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Small curvature concentration and Ricci flow smoothing

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ABSTRACT

We show that a complete Ricci flow of bounded curvature which begins from a manifold with a Ricci lower bound, local entropy bound, and small local scale-invariant integral curvature control will have global point-wise curvature control at positive times which only depends on the initial almost Euclidean structure. As applications, we use the Ricci flows to study the diffeomorphism type of manifolds and the regularity of Gromov-Hausdorff limit of manifolds with small curvature concentration.

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1. Introduction

The Ricci flow deforms a metric g on a Riemannian manifold (M^n, g) according to the equation

$$\frac{\partial}{\partial t} g(t) = -2\mathrm{Ric}_{g(t)}.$$

Since its introduction by Hamilton [23], the Ricci flow has been used in a wide variety of settings to regularize metrics. One sense in which this occurs is described by Perelman's pseudolocality theorem [43], which has played a crucial role in work on the short time existence of the Ricci flow, especially in settings where the initial data lacks bounded curvature or completeness [50,27].

Theorem 1.1. *[43, Theorem 10.1] For any $\alpha > 0$, there exist positive constant ε and δ such that if $(M^n, g(t))$, $t \in [0, \varepsilon r_0]$ is a solution to the Ricci flow for some $r_0 > 0$ and in addition*

- *$R \geq -r_0^{-2}$ on $B_0(x_0, r_0)$;*
- *$|\partial\Omega|^n \geq (1-\delta) c_n |\Omega|^{n-1}$, for any open set in $B_0(x_0, r_0)$, where c_n is the isoperimetric constant of $\mathbb{R}^n$,*

then for any $t \in [0, (\varepsilon r_0)^2]$ and $x \in B_t(x_0, \varepsilon r_0)$

$$|\mathrm{Rm}|(x,t) \leq \alpha t^{-1} + (\varepsilon r_0)^{-2}.$$

Thus, Perelman's pseudolocality tells us that given a lower Ricci bound on an almost Euclidean region, we can deduce regularization in the sense of curvature control along the Ricci flow for short times. Since Perelman's work, many extensions have been developed in a variety of settings [15,8,49,53].

Related regularization results for the Ricci flow under critical $L^{n/2}$ bounds of Rm have previously been studied in [56,38] assuming also pointwise two-sided bounds on |Ric|, and in [51] assuming alternatively a supercritical $\|\mathrm{Ric}\|_p$, $p > n/2$ bound. In this note, we will consider conditions which are scaling invariant. In particular, we study the flow under local critical $L^{n/2}$ bounds of Rm, but will instead do so in combination with a Ricci lower bound and control of the local entropy, a localization of Perelman's entropy introduced by Wang [52].

Below we state our main result, referring to the beginning of Section 2 for most of the associated notation. Throughout, we will use $a \wedge b$ to denote $\min\{a, b\}$.

Theorem 1.2. *For all $A, \lambda > 0$, there are $C_0(n, A, \lambda)$, $\sigma(n, A)$ and $\hat{T}(n, A, \lambda) > 0$ such that the following holds. Suppose $(M^n, g(t))$ is a complete Ricci flow of bounded curvature on $[0, T]$ and the initial metric $g(0)$ satisfies the followings:*

(a) $\mathrm{Ric}(g(0)) \geq -\lambda$;
(b) $\nu(B_{g(0)}(p,5), g(0), 1) \geq -A$;
(c) $\left(\int_{B_{g(0)}(p,2)} |\mathrm{Rm}(g_0)|^{n/2} d\mu_0\right)^{2/n} \leq \varepsilon$ *for some* $\varepsilon < \sigma$,

for all $p \in M$. *Then we have for any* $x \in M$ *and* $t \in (0, T \wedge \hat{T}]$,

$$|\mathrm{Rm}|(x,t) \leq \frac{C_0 \varepsilon}{t} \quad \textit{and} \quad \mathrm{inj}_{g(t)}(x) \geq C_0^{-1}\sqrt{t}. \tag{1.1}$$

Moreover, we have $\left(\int_{B_t(x,1)} |\mathrm{Rm}(g(t))|^{n/2} d\mu_t\right)^{2/n} \leq C_0 \varepsilon$ *for all* $x \in M$. *In particular, the Ricci flow must exist up to* $\hat{T}$.

Remark 1.1. In the statement above, we choose the scale 1 in the local entropy condition only for convenience.

Theorem 1.2 is a smoothing result also based on an initial "almost Euclidean" assumption. However, instead of characterizing this using the isoperimetric constant as in Theorem 1.1, we instead use rough non-collapsing and small curvature concentration.

Using ideas related to those used to prove Theorem 1.2 and point-picking technique, we prove a gap result for steady and shrinking gradient Ricci solitons without assuming curvature boundedness.

Theorem 1.3. *For all* $A > 0$, *there is* $\varepsilon(n, A) > 0$ *such that if* (M^n, g, f) *is a complete shrinking or steady gradient Ricci soliton satisfying*

(a) $\nu(M, g) \geq -A$;
(b) $\left(\int_M |\mathrm{Rm}|^{n/2} d\mu\right)^{2/n} \leq \varepsilon$,

then (M, g) *is isometric to the standard Euclidean space* $\mathbb{R}^n$.

Remark 1.2. The smallness condition (b) in Theorem 1.3 is necessary. Indeed, as pointed out by Chen-Zhu [12], the Eguchi-Hanson metric (M, g) is Ricci flat with $|\mathrm{Rm}(x)| \leq C(d_g(x,p)+1)^{-6}$ and has Euclidean volume growth. Therefore it satisfies $||\mathrm{Rm}||_{L^{n/2}} < \infty$ and $\nu(M, g) > -\infty$ by Remark 1.4 and a scaling argument. However, it is not isometric to Euclidean flat metric.

Gap results for gradient Ricci solitons have been previously studied for instance in [58,21,60] under global assumptions on the potential function f, sometimes along with pointwise curvature control (see also [42,16,20,7]).

Theorem 1.2 lends itself to several applications. First, we have a gap result for Ricci-nonnegative Riemannian manifolds with $\|\mathrm{Rm}\|_{L^{n/2}}$ small.

Corollary 1.1. *For all $A > 0$, there is $\sigma(n, A) > 0$ such that if (M^n, g) is a complete noncompact manifold with*

1. *bounded curvature*
2. $\mathrm{Ric} \geq 0$*;*
3. $\nu(M, g) \geq -A$*;*
4. $\left(\int_M |\mathrm{Rm}|^{n/2} d\mu_g\right)^{2/n} \leq \sigma$.

Then g is of Euclidean volume growth. Moreover, M^n is diffeomorphic to $\mathbb{R}^n$.

There is a large body of work on Ricci-nonnegative noncompact manifolds, and several results show that under some additional assumptions (such as almost Euclidean volume growth), such manifolds must be diffeomorphic to $\mathbb{R}^n$ [4,10,54]. Corollary 1.1 is related to two results of this kind by Ledoux and Xia [35,55], which assert that a complete, Ricci-nonnegative manifold with Euclidean-type Sobolev constant close to that of Euclidean space must be diffeomorphic to $\mathbb{R}^n$. Indeed, Condition (3) of Corollary 1.1 above can be seen as a weakening of this requirement, since it holds as long as there is some constant which makes the Euclidean-type Sobolev inequality valid. This is compensated for by Condition (4) on the smallness of the total scale-invariant curvature.

Carron has also pointed out to us that Corollary 1.1 is related to the following statement which can be derived from works of Cheeger [9] and Cheeger-Colding [10]: If a complete noncompact manifold with $\mathrm{Ric} \geq 0$ has $\int_M |\mathrm{Rm}|^{n/2} d\mu_g$ sufficiently small relative its asymptotic volume ratio (assumed nonzero), then it must be diffeomorphic to $\mathbb{R}^n$. Indeed, these hypotheses ensure by [9, Theorem 4.32] that the manifold must have asymptotic volume ratio close to one, from which one can conclude that the manifold must be diffeomorphic to $\mathbb{R}^n$ by [10, Theorem A.1.11.] (see also [53, Theorem 5.7] for a recent proof via Ricci flow). Our assumptions in Corollary 1.1 differ slightly from this statement's because our required smallness of σ is relative to a lower bound on entropy. Although we will prove below that under bounded curvature, bounded entropy, and small $||\mathrm{Rm}||_{L^{n/2}}$ we indeed have almost Euclidean volume growth, it is unclear to us whether an entropy lower bound implies an asymptotic volume ratio lower bound in general. The method here relies on the existence of Ricci flow with uniform estimates. It will be interesting to know if the curvature boundedness is necessary to ensure its existence in general.

We can also apply Theorem 1.2 to obtain a finite diffeomorphism-type result and a Gromov-Hausdorff compactness result in the setting of length spaces.

Corollary 1.2. *For all $A > 0$, there is $\sigma(n, A) > 0$ such that for C_1, C_2, the space of compact manifolds (M, g) satisfying*

(a) $\mathrm{Ric}(g) \geq -C_1$*;*
(b) $Vol_g(M) \leq C_2$*;*

(c) $\inf_{p\in M} \nu(B_g(p,5),g,1) \geq -A$;
(d) $\sup_{p\in M} \left(\int_{B_g(p,2)} |\mathrm{Rm}|^{n/2} d\mu_g\right)^{2/n} \leq \sigma$

contains finitely many diffeomorphism types.

Corollary 1.2 follows via an argument analogous to the that in the proof of [32, Theorem 37.1], which was proved by Perelman [43, Remark 10.5] using Perelman's pseudolocality. In our case, the use of Perelman's pseudolocality is replaced by Theorem 1.2.

Theorem 1.4. *For any positive integer $n \geq 3$ and constant $A \geq 1000n$, there exists constant $\varepsilon_0(n,A)$ such that the following holds. Suppose (M_i^n, g_i, p_i) is a pointed sequence of Riemannian manifolds with the following properties:*

(a) *(M_i, g_i) has bounded curvature;*
(b) $\mathrm{Ric}(g_i) \geq -\lambda$;
(c) $\left(\int_{B_{g_i}(q,2)} |\mathrm{Rm}(g_i)|^{n/2} d\mu_i\right)^{2/n} \leq \varepsilon_0$ *for all q in M_i;*
(d) $\nu(B_{g_i}(q,5),g_i,1) \geq -A$, *for all q in M_i.*

Then there exists a smooth manifold M_∞ and a complete distance metric d_∞ on M_∞ generating the same topology as M_∞ such that after passing to sub-sequence, (M_i, d_{g_i}, p_i) converges in pointed Gromov Hausdorff sense to $(M_\infty, d_{g_\infty}, p_\infty)$.

Remark 1.3. The Ricci lower bound assumption on the initial metric can in fact be weakened to a scalar curvature lower bound and uniform volume doubling for some fixed scale. But we feel it is more natural to state the result with the Ricci assumption.

Remark 1.4. The initial Ricci curvature lower bound in fact gives a Sobolev inequality on the geodesic balls in M, which in turn implies a Log Sobolev inequality and provides a lower bound for the local entropy in terms of its volume. Hence the local entropy $\nu(B_{g_0}(p,5),g_0,1)$ lower bound condition in the above theorem can be replaced by a uniform volume lower bound condition for the geodesic balls on M, namely,

$$\mathrm{Vol}_{g_0}(B_{g_0}(p,5)) \geq v_0, \tag{1.2}$$

for some positive constant v_0, for all $p \in M$. In that case, the constants ε, C and $\hat{T}$ also depend on v_0. In particular, the global entropy $\nu(M,g)$'s lower bound can be replaced by $\mathrm{Ric} \geq 0$ and a lower bound on the asymptotic volume ratio.

There have been many studies of compactness under scale-invariant integral curvature bounds, notably Anderson–Cheeger's diffeomorphism finiteness result [2]. Orbifold compactness results under $\|\mathrm{Rm}\|_{L^{n/2}}$ bounds have also been obtained for Einstein manifolds as well as for both compact and noncompact gradient Ricci solitons [1,6,26]. In

comparison, Theorem 1.4 does not impose such analytic conditions on (M_i, g_i), but does require sufficient smallness of the local scale-invariant curvature concentration.

Theorem 1.4 is also a smoothing result for limit spaces of manifolds with lower curvature bounds, achieved via distance distortion estimates and pseudolocality-type estimates of the Ricci flow. There has been much recent work in this direction in many different settings [3,40,41,46,47,36,28,33,30,31].

The structure of the rest of this paper is as follows. In Section 2, we prove our main smoothing result, Theorem 1.2. In Section 3, we prove our gap result for gradient Ricci soliton, Theorem 1.3. In Section 4, we prove our gap result for complete noncompact Ricci nonnegative manifolds, Corollary 1.1. Finally in Section 5, we prove our Gromov-Hausdorff compactness result, Theorem 1.4.

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2. Curvature estimates of Ricci flows

In this section, we will prove the semi-local estimates of the Ricci flow. We begin by fixing some notation below.

Suppose (M^n, g) is an n dimensional complete (not necessarily compact) Riemannian manifold and Ω is a connected domain on M with smooth boundary (boundaryless if $M = \Omega$). Hereinafter, we shall reserve the positive integer n for the dimension of M. Wang [52] localized Perelman's entropy and proved an almost monotonicity in local entropy when Ω is bounded, generalizing the result in [43]. Using his notation, we have:

$$D_g(\Omega) := \left\{ u : u \in W_0^{1,2}(\Omega), u \geq 0 \text{ and } \|u\|_2 = 1 \right\}, \tag{2.1}$$

$$W(\Omega, g, u, \tau) := \int_\Omega \tau(Ru^2 + 4|\nabla u|^2) - 2u^2 \log u d\mu - \frac{n}{2}\log(4\pi\tau) - n, \tag{2.2}$$

$$\nu(\Omega, g, \tau) := \inf_{u \in D_g(\Omega), s \in (0,\tau]} W(\Omega, g, u, s), \tag{2.3}$$

$$\nu(\Omega, g) := \inf_{\tau \in (0,\infty)} \nu(\Omega, g, \tau). \tag{2.4}$$

In order to prove the curvature estimate of Theorem 1.2, we first show that it can be reduced to the preservation of local $L^{n/2}$ control of $\mathrm{Rm}(g(t))$.

Proposition 2.1. *For all $A > 0$, there is $c_0(n, A) > 0$ such that the following holds. Suppose $(M, g(t)), t \in [0, T]$ is a complete Ricci flow with bounded curvature such that for all $(x, t) \in M \times [0, T]$, the following holds:*

(a) $\nu(B_{g_0}(x, 10^6 n\sqrt{T}), g_0, 2T) \geq -A$;
(b) $\left(\int_{B_{g(t)}(x,\sqrt{t})} |\mathrm{Rm}(g(t))|^{n/2} d\mu_{g(t)}\right)^{n/2} \leq c_0\varepsilon$ *for* $\varepsilon < 1$,

then we have

$$\sup_M |\mathrm{Rm}(x, t)| < \varepsilon t^{-1} \tag{2.5}$$

for all $t \in (0, T]$.

Proof. By rescaling, we may assume $T = 1$. Suppose on the contrary that the result is not true. Then for some $A, \varepsilon > 0$, we can find sequences of $\delta_i = c_i\varepsilon_i$ with $c_i \to 0$, $\varepsilon_i \in (0, 1)$ and $\{(M_i, g_i(t), p_i)\}$ with bounded curvature such that

1. $\nu(B_{g_i(0)}(x, 10^6 n), g_i(0), 2) \geq -A$;
2. $\left(\int_{B_t(x,\sqrt{t})} |\mathrm{Rm}(g_i(t))|^{n/2} d\mu_{i,t}\right)^{2/n} \leq \delta_i \to 0$ for all $(x, t) \in M_i \times [0, 1]$

but for some $(x_i, t_i) \in M_i \times (0, 1]$,

$$|\mathrm{Rm}_i(x_i, t_i)| = \varepsilon_i t_i^{-1}.$$

We may choose $t_i > 0$ such that for all $(y, s) \in M_i \times (0, t_i)$,

$$|\mathrm{Rm}_i(y, s)| < 2\varepsilon_i s^{-1}. \tag{2.6}$$

This can be done since the upper bound of curvature varies continuously by boundedness of curvature. Let $Q_i = t_i^{-1} \geq 1$. Consider the rescaled Ricci flow $\tilde{g}_i(t) = Q_i g_i(Q_i^{-1} t)$ for $t \in [0, 1]$ which satisfies

(a) $\nu(B_{g_i(0)}(y, 10^6 n), \tilde{g}_i(0), 2) \geq -A$ for all $y \in M_i$;
(b) $\left(\int_{B_{\tilde{g}_i(t)}(y,\sqrt{t})} |\mathrm{Rm}(\tilde{g}_i(t))|^{n/2} d\tilde{\mu}_{i,t}\right)^{2/n} \leq \delta_i \to 0$ for all $(y, t) \in M_i \times [0, 1]$;
(c) $|\mathrm{Rm}_{\tilde{g}_i}(y, s)| < s^{-1}$ on $M_i \times (0, 1)$;
(d) $|\mathrm{Rm}_{\tilde{g}_i}(x_i, 1)| = \varepsilon_i$.

By (a) and [52, Theorem 5.4], we have uniform lower bound of the entropy $\nu(B_{\tilde{g}_i(t)}(y, 1), \tilde{g}_i(t), 1)$. Now (c) and [52, Theorem 3.3] implies an uniform lower bound of the volume of $B_{\tilde{g}_i(t)}(x_i, r)$ which depends only on A and n for any r, $t \in [1/2, 1]$.

By the curvature bound (c), we also have uniform Sobolev inequality on $B_{\tilde{g}_i(t)}(x_i, 1)$, for $t \in [1/2, 1]$ (see [44] and [37]). It follows from Kato's inequality and the evolution equation of Rm under the Ricci flow that

$$\left(\frac{\partial}{\partial t} - \Delta\right) |\mathrm{Rm}| \leq 8|\mathrm{Rm}|^2. \tag{2.7}$$

Since the curvature is uniformly bounded on $[\frac{1}{2}, 1]$, we may apply the Moser iteration argument [37], together with (b) and Hölder inequality to show that

$$\begin{aligned}
\varepsilon_i &= |\mathrm{Rm}_{\tilde{g}_i}(x_i, 1)| \\
&\leq C(n, A) \int_{1/2}^{1} \fint_{B_{\tilde{g}_i(t)}(x_i, 1/2)} |\mathrm{Rm}_{\tilde{g}_i}| d\mu_s ds \\
&\leq C'(n, A) \left(\int_{1/2}^{1} \fint_{B_{\tilde{g}_i(t)}(x_i, 1/2)} |\mathrm{Rm}_{\tilde{g}_i}|^{n/2} d\mu_s ds \right)^{2/n} \\
&\leq C'(n, A) c_i \varepsilon_i \to 0 \text{ as } i \to \infty,
\end{aligned} \tag{2.8}$$

which is impossible if c_i is too small. This completes the proof of the lemma. □

Remark 2.1. The a-priori curvature boundedness and completeness is in fact unnecessary, see Section 3.

Next, we will show that if the initial local $L^{n/2}$ is small enough, then it is preserved in some semi-local sense. We first begin with the energy evolution of $L^{n/2}$ norm.

Lemma 2.1. *Suppose $n \geq 3$ and $(M, g(t))$ is a complete solution to the Ricci flow, $t \in [0, T]$. Then for any $\alpha \geq \frac{n}{4}$, $\beta > 0$ and $\phi(x, t)$ compactly supported function in spacetime, there exist positive constants $C(\alpha)$ and $C'(n, \alpha)$ such that*

$$\begin{aligned}
\frac{d}{dt} \int_M \phi^2 (|\mathrm{Rm}|^2 + \beta)^\alpha d\mu_t \leq & - C(\alpha) \int_M |\nabla(\phi(|\mathrm{Rm}|^2 + \beta)^{\alpha/2})|^2 d\mu_t \\
& + C'(n, \alpha) \int_M \phi^2 (|\mathrm{Rm}|^2 + \beta)^{\alpha + 1/2} d\mu_t \\
& + C'(n, \alpha) \int_M |\nabla \phi|^2 (|\mathrm{Rm}|^2 + \beta)^\alpha d\mu_t. \\
& + \int_M 2\phi \Box \phi (|\mathrm{Rm}|^2 + \beta)^\alpha d\mu_t,
\end{aligned} \tag{2.9}$$

where $\Box = \frac{\partial}{\partial t} - \Delta_{g(t)}$.

Proof. We compute the time derivative of the integral norm as in [17]. Using $\frac{\partial}{\partial t} d\mu_t = -R d\mu_t \leq c(n)|\mathrm{Rm}| d\mu_t$, we have

$$\begin{aligned} \frac{d}{dt}\int_M \phi^2(|\mathrm{Rm}|^2+\beta)^\alpha d\mu_t \leq & \int_M \frac{\partial}{\partial t}\left(\phi^2(|\mathrm{Rm}|^2+\beta)^\alpha\right) d\mu_t \\ & + c(n)\int_M \phi^2(|\mathrm{Rm}|^2+\beta)^{\alpha+1/2} d\mu_t. \end{aligned} \tag{2.10}$$

For the first term on the R.H.S.,

$$\begin{aligned} & \int_M \frac{\partial}{\partial t}\left(\phi^2(|\mathrm{Rm}|^2+\beta)^\alpha\right) d\mu_t \\ = & \int_M \Box\left(\phi^2(|\mathrm{Rm}|^2+\beta)^\alpha\right) d\mu_t \\ = & \int_M 2\phi\Box\phi(|\mathrm{Rm}|^2+\beta)^\alpha d\mu_t - \int_M 2|\nabla\phi|^2(|\mathrm{Rm}|^2+\beta)^\alpha d\mu_t \\ & + \int_M \alpha\phi^2(|\mathrm{Rm}|^2+\beta)^{\alpha-1}\Box|\mathrm{Rm}|^2 d\mu_t \\ & - \int_M 4\alpha(\alpha-1)\phi^2(|\mathrm{Rm}|^2+\beta)^{\alpha-2}|\mathrm{Rm}|^2|\nabla|\mathrm{Rm}||^2 d\mu_t \\ & - \int_M 8\alpha\phi\langle\nabla\phi,\nabla|\mathrm{Rm}|\rangle|\mathrm{Rm}|(|\mathrm{Rm}|^2+\beta)^{\alpha-1} d\mu_t, \end{aligned} \tag{2.11}$$

where $\Box = \frac{\partial}{\partial t} - \Delta_{g(t)}$. To proceed, we apply the evolution equation of $|\mathrm{Rm}|^2$ (see [17] and ref. therein)

$$\Box|\mathrm{Rm}|^2 \leq -2|\nabla\mathrm{Rm}|^2 + 16|\mathrm{Rm}|^3. \tag{2.12}$$

It follows from (2.12) that

$$\begin{aligned} \int_M \alpha\phi^2(|\mathrm{Rm}|^2+\beta)^{\alpha-1}\Box|\mathrm{Rm}|^2 d\mu_t \leq & -2\alpha\int_M \phi^2(|\mathrm{Rm}|^2+\beta)^{\alpha-1}|\nabla\mathrm{Rm}|^2 d\mu_t \\ & + 16\alpha\int_M \phi^2(|\mathrm{Rm}|^2+\beta)^{\alpha+1/2} d\mu_t. \end{aligned} \tag{2.13}$$

Hence by Kato's inequality and Hölder inequality,

$$
\begin{aligned}
&\int_M \alpha\phi^2(|\mathrm{Rm}|^2+\beta)^{\alpha-1}\Box|\mathrm{Rm}|^2 d\mu_t \\
&-\int_M 4\alpha(\alpha-1)\phi^2(|\mathrm{Rm}|^2+\beta)^{\alpha-2}|\mathrm{Rm}|^2|\nabla|\mathrm{Rm}||^2 d\mu_t \\
&-\int_M 8\alpha\phi\langle\nabla\phi,\nabla|\mathrm{Rm}|\rangle|\mathrm{Rm}|(|\mathrm{Rm}|^2+\beta)^{\alpha-1} d\mu_t \\
\leq\ & -C(\alpha)\int_M \phi^2(|\mathrm{Rm}|^2+\beta)^{\alpha-1}|\nabla|\mathrm{Rm}||^2 d\mu_t \\
&+16\alpha\int_M \phi^2(|\mathrm{Rm}|^2+\beta)^{\alpha+1/2} d\mu_t \\
&+\int_M 8\alpha\phi|\nabla\phi||\nabla|\mathrm{Rm}||\mathrm{Rm}|(|\mathrm{Rm}|^2+\beta)^{\alpha-1} d\mu_t \\
\leq\ & -C'(\alpha)\int_M \phi^2(|\mathrm{Rm}|^2+\beta)^{\alpha-2}|\mathrm{Rm}|^2|\nabla|\mathrm{Rm}||^2 d\mu_t \\
&+16\alpha\int_M \phi^2(|\mathrm{Rm}|^2+\beta)^{\alpha+1/2} d\mu_t \\
&+C''(\alpha)\int_M |\nabla\phi|^2(|\mathrm{Rm}|^2+\beta)^{\alpha} d\mu_t.
\end{aligned}
$$

We also have by Cauchy Schwarz inequality

$$
\begin{aligned}
&C'(\alpha)\int_M \phi^2(|\mathrm{Rm}|^2+\beta)^{\alpha-2}|\mathrm{Rm}|^2|\nabla|\mathrm{Rm}||^2 d\mu_t \\
=\ & C'''(\alpha)\int_M \phi^2|\nabla(|\mathrm{Rm}|^2+\beta)^{\frac{\alpha}{2}}|^2 d\mu_t \\
\geq\ & \frac{C'''(\alpha)}{2}\int_M |\nabla(\phi(|\mathrm{Rm}|^2+\beta)^{\frac{\alpha}{2}})|^2 d\mu_t - C'''(\alpha)\int_M |\nabla\phi|^2(|\mathrm{Rm}|^2+\beta)^{\alpha} d\mu_t.
\end{aligned}
\tag{2.14}
$$

All in all,

$$\begin{aligned}\frac{d}{dt}\int_M \phi^2(|\mathrm{Rm}|^2+\beta)^\alpha d\mu_t \le C(n,\alpha)\int_M \phi^2(|\mathrm{Rm}|^2+\beta)^{\alpha+1/2}d\mu_t \\ + \int_M 2\phi\Box\phi(|\mathrm{Rm}|^2+\beta)^\alpha d\mu_t \\ + C(\alpha)\int_M 2|\nabla\phi|^2(|\mathrm{Rm}|^2+\beta)^\alpha d\mu_t \\ - \frac{C'''(\alpha)}{2}\int_M |\nabla(\phi(|\mathrm{Rm}|^2+\beta)^{\frac{\alpha}{2}})|^2 d\mu_t.\end{aligned} \tag{2.15}$$

Our desired inequality (2.9) then follows. □

We also need the following lemma showing that the local entropy implies local Sobolev inequality.

Lemma 2.2. *For all $A \ge 1000n, \lambda > 0$ and $\delta > 0$, there are positive constants $C_S(n,A,\lambda,\delta)$ and $\hat{T}(n,A,\lambda,\delta)$ such that the following holds. Suppose $(M,g(t))$ is a complete Ricci flow with bounded curvature on $[0,T]$ and for all $p \in M$ and all $t \in (0,T]$, all of the following conditions are satisfied*

1. $R_{g(0)} \ge -n\lambda$;
2. $\nu(B_0(p,5),g(0),1) \ge -A$;
3. $\mathrm{Ric}(p,t) \le \delta t^{-1}$.

Then we have for any $p \in M$ and $t \in (0, \min\{T,\hat{T}\}]$,

$$\nu(B_t(p,2),g(t),32^{-1}) \ge -2A \tag{2.16}$$

and for any $\varphi \in C_0^\infty(B_t(p,2))$

$$\left(\int_{B_t(p,2)} |\varphi|^{\frac{2n}{n-2}} d\mu_t\right)^{\frac{n-2}{n}} \le C_S\left(\int_{B_t(p,2)} |\nabla\varphi|^2 + (R + c_n\lambda + 1)\varphi^2 d\mu_t\right). \tag{2.17}$$

Proof. For (2.16), we apply [52, Theorem 5.4] to get for all small $t \le \min\{\hat{T}(n,A,\delta),T\}$

$$\begin{aligned}\nu(B_t(p,2),g(t),32^{-1}) &\ge \nu(B_0(p,5),g(0),t+32^{-1}) - 16t \\ &\ge \nu(B_0(p,5),g(0),1) - A \\ &\ge -2A\end{aligned} \tag{2.18}$$

This completes the proof of (2.16). Since the Ricci flow has bounded curvature, it follows from the maximum principle that there exists a dimensional constant c_n such that for all $t \leq \min\{\hat{T}(n, A, \delta), T\}$

$$R(x,t) \geq -c_n\lambda. \tag{2.19}$$

By the definition of local ν entropy (2.2) and (2.16), we have a uniform Log Sobolev inequality: for any $\tau \in (0, 32^{-1})$, $u \in W_0^{1,2}(B_t(p,2))$, with $\|u\|_{g(t),2} = 1$,

$$\begin{aligned} \int\limits_{B_t(p,2)} u^2 \log u^2 d\mu_t \leq & \tau \int\limits_{B_t(p,2)} 4|\nabla u|^2 + Ru^2 d\mu_t \\ & - \frac{n}{2}\log(4\pi\tau) - n + 2A. \end{aligned} \tag{2.20}$$

The uniform Log Sobolev inequality then implies a uniform Sobolev inequality along the Ricci flow as first described in [59] (see also [17,57] and Theorems 2.1 and 2.2). Indeed, when $\partial B_t(p,2)$ is nonempty, the same arguments as in [19,57] will give us Theorems 2.1 and 2.2 below for the Dirichlet Sobolev inequality, and these together with (2.20) imply (2.17), finishing the proof. □

We shall now state Theorems 2.1 and 2.2 without mentioning the proofs, since they are essentially the same as those found in [19,57]. Let (N, h) be a smooth compact Riemannian manifold with metric h and smooth boundary ∂N, H the elliptic operator $= -\Delta + 4^{-1}(R + c_n\lambda)$, where λ and c_n are non-negative constants such that $R \geq -c_n\lambda$ on N,

$$Q(u,v) = \int\limits_N \nabla u \cdot \nabla v + 4^{-1}(R + c_n\lambda)u \cdot v d\mu_h \tag{2.21}$$

and write $Q(u,u)$ as $Q(u)$. For $t > 0$, consider the semigroup e^{-tH} of the operator H. For any $u_0 \in L^2(N,h)$, the function $u(t) := e^{-tH}u_0$ is the solution to the Dirichlet evolution equation

$$\begin{cases} \frac{\partial u}{\partial t} = -Hu \\ u(0) = u_0 \\ u = 0 \text{ on } \partial N. \end{cases} \tag{2.22}$$

Theorem 2.1 *([19,57]). Let $\sigma^* \in (0, \infty]$. Suppose that for all $\sigma \in (0, \sigma^*)$,*

$$\int\limits_N u^2 \log u^2 d\mu \leq \sigma \int\limits_N |\nabla u|^2 + 4^{-1}(R + c_n\lambda)u^2 d\mu + \beta(\sigma) \tag{2.23}$$

is true for any $u \in W_0^{1,2}(N)$ *with* $\|u\|_2 = 1$, *where* β *is a continuous nonincreasing function and* $R + c_n\lambda \geq 0$. *If in addition the function,*

$$\tau(t) := \frac{1}{2t}\int_0^t \beta(s)ds \tag{2.24}$$

is finite for any $t \in (0, \sigma^*)$. *Then for each* $u \in L^2(N)$

$$\|e^{-tH}u\|_\infty \leq e^{\tau(t)}\|u\|_2 \tag{2.25}$$

for $t \in (0, \sigma^*/4)$. *Moreover, for all* $u \in L^1(N)$

$$\|e^{-tH}u\|_\infty \leq e^{2\tau(\frac{t}{2})}\|u\|_1 \tag{2.26}$$

for $t \in (0, \sigma^*/4)$.

Theorem 2.2 *([19,57]). Suppose there exist positive constants* c_1 *and* t_1 *such that for all* $t \in (0, t_1)$ *and* $u \in L^2(N)$

$$\|e^{-tH}u\|_\infty \leq c_1 t^{-\frac{n}{4}}\|u\|_2. \tag{2.27}$$

Set $H_0 = H + 1$. *Then for some constant* $C(c_1, t_1, n)$,

$$\|H_0^{-1/2}u\|_{\frac{2n}{n-2}} \leq C\|u\|_2, \tag{2.28}$$

for any $u \in L^2(N)$. *In particular,*

$$\begin{aligned}\|u\|^2_{\frac{2n}{n-2}} &\leq C^2\|H_0^{1/2}u\|_2^2 \\ &\leq C^2(Q(u) + \|u\|_2^2),\end{aligned} \tag{2.29}$$

for all $u \in W_0^{1,2}(N)$, *where* $H_0^{1/2}$ *and* $H_0^{-1/2}$ *denote the fractional operator of* H_0 *and its inverse respectively (see [19,57]).*

With Proposition 2.1 and Lemmas 2.1 and 2.2 in hand, we can now prove Theorem 1.2 to conclude this section.

Proof of Theorem 1.2. Let Λ be a constant to be chosen later. Since $g(t)$ has bounded curvature, we may let $\hat{T}$ be the maximal time such that for all $(x,t) \in M \times [0, \hat{T} \wedge T)$, we have

$$\left(\int_{B_{g(t)}(x,1)} |\mathrm{Rm}(g(t))|^{n/2} d\mu_t\right)^{2/n} \leq \Lambda\varepsilon. \tag{2.30}$$

By [52, Theorem 3.3], Proposition 2.1 and the injectivity radius estimates in [11], it suffices to show that $\hat{T}$ is bounded from below uniformly if ε and Λ are chosen appropriately.

We may choose ε small enough so that $\Lambda\varepsilon = \delta < 1$ is small. Therefore, the Ricci flow $g(t)$ satisfies

1. $\left(\int_{B_{g_0}(x,2)} |\mathrm{Rm}(g(0))|^{n/2} d\mu_0\right)^{2/n} \leq \varepsilon$;
2. $\left(\int_{B_{g(t)}(x,1)} |\mathrm{Rm}(g(t))|^{n/2} d\mu_t\right)^{2/n} \leq \delta$;
3. $\nu(B_{g_0}(x,5), g_0, 1) \geq -A$;
4. $\mathrm{Ric}(g_0) \geq -\lambda$

for all $(x,t) \in M \times [0, T \wedge \hat{T})$. By the monotonicity of the local entropy over domain and scale, we may assume $\hat{T} < 25n^{-2}10^{-12}$ so that applying Proposition 2.1 on $g(t), t \in [0, T \wedge \hat{T})$ yields

$$\sup_M |\mathrm{Rm}(g(t))| \leq c(n,A)\delta t^{-1} < t^{-1}. \tag{2.31}$$

Now we are ready to estimate $\hat{T}$. For any $x_0 \in M$, we let $\eta(x,t) = d_t(x,x_0) + c_n\sqrt{t}$ and define $\phi(x,t) = e^{-10t}\varphi(\eta(x,t))$ where $\varphi(s)$ is a cutoff function on $\mathbb{R}$ so that $\varphi \equiv 1$ on $(-\infty, \frac{1}{2}]$, $\varphi \equiv 0$ outside $(-\infty, 1]$ and satisfies $\varphi'' \geq -10\varphi$, $0 \geq \varphi' \geq -10\sqrt{\varphi}$. By choosing c_n large enough, we have from [43, Lemma 8.3] that

$$\left(\frac{\partial}{\partial t} - \Delta\right)\phi \leq 0 \tag{2.32}$$

in the sense of barrier and hence in the sense of distribution, see [39, Appendix A].

Using Lemma 2.1 with the above choice of ϕ and $\alpha = n/4$, we conclude that

$$\begin{aligned}\frac{d}{dt}\int_M \phi^2(|\mathrm{Rm}|^2+\beta)^{n/4} d\mu_t \leq & -C_n^{-1}\int_M |\nabla(\phi(|\mathrm{Rm}|^2+\beta)^{n/8})|^2 d\mu_t \\ & + C_n\int_M \phi^2(|\mathrm{Rm}|^2+\beta)^{n/4+1/2} d\mu_t \\ & + C_n\int_{supp(\phi)} (|\mathrm{Rm}|^2+\beta)^{n/4} d\mu_t.\end{aligned} \tag{2.33}$$

Noted that ϕ is supported on $B_t(x_0,1)$, By Lemma 2.2, the first term on the right can be estimated as

$$\begin{aligned} &C_n^{-1}\int\limits_M |\nabla(\phi(|\mathrm{Rm}|^2+\beta)^{n/8})|^2 d\mu_t \\ \geq\ & C_1(n,A)\left(\int\limits_M |(\phi^2(|\mathrm{Rm}|^2+\beta)^{n/4})|^{\frac{n}{n-2}} d\mu_t\right)^{\frac{n-2}{n}} \\ &- C_1(n,A)\int\limits_M \phi^2(R+c_n\lambda+1)(|\mathrm{Rm}|^2+\beta)^{n/4} d\mu_t \end{aligned} \tag{2.34}$$

while the second term can be estimated by

$$\begin{aligned} &C_n\int\limits_M \phi^2(|\mathrm{Rm}|^2+\beta)^{n/4+1/2} d\mu_t \\ \leq\ & C_n\left(\int\limits_{supp(\phi)} (|\mathrm{Rm}|^2+\beta)^{\frac{n}{4}} d\mu_t\right)^{\frac{2}{n}} \left(\int\limits_M \left[\phi^2(|\mathrm{Rm}|^2+\beta)^{\frac{n}{4}}\right]^{\frac{n}{n-2}} d\mu_t\right)^{\frac{n-2}{n}} \\ \leq\ & C_n\delta\left(\int\limits_M \left[\phi^2(|\mathrm{Rm}|^2+\beta)^{\frac{n}{4}}\right]^{\frac{n}{n-2}} d\mu_t\right)^{\frac{n-2}{n}} \end{aligned} \tag{2.35}$$

as $\beta \to 0$. We can apply the same argument to $\int_M \phi^2 R(|\mathrm{Rm}|^2+\beta)^{n/4} d\mu_t$ to deduce the same upper bound. Therefore, we conclude that if $\delta \leq \sigma(n,A) << 1$, then as $\beta \to 0$ we have

$$\begin{aligned} \frac{d}{dt}\int\limits_M \phi^2(|\mathrm{Rm}|^2+\beta)^{n/4} d\mu_t \leq & C(n,A,\lambda)\int\limits_{B_t(x_0,1)} (|\mathrm{Rm}|^2+\beta)^{n/4} d\mu_t \\ \leq & C(n,A,\lambda)\delta^{\frac{n}{2}} \end{aligned} \tag{2.36}$$

By letting $\beta \to 0$ together with the assumption on the initial metric, we conclude that for all $(x_0,t) \in M \times [0, T\wedge\hat{T})$,

$$\int\limits_{B_t(x_0,\frac{1}{4})} |\mathrm{Rm}|^{n/2} d\mu_t \leq e^{10\hat{T}}(C_1(n,A,\lambda)\Lambda^{\frac{n}{2}}t+1)\varepsilon^{\frac{n}{2}}. \tag{2.37}$$

Now we claim that there is $\tilde{T}(n,A)$ depending only on n, A such that for all $(y,t) \in M \times [0, T\wedge\hat{T}]$, if $T\wedge\hat{T} \leq \tilde{T}(n,A)$, then we have

$$B_t(y,1) \subset \bigcup_{i=1}^N B_t(x_i, \frac{1}{4}) \tag{2.38}$$

for some $N(n, A, \lambda) \in \mathbb{N}$. If the claim is true, then we conclude that for all $(y, t) \in M \times [0, T \wedge \hat{T} \wedge \tilde{T}(n, A)]$,

$$\begin{aligned} \int\limits_{B_t(y,1)} |\mathrm{Rm}|^{n/2} d\mu_t &\leq \sum_{i=1}^{N} \int\limits_{B_t(x_i, \frac{1}{4})} |\mathrm{Rm}|^{n/2} d\mu_t \\ &\leq N e^{10\tilde{T}} (C_1 \Lambda^{\frac{n}{2}} \tilde{T} + 1) \varepsilon^{\frac{n}{2}}. \end{aligned} \tag{2.39}$$

Therefore if we choose $\Lambda^{n/2} = 4N$ and further require $\tilde{T} \leq (4^{\frac{n}{2}} N^{\frac{n}{2}} C_1)^{-1}$, then we have contradiction and hence $\hat{T} \geq \tilde{T}(n, A, \lambda)$. This will complete the proof. Hence, it remains to establish the uniform covering.

For each $(y, t) \times M \times [0, T \wedge \hat{T} \wedge \tilde{T})$, we let $\{x_i\}_{i=1}^{N}$ be a maximal set of points in $B_t(y, 1)$ such that $B_t(x_i, \frac{1}{8})$ are disjoint from each other and (2.38) holds. By (2.31) and distance distortion estimates [43, Lemma 8.3], we have $B_t(y, 1) \subset B_0(y, 2)$ if $\tilde{T}$ is small. At the same time, by choosing δ sufficiently small, we may apply the proof of [29, Lemma 2.4] (see also [34, Lemma 2.2]) to show that $B_t(x_i, \frac{1}{8}) \supset B_0(x_i, r_0)$ for some uniformly small r_0. Therefore,

$$\begin{aligned} \sum_{i=1}^{N} \mathrm{Vol}_{g_0} \left(B_0(x_i, r_0) \right) &\leq \sum_{i=1}^{N} \mathrm{Vol}_{g_0} \left(B_t(x_i, \frac{1}{8}) \right) \\ &\leq \mathrm{Vol}_{g_0} \left(B_t(y, 1) \right) \\ &\leq \mathrm{Vol}_{g_0} \left(B_0(y, 2) \right). \end{aligned} \tag{2.40}$$

Since $x_i \in B_t(y, 1) \subset B_0(y, 2)$, the estimates on N then follow from $\mathrm{Ric}(g_0)$ lower bound and volume comparison. The desired result follows by re-labeling the constants. □

3. Gap theorem of Ricci solitons

In this section we will prove Theorem 1.3, a gap theorem for shrinking and steady gradient Ricci solitons. We do not assume a-priori bounds on the curvature. The novel idea is to obtain local curvature control under the small $L^{n/2}$ curvature and local entropy bound (see also [22]). We first prove the following result, from which Theorem 1.3 shall follow.

Theorem 3.1. *For all $A \geq 1000n$, there is $\varepsilon(n, A), C(n, A), \hat{T}(n, A) > 0$ such that the following holds. Suppose $(M, g(t))$ is a Ricci flow on $[0, T]$ and $p \in M$ be a point such that for all $t \in (0, T]$,*

1. *$B_t(p, 1) \Subset M$;*
2. *$\left(\int_{B_t(p, 4A\sqrt{t})} |\mathrm{Rm}|^{n/2} d\mu_t \right)^{2/n} \leq \varepsilon_0$ for some $\varepsilon_0 < \varepsilon$;*
3. *$\nu(B_t(p, 4A\sqrt{t}), g(t), t) \geq -A$;*

then we have

$$\begin{cases} |\mathrm{Rm}|(x,t) \leq C(n,A)\varepsilon_0 t^{-1} \\ \mathrm{inj}(x,t) \geq C(n,A)^{-1}\sqrt{t} \end{cases} \tag{3.1}$$

for all $x \in B_t(p, \frac{1}{4}A\sqrt{t}), t \leq T \wedge \hat{T}$.

Proof. We will split the proof into three parts.

Step 1. Rough estimates under stronger assumption. We first prove the rough curvature estimate: $|\mathrm{Rm}(x,t)| \leq C(n,A)t^{-1}$ on $B_t(p, \frac{1}{2}A\sqrt{t})$ under an extra assumption:

$$\star \qquad \mathrm{Ric}(x,t) \leq t^{-1}, \ \text{ on } \ B_t(p,\sqrt{t}), t \in (0,T].$$

The injectivity radius estimates will follow from the work of [11] by [52, Theorem 3.3]. For $x \in B_t(p, A\sqrt{t}), t < T$, we define

$$r(x,t) = \sup\left\{0 < r < A\sqrt{t} - d_t(x,p) : \sup_{P(x,t,r)} |\mathrm{Rm}| \leq r^{-2}\right\} \tag{3.2}$$

where $P(x,t,r) = \{(y,s) : y \in B_s(x,r), s \in [t-r^2, t] \cap (0,T]\}$. We claim that there are $\varepsilon, c_0, T_0 > 0$ depending only on n, A such that if assumptions hold for $\varepsilon_0 < \varepsilon$, then for all $x \in B_t(p, A\sqrt{t})$ and $t < T \wedge T_0$,

$$F(x,t) = \frac{r(x,t)}{A\sqrt{t} - d_t(x,p)} \geq c_0. \tag{3.3}$$

The rough curvature estimate then follows immediately from the claim since for any $x \in B_t(p, \frac{1}{2}A\sqrt{t})$ and $t < T \wedge T_0$,

$$\begin{aligned} |\mathrm{Rm}|(x,t) &\leq \frac{1}{r^2(x,t)} \\ &\leq \frac{1}{c_0^2(A\sqrt{t} - d_t(p,x))^2} \\ &\leq \frac{4}{c_0^2 A^2 t}. \end{aligned} \tag{3.4}$$

Suppose on the contrary that the claim is not true for some A and n, we can find a sequence of Ricci flow $\{(M_i, g_i(t), p_i)\}_{i=1}^{\infty}$, $t_i, \varepsilon_i \to 0$ such that

- $\mathrm{Ric}_i(x,t) < t^{-1}$ for all $x \in B_t(p_i, \sqrt{t})$, $t < t_i$;
- $\displaystyle\int_{B_{g_i(t)}(p_i, 4A\sqrt{t})} |\mathrm{Rm}_i|^{n/2} d\mu_{i,t} \leq \varepsilon_i$ for all $t < t_i$;
- $\nu(B_{g_i(t)}(p, 4A\sqrt{t}), g_i(t), t) \geq -2A$ for $t < t_i$,

but for some sequence $x_i \in B_t(p_i, A\sqrt{t})$, we have

$$F_i(x_i, t_i) = \min\{F_i(y, s) : s \in (0, t_i), y \in B_s(p_i, A\sqrt{s})\} \to 0. \tag{3.5}$$

Re-scale the flow by $\tilde{g}_i(t) = Q_i g_i(t_i + Q_i^{-1}t)$, $-Q_i t_i \leq t \leq 0$ where $Q_i^{-1/2} = r_i(x_i, t_i)$ so that $\tilde{r}_i(x_i, 0) = 1$. Then by (3.5)

$$\begin{aligned} d_{\tilde{g}_i(0)}(x_i, \partial B_{g_i(t_i)}(p_i, A\sqrt{t_i})) &= \frac{d_{g_i(t_i)}(x_i, \partial B_{g_i(t_i)}(p_i, A\sqrt{t_i}))}{r(x_i, t_i)} \\ &\geq \frac{A\sqrt{t_i} - d_{t_i}(x_i, p_i)}{r(x_i, t_i)} \\ &= F_i(x_i, t_i)^{-1} \\ &\to +\infty. \end{aligned} \tag{3.6}$$

That is to say the pointed Cheeger-Gromov limit of the flow centred at x_i is complete provided it exists. Furthermore, we may invoke (3.5) again to see that

$$\begin{aligned} Q_i t_i &= \frac{t_i}{r_i(x_i, t_i)^2} \\ &> \frac{1}{A^2}\left(\frac{A\sqrt{t_i} - d_{t_i}(x_i, p_i)}{r_i(x_i, t_i)}\right)^2 \\ &\to +\infty. \end{aligned} \tag{3.7}$$

Next, we would like to show that after passing to a sub-sequence, the flows converge in Cheeger-Gromov sense. The two key ingredients are uniform curvature bound in i on compact sets in spacetime and the injectivity radius lower bound at x_i w.r.t. $\tilde{g}_i(0)$.

Let $r > 0$ and $y \in \tilde{B}_s(x_i, r), s \in (-1, 0]$. Using $r_i(x_i, t_i) = Q_i^{-1/2} << \sqrt{t_i}$ and assumption $\star$, we apply Hamilton-Perelman's distance estimates [43, Lemma 8.3] (see also [25]) with $r_0 = Q_i^{-1/2}$ and $K = Q_i$ so that

$$\begin{aligned} d_{Q_i^{-1}s+t_i}(x_i, p_i) &\leq d_{t_i}(x_i, p_i) + C_n Q_i^{1/2} \cdot (t_i - Q_i^{-1}s - t_i) \\ &\leq d_{t_i}(x_i, p_i) + C_n Q_i^{-1/2}. \end{aligned} \tag{3.8}$$

Hence

$$\begin{aligned} d_{Q_i^{-1}s+t_i}(y, p_i) &\leq Q_i^{-\frac{1}{2}} r + d_{Q_i^{-1}s+t_i}(x_i, p_i) \\ &\leq C_n Q_i^{-\frac{1}{2}} r + d_{t_i}(x_i, p_i) \end{aligned} \tag{3.9}$$

It follows from (3.5) and (3.7) that for all large $i > N(n, A, r)$,

$$\begin{aligned} A\sqrt{Q_i t_i + s} - Q_i^{\frac{1}{2}} d_{t_i}(x_i, p_i) &\geq A\sqrt{Q_i t_i - 1} - Q_i^{\frac{1}{2}} d_{t_i}(x_i, p_i) \\ &\geq A\sqrt{Q_i t_i} - Q_i^{\frac{1}{2}} d_{t_i}(x_i, p_i) - A \\ &= F(x_i, t_i)^{-1} - A \\ &> C_n r. \end{aligned} \tag{3.10}$$

Hence by (3.9), $y \in B_{Q_i^{-1}s+t_i}(p_i, A\sqrt{Q_i^{-1}s + t_i})$. We have by (3.5), (3.9) and (3.10) that

$$\begin{aligned} \tilde{r}_i(y, s) &= \frac{r_i(y, Q_i^{-1}s + t_i)}{r_i(x_i, t_i)} \\ &= \frac{F_i(y, Q_i^{-1}s + t_i)}{F_i(x_i, t_i)} \cdot \frac{A\sqrt{Q_i^{-1}s + t_i} - d_{Q_i^{-1}s+t_i}(y, p_i)}{A\sqrt{t_i} - d_{t_i}(x_i, p_i)} \\ &\geq \frac{A\sqrt{Q_i^{-1}s + t_i} - d_{Q_i^{-1}s+t_i}(y, p_i)}{A\sqrt{t_i} - d_{t_i}(x_i, p_i)} \\ &\geq \frac{A\sqrt{Q_i^{-1}s + t_i} - d_{Q_i^{-1}s+t_i}(x_i, p_i) - Q_i^{-1/2} r}{A\sqrt{t_i} - d_{t_i}(x_i, p_i)} \\ &\geq \frac{F(x_i, t_i)^{-1} - A - C_n r}{F(x_i, t_i)^{-1}}. \end{aligned} \tag{3.11}$$

Thus for all $y \in \tilde{B}_s(x_i, r), s \in (-1, 0]$, $r > 0$ and $i > N(n, A, r)$, we have

$$\tilde{r}_i(y, s) > \frac{1}{2}. \tag{3.12}$$

This gives the curvature estimates on any compact subset in space-time. By our assumptions, for any $r > 0$ and $i > N(n, A, r)$, the entropy satisfies

$$\begin{aligned} &\nu(\widetilde{B}_t(x_i, r), \tilde{g}_i(t), Q_i t_i + t) \\ &\geq \nu(\widetilde{B}_t(p_i, 4A\sqrt{Q_i t_i + t}), \tilde{g}_i(t), Q_i t_i + t) \\ &= \nu(B_t(p_i, 4A\sqrt{t_i + Q_i^{-1}t}), g_i(t_i + Q_i^{-1}t), t_i + Q_i^{-1}t) \\ &\geq -2A. \end{aligned} \tag{3.13}$$

By virtue of (3.7), (3.12), (3.13) and [52, Theorem 3.3], the volume ratios $\frac{\widetilde{V}_0(\widetilde{B}_0(x_i, r))}{r^n}$ are uniformly bounded from below in i for any all $r \in (0, 1/2]$. Thanks to (3.12) and Cheeger-Gromov-Taylor injectivity radius estimate [11], the injectivity radius at x_i w.r.t. $\tilde{g}_i(0)$ have a uniform positive lower bound in i. Hence by Hamilton's compactness theorem (see [24], [18]), we can pass $\tilde{g}_i(t)$ to a complete limiting Ricci flow $(M_\infty, \tilde{g}_\infty(t), x_\infty)$, $t \in (-1, 0]$ which is a complete solution with bounded curvature. By the choice of $Q_i^{-1/2}$

and (3.9), we have $B_{t_i}(x_i, Q_i^{-1/2}) \Subset B_{t_i}(p_i, A\sqrt{t_i})$. Therefore, for all $-1 < s < 0$, if $d_{Q_i^{-1}s+t_i}(x_i, y) < r$, then by (3.8)

$$\begin{aligned} d_{Q_i^{-1}s+t_i}(y, p_i) &\leq d_{Q_i^{-1}s+t_i}(y, x_i) + d_{Q_i^{-1}s+t_i}(x_i, p_i) \\ &\leq r + d_{t_i}(x_i, p_i) + C_n Q_i^{-1/2} \\ &\leq r + 2A\sqrt{t_i}. \end{aligned} \tag{3.14}$$

Therefore, if $r = \sqrt{t_i}$, then we have

$$B_{Q_i^{-1}s+t_i}(x_i, \sqrt{t_i}) \Subset B_{Q_i^{-1}s+t_i}(p_i, 3A\sqrt{Q_i^{-1}s + t_i})$$

for all $i \to +\infty$. This together with the assumption implies

$$\int_{M_\infty} |\widetilde{\mathrm{Rm}}|^{n/2} d\tilde{\mu}_s \equiv 0 \tag{3.15}$$

for all $-1 < s \leq 0$ and hence is a flat solution. Moreover, by the monotonicity of local entropy over domain, (3.13) and the proof of Lemma 6.28 in [18], we have

$$\nu(M_\infty, \tilde{g}_\infty(t)) \geq -2A \tag{3.16}$$

for all $-1 < t \leq 0$. Recall that we have $\tilde{r}_i(x_i, 0) = 1$. As the entropy is bounded from below for all scales, the manifolds must be of maximum volume growth which implies that $\tilde{g}_\infty(t)$ is the static flat Euclidean metric. This contradicts with the curvature radius at $(x_\infty, 0)$ and completes the proof under the assumption $\star$. Now the injectivity radius estimate follows from the curvature estimate and the work of [11].

Step 2. Removing assumption $\star$ in Step 1. Since $B_t(p, 1) \Subset M$ for $t \leq T$, by smoothness of solution we may find $\tilde{T} \leq T$ such that $|\mathrm{Ric}| < t^{-1}$ for $x \in B_t(p, \sqrt{t})$, $t \in (0, \tilde{T})$. W.L.O.G., we may assume that $\tilde{T}$ to be small uniformly, otherwise the required estimate on $|\mathrm{Rm}|$ follows by Step 1. Hence the result under $\star$ gives the curvature estimates over a smaller ball, i.e. for some $\hat{T}(n, A)$,

$$|\mathrm{Rm}|(x, t) \leq C(n, A)t^{-1} \tag{3.17}$$

for all $x \in B_t(p, \frac{1}{2}A\sqrt{t})$, $t \leq \min\{\tilde{T}, \hat{T}(n, A)\}$.

We claim that $\tilde{T} \geq T \wedge \hat{T}(n, A)$. Suppose that is not the case, denote $s = \tilde{T}$, then by the maximality of $\tilde{T}$ there is $\bar{x} \in \overline{B_s(p, \sqrt{s})}$ such that $|\mathrm{Ric}|(\bar{x}, s) = s^{-1}$. By considering the flow $s^{-1}g(st), t \in [0, 1]$, we may wlog assume $s = 1$. By the estimates of $\mathrm{inj}(x, t)$, (3.17), Theorem 3.3 in [52], $\mathrm{Vol}_{g(s)}\left(B_s(\bar{x}, \frac{1}{4}A\sqrt{s})\right)$ is uniformly bounded from below for any $s \in [1/2, 1]$. Together with a result of Saloff-Coste [44], we get a uniform Sobolev

inequality on $B_s(\bar{x}, \frac{1}{4}A\sqrt{s})$ for any $s \in [1/2, 1]$. Then the Moser iteration argument [37, Chapter 19] on $B_s(\bar{x}, \frac{1}{4}A\sqrt{s})$ and the Hölder inequality would imply

$$\begin{aligned} 1 &= |\mathrm{Ric}|(\bar{x}, 1) \\ &\leq c(n)|\mathrm{Rm}|(\bar{x}, 1) \\ &\leq C'(n, A)\left(\int_{1/2}^{1} \fint_{B_s(\bar{x}, \frac{1}{4}A\sqrt{s})} |\mathrm{Rm}|^{n/2} d\mu_s ds\right)^{2/n} \\ &\leq C''(n, A)\varepsilon_0, \end{aligned} \tag{3.18}$$

which is impossible if $\varepsilon_0 \leq \varepsilon(n, A)$ is sufficiently small. Hence $\tilde{T} \geq T \wedge \hat{T}(n, A)$. This implies the curvature estimate for $|\mathrm{Rm}|$ on $B_t(p, \frac{1}{4}A\sqrt{t})$ by Step 1.

Step 3. Improved curvature estimates. At this point we have already obtained a rough curvature estimate on $B_t(p, \frac{1}{2}A\sqrt{t}), t \in [0, T \wedge \hat{T}]$. For each $s \in [0, T \wedge \hat{T}]$, we may consider $\tilde{g}(t) = s^{-1}g(st), t \in [0, 1]$. Since we have curvature bound on $[\frac{1}{2}, 1]$ and entropy lower bound, with the scaling invariant $L^{n/2}$ assumption we can apply iteration [37] again to show that

$$\begin{aligned} &|\mathrm{Rm}(\tilde{g}(x, 1))| \\ &\leq C(n, A)\left(\int_{1/2}^{1} \fint_{B_{\tilde{g}(s)}(x, \frac{1}{4}A\sqrt{s})} |\mathrm{Rm}(\tilde{g}(t))|^{n/2} d\mu_s ds\right)^{2/n} \\ &\leq C(n, A)\varepsilon_0. \end{aligned} \tag{3.19}$$

This gives an improved coefficient on curvature decay by rescaling it back to $g(t)$. □

We now show how our gap theorem for complete shrinking and steady gradient Ricci solitons with small $||\mathrm{Rm}||_{L^{n/2}}$, Theorem 1.3, follows from Theorem 3.1. Recall that a complete Riemannian manifold (M, g) is said to be a shrinking (steady) gradient Ricci soliton if there exists a smooth function f such that

$$\mathrm{Ric} + \nabla^2 f = \lambda g, \tag{3.20}$$

where the constant $\lambda = \frac{1}{2}$ ($= 0$ resp.).

Proof of Theorem 1.3. Let $\lambda = 1/2$ or 0 be the constant as in (3.20). We consider the flow ϕ_t of the vector field $\frac{\nabla f}{1-2\lambda t}$ with ϕ_0 being the identity map. It is known that $g(t) := (1-2\lambda t)\phi_t^* g$ is an ancient solution to the Ricci flow on M with $g(0) = g$ and $t \in (-\infty, \frac{1}{2\lambda})$ ($= \mathbb{R}$ if $\lambda = 0$, see [18,61]). By the reparametrization and the scaling invariance of Conditions 1 and 2 in Theorem 1.3, we have for all $t \in (-\infty, \frac{1}{2\lambda})$:

1. $\nu(M, g(t)) \geq -A$;
2. $\int_M |\mathrm{Rm}|^{n/2}_{g(t)} d\mu_{g(t)} \leq \varepsilon$.

We are going to show something slightly more general, namely if $(M, g(t))$ is a complete ancient solution to the Ricci flow on $(-\infty, 0]$ such that $g(t)$ satisfies the above two conditions for each $t \in (-\infty, 0]$, then $(M, g(t))$ is isometric to $\mathbb{R}^n$. For any $Q > 1$ and $\tau \leq 0$, we consider the rescaled solution $h(t) := (\frac{Q}{\hat{T}})^{-1} g(\frac{Q}{\hat{T}} t - Q + \tau)$, where $t \in [0, \hat{T}]$ and $\hat{T}$ is the constant as in Theorem 3.1. It is not difficult to see that $h(t)$ also satisfies the two conditions in Theorem 1.3. Hence we may apply Theorem 3.1 for all sufficiently small ε to get for any $x \in M$

$$\begin{aligned} Q|Rm|_g(x, \tau) &= \hat{T}|Rm|_h(x, \hat{T}) \\ &\leq C(n, A)\varepsilon. \end{aligned}$$

By letting $Q \to \infty$, we have $g(\tau)$ is flat. The entropy lower bound at all scales then implies the maximal volume growth of $g(\tau)$ and thus it is isometric to $\mathbb{R}^n$. □

4. Gap theorem with small $||\mathrm{Rm}||_{L^{n/2}}$

In this section, we will use Ricci flow to discuss Riemannian manifolds with $\mathrm{Ric} \geq 0$ and with small $||\mathrm{Rm}||_{L^{n/2}}$ which are non-collapsed in term of entropy. We first show that under the assumption of Corollary 1.1, we have a long-time solution of the Ricci flow and g_0 has maximal volume growth.

Theorem 4.1. *For any $A > 0$, there is $\sigma(n, A), C_1(n, A) > 0$ such that the following holds. Suppose (M, g_0) is a complete non-compact Riemannian manifold with bounded curvature such that*

1. $\mathrm{Ric}(g_0) \geq 0$;
2. $\nu(M, g_0) \geq -A$;
3. $\left(\int_M |\mathrm{Rm}(g_0)|^{n/2} d\mu_{g_0}\right)^{2/n} \leq \varepsilon$ *for some* $\varepsilon < \sigma$.

Then there is a Ricci flow $g(t)$ starting from g_0 on $M \times [0, \infty)$ such that for all $t > 0$,

$$\begin{cases} \sup_M |\mathrm{Rm}(g(t))| \leq C_1 \varepsilon t^{-1} \\ \left(\int_M |\mathrm{Rm}(g(t))|^{n/2} d\mu_t\right)^{2/n} \leq C_1 \varepsilon \end{cases} \tag{4.1}$$

Moreover, g_0 is of maximal volume growth.

Remark 4.1. The assumption on the global entropy of all scale can also be implied by maximal volume growth.

Proof. For $R > 0$, we let $g_{R,0} = R^{-2}g_0$ which still satisfies the assumptions of the Theorem, which are scaling invariant. Therefore we can run Shi's Ricci flow $g_R(t)$ [45] for a short-time with initial metric $g_{R,0}$. By Theorem 1.2, if σ is sufficiently small, $g_R(t)$ exists on $M \times [0, T(n, A)]$ and satisfies

$$\begin{cases} |\mathrm{Rm}(g_R(t))| \leq C_1 \varepsilon t^{-1} \\ \left(\int_{B_{g_R(t)}(x,1)} |\mathrm{Rm}(g_R(t))|^{n/2} d\mu_{R,t} \right)^{2/n} \leq C_1 \varepsilon \end{cases} \tag{4.2}$$

for all $(x, t) \in M \times [0, T]$. By re-scaling it back and the uniqueness of Ricci flow [14], we obtain a Ricci flow $g(t)$ on $[0, TR^2)$ with $|\mathrm{Rm}| \leq C_1 \varepsilon t^{-1}$ and $g(0) = g_0$. Moreover, we have for all $R, t > 0$,

$$\left(\int\limits_{B_t(x,R)} |\mathrm{Rm}(g(t))|^{n/2} d\mu_t \right)^{2/n} \leq C_1 \varepsilon. \tag{4.3}$$

The global integral estimate then follows by letting $R \to +\infty$.

To see that g_0 is of maximal volume growth, thanks to the improved regularity on curvature and monotonicity of entropy ν, the re-scaled Ricci flow $g_R(t)$ satisfies

$$\mathrm{Vol}_{g_R(1)}(B_{g_R(1)}(x, 1)) \geq c. \tag{4.4}$$

Since the lower bound of scalar curvature is preserved along the Ricci flow, together with [46, Corollary 3.3], we have, if σ is sufficiently small, that

$$\begin{aligned} c &\leq \mathrm{Vol}_{g_R(1)}(B_{g_R(1)}(x, 1)) \\ &\leq \mathrm{Vol}_{g_R(0)}(B_{g_R(0)}(x, 2)) \\ &= \frac{\mathrm{Vol}_{g_0}(B_{g_0}(x, 2R))}{R^n}. \end{aligned} \tag{4.5}$$

Since R is arbitrarily large, this completes the proof. □

Before we prove the Corollary 1.1, we will show that the asymptotic volume ratio can be improved to be almost Euclidean if we further shrink the integral curvature and hence is almost Euclidean in the sense of local entropy [53, Lemma 4.10]. This is in spirit similar to the gap theorem proved by Cheeger [9, Theorem 4.32].

Theorem 4.2. *For all $A, \lambda, \delta > 0$, there are $\sigma(n, A, \lambda, \delta), r(n, A, \lambda, \delta) > 0$ such that if (M, g) is a complete Riemannian manifold of bounded curvature so that for all $p \in M$,*

1. $\mathrm{Ric}(g) \geq -\lambda$;
2. $\nu\left(B_g(p, 5), g, 1\right) \geq -A$;

3. $\left(\int_{B_g(p,2)} |\mathrm{Rm}(g)|^{n/2} d\mu_g\right)^{2/n} < \sigma$.

Then for all $p \in M$,

$$\mathrm{Vol}_g\left(B_g(p,r)\right) \geq (1-\delta)\omega_n r^n. \tag{4.6}$$

Proof. Let $g(t), t \in [0,\hat{T}]$ be the Ricci flow solution obtained from Theorem 1.2 and Shi's Ricci flow [45]. We claim that for given δ, there are constants $T(n,A,\lambda,\delta)$ and $\hat{\sigma}(n,A,\lambda,\delta)$ such that if $\sigma < \hat{\sigma}$, then for $(x,t) \in M \times [0,T]$,

$$\mathrm{Vol}_{g(t)}\left(B_{g(t)}(x,\sqrt{t})\right) \geq (1-\delta)\omega_n t^{n/2}. \tag{4.7}$$

Suppose on the contrary, we can find a sequence of $g_i(t), t \in [0,\hat{T}]$ such that $g_i(0)$ satisfies the same assumptions as in Theorem 1.2 with $\varepsilon_i \to 0$ now but for some $\hat{x}_i \in M_i$ and $0 < \sqrt{t_i} \to 0$,

$$\mathrm{Vol}_{g_i(t_i)}\left(B_{g_i(t_i)}(\hat{x}_i,\sqrt{t_i})\right) < (1-\delta)\omega_n t_i^{n/2}. \tag{4.8}$$

We note here that since $\varepsilon_i \to 0$, the existence time $\hat{T}$ can be chosen to be independent of $i \to +\infty$ by Theorem 1.2.

Consider the rescaled Ricci flow $\tilde{g}_i(t) = t_i^{-1} g(t_i t)$ on $M_i \times [0,1]$. The original estimates from Theorem 1.2 imply that for all sufficiently large i and all $(x,t) \in M_i \times (0,1]$,

$$|\mathrm{Rm}(\tilde{g}_i(t))| \leq C_0(n,A,\lambda)\varepsilon_i t^{-1} \quad \text{and} \quad \mathrm{inj}_{\tilde{g}_i(t)} \geq c_0(n,A,\lambda)\sqrt{t}$$

which enable us to pass $(M_i, \tilde{g}_i(t), \hat{x}_i)$ to a sub-sequential limit $(M_\infty, \tilde{g}_\infty(t), \hat{x}_\infty)$ in smooth Cheeger-Gromov sense by Hamilton's compactness [24]. In particular, $\tilde{g}_\infty(t)$ is flat for $t \in (0,1]$ since $\varepsilon_i \to 0$. On the other hand, since $t_i \to 0$, we may apply the local monotonicity of entropy in [52, Theorem 5.4] again to show that $\nu(M_\infty, \tilde{g}_\infty(t)) \geq -2A$ which implies $\tilde{g}_\infty(t)$ is of Euclidean volume growth by Theorem [52, Theorem 3.3] and hence $(M_\infty, \tilde{g}_\infty(1)) \equiv (\mathbb{R}^n, g_{euc})$ which contradicts (4.8).

After relabeling the constants, (4.7), together with volume comparison implies that for all $t \in [0,T]$ and $r < \sqrt{t}$,

$$\mathrm{Vol}_{g(t)}\left(B_{g(t)}(x,r)\right) \geq \left(1 - \frac{\delta}{2}\right)\omega_n r^n. \tag{4.9}$$

By the scalar curvature lower bound of $g(t)$ and [46, Corollary 3.3],

$$\begin{aligned}\mathrm{Vol}_{g_0}\left(B_{g_0}(x,r)\right) &\geq e^{-n\lambda t}\cdot \mathrm{Vol}_{g(t)}\left(B_{g_0}(x,r)\right)\\ &\geq e^{-n\lambda t}\cdot \mathrm{Vol}_{g(t)}\left(B_{g(t)}(x,r-c_n\sqrt{\sigma t})\right)\\ &\geq e^{-n\lambda t}\left(1-\frac{\delta}{2}\right)\omega_n\left(r-c_n\sqrt{\sigma t}\right)^n\\ &\geq (1-\delta)\,\omega_n r^n\end{aligned} \tag{4.10}$$

if we choose t,σ small enough and $r=\frac{1}{2}\sqrt{t}$. This completes the proof. □

Proof of Corollary 1.1. Theorem 4.1 implies that g_0 is of maximal volume growth. By [10, Theorem A.1.11] (see also [53, Theorem 5.7]), it suffices to show that the asymptotic volume growth can be made arbitrarily close to the Euclidean one if we shrink σ. This follows from Theorem 4.2 and the rescaling argument as in the proof of Theorem 4.1.

Alternatively, we can also prove the homeomorphism by showing $M=\bigcup_{i=1}^{\infty}U_i$ where U_i is diffeomorphic to a Euclidean ball and $U_i\subset U_{i+1}$ for all i using the expansion of injectivity radius, curvature estimate $|\mathrm{Rm}(x,t)|\leq \varepsilon t^{-1}$ from Theorem 4.1. Then the homeomorphism will follow from the main result of [5], see also [13, Section 3]. Notice that Gompf's result says that among the Euclidean spaces only $\mathbb{R}^4$ has exotic differential structures. So for $n>4$, the homeomorphisms can be made to be diffeomorphisms (see [48]). □

5. Regularity of Gromov-Hausdorff limit

In this section, we discuss the compactness of Riemannian manifolds satisfying small $L^{n/2}$ bound. We remark here that the Gromov-Hausdorff limit follows from Ricci lower bound directly. The key part is to construct the differentiable structure on the limit using the pseudolocality of Ricci flows.

Proof of Theorem 1.4. By Shi's Ricci flow existence [45] and Theorem 1.2, by choosing ε_0 small enough we can find a sequence of Ricci flow $g_i(t)$ on $M_i\times[0,T(n,A)]$ such that

1. $\mathrm{Ric}(g_i(0))\geq -\lambda$;
2. $\nu(B_{g_i(t)}(x,1),g_i(t),\frac{1}{32})\geq -2A$;
3. $|\mathrm{Rm}(g_i(t))|\leq \frac{C\varepsilon_0}{t}$

for all $(x,t)\in M_i\times(0,T]$. By [52, Theorem 3.3] and [11], we can apply Hamilton's compactness to pass $(M_i,g_i(t),p_i)$ to $(M_\infty,g_\infty(t),p_\infty)$ for $t\in(0,T]$ in the smooth Cheeger-Gromov sense after passing to sub-sequence. More precisely, there is an exhaustion $\{\Omega_i\}_{i=1}^{\infty}$ of M_∞ and a sequence of diffeomorphism $F_i:\Omega_i\to M_i$ onto its image

such that for any compact subset $\Omega \times [a,b] \Subset M_\infty \times (0,T]$, we have $F_i^* g_i(t) \to g_\infty(t)$ in $C^\infty_{loc}(\Omega \times [a,b])$.

We now construct the Gromov-Hausdorff limit of g_i using F_i in a more precise way so that its relation to M_∞'s topology is clearer. This essentially follows from the proofs of Gromov's compactness theorem and the distance distortion estimates. Since M_∞ is a smooth manifold, we let $\{x_k\}_{k=1}^\infty$ be a countable dense set with respect to $g_\infty(1)$. Then for each k,l, we have $x_k, x_l \in B_{g_\infty(1)}(p_\infty, R_{k,l})$ and hence by distance distortion estimates [46, Corollary 3.3] using curvature estimates above, we have

$$d_{F_i^* g_i}(x_k, x_l) \leq d_{F_i^* g_i(1)}(x_k, x_l) + C_n \leq C(k,l) \tag{5.1}$$

as $i \to +\infty$. Here we have used the fact that $F_i^* g_i(1)$ converges locally uniformly to $g_\infty(1)$. Therefore, $\lim_{i\to+\infty} d_{F_i^* g_i}(x_k, x_l)$ exists after we pass it to some sub-sequence which we denote it as $d_\infty(x_k, x_l)$. Repeating the process for each k,l, we define d_∞ on the dense set. For general $x, y \in M_\infty$, we define $d_\infty(x,y)$ using the density of $\{x_k\}$. This is well defined since if there are two sequences $x_i, x_i' \to x \in M_\infty$ and $y_i, y_i' \to y \in M_\infty$ with respect to $g_\infty(1)$, then for i sufficiently large,

$$\begin{aligned} d_\infty(x_i, y_i) &\leq d_\infty(x_i', y_i') + d_\infty(x_i, x_i') + d_\infty(y_i, y_i') \\ &\leq d_\infty(x_i', y_i') + C\left(d_{g_\infty(1)}(x_i, x_i')\right)^{1/2} + C\left(d_{g_\infty(1)}(y_i, y_i')\right)^{1/2} \\ &= d_\infty(x_i', y_i') + o(1), \end{aligned} \tag{5.2}$$

by using [29, Lemma 2.4] and [46, Corollary 3.3]. By passing $i \to +\infty$ and switching the sequences, we have the uniqueness of the limit. In other words, we have

$$\lim_{i\to+\infty} d_{F_i^* g_i}(x,y) = d_\infty(x,y) \tag{5.3}$$

for all $x, y \in M_\infty$.

Now we claim that $d_\infty(\cdot,\cdot)$ is in fact a distance defined on $M_\infty \times M_\infty$. To see this, let $y, z \in M_\infty$ be such that $d_\infty(z,y) = 0$. If $y \neq z$, then we have $d_{g_\infty(1)}(z,y) > r$ for some $r > 0$. For any $\varepsilon > 0$, we can find $y', z' \in \{x_i\}_{i=1}^\infty$ such that $d_{g_\infty(1)}(y,y') + d_{g_\infty(1)}(z,z') + d_\infty(y',z') < \varepsilon$ and therefore we can find $N \in \mathbb{N}$ such that for $i > N$, $d_{F_i^* g_i}(y',z') < 3\varepsilon$. Applying [29, Lemma 2.4] again, we deduce

$$d_{F_i^* g_i(1)}(y',z') \leq C(n,\lambda)\varepsilon^{2/3}. \tag{5.4}$$

Here we note that although [29, Lemma 2.4] is stated globally, it is easy to see that the proof holds locally and only require the curvature bound in form of εt^{-1} for ε small enough and an initial Ricci lower bound which is available in our situation. Therefore, if ε is sufficiently small, it will violate the fact that $d_{g_\infty(1)}(y,z) > r$. This shows that d_∞ defines a distance metric on M_∞.

To see that d_∞ generates the same topology as M_∞, it suffices to point out that [29, Lemma 2.4] together with a limiting argument implies that for $d_\infty(x,y) < 1$, we have

$$C_n^{-1} d_{g_\infty(1)}(x,y)^{3/2} \leq d_\infty(x,y) \leq C_n d_{g_\infty(1)}(x,y)^{1/2} \tag{5.5}$$

and hence all small open balls are uniformly comparable. Moreover by [34, Lemma 2.2], we also see that $\{B_{d_\infty}(p_\infty, k)\}_{k=1}^\infty$ is an exhaustion of M_∞. By the construction, (5.3), and (5.5), the pointed Gromov-Hausdorff convergence is straight forward with F_i being the Gromov-Hausdorff approximation on each compact set $\Omega \Subset M_\infty$. □

References

[1] M. Anderson, Ricci curvature bounds and Einstein metrics on compact manifolds, J. Am. Math. Soc. 2 (3) (1989) 455–490.
[2] M.T. Anderson, J. Cheeger, Diffeomorphism finiteness for manifolds with Ricci curvature and $L^{n/2}$-norm of curvature bounded, Geom. Funct. Anal. 1 (3) (1991) 231–252.
[3] R.H. Bamler, E. Cabezas-Rivas, B. Wilking, The Ricci flow under almost non-negative curvature conditions, Invent. Math. 217 (2019) 95–126.
[4] S. Bando, A. Kasue, H. Nakajima, On a construction of coordinates at infinity on manifolds with fast curvature decay and maximal volume growth, Invent. Math. 97 (2) (1989) 313–349.
[5] M. Brown, The monotone union of open n-cells is an open n-cell, Proc. Am. Math. Soc. 12 (1961) 812–814.
[6] H.-D. Cao, N. Šešum, A compactness result for Kähler Ricci solitons, Adv. Math. 211 (2) (2007) 794–818.
[7] P.-Y. Chan, Curvature estimates and gap theorems for expanding Ricci solitons, arXiv:2001.11487, Int. Math. Res. Not. (2022), https://doi.org/10.1093/imrn/rnab257, in press.
[8] A. Chau, L.-F. Tam, C. Yu, Pseudolocality for the Ricci flow and applications, Can. J. Math. 63 (1) (2011) 55–85.
[9] J. Cheeger, Integral bounds on curvature elliptic estimates and rectifiability of singular sets, Geom. Funct. Anal. 13 (1) (2003) 20–72.
[10] J. Cheeger, T.H. Colding, On the structure of spaces with Ricci curvature bounded below. I, J. Differ. Geom. 46 (3) (1997) 406–480.
[11] J. Cheeger, M. Gromov, M. Taylor, Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds, J. Differ. Geom. 17 (1) (1982) 15–53.
[12] B.-L. Chen, X.-P. Zhu, A gap theorem for complete noncompact manifolds with nonnegative Ricci curvature, Commun. Anal. Geom. 10 (1) (2002) 217–239.
[13] B.-L. Chen, X.-P. Zhu, On complete noncompact Kähler manifolds with positive bisectional curvature, Math. Ann. 327 (2003) 1–23.
[14] B.-L. Chen, X.-P. Zhu, Uniqueness of the Ricci flow on complete noncompact manifolds, J. Differ. Geom. 74 (2006) 119–154.
[15] B.L. Chen, Strong uniqueness of Ricci flow, J. Differ. Geom. 82 (2) (2009) 362–382.
[16] C.-W. Chen, A. Deruelle, Structure at infinity of expanding gradient Ricci soliton, Asian J. Math. 19 (5) (2015) 933–950.
[17] E. Chen, Convergence of the Ricci flow on asymptotically flat manifolds with integral curvature pinching, arXiv:1907.13189, Ann. Sc. Norm. Super. Pisa, Cl. Sci. (2022), https://doi.org/10.2422/2036-2145.202001_007, in press.
[18] B. Chow, et al., The Ricci Flow: Techniques and Applications, Part I. Geometric Aspects, Math. Survey and Monographs, vol. 135, Amer. Math. Soc., Providence, RI, 2007.
[19] E.B. Davies, Heat Kernel and Spectral Theory, Cambridge University Press, 1989.
[20] Y.-X. Deng, X.-H. Zhu, Complete noncompact gradient Ricci solitons with nonnegative Ricci curvature, Math. Z. 279 (2015) 211–226.
[21] M. Fernández-López, E. García-Río, Some gap theorems for gradient Ricci solitons, Int. J. Math. 23 (7) (2012) 1250072, 9 pp.

[22] H.-B. Ge, W.-S. Jiang, ε-regularity for shrinking Ricci solitons and Ricci flows, Geom. Funct. Anal. 27 (5) (2017) 1231–1256.
[23] R. Hamilton, Three manifolds with positive Ricci curvature, J. Differ. Geom. 17 (1982) 255–306.
[24] R. Hamilton, A compactness property for solutions of the Ricci flow, Am. J. Math. 117 (3) (1995) 545–572.
[25] R. Hamilton, The Formation of Singularities in the Ricci Flow, Surveys in Differential Geometry, vol. 2, International Press, 1995, pp. 7–136.
[26] R. Haslhofer, R. Müller, A compactness theorem for complete Ricci shrinkers, Geom. Funct. Anal. 21 (5) (2011) 1091–1116.
[27] F. He, Existence and applications of Ricci flows via pseudolocality, arXiv:1610.01735.
[28] R. Hochard, Théorèmes d'existence en temps court du flot de Ricci pour des variétés non-complètes, non-éffondrées, à courbure minorée, PhD thesis, Université de Bordeaux, 2019.
[29] H.-Z. Huang, L.-L. Kong, X.-C. Rong, S.-C. Xu, Collapsed manifolds with Ricci bounded covering geometry, Trans. Am. Math. Soc. 373 (11) (2020) 8039–8057.
[30] S. Huang, X. Rong, B. Wang, Collapsing geometry with Ricci curvature bounded below and Ricci flow smoothing, SIGMA 16 (2020) 123, 25 pp.
[31] S. Huang, B. Wang, Ricci flow smoothing for locally collapsing manifolds, arXiv:2008.09956 [math. DG].
[32] B. Kleiner, J. Lott, Notes on Perelman's papers, Geom. Topol. 12 (5) (2008) 2587–2855.
[33] M.-C. Lee, L.-F. Tam, Kähler manifolds with almost non-negative curvature, Geom. Topol. 25 (4) (2021) 1979–2015.
[34] M.-C. Lee, L.-F. Tam, Some local maximum principles along Ricci flows, Can. J. Math. (2020) 1–20, https://doi.org/10.4153/S0008414X20000772.
[35] M. Ledoux, On manifolds with non-negative Ricci curvature and Sobolev inequalities, Commun. Anal. Geom. 7 (2) (1999) 347–353.
[36] Y. Lai, Ricci flow under local almost non-negative curvature conditions, Adv. Math. 343 (2019) 353–392.
[37] P. Li, Geometric Analysis, Cambridge University Press, 2012.
[38] Y. Li, Smoothing Riemannian metrics with bounded Ricci curvatures in dimension four, II, Ann. Glob. Anal. Geom. 41 (4) (2012) 407–421.
[39] C. Mantegazza, G. Mascellani, G. Uraltsev, On the distributional Hessian of the distance function, Pac. J. Math. 270 (1) (2014) 151–166.
[40] A.-D. McLeod, P.-M. Topping, Global regularity of three-dimensional Ricci limit spaces, arXiv: 1803.00414, Trans. Am. Math. Soc. (2022), in press.
[41] A.-D. McLeod, P.-M. Topping, Pyramid Ricci flow in higher dimensions, Math. Z. 296 (1–2) (2020) 511–523.
[42] O. Munteanu, M.-T. Wang, The curvature of gradient Ricci solitons, Math. Res. Lett. 18 (6) (2011) 1051–1069.
[43] G. Perelman, The entropy formula for the Ricci flow and its geometric applications, arXiv:math. DG/0211159.
[44] L. Saloff-Coste, Uniformly elliptic operators on Riemannian manifolds, J. Differ. Geom. 36 (2) (1992) 417–450.
[45] W.-X. Shi, Deforming the metric on complete Riemannian manifold, J. Differ. Geom. 30 (1989) 223–301.
[46] M. Simon, P.-M. Topping, Local control on the geometry in 3D Ricci flow, arXiv:1611.06137, 2016, J. Diff. Geom., in press.
[47] M. Simon, P.-M. Topping, Local mollification of Riemannian metrics using Ricci flow, and Ricci limit spaces, Geom. Topol. 25 (2) (2021) 913–948.
[48] J. Stallings, The piecewise-linear structure of Euclidean space, Proc. Camb. Philos. Soc. 58 (1962) 481–488.
[49] G. Tian, B. Wang, On the structure of almost Einstein manifolds, J. Am. Math. Soc. 28 (4) (2015) 1169–1209.
[50] P. Topping, Ricci flow compactness via pseudolocality, and flows with incomplete initial metrics, J. Eur. Math. Soc. 12 (6) (2010) 1429–1451.
[51] Y. Wang, Pseudolocality of the Ricci flow under integral bound of curvature, J. Geom. Anal. 23 (1) (2013) 1–23.
[52] B. Wang, The local entropy along Ricci flow Part A: the no-local-collapsing theorems, Camb. J. Math. 6 (3) (2018) 267–346.
[53] B. Wang, The local entropy along Ricci flow Part B: the pseudo-locality theorems, arXiv:2010.09981.

[54] C. Xia, Open manifolds with nonnegative Ricci curvature and large volume growth, Comment. Math. Helv. 74 (1999) 456–466.
[55] C. Xia, Complete manifolds with nonnegative Ricci curvature and almost best Sobolev constant, Ill. J. Math. 45 (4) (2001) 1253–1259.
[56] Y.-Y. Yang, Smoothing metrics on closed Riemannian manifolds through the Ricci flow, Ann. Glob. Anal. Geom. 40 (4) (2011) 411–425.
[57] R.-G. Ye, The logarithmic Sobolev and Sobolev inequalities along the Ricci flow, Commun. Math. Stat. 3 (1) (2015) 1–36.
[58] T. Yokota, Perelman's reduced volume and a gap theorem for the Ricci flow, Commun. Anal. Geom. 17 (2) (2009) 227–263.
[59] Q. Zhang, A uniform Sobolev inequality under Ricci flow, Int. Math. Res. Not. (2007) rnm056.
[60] S. Zhang, A gap theorem on complete shrinking gradient Ricci solitons, Proc. Am. Math. Soc. 146 (1) (2018) 359–368.
[61] Z.-H. Zhang, On the completeness of gradient Ricci solitons, Proc. Am. Math. Soc. 137 (2009) 2755–2759.