic pictures clarify how the convex hull of the space curve γ with totally positive torsion is parametrized. If n is even then the *upper hull* is described by convex combination of $\frac{n}{2}+1$ points of γ, where among these points ~~,~~ $\frac{n}{2}$ are *free*, ~~i.e.~~that is, they are chosen in an arbitrary way on the space curve, and the last point $\gamma(b)$ is always fixed. For the *lower hull* $\gamma(a)$ is fixed instead of $\gamma(b)$. If n is odd, the picture is asymmetric. In this case the *upper hull* fixes ~~2~~ two endpoints $\gamma(a)$ and $\gamma(b)$ and has $\frac{n-1}{2}$ free points. The lower hull has $\frac{n+1}{2}$ free points, and no fixed points. The case $n=0$ (the convex hull of an interval), not mentioned in Theorem 2.1, was helpful to guess the construction in higher dimensions, it has two fixed points $\gamma(a)$ and $\gamma(b)$. Compare with the exact pictures for the cases $n+1=2,3,4$ shown in Figures 3,4, and 2.

The next theorem answers Question 2 ~~,~~ and also provides us with optimizers, ~~i.e.~~that is, the random variables Y ~~which~~ that attain supremum (infimum) in Question 2.

Theorem 2.3. *Let $\gamma : [a,b] \to \mathbb{R}^{n+1}$, $\gamma \in C([a,b]) \cap C^{n+1}((a,b))$ be such that all the leading principal minors of the $(n+1)\times(n+1)$ matrix $(\gamma'(t),\ldots,\gamma^{(n+1)}(t))$ are positive for all $t \in (a,b)$. Then*

$$\sup_{a\le Y\le b}\{\mathbb{E}\gamma_{n+1}(Y) \,:\, \mathbb{E}\bar{\gamma}(Y)=x\} = B^{\sup}(x), \tag{2.14}$$

$$\inf_{a\le Y\le b}\{\mathbb{E}\gamma_{n+1}(Y) \,:\, \mathbb{E}\bar{\gamma}(Y)=x\} = B^{\inf}(x), \tag{2.15}$$

hold for all $x \in \mathrm{conv}(\bar{\gamma}([a,b]))$, where $B^{\sup}$ and $B^{\inf}$ are given by (2.1) and (2.2). Moreover, given $x \in \mathrm{conv}(\bar{\gamma}([a,b]))$, the supremum in (2.14) (infimum in (2.15)) is attained by the random variable $\zeta(x)$ (the random variable $\xi(x)$) defined as follows~~:~~.

Case 1: $n=2\ell-1$. Then by (2.6) and (2.7), $x = (1-\sum_{j=1}^{\ell}\beta_j)\bar{\gamma}(a) + \sum_{j=2}^{\ell}\beta_j\bar{\gamma}(x_j) + \beta_1\bar{\gamma}(b)$ for some $(\beta_1,\ldots,\beta_\ell,x_2,\ldots,x_\ell) \in \Delta_c^{\ell}\times\Delta_*^{\ell-1}$. Set $\mathbb{P}(\zeta(x)=a) = 1-\sum_{j=1}^{\ell}\beta_j$, $\mathbb{P}(\zeta(x)=b)=\beta_1$, and $\mathbb{P}(\zeta(x)=x_j)=\beta_j$ for $j=2,\ldots,\ell$. Also, by (2.6) and (2.8), $x=(1-\sum_{j=2}^{\ell}\lambda_j)\bar{\gamma}(y_1)+\sum_{j=2}^{\ell}\lambda_j\bar{\gamma}(y_j)$ for some $(\lambda_2,\ldots,\lambda_\ell,y_1,\ldots,y_\ell) \in \Delta_c^{\ell-1}\times\Delta_*^{\ell}$. Set $\mathbb{P}(\xi(x)=y_1)=1-\sum_{j=2}^{\ell}\lambda_j$ ~~,~~ and $\mathbb{P}(\xi(x)=y_j)=\lambda_j$ for $j=2,\ldots,\ell$.

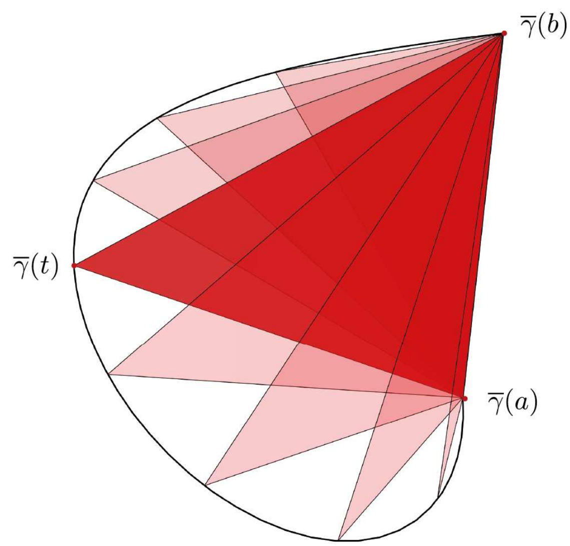

FIGURE 2. For $n+1=3+1$, the set $\mathrm{conv}(\overline{\gamma}([a,b]))$ is foliated by triangles (simplices) with vertices $\overline{\gamma}(a), \overline{\gamma}(b)$, and $\overline{\gamma}(t)$ for each $t \in (a,b)$. The function $B^{\sup}$ is linear on each such triangle and $B^{\sup}(\overline{\gamma}) = \gamma_4$. Also $B^{\sup} = B^{\inf}$ on the edges of each triangle.

Case 2: $n = 2\ell$. *Then, by (2.3) and (2.4),* $x = \sum_{j=1}^{\ell} \beta_j \overline{\gamma}(x_j) + (1 - \sum_{j=1}^{\ell} \beta_j)\overline{\gamma}(b)$ *for some* $(\beta_1, \dots, \beta_\ell, x_1, \dots, x_\ell) \in \Delta_c^\ell \times \Delta_*^\ell$. *Set* $\mathbb{P}(\zeta(x) = b) = 1 - \sum_{j=1}^{\ell} \beta_j$ ~~,~~ *and* $\mathbb{P}(\zeta(x) = x_j) = \beta_j$ *for* $j = 1, \dots, \ell$. *Also, by (2.3) and (2.5),* $x = (1 - \sum_{j=1}^{\ell} \lambda_j)\overline{\gamma}(a) + \sum_{j=1}^{\ell} \lambda_j \overline{\gamma}(y_j)$ *for some* $(\lambda_1, \dots, \lambda_\ell, y_1, \dots, y_\ell) \in \Delta_c^\ell \times \Delta_*^\ell$. *Set* $\mathbb{P}(\xi(x) = a) = 1 - \sum_{j=1}^{\ell} \lambda_j$ ~~,~~ *and* $\mathbb{P}(\xi(x) = y_j) = \lambda_j$ *for* $j = 1, \dots, \ell$.

The next corollary recovers the result of ~~Karlin–Sharpley [13], i.e., the~~ Karlin and Sharpley [13], that is, equations (1.8) in case of the moment curve.

Corollary 2.4. *Let* $\gamma(t) = (t, \dots, t^n, t^{n+1}) : [0,1] \to \mathbb{R}^{n+1}$. *If* $x = (x_1, \dots, x_n) \in \mathrm{int}(\mathrm{conv}(\overline{\gamma}([0,1])))$ *then* $B^{\sup}(x)$ *and* $B^{\inf}(x)$ *are the unique solutions* x_{n+1} *of the equations* $K_{n+1} = 0$ *and* $S_{n+1} = 0$ *correspondingly, where* K_{n+1} *and* S_{n+1} *are defined by (1.9) and (1.10).*

In the next corollary we give a sufficient local description of convex curves. Recall that a curve $\gamma : [a,b] \to \mathbb{R}^n$ is called *convex* if no $n+1$ its different points lie in a single affine hyperplane.

Corollary 2.5. *Let* $\gamma : [a,b] \to \mathbb{R}^n$, $\gamma \in C([a,b]) \cap C^n((a,b))$ *be such that all the leading principal minors of the* $n \times n$ *matrix* $(\gamma'(t), \dots, \gamma^{(n)}(t))$ *are positive for all* $t \in (a,b)$. *Then* γ *is convex. In particular, for any integer* k, $1 \leq k \leq n$, *the equation* $c_0 + c_1\gamma_1(t) + \ldots + c_k\gamma_k(t) = 0$ *has at most* k *roots on* $[a,b]$ *provided that* $(c_0, \dots, c_k) \neq (0, \dots, 0)$.

Recall the definition of Carathéodory number $b(\gamma)$ of a curve $\gamma : [a,b] \to \mathbb{R}^n$, ~~i.e.~~that is, the smallest integer k such that any point of $\mathrm{conv}(\gamma([a,b]))$ can be represented as a convex combination of at most k points of $\gamma([a,b])$, see (1.6). The next corollary directly follows from Theorem 2.1 (parts (2.6), (2.8), (2.3), and (2.5)).

Corollary 2.6. *Let* $\gamma : [a,b] \to \mathbb{R}^n$, $\gamma \in C([a,b]) \cap C^n((a,b))$ *be a curve with totally positive torsion. Then its Carathéodory number equals* ~~*to*~~ $\lfloor \frac{n+2}{2} \rfloor$.

In the next corollary we obtain formulas for the volumes of the convex hulls of a space curve having totally positive torsion both in even and odd dimensions.

Corollary 2.7. *Let $\gamma : [a,b] \to \mathbb{R}^n$, $\gamma \in C([a,b]) \cap C^n((a,b))$ be a curve with totally positive torsion. If $n = 2\ell$ then*

$$\begin{aligned}
&\mathrm{Vol}(\mathrm{conv}(\gamma([a,b])))\\
&\quad= \frac{(-1)^{\frac{\ell(\ell-1)}{2}}}{(2\ell)!}\int_{a\le x_1\le\ldots\le x_\ell\le b}\det(\gamma(x_1)-\gamma(a),\ldots,\gamma(x_\ell)-\gamma(a),\gamma'(x_1),\ldots,\gamma'(x_\ell))\,dx\\
&\quad= \frac{(-1)^{\frac{\ell(\ell-1)}{2}}}{(2\ell)!}\int_{a\le x_1\le\ldots\le x_\ell\le b}\det(\gamma(x_1)-\gamma(b),\ldots,\gamma(x_\ell)-\gamma(b),\gamma'(x_1),\ldots,\gamma'(x_\ell))\,dx.
\end{aligned}$$

If $n = 2\ell - 1$ then

$$\begin{aligned}
&\mathrm{Vol}(\mathrm{conv}(\gamma([a,b])))\\
&= \frac{(-1)^{\frac{(\ell-1)(\ell-2)}{2}}}{(2\ell-1)!}\int_{a\le x_2\le\ldots\le x_\ell\le b}\det(\gamma(b)-\gamma(a),\gamma(x_2)-\gamma(a),\ldots,\gamma(x_\ell)-\gamma(a),\gamma'(x_2),\ldots,\gamma'(x_\ell))\,dx\\
&= \frac{(-1)^{\frac{\ell(\ell-1)}{2}}}{(2\ell-1)!}\int_{a\le x_1\le\ldots\le x_\ell\le b}\det(\gamma(x_2)-\gamma(x_1),\ldots,\gamma(x_\ell)-\gamma(x_1),\gamma'(x_1),\ldots,\gamma'(x_\ell))\,dx.
\end{aligned}$$

Let Area denote n dimensional Lebesgue measure in $\mathbb{R}^{n+1}$, and let A^{Tr} be the transpose of a matrix A.

Corollary 2.8. *Let $\gamma : [a,b] \to \mathbb{R}^{n+1}$, $\gamma \in C^1([a,b]) \cap C^{n+1}((a,b))$ be a curve with totally positive torsion. If $n = 2\ell$ then*

$$\mathrm{Area}(\partial\,\mathrm{conv}(\gamma([a,b]))) = \frac{1}{n!}\int_{a\le x_1\le\ldots\le x_\ell\le b}\left(\sqrt{\det S_a^{\mathrm{Tr}}S_a}+\sqrt{\det S_b^{\mathrm{Tr}}S_b}\right)dx,$$

where $S_r = (\gamma(x_1)-\gamma(r),\ldots,\gamma(x_\ell)-\gamma(r),\gamma'(x_1),\ldots,\gamma'(x_\ell))$ is $(2\ell+1)\times 2\ell$ matrix, and dx is ℓ dimensional Lebesgue measure.

If $n = 2\ell - 1$ then

$$\mathrm{Area}(\partial\,\mathrm{conv}(\gamma([a,b]))) = \frac{1}{n!}\int_{a\le x_2\le\ldots\le x_\ell\le b}\sqrt{\det \Psi^{\mathrm{Tr}}\Psi}\,d\tilde{x} + \frac{1}{n!}\int_{a\le x_1\le\ldots\le x_\ell\le b}\sqrt{\det \Phi^{\mathrm{Tr}}\Phi}\,dx,$$

where $\Psi = (\gamma(b)-\gamma(a),\gamma(x_2)-\gamma(a),\ldots,\gamma(x_\ell)-\gamma(a),\gamma'(x_2),\ldots,\gamma'(x_\ell))$, $\Phi = (\gamma(x_2)-\gamma(x_1),\ldots,\gamma(x_\ell)-\gamma(x_1),\gamma'(x_1),\ldots,\gamma'(x_\ell))$ are $2\ell\times(2\ell-1)$ size matrices, and $d\tilde{x}$ denotes $\ell-1$ dimensional Lebesgue measure.

3. The ~~proof~~ Proof of ~~main results~~the Main Results

Sometimes we will omit the index n and simply write U, L instead of U_n, L_n, and it will be clear from the context what is the corresponding number n. Before we start proving Theorem 2.1, first let us state several lemmas that will be helpful throughout the rest of the paper. The next lemma illustrates *local to global* principle.

Lemma 3.1. *If the torsion of γ is totally positive on (a,b) then*

$$\det(\gamma'(x_1),\gamma'(x_2),\ldots\gamma'(x_{n+1})) > 0 \tag{3.1}$$

for all $a < x_1 < \ldots < \ldots < x_{n+1} < b$.

Proof. Without loss of generality assume $[a,b] = [0,1]$. The lemma can be derived from ~~the~~ identity (9) obtained in [4]. ~~As~~ Since the lemma is an important step in the proofs of the main results stated in this paper, for the readers~~convenience~~' convenience, we decided to include the proof of the lemma without invoking the identity from [4].

We have

$$\det\begin{pmatrix} \gamma_1'(x_1) & \gamma_1'(x_2) & \dots & \gamma_1'(x_{n+1}) \\ \gamma_2'(x_1) & \gamma_2'(x_2) & \dots & \gamma_2'(x_{n+1}) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{n+1}'(x_1) & \gamma_{n+1}'(x_2) & \dots & \gamma_{n+1}'(x_{n+1}) \end{pmatrix} =$$

$$\det\begin{pmatrix} 1 & 1 & \dots & 1 \\ \frac{\gamma_2'(x_1)}{\gamma_1'(x_1)} & \frac{\gamma_2'(x_2)}{\gamma_1'(x_2)} & \dots & \frac{\gamma_2'(x_{n+1})}{\gamma_1'(x_{n+1})} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\gamma_{n+1}'(x_1)}{\gamma_1'(x_1)} & \frac{\gamma_{n+1}'(x_2)}{\gamma_1'(x_2)} & \dots & \frac{\gamma_{n+1}'(x_{n+1})}{\gamma_1'(x_{n+1})} \end{pmatrix} \prod_{j=1}^{n+1} \gamma_1'(x_j) =$$

$$\det\begin{pmatrix} 1 & 0 & \dots & 0 \\ \frac{\gamma_2'(x_1)}{\gamma_1'(x_1)} & \frac{\gamma_2'(x_2)}{\gamma_1'(x_2)} - \frac{\gamma_2'(x_1)}{\gamma_1'(x_1)} & \dots & \frac{\gamma_2'(x_{n+1})}{\gamma_1'(x_{n+1})} - \frac{\gamma_2'(x_1)}{\gamma_1'(x_1)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\gamma_{n+1}'(x_1)}{\gamma_1'(x_1)} & \frac{\gamma_{n+1}'(x_2)}{\gamma_1'(x_2)} - \frac{\gamma_{n+1}'(x_1)}{\gamma_1'(x_1)} & \dots & \frac{\gamma_{n+1}'(x_{n+1})}{\gamma_1'(x_{n+1})} - \frac{\gamma_{n+1}'(x_1)}{\gamma_1'(x_1)} \end{pmatrix} \prod_{j=1}^{n+1} \gamma_1'(x_j) =$$

$$\det\begin{pmatrix} \frac{\gamma_2'(x_2)}{\gamma_1'(x_2)} - \frac{\gamma_2'(x_1)}{\gamma_1'(x_1)} & \dots & \frac{\gamma_2'(x_{n+1})}{\gamma_1'(x_{n+1})} - \frac{\gamma_2'(x_1)}{\gamma_1'(x_1)} \\ \vdots & \ddots & \vdots \\ \frac{\gamma_{n+1}'(x_2)}{\gamma_1'(x_2)} - \frac{\gamma_{n+1}'(x_1)}{\gamma_1'(x_1)} & \dots & \frac{\gamma_{n+1}'(x_{n+1})}{\gamma_1'(x_{n+1})} - \frac{\gamma_{n+1}'(x_1)}{\gamma_1'(x_1)} \end{pmatrix} \prod_{j=1}^{n+1} \gamma_1'(x_j) \overset{(*)}{=}$$

$$\det\begin{pmatrix} \frac{\gamma_2'(x_2)}{\gamma_1'(x_2)} - \frac{\gamma_2'(x_1)}{\gamma_1'(x_1)} & \dots & \frac{\gamma_2'(x_{n+1})}{\gamma_1'(x_{n+1})} - \frac{\gamma_2'(x_n)}{\gamma_1'(x_n)} \\ \vdots & \ddots & \vdots \\ \frac{\gamma_{n+1}'(x_2)}{\gamma_1'(x_2)} - \frac{\gamma_{n+1}'(x_1)}{\gamma_1'(x_1)} & \dots & \frac{\gamma_{n+1}'(x_{n+1})}{\gamma_1'(x_{n+1})} - \frac{\gamma_{n+1}'(x_n)}{\gamma_1'(x_n)} \end{pmatrix} \prod_{j=1}^{n+1} \gamma_1'(x_j) =$$

$$\int_{x_1}^{x_2}\int_{x_2}^{x_3}\cdots\int_{x_n}^{x_{n+1}} \det\begin{pmatrix} \left(\frac{\gamma_2'(y_1)}{\gamma_1'(y_1)}\right)' & \dots & \left(\frac{\gamma_2'(y_n)}{\gamma_1'(y_n)}\right)' \\ \vdots & \ddots & \vdots \\ \left(\frac{\gamma_{n+1}'(y_1)}{\gamma_1'(y_1)}\right)' & \dots & \left(\frac{\gamma_{n+1}'(y_n)}{\gamma_1'(y_n)}\right)' \end{pmatrix} dy_1 dy_2 \dots dy_n \prod_{j=1}^{n+1} \gamma_1'(x_j),$$

where in the equality $(*)$ we used the property of the determinant that if $v_1, \dots, v_k$ are column vectors in $\mathbb{R}^k$ then $\det(v_2 - v_1, v_3 - v_1, \dots, v_k - v_1)) = \det(v_2 - v_1, v_3 - v_2, \dots, v_k - v_{k-1})$ by subtracting the columns from each other.

The leading principal minors of the matrix $(\gamma', \gamma'', \dots, \gamma^{(n+1)})$ are positive. In particular γ_1' is positive on $(0,1)$, and hence the factor $\prod_{j=1}^{n+1} \gamma'(x_j) > 0$. To verify (3.1), it suffices to show

$$\det\begin{pmatrix} \left(\frac{\gamma_2'(y_1)}{\gamma_1'(y_1)}\right)' & \dots & \left(\frac{\gamma_2'(y_n)}{\gamma_1'(y_n)}\right)' \\ \vdots & \ddots & \vdots \\ \left(\frac{\gamma_{n+1}'(y_1)}{\gamma_1'(y_1)}\right)' & \dots & \left(\frac{\gamma_{n+1}'(y_n)}{\gamma_1'(y_n)}\right)' \end{pmatrix} > 0 \quad \text{for all} \quad 0 < y_1 < y_2 < \dots < y_n < 1. \tag{3.2}$$

We will repeat the same computation as before but now for the determinant in (3.2), and, eventually, we will see that the proof of the lemma will be just n times the application of the previous computation together with an identity for determinants that we have not described yet.

Before we proceed, let us make a couple of observations. We started with the determinant of $(n+1) \times (n+1)$ matrix. Next, we divided the columns by the entries in the first row, which consist of $\gamma_1' > 0$, and after the Gaussian elimination and the fundamental theorem of calculus we ended up with the integral of the determinant of $n \times n$, and we also acquired the factor $\prod_{j=1}^{n+1} \gamma_1'(x_j) > 0$. To repeat the same computation for the determinant in (3.2) and the ones that we obtain in a similar manner, we should verify that the entries in the first row of all such new matrices (of smaller sizes) are positive. Such entries are changed as follows:

$$\gamma_1' \overset{\text{step 1}}{\mapsto} \left(\frac{\gamma_2'}{\gamma_1'}\right)' \overset{\text{step 2}}{\mapsto} \left(\frac{\left(\frac{\gamma_3'}{\gamma_1'}\right)'}{\left(\frac{\gamma_2'}{\gamma_1'}\right)'}\right)' \overset{\text{step 3}}{\mapsto} \left(\frac{\left(\frac{\left(\frac{\gamma_4'}{\gamma_1'}\right)'}{\left(\frac{\gamma_2'}{\gamma_1'}\right)'}\right)'}{\left(\frac{\left(\frac{\gamma_3'}{\gamma_1'}\right)'}{\left(\frac{\gamma_2'}{\gamma_1'}\right)'}\right)'}\right)' \overset{\text{step 4}}{\mapsto} \dots . \tag{3.3}$$

We claim that after the k'th step, $1 \le k \le n$, the obtained entry is of the form $\frac{\Delta_{k+1}\Delta_{k-1}}{\Delta_k^2}$, where Δ_ℓ denotes the leading $\ell \times \ell$ principal minor of the matrix $(\gamma', \gamma'', \dots, \gamma^{(n+1)})$ (by definition we set $\Delta_0 := 1$). Assuming the claim, Lemma 3.1 follows immediately because of the condition $\Delta_\ell > 0$ on $(0, 1)$ for all $0 \le \ell \le n+1$.

To verify the claim, we set $T = (\gamma', \gamma'', \dots, \gamma^{(n+1)})$. Given subsets $I, J \subset \{1, \dots, n+1\}$, we define $T_{I\times J}$ to be the determinant of the submatrix of T formed by choosing the rows of the index set I and the columns of index set J. We have

$$\left(\frac{\gamma_2'}{\gamma_1'}\right)' = \frac{\gamma_2''\gamma_1' - \gamma_1''\gamma_2'}{\gamma_1'} = \frac{T_{\{1,2\}\times\{1,2\}}}{T^2_{\{1\}\times\{1\}}},$$

$$\left(\frac{\gamma_\ell'}{\gamma_1'}\right)' = \frac{T_{\{1,\ell\}\times\{1,2\}}}{T^2_{\{1\}\times\{1\}}}, \quad \text{for all} \quad \ell \ge 2;$$

$$\left(\frac{\left(\frac{\gamma_\ell'}{\gamma_1'}\right)'}{\left(\frac{\gamma_2'}{\gamma_1'}\right)'}\right)' = \left(\frac{T_{\{1,\ell\}\times\{1,2\}}}{T_{\{1,2\}\times\{1,2\}}}\right)' \overset{(*)}{=} \frac{T_{\{1,\ell\}\times\{1,3\}}T_{\{1,2\}\times\{1,2\}} - T_{\{1,\ell\}\times\{1,2\}}T_{\{1,2\}\times\{1,3\}}}{T^2_{\{1,2\}\times\{1,2\}}}$$

$$\overset{(**)}{=} \frac{T_{\{1,2,\ell\}\times\{1,2,3\}}\, T_{\{1\}\times\{1\}}}{T^2_{\{1,2\}\times\{1,2\}}}, \quad \text{for all} \quad \ell \ge 3, \tag{3.4}$$

where $(*)$ follows from the identity $(T_{I\times\{1,2,\dots,k-1,k\}})' = T_{I\times\{1,2,\dots,k-1,k+1\}}$, and $(**)$ follows from the following general identity for determinants:

$$T_{\{[k-2],\ell\}\times\{[k-2],k\}}T_{[k-1]\times[k-1]} - T_{\{[k-2],\ell\}\times[k-1]}T_{[k-1]\times\{[k-2],k\}} = T_{\{[k-1],\ell\}\times[k]}T_{[k-2]\times[k-2]} \tag{3.5}$$

for all $k, 3 \le k \le n+1$, where we set $[d] := \{1, 2, \dots, d\}$ for a positive integer d. Before we verify the identity (3.5), notice that it also implies

$$\left(\frac{T_{\{[k-2],\ell\}\times[k-1]}}{T_{[k-1]\times[k-1]}}\right)' = \frac{T_{\{[k-2],\ell\}\times\{[k-2],k\}}T_{[k-1]\times[k-1]} - T_{\{[k-2],\ell\}\times[k-1]}T_{[k-1]\times\{[k-2],k\}}}{T^2_{[k-1]\times[k-1]}}$$

$$= \frac{T_{\{[k-1],\ell\}\times[k]}T_{[k-2]\times[k-2]}}{T^2_{[k-1]\times[k-1]}}, \tag{3.6}$$

for all k, ℓ such that $3 \le k \le n+1$ and $k-1 \le \ell \le n+1$. Therefore

$$\left(\frac{\left(\frac{\left(\frac{\gamma_\ell'}{\gamma_1'}\right)'}{\left(\frac{\gamma_2'}{\gamma_1'}\right)'}\right)'}{\left(\frac{\left(\frac{\gamma_3'}{\gamma_1'}\right)'}{\left(\frac{\gamma_2'}{\gamma_1'}\right)'}\right)'}\right)' \overset{(3.4)}{=} \left(\frac{T_{\{1,2,\ell\}\times\{1,2,3\}}}{T_{\{1,2,3\}\times\{1,2,3\}}}\right)' \overset{(3.6)}{=} \frac{T_{\{[3],\ell\}\times[4]}T_{[2]\times[2]}}{T^2_{[3]\times[3]}}.$$

In particular, after step 3, the entry in (3.3) becomes $\frac{T_{[4]\times[4]}T_{[2]\times[2]}}{T^2_{[3]\times[3]}} > 0$ because $T_{[k]\times[k]} = \Delta_k$. It then follows that after step k, the entry in (3.3) takes the form

$$\left(\frac{T_{\{[k-1],k+1\}\times[k]}}{T_{[k]\times[k]}}\right)' \overset{(3.5)}{=} \frac{T_{[k+1]\times[k+1]}T_{[k-1]\times[k-1]}}{T^2_{[k]\times[k]}} = \frac{\Delta_{k+1}\Delta_{k-1}}{\Delta_k} > 0,$$

for all $1 \le k \le n$. Thus the proof of Lemma 3.1 is complete provided that the determinant identity (3.5) is verified. Let Δ be an invertible $(k-2)\times(k-2)$ matrix, $p, w, u, q \in \mathbb{R}^{k-2}$, and let $a, b, c, d \in \mathbb{R}$. To verify ~~the~~ identity (3.5), it suffices to show that

$$\det\begin{pmatrix}\Delta & q^T\\ w & a\end{pmatrix}\det\begin{pmatrix}\Delta & u^T\\ p & b\end{pmatrix} - \det\begin{pmatrix}\Delta & u^T\\ w & c\end{pmatrix}\det\begin{pmatrix}\Delta & q^T\\ p & d\end{pmatrix} = \det\begin{pmatrix}\Delta & u^T & q^T\\ p & b & d\\ w & c & a\end{pmatrix}\det\Delta. \tag{3.7}$$

Since $\det\begin{pmatrix}A & B\\ C & D\end{pmatrix} = \det A\,\det(D - CA^{-1}B)$ for an invertible $m\times m$ matrix A, and arbitrary $n\times n$ matrix D, $n\times m$ matrix B, and $m\times n$ matrix C, we see that (3.7) simplifies to

$$(\det\Delta)^2\left[(a - w\Delta^{-1}q^T)(b - p\Delta^{-1}u^T) - (c - w\Delta^{-1}u^T)(d - p\Delta^{-1}q^T)\right] =$$
$$(\det\Delta)^2\det\left(\begin{pmatrix}b & d\\ c & a\end{pmatrix} - \begin{pmatrix}p\\ w\end{pmatrix}\Delta^{-1}\left(u^T q^T\right)\right),$$

which holds because $\begin{pmatrix}p\\ w\end{pmatrix}\Delta^{-1}\left(u^T q^T\right) = \begin{pmatrix}p\Delta^{-1}u^T & p\Delta^{-1}q^T\\ w\Delta^{-1}u^T & w\Delta^{-1}q^T\end{pmatrix}$. The lemma is proved. □

Corollary 3.2. *Let $a < b$, and let $\beta : [a,b] \to \mathbb{R}^m$ be a curve $\beta \in C([a,b]) \cap C^m((a,b))$ with totally positive torsion. Choose any $a \le z_1 < \ldots < z_m \le b$ and $r \in [0,1] \setminus \{z_1, \ldots, z_m\}$. Then the vectors $\beta(z_1) - \beta(r), \ldots, \beta(z_m) - \beta(r)$ are linearly independent in $\mathbb{R}^m$.*

Proof. Let ν, $0 \le \nu \le m$, be chosen in such a way that $r \in [z_\nu, z_{\nu+1}]$. Here we set $z_0 := a$ ~~,~~ and $z_{m+1} := b$. We have

$$\det(\beta(z_1) - \beta(r), \ldots, \beta(z_m) - \beta(r)) =$$
$$\pm\det(\beta(z_2) - \beta(z_1), \ldots, \beta(r) - \beta(z_\nu), \beta(z_{\nu+1}) - \beta(r), \ldots, \beta(z_m) - \beta(z_{m-1})) =$$
$$\pm\int_{z_{m-1}}^{z_m}\cdots\int_r^{z_{\nu+1}}\int_{z_\nu}^r\cdots\int_{z_1}^{z_2}\det(\beta'(s_1), \ldots, \beta'(s_\nu), \beta'(s_{\nu+1}), \ldots \beta'(s_m))ds_1\ldots ds_m \ne 0$$

by Lemma 3.1. □

Certain parts of the proof of Theorem 2.1 will require induction on the dimension $n+1$. In particular, we will need to verify the base cases when $n = 1$ (the odd case) and $n = 2$ (even case).

In what follows, without loss of generality we assume $[a,b] = [0,1]$, and $\gamma(0) = 0$.

3.1. The ~~proof~~ Proof of Theorem 2.1 in ~~dimension~~ Dimension 1+1. This case is trivial and Theorem 2.1 essentially follows by looking at ~~Fig.~~ Figure 3.

If we reparametrize the curve γ as $\tilde\gamma(t) := \gamma(\gamma_1^{-1}(t))$, $t \in (0, \gamma_1(1))$, then $\tilde\gamma$ has totally positive torsion. So $\tilde\gamma(t) = (t, g(t)), t \in (0, \gamma_1(1))$ where $g(0) = 0$, and $\frac{d^2}{dt^2}g(t) > 0$ for all $t \in (0, \gamma_1(1))$. We have $U_1(\beta_1) = \beta_1\gamma(1), \beta_1 \in [0,1]$ ~~,~~ is the line joining the endpoints of $\tilde\gamma$. Also $L_1(x_1) = \gamma(x_1), x_1 \in [0,1]$ ~~,~~ is the curve coinciding with $\tilde\gamma$. It is easy to see that in this case Theorem 2.1 holds true.

3.2. The ~~proof~~ Proof of Theorem 2.1 in ~~dimension~~ Dimension 2+1.

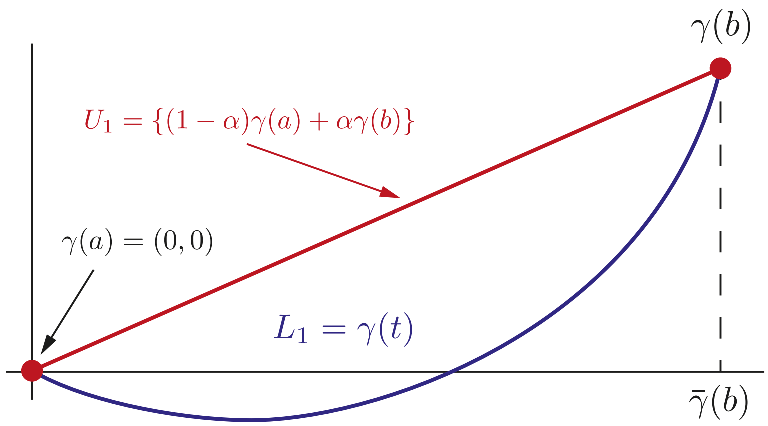

FIGURE 3. Proof of Theorem 2.1 for dimension $n+1 = 1+1$.

3.2.1. *The ~~lower hull~~Lower Hull.* Recall that

$$\overline{L}_2 : \Delta^1_c \times \Delta^1_* = [0,1]^2 \ni (\alpha, x) \mapsto \alpha\overline{\gamma}(x).$$

We claim

$$\overline{L}_2(\partial([0,1]^2)) = \partial(\mathrm{conv}(\overline{\gamma}([0,1]))); \tag{3.8}$$

$$\overline{L}_2 : \mathrm{int}([0,1]^2) \mapsto \mathrm{int}(\mathrm{conv}(\overline{\gamma}([0,1]))) \quad \text{is diffeomorphism.} \tag{3.9}$$

To verify (3.8), it suffices to show that $\overline{\gamma}$ is the convex curve in $\mathbb{R}^2$. Convexity of $\overline{\gamma}$ can be verified in a similar way as in Section 3.1. However, here we present one more proof, which later will be adapted to higher dimensions too. Assume contrary, ~~i.e.~~that is, there exists $0 \le a < b < c \le 1$ such that $\overline{\gamma}(a), \overline{\gamma}(b), \overline{\gamma}(c)$ lie on the same line, ~~i.e.~~that is,

$$0 = \det(\overline{\gamma}(b) - \overline{\gamma}(a), \overline{\gamma}(c) - \overline{\gamma}(b)) = \int_a^b \int_b^c \det(\overline{\gamma}'(y_1), \overline{\gamma}'(y_2)) dy_1 dy_2. \tag{3.10}$$

~~The equation~~ Equation (3.10) is in contradiction with Lemma 3.1 applied to $\overline{\gamma}$.

To verify (3.9), by ~~Hadamard-Caccioppoli~~ Hadamard–Caccioppoli theorem it suffices to check that the differential of $\overline{L} := \overline{L}_2$ at the interior of $[0,1]^2$ has full rank, and the map $\overline{L}_2$ is injection. The injectivity will be verified later in all dimensions simultaneously (see the section on proofs of (2.7), (2.8), (2.4), and (2.5)). To verify the full rank property, we have $D\overline{L} = (\overline{L}_\alpha, \overline{L}_x) = \alpha \det(\overline{\gamma}(x), \overline{\gamma}'(x))$. On the other hand,

$$\det(\overline{\gamma}(x), \overline{\gamma}'(x)) = \int_0^x \det(\overline{\gamma}'(y_1), \overline{\gamma}'(x)) dy_1 \overset{\text{Lemma 3.1}}{>} 0. \tag{3.11}$$

Thus, see ~~Fig.~~Figure 4,

$$L_2 : \Delta^1_c \times \Delta^1_* = [0,1]^2 \ni (\alpha, x) \mapsto \alpha\gamma(x)$$

parametrizes a surface in $\mathbb{R}^3$, which is a graph of a function B^{inf} defined on $\mathrm{conv}(\overline{\gamma}([0,1]))$ as follows:

$$B^{\mathrm{inf}}(\alpha\overline{\gamma}(x)) = \alpha\gamma_3(x), \quad \text{for all} \quad (\alpha, x) \in [0,1]^2.$$

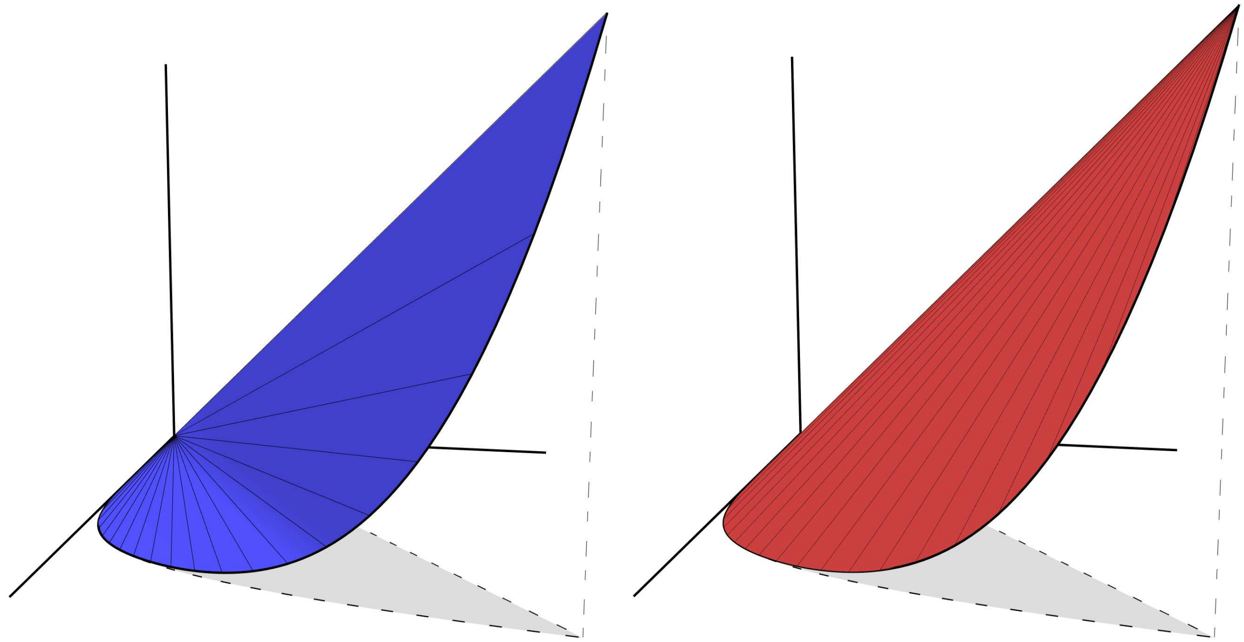

FIGURE 4. Two pieces of the boundary of the convex hull of γ: the lower hull L_2 (left) and the upper hull U_2

Let us check that $B^{\inf}$ is convex. Indeed, at any point $(\alpha_0, x_0) \in \mathrm{int}([0,1]^2)$ the set of points $\xi \in \mathbb{R}^3$ belonging to the tangent plane at point $L_2(\alpha_0, x_0)$ is found as the solution of the equation

$$\det(L_\alpha(\alpha_0, x_0), L_x(\alpha_0, x_0), \xi - L(\alpha_0, x_0)) = \alpha_0 \det(\gamma(x_0), \gamma'(x_0), \xi) = 0. \tag{3.12}$$

For $\xi = e_3$, where $e_3 = (0,0,1)$, we have

$$\det(\gamma(x_0), \gamma'(x_0), e_3) = \det(\overline{\gamma}(x_0), \overline{\gamma}'(x_0)) \overset{(3.11)}{>} 0.$$

Therefore, to verify the convexity of $B^{\inf}$, ~~i.e.~~that is, the surface $L([0,1]^2)$ lies above the tangent plane at point $L(\alpha_0, x_0)$, it suffices to show that

$$\det(\gamma(x_0), \gamma'(x_0), L(\alpha, x)) = \alpha \det(\gamma(x_0), \gamma'(x_0), \gamma(x)) \geq 0.$$

If $x = x_0$, there is nothing to prove. If $x > x_0$ then

$$\det(\gamma(x_0), \gamma'(x_0), \gamma(x)) = \int_0^{x_0} \int_{x_0}^{x} \det(\gamma'(y_1), \gamma'(x_0), \gamma'(y_3)) dy_1 dy_3 \overset{\text{Lemma 3.1}}{>} 0.$$

Similarly, if $x < x_0$, by Lemma 3.1 we have

$$\det(\gamma(x_0), \gamma'(x_0), \gamma(x)) = \int_x^{x_0} \int_0^{x} \det(\gamma'(y_1), \gamma'(x_0), \gamma'(y_3)) dy_1 dy_3 > 0.$$

To verify that $B^{\inf}$ is the maximal convex function defined on $\mathrm{conv}(\overline{\gamma}([0,1]))$ such that $B(\overline{\gamma}(s)) = \gamma_3(s)$, notice that since every point $(\xi, B^{\inf}(\xi))$, where $\xi \in \mathrm{conv}(\overline{\gamma}([0,1]))$, is the convex combination of some points of γ, it follows that any other candidate $\tilde{B}$ would be smaller than B by convexity.

3.2.2. *The ~~upper hull~~Upper Hull.* Consider the map

$$\overline{U}_2 : \Delta_c^1 \times \Delta_*^1 = [0,1]^2 \ni (\alpha, x) \mapsto \alpha \overline{\gamma}(x) + (1-\alpha)\overline{\gamma}(1).$$

Similarly as before, Φ satisfies (3.8) and (3.9). ~~The property~~ Property (3.8) follows from ~~from~~ the convexity of $\overline{\gamma}$. ~~The property~~ Property (3.9) follows from

$$\det(\overline{U}_\alpha, \overline{U}_x) = \alpha \det(\overline{\gamma}(x) - \overline{\gamma}(1), \overline{\gamma}'(x)) = \int_x^1 \det(\overline{\gamma}'(x), \overline{\gamma}'(y_2))dy_2 \neq 0$$

for all $(\alpha, x) \in \operatorname{int}([0,1]^2)$ by Lemma 3.1 applied to $\overline{\gamma}$.

Next, we show that

$$B^{\sup}(\alpha\overline{\gamma}(x) + (1-\alpha)\overline{\gamma}(1)) = \alpha\gamma_3(x) + (1-\alpha)\gamma_3(1)$$

defines a minimal concave function on $\operatorname{conv}(\overline{\gamma}([0,1]))$ with the property $B^{\sup}(\overline{\gamma}) = \gamma_3$, see ~~Fig~~ Figure 3. Let $U(\alpha, x) = \alpha\gamma(x) + (1-\alpha)\gamma(1)$. The equation of the tangent plane at point $U(\alpha_0, x_0)$, where $(\alpha_0, x_0) \in \operatorname{int}([0,1]^2)$, is given by

$$\det(U_\alpha(\alpha_0,x_0), U_x(\alpha_0,x_0), \xi - U(\alpha_0,x_0)) = \alpha_0 \det(\gamma(x_0)-\gamma(1), \gamma'(x_0), \xi - \alpha_0(\gamma(x_0)-\gamma(1)) - \gamma(1))$$
$$= \alpha_0 \det(\gamma(x_0)-\gamma(1), \gamma'(x_0), \xi - \gamma(1)) = 0.$$

For $\xi = \lambda e_3$ with $\lambda \to +\infty$, we have

$$\operatorname{sign}[\det(\gamma(x_0)-\gamma(1), \gamma'(x_0), \lambda e_3 - \gamma(1))] = \operatorname{sign}[\det(\overline{\gamma}(x_0)-\overline{\gamma}(1), \overline{\gamma}'(x_0)]$$
$$= \operatorname{sign}\left[\int_{x_0}^1 \det(\overline{\gamma}'(x_0), \overline{\gamma}(y_2))dy_2\right] > 0$$

by Lemma 3.1 applied to $\overline{\gamma}$. Therefore, the concavity of $B^{\sup}$ would follow from the following inequality:

$$\det(\gamma(x_0)-\gamma(1), \gamma'(x_0), U(\alpha,x)-\gamma(1)) = \alpha \det(\gamma(x_0)-\gamma(1), \gamma'(x_0), \gamma(x)-\gamma(1)) \le 0$$

for all $x_0, \alpha, x \in [0,1]$. If $x = x_0$, there is nothing to prove. Consider $x > x_0$ (the case $x < x_0$ is similar). Then

$$\det(\gamma(x_0)-\gamma(1), \gamma'(x_0), \gamma(x)-\gamma(1)) = \det(\gamma(x_0)-\gamma(1), \gamma'(x_0), \gamma(x)-\gamma(x_0)) =$$
$$-\det(\gamma(x_0)-\gamma(x), \gamma'(x_0), \gamma(1)-\gamma(x_0)) = -\int_x^{x_0}\int_{x_0}^1 \det(\gamma'(y_1), \gamma'(x_0), \gamma'(y_2))dy_2dy_1 < 0$$

by Lemma 3.1.

~~The properties~~ Properties (2.12) and (2.13) will be verified in ~~sections~~ Sections 3.3.3 and 3.3.4.

3.3. The ~~proof~~ Proof of Theorem 2.1 in an ~~arbitrary dimension~~ Arbitrary Dimension $n+1$.

Proof. Since Theorem 2.1 contains several statements, the whole proof will be split into several parts.

The proof of claims (2.6) and (2.3).

The proof will be by induction on n. We have checked the statement for $n = 1, 2$. First we consider the case when $n = 2\ell - 1$. We shall verify the claim (2.6) by showing that $\overline{U}_{2\ell-1}|_{\partial(\Delta_c^\ell \times \Delta_*^{\ell-1})}$, ~~i.e.~~ that is, the restriction of $\overline{U}_{2\ell-1}$ on $\partial(\Delta_c^\ell \times \Delta_*^{\ell-1})$ ~~,~~ coincides with maps $U_{2\ell-2}$ and $L_{2\ell-2}$ (similarly for $\overline{L}_{2\ell-1}|_{\partial(\Delta_c^{\ell-1} \times \Delta_*^\ell)}$). Since by the induction the union of the images of $U_{2\ell-2}$ and $L_{2\ell-2}$ coincides with the boundary of the convex hull of $\overline{\gamma}([0,1])$, see (2.13), we obtain the claim.

Recall that

$$\overline{U}_{2\ell-1} : \Delta_c^\ell \times \Delta_*^{\ell-1} \ni (\beta_1, \ldots, \beta_\ell, y_2, \ldots, y_\ell) \mapsto \beta_1\overline{\gamma}(1) + \sum_{j=2}^{\ell} \beta_j \overline{\gamma}(y_j),$$

and

$$U_{2\ell-2} : \Delta_c^{\ell-1} \times \Delta_*^{\ell-1} \ni (\lambda_1, \dots, \lambda_{\ell-1}, x_1, \dots, x_{\ell-1}) \mapsto \sum_{j=1}^{\ell-1} \lambda_j \overline{\gamma}(x_j) + (1 - \sum_{j=1}^{\ell-1} \lambda_j)\overline{\gamma}(1),$$

$$L_{2\ell-2} : \Delta_c^{\ell-1} \times \Delta_*^{\ell-1} \ni (\lambda_1, \dots, \lambda_{\ell-1}, z_1, \dots, z_{\ell-1}) \mapsto \sum_{j=1}^{\ell-1} \lambda_j \overline{\gamma}(z_j).$$

If $\beta_1 = 0$ then $\overline{U}_{2\ell-1}$ coincides with $L_{2\ell-2}$. If $\sum_{j=1}^{n} \beta_j = 1$, ~~i.e.~~that is, $\beta_1 = 1 - \sum_{j=2}^{\ell} \beta_j$, then $\overline{U}_{2\ell-1}$ coincides with $U_{2\ell-2}$. Thus, we have

$$\partial \operatorname{conv}(\overline{\gamma}([0,1])) \overset{\text{induction}}{=} U_{2\ell-2}(\Delta_c^{\ell-1} \times \Delta_*^{\ell-1}) \cup L_{2\ell-2}(\Delta_c^{\ell-1} \times \Delta_*^{\ell-1}) \subset \overline{U}_{2\ell-1}(\partial\,(\Delta_c^{\ell} \times \Delta_*^{\ell-1})).$$

On the other hand, if $\beta_p = 0$ for some $p \in \{2, \dots, \ell\}$, then $\overline{U}_{2\ell-1}$ coincides with $L_{2\ell-2}$ restricted to $z_1 = 1$. If at least one of the following conditions hold: a) $y_2 = 0$; b) $y_s = y_{s+1}$ for some $s \in \{2, \dots, \ell-1\}$; c) $y_\ell = 1$, then $\overline{U}_{2\ell-1}$ coincides with $U_{2\ell-2}$ restricted to $x_1 = 0$. Thus we obtain $\partial \operatorname{conv}(\overline{\gamma}([0,1])) = \overline{U}_{2\ell-1}(\partial\,(\Delta_c^{\ell} \times \Delta_*^{\ell-1}))$.

Next, we verify that $\partial \operatorname{conv}(\overline{\gamma}([0,1])) = \overline{L}_{2\ell-1}(\partial\,(\Delta_c^{\ell-1} \times \Delta_*^{\ell}))$. We recall

$$\overline{L}_{2\ell-1} : \Delta_c^{\ell-1} \times \Delta_*^{\ell} \ni (\beta_2, \dots, \beta_\ell, y_1, \dots, y_\ell) \mapsto \sum_{j=2}^{\ell} \beta_j \overline{\gamma}(y_j) + (1 - \sum_{j=2}^{\ell} \beta_j)\overline{\gamma}(y_1).$$

If $y_\ell = 1$ then $\overline{L}_{2\ell-1}$ coincides with $U_{2\ell-2}$. If $y_1 = 0$ then $\overline{L}_{2\ell-1}$ coincides with $L_{2\ell-2}$. Thus, by induction $\partial \operatorname{conv}(\overline{\gamma}([0,1])) \subset \overline{L}_{2\ell-1}(\partial\,(\Delta_c^{\ell-1} \times \Delta_*^{\ell}))$.

Next, if $y_s = y_{s+1}$ for some $s \in \{1, \dots, \ell-1\}$, then $\overline{L}_{2\ell-1}$ coincides with $L_{2\ell-2}$ restricted to $\lambda_1 = 1 - \sum_{j=2}^{\ell-1} \lambda_j$. Also, if $\sum_{j=2}^{\ell} \beta_j = 1$ then $\overline{L}_{2\ell-1}$ coincides with $L_{2\ell-2}$. Finally, if $\beta_s = 0$ for some $s \in \{2, \dots, \ell\}$, then $\overline{L}_{2\ell-1}$ coincides with $L_{2\ell-2}$ restricted to $\sum_{j=1}^{\ell-1} \lambda_j = 1$. Thus we obtain $\partial \operatorname{conv}(\overline{\gamma}([0,1])) = \overline{L}_{2\ell-1}(\partial\,(\Delta_c^{\ell-1} \times \Delta_*^{\ell}))$.

Next, we assume $n = 2\ell$. First we verify (2.3). As before, we claim that the restriction of $\overline{U}_{2\ell}$ on $\partial(\Delta_c^{\ell} \times \Delta_*^{\ell})$ coincides with maps $U_{2\ell-1}$ and $L_{2\ell-1}$ (similarly for $\overline{L}_{2\ell}$). Since by ~~the~~ induction the union of the images of $U_{2\ell-1}$ and $L_{2\ell-1}$ coincide with the boundary of the convex hull of $\overline{\gamma}([0,1])$, see (2.13), we obtain the claim.

We recall that

$$\overline{U}_{2\ell} : \Delta_c^{\ell} \times \Delta_*^{\ell} \ni (\lambda_1, \dots, \lambda_\ell, x_1, \dots, x_\ell) \mapsto \sum_{j=1}^{\ell} \lambda_j \overline{\gamma}(x_j) + (1 - \sum_{j=1}^{\ell} \lambda_j)\overline{\gamma}(1);$$

and

$$U_{2\ell-1} : \Delta_c^{\ell} \times \Delta_*^{\ell-1} \ni (\beta_1, \dots, \beta_\ell, y_2, \dots, y_\ell) \mapsto \beta_1 \overline{\gamma}(1) + \sum_{j=2}^{\ell} \beta_j \overline{\gamma}(y_j);$$

$$L_{2\ell-1} : \Delta_c^{\ell-1} \times \Delta_*^{\ell} \ni (\beta_2, \dots, \beta_\ell, z_1, \dots, z_\ell) \mapsto (1 - \sum_{j=2}^{\ell} \beta_j)\overline{\gamma}(z_1) + \sum_{j=2}^{\ell} \beta_j \overline{\gamma}(z_j).$$

Notice that if $\sum_{j=1}^{\ell} \lambda_j = 1$ then $\overline{U}_{2\ell}$ coincides with $L_{2\ell-1}$. On the other hand, if $x_1 = 0$ then $\overline{U}_{2\ell}$ coincides with $U_{2\ell-1}$. Thus, by induction we have $\partial \operatorname{conv}(\overline{\gamma}([0,1])) \subset \overline{U}_{2\ell}(\partial\,(\Delta_c^{\ell} \times \Delta_*^{\ell}))$. Also notice that if $\lambda_p = 0$ for some $p \in \{1, \dots, \ell\}$ (or $x_s = x_{s+1}$ for some $s \in \{1, \dots, \ell-1\}$, or $x_\ell = 1$) then $\overline{U}_{2\ell}$ coincides with $U_{2\ell-1}$ restricted to the boundary of $\Delta_c^{\ell-1} \times \Delta_*^{\ell}$ (if $\lambda_p = 0$ or $x_\ell = 1$ take $\beta_1 = 1 - \sum_{j=2}^{\ell} \beta_j$, if $x_s = x_{s+1}$ take $\beta_1 = 1 - \sum_{j=2}^{\ell} \beta_j$). Thus we obtain $\partial \operatorname{conv}(\overline{\gamma}([0,1])) = \overline{U}_{2\ell}(\partial\,(\Delta_c^{\ell} \times \Delta_*^{\ell}))$.

Next, we verify the claim $\overline{L}_{2\ell}(\partial\,(\Delta_c^\ell \times \Delta_*^\ell)) = \partial\,\mathrm{conv}(\overline{\gamma}([0,1]))$. We recall that

$$\overline{L}_{2\ell} : \Delta_c^\ell \times \Delta_*^\ell \ni (\lambda_1, \ldots, \lambda_\ell, x_1, \ldots, x_\ell) \mapsto \sum_{j=1}^{\ell} \lambda_j \overline{\gamma}(x_j),.$$

If $\sum_{j=1}^{\ell} \lambda_j = 1$ then $\overline{L}_{2\ell}$ coincides with $L_{2\ell-1}$. If $x_\ell = 1$ then $\overline{L}_{2\ell}$ coincides with $U_{2\ell-1}$. Thus by induction we have $\partial\,\mathrm{conv}(\overline{\gamma}([0,1])) \subset \overline{L}_{2\ell}(\partial\,(\Delta_c^\ell \times \Delta_*^\ell))$ $\partial\,\mathrm{conv}(\overline{\gamma}([0,1])) \subset \overline{L}_{2\ell}(\partial\,(\Delta_c^\ell \times \Delta_*^\ell))$.

If $\lambda_p = 0$ for some $p \in \{1, \ldots, \ell\}$, or $x_1 = 0$, then $\overline{L}_{2\ell}$ coincides with $U_{2\ell-1}$ if we choose $\beta_1 = 0$. Finally, if $x_s = x_{s+1}$ for some $s \in \{1, \ldots, \ell-1\}$, then $\overline{L}_{2\ell}$ coincides with $U_{2\ell-1}$ if we choose $\beta_1 = 0$, and $\beta_{s+1} = \lambda_s + \lambda_{s+1}$. Therefore, we have $\overline{L}_{2\ell}(\partial\,(\Delta_c^\ell \times \Delta_*^\ell)) \subset \partial\,\mathrm{conv}(\overline{\gamma}([0,1]))$, and the claim (2.3) is verified.

The proof of claims (2.7), (2.8), (2.4), and (2.5).

We start by showing that the Jacobian of the map $\overline{U}_n$ has full rank at the interior points of its domain. Hence the map is local diffeomorphism by the inverse function theorem. Therefore, the map is surjective, otherwise the image of its domain would have a boundary in the interior of the codomain (boundary goes to boundary by (2.3) and (2.6)) and this would contradict the local diffeomoerphismdiffeomorphism. Next, we show that the map $\overline{U}_n$ is injective, and hence proper. So we conclude that $\overline{U}_n$ is diffeomorphism. Similar reasoning will be done for $\overline{L}_n$.

First we verify that the Jacobian matrices $\nabla\overline{U}_n$ and $\nabla\overline{L}_n$ have full rank at the interior points of their domains.

Assume $n = 2\ell - 1$. We have

$$\begin{aligned}
&\det(\nabla\overline{U}_{2\ell-1}) = \det(\overline{\gamma}(1), \overline{\gamma}(x_2), \ldots, \overline{\gamma}(x_\ell), \beta_2\overline{\gamma}'(x_2), \ldots, \beta_\ell\overline{\gamma}'(x_\ell)) \\
&= \pm\det(\overline{\gamma}(x_2), \overline{\gamma}'(x_2), \overline{\gamma}(x_3), \overline{\gamma}'(x_3), \ldots, \overline{\gamma}(x_\ell), \overline{\gamma}'(x_\ell), \overline{\gamma}(1)) \prod_{j=2}^{\ell} \beta_j \\
&= \pm\det(\overline{\gamma}(x_2) - \overline{\gamma}(0), \overline{\gamma}'(x_2), \overline{\gamma}(x_3) - \overline{\gamma}(x_2), \overline{\gamma}'(x_3), \ldots, \overline{\gamma}(x_\ell) - \overline{\gamma}(x_{\ell-1}), \overline{\gamma}'(x_\ell), \overline{\gamma}(1) - \overline{\gamma}(x_\ell)) \prod_{j=2}^{\ell} \beta_j \\
&= \pm \prod_{j=2}^{\ell} \beta_j \int_{x_\ell}^{1} \cdots \int_{x_2}^{x_3} \int_0^{x_2} \det(\overline{\gamma}'(s_1), \overline{\gamma}'(x_2), \overline{\gamma}'(s_2), \ldots, \overline{\gamma}'(x_\ell), \overline{\gamma}'(s_\ell)) ds_1 ds_2 \ldots ds_\ell.
\end{aligned}$$

Thus $\det(\nabla\overline{U}_{2\ell-1})$ is nonzero by Lemma 3.1.

Next, we verify that $\det(\nabla \overline{L}_{2\ell-1}) \neq 0$. Indeed,

$$\det(\nabla \overline{L}_{2\ell-1}) =$$

$$\det(\overline{\gamma}(x_2)-\overline{\gamma}(x_1), \overline{\gamma}(x_3)-\overline{\gamma}(x_1), \dots, \overline{\gamma}(x_\ell)-\overline{\gamma}(x_1), \overline{\gamma}'(x_1), \dots, \overline{\gamma}'(x_\ell))(1-\sum_{j=2}^{\ell}\beta_j)\prod_{j=2}^{\ell}\beta_j =$$

$$\det(\overline{\gamma}(x_2)-\overline{\gamma}(x_1), \overline{\gamma}(x_3)-\overline{\gamma}(x_2), \dots, \overline{\gamma}(x_\ell)-\overline{\gamma}(x_{\ell-1}), \overline{\gamma}'(x_1), \dots, \overline{\gamma}'(x_\ell))(1-\sum_{j=2}^{\ell}\beta_j)\prod_{j=2}^{\ell}\beta_j =$$

$$\pm\det(\overline{\gamma}'(x_1), \overline{\gamma}(x_2)-\overline{\gamma}(x_1), \overline{\gamma}'(x_2), \overline{\gamma}(x_3)-\overline{\gamma}(x_2), \dots, \overline{\gamma}(x_\ell)-\overline{\gamma}(x_{\ell-1}), \overline{\gamma}'(x_\ell))(1-\sum_{j=2}^{\ell}\beta_j)\prod_{j=2}^{\ell}\beta_j =$$

$$\pm(1-\sum_{j=2}^{\ell}\beta_j)\prod_{j=2}^{\ell}\beta_j \times$$

$$\int_{x_{\ell-1}}^{x_\ell}\dots\int_{x_2}^{x_3}\int_{x_1}^{x_2}\det(\overline{\gamma}'(x_1), \overline{\gamma}'(s_1), \overline{\gamma}'(x_2), \overline{\gamma}'(s_2), \dots, \overline{\gamma}'(s_{\ell-1}), \overline{\gamma}'(x_\ell))ds_1 ds_2 \dots ds_{\ell-1} \neq 0$$

by Lemma 3.1.

Assume $n = 2\ell$. We have

$$\det(\nabla \overline{U}_{2\ell}) = \det(\overline{\gamma}(x_1)-\overline{\gamma}(1), \dots, \overline{\gamma}(x_\ell)-\overline{\gamma}(1), \overline{\gamma}'(x_1), \dots, \overline{\gamma}'(x_\ell))\prod_{j=1}^{\ell}\lambda_j =$$

$$\pm\det(\overline{\gamma}'(x_1), \overline{\gamma}(x_1)-\overline{\gamma}(x_2), \overline{\gamma}'(x_2), \overline{\gamma}(x_2)-\overline{\gamma}(x_3), \dots, \overline{\gamma}'(x_\ell), \overline{\gamma}(x_\ell)-\overline{\gamma}(1))\prod_{j=1}^{\ell}\lambda_j =$$

$$\pm\int_{x_\ell}^{1}\dots\int_{x_2}^{x_3}\int_{x_1}^{x_2}\det(\overline{\gamma}'(x_1), \overline{\gamma}'(s_1), \overline{\gamma}'(x_2), \overline{\gamma}'(s_2), \dots, \overline{\gamma}'(x_\ell), \overline{\gamma}'(s_\ell))ds_1 ds_2 \dots ds_\ell \prod_{j=1}^{\ell}\lambda_j,$$

which is nonzero by Lemma 3.1.

Finally, we verify $\det(\nabla \overline{L}_{2\ell}) \neq 0$. We have

$$\det(\nabla \overline{L}_{2\ell}) = \det(\overline{\gamma}(x_1), \dots, \overline{\gamma}(x_\ell), \overline{\gamma}'(x_1), \dots, \overline{\gamma}'(x_\ell))\prod_{j=1}^{\ell}\lambda_j =$$

$$\pm\det(\overline{\gamma}(x_1)-\overline{\gamma}(0), \overline{\gamma}'(x_1), \overline{\gamma}(x_2)-\overline{\gamma}(x_1), \overline{\gamma}'(x_2), \dots, \overline{\gamma}(x_\ell)-\overline{\gamma}(x_{\ell-1}), \overline{\gamma}'(x_\ell))\prod_{j=1}^{\ell}\lambda_j =$$

$$\pm\int_{x_{\ell-1}}^{x_\ell}\dots\int_{x_1}^{x_2}\int_{0}^{x_1}\det(\overline{\gamma}'(s_1), \overline{\gamma}'(x_1), \overline{\gamma}'(s_2), \overline{\gamma}'(x_2), \dots, \overline{\gamma}'(s_\ell), \overline{\gamma}'(x_\ell))ds_1 ds_2 \dots ds_\ell \prod_{j=1}^{\ell}\lambda_j.$$

Thus $\det(\nabla \overline{L}_{2\ell}) \neq 0$ by Lemma 3.1.

Next, we show that the map $\overline{U}_n$ is injective in the interior of its domain. Assume $n = 2\ell$. Let $(\lambda_1, \dots, \lambda_\ell, x_1, \dots, x_\ell)$ and $(\beta_1, \dots, \beta_\ell, y_1, \dots, y_\ell)$ be two different points in $\mathrm{int}(\Delta_c^\ell \times \Delta_*^\ell)$ such that $\overline{U}_\ell$ takes the same values on these points. Then

$$\sum_{j=1}^{\ell}\lambda_j(\overline{\gamma}(x_j)-\overline{\gamma}(1)) - \sum_{k=1}^{\ell}\beta_k(\overline{\gamma}(y_k)-\overline{\gamma}(1)) = 0. \tag{3.13}$$

We claim that (3.13) holds if and only if $x_j = y_j$ and $\lambda_j = \beta_j$ for all $j = 1, \dots, \ell$. Indeed, we need the following lemma.

Lemma 3.3. *For any numbers z_j, $1 \le j \le 2\ell$, such that $0 < z_1 < z_2 < \ldots < z_{2\ell} \le 1$, and any $r \in [0,1] \setminus \{z_1, \ldots, z_{2\ell}\}$, the vectors $\overline{\gamma}(z_1) - \overline{\gamma}(r), \ldots, \overline{\gamma}(z_{2\ell}) - \overline{\gamma}(r)$ are linearly independent in $\mathbb{R}^{2\ell}$.*

Proof. The lemma follows from Corollary 3.2 applied to $\beta = \overline{\gamma}$. □

Let N be the cardinality of the set $Q = \{x_1, \ldots, x_\ell\} \cap \{y_1, \ldots, y_\ell\}$. If $N = \ell$ then necessarily $x_j = y_j$ for all $j = 1, \ldots, \ell$, and ~~the~~ equation (3.13) combined with Lemma 3.3 implies that $\lambda_j = \beta_j$ for all $j = 1, \ldots, \ell$. Therefore, assume $N < \ell$. Then we can split the sum (3.13) into the sum of ~~3~~ three terms: the sum of $\lambda_j(\overline{\gamma}(x_j) - \overline{\gamma}(1))$ where $x_j \notin Q$; the sum $(\lambda_j - \beta_{i_j})(\overline{\gamma}(x_j) - \overline{\gamma}(1))$ where $x_j \in Q$; and the sum $\beta_j(\overline{\gamma}(y_j) - \overline{\gamma}(1))$ where $y_j \notin Q$. Since β_j and λ_j cannot be zero, then applying Lemma 3.3 with $r = 1$ we get a contradiction.

Next, we verify the injectivity of $\overline{L}_{2\ell}$ on the interior of its domain. Let $(\lambda_1, \ldots, \lambda_\ell, x_1, \ldots, x_\ell)$ and $(\beta_1, \ldots, \beta_\ell, y_1, \ldots, y_\ell)$ belong to $\operatorname{int}(\Delta_c^\ell \times \Delta_*^\ell)$ and satisfy

$$\sum_{j=1}^{\ell} \lambda_j \overline{\gamma}(x_j) - \sum_{k=1}^{\ell} \beta_k \overline{\gamma}(y_k) = 0.$$

By applying Lemma 3.3 with $r = 0$ and invoking the set Q as before, we obtain $x_j = y_j$, $\lambda_j = \beta_j$ for all $j = 1, \ldots, \ell$.

Assume $n = 2\ell - 1$. To verify the injectivity of $\overline{U}_{2\ell-1}$ on the interior of $\Delta_c^\ell \times \Delta_*^{\ell-1}$, we pick points $(\lambda_1, \ldots, \lambda_\ell, x_2, \ldots, x_\ell)$ and $(\beta_1, \ldots, \beta_\ell, y_2, \ldots, y_\ell)$ from $\operatorname{int}(\Delta_c^\ell \times \Delta_*^{\ell-1})$, and we assume

$$(\lambda_1 - \beta_1)\overline{\gamma}(1) + \sum_{j=2}^{\ell} \lambda_j \overline{\gamma}(x_j) - \sum_{j=2}^{\ell} \beta_j \overline{\gamma}(y_j) = 0. \tag{3.14}$$

Lemma 3.4. *For any numbers $0 < z_1 < \ldots < z_{2\ell-2} < 1$, the vectors $\overline{\gamma}(z_1), \ldots, \overline{\gamma}(z_{2\ell-2}), \overline{\gamma}(1)$ are linearly independent in $\mathbb{R}^{2\ell-1}$.*

Proof. The lemma follows from Corollary 3.2 applied to $\beta = \overline{\gamma}$, $z_{2\ell-1} = 1$, and $r = 0$. □

Invoking the set Q ~~,~~ and repeating the same reasoning as in the case of injectivity of $\overline{U}_{2\ell}$, we see that ~~the~~ equality (3.14) combined with Lemma 3.4 implies $x_j = y_j$ for all $j = 2, \ldots, \ell$, and $\lambda_j = \beta_j$ for all $j = 1, \ldots, \ell$.

To verify the injectivity of $\overline{L}_{2\ell-1}$ on the interior of $\Delta_c^{\ell-1} \times \Delta_*^\ell$, we pick points $(\lambda_2, \ldots, \lambda_\ell, x_1, \ldots, x_\ell)$ and $(\beta_2, \ldots, \beta_\ell, y_1, \ldots, y_\ell)$ from $\operatorname{int}(\Delta_c^{\ell-1} \times \Delta_*^\ell)$, and we assume

$$(1 - \sum_{j=2}^{\ell} \lambda_j)\overline{\gamma}(x_1) + \sum_{j=2}^{\ell} \lambda_j \overline{\gamma}(x_j) = (1 - \sum_{j=2}^{\ell} \beta_j)\overline{\gamma}(y_1) + \sum_{j=2}^{\ell} \beta_j \overline{\gamma}(y_j). \tag{3.15}$$

Without loss of generality, assume $y_1 \le x_1$. We rewrite (3.15) as follows:

$$(1 - \sum_{j=2}^{\ell} \lambda_j)(\overline{\gamma}(x_1) - \overline{\gamma}(y_1)) + \sum_{j=2}^{\ell} \lambda_j(\overline{\gamma}(x_j) - \overline{\gamma}(y_1)) - \sum_{j=2}^{\ell} \beta_j(\overline{\gamma}(y_j) - \overline{\gamma}(y_1)) = 0. \tag{3.16}$$

Notice that if the points $x_1, \ldots, x_\ell, y_1, \ldots, x_\ell$ are different from each other, and they belong to the interval $(0,1)$, then the vectors $\overline{\gamma}(x_1) - \overline{\gamma}(y_1), \ldots, \overline{\gamma}(x_\ell) - \overline{\gamma}(y_1), \overline{\gamma}(y_2) - \overline{\gamma}(y_1), \ldots, \overline{\gamma}(y_\ell) - \overline{\gamma}(y_1)$ are linearly independent. The proof of the linear independence proceeds absolutely in the same way as the proof of Lemma 3.3, therefore we omit the proof to avoid the repetitions. Let $Q =$

$\{x_2,\ldots,x_\ell\}\cap\{y_2,\ldots,y_\ell\}$, $X=\{x_2,\ldots,x_\ell\}$, and $Y=\{y_2,\ldots,y_\ell\}$. Then (3.16) takes the form

$$\begin{aligned}&(1-\sum_{j=2}^{\ell}\lambda_j)(\overline{\gamma}(x_1)-\overline{\gamma}(y_1))+\sum_{j\,:\,x_j\in X\setminus Q}\lambda_j(\overline{\gamma}(x_j)-\overline{\gamma}(y_1))+\\ &\sum_{j\,:\,x_j\in Q}(\lambda_j-\beta_{k_j})(\overline{\gamma}(x_j)-\overline{\gamma}(y_1))-\sum_{j\,:\,y_j\in Y\setminus Q}\beta_j(\overline{\gamma}(y_j)-\overline{\gamma}(y_1))=0.\end{aligned}\tag{3.17}$$

If $y_1<x_1$ then from the linear independence we obtain that $x_j=y_j$ for all $j=1,\ldots,\ell$, and $\lambda_j=\beta_j$ for all $j=2,\ldots,\ell$. In what follows we assume $y_1<x_1$.

Notice that if for any $y\in Y\setminus Q$ we have $y\neq x_1$, then (3.17) contradicts ~~to~~ the linear independence. On the other hand, if for some $y_{j^*}\in Y\setminus Q$ we have $y_{j^*}=x_1$ (we remark that there can be only one such y_{j^*} in $Y\setminus Q$, moreover, $y_{j^*}\notin Q$) then (3.17) we can rewrite as

$$\begin{aligned}&(1-\beta_j^*-\sum_{j=2}^{\ell}\lambda_j)(\overline{\gamma}(x_1)-\overline{\gamma}(y_1))+\sum_{j\,:\,x_j\in X\setminus Q}\lambda_j(\overline{\gamma}(x_j)-\overline{\gamma}(y_1))+\\ &\sum_{j\,:\,x_j\in Q}(\lambda_j-\beta_{k_j})(\overline{\gamma}(x_j)-\overline{\gamma}(y_1))-\sum_{j\,:\,y_j\in Y\setminus Q,\,y_j\neq y_{j^*}}\beta_j(\overline{\gamma}(y_j)-\overline{\gamma}(y_1))=0.\end{aligned}\tag{3.18}$$

Invoking the linear independence, we must have $1-\beta_j^*-\sum_{j=2}^{\ell}\lambda_j=0$. Since $\lambda_j,\beta_j>0$, we have $X\setminus Q$ and $Y\setminus(Q\cup\{y_{j^*}\})$ are empty. Then Q has cardinality $\ell-1$ and Q does not contain y_{j^*}, which is a contradiction.

3.3.1. *The ~~proof~~ Proof of (2.9).* Assume $n=2\ell$. Since $\overline{U}_n$ and $\overline{L}_n$ are diffeomorphisms between $\mathrm{int}(\Delta_c^\ell\times\Delta_*^\ell)$ and $\mathrm{int}(\mathrm{conv}(\overline{\gamma}([0,1])))$, we see that the equations

$$B^{\sup}(\overline{U}(t))=U^z(t),\tag{3.19}$$

$$B^{\inf}(\overline{L}(t))=L^z(t)\tag{3.20}$$

for all $t\in\mathrm{int}(\Delta_c^\ell\times\Delta_*^\ell)$ define functions $B^{\sup}$ and $B^{\inf}$ uniquely on $\mathrm{int}(\mathrm{conv}(\overline{\gamma}([0,1])))$. We would like to extend the definitions of $B^{\sup}$ and $B^{\inf}$ to the boundary of $\mathrm{conv}(\overline{\gamma}([0,1]))$ just by taking $t\in\partial(\Delta_c^\ell\times\Delta_*^\ell)$ in (3.19) and (3.20). To make sure that the choice $t\in\partial(\Delta_c^\ell\times\Delta_*^\ell)$ in (3.19) defines $B^{\sup}$ (and $B^{\inf}$) uniquely and continuously on $\mathrm{conv}(\overline{\gamma}([0,1]))$, we shall verify the following.

Lemma 3.5. *If $\overline{U}(t_1)=\overline{U}(t_2)$ for some $t_1,t_2\in\Delta_c^\ell\times\Delta_*^\ell$, then $U^z(t_1)=U^z(t_2)$. Similarly, if $\overline{L}(t_1)=\overline{L}(t_2)$ for some $t_1,t_2\in\Delta_c^\ell\times\Delta_*^\ell$, then $L^z(t_1)=L^z(t_2)$.*

Proof. Without loss of generality we can assume that $t_1,t_2\in\partial(\Delta_c^\ell\times\Delta_*^\ell)$; otherwise the lemma follows from (2.3), (2.4), and (2.5).

First we show that $\overline{L}(t_1)=\overline{L}(t_2)$ for some $t_1,t_2\in\partial(\Delta_c^\ell\times\Delta_*^\ell)$ implies $L^z(t_1)=L^z(t_2)$. If $t_1=t_2$, there is nothing to prove~~,~~; therefore, we assume $t_1\neq t_2$. For $t_1=(\lambda_1,\ldots,\lambda_\ell,x_1,\ldots,x_\ell)\in\partial(\Delta_c^\ell\times\Delta_*^\ell)$, we have

$$\overline{L}_{2\ell}(t_1)=\sum_{j=1}^{\ell}\lambda_j\overline{\gamma}(x_j).$$

Among $\lambda_1,\ldots,\lambda_\ell$ many of them can be zero, so we reduce the sum into $\sum_{j=1}^{\ell_1}\lambda_{q_j}\overline{\gamma}(x_{q_j})$ where $\lambda_{q_j}>0$, $\ell_1\le\ell$, and $0\le x_{q_1}\le\ldots\le x_{q_{\ell_1}}\le 1$. Next, among $x_{q_1},\ldots,x_{q_{\ell_1}}$ many can be equal to each other. Those x_{q_j} that are equal to each other we group them together, and those x_j ~~'s which~~ that are zero we remove from the sum by reducing the sum if necessary. This brings ~~as~~ us to the following expression:

$$\overline{L}_{2\ell}(t_1) = \sum_{k=1}^{m} \lambda_{I_k} \overline{\gamma}(x_{I_k}),$$

where $I_k \subset \{1, \ldots, \ell\}$, the sets I_k are disjoint for all $k = 1, \ldots, m$. Here, $0 < x_{I_1} < \ldots < x_{I_m} \leq 1$; for any $k, 1 \leq k \leq m$, we have $x_j = x_{I_k}$ for all $j \in I_k$; for any k, $1 \leq k \leq m$, we set $0 < \lambda_{I_k} := \sum_{j \in I_k} \lambda_j$. We remark that if $I_k = \emptyset$ then the term $\lambda_{I_k} \overline{\gamma}(x_{I_k})$ is zero by definition.

Similarly, for $t_2 = (\beta_1, \ldots, \beta_\ell, y_1, \ldots, y_\ell) \in \partial(\Delta_c^\ell \times \Delta_*^\ell)$, we can write

$$\overline{L}_{2\ell}(t_2) = \sum_{k=1}^{v} \beta_{J_k} \overline{\gamma}(y_{J_k})$$

with $v \leq \ell$.

As in the previous section, from linear independence of the vectors $\overline{\gamma}(z_1), \ldots, \overline{\gamma}(z_{2\ell})$, where $0 < z_1 < \ldots < z_{2\ell} \leq 1$, it follows that $\overline{L}_{2\ell}(t_1) = \overline{L}_{2\ell}(t_2)$ holds if and only if $v = m$, $x_{I_k} = y_{J_k}$, and $\lambda_{I_k} = \beta_{J_k}$ for all $k = 1, \ldots, m$. Hence $L^z_{2\ell}(t_1) = L^z_{2\ell}(t_2)$.

The proof for the map $\overline{U}_{2\ell}$ proceeds in the same way as for $\overline{L}_{2\ell}$. Indeed, the equality $\overline{U}_{2\ell}(t_1) = \overline{U}_{2\ell}(t_2)$ implies $\sum_{j=1}^{\ell} \lambda_j(\overline{\gamma}(x_j) - \overline{\gamma}(1)) = \sum_{j=1}^{\ell} \beta_j(\overline{\gamma}(y_j) - \overline{\gamma}(1))$. By removing zero terms , and grouping the similar terms inside the sums as before, we obtain the equation

$$\sum_{k=1}^{m} \lambda_{I_k}(\overline{\gamma}(x_{I_k}) - \overline{\gamma}(1)) = \sum_{k=1}^{v} \beta_{J_k}(\overline{\gamma}(y_{J_k}) - \overline{\gamma}(1)),$$

where we also removed the terms containing those x_j and y_i which that are equal to 1. Applying Lemma 3.3 with $r = 1$, we obtain that $v = m$ and $x_{I_k} = y_{J_k}$ for all $k = 1, \ldots, m$, and $\lambda_{I_k} = \beta_{J_k}$. Hence $U^z_{2\ell}(t_1) = U^z_{2\ell}(t_2)$. □

Next, we prove the analog of Lemma 3.5 for $n = 2\ell - 1$.

Lemma 3.6. *If $\overline{U}(t_1) = \overline{U}(t_2)$ for some $t_1, t_2 \in \Delta_c^\ell \times \Delta_*^{\ell-1}$, then $U^z(t_1) = U^z(t_2)$. Similarly, if $\overline{L}(t_1) = \overline{L}(t_2)$ for some $t_1, t_2 \in \Delta_c^{\ell-1} \times \Delta_*^\ell$, then $L^z(t_1) = L^z(t_2)$.*

Proof. Without loss of generality we can assume that $t_1, t_2 \in \partial(\Delta_c^\ell \times \Delta_*^{\ell-1})$ (similarly, $t_1, t_2 \in \partial(\Delta_c^{\ell-1} \times \Delta_*^\ell)$ in the second claim of the lemma); otherwise the lemma follows from (2.6), (2.7), and (2.8).

We show that the equality $\overline{U}(t_1) = \overline{U}(t_2)$ for some $t_1 = (\lambda_1, \ldots, \lambda_\ell, x_2, \ldots, x_\ell)$, and $t_2 = (\beta_1, \ldots, \beta_\ell, y_2, \ldots, y_\ell)$ in $\partial(\Delta_c^\ell \times \Delta_*^{\ell-1})$ implies $L^z(t_1) = L^z(t_2)$. We can further assume $t_1 \neq t_2$; otherwise there is nothing to prove. We have

$$\lambda_1 \overline{\gamma}(1) + \sum_{j=2}^{\ell} \lambda_j \overline{\gamma}(x_j) = \beta_1 \overline{\gamma}(1) + \sum_{j=2}^{\ell} \beta_j \overline{\gamma}(y_j). \tag{3.21}$$

As in the previous lemma, in the left hand left-hand side of (3.21) we reduce the sum by removing those λ_j 's which that are equal to zero. We further reduce the sum by considering only positive x_j's. Next, among the numbers $0 \leq x_2 \leq \ldots \leq x_\ell \leq 1$, those who we group those that are equal to each other we group them together, and those x_j 's which that are equal to 1 we group with $\lambda_1 \overline{\gamma}(1)$. Eventually, the left hand left-hand side of (3.21) takes the form $\lambda_{I_0}\overline{\gamma}(1) + \sum_{j=1}^{m} \lambda_{I_j} \overline{\gamma}(x_j)$, where $m \leq \ell - 1$, $0 < x_{I_1} < \ldots < x_{I_m} < 1$, and $\lambda_{I_j} = \sum_{j \in I_j} \lambda_j$ with $\lambda_{I_0} \geq 0$ and $\lambda_{I_j} > 0$ for all $j = 1, \ldots, m$. Making a similar reduction in the right hand right-hand side of (3.21), we see that (3.21) takes the form

$$(\lambda_{I_0} - \beta_{J_0})\overline{\gamma}(1) + \sum_{j=1}^{m} \lambda_{I_j} \overline{\gamma}(x_{I_j}) - \sum_{j=1}^{v} \beta_{J_j} \overline{\gamma}(y_{J_j}) = 0. \tag{3.22}$$

Since $1+m+v \le 2\ell-1$, it follows from Lemma 3.4 that (3.22) holds if and only if $m=v$, $\lambda_{I_j}=\beta_{J_j}$ for all $j=2,\dots,m$, and $x_{I_j}=y_{J_j}$ for all $j=1,\dots,m$. It then follows that $U^z(t_1)=U^z(t_2)$.

Next, we show that the equality $\overline{L}(t_1)=\overline{L}(t_2)$ for some $t_1=(\lambda_2,\dots,\lambda_\ell,x_1,\dots,x_\ell)$, and $t_2=(\beta_2,\dots,\beta_\ell,y_1,\dots,y_\ell)$ in $\partial(\Delta_c^{\ell-1}\times\Delta_*^{\ell})$ implies $L^z(t_1)=L^z(t_2)$. Without loss of generality assume $t_1\neq t_2$ and $y_1\le x_1$. The equality $\overline{L}(t_1)=\overline{L}(t_2)$ implies

$$(1-\sum_{j=2}^{\ell}\lambda_j)\overline{\gamma}(x_1)+\sum_{j=2}^{\ell}\lambda_j\overline{\gamma}(x_j)=(1-\sum_{j=2}^{\ell}\beta_j)\overline{\gamma}(y_1)+\sum_{j=2}^{\ell}\beta_j\overline{\gamma}(y_j),$$

which we can rewrite as

$$(1-\sum_{j=2}^{\ell}\lambda_j)(\overline{\gamma}(x_1)-\overline{\gamma}(y_1))+\sum_{j=2}^{\ell}\lambda_j(\overline{\gamma}(x_j)-\overline{\gamma}(y_1))-\sum_{j=2}^{\ell}\beta_j(\overline{\gamma}(y_j)-\overline{\gamma}(y_1))=0. \tag{3.23}$$

We would like to show $L^z(t_1)-L^z(t_2)=0$. Notice that

$$L^z(t_1)-L^z(t_2)=(1-\sum_{j=2}^{\ell}\lambda_j)(\gamma_{n+1}(x_1)-\gamma_{n+1}(y_1))+\sum_{j=2}^{\ell}\lambda_j(\gamma_{n+1}(x_j)-\gamma_{n+1}(y_1))- \\ \sum_{j=2}^{\ell}\beta_j(\gamma_{n+1}(y_j)-\gamma_{n+1}(y_1)). \tag{3.24}$$

Rearranging and grouping equal terms in (3.23) as in the previous arguments, we can rewrite (3.23) as

$$(1-\sum_{j=1}^{m_1}\lambda_{I_j^1}-\beta_{I_0})(\overline{\gamma}(x_1)-\overline{\gamma}(y_1))+\sum_{j=1}^{m_2}\lambda_{I_j^2}(\overline{\gamma}(x_{I_j^2})-\overline{\gamma}(y_1)) \\ +\sum_{j=1}^{m_3}(\lambda_{I_j^3}-\beta_{J_j^1})(\overline{\gamma}(x_{I_j^3})-\overline{\gamma}(y_1))-\sum_{j=1}^{m_4}\beta_{J_j^2}(\overline{\gamma}(y_{J_j^2})-\overline{\gamma}(y_1))=0, \tag{3.25}$$

where m_1,m_2,m_4 are ~~non-negative~~ nonnegative integers with $1+m_2+m_3+m_4\le 2\ell-1$ (if $m_k=0$ then the corresponding sum is set to be zero), $I_j^1,I_j^2,I_j^3,J_j^1,J_j^2$ are subsets of $\{2,\dots,\ell\}$, $\beta_{I_0}\ge 0$, $\lambda_{I_j^k}=\sum_{j\in I_j^k}\lambda_j>0$, $\beta_{J_j^k}=\sum_{j\in J_j^k}\beta_j>0$, $\lambda_{I_j^3}\neq\beta_{J_j^1}$, and the points $x_1,\{x_{I_j^2}\}_{j=1}^{m_2},\{x_{I_j^3}\}_{j=1}^{m_3},\{y_{J_j^2}\}_{j=1}^{m_4}$ are different from each other, none of them (except of x_1) coincides with y_1, and all of them (except of x_1) belong to $(0,1]$. We remark that x_1 can be equal to y_1. In a similar way we can rewrite (3.24) as (3.25), ~~i.e.~~ that is,

$$L^z(t_1)-L^z(t_2)= \\ (1-\sum_{j=1}^{m_1}\lambda_{I_j^1}-\beta_{I_0})(\gamma_{n+1}(x_1)-\gamma_{n+1}(y_1))+\sum_{j=1}^{m_2}\lambda_{I_j^2}(\gamma_{n+1}(x_{I_j^2})-\gamma_{n+1}(y_1)) \\ +\sum_{j=1}^{m_3}(\lambda_{I_j^3}-\beta_{J_j^1})(\gamma_{n+1}(x_{I_j^3})-\gamma_{n+1}(y_1))-\sum_{j=1}^{m_4}\beta_{J_j^2}(\gamma_{n+1}(y_{J_j^2})-\gamma_{n+1}(y_1)).$$

The next lemma follows from Corollary 3.2.

Lemma 3.7. *For any numbers z_j, $1\le j\le 2\ell-1$, such that $0<z_1<z_2<\ldots<z_{2\ell}\le 1$, and any $r\in[0,1]\setminus\{z_1,\dots,z_{2\ell}\}$, the vectors $\overline{\gamma}(z_1)-\overline{\gamma}(r),\dots,\overline{\gamma}(z_{2\ell-1})-\overline{\gamma}(r)$ are linearly independent in $\mathbb{R}^{2\ell-1}$.*

If $y_1=x_1$ then $L^z(t_1)-L^z(t_2)=0$ follows from (3.25) and Lemma 3.7. If $y_1<x_1$, then applying Lemma 3.7 to (3.25) we see that $1-\sum_{j=1}^{m_1}\lambda_{I_j^1}-\beta_{I_0}=0$ and $m_2=m_3=m_4=0$, which implies that $L^z(t_1)-L^z(t_2)=0$. Lemma 3.6 is proved. □

3.3.2. *The ~~proof~~ Proof of (2.10) and (2.11).* We start with (2.10). Assume $n = 2\ell - 1$. First we show that $B^{\sup}(\overline{\gamma}) = \gamma_{n+1}$. We remind that

$$B^{\sup}(\beta_1\overline{\gamma}(1) + \sum_{j=2}^{\ell} \beta_j \overline{\gamma}(x_j)) = \beta_1\gamma_{n+1}(1) + \sum_{j=2}^{\ell} \beta_j \gamma_{n+1}(x_j),$$

holds for all $(\beta_1, \ldots, \beta_\ell, x_2, \ldots, x_\ell) \in \Delta_c^\ell \times \Delta_*^{\ell-1}$. We claim that if $\beta_1\overline{\gamma}(1) + \sum_{j=2}^{\ell} \beta_j\overline{\gamma}(x_j) = \overline{\gamma}(y)$ for some $y \in [0,1]$ then $\beta_1\gamma_{n+1}(1) + \sum_{j=2}^{\ell} \beta_j\gamma_{n+1}(x_j) = \gamma_{n+1}(y)$. Indeed, $\overline{\gamma}(y) = \overline{U}(t_2)$ with $t_2 = (1, 0, \ldots, 0, y) \in \Delta_c^\ell \times \Delta_*^{\ell-1}$, and $\beta_1\gamma_{n+1}(1) + \sum_{j=2}^{\ell} \beta_j\gamma_{n+1}(x_j) = \overline{U}(t_1)$ with $t_1 = (\beta_1, \ldots, \beta_\ell, x_1, \ldots, x_\ell) \in \Delta_c^\ell \times \Delta_*^{\ell-1}$. Thus the claim follows from Lemma 3.6.

Next, we show that $B^{\sup}$ is concave on $\mathrm{conv}(\overline{\gamma}([0,1]))$. ~~As~~ Since the surface parametrized by $U_n(t)$, $t \in \Delta_c^\ell \times \Delta_*^{\ell-1}$ ~~,~~ coincides with the graph $\{(x, B^{\sup}(x)), x \in \mathrm{conv}(\overline{\gamma}([0,1]))\}$, and $B^{\sup} \in C(\mathrm{conv}(\overline{\gamma}([0,1])))$, it suffices to show that the tangent plane T at $U_n(s)$ ~~,~~ for any $s = (\lambda_1, \ldots, \lambda_\ell, y_2, \ldots, y_\ell) \in \mathrm{int}(\Delta_c^\ell \times \Delta_*^{\ell-1})$ ~~,~~ lies *above* the surface U_n. The equation of the tangent plane T at $U(s) := U_n(s)$ is given as

$$T(x) := \det(U_{\beta_1}(s), \ldots, U_{\beta_\ell}(s), U_{x_2}(s), \ldots, U_{x_\ell}(s), x - U(s)) = 0, \quad x \in \mathbb{R}^{n+1}.$$

We have

$$T(x) = \lambda_1 \cdots \lambda_\ell \det(\gamma(1), \gamma(y_2), \ldots, \gamma(y_\ell), \gamma'(y_2), \ldots, \gamma'(y_\ell), x).$$

To show that the tangent plane T lies *above* the surface, first we should find the sign of $T(\lambda e_{n+1})$ as $\lambda \to \infty$, where $e_{n+1} = (\underbrace{0, \ldots, 0, 1}_{n+1})$. For sufficiently large positive λ, we have

$$\mathrm{sign}(T(\lambda e_{n+1})) = \mathrm{sign}(\det(\overline{\gamma}(1), \overline{\gamma}(y_2), \ldots, \overline{\gamma}(y_\ell), \overline{\gamma}'(y_2), \ldots, \overline{\gamma}'(y_\ell))).$$

On the other hand, we have

$$\begin{aligned}
&\det(\overline{\gamma}(1), \overline{\gamma}(y_2), \ldots, \overline{\gamma}(y_\ell), \overline{\gamma}'(y_2), \ldots, \overline{\gamma}'(y_\ell)) = \\
&(-1)^{\frac{(\ell-1)(\ell-2)}{2}} \det(\overline{\gamma}(y_2), \overline{\gamma}'(y_2), \ldots, \overline{\gamma}(y_\ell), \overline{\gamma}'(y_\ell), \overline{\gamma}(1)) = \\
&(-1)^{\frac{(\ell-1)(\ell-2)}{2}} \det(\overline{\gamma}(y_2) - \overline{\gamma}(0), \overline{\gamma}'(y_2), \ldots, \overline{\gamma}(y_\ell) - \overline{\gamma}(y_{\ell-1}), \overline{\gamma}'(y_\ell), \overline{\gamma}(1) - \overline{\gamma}(y_\ell)) = \\
&(-1)^{\frac{(\ell-1)(\ell-2)}{2}} \int_{y_\ell}^1 \int_{y_{\ell-1}}^{y_\ell} \cdots \int_0^{y_2} \det(\overline{\gamma}'(v_2), \overline{\gamma}'(y_2), \ldots, \overline{\gamma}'(v_\ell), \overline{\gamma}'(y_\ell), \overline{\gamma}'(v_{\ell+1})) dv_2 \ldots dv_\ell dv_{\ell+1}.
\end{aligned}$$

Thus, Lemma 3.1 applied to $\overline{\gamma}$ shows that $\mathrm{sign}(T(\lambda e_{n+1}))$ ~~,~~ for sufficiently large λ ~~,~~ coincides with $(-1)^{\frac{(\ell-1)(\ell-2)}{2}}$. Therefore, the surface $U(t)$ being *below* the tangent plane T simply means that $(-1)^{\frac{(\ell-1)(\ell-2)}{2}} T(U(t)) \leq 0$ for all $t = (\beta_1, \ldots, \beta_\ell, x_2, \ldots, x_\ell) \in \Delta_c^\ell \times \Delta_*^{\ell-1}$. We have

$$T(U(t)) = \sum_{j=2}^{\ell} \beta_j \det(\gamma(1), \gamma(y_2), \ldots, \gamma(y_\ell), \gamma'(y_2), \ldots, \gamma'(y_\ell), \gamma(x_j)) \prod_{k=1}^{\ell} \lambda_k.$$

It suffices to verify that

$$(-1)^{\frac{(\ell-1)(\ell-2)}{2}} \det(\gamma(1), \gamma(y_2), \ldots, \gamma(y_\ell), \gamma'(y_2), \ldots, \gamma'(y_\ell), \gamma(u)) \leq 0 \tag{3.26}$$

for all $u \in [0,1]$. We have

$$\begin{aligned}
&(-1)^{\frac{(\ell-1)(\ell-2)}{2}} \det(\gamma(1), \gamma(y_2), \ldots, \gamma(y_\ell), \gamma'(y_2), \ldots, \gamma'(y_\ell), \gamma(u)) \\
&= \det(\gamma(y_2), \gamma'(y_2), \ldots, \gamma(y_\ell), \gamma'(y_\ell), \gamma(1), \gamma(u)).
\end{aligned} \tag{3.27}$$

If $u \in [y_\ell, 1]$, then

$$\det(\gamma(y_2), \gamma'(y_2), \dots, \gamma(y_\ell), \gamma'(y_\ell), \gamma(1), \gamma(u)) =$$
$$-\det(\gamma(y_2), \gamma'(y_2), \dots, \gamma(y_\ell), \gamma'(y_\ell), \gamma(u), \gamma(1)) =$$
$$-\det(\gamma(y_2) - \gamma(0), \gamma'(y_2), \dots, \gamma(y_\ell) - \gamma(y_{\ell-1}), \gamma'(y_\ell), \gamma(u) - \gamma(y_\ell), \gamma(1) - \gamma(u)) =$$
$$-\int_u^1 \int_{y_\ell}^{x_j} \int_{y_{\ell-1}}^{y_\ell} \dots \int_0^{y_2} \det(\gamma'(v_2), \gamma'(y_2), \dots, \gamma'(v_\ell), \gamma'(y_\ell), \gamma'(v_{\ell+1}), \gamma'(v_{\ell+2})) dv_2 \dots dv_\ell dv_{\ell+1} dv_{\ell+2}$$

is ~~non-positive~~ nonpositive by Lemma 3.1.

If $u \in [0, y_2]$ we again use (3.27). Next, we move the column $\gamma(u)$ to the left of the column $\gamma(y_2)$. Notice that we will acquire the negative sign because passing the couples $\gamma(y_i), \gamma'(y_i)$ does not change the sign of the determinant, the negative sign arises by passing $\gamma(1)$. Using the similar integral representation as before together with Lemma 3.1, we see that ~~the~~ inequality (3.26) holds true in the case $u \in [0, \gamma(y_2)]$. The case $u \in [y_i, y_{i+1}]$ for some $i \in \{2, \dots, \ell-1\}$ ~~,~~ is similar to the previous case. Indeed, first we apply (3.27), then we place the column $\gamma(u)$ between the columns $\gamma'(y_i), \gamma(y_{i+1})$ (thus we acquire the negative sign), we use the similar integral representation as before together with Lemma 3.1 to conclude that (3.26) holds true in this case too. This finishes the proof of concavity of $B^{\sup}$ on $\mathrm{conv}(\overline{\gamma}([0,1]))$.

Next, we show that $B^{\sup}$ is the minimal concave function in a family of concave functions G on $\mathrm{conv}(\overline{\gamma}([0,1]))$ with the obstacle condition $G(\overline{\gamma}(s)) \geq \gamma_{n+1}(s)$ for all $s \in [0,1]$. Indeed, pick an arbitrary point $x \in \mathrm{conv}(\overline{\gamma}([0,1]))$. We would like to show $G(x) \geq B^{\sup}(x)$. There exists $(\lambda_1, \dots, \lambda_\ell, y_2, \dots, y_\ell) \in \Delta_c^\ell \times \Delta_*^{\ell-1}$ such that $x = \lambda_1 \overline{\gamma}(1) + \sum_{j=2}^\ell \lambda_j \overline{\gamma}(y_j)$. Therefore

$$B^{\sup}(x) = \lambda_1 \gamma_{n+1}(1) + \sum_{j=2}^{\ell} \lambda_j \gamma_{n+1}(y_j) \leq \lambda_1 G(\overline{\gamma}(1)) + \sum_{j=2}^{\ell} \lambda_j G(\overline{\gamma}(y_j)) \leq G(x).$$

Next we consider $B^{\sup}$ when $n = 2\ell$. We only check the concavity of $B^{\sup}$ because the remaining properties (minimality and the obstacle condition $B^{\sup}(\overline{\gamma}) = \gamma_{n+1}$) are verified similarly as in the dimension $n = 2\ell - 1$. The equation of the tangent plane T at point

$$U(s) := U_n(s) = \sum_{j=1}^{\ell} \beta_j \gamma(y_j) + (1 - \sum_{j=1}^{\ell} \beta_j) \gamma(1),$$

where $s = (\beta_1, \dots, \beta_\ell, y_1, \dots, y_\ell) \in \mathrm{int}(\Delta_c^\ell \times \Delta_*^\ell)$, is given as

$$T(x) := \det(U_{\beta_1}, \dots, U_{\beta_\ell}, U_{y_1}, \dots, U_{y_\ell}, x - U(s)) = 0, \quad x \in \mathbb{R}^{n+1}.$$

We have

$$\mathrm{sign}(T(x)) = \mathrm{sign}(\det(\gamma(y_1) - \gamma(1), \dots, \gamma(y_\ell) - \gamma(1), \gamma'(y_1), \dots, \gamma'(y_\ell), x - \gamma(1))).$$

Next,

$$\mathrm{sign}(T(\lambda e_{n+1})) = \mathrm{sign}(\det(\overline{\gamma}(y_1) - \overline{\gamma}(1), \dots, \overline{\gamma}(y_\ell) - \overline{\gamma}(1), \overline{\gamma}'(y_1), \dots, \overline{\gamma}'(y_\ell)))$$

as $\lambda \to +\infty$. On the other hand, we have

$$\det(\overline{\gamma}(y_1) - \overline{\gamma}(1), \dots, \overline{\gamma}(y_\ell) - \overline{\gamma}(1), \overline{\gamma}'(y_1), \dots, \overline{\gamma}'(y_\ell)) =$$
$$(-1)^\ell \det(\overline{\gamma}(y_2) - \overline{\gamma}(y_1), \dots, \overline{\gamma}(y_\ell) - \overline{\gamma}(y_{\ell-1}), \overline{\gamma}(1) - \overline{\gamma}(y_\ell), \overline{\gamma}'(y_1), \dots, \overline{\gamma}'(y_\ell)) =$$
$$(-1)^{\frac{\ell(\ell-1)}{2}} \det(\overline{\gamma}'(y_1), \overline{\gamma}(y_2) - \overline{\gamma}(y_1), \dots, \overline{\gamma}'(y_\ell), \overline{\gamma}(1) - \overline{\gamma}(y_\ell)) =$$
$$(-1)^{\frac{\ell(\ell-1)}{2}} \int_{y_\ell}^1 \dots \int_{y_1}^{y_2} \det(\overline{\gamma}'(y_1), \overline{\gamma}'(x_1), \dots, \overline{\gamma}'(y_\ell), \overline{\gamma}'(x_\ell)) dx_1 \dots dx_\ell.$$

Thus, it follows from Lemma 3.1 that $\operatorname{sign}(T(\lambda e_{n+1})) = (-1)^{\frac{\ell(\ell-1)}{2}}$ as $\lambda \to \infty$. Therefore, to verify the concavity of $B^{\sup}$, it suffices to show $(-1)^{\frac{\ell(\ell-1)}{2}} T(U(t)) \le 0$ for all $t = (\lambda_1, \dots, \lambda_\ell, x_1, \dots, x_\ell) \in \Delta_c^\ell \times \Delta_*^\ell$. We have

$$T(U(t)) = \sum_{j=1}^{\ell} \lambda_j \det(\gamma(y_1) - \gamma(1), \dots, \gamma(y_\ell) - \gamma(1), \gamma'(y_1), \dots, \gamma'(y_\ell), \gamma(x_j) - \gamma(1)) \prod_{i=1}^{\ell} \beta_i.$$

It suffices to show that $(-1)^{\frac{\ell(\ell-1)}{2}} \det(\gamma(y_1)-\gamma(1), \dots, \gamma(y_\ell)-\gamma(1), \gamma'(y_1), \dots, \gamma'(y_\ell), \gamma(u)-\gamma(1)) \le 0$ for all $u \in [0,1]$. Assume $u \in [y_i, y_{i+1}]$ for some $i \in \{1, \dots, \ell-1\}$. We have

$$\begin{aligned}
&(-1)^{\frac{\ell(\ell-1)}{2}} \det(\gamma(y_1) - \gamma(1), \dots, \gamma(y_\ell) - \gamma(1), \gamma'(y_1), \dots, \gamma'(y_\ell), \gamma(u) - \gamma(1)) = \\
&- \det(\gamma'(y_1), \gamma(1) - \gamma(y_1), \dots, \gamma'(y_\ell), \gamma(1) - \gamma(y_\ell), \gamma(1) - \gamma(u)) = - \\
&\det(\gamma'(y_1), \gamma(1) - \gamma(y_1), \dots, \gamma'(y_i), \gamma(1) - \gamma(y_i), \gamma(1) - \gamma(u), \gamma'(y_{i+1}), \gamma(1) - \gamma(y_{i+1}), \dots, \gamma(1) - \gamma(y_\ell)) = \\
&- \det(\gamma'(y_1), \gamma(y_2) - \gamma(y_1), \dots, \gamma'(y_i), \gamma(u) - \gamma(y_i), \gamma(y_{i+1}) - \gamma(u), \gamma'(y_{i+1}), \gamma(y_{i+2}) - \gamma(y_{i+1}), \dots, \gamma(1) - \gamma(y_\ell)) \\
&= - \int_{y_\ell}^{1} \dots \int_{u}^{y_{i+1}} \int_{y_i}^{u} \dots \int_{y_1}^{y_2} \det(\gamma'(y_1), \gamma'(v_1), \dots, \gamma'(y_i), \gamma'(w), \gamma'(v_i), \gamma'(y_{i+1}), \dots, \gamma'(v_\ell)) dv_1 \dots dw dv_i \dots dv_\ell,
\end{aligned}$$

which has a nonpositive sign by Lemma 3.1 (here y_{i+2} for $i = \ell - 1$ is set to be 1). The cases $u \in [0, y_1]$ ~~,~~ and $u \in [y_\ell, 1]$ are treated similarly.

Next, we verify (2.11). The obstacle condition $B^{\inf}(\overline{\gamma}) = \gamma_{n+1}$ and the minimality (assuming $B^{\inf}$ is convex) are verified ~~similarly~~ similar as in the case $B^{\sup}$. So, in what follows we only verify the convexity of $B^{\inf}$.

Assume $n = 2\ell - 1$. The equation of the tangent plane T at point

$$L(s) := L_n(s) = (1 - \sum_{j=2}^{\ell} \beta_j)\gamma(y_1) + \sum_{j=2}^{\ell} \beta_j \gamma(y_j),$$

where $s = (\beta_2, \dots, \beta_\ell, y_1, \dots, y_\ell) \in \operatorname{int}(\Delta_c^{\ell-1} \times \Delta_*^\ell)$, is given by

$$\begin{aligned}
&T(x) := \det(L_{\beta_2}, \dots, L_{\beta_\ell}, L_{y_1}, \dots, L_{y_\ell}, x - L(s)) = \\
&\det(\gamma(y_2) - \gamma(y_1), \dots, \gamma(y_\ell) - \gamma(y_1), \gamma'(y_1), \dots, \gamma'(y_\ell), x - \gamma(y_1)) \, (1 - \sum_{j=2}^{\ell} \beta_j) \prod_{j=2}^{\ell} \beta_j.
\end{aligned}$$

We have

$$\operatorname{sign}(T(\lambda e_{n+1})) = \operatorname{sign}(\det(\overline{\gamma}(y_2) - \overline{\gamma}(y_1), \dots, \overline{\gamma}(y_\ell) - \overline{\gamma}(y_1), \overline{\gamma}'(y_1), \dots, \overline{\gamma}'(y_\ell)))$$

as $\lambda \to +\infty$. On the other hand,

$$\begin{aligned}
&\det(\overline{\gamma}(y_2) - \overline{\gamma}(y_1), \dots, \overline{\gamma}(y_\ell) - \overline{\gamma}(y_1), \overline{\gamma}'(y_1), \dots, \overline{\gamma}'(y_\ell)) = \\
&(-1)^{\frac{\ell(\ell-1)}{2}} \det(\overline{\gamma}'(y_1), \overline{\gamma}(y_2) - \overline{\gamma}(y_1), \dots, \overline{\gamma}'(y_{\ell-1}), \overline{\gamma}(y_\ell) - \overline{\gamma}(y_{\ell-1}), \overline{\gamma}'(y_\ell)) = \\
&(-1)^{\frac{\ell(\ell-1)}{2}} \int_{y_{\ell-1}}^{y_\ell} \dots \int_{y_1}^{y_2} \det(\overline{\gamma}'(y_1), \overline{\gamma}'(v_2), \dots, \overline{\gamma}'(y_{\ell-1}), \overline{\gamma}'(v_\ell), \overline{\gamma}'(y_\ell)) dv_2 \dots dv_\ell.
\end{aligned}$$

Thus $\operatorname{sign}(T(\lambda e_{n+1})) = (-1)^{\frac{\ell(\ell-1)}{2}}$ by Lemma 3.1 as $\lambda \to +\infty$. Therefore, $B^{\inf}$ is convex if $(-1)^{\frac{\ell(\ell-1)}{2}} T(L(t)) \geq 0$ for all $t = (\lambda_2, \dots, \lambda_\ell, x_1, \dots, x_\ell) \in \Delta_c^{\ell-1} \times \Delta_*^\ell$. We have

$$T(L(t)) = \det(\gamma(y_2)-\gamma(y_1), \dots, \gamma(y_\ell)-\gamma(y_1), \gamma'(y_1), \dots, \gamma'(y_\ell), L(t)-\gamma(y_1))\,(1-\sum_{j=2}^{\ell}\beta_j)\prod_{j=2}^{\ell}\beta_j =$$

$$(1-\sum_{k=2}^{\ell}\lambda_k)\det(\gamma(y_2)-\gamma(y_1), \dots, \gamma(y_\ell)-\gamma(y_1), \gamma'(y_1), \dots, \gamma'(y_\ell), \gamma(x_1)-\gamma(y_1))\,(1-\sum_{j=2}^{\ell}\beta_j)\prod_{j=2}^{\ell}\beta_j$$

$$+\sum_{k=2}^{\ell}\lambda_k \det(\gamma(y_2)-\gamma(y_1), \dots, \gamma(y_\ell)-\gamma(y_1), \gamma'(y_1), \dots, \gamma'(y_\ell), \gamma(x_k)-\gamma(y_1))\,(1-\sum_{j=2}^{\ell}\beta_j)\prod_{j=2}^{\ell}\beta_j.$$

Thus, to verify the convexity of $B^{\inf}$, it suffices to show

$$(-1)^{\frac{\ell(\ell-1)}{2}}\det(\gamma(y_2)-\gamma(y_1), \dots, \gamma(y_\ell)-\gamma(y_1), \gamma'(y_1), \dots, \gamma'(y_\ell), \gamma(u)-\gamma(y_1)) \geq 0$$

for all $u \in [0,1]$. Notice that

$$(-1)^{\frac{\ell(\ell-1)}{2}}\det(\gamma(y_2)-\gamma(y_1), \dots, \gamma(y_\ell)-\gamma(y_1), \gamma'(y_1), \dots, \gamma'(y_\ell), \gamma(u)-\gamma(y_1)) =$$
$$\det(\gamma'(y_1), \gamma(y_2)-\gamma(y_1), \dots, \gamma'(y_{\ell-1}), \gamma(y_\ell)-\gamma(y_1), \gamma'(y_\ell), \gamma(u)-\gamma(y_1)).$$

Next, assume $u \in [y_i, y_{i+1}]$ for some $i \in \{1, \dots, \ell-1\}$ (the cases $u \in [0, y_1]$ and $u \in [y_\ell, 1]$ are considered ~~similarly~~similar). We have

$$\det(\gamma'(y_1), \gamma(y_2)-\gamma(y_1), \dots, \gamma'(y_{\ell-1}), \gamma(y_\ell)-\gamma(y_1), \gamma'(y_\ell), \gamma(u)-\gamma(y_1)) =$$
$$\det(\gamma'(y_1), \gamma(y_2)-\gamma(y_1), \dots, \gamma'(y_i), \gamma(u)-\gamma(y_1), \gamma(y_{i+1})-\gamma(y_1), \gamma'(y_{i+1}), \dots, \gamma(y_\ell)-\gamma(y_1), \gamma'(y_\ell)) =$$
$$\det(\gamma'(y_1), \gamma(y_2)-\gamma(y_1), \dots, \gamma'(y_i), \gamma(u)-\gamma(y_i), \gamma(y_{i+1})-\gamma(u), \gamma'(y_{i+1}), \dots, \gamma(y_\ell)-\gamma(y_{\ell-1}), \gamma'(y_\ell))$$
$$= \int_{y_{\ell-1}}^{y_\ell} \cdots \int_u^{y_{i+1}} \int_{y_i}^{u} \cdots \int_{y_1}^{y_2}$$
$$\det(\gamma'(y_1), \gamma'(v_1), \dots, \gamma'(y_i), \gamma'(w), \gamma'(v_i), \gamma'(y_{i+1}), \dots, \gamma'(v_{\ell-1}), \gamma'(y_\ell))dv_1 \dots dv_i dw \dots dv_{\ell-1}.$$

Thus $T(L(t)) \geq 0$ by Lemma 3.1.

Next, we consider $B^{\inf}$ when $n = 2\ell$. As in the previous cases, we only verify the convexity of $B^{\inf}$ (minimality and the obstacle condition $B^{\inf}(\overline{\gamma}) = \gamma_{n+1}$ are verified easily).

The equation of the tangent plane T at point

$$L(s) := L_n(s) = \sum_{j=1}^{\ell}\beta_j\gamma(y_j),$$

where $s = (\beta_1, \dots, \beta_\ell, y_1, \dots, y_\ell) \in \operatorname{int}(\Delta_c^\ell \times \Delta_*^\ell)$, is given by

$$T(x) := \det(L_{\beta_1}, \dots, L_{\beta_\ell}, L_{y_1}, \dots, L_{y_\ell}, x - L(s)) = \det(\gamma(y_1), \dots, \gamma(y_\ell), \gamma'(y_1), \dots, \gamma'(y_\ell), x)\prod_{j=1}^{\ell}\beta_j.$$

We have

$$\operatorname{sign}(T(\lambda e_{n+1})) = \operatorname{sign}(\det(\overline{\gamma}(y_1), \dots, \overline{\gamma}(y_\ell), \overline{\gamma}'(y_1), \dots, \overline{\gamma}'(y_\ell)))$$

as $\lambda \to +\infty$. On the other hand,

(3.28)
$$\det(\overline{\gamma}(y_1), \dots, \overline{\gamma}(y_\ell), \overline{\gamma}'(y_1), \dots, \overline{\gamma}'(y_\ell)) =$$
$$(-1)^{\frac{\ell(\ell-1)}{2}} \det(\overline{\gamma}(y_1) - \overline{\gamma}(0), \overline{\gamma}'(y_1), \dots, \overline{\gamma}(y_\ell) - \overline{\gamma}(y_{\ell-1}), \overline{\gamma}'(y_\ell)) =$$
$$(-1)^{\frac{\ell(\ell-1)}{2}} \int_{y_{\ell-1}}^{y_\ell} \dots \int_0^{y_1} \det(\overline{\gamma}'(v_1), \overline{\gamma}'(y_1), \dots, \overline{\gamma}'(v_\ell), \overline{\gamma}'(y_\ell)) dv_1 \dots dv_\ell.$$

Thus $\mathrm{sign}(T(\lambda e_{n+1})) = (-1)^{\frac{\ell(\ell-1)}{2}}$ by Lemma 3.1 as $\lambda \to +\infty$. Therefore, $B^{\inf}$ is convex if $(-1)^{\frac{\ell(\ell-1)}{2}} T(L(t)) \geq 0$ for all $t = (\lambda_1, \dots, \lambda_\ell, x_1, \dots, x_\ell) \in \Delta_c^\ell \times \Delta_*^\ell$. We have

$$T(L(t)) = \sum_{k=1}^{\ell} \lambda_k \det(\gamma(y_1), \dots, \gamma(y_\ell), \gamma'(y_1), \dots, \gamma'(y_\ell), \gamma(x_k)) \prod_{j=1}^{\ell} \beta_j.$$

Thus, to verify the convexity of $B^{\inf}$, it suffices to show

$$(-1)^{\frac{\ell(\ell-1)}{2}} \det(\gamma(y_1), \dots, \gamma(y_\ell), \gamma'(y_1), \dots, \gamma'(y_\ell), \gamma(u)) \geq 0 \quad \text{for all} \quad u \in [0,1].$$

Notice that

$$(-1)^{\frac{\ell(\ell-1)}{2}} \det(\gamma(y_1), \dots, \gamma(y_\ell), \gamma'(y_1), \dots, \gamma'(y_\ell), \gamma(u)) = \det(\gamma(y_1), \gamma'(y_1), \dots, \gamma(y_\ell), \gamma'(y_\ell), \gamma(u)).$$

Next, assume $u \in [y_i, y_{i+1}]$ for some $i \in \{1, \dots, \ell-1\}$ (the cases $u \in [0, y_1]$ or $u \in [y_\ell, 1]$ are considered similarlysimilar). Set $y_0 = 0$. We have

$$\det(\gamma(y_1), \gamma'(y_1), \dots, \gamma(y_\ell), \gamma'(y_\ell), \gamma(u)) =$$
$$\det(\gamma(y_1), \gamma'(y_1), \dots, \gamma(y_i), \gamma'(y_i), \gamma(u), \gamma(y_{i+1}), \gamma'(y_{i+1}), \dots) =$$
$$\det(\gamma(y_1) - \gamma(0), \gamma'(y_1), \dots, \gamma(y_i) - \gamma(y_{i-1}), \gamma'(y_i), \gamma(u) - \gamma(y_i), \gamma(y_{i+1}) - \gamma(u), \gamma'(y_{i+1}), \dots) =$$
$$\int_{y_{\ell-1}}^{y_\ell} \dots \int_u^{y_{i-1}} \int_{y_i}^{u} \int_{y_{i-1}}^{y_i} \dots \int_0^{y_1}$$
$$\det(\gamma'(v_1), \gamma'(y_1), \dots, \gamma'(v_i), \gamma'(y_i), \gamma'(w), \gamma'(v_{i+1}), \gamma'(y_{i+1}), \dots, \gamma'(v_\ell), \gamma'(y_\ell)) dv_1 \dots dv_i dw dv_{i+1} \dots dv_\ell.$$

Thus $T(L(t)) \geq 0$ by Lemma 3.1.

3.3.3. *The ~~proof~~ Proof of (2.12).* First we show the implication $B^{\sup}(u) = B^{\inf}(u) \Rightarrow u \in \partial\,\mathrm{conv}(\overline{\gamma}([0,1]))$. Consider the case $n = 2\ell$. Assume contrary, i.e.that is, $u \in \mathrm{int}(\mathrm{conv}(\overline{\gamma}([0,1])))$. Then, using (2.4), (2.5), we can find $t = (\lambda_1, \dots, \lambda_\ell, x_1, \dots, x_\ell)$ and $s = (\beta_1, \dots, \beta_\ell, y_1, \dots, y_\ell)$, both in $\mathrm{int}(\Delta_c^\ell \times \Delta_*^\ell)$, such that

$$u = \sum_{j=1}^{\ell} \lambda_j \overline{\gamma}(x_j) + (1 - \sum_{j=1}^{\ell} \lambda_j) \overline{\gamma}(1) = \sum_{j=1}^{\ell} \beta_j \overline{\gamma}(y_j).$$

The equality $B^{\sup}(u) = B^{\inf}(u)$ implies (see (3.19), (3.20))

$$\sum_{j=1}^{\ell} \lambda_j \gamma(x_j) + (1 - \sum_{j=1}^{\ell} \lambda_j) \gamma(1) = \sum_{j=1}^{\ell} \beta_j \gamma(y_j).$$

We see that $\gamma(1)$ is a linear combination of 2ℓ vectors $\gamma(x_j), \gamma(y_j), j = 1, \dots, \ell$, which leads us to a contradiction with Corollary 3.2. Thus $u \in \partial\,\mathrm{conv}(\overline{\gamma}([0,1]))$.

Next, consider the case $n = 2\ell - 1$ and assume the contrary, i.e.that is, $u \in \mathrm{int}(\mathrm{conv}(\overline{\gamma}([0,1])))$. Similarly as before, we have

$$\lambda_1 \gamma(1) + \sum_{j=2}^{\ell} \lambda_j \gamma(x_j) = (1 - \sum_{j=2}^{\ell} \beta_j) \gamma(y_1) + \sum_{j=2}^{\ell} \beta_j \gamma(y_j) \tag{3.29}$$

for some $t = (\lambda_1, \dots, \lambda_\ell, x_2, \dots, x_\ell) \in \mathrm{int}(\Delta_c^\ell \times \Delta_*^{\ell-1})$ and $s = (\beta_2, \dots, \beta_\ell, y_1, \dots, y_\ell) \in \mathrm{int}(\Delta_c^\ell \times \Delta_*^{\ell-1})$. ~~The equality~~ Equality (3.29) shows that $\gamma(1)$ is a linear combination of $2\ell - 1$ vectors $\{\gamma(x_j)\}_{j=2}^{\ell}$, $\{\gamma(y_j)\}_{j=1}^{\ell}$ ~~which contradicts to~~ , which contradicts Corollary 3.2.

Next we show the implication $u \in \partial \operatorname{conv}(\overline{\gamma}([0,1])) \Rightarrow B^{\sup}(u) = B^{\inf}(u)$. Consider $n = 2\ell$. Suppose

$$\overline{U}(t) \stackrel{\text{def}}{=} \sum_{j=1}^{\ell} \lambda_j \overline{\gamma}(x_j) + (1 - \sum_{j=1}^{\ell} \lambda_j)\overline{\gamma}(1) = \sum_{j=1}^{\ell} \beta_j \overline{\gamma}(y_j) \stackrel{\text{def}}{=} \overline{L}(s)$$

for some $t = (\lambda, \dots, \lambda_\ell, x_1, \dots, x_\ell)$ and $s = (\beta_1, \dots, \beta_\ell, y_1, \dots, y_\ell)$, both in $\partial(\Delta_c^\ell \times \Delta_*^\ell)$. The goal is to show that

$$U^z(t) \stackrel{\text{def}}{=} \sum_{j=1}^{\ell} \lambda_j \gamma_{n+1}(x_j) + (1 - \sum_{j=1}^{\ell} \lambda_j)\gamma_{n+1}(1) = \sum_{j=1}^{\ell} \beta_j \gamma_{n+1}(y_j) \stackrel{\text{def}}{=} L^z(s). \tag{3.30}$$

We claim that (3.30) follows from the second part of Lemma 3.5. For this, it suffices to show that any point $\overline{U}(t)$, $t \in \partial(\Delta_c^\ell \times \Delta_*^\ell)$, can be written as $\overline{L}(s_1)$ for some $s_1 = (\beta_1', \dots, \beta_\ell', y_1', \dots, y_\ell') \in \partial(\Delta_c^\ell \times \Delta_*^\ell)$. Indeed, as $t \in \partial(\Delta_c^\ell \times \Delta_*^\ell)$, several cases can happen. 1) If $\sum_{j=1}^{\ell} \lambda_j = 1$, then choose $\beta_j' = \lambda_j$, $j = 1, \dots, \ell - 1$, $\beta_\ell' = 1 - \sum_{j=1}^{\ell-1} \lambda_j$, and $y_j' = x_j$, $j = 1, \dots, \ell$. Then

$$L^z(s) = B^{\inf}(\overline{L}(s)) \stackrel{\text{Lemma 3.5}}{=} B^{\inf}(\overline{L}(s_1)) = \sum_{j=1}^{\ell} \beta_j' \gamma_{n+1}(y_j') = U^z(t). \tag{3.31}$$

Next, 2) if at least one $\lambda_j = 0$, say $\lambda_p = 0$ for some $p \in \{1, \dots, \ell\}$, then take $\beta_1' = \lambda_1, \dots, \beta_{p-1}' = \lambda_{p-1}, \beta_p = \lambda_{p+1}, \dots, \beta_{\ell-1}' = \lambda_\ell, \beta_\ell = 1 - \sum_{j=1}^{\ell} \lambda_j$, and $y_1' = x_1, \dots, y_{p-1}' = x_{p-1}, y_p' = x_{p+1}, \dots, y_{\ell-1}' = x_\ell, y_\ell' = 1$ and repeat (3.31). Next 3) if $x_\ell = 1$, choose $(\beta_j', y_j') = (\lambda_j, x_j)$ for $j = 1, \dots, \ell - 1$, and $(\beta_\ell', y_\ell') = (\sum_{j=1}^{\ell-1} \lambda_j, 1)$ and repeat (3.31). 4) If $x_p = x_{p+1}$ for some $p \in \{1, \dots, \ell - 1\}$ then take $y_j' = x_j$ for $j = 1, \dots, p$; $y_j' = x_{j+1}$ for $j = p+1, \dots, \ell - 1$; $y_\ell' = 1$; $\beta_1' = \lambda_1, \dots, \beta_p' = \lambda_p + \lambda_{p+1}$, $\beta_{p+1}' = \lambda_{p+2}, \dots, \beta_{\ell-1}' = \lambda_\ell$, $\beta_\ell' = 1 - \sum_{j=1}^{\ell} \lambda_j$, and repeat (3.31). Finally, 5) if $x_1 = 0$ choose $\beta_j' = \lambda_{j+1}$, $j = 1, \dots, \ell - 1$; $\beta_\ell' = 1 - \sum_{j=1}^{\ell} \lambda_j$; $y_j' = x_{j+1}$, $j = 1, \dots, \ell - 1$; $y_\ell' = 1$, and apply (3.31).

Next, consider $n = 2\ell - 1$. Suppose

$$\overline{U}(t) \stackrel{\text{def}}{=} \sum_{j=2}^{\ell} \beta_j \overline{\gamma}(x_j) + \beta_1 \overline{\gamma}(1) = (1 - \sum_{j=2}^{\ell} \lambda_j)\overline{\gamma}(y_1) + \sum_{j=2}^{\ell} \lambda_j \overline{\gamma}(y_j) \stackrel{\text{def}}{=} \overline{L}(s)$$

for some $t = (\beta_1, \dots, \beta_\ell, x_2, \dots, x_\ell) \in \partial(\Delta_c^\ell \times \Delta_*^{\ell-1})$ and $s = (\lambda_2, \dots, \lambda_\ell, y_1, \dots, y_\ell) \in \partial(\Delta_c^{\ell-1} \times \Delta_*^\ell)$. We would like to show

$$U^z(t) \stackrel{\text{def}}{=} \sum_{j=2}^{\ell} \beta_j \gamma_{n+1}(x_j) + \beta_1 \gamma_{n+1}(1) = (1 - \sum_{j=2}^{\ell} \lambda_j)\gamma_{n+1}(y_1) + \sum_{j=2}^{\ell} \lambda_j \gamma_{n+1}(y_j) \stackrel{\text{def}}{=} L^z(s). \tag{3.32}$$

As in the case $n = 2\ell - 1$, we claim that (3.32) follows from Lemma 3.6. It suffices to show that for any point $\overline{U}(t)$, $t \in \partial(\Delta_c^\ell \times \Delta_*^{\ell-1})$, there exists a point $s_1 = (\lambda_2', \dots, \lambda_\ell', y_1', \dots, \lambda_\ell') \in \partial(\Delta_c^{\ell-1} \times \Delta_*^\ell)$ such that $\overline{U}(t) = \overline{L}(s_1)$. Several instances may happen. 1) ~~if~~ If $\sum_{j=1}^{\ell} \beta_j = 1$. Let

$$(\lambda_2', \dots, \lambda_{\ell-1}', \lambda_\ell', y_1', \dots, y_{\ell-1}', y_\ell') = (\beta_3, \dots, \beta_\ell, \beta_1, x_2, \dots, x_\ell, 1).$$

Notice that $1 - \sum_{j=2}^{\ell} \lambda_j' = \beta_2$. 2) ~~if~~ If $\beta_p = 0$ for some $p \in \{1, \dots, \ell - 1\}$ then let

$$(\lambda_2', \dots, \lambda_{p-1}', \lambda_p', \dots, \lambda_{\ell-1}', \lambda_\ell', y_1', y_2', \dots, y_{p-1}', y_p', \dots, y_{\ell-1}', y_\ell') =$$
$$(\beta_2, \dots, \beta_{p-1}, \beta_{p+1}, \dots, \beta_\ell, \beta_1, 0, x_2, \dots, x_{p-1}, x_{p+1}, \dots, x_\ell, 1).$$

3) if If $\beta_1 = 0$ then we choose $y_1' = 0$ and

$$(\lambda_2', \ldots, \lambda_\ell', y_2', \ldots, y_\ell') = (\beta_2, \ldots, \beta_\ell, x_2, \ldots, x_\ell).$$

4) if If $x_2 = 0$ then we choose $y_1 = 0$ and

$$(\lambda_2', \ldots, \lambda_{\ell-1}', \lambda_\ell', y_2', \ldots, y_{\ell-1}', y_\ell') = (\beta_3, \ldots, \beta_\ell, \beta_1, x_3, \ldots, x_\ell, 1).$$

5) if If $x_\ell = 1$ then let $y_1 = 0$ and

$$(\lambda_2', \ldots, \lambda_{\ell-1}', \lambda_\ell', y_2', \ldots, y_{\ell-1}', y_\ell') = (\beta_2, \ldots, \beta_{\ell-1}, \beta_\ell + \beta_1, x_2, \ldots, x_{\ell-1}, 1).$$

Finally, 6) if $x_p = x_{p+1}$ for some $p \in \{2, \ldots, \ell-1\}$, take $y_1 = 0$ and

$$\begin{aligned}&(\lambda_2', \ldots, \lambda_{p-1}', \lambda_p', \lambda_{p+1}', \ldots, \lambda_{\ell-1}', \lambda_\ell', y_2', \ldots, y_p', y_{p+1}', \ldots, y_{\ell-1}', y_\ell') = \\ &(\beta_2, \ldots, \beta_{p-1}, \beta_p + \beta_{p+1}, \beta_{p+2}, \ldots, \beta_\ell, \beta_1, x_2, \ldots, x_p, x_{p+2}, \ldots x_\ell, 1).\end{aligned}$$

Under such choices we have

$$L^z(s) = B^{\inf}(\overline{L}(s)) \overset{\text{Lemma 3.6}}{=} B^{\inf}(\overline{L}(s_1)) = (1 - \sum_{j=2}^{\ell} \lambda_j')\gamma_{n+1}(y_1') + \sum_{j=2}^{\ell} \lambda_j' \gamma_{n+1}(y_j') = U^z(t).$$

This finishes the proof of (2.12).

3.3.4. *The proof Proof of (2.13).* The inclusion

$$\{(x, B^{\sup}(x)), x \in \operatorname{conv}(\bar{\gamma}([0,1]))\} \cup \{(x, B^{\inf}(x)), x \in \operatorname{conv}(\bar{\gamma}([0,1]))\} \subset \partial \operatorname{conv}(\gamma([0,1]))$$

is trivial. Indeed, it follows from (3.19) that the point $(x, B^{\sup}(x))$ is a convex combination of some points of $\gamma([0,1])$, ; therefore, $(x, B^{\sup}(x)) \in \operatorname{conv}(\gamma([0,1]))$. On the other hand, no point of the form (x, s), where $s > B^{\sup}(x)$, belongs to $\operatorname{conv}(\gamma([0,1]))$. Indeed, otherwise $(x, s) = \sum_{j=1}^m \lambda_j \gamma(t_j)$ for some $t_j \in [0,1]$ and nonnegative λ_j such that $\sum_{j=1}^m \lambda_j = 1$. Then

$$B^{\sup}(x) = B^{\sup}\left(\sum \lambda_j \bar{\gamma}(t_j)\right) \overset{(2.10)}{\geq} \sum \lambda_j B^{\sup}(\bar{\gamma}(t_j)) \overset{(2.10)}{=} \sum \lambda_j \gamma_{n+1}(t_j) = s$$

gives a contradiction. Thus $(x, B^{\sup}(x)) \in \partial \operatorname{conv}(\gamma([0,1]))$. In a similar way, we have $(x, B^{\inf}(x)) \in \partial \operatorname{conv}(\gamma([0,1]))$ for $x \in \operatorname{conv}(\bar{\gamma}([0,1]))$.

To verify the inclusion

$$\partial \operatorname{conv}(\gamma([0,1])) \subset \{(x, B^{\sup}(x)), x \in \operatorname{conv}(\bar{\gamma}([0,1]))\} \cup \{(x, B^{\inf}(x)), x \in \operatorname{conv}(\bar{\gamma}([0,1]))\},$$

we pick a point $(x,t) \in \partial \operatorname{conv}(\gamma([0,1]))$ $(x,t) \in \partial \operatorname{conv}(\gamma([0,1]))$, where $x \in \mathbb{R}^n$, i.e. that is, $x \in \operatorname{conv}(\bar{\gamma}([0,1]))$. Clearly, $B^{\inf}(x) \leq t \leq B^{\sup}(x)$. Assume contrary that $B^{\inf}(x) < t < B^{\sup}(x)$. If $x \in \partial \operatorname{conv}(\bar{\gamma}([0,1]))$ then by (2.12) we have $B^{\inf}(x) = B^{\sup}(x)$, ; therefore, we get a contradiction. If $x \in \operatorname{int}(\operatorname{conv}(\bar{\gamma}([0,1])))$ then (2.12) and the continuity of $B^{\sup}$ and $B^{\inf}$ imply that there exists a ball $U_\varepsilon(x)$ or of radius $\varepsilon > 0$ centered at point x such that $U_\varepsilon(x) \subset \operatorname{int}(\operatorname{conv}(\bar{\gamma}([0,1])))$ and $B^{\inf}(s) < t - \delta < t + \delta < B^{\sup}(s)$ for all $s \in U_\varepsilon(x)$ and some $\delta > 0$. Then

$$\begin{aligned}&(x,t) \in U_{\min\{\varepsilon,\delta\}}((x,t)) \subset \{(s,y) \,:\, B^{\inf}(s) \leq y \leq B^{\sup}(s), s \in U_{\min\{\varepsilon,\delta\}}(x)\} = \\ &\operatorname{conv}(\{(s, B^{\inf}(s)),\ s \in U_{\min\{\varepsilon,\delta\}}(x)\} \cup \{(s, B^{\sup}(s)),\ s \in U_{\min\{\varepsilon,\delta\}}(x)\}) \subset \operatorname{conv}(\gamma([0,1])),\end{aligned}$$

where $U_{\min\{\varepsilon,\delta\}}((x,t))$ is the ball in $\mathbb{R}^{n+1}$ centered at (x,t) with radius $\min\{\varepsilon,\delta\}$. We obtain a contradiction with the assumption that (x,t) belongs to the boundary of $\operatorname{conv}(\gamma([0,1]))$.

The proof of Theorem 2.1 is complete. □

3.4. **The ~~proof~~ Proof of Proposition 2.2.** Take $\gamma(t) = (t, t^4, -t^3)$ on $[-1,1]$. We have

$$(\gamma', \gamma'', \gamma''') = \begin{pmatrix} 1 & 0 & 0 \\ 4t^3 & 12t^2 & 24t \\ -3t^2 & -6t & -6 \end{pmatrix}.$$

All the leading principal minors of the matrix $(\gamma', \gamma'', \gamma''')$ are positive on $[-1,1]\setminus\{0\}$, and we notice that 2×2 and 3×3 ~~the~~ leading principal minors vanish at $t=0$. Assume contrary to Proposition 2.2 that the map $B^{\sup}(x,y)$ defined on $\mathrm{conv}(\overline{\gamma}([-1,1]))$ by (2.1) is concave. We have

$$B(\lambda(a,a^4) + (1-\lambda)(1,1)) = -\lambda a^3 - (1-\lambda), \lambda \in [0,1], a \in (-1,1). \tag{3.33}$$

In particular, ~~$g(y) := B(0,y), y \in [0,1],$~~ $g(y) := B(0,y), y \in [0,1]$ must be concave. The restriction $\lambda a + (1-\lambda) = 0$ implies $\lambda = \frac{1}{1-a}$. Therefore

$$\lambda a^4 + (1-\lambda) = a^3 + a^2 + a \quad \text{and} \quad -\lambda a^3 - (1-\lambda) = a^2 + a.$$

Since $-a^3 - a^2 - a = y \in [0,1]$, we must have $a \in [-1,0]$. Thus $g(-a^3 - a^2 - a) = a^2 + a$ for $a \in [-1,0]$. ~~differentiating~~ Differentiating both sides in a two times, we obtain

$$g'(-a^3 - a^2 - a) = -\frac{2a+1}{3a^2 + 2a + 1},$$
$$g''(-a^3 - a^2 - a) = \frac{-6a(a+1)}{(3a^2+2a+1)^3} > 0 \quad \text{for} \quad a \in [-1,0).$$

Thus $g'' > 0$ gives a contradiction.

3.5. **The ~~proof~~ Proof of Theorem 2.3.** We verify (2.14). The verification of (2.15) is similar. Denote

$$M^{\sup}(x) := \sup_{a\le Y\le b}\{\mathbb{E}\gamma_{n+1}(Y) : \mathbb{E}\overline{\gamma}(Y) = x\}, \quad x \in \mathrm{conv}(\overline{\gamma}([a,b])).$$

First we show the inequality $M^{\sup} \le B^{\sup}$ on $\mathrm{conv}(\overline{\gamma}([a,b]))$. Indeed, let $x \in \mathrm{conv}(\overline{\gamma}([a,b]))$. Pick an arbitrary random variable Y with values in $[a,b]$ ~~,~~ such that $\mathbb{E}\overline{\gamma}(Y) = x$. Then

$$\mathbb{E}\gamma_{n+1}(Y) \overset{(2.10)}{=} \mathbb{E}B^{\sup}(\overline{\gamma}(Y)) \overset{(2.10)+\text{Jensen}}{\le} B^{\sup}(\mathbb{E}\overline{\gamma}(Y)) = B^{\sup}(x).$$

Taking the supremum over all Y, $a \le Y \le b$, such that $\mathbb{E}\overline{\gamma}(Y) = x$, gives the inequality $M^{\sup}(x) \le B^{\sup}(x)$.

To verify the reverse inequality $M^{\sup}(x) \ge B^{\sup}(x)$, it suffices to construct at least one random variable $Y = Y(x)$, $a \le Y \le b$, such that $\mathbb{E}\overline{\gamma}(Y) = x$ and $\mathbb{E}\gamma_{n+1}(Y) = B^{\sup}(x)$. Notice that $Y = \zeta(x)$, where $\zeta(x)$ is defined in Theorem 2.3, satisfies $a \le \zeta(x) \le b$, $\mathbb{E}\overline{\gamma}(\zeta(x)) = x$. It also follows from (2.1) that $\mathbb{E}\gamma_{n+1}(\zeta(x)) = B^{\sup}(x)$.

3.6. **The ~~proof~~ Proof of Corollary 2.4.** . The moment curve γ has totally positive torsion on $[0,1]$, hence, Theorem 2.1 applies.

First we work with $B^{\sup}(x) = x_{n+1}$. Consider the case $n = 2\ell$. By Theorem 2.1 there exists a unique point $(\lambda_1, \ldots, \lambda_\ell, y_1, \ldots, y_\ell) \in \mathrm{int}(\Delta_c^\ell \times \Delta_*^\ell)$ such that $\sum_{j=1}^{\ell}\lambda_j\overline{\gamma}(y_j) + (1-\sum_{j=1}^{\ell}\lambda_j)\overline{\gamma}(1) = x$, then the value $x_{n+1} := B^{\sup}(x)$ equals ~~to~~ $\sum_{j=1}^{\ell}\lambda_j y_j^{2\ell+1} + (1-\sum_{j=1}^{\ell}\lambda_j)$. We would like to show that the linear equation

$$\det\begin{pmatrix} a_0 & a_1 & \ldots & a_\ell \\ \vdots & & & \\ a_\ell & a_{\ell+1} & \ldots & a_{2\ell} \end{pmatrix} = 0, \tag{3.34}$$

where $a_k := x_k - x_{k+1}$, $k = 0, \ldots, 2\ell$, $x_0 := 1$, has a unique solution in x_{n+1} ~~which equals to~~ , which equals $\sum_{j=1}^{\ell}\lambda_j y_j^{2\ell+1} + (1-\sum_{j=1}^{\ell}\lambda_j)$. First we check why $x_{n+1} = \sum_{j=1}^{\ell}\lambda_j y_j^{2\ell+1} + (1-\sum_{j=1}^{\ell}\lambda_j)$ solves

(3.34). Notice that $a_k = \langle y^k, \beta\rangle$, where $y^k := (y_1^k, \dots, y_\ell^k)$, and $\beta := (\lambda_1(1-y_1), \dots, \lambda_\ell(1-y_\ell))$. The j'th column of the matrix in (3.34), call it w_j, $j = 0, \dots, \ell$, we can write as $w_j = AD^j\beta^T$, where A is $(\ell+1)\times\ell$ matrix with m'th column $(1, y_m, \dots, y_m^\ell)^T$, and D is $\ell\times\ell$ diagonal matrix with diagonal entries $y_1, \dots, y_\ell$. Since there exists a nonzero vector $(z_0, \dots, z_\ell) \in \mathbb{R}^{\ell+1}$ such that $z_0D^0 + \dots + z_\ell D^\ell = 0$ (the number of variables z_j is greater than the number of equations, ~~i.e.~~that is, ℓ), it follows that the vectors $\{w_0, \dots, w_\ell\}$ are linearly dependent, so (3.34) holds true.

To show the uniqueness of the solution x_{n+1}, it suffices to show that the leading $\ell\times\ell$ principal minor R of the matrix in (3.34) has nonzero determinant. Notice that $R = \det(\tilde{w}_0, \dots, \tilde{w}_{\ell-1})$, where $\tilde{w}_j = \tilde{A}D^j\beta^T$ and $\tilde{A}$ is obtained from A by removing the last row. Assume contrary that $R = 0$. Then there exists a nonzero vector $(z_0, \dots, z_{\ell-1}) \in \mathbb{R}^\ell$ such that $\tilde{A}(z_0D^0 + \dots + z_{\ell-1}D^{\ell-1})\beta^T = 0$. As $\det(\tilde{A}) \neq 0$ (Vandermonde matrix), we have $(z_0D^0 + \dots + z_{\ell-1}D^{\ell-1})\beta^T = 0$. Since the entries of β^T are nonzero and the matrix $(z_0D^0 + \dots + z_{\ell-1}D^{\ell-1})$ is diagonal, we must have $z_0D^0 + \dots + z_{\ell-1}D^{\ell-1} = 0$. The last equation rewrites as $\tilde{A}^Tz^T = 0$, where $z = (z_0, \dots, z_{\ell-1}) \neq 0$, which is a contradiction.

Next, consider $n = 2\ell-1$. In this case $x = (1-\sum_{j=1}^\ell \lambda_j)\overline{\gamma}(0) + \sum_{j=2}^\ell \lambda_j\overline{\gamma}(y_j) + \lambda_1\overline{\gamma}(1)$ for a unique $(\lambda_1, \dots, \lambda_\ell, y_2, \dots, y_\ell) \in \mathrm{int}(\mathrm{conv}(\overline{\gamma}([0,1])))$, and the value $x_{n+1} := B^{\sup}(x)$ is $(1-\sum_{j=1}^\ell \lambda_j)\gamma_{n+1}(0) + \sum_{j=2}^\ell \lambda_j\gamma_{n+1}(y_j) + \lambda_1\gamma_{n+1}(1)$. Set $b_k := x_k - x_{k+1}$, $k = 1, \dots, 2\ell-1$. As before we would like to show that the linear equation

$$\det\begin{pmatrix} b_1 & b_2 & \dots & b_\ell \\ \vdots & & & \\ b_\ell & b_{\ell+1} & \dots & b_{2\ell-1}\end{pmatrix} = 0, \tag{3.35}$$

has a unique solution in x_{n+1}~~which equals to~~, which equals $(1-\sum_{j=1}^\ell \lambda_j)\gamma_{n+1}(0) + \sum_{j=2}^\ell \lambda_j\gamma_{n+1}(y_j) + \lambda_1\gamma_{n+1}(1)$. To check that such a choice for x_{n+1} solves (3.35), notice that $b_k = \langle y^k, \beta\rangle$, where $y^k = (y_2^k, \dots, y_\ell^k)$ and $\beta = (\lambda_2(1-y_2), \dots, \lambda_\ell(1-y_\ell))$.

The j'th column of the matrix in (3.35), call it w_j, $j = 1, \dots, \ell$, we can write as $w_j = AD^j\beta^T$, where A is $\ell\times(\ell-1)$ matrix with m'th column $(y_m, \dots, y_m^\ell)^T$, $m = 2, \dots, \ell$, and D is $(\ell-1)\times(\ell-1)$ diagonal matrix with diagonal entries $y_2, \dots, y_\ell$. Since there exists a nonzero vector $(z_1, \dots, z_\ell) \in \mathbb{R}^\ell$ such that $z_1D + \dots + z_\ell D^\ell = 0$ (the number of variables z_j is greater than the number of equations, ~~i.e.~~that is, $\ell-1$), it follows that the vectors $\{w_1, \dots, w_\ell\}$ are linearly dependent, so (3.35) holds true.

To show the uniqueness of the solution x_{n+1}, it suffices to show that the leading $(\ell-1)\times(\ell-1)$ principal minor R of the matrix in (3.35) has nonzero determinant. Notice that $R = \det(\tilde{w}_1, \dots, \tilde{w}_{\ell-1})$, where $\tilde{w}_j = \tilde{A}D^j\beta^T$, and $\tilde{A}$ is obtained from A by removing the last row. Assume contrary that $R = 0$. Then there exists nonzero vector $(z_1, \dots, z_{\ell-1}) \in \mathbb{R}^{\ell-1}$ such that $\tilde{A}(z_1D + \dots + z_{\ell-1}D^{\ell-1})\beta^T = 0$. As $\det(\tilde{A}) \neq 0$ (Vandermonde matrix), we have $(z_1D + \dots + z_{\ell-1}D^{\ell-1})\beta^T = 0$. Since the entries of β^T are nonzero and the matrix $(z_1D + \dots + z_{\ell-1}D^{\ell-1})$ is diagonal, we must have $z_1D + \dots + z_{\ell-1}D^{\ell-1} = 0$. The last equation rewrites as $\tilde{A}^Tz^T = 0$ where $z = (z_1, \dots, z_{\ell-1}) \neq 0$, which is a contradiction.

Next we work with $B^{\inf}(x)$. Consider $n = 2\ell$. There is a unique point $(\lambda_1, \dots, \lambda_\ell, y_1, \dots, y_\ell) \in \mathrm{int}(\Delta_c^\ell \times \Delta_*^\ell)$ such that $\sum_{j=1}^\ell \lambda_j\overline{\gamma}(y_j) = x$. It suffices to show that the linear equation

$$\det\begin{pmatrix} x_1 & x_2 & \dots & x_{\ell+1} \\ \vdots & & & \\ x_{\ell+1} & x_{\ell+2} & \dots & x_{2\ell+1}\end{pmatrix} = 0, \tag{3.36}$$

has a unique solution $x_{2\ell+1} = \sum_{j=1}^\ell \lambda_j\gamma_{n+1}(y_j)$. The j'th column of the matrix in (3.36), call it w_j, $j = 1, \dots, \ell+1$, we can write as $w_j = AD^j\lambda^T$, where A is $(\ell+1)\times\ell$ matrix with m'th column $(y_m, \dots, y_m^{\ell+1})^T$, $m = 1, \dots, \ell$, D is $\ell\times\ell$ diagonal matrix with diagonal entries $y_1, \dots, y_\ell$, and $\lambda = (\lambda_1, \dots, \lambda_\ell)$. The rest of the reasoning (including the uniqueness of the solution x_{n+1}) is similar to the one we just discussed for $B^{\sup}$ and $n = 2\ell$.

Finally, consider $n = 2\ell - 1$. There exists a unique point $(\beta_2, \dots, \beta_\ell, y_1, \dots, y_\ell) \in \mathrm{int}(\Delta_c^{\ell-1} \times \Delta_*^\ell)$ such that $\sum_{j=1}^{\ell} \beta_j \gamma(y_j) = x$, where $\beta_1 := 1 - \sum_{j=2}^{\ell} \beta_j$. It suffices to show that the linear equation

$$\det \begin{pmatrix} 1 & x_1 & \dots & x_\ell \\ \vdots & & & \\ x_\ell & x_{\ell+1} & \dots & x_{2\ell} \end{pmatrix} = 0, \tag{3.37}$$

has a unique solution $x_{2\ell} = \sum_{j=1}^{\ell} \beta_j \gamma_{n+1}(y_j)$. The j'th column of the matrix in (3.36), call it w_j, $j = 1, \dots, \ell+1$, we can write as $w_j = AD^{j-1}\beta^T$, where A is $(\ell+1) \times \ell$ matrix with m'th column $(1, y_m, \dots, y_m^\ell)^T$, $m = 1, \dots, \ell$, D is $\ell \times \ell$ diagonal matrix with diagonal entries $y_1, \dots, y_\ell$, and $\beta = (\beta_1, \dots, \beta_\ell)$. The rest of the reasoning (including the uniqueness of the solution x_{n+1}) is similar to the one we just discussed for $B^{\sup}$ and $n = 2\ell$.

3.7. The ~~proof~~ Proof of Corollary 2.5. Assume contrary that there exist $n+1$ points, $\gamma(t_1), \dots, \gamma(t_{n+1})$, where $a \le t_1 < \dots < t_{n+1} \le b$, which lie in a single affine hyperplane. In particular, we have

$$\det(\gamma(t_2) - \gamma(t_1), \gamma(t_3) - \gamma(t_1), \dots, \gamma(t_{n+1}) - \gamma(t_1)) = 0. \tag{3.38}$$

On the other hand, we have

$$\begin{aligned} &\det(\gamma(t_2) - \gamma(t_1), \gamma(t_3) - \gamma(t_1), \dots, \gamma(t_{n+1}) - \gamma(t_1)) = \\ &\det(\gamma(t_2) - \gamma(t_1), \gamma(t_3) - \gamma(t_2), \dots, \gamma(t_{n+1}) - \gamma(t_n)) = \\ &\int_{t_n}^{t_{n+1}} \dots \int_{t_2}^{t_3} \int_{t_1}^{t_2} \det(\gamma'(s_1), \gamma'(s_2) \dots, \gamma'(s_n)) ds_1 ds_2 \dots ds_n > 0 \end{aligned}$$

by Lemma 3.1. Thus we have a contradiction with (3.38).

3.8. The ~~proof~~ Proof of Corollary 2.7. To prove the formulas for the volume, we apply Theorem 2.1, where γ in Corollary 2.7 will be used as $\overline{\gamma}$ in Theorem 2.1. Let $n = 2\ell$. To verify

$$\begin{aligned} &\mathrm{Vol}(\mathrm{conv}(\gamma([a,b]))) \\ &\quad = \frac{(-1)^{\frac{\ell(\ell-1)}{2}}}{(2\ell)!} \int_{a \le x_1 \le \dots \le x_\ell \le b} \det(\gamma(x_1) - \gamma(a), \dots, \gamma(x_\ell) - \gamma(a), \gamma'(x_1), \dots, \gamma'(x_\ell)) dx, \end{aligned} \tag{3.39}$$

notice that according to Theorem 2.1 the map $U := U_{2\ell}$, where

$$U_{2\ell} : \Delta_c^\ell \times \Delta_*^\ell \ni (\lambda_1, \dots, \lambda_\ell, x_1, \dots, x_\ell) \mapsto (1 - \sum_{j=1}^{\ell} \lambda_j)\gamma(a) + \sum_{j=1}^{\ell} \lambda_j \gamma(x_j),$$

is diffeomorphism between $\mathrm{int}(\Delta_c^\ell \times \Delta_*^\ell)$ and $\mathrm{int}(\mathrm{conv}(\gamma([a,b])))$. In particular, by the change of variables formula, we have

$$\begin{aligned} &\mathrm{Vol}(\mathrm{conv}(\gamma([a,b]))) = \int_{\Delta_c^\ell} \int_{\Delta_*^\ell} |\det(U_{\lambda_1}, \dots, U_{\lambda_\ell}, U_{x_1}, \dots, U_{x_\ell})| d\lambda\, dx = \\ &\int_{\Delta_c^\ell} \lambda_1 \dots \lambda_\ell d\lambda \int_{\Delta_*^\ell} |\det(\gamma(x_1) - \gamma(a), \dots, \gamma(x_\ell) - \gamma(a), \gamma'(x_1), \dots, \gamma'(x_\ell))| dx. \end{aligned}$$

Next, using the identity

$$\int_{\Delta_c^\ell} \lambda_1^{p_1-1} \dots \lambda_\ell^{p_\ell-1} (1 - \sum_{j=1}^{\ell} \lambda_j)^{p_0-1} d\lambda = \frac{\prod_{j=0}^{\ell} \Gamma(p_j)}{\Gamma(\sum_{j=0}^{\ell} p_j)} \tag{3.40}$$

valid for all $p_0, \dots, p_\ell > 0$ (see Dirichlet distribution in [17]) ~~,~~ and the property

$$|\det(\gamma(x_1) - \gamma(a), \dots, \gamma(x_\ell) - \gamma(a), \gamma'(x_1), \dots, \gamma'(x_\ell))|$$
$$= (-1)^{\frac{\ell(\ell-1)}{2}} \det(\gamma(x_1) - \gamma(a), \dots, \gamma(x_\ell) - \gamma(a), \gamma'(x_1), \dots, \gamma'(x_\ell))$$

whenever $a < x_1 < \dots < x_\ell < b$, see (3.28), we recover (3.39). The other three identities in Corollary 2.7 are obtained in the same way by repeating the computations with $L_{2\ell}$, and in the case of odd dimensions with $U_{2\ell-1}$ and $L_{2\ell-1}$.

3.9. **The ~~proof~~ Proof of Corollary 2.8.** Let $n = 2\ell$ (the case $n = 2\ell - 1$ is similar and will be omitted), and let us verify the identity

$$\mathrm{Area}(\partial\, \mathrm{conv}(\gamma([a,b]))) = \frac{1}{n!} \int_{a \le x_1 \le \dots \le x_\ell \le b} \left(\sqrt{\det S_a^{\mathrm{Tr}} S_a} + \sqrt{\det S_b^{\mathrm{Tr}} S_b} \right) dx,$$

where $S_r = (\gamma(x_1) - \gamma(r), \dots, \gamma(x_\ell) - \gamma(r), \gamma'(x_1), \dots, \gamma'(x_\ell))$. By (2.13) we have

$$\partial\, \mathrm{conv}(\gamma([a,b])) = \{(x, B^{\sup}(x)), x \in \mathrm{conv}(\bar\gamma([a,b]))\} \cup \{(x, B^{\inf}(x)), x \in \mathrm{conv}(\bar\gamma([a,b]))\}.$$

On the other hand, by (2.12) and (2.3) the set $\{(x, B^{\sup}(x)), x \in \mathrm{conv}(\bar\gamma([a,b]))\} \cap \{(x, B^{\inf}(x)), x \in \mathrm{conv}(\bar\gamma([a,b]))\}$ is contained in the image of C^1 map of the set $\partial(\Delta_c^\ell \times \Delta_*^\ell)$, which has zero n dimensional Lebesgue measure. Therefore, it follows from (2.4) and (2.5) that

$$\mathrm{Area}(\partial\, \mathrm{conv}(\gamma([a,b]))) =$$
$$\mathrm{Area}(\{(x, B^{\sup}(x)), x \in \mathrm{conv}(\bar\gamma([a,b]))\}) + \mathrm{Area}(\{(x, B^{\inf}(x)), x \in \mathrm{conv}(\bar\gamma([a,b]))\}) =$$
$$\int_{\Delta_c^\ell \times \Delta_*^\ell} \sqrt{\det A^{\mathrm{Tr}} A}\, dx d\lambda + \int_{\Delta_c^\ell \times \Delta_*^\ell} \sqrt{\det C^{\mathrm{Tr}} C}\, dx d\lambda,$$

where $A = (U_{\lambda_1}, \dots, U_{\lambda_\ell}, U_{x_1}, \dots, U_{x_\ell})$ with $U := U_n$, and $C = (L_{\lambda_1}, \dots, L_{\lambda_\ell}, L_{x_1}, \dots, L_{x_\ell})$ with $L := L_n$. Notice that $A^{\mathrm{Tr}} A = R S_b^{\mathrm{Tr}} S_b R$, where R is $2\ell \times 2\ell$ diagonal matrix with diagonal entries $r_1 = \dots = r_\ell = 1$, and $r_{\ell+1} = \lambda_1, \dots, r_{\ell+\ell} = \lambda_\ell$. Similarly, $C^{\mathrm{Tr}} C = R S_a^{\mathrm{Tr}} S_a R$. Therefore,

$$\int_{\Delta_c^\ell \times \Delta_*^\ell} \sqrt{\det A^{\mathrm{Tr}} A}\, dx d\lambda + \int_{\Delta_c^\ell \times \Delta_*^\ell} \sqrt{\det C^{\mathrm{Tr}} C}\, dx d\lambda =$$
$$\int_{\Delta_c^\ell} \lambda_1 \cdots \lambda_\ell d\lambda \int_{\Delta_*^\ell} \sqrt{\det S_b^{\mathrm{Tr}} S_b} dx + \int_{\Delta_c^\ell} \lambda_1 \cdots \lambda_\ell d\lambda \int_{\Delta_*^\ell} \sqrt{\det S_a^{\mathrm{Tr}} S_a} dx \overset{(3.40)}{=}$$
$$\frac{1}{(2\ell)!} \int_{\Delta_*^\ell} \left(\sqrt{\det S_b^{\mathrm{Tr}} S_b} + \sqrt{\det S_a^{\mathrm{Tr}} S_a} \right) dx.$$

This finishes the proof of Corollary 2.8.

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