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Exponential stability almost surely of linear dynamical systems on stochastic pulse time scales

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Abstract— **In this paper, we expand the stability theory of dynamical systems on stochastic time scales to the case of *stochastic pulse time scales*. The class of systems considered here evolve on nonuniform time-domains that consist of a union of disjoint closed intervals with stochastic lengths, followed by random step sizes. Necessary and sufficient conditions for exponential stability almost surely are derived. The approach is based on determining the region of exponential stability almost surely. An illustrative numerical example is presented to validate the results. The class of systems considered in this paper has important applications for example, control networks subject to communications failures, population dynamics, signal processing with variable sampling, consensus multi-agents systems and wide-area power system controls.**

Keywords: Time scales; Stability analysis; Exponential stability almost surely, Mean square stability.

I. Introduction

Most of the existing methods for analyzing the stability of dynamical systems are applied to systems operating on a continuous-time domain or a uniform discrete-time domain. However, in engineering and several areas of industry, there are many dynamical systems which operate on a non-uniform time domain that can be either discrete non-uniform or a combination of discrete points and variable continuous intervals. For example, impulsive systems in which the state jumps are not instantaneous, a set of discrete-time controllers, signal processing with variable sampling, population model, dynamic programming, and failure of communication in control networks. Time scales theory was developed to study such complex systems because it captures the interplay between continuous and discrete analysis. It leads to understanding and analyzing dynamical systems evolving on any non-uniform time domain, and to extends these theories to more general classes of dynamical systems on a non-uniform time domain, denoted $\mathbb{T}$. Time scales theory advanced quickly, culminating in the advanced monograph [1]. Recently, the application of time scales theory in systems and control has gained attention in the literature. The stability analysis was studied for linear and nonlinear systems evolving on an arbitrary time scale in several works [2], [3], [4], [5]. This analysis was extended to systems evolving on a stochastic non-uniform discrete time scale in [6], [7], [8]. In [9], [10], [11], the authors extended the study to a switched systems. In [12], the authors considered a special class of switched systems evolving on the time scale $\mathbb{T} = \cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}], \sigma(t_k) > t_k$, formed by a union of disjoint closed intervals with variable length, followed by a variable step size. The exponential stability of this class of switched systems has been studied in [13], [14] by considering that unstable modes may exist. In [15], the authors derived a necessary and sufficient condition for exponential stability by introducing a region of exponential stability $\mathcal{S}(\mathbb{T})$. The problem of intermittent information transmission in communication networks due to temporary sensor failures or the presence of communication obstacles was converted to this special class of switched systems (continuous/discrete) and studied in [16], [14] (for the consensus of multi-agent systems problem), and in [17] (for the problem of wide-area power system controls).

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In this paper, the stability of linear systems on a stochastic time scale $\mathbb{T} = \cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}]$ is analyzed. The underlying assumption is that the system runs in continuous-time for a random amount of time τ, followed by a disruption of stochastic length μ. The length of each continuous interval and the length of each disruption period is drawn from one of two separate stationary statistical distributions. Moreover, we assume that the length of each continuous interval is independent of the length of each disruption. Such a situation could arise, for example, in the leader-follower consensus problem with variable uptime and downtime [16], [18], since communication uptime and downtime are stochastic in nature. This situation could also arise in network control systems with variable sampling times [19]. To approach this subject, we utilize the theory of dynamic equations on stochastic time scales. In particular, necessary ad sufficient conditions of exponential stability almost surely are derived by introducing region of exponential stability almost surely.

II. Preliminaries

A time scale $\mathbb{T}$ is any nonempty closed subset of the real numbers $\mathbb{R}$, as the usual integer subsets ($h\mathbb{Z}$ or $\mathbb{N}$), any discrete set with variable steps sizes, or any combination of discrete points unioned with closed intervals [1]. $\forall t \in \mathbb{T}$, the *forward jump operator* is given by $\sigma(t) := \inf_{s\in\mathbb{T}}\{s > t\}$, the *backward jump operator* by $\rho(t) := \sup_{s\in\mathbb{T}}\{s < t\}$, and the *graininess function* by $\mu(t) := \sigma(t) - t$ which measure the distance between any two consecutive times. If $\mathbb{T}$ has a left-scattered, then the set $\mathbb{T}^{\kappa} = \mathbb{T} - \{m\}$; otherwise $\mathbb{T}^{\kappa} = \mathbb{T}$. The time scale Δ-derivative of $x : \mathbb{T}^{\kappa} \to \mathbb{R}$ is defined as

$$x^{\Delta}(t) := \lim_{s \searrow t} \frac{x(\sigma(t)) - x(s)}{\sigma(t) - s}.$$

One can notice that if $\mathbb{T} = \mathbb{R}$, we have $\sigma(t) = t$, and $x^{\Delta}(t) = \dot{x}(t)$, which is the usual derivative of x. If $\mathbb{T} = h\mathbb{Z}$, we have $\sigma(t) = t + h$, then $x^{\Delta}(t) = \frac{x(t+h)-x(t)}{h}$, which is the usual difference equation. A function $\lambda : \mathbb{T} \to \mathbb{R}$ is *regressive* (resp. *positively regressive*) if $1 + \mu(t)\lambda(t) \neq 0$ (resp. $1 + \mu(t)\lambda(t) > 0$), $\forall t \in \mathbb{T}^{\kappa}$ and it is said to be *uniformly regressive*, if $\exists \gamma > 0$ such that $\gamma^{-1} \leq |1 + \mu(t)\lambda(t)|, \forall t \in \mathbb{T}^{\kappa}$ [4]. We denote the set of regressive (resp. positively regressive) functions by $\mathcal{R}$ (resp. $\mathcal{R}^{+}$). A matrix function $A : \mathbb{T} \to \mathbb{R}^{n\times n}$ is called *regressive* if and only if all its eigenvalues are regressive (see [1]). Consider the linear dynamical system on an arbitrary time scale $\mathbb{T}$

$$x^{\Delta} = Ax, \quad x(t_0) = x_0, \quad A \in \mathcal{R}. \tag{1}$$

The unique solution of (1) is $x(t) = e_A(t, t_0)x_0$, where the *generalized exponential function* $e_A(t, t_0)$ is given by

$$e_A(t,s) = \begin{cases} \exp\left(\int_s^t \dfrac{\log(1+\mu(\tau)A(\tau))}{\mu(\tau)}\Delta\tau\right); & \mu(\tau) \neq 0 \\ \exp(\int_s^t A(\tau)d\tau); & \mu(\tau) = 0, \end{cases} \tag{2}$$

where the Δ-integral is used [1]. For $f : \mathbb{T} \to \mathbb{R}$, we have

$$\int_t^{\sigma(t)} f(s)\Delta s = \mu(t)f(t). \tag{3}$$

Let us present some algebraic results on matrices on time scales theory that are needed for this paper. Let $A \in \mathbb{R}^{n\times n}$ be a regressive matrix, there always exists an invertible matrix $Q \in \mathbb{C}^{n\times n}$ such that $e_A(t, s)$ is given, for $t, s \in \mathbb{T}^{\kappa}$, by

$$e_A(t,s) = Q \begin{pmatrix} e_{J_1}(t,s) & & \\ & \ddots & \\ & & e_{J_l}(t,s) \end{pmatrix} Q^{-1}, \tag{4}$$

$$\text{with } J = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_l \end{pmatrix}, \text{ such that } J_k \in \mathbb{C}^{d_k\times d_k} \tag{5}$$

$$\text{and } J_k = \lambda_k I + N = \begin{pmatrix} \lambda_k & 1 & 0 & \dots & 0 \\ & \lambda_k & 1 & \dots & 0 \\ & & \ddots & & \vdots \\ & & & & \lambda_k \end{pmatrix}, \tag{6}$$

where $k = 1, 2, \dots, l$ and d_k is the algebraic order of λ_k, such that $d_1 + d_2 + \ldots + d_l = n$. The matrix $N \in \mathbb{C}^{d_k\times d_k}$ denotes a square matrix which satisfies $N^{d_k} = 0$. We have $spec(J_k) = \{\lambda_k\}$, whence $spec(A) = \{\lambda_1, \lambda_2, \dots, \lambda_l\}$, the set of the eigenvalues of A. Each Jordan block J_k has only one independent eigenvector. If A has distinct eigenvalues, then $d_k = 1$ and $N = 0$ with $J = diag(\lambda_1, \lambda_2, \dots, \lambda_n)$. We have the following property:

$$e_{J_k}(t,s) = e_{\lambda_k}(t,s) \begin{pmatrix} 1 & m^1_{\lambda_k}(t,s) & \dots & m^{d_k-1}_{\lambda_k}(t,s) \\ & 1 & \dots & m^{d_k-2}_{\lambda_k}(t,s) \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix}, \tag{7}$$

such that, the mappings $m^n_{\lambda} : \mathbb{T}^{\kappa} \times \mathbb{T}^{\kappa} \to \mathbb{C}$ are monomials of degree n, and recursively defined, for a regressive function $\lambda : \mathbb{T}^{\kappa} \to \mathbb{C}$, by

$$m^0_{\lambda}(t,s) = 1, \quad m^{n+1}_{\lambda}(t,s) = \int_s^t \frac{m^n_{\lambda}(\tau,s)}{1+\mu(\tau)\lambda(\tau)}\Delta\tau, \ \forall n \in \mathbb{N}. \tag{8}$$

- For $\mathbb{T} = \mathbb{R}$ and $\lambda \in \mathbb{C}$, $m^n_{\lambda}(t,s) = \frac{(t-s)^n}{n!}$, for $t, s \in \mathbb{R}$.
- For $\mathbb{T} = h\mathbb{Z}$ and a regressive constant $\lambda \in \mathbb{C}$, $m^n_{\lambda}(t,s) = \frac{(t-s)^n}{n!(1+h\lambda)^n}$, for $t, s \in h\mathbb{Z}$.

Lemma II.1. *[4] Let λ be uniformly regressive, then the bound $|m^n_{\lambda}(t,s)| \leq \gamma^n(t-s)^n$ holds for $t \geq s$ and $n \in \mathbb{N}^*(=\mathbb{N}/\{0\})$.*

The dynamical system (1) is exponentially stable, if there exists a constant $\beta \geq 1$ and a constant $\alpha < 0$, with $\alpha \in \mathcal{R}^{+}$, such that the corresponding solutions satisfies

$$\|x(t)\| \leq \beta\|x_0\|e_{\alpha}(t,t_0), \quad \forall t_0, t \in \mathbb{T}, \ t_0 \leq t.$$

Specifically, the condition that $\alpha \in \mathcal{R}^{+}$ reduce to $\alpha < 0$ for $\mathbb{T} = \mathbb{R}$, and to $\frac{-1}{h} < \alpha < 0$, for $\mathbb{T} = h\mathbb{Z}$. In [4], the authors classified the exponential stability of (1) by determining a region of exponential stability $\mathcal{S}_{\mathbb{C}}$ as follows:

$$\mathcal{S}_{\mathbb{C}}(\mathbb{T}) := \{\lambda \in \mathbb{C} : \limsup_{T\to\infty} \frac{1}{T-t_0}\int_{t_0}^T \lim_{s\to\mu(t)} \frac{\log|1+s\lambda|}{s}\Delta t < 0\}, \tag{9}$$

and the dynamical system (1) is exponentially stable if and only if the eigenvalues of A are uniformly regressive and are all inside $\mathcal{S}_{\mathbb{C}}$. In [20], the authors simplified the computation of the set $\mathcal{S}_{\mathbb{C}}(\mathbb{T})$ for purely discrete time scales where the graininess function has a finite range by introducing the concept of asymptotic graininess. In [21], the computation of this set was generalized to a discrete stochastic time scale. Note that, the computation of this set for an arbitrary $\mathbb{T}$ can be difficult. Therefore, the *Hilger circle* is defined as [1]

$$\mathcal{H}_{\mu(t)} := \{z \in \mathbb{C}/\{-1/\mu(t)\} : |1+\mu(t)z| < 1\}, \ \mathcal{H}_0 = \mathbb{C}^{-} \tag{10}$$

The authors in [22] showed that for any time scale, the system (1) is exponentially stable if the eigenvalues of A are inside the Hilger's discs $\mathcal{H}_{\min}$ corresponding to $\mu(t) = \mu_{\max} = \sup_{t\in\mathbb{T}}\mu(t) < \infty$, which is a subset of $\mathcal{S}_{\mathbb{C}}$. The sudy of stability of dynamical systems on an arbitrary time scale with variable $\mu(t)$ (deterministic or stochastic) is a very interesting question that has garnered much attention [20], [4], [6], [21], [15]. Consider now the discrete stochastic time scale $\tilde{\mathbb{T}}$ generated by a graininess function μ, which is considered as a random variable and determined by the sequence $\mu = \{\mu_k\}_{k\in\mathbb{N}}$, such that (see Fig. 1)

$$\tilde{\mathbb{T}} := \{t_0\} \cup \left\{t_0 + \sum_{i=0}^{n}\mu_i : \ n \in \mathbb{N}_0\right\}, \tag{11}$$

where μ_i are independent and identically distributed for all $i \in \mathbb{N}$. We define the concepts of exponential stability almost surely of the dynamical system (1) on $\tilde{\mathbb{T}}$. We say that the

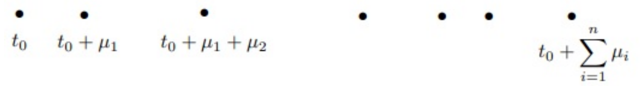

Fig. 1: Generating a sequence of representative values from a discrete stochastic pulse time scale $\tilde{\mathbb{T}}$

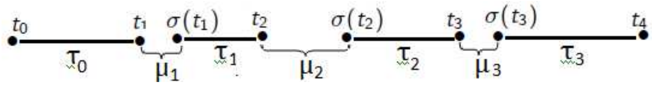

Fig. 2: Generating a sequence of representative values from a stochastic pulse time scale, $\hat{\mathbb{T}}_{\tau,\mu}$

dynamical system (1) is exponentially stable almost surely on $\tilde{\mathbb{T}}$, if and only if

$$\mathbb{P}[\exists\alpha>0,\ \text{s.t}\ \forall t_0, t_k\in\tilde{\mathbb{T}},\ \exists K=K(t_0)\geq 1:\\ |e_A(t_k,t_0)|\leq Ke^{-\alpha(t_k-t_0)}]=1.$$

where $\mathbb{P}$ denotes the probability. On $\tilde{\mathbb{T}}$, a region of exponential stability almost surely is determined in [21] for a scalar dynamical system, which is analogous to the set (9), as follows.

Theorem II.2 ([21])**.** *Let $\lambda\in\mathbb{C}$. Assume $\lambda\neq -1/\mu$ almost surely. Then the scalar dynamic equation $x^\Delta=\lambda x$, is exponentially stable almost surely on $\tilde{\mathbb{T}}$ if*

$$\mathbb{E}[\ln|1+\lambda\mu|]<0, \qquad (\mathbb{E}\ \textit{denotes the expectation}),$$

and the region of exponential stability almost surely is given by the set $\hat{\mathcal{S}}=\{\lambda\in\mathbb{C}:\mathbb{E}[\ln|1+\lambda\mu|]<0\}$.

III. Problem statement

The objective of this work is to extends the results of the exponential stability almost surely, presented in [21], for the discrete stochastic time scales $\tilde{\mathbb{T}}$ defined in (11) to a new class of stochastically generated time scale. Let $A\in\mathbb{R}^{n\times n}$ be a regressive matrix and consider the dynamical system

$$x^\Delta(t)=Ax(t), \quad x(t_0)=x_0, \tag{12}$$

on the time scale $\mathbb{T}=\cup_{k=0}^{\infty}[\sigma(t_k),t_{k+1}]$ formed by a union of disjoints closed intervals, with $t_0=\sigma(t_0)=0$. Denote the length of the continuous intervals by $\tau_k:=t_{k+1}-\sigma(t_k),\ k\in\mathbb{N}$ and the discrete gap sizes by $\mu_k:=\sigma(t_k)-t_k,\ k\in\mathbb{N}$, which are considered as random variables. This structure is visualized in Figure 2. We call the time scale generated by these random variables, a *stochastic pulse time scale*. The system (12) can be converted to a discrete switched system evolving on the discrete stochastic time scale $\hat{\mathbb{T}}_{\tau,\mu}=\cup_{k=0}^{\infty}(\{t_k\}\cup\{\sigma(t_k)\})$, such that its forward jump operator $\hat{\sigma}:\hat{\mathbb{T}}_{\tau,\mu}\to\mathbb{R}$, is determined by $\hat{\sigma}(t_k)=\sigma(t_k)$ and $\hat{\sigma}(\sigma(t_k))=t_{k+1}$. For $\sigma(t_k)\leq t\leq t_{k+1},\ k\in\mathbb{N}$, we have

$$x(t)=e^{A(t-\sigma(t_k))}x(\sigma(t_k)).$$

For $t=t_{k+1}$, $\quad x(t_{k+1})=e^{A(t_{k+1}-\sigma(t_k))}x(\sigma(t_k)),\ \forall k\in\mathbb{N}$. From the above equation, we get

$$\begin{aligned}\frac{x(t_{k+1})-x(\sigma(t_k))}{t_{k+1}-\sigma(t_k)} &= \frac{e^{A(t_{k+1}-\sigma(t_k))}x(\sigma(t_k))-x(\sigma(t_k))}{\tau_k}\\ &=\left(\frac{e^{A\tau_k}-I_n}{\tau_k}\right)x(\sigma(t_k)).\end{aligned} \tag{13}$$

Note that, for $t=\sigma(t_k)\in\hat{\mathbb{T}}_{\tau,\mu},\ k\in\mathbb{N}$, the discrete Δ-derivative of $x(t)$ is given by

$$x^\Delta(\sigma(t_k))=\frac{x(\hat{\sigma}(\sigma(t_k)))-x(\sigma(t_k))}{\hat{\sigma}(\sigma(t_k))-\sigma(t_k)}=\frac{x(t_{k+1})-x(\sigma(t_k))}{t_{k+1}-\sigma(t_k)}. \tag{14}$$

Finally, according to (13) and (14) we can convert (12) to the following discrete switched linear system on $\hat{\mathbb{T}}_{\tau,\mu}$

$$x^\Delta(t)=\begin{cases}\left(\dfrac{e^{A\tau_k}-I}{\tau_k}\right)x(t) & ;\quad t\in\cup_{k=0}^{\infty}\{\sigma(t_k)\}\\ Ax(t) & ;\quad t\in\cup_{k=0}^{\infty}\{t_{k+1}\}.\end{cases} \tag{15}$$

Let us define the graininess function of $\hat{\mathbb{T}}_{\tau,\mu}$

$$\hat{\mu}(t)=\hat{\sigma}(t)-t=\begin{cases}\tau_k & ;\quad t\in\cup_{k=0}^{\infty}\{\sigma(t_k)\}\\ \mu_k & ;\quad t\in\cup_{k=0}^{\infty}\{t_{k+1}\}.\end{cases}$$

We will consider that the sequences of continuous intervals length $\tau=\{\tau_k\}_{k\in\mathbb{N}}$ and the discrete gaps $\mu=\{\mu_k\}_{k\in\mathbb{N}}$ form a sequence of independent and identically distributed (i.i.d) random variables. Note that, the region of exponential stability $\mathcal{S}_{\mathbb{C}}$ of (12) has been derived in [4] for a particular $\mu(t)$ of $\mathbb{T}=\cup_{k=0}^{\infty}[\sigma(t_k),t_{k+1}]$. This result was extended in [15] for a class of switched systems between a continuous-time dynamic on $\cup_{k=0}^{\infty}[\sigma(t_k),t_{k+1}]$ and a discrete-time dynamic at times $\cup_{k=0}^{\infty}\{t_{k+1}\}$. The aims of this paper is to determine a region of exponential stability almost surely of the system (12) using a stochastic approach by considering the converted switched system (15), evolving on the purely discrete time scale $\hat{\mathbb{T}}_{\tau,\mu}$, and then generalize the results to any switched system.

IV. Main results

First, we shall study the scalar case of the class of switched systems (15) by determining the region of exponential stability and deriving the conditions for exponential stability almost surely. Next we will generalize the results to the matricial case.

A. Scalar switched system

1) Region of exponential stability on random time scales:

Consider the scalar dynamical system

$$x^\Delta(t)=\lambda x(t), \quad x(t_0)=x_0, \quad \lambda\in\mathbb{C} \tag{16}$$

on $\mathbb{T}=\cup_{k=0}^{\infty}[\sigma(t_k),t_{k+1}]$, with $\lambda\in\mathcal{R}$. As previously, the system (16) can be converted to the scalar switched system on the discrete time scale $\hat{\mathbb{T}}_{\tau,\mu}$, as follows:

$$x^\Delta(t)=\begin{cases}\left(\dfrac{e^{\lambda\tau_k}-1}{\tau_k}\right)\ x(t) & ;\quad t\in\cup_{k=0}^{\infty}\{\sigma(t_k)\}\\ \lambda\ x(t) & ;\quad t\in\cup_{k=0}^{\infty}\{t_{k+1}\}\end{cases} \tag{17}$$

In the deterministic case, $\hat{\mathbb{T}}_{\tau,\mu}$ is considered as a discrete time scale generated by the gaps $\{\tau_k\}_{k\in\mathbb{N}}$ and $\{\mu_k\}_{k\in\mathbb{N}}$ which are variable in time. We determine the region of exponential stability of (17) in the following Proposition.

Proposition IV.1. *Let the switched system (17) on $\hat{\mathbb{T}}_{\tau,\mu}$. The region of exponential stability of (17) is given by the set*

$$\begin{aligned}\mathcal{S}_{\tau,\mu}(\hat{\mathbb{T}}) = \{&\lambda \in \mathbb{C} : \\ &\limsup_{t\to\infty} \tfrac{1}{(t-t_0)} \textstyle\int_{t_0}^{t} \tfrac{\log|1+\hat{\mu}(s)\lambda|}{\hat{\mu}(s)} \Delta s < 0\},\end{aligned} \tag{18}$$

for $t_0, t \in \hat{\mathbb{T}}_{\tau,\mu}$, and $t_0 < t$, with

$$\frac{\log|1+\hat{\mu}(s)\lambda|}{\hat{\mu}(s)} = \begin{cases} \Re(\lambda)\tau_k; & s \in \cup_{k=0}^{\infty}\{\sigma(t_k)\} \\ \dfrac{\log|1+\mu_k\lambda|}{\mu_k}; & s \in \cup_{k=0}^{\infty}\{t_{k+1}\} \end{cases} \tag{19}$$

where $\Re(.)$ is the real part and $|.|$ is the modulus. The system (17) is exponentially stable if and only if $\lambda \in \mathcal{S}_{\tau,\mu}(\hat{\mathbb{T}})$.

Proof. The proof follows by a modification of the proof of Proposition 1 in [15] (see Appendix). □

2) Exponential stability almost surely:
Consider the switched system (17) on $\hat{\mathbb{T}}_{\tau,\mu}$ which is considered as an i.i.d stochastic time scale generated by the random variables $\{\tau_k\}_{k\in\mathbb{N}}$ and $\{\mu_k\}_{k\in\mathbb{N}}$. We determine a necessary and sufficient condition for exponential stability almost surely in the following Theorem.

Theorem IV.2. *The scalar switched system (17) is exponentially stable almost surely on $\hat{\mathbb{T}}_{\tau,\mu}$ if and only if*

$$\mathbb{E}[\Re(\lambda)\tau_k + \log(|1+\mu_k\lambda|)] < 0. \tag{20}$$

Proof. From the system (17) on $\hat{\mathbb{T}}_{\tau,\mu}$, we have

$$x(t_{k+1}) = e^{\lambda\tau_k}(1+\mu_k\lambda)x(t_k), \quad \forall k \in \mathbb{N}. \tag{21}$$

By applying the Theorem of Bitmead [23] which state that the stochastic difference equation $x_{n+1} = a_n x_n$, where $\{a_n\}$ is a sequence of ergodic scalar random variables, is exponentially stable almost surely if and only if $\mathbb{E}[\log(|a_n|)] < 0$. Since $\{\mu_k\}_{k\in\mathbb{N}}$ and $\{\tau_k\}_{k\in\mathbb{N}}$ are sequence of mutually independent random variables, so $\{e^{\lambda\tau_k}(1+\mu_k\lambda)\}_{k\in\mathbb{N}}$ is a sequence of ergodic scalar random variables and we have

$$\mathbb{E}[\log(|e^{\lambda\tau_k}(1+\mu_k\lambda)|)] = \mathbb{E}[\Re(\lambda)\tau_k + \log(|1+\mu_k\lambda|)] < 0. \tag{22}$$

□

From Proposition IV.1 and Theorem IV.2, we can derive the following relationship between the exponential stability almost surely and the region of exponential stability $\mathcal{S}_{\tau,\mu}(\hat{\mathbb{T}})$.

Corollary IV.2.1. *The switched system (17) is exponentially stable almost surely if and only if $\lambda \in \mathcal{S}_{\tau,\mu}(\hat{\mathbb{T}})$.*

Proof. Let $\tau = \{\tau_k\}_{k\in\mathbb{N}}$ and $\mu = \{\mu_k\}_{k\in\mathbb{N}}$. By the strong law of large numbers [24], we have

$$\mathbb{E}[\Re(\lambda)\tau + \log(|1+\mu\lambda|)] = \lim_{k\to\infty} \frac{\sum_{i=0}^{k-1}[\Re(\lambda)\tau_i + \log|1+\mu_i\lambda|]}{k} \tag{23}$$

According to (3), we have the two relationships

$$\Re(\lambda)\tau_i = \int_{\sigma(t_i)}^{t_{i+1}} \Re(\lambda)\Delta s \tag{24}$$

$$\log|1+\mu_i\lambda| = \int_{t_i}^{\sigma(t_i)} \frac{\log|1+\mu(s)\lambda|}{\mu(s)}\Delta s \tag{25}$$

From (24), (25) and (19), we get

$$\Re(\lambda)\tau_i + \log|1+\mu_i\lambda| = \int_{t_i}^{t_{i+1}} \frac{\log|1+\hat{\mu}(s)\lambda|}{\hat{\mu}(s)}\Delta s \tag{26}$$

From (23) and (26), we have

$$\begin{aligned}
&\mathbb{E}[\Re(\lambda)\tau + \log(|1+\mu\lambda|)] \\
&= \lim_{k\to\infty} \frac{1}{k} \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \frac{\log|1+\hat{\mu}(s)\lambda|}{\hat{\mu}(s)}\Delta s \\
&= \lim_{k\to\infty} \frac{(t_{k-1}-t_0)}{k(t_{k-1}-t_0)} \int_{t_0}^{t_{k-1}} \frac{\log|1+\hat{\mu}(s)\lambda|}{\hat{\mu}(s)}\Delta s \\
&= \lim_{k\to\infty} \frac{\sum_{i=0}^{k-1}(\tau_i+\mu_i)}{k(t_{k-1}-t_0)} \int_{t_0}^{t_{k-1}} \frac{\log|1+\hat{\mu}(s)\lambda|}{\hat{\mu}(s)}\Delta s \\
&= (\mathbb{E}[\tau]+\mathbb{E}[\mu]) \lim_{k\to\infty} \frac{1}{(t_{k-1}-t_0)} \int_{t_0}^{t_{k-1}} \frac{\log|1+\hat{\mu}(s)\lambda|}{\hat{\mu}(s)}.
\end{aligned}$$

Since $(\mathbb{E}[\tau]+\mathbb{E}[\mu]) > 0$, so $\mathbb{E}[\Re(\lambda)\tau + \log(|1+\mu\lambda|)] < 0$, if and only if $\lambda \in \mathcal{S}_{\tau,\mu}(\hat{\mathbb{T}})$. Suppose now that $\mathbb{E}[\Re(\lambda)\tau + \log(|1+\mu\lambda_j|)] > 0$, so from the above analysis, we have

$$(\mathbb{E}[\mu]+\mathbb{E}[\tau]) \lim_{k\to\infty} \frac{1}{(t_k-t_0)} \int_{t_0}^{t_k} \frac{\log|1+s\lambda|}{s}\Delta s > 0$$

which implies that $\lambda \notin \mathcal{S}_{\mu,\tau}(\hat{\mathbb{T}})$ and from Theorem IV.2, the switched system is not exponentially stable almost surely. □

B. Generalization to linear matrix switched systems

Consider now the switched dynamical system (15). Based on a result of Theorem 2 in [15], we will characterize the exponential stability of this class of switched systems.

Theorem IV.3. *(Characterization of exponential stability) Let the discrete time scale $\hat{\mathbb{T}}_{\tau,\mu}$ and a regressive matrix $A \in \mathbb{R}^{n\times n}$. The following properties are satisfied:*

(i) *If the switched system (15) is exponentially stable, then for any eigenvalue λ_j of A, we have $\lambda_j \in \mathcal{S}_{\tau,\mu}(\hat{\mathbb{T}})$.*
(ii) *If the eigenvalues λ_j of A are uniformly regressive, and if for any eigenvalue λ_j of A we have $\lambda_j \in \mathcal{S}_{\mu,\tau}(\hat{\mathbb{T}})$, then the system (15) is exponentially stable.*

Proof.

(i) The general solution of (15) is given by (see [12]):

$$x(t) = \left[\prod_{i=k-1}^{0} e^{A\tau_i}(I+\mu_i A)\right] x(t_0) = e_A(t,t_0)\, x(t_0). \tag{27}$$

Note that, $x(t) = \prod_{i=k-1}^{0}[e^{\lambda_j\tau_i}(1+\lambda_j\mu_i)]\ V_j,\ \forall 1 \le j \le n$ is a solution of the system (15) where V_j is the eigenvector associated to λ_j (see [1] Chapter 5 and [12] for further details). Let $e_{\lambda_j}(t,t_0) = \prod_{i=k-1}^{0} e^{\lambda_j \tau_i}(1+\lambda_j \mu_i)$. If (15) is exponentially stable, there exist constants $K = K(t_0)$ and $0 > \alpha \in \mathcal{R}^+$ such that

$$|e_{\lambda_j}(t,t_0)| \le K e_\alpha(t,t_0), \quad \text{for } t \ge t_0$$

for all eigenpairs (λ_j, V_j) of A. On the evidence of the proposition IV.1, we conclude that $\lambda_j \in \mathcal{S}_{\tau,\mu}(\hat{\mathbb{T}})$.

(ii) From (4) and (5), there exist an invertible matrix Q such that $A = QJQ^{-1}$, where J is the associated Jordan matrix of A and defined as in (6), such that

$$\begin{aligned} x(t) &= Q\left[\prod_{i=k-1}^{0} e^{J\tau_i}(I+\mu_i J)\right] Q^{-1}\, x(t_0) \\ &= Q e_J(t,t_0) Q^{-1}\, x(t_0), \text{ and} \end{aligned} \tag{28}$$

$$\prod_{i=k-1}^{0} e^{J_j\tau_i}(I_j+\mu_i J_j) = \prod_{i=k-1}^{0} e^{\lambda_j\tau_i}(1+\mu_i\lambda_j) \times \begin{pmatrix} 1 & m^1_{\lambda_j}(t,t_0) & \dots & m^{d_j-1}_{\lambda_j}(t,t_0) \\ 0 & 1 & \dots & m^{d_j-2}_{\lambda_j}(t,t_0) \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix},$$

which implies that

$$e_J(t,t_0) = \begin{pmatrix} e_{\lambda_1}(t,t_0)M_1 & & \\ & \ddots & \\ & & e_{\lambda_l}(t,t_0)M_l, \end{pmatrix}, \tag{29}$$

where $M_j,\ 1 \le j \le l \le n$ are matrices which depend on the monomiales $m^{(.)}_{\lambda_j}$. Note that, if A is diagonalizable, so M_j are the identity matrices.

For $j \in \{1,\dots,l\}$, we have

$$|e_{\lambda_j}(t,t_0)| = e^{\int_{t_0}^{t} \frac{\log|1+\hat{\mu}(s)\lambda_j|}{\hat{\mu}(s)}\Delta s}, \quad t \ge t_0.$$

If we consider that all the eigenvalue λ_j of A, satisfy $\lambda_j \in \mathcal{S}_{\tau,\mu}(\hat{\mathbb{T}})$, then

$$\limsup_{t\to\infty} \frac{1}{t-t_0}\int_{t_0}^{t} \frac{\log|1+\hat{\mu}(s)\lambda_j|}{\hat{\mu}(s)}\Delta s = \alpha_j < 0$$

with α_j is a negative positively regressive constant. Therefore, we have the following estimate

$$|e_{\lambda_j}(t,t_0)| \le K_1\ e^{\alpha_j(t-t_0)} \le K_1\ e^{\alpha(t-t_0)}, \quad t \ge t_0$$

with $K_1 = K_1(t_0) \ge 1$ and $\alpha = \min_{1\le j\le n}\{\alpha_j\}$ (see Appendix). From (27), (28) and (29), we have

$$\|e_A(t,t_0)\| \le \|Q\|\ \|Q^{-1}\|\ \|e_J(t,t_0)\|, \text{ with}$$

$$\|e_J(t,t_0)\| \le K_2 \max_{1\le j\le n} |e_{\lambda_j}(t,t_0)\ m^{\eta_j}_{\lambda_j}(t,t_0)|,$$

for $0 \le \eta_j \le d_j$. Since the eigenvalues of A are assumed to be uniformly regressive, and from Lemma II.1 we get

$$|m^{\eta_j}_{\lambda_j}(t,t_0)\ e_{\lambda_j}(t-t_0))| \le K_3\ \gamma_j^{\eta_j}\ (t-t_0)\ e^{\alpha_j(t-t_0)}.$$

From the above inequalities, we conclude that there are $K = K(K_1, K_2, K_3, \alpha, t_0, \eta_j) \ge 1$, such that

$$\|e_A(t,t_0)\| \le K\ e^{\alpha(t-t_0)} \quad \text{for all } t \ge t_0,$$

which shows the exponential stability of system (15). □

Theorem IV.4. *(Exponential stability almost surely) Consider $\hat{\mathbb{T}}_{\tau,\mu}$ the i.i.d stochastic time scale generated by $\mu = \{\mu_k\}_{k\in\mathbb{N}}$ and $\tau = \{\tau_k\}_{k\in\mathbb{N}}$, and let the switched system (15) such that all the eigenvalues λ_j of A are uniformly regressive with respect to τ and μ. The system (15) is exponentially stable almost surely on $\hat{\mathbb{T}}_{\tau,\mu}$, if and only if*

$$\mathbb{E}[\Re(\lambda_j)\tau + \log(|1+\mu\lambda_j|)] < 0, \quad \forall 1 \le j \le n. \tag{30}$$

Proof.
It is a direct result from Corollary IV.2.1 and Theorem IV.3. The condition (30) is satisfied if and only if $\lambda_j \in \mathcal{S}_{\mu,\tau}(\hat{\mathbb{T}}),\ \forall 1 \le j \le n$ according to Corollary IV.2.1, and from Theorem IV.3 we conclude that the switched system (15) is exponentially stable almost surely on $\hat{\mathbb{T}}_{\mu,\tau}$. □

V. Numerical Example

Let the switched system (15) on $\hat{\mathbb{T}}_{\tau,\mu}$, with $A = \begin{pmatrix} -1 & 5.3 \\ -4.2 & -2 \end{pmatrix}$. The eigenvalues of A are $\lambda_{1,2} = \frac{-3}{2} \pm i4.7$. Suppose that $\mu = \{\mu_k\}$ and $\tau = \{\tau_k\}$ follow the uniform distribution. Consider these different cases:

1) $\mu \to U(0.3,1)$ and $\tau \to U(0.3,1)$.
2) $\mu \to U(0.5,3)$ and $\tau \to U(0.5,3)$.
3) $\mu \to U(0.5,3)$ and $\tau \to U(0.3,1)$.
4) $\mu \to U(0.3,1)$ and $\tau \to U(0.5,3)$.

The region of exponential stability almost surely is given by the set $\mathcal{S}_{\tau,\mu} = \{\lambda \in \mathbb{C} : \mathbb{E}[\Re(\lambda)\tau_k + \log(|1+\mu_k\lambda|)] < 0\}$. These regions are plotted for the above different cases in Figures 3 and 4. We can see that the eigenvalues of A are not in the region of exponential stability for $\mu, \tau \to U(0.3,1)$ and for $\mu \to U(0.5,3)$, $\tau \to U(0.3,1)$, and the system is unstable. When $\mu, \tau \to U(0.5,3)$ and $\mu \to U(0.3,1)$, $\tau \to U(0.5,3)$, the region of exponential stability contains the eigenvalues of A and the system is exponentially stable almost surely.

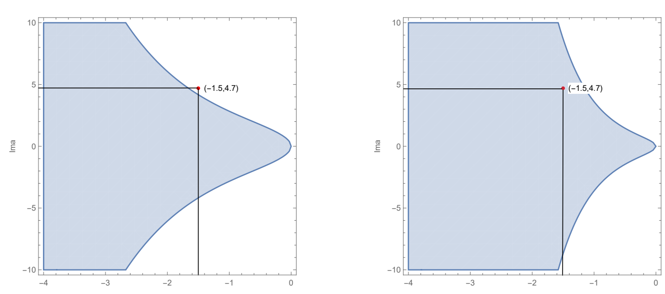

Fig. 3: Region of exponential stability almost surely on $\hat{\mathbb{T}}_{\tau,\mu}$. Left: μ and $\tau \to U(0.3,1)$. Right: μ and $\tau \to U(0.5,3)$

VI. Conclusion

In this work, exponential stability almost surely was studied for a linear dynamical system on random time scales. The system evolves on a particular non-uniform time scale formed by a union of disjoint closed intervals with

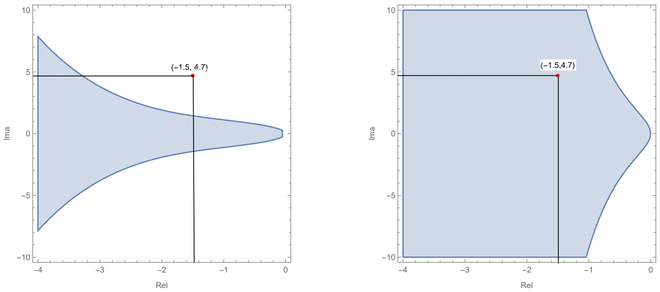

Fig. 4: Region of exponential stability almost surely on $\hat{\mathbb{T}}_{\tau,\mu}$. Left: $\mu \to U(0.5, 3)$, $\tau \to U(0.3, 1)$. Right: $\mu \to U(0.3, 1)$, $\tau \to U(0.5, 3)$.

stochastic length and separated by stochastic gaps. Condition are derived by determining a region of exponential stability almost surely. This work provides preliminary results for future works where determining regions of mean square stability and generalizations to switched systems with non-commutative state matrices are of interest.

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VII. APPENDIX

Proof. (Proposition IV.1) It is assumed that system (17) is exponentially stable, such that

$$|e_\lambda(t_k, t_0)| \leq K \ e_\alpha(t_k, t_0) \quad \text{for } t \geq t_0,\ t_k, t_0 \in \hat{\mathbb{T}}_{\tau,\mu},$$

with $K \geq 1$ and $0 > \alpha \in \mathcal{R}^+$. The explicit modulus of the generalized exponential function of (17) is given by

$$|e_\lambda(t_k, t_0)| = e^{\int_{t_0}^{t_k} \frac{\log|1+\hat{\mu}(s)\lambda|}{\hat{\mu}(s)} \Delta s}, \quad \text{for } t_k \geq t_0,\ t_k, t_0 \in \hat{\mathbb{T}}_{\tau,\mu}.$$

It implies that

$$\begin{aligned} \int_{t_0}^{t_k} \frac{\log|1+\hat{\mu}(s)\lambda|}{\hat{\mu}(s)} \Delta s \quad & \leq \log(K) + \int_{t_0}^{t_k} \frac{\log(1+\hat{\mu}(s)\alpha)}{\hat{\mu}(s)} \Delta s \\ & = \log(K) + \sum_{i=0}^{k-1} \mu_i \left(\frac{\log(1+\mu_i\alpha)}{\mu_i} \right) \\ & \quad + \sum_{i=0}^{k-1} \tau_i \alpha \\ & \leq \log(K) + \sum_{i=0}^{k-1} \mu_i\alpha + \sum_{i=0}^{k-1} \tau_i\alpha \\ & = \log(K) + \alpha(t_k - t_0) \end{aligned}$$

One gets: $\limsup_{t_k\to\infty} \frac{1}{t_k - t_0} \int_{t_0}^{t_k} \frac{\log|1+\hat{\mu}(s)\lambda|}{\hat{\mu}(s)} \Delta s \leq \alpha < 0$.

In the other hand, suppose that $\lambda \in \mathcal{S}_{\tau,\mu}(\hat{\mathbb{T}})$ such that

$$\limsup_{t_k\to\infty} \frac{1}{t_k - t_0} \int_{t_0}^{t_k} \frac{\log|1+\hat{\mu}(s)\lambda|}{\hat{\mu}(s)} \Delta s = \eta < 0. \quad (31)$$

Then, for all $0 < \varepsilon < -\eta$ there exists a constant $K = K(t_0) \geq 1$ such that, for $t_k \geq t_0$, we have

$$|e_\lambda(t_k, t_0)| \leq K e^{(\eta+\varepsilon)(t_k - t_0)} = K e^{\alpha(t_k - t_0)},$$

which implies the exponential stability of (17). ☐

REFERENCES

[1] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*. Birkhäuser Boston, Inc., Boston, MA, 2001.

[2] J. DaCuhna, "Stability for time varying linear dynamic systems on time scales," *J. Comput. Appl. Math.*, vol. 176, pp. 381–410, 2005.

[3] A. Peterson and Y. Raffoul, "Exponential stability of dynamic equations on time scales," *Adv. Differ. Equ-Ny*, vol. 2, pp. 133–144, 2005.

[4] C. Pötzsche, S. Siegmund, and F. Wirth, "A spectral characterization of exponential stability for linear time-invariant systems on time scales," *Discrete and Continuous Dynamical Systems*, vol. 9, no. 5, pp. 1223–1242, 2003.

[5] Z. Bartosiewicz and E. Piotrowska, "On stabilisability of nonlinear systems on time scales," *Int. J. Control*, vol. 86, pp. 139–145, 2013.

[6] D. Poulsen, J. Davis, I. Gravagne, and R. Marks, "On the stability of mu-varying dynamic equations on stochastically generated time scales," *IEEE SSST*, pp. 18–23, March 2012.

[7] D. Poulsen, J. Davis, and I. Gravagne, "Observer based feedback controllers on stochastic time scales," in *2013 IEEE 45th Southeastern Symposium on System Theory*, pp. 104–107, March 2013.

[8] D. Poulsen, J. Davis, and I. Gravagne, "Stochastic time scales: Quadratic Lyapunov functions and probabilistic regions of stability," in *2013 IEEE 45th Southeastern Symposium on System Theory*, pp. 98–103, March 2013.

[9] M. Davis, J., A. Gravagne, I., and A. Ramos, A., "Stability of switched linear systems on non-uniform time domains," *IEEE SSST, Texas, USA*, 2010.

[10] A. Gravagne, I., E. Johns, and M. Davis, J., "Switched linear systems on time scales with relaxed commutativity constraints," *IEEE SSST, Auburn University, USA*, pp. 733–738, October 2011.

[11] G. Eisenbarth, M. Davis, J., and A. Gravagne, I., "Stability of simultaneous triangularizable switched systems on hybrid domains," *Electronic Journal of Differential Equations*, vol. 63, pp. 1–19, June 2014.

[12] F. Taousser, M. Defoort, and M. Djemai, "Stability analysis of a class of switched linear systems on non-uniform time domains," *Syst. Control Lett.*, vol. 74, pp. 24–31, 2014.

[13] F. Taousser, M. Defoort, and M. Djemai, "Stability analysis of a class of uncertain switched systems on time scale using lyapunov functions," *Nonlinear Anal.-Hybri.*, vol. 16, pp. 13–23, 2015.

[14] F. Z. Taousser, S. M. Djouadi, K. Tomsovic, and M. Djemai, "Stability of nonlinear switched systems on non-uniform time domains with application to multi-agents consensus," in *American Control Conference (ACC), Colorado*, 2020.

[15] F. Taousser, M. Defoort, and M. Djemai, "Region of exponential stability of switched linear systems on time scales," *IFAC-PapersOnLine*, vol. 48, no. 27, pp. 93–98, 2015.

[16] F. Taousser, M. Defoort, and M. Djemai, "Consensus for linear multi-agent system with intermittent information transmissions using the time-scale theory," *International Journal of Control*, vol. 89, no. 1, pp. 210–220, 2016.

[17] F. Z. Taousser, M. E. Raoufat, K. Tomsovic, and S. M. Djouadi, "Stability of wide-area power system controls with intermittent information transmission," *IEEE Transactions on Power Systems*, vol. 34, pp. 3494–3503, 2019.

[18] G. Wen, Z. Duan, and G. Chen, "Distributed consensus of multi-agent systems with general linear node dynamics through intermittent communications," in *Control and Decision Conference (CCDC), 2012 24th Chinese*, pp. 1–5, IEEE, 2012.

[19] J. P. Hespanha, P. Naghshtabrizi, and Y. Xu, "A survey of recent results in networked control systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 138–162, 2007.

[20] J. Davis, I. Gravagne, B. Jackson, and R. Marks, "Regions of exponential stability for LTI systems on nonuniform domains," in *2011 IEEE 43rd Southeastern Symposium on System Theory*, pp. 37–42, 2011.

[21] D. Poulsen, *Stability and Control on Stochastic Time Scales*. PhD thesis, Baylor University, 2015.

[22] T. Gard and J. Hoffacker, "Asymptotic behavior of natural growth on time scales," *Dynamic Systems and Applicationsc*, vol. 12, 2002.

[23] R. Bitmead, "Convergence properties of LMS adaptive estimators with unbounded dependent inputs," in *1981 20th IEEE Conference on Decision and Control including the Symposium on Adaptive Processes*, pp. 607–612, Dec 1981.

[24] L. Koralov and Y. Sinai, *Theory of Probability and Random Processes*. Universitext, Springer Berlin Heidelberg, 2007.