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Ricci flow and a sphere theorem for $L^{n/2}$-pinched Yamabe metrics

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ABSTRACT

We obtain a differential sphere and Ricci flow convergence theorem for positive scalar curvature Yamabe metrics with $L^{n/2}$-pinched curvature in general dimensions n. Previously, E. Hebey and M. Vaugon obtained in [9] a corresponding result for L^p-pinching with $p > n/2$.

1. Introduction

Let M^n be a compact smooth manifold of dimension $n \geq 3$ and g a smooth Riemannian metric on M. Recall that the Yamabe invariant of the conformal class of g, $[g] = \{e^u g, u \in C^\infty(M)\}$, is defined to be

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$$Y([g]) \equiv Y(M,[g]) = \inf_{\substack{\tilde{g}\in[g],\\ \mathrm{vol}_{\tilde{g}}(M)=1}} \int_M R_{\tilde{g}}\, d\,\mathrm{vol}_{\tilde{g}},$$

where R denotes the scalar curvature. We will also call $Y(M,[g])$ the Yamabe invariant of (M,g) or g and denote it by $Y(M,g)$ or $Y(g)$. Recall also that the σ-invariant $\sigma(M)$ of M is defined to be

$$\sigma(M) = \sup\{Y(M,g) : g \text{ is a smooth metric on } M\}.$$

By the solution of the Yamabe problem [17,16,1,14], we know that the infimum $Y(M,[g])$ is achieved in the conformal class $[g]$ for any smooth metric g on M. The minimizers, i.e. the infimum achieving metrics, have constant scalar curvature of the same sign as that of $Y(M,[g])$ and are referred to as Yamabe metrics. (Note that the corresponding result in dimension 2 is the classical uniformization theorem and Yamabe metrics there have constant Gauss curvature.) In this paper we will focus on Yamabe metrics with positive scalar curvature and establish a sphere and Ricci flow convergence theorem for such metrics (i.e. for Riemannian manifolds whose metrics are Yamabe metrics) with $L^{n/2}$-pinched curvature.

Let

$$\mathcal{Y}^+(M) = \{\text{Yamabe metrics } g \text{ on } M:\ R_g > 0,\ \mathrm{vol}_g(M) = 1\}.$$

Let Rm denote the Riemann curvature tensor and Z the *concircular curvature tensor*:

$$Z = Rm - \frac{R}{2n(n-1)} g \owedge g.$$

Note that $Z \equiv 0$ if and only if the metric g has constant sectional curvature. Employing the Ricci flow, G. Huisken, C. Margerin and S. Nishikawa obtained differential sphere theorems for Riemannian manifolds satisfying pointwise pinching conditions of the form $|Z| < c(n)R$ [10,11,13], where the scalar curvature is assumed to be positive and $c(n)$ is a sufficiently small positive constant depending only on the dimension n. Subsequently, R. Ye obtained a differential sphere and Ricci flow convergence theorem for Riemannian manifolds satisfying an integral pinching condition in terms of the concircular curvature tensor Z [19]. In [19], the L^2-norm is employed for integral curvature pinching, but a pointwise upper bound for $|Rm|$ is assumed. An integral pinching condition allows large deviation of Z from 0 in some places, and hence is significant geometrically. On the other hand, although the pointwise upper bound assumed for $|Rm|$ in [19] is natural, it is not an integral condition. It is very desirable to remove this pointwise curvature bound.

In [9] a differential sphere and Ricci flow convergence theorem for L^p-integral pinching with $p > n/2$ without involving pointwise curvature bounds was obtained for Yamabe metrics. An $L^{n/2}$-pinching theorem was also obtained in [9], but it is valid only under the assumption that the given conformal class is locally conformally flat or contains an Einstein metric. Note that the $L^{n/2}$ formulation with the critical exponent $n/2$ is most natural from a geometrical point of view, as the integral $\int_M |Z|^{n/2} dvol$ is scaling invariant. In contrast, the integral $\int_M |Z|^p dvol$ with $p > n/2$ needs to be multiplied e.g. by a power of the volume to become scaling invariant. The work [8] contains a sphere theorem (Theorem B) for 4-dimensional manifolds with positive Euler characteristic, where the metric is assumed to have L^2-pinched Weyl curvature tensor W, which is a part of the concircular curvature tensor Z. Note that this pinching condition is a conformally invariant one. A Ricci flow convergence result for Yamabe metrics under the said conditions follows from the proof of Theorem B in [8].

We would also like to mention that A. Chang, M. Gursky and P. Yang obtained a deep differential sphere theorem for 4-dimensional Riemannian manifolds which have positive Yamabe invariant and satisfy a sharp L^2-pinching condition for the Weyl tensor [4]. Previously, Margerin obtained a sphere theorem in dimension 4 with a sharp pointwise pinching condition [12]. Sphere theorems in dimension 4 for $\int |\mathrm{Rm}|^2$- as well as $\int |\mathrm{Z}|^2$-pinching were obtained in [15] and [2] respectively, in which the gradient flows of the corresponding functionals were employed. See [3] for a related sphere theorem in dimension 3.

Our main result is as follows.

Theorem 1.1. *For each $n \geq 3$ there exists a constant $\Lambda(n) > 0$ depending only on n such that if (M^n, g_0) is a compact Riemannian manifold of dimension n with $g_0 \in \mathcal{Y}^+(M)$ and*

$$\|Z\|_{\frac{n}{2}} < \Lambda(n) R_{g_0}, \tag{1.1}$$

then M^n is diffeomorphic to an isometric quotient of $\mathbb{S}^n$. Moreover, the volume-normalized Ricci flow starting from (M^n, g_0) exists for all positive times and converges smoothly to a spherical space form at the time infinity.

The 4-dimensional case of this theorem can also be derived from Theorem B and its proof in [8], as the $\int |Z|^2$-pinching implies positivity of the Euler characteristic in this dimension via Chern-Gauss-Bonnet theorem.

There are several good reasons for considering Yamabe metrics in the set-up. These metrics are important because of their minimizing and constant scalar curvature property. They are also abundant as they exist in every conformal class of metrics. Indeed, they can be used to represent conformal classes. (In general Yamabe metrics are not unique in a conformal class. But one can consider all Yamabe metrics in a conformal class.) In this regard, our main theorem can be reformulated as a differential sphere theorem about conformal classes, in which the condition is that a given conformal class contains a Yamabe metric satisfying the conditions of the above main theorem. We would like to point out that the 4-dimensional hypotheses in [8,4] are formulated for general metrics in a given conformal class. Note however that in [8] Yamabe metrics or nearly

Yamabe metrics serve as the key bridge leading to the conclusion about the conformal class. In [4], instead of Yamabe metrics some other geometrically special metrics are employed.

Another point about Yamabe metrics is their role in the study of the σ-invariant. Thus our main theorem can also be related to this topic. Furthermore, the limit behavior of a sequence of Yamabe metrics g_k on a manifold M with $Y(g_k) \to \sigma(M)$ can be exploited to understand the existence of Einstein metrics (possibly with singularities) on M. We would also like to note that in dimensions higher than 4 one does not expect the smallness of $\int_M |Z|^{n/2} dvol$ to be sufficient for a sphere theorem to hold true for general metrics, as this integral condition alone does not imply enough control of geometrical structures. Finally, the assumption of Yamabe metrics is very natural from the viewpoint of geometrical analysis, as Yamabe metrics enjoy a natural Sobolev inequality, which plays a crucial role in our main result.

Recently the first named author [5] showed that the Ricci flow deforms an asymptotically flat metric with $L^{n/2}$-pinching of curvature to a Euclidean metric. The present paper is in part inspired by that work. A key tool is a Sobolev inequality along the Ricci flow derived from [20], see Theorem 2.2. Along related lines we have obtained in [6] an extension of Gromov-Ruh almost flat manifold theorem to "$L^{n/2}$-almost flat manifolds," and derived in [7] sphere theorems and other space form theorems under $L^{n/2}$-pinching.

2. Preliminaries

Our starting point is that if $g_0 \in \mathcal{Y}^+(M^n)$, then it satisfies the Sobolev inequality

$$\|u\|^2_{\frac{2n}{n-2}} \le \frac{c(n)}{R_0} \int |\nabla u|^2 \, dV_0 + \int u^2 \, dV_0 \quad \text{for all } u \in C^\infty(M^n), \tag{2.1}$$

where $c(n) = 4\frac{n-1}{n-2}$, $R_0 = R_{g_0}$ and $dV_0 = d\,\mathrm{vol}_{g_0}$. This was used in [9], and is an easy consequence of the minimizing property of g_0 and the elementary formulae

$$\int_M R_g d\,\mathrm{vol}_g = \int_M (c(n)|\nabla_{g_0} u|^2_{g_0} + R_{g_0} u^2) d\,\mathrm{vol}_{g_0}, \mathrm{vol}_g = \int_M u^{\frac{2n}{n-2}} d\,\mathrm{vol}_{g_0} \tag{2.2}$$

for $g = u^{\frac{4}{n-2}} g_0$.

We will also use the fact that for any Riemannian metric g,

$$Y(M^n, [g]) \le Y(\mathbb{S}^n) = n(n-1)\omega_n^{2/n}, \tag{2.3}$$

where ω_n is the volume of $\mathbb{S}^n$, see [1].

Recall that the Ricci flow and normalized Ricci flow starting from (M^n, g_0) are respectively given by

$$\begin{cases} \frac{\partial}{\partial t} g = -2\mathrm{Ric}_g, \\ g(0) = g_0, \end{cases} \qquad \begin{cases} \frac{\partial}{\partial t} g = -2\mathrm{Ric}_g + \frac{2}{n}\bar{R} g, \\ g(0) = g_0, \end{cases} \tag{2.4}$$

where $\bar{R}$ is the average of the scalar curvature R.

From the Sobolev inequality (2.1) we can argue as in [20] to deduce the following log-Sobolev inequality.

Proposition 2.1. *Let (M, g_0) be a compact manifold, and $g_0 \in \mathcal{Y}^+(M)$. Then there exists $d(n) > 0$ such that the following log-Sobolev inequality holds for any $u \in W^{1,2}$ with $\int u^2 \, dV_t = 1$ and all $\sigma > 0$ along the Ricci flow starting from (M^n, g_0):*

$$\int_M u^2 \log u^2 \, dV_{g(t)} \le \sigma \int_M \left(|\nabla u|^2 + \frac{R}{4} u^2 \right) dV_{g(t)} - \frac{n}{2} \log \sigma + \frac{n}{2} \log \frac{c(n)}{R_0} + d(n). \tag{2.5}$$

Proof. In what follows we refer extensively to notations and results of [20]. Since $g_0 \in \mathcal{Y}^+(M^n)$, we have the Sobolev inequality (2.1) which implies that in the notation of [20], we have $C_S^2(M, g_0) = \frac{c(n)}{R_0}$ and $\tilde{C}_S^2(M, g_0) = \max\{\frac{c(n)}{R_0}, 1\}$. By the resolution of the Yamabe problem we have two cases:

(i) $c(n) \le R_0 \le Y(S^n, [g_0])$; then $\tilde{C}_S(M, g_0) = 1$.
(ii) $0 < R_0 \le c(n)$; then $\tilde{C}_S(M, g_0) = \frac{c(n)}{R_0}$.

One may check explicitly that both cases are possible. We will now go through the log Sobolev inequalities of [20, Theorems 1.1, 1.2], in our particular situation when $g_0 \in \mathcal{Y}^+(M)$.

By [20, Theorem 1.1], in the two cases described above, we have

(i) For each $\sigma > 0$ and $t \in [0, T)$, there holds

$$\int_M u^2 \log u^2 \, dV_t \le \sigma \int_M \left(|\nabla u|^2 + \frac{R}{4} u^2 \right) dV_t - \frac{n}{2} \log \sigma + 4 \frac{R_0}{c(n)} \left(t + \frac{\sigma}{4} \right) + \frac{n}{2} (\log n - 1). \tag{2.6}$$

(ii) For each $\sigma > 0$ and $t \in [0, T)$, there holds

$$\int_M u^2 \log u^2 \, dV_t \le \sigma \int_M \left(|\nabla u|^2 + \frac{R}{4} u^2 \right) dV_t - \frac{n}{2} \log \sigma + 4 \frac{R_0}{c(n)} \left(t + \frac{\sigma}{4} \right) + \frac{n}{2} \log \frac{c(n)}{R_0} + \frac{n}{2} (\log n - 1). \tag{2.7}$$

To apply [20, Theorem 1.2], first we note that $\lambda_0 \geq \frac{R_0}{4}$, where λ_0 is the first eigenvalue of $-\Delta + \frac{R_0}{4}$, and that

$$\delta_0 = \frac{1}{1+\lambda_0 \frac{c(n)}{R_0}} \leq \frac{1}{1+\frac{c(n)}{4}},$$
$$\sigma_0 = \frac{n}{2}\left[\log\left(1+\frac{R_0}{\lambda_0 c(n)}\right) - 1\right] \leq \frac{n}{2}\log\left(1+\frac{4}{c(n)}\right) - \frac{n}{2}.$$

In particular, δ_0 and σ_0 are bounded by dimensional constants. Thus in both cases described above we obtain the same statement:

Let $t \in [0,T)$ and $\sigma > 0$ satisfy $t+\sigma \geq \frac{n}{8}C_S(M,g_0)^2\delta_0$. Then there holds

$$\int_M u^2 \log u^2 \, dV_t \leq \sigma \int_M \left(|\nabla u|^2 + \frac{R}{4}u^2\right) dV_t - \frac{n}{2}\log\sigma + \frac{n}{2}\log n + \frac{n}{2}\log\frac{c(n)}{R_0} + \sigma_0. \tag{2.8}$$

We can now conclude Proposition 2.1, which can be considered as a specialized version of [20, Theorem 1.3] for the case $g_0 \in \mathcal{Y}^+(M^n)$ and brings out the analytic role of the scalar curvature R_0. Note that $\frac{n}{8}C_S(M,g_0)^2\delta_0 \leq \frac{nc(n)}{8R_0}\delta_0$. For any fixed t, suppose that $t+\sigma < \frac{nc(n)}{8R_0}\delta_0$. Then by putting (2.6) and (2.8) together in the first case, and (2.7) and (2.8) together in the second case, we obtain the result. Here we are using the fact that $\log\frac{c(n)}{R_0}$ can be bounded from below by the dimensional constant $\log\frac{c(n)}{Y(S^n,[g_0])}$ in order to deal with (2.6) where the term $\frac{n}{2}\log\frac{c(n)}{R_0}$ does not appear. □

Now (2.5) combined with the general Sobolev inequality in [20] gives the following Sobolev inequality along Ricci flow, which is our main analytic tool. Note that the explicit dependence on the coefficients is very important here. See also [21] for related more general results.

Theorem 2.2. *Let (M^n, g_0) be a compact manifold, and $g_0 \in \mathcal{Y}^+(M)$. Then there exists $C(n) > 0$ such that the following Sobolev inequality holds for any $u \in C^\infty(M)$ along the Ricci flow starting from (M^n, g_0):*

$$\left(\int u^{\frac{2n}{n-2}} \, dV_{g(t)}\right)^{\frac{n-2}{n}} \leq \frac{C(n)}{R_{g(0)}}\left(\int |\nabla u|^2 + R_{g(t)}u^2 \, dV_{g(t)}\right) \tag{2.9}$$

Proof of Theorem 2.2. Again we employ the notations and results in [20]. Dimensional constants below may change in size at different points of the argument. Apply [20, Theorem 5.5] to the log Sobolev inequality (2.5) of Proposition 2.1 and take $\sigma^* \to \infty$. To conclude it suffices to check that the constant $C(\overline{C},\mu)$ in [20, (5.18)] becomes $C(n)\frac{1}{R_0}$, with $\mu = n$. From [20, (5.22)] we check that from (2.5), we have $\overline{C} = C(n)R_0^{-\frac{n}{4}}$. Then from [20, (9.32)], we check that $C(\overline{C}, n) = K(n)\overline{C}^{\frac{4}{n}}$, which gives us the conclusion. □

We will also use a standard parabolic Moser iteration argument. In particular, we will use the following formulation due to D. Yang [18], in which the dependence of estimates on uniform Sobolev constants and integral bounds is made explicit.

Theorem 2.3 *([18, Theorem 4]). Let f, b be smooth nonnegative functions satisfying on $M \times [0, T]$,*

$$\frac{\partial}{\partial t} f \le \Delta f + bf,$$

where Δ is the Laplace-Beltrami operator of the metric g_t, and suppose $\frac{\partial}{\partial t} dV_{g_t} = h_t dV_{g_t}$. Let $A, B > 0$ be such that

$$\|u\|^2_{\frac{2n}{n-2}} \le A\|\nabla u\|_2^2 + B\|u\|_2^2,$$

for all $u \in C^\infty(M)$ and for all $t \in [0, T]$, and assume that for some $q > n/2$,

$$\max_{0 \le t \le T} (\|b\|_q + \|h_t\|_q) \le \beta.$$

Then given $p_0 > 1$, there exists a constant $C = C(n, q, p_0)$ such that for all $x \in M$ and $t \in (0, T]$,

$$|f(x,t)| \le CA^{\frac{n}{2p_0}} \left[\frac{B}{A} + A^{\frac{n}{2q-n}} \beta^{\frac{2q}{2q-n}} + \frac{1}{t}\right]^{\frac{n+2}{2p_0}} \left(\int_0^t \int f^{p_0} \, dV_t \, dt\right)^{\frac{1}{p_0}}. \tag{2.10}$$

Finally we will need the following pointwise pinching result.

Theorem 2.4 *([11], see also [10,13]). For a compact Riemannian manifold (M^n, g), if*

$$|Z| < \sqrt{\frac{1}{2n(n-1)(n-2)}}\, R, \tag{2.11}$$

then the normalized Ricci flow starting from (M^n, g) exists for all positive times and converges to a spherical space form at the time infinity.

3. Proof of Theorem 1.1

The proof of Theorem 1.1 is divided into several steps. The first key step is to establish that for the Ricci flow starting from (M^n, g_0) with pinching condition (1.1), the pinching is controlled for a uniform time whenever Ricci flow exists. Compare to [5, Lemma 4.2], where it is shown the pinching is non-increasing along the flow, which is not the case here.

For simplicity we often use subscripts of 0 and t to denote geometric quantities at certain times along the flow as before; for instance R_0 denotes $R_{g(0)}$. Below, "$\lesssim$" indicates an inequality which holds when the terms involved are multiplied by appropriate positive dimensional constants.

Lemma 3.1. *There exist $\Lambda(n) > 0$ and $\delta(n) > 0$ such that if (M^n, g_0) is a compact manifold satisfying the hypotheses of Theorem 1.1, and the Ricci flow of (M^n, g_0) exists on $[0, T_0)$, then on $[0, T_0) \cap [0, \delta(n) R_0^{-1}]$ along the flow we have the bound*

$$\||Z| + (R - R_0)\|_{\frac{n}{2}} < 2\Lambda(n) R_0. \tag{3.1}$$

Proof. Under the Ricci flow, we have [9]

$$\begin{aligned} \frac{\partial}{\partial t}|Z| &\lesssim \Delta|Z| + |R||Z| + |Z|^2, \\ \frac{\partial}{\partial t}R &= \Delta R + 2|Ric|^2 \lesssim \Delta R + |Z|^2 + R^2. \end{aligned}$$

Moreover, the evolution equation for R together with the maximum principle implies $R - R_0 \geq 0$, and we have

$$\frac{\partial}{\partial t}(R - R_0) \lesssim \Delta(R - R_0) + |Z|^2 + (R - R_0)^2 + R_0^2.$$

Let $P = |Z| + (R - R_0)$, then

$$\frac{\partial}{\partial t}P \lesssim \Delta P + P^2 + R_0^2. \tag{3.2}$$

Since

$$\frac{\partial}{\partial t}(dV_t) = -R_t \, dV_t,$$

and $R_t > 0$, we have

$$\frac{d}{dt}\int P^{\frac{n}{2}} \, dV_t \lesssim -\int |\nabla P^{\frac{n}{4}}|^2 \, dV_t + \int P^{\frac{n}{2}+1} + R_0^2 P^{\frac{n}{2}-1} \, dV_t. \tag{3.3}$$

We have that $\|P_0\|_{\frac{n}{2}} < \Lambda(n) R_0$. Let $T \in (0, T_0)$ be such that $\|P\|_{\frac{n}{2}} < 2\Lambda(n) R_0$ on $[0, T]$. Then applying (2.9) with $u = P^{\frac{n}{4}}$, we have

$$\begin{aligned}\int |\nabla P^{\frac{n}{4}}|^2 \, dV_t &\gtrsim R_0 \left(\int P^{\frac{n}{2}\frac{n}{n-2}} \, dV_t \right)^{\frac{n-2}{n}} - \int R_t P^{\frac{n}{2}} \, dV_t \\ &\geq R_0 \left(\int P^{\frac{n}{2}\frac{n}{n-2}} \, dV_t \right)^{\frac{n-2}{n}} - \int P^{\frac{n}{2}+1} \, dV_t - R_0 \int P^{\frac{n}{2}} \, dV_t.\end{aligned}$$

By Hölder's inequality

$$\begin{aligned}\int P^{\frac{n}{2}+1} &\leq \|P\|_{\frac{n}{2}} \left(\int P^{\frac{n}{2}\frac{n}{n-2}} \, dV_t \right)^{\frac{n-2}{n}} \\ &\leq 2\Lambda(n) R_0 \left(\int P^{\frac{n}{2}\frac{n}{n-2}} \, dV_t \right)^{\frac{n-2}{n}}.\end{aligned}$$

Plugging this into (3.3) we find that for $\Lambda(n) > 0$ sufficiently small,

$$\begin{aligned}\frac{d}{dt} \int P^{\frac{n}{2}} \, dV_t &\lesssim -R_0 \left(\int P^{\frac{n}{2}\frac{n}{n-2}} \, dV_t \right)^{\frac{n-2}{n}} + R_0 \int P^{\frac{n}{2}} \, dV_t + R_0^2 \int P^{\frac{n}{2}-1} \, dV_t \qquad (3.4) \\ &\leq R_0 \int P^{\frac{n}{2}} \, dV_t + R_0^2 \mathrm{vol}(g_t)^{\frac{2}{n}} \left(\int P^{\frac{n}{2}} \, dV_t \right)^{\frac{n-2}{n}} \\ &\lesssim R_0 \left(\int P^{\frac{n}{2}} \, dV_t + R_0^{\frac{n}{2}} \right),\end{aligned}$$

where we have applied Hölder's and Young's inequalities in the second and third lines above, respectively, and used the fact that $\mathrm{vol}(g_t) \leq \mathrm{vol}(g_0) = 1$. Hence

$$\int P^{\frac{n}{2}} \, dV_t + R_0^{\frac{n}{2}} \leq \left(\int P^{\frac{n}{2}} \, dV_0 + R_0^{\frac{n}{2}} \right) \exp\left(C(n) R_0 t \right), \qquad (3.5)$$

for some dimensional constant $C(n) > 0$. We therefore conclude that there exists a $\delta(n) > 0$ such that $\|P\|_{\frac{n}{2}} < 2\Lambda(n) R_0$ on $[0, T_0) \cap [0, \delta(n) R_0^{-1}]$. Recall that by (2.3) $R_0 = Y(M^n, [g_0])$ is bounded above by the dimensional constant $Y(\mathbb{S}^n)$, hence R_0^{-1} is uniformly bounded below by a positive dimensional constant. □

Remark. The constant of the Sobolev inequality (2.9) is $\frac{C(n)}{R_0}$, and hence becomes large when R_0 is small. On the other hand, the larger the Sobolev constant is, the worse the estimate in Theorem 2.3 becomes. So the time $\delta(n) R_0^{-1}$ in the above lemma seems to be in conflict with the Sobolev constant $\frac{C(n)}{R_0}$ and in fact has exactly the right dependence on R_0 to allow us to prove Theorem 1.1. (Note that this time leads to a similar consequence for the existence time of the Ricci flow.) This delicate phenomenon seems to be new and is very interesting.

With the pinching assumption (3.1) the Sobolev inequality (2.9) immediately gives a uniform Sobolev inequality without the scalar curvature term by an application of Hölder's inequality.

Lemma 3.2. *Let (M^n, g_0) be a compact manifold with $g_0 \in \mathcal{Y}^+(M)$. Suppose that the Ricci flow starting from (M^n, g_0) exists on $[0, T]$ so that the bound (3.1) holds along $[0, T]$. Then the following uniform Sobolev inequality holds along the flow for all $u \in C^\infty(M)$ along the Ricci flow starting from (M^n, g_0):*

$$\left(\int u^{\frac{2n}{n-2}} \, dV_{g(t)}\right)^{\frac{n-2}{n}} \leq \frac{C(n)}{R_0} \int |\nabla u|^2 \, dV_t + C(n) \int u^2 \, dV_t, \tag{3.6}$$

where $C(n) > 0$ is a dimensional constant which is different from the constant of (2.9) of Theorem 2.2.

With the Sobolev inequality (3.6) and the $L^{n/2}$ smallness of $P = |Z| + (R - R_0)$, since P satisfies the nonlinear evolution equation (3.2) it is known that we can improve the estimate and get an $L^{n/2+1}$ (higher power) bound of P, in fact also small.

Lemma 3.3. *Let (M^n, g_0) be a compact manifold with $g_0 \in \mathcal{Y}^+(M)$, and the bound (3.1) holds along the Ricci flow starting from (M^n, g_0) in the interval $[0, T]$. Then for $t \in [0, T]$,*

$$\int P^{\frac{n}{2}+1} \, dV_t \lesssim \Lambda(n)^{\frac{n}{2}} \left(R_0 t + 1\right) \left(R_0 + \frac{1}{t}\right) R_0^{\frac{n}{2}}. \tag{3.7}$$

As mentioned before the bootstrap to get some bound is standard. But since we need an explicit dependence of the bound on R_0 later, which is not in the literature, we present a proof. Compare [8, Lemma 4.2].

Proof. By Lemma 3.2, Sobolev inequality (3.6) holds.

From (3.2) we have (see (3.3))

$$\frac{1}{q} \frac{d}{dt} \int P^q \, dV_t \leq -C_1 \frac{4(q-1)}{q^2} \int |\nabla P^{\frac{q}{2}}|^2 \, dV_t + C_2 \int P^{q+1} + R_0^2 P^{q-1} \, dV_t,$$

where $C_1, C_2 > 0$ are dimensional constants independent of q. Using this for $q = \frac{n}{2}$ as before as well as for $q = \frac{n}{2} + 1$, we find as for (3.4) by using (3.6) that for $\Lambda(n) > 0$ sufficiently small,

$$\frac{d}{dt} \int P^{\frac{n}{2}} \, dV_t + R_0 \left(\int P^{\frac{n}{2} \frac{n}{n-2}} \, dV_t\right)^{\frac{n-2}{n}} \lesssim \int R_0 P^{\frac{n}{2}} + R_0^2 P^{\frac{n}{2}-1} \, dV_t$$

$$\frac{d}{dt} \int P^{\frac{n}{2}+1} \, dV_t + R_0 \left(\int P^{(\frac{n}{2}+1)\frac{n}{n-2}} \, dV_t\right)^{\frac{n-2}{n}} \lesssim \int R_0 P^{\frac{n}{2}+1} + R_0^2 P^{\frac{n}{2}} \, dV_t.$$

Thus, if we take $0 < \tau < \tau' < T$, and define ψ by

$$\psi(t) = \begin{cases} 0, & 0 \le t \le \tau, \\ \frac{t-\tau}{\tau'-\tau}, & \tau \le t \le \tau', \\ 1, & \tau' \le t \le T. \end{cases}$$

Multiply above inequalities by $\psi(t)$ and integrate from τ to $t_0 \in [\tau', T]$, we have

$$\begin{aligned} &\int P^{\frac{n}{2}} \, dV_{t_0} + R_0 \int_{\tau'}^{t_0} \left(\int P^{\frac{n}{2}\frac{n}{n-2}} \, dV_t \right)^{\frac{n-2}{n}} \\ &\lesssim \left(R_0 + \frac{1}{\tau'-\tau} \right) \int_{\tau}^{t_0} \int P^{\frac{n}{2}} + R_0 P^{\frac{n}{2}-1} \, dV_t \, dt, \end{aligned} \tag{3.8}$$

$$\begin{aligned} &\int P^{\frac{n}{2}+1} \, dV_{t_0} + R_0 \int_{\tau'}^{t_0} \left(\int P^{(\frac{n}{2}+1)\frac{n}{n-2}} \, dV_t \right)^{\frac{n-2}{n}} \\ &\lesssim \left(R_0 + \frac{1}{\tau'-\tau} \right) \int_{\tau}^{t_0} \int P^{\frac{n}{2}+1} + R_0 P^{\frac{n}{2}} \, dV_t \, dt. \end{aligned} \tag{3.9}$$

Let $t_0 = t$, and $\tau = \frac{t}{2}, \tau' = \frac{3}{4}t$ in (3.9) and $\tau = \frac{t}{4}, \tau' = \frac{t}{2}$ in (3.8), we obtain

$$\begin{aligned} \int P^{\frac{n}{2}+1} \, dV_t &\lesssim \left(R_0 + \frac{1}{t} \right) \left(\sup_{s \in [\frac{t}{2}, t]} \left(\int P^{\frac{n}{2}} \, dV_s \right)^{\frac{2}{n}} \right) \int_{\frac{t}{2}}^{t} \left(\int P^{\frac{n}{2}\frac{n}{n-2}} \, dV_s \right)^{\frac{n-2}{n}} ds \\ &\quad + \left(R_0 + \frac{1}{t} \right) \int_{\frac{t}{2}}^{t} \int R_0 P^{\frac{n}{2}} \, dV_s \, ds \\ &\lesssim \Lambda(n) \left(R_0 + \frac{1}{t} \right)^2 \int_{\frac{t}{4}}^{t} \int P^{\frac{n}{2}} + R_0 P^{\frac{n}{2}-1} \, dV_s \, ds \\ &\quad + (R_0 t + 1) \Lambda(n)^{\frac{n}{2}} R_0^{\frac{n}{2}+1} \\ &\lesssim \Lambda(n)^{\frac{n}{2}} (R_0 t + 1) \left(R_0 + \frac{1}{t} + R_0 \right) R_0^{\frac{n}{2}}, \end{aligned}$$

which is the desired estimate. □

With this $L^{n/2+1}$ bound we can use Theorem 2.3 to get an L^∞ bound of $|Z| + (R - R_0)$ and a uniform lower bound on the time of existence of the Ricci flow.

Proposition 3.4. *Let (M^n, g_0) be as in the hypotheses of Theorem 1.1. Then the Ricci flow starting from (M^n, g_0) exists on the interval of time $[0, \delta(n)R_0^{-1}]$, where $\delta(n) > 0$ is as in Lemma 3.1.*

Proof. Let (M^n, g_0) be as in the hypotheses of Theorem 1.1, and suppose the Ricci flow starting from (M^n, g_0) exists on $[0, T_0)$. We first claim that in the notation of Lemma 3.1, $\delta(n)R_0^{-1} < T_0$. Suppose not; then $T_0 \le \delta(n)R_0^{-1}$, so on $[0, T_0)$,

$$\||Z| + (R - R_0)\|_{\frac{n}{2}} < 2\Lambda(n)R_0. \tag{3.10}$$

Therefore by Lemma 3.3, we see that $\||Z| + (R - R_0)\|_{\frac{n}{2}+1}$ is uniformly bounded (by some constant depending on n, R_0 and T_0) on $[\frac{T_0}{2}, T_0)$. Thus, $\|Rm\|_{\frac{n}{2}+1}$ is uniformly bounded on $[\frac{T_0}{2}, T_0)$. Also by Lemma 3.2 we have the uniform Sobolev inequality (3.6). Recall that under the Ricci flow, $|Rm|$ satisfies

$$\frac{\partial}{\partial t}|Rm| \le \Delta|Rm| + D(n)|Rm|^2$$

for some $D(n) > 0$. Therefore we can apply Moser's weak maximum principle as stated in Theorem 2.3: let $f = |\mathrm{Rm}|$, $b = D(n)|\mathrm{Rm}|$, $h_t = -R_t$, and take $q = \frac{n}{2} + 1$, $p_0 = \frac{n}{2}$. Since $\left(\int_{\frac{T_0}{4}}^{T_0} \int |\mathrm{Rm}|^{\frac{n}{2}}\, dV_t\, dt\right)^{\frac{2}{n}} < \infty$ by (3.10), we conclude that $|\mathrm{Rm}|$ must be bounded by a uniform constant in $\left[\frac{T_0}{2}, T_0\right)$, and hence does not blow up as $t \to T_0$, which contradicts the maximality of T_0. Hence $\delta(n)R_0^{-1} < T_0$ and the Ricci flow of (M^n, g_0) exists as claimed on the uniform time interval $[0, \delta(n)R_0^{-1}]$. $\square$

We can now use Theorem 2.3 a second time to control $|Z|$ uniformly along the Ricci flow starting from (M^n, g_0) and thus prove Theorem 1.1.

Proof of Theorem 1.1. Let $f = |Z|$; then we have $\frac{\partial}{\partial t} f \le \Delta f + |\mathrm{Rm}| f$, and by Lemma 3.3 we have that $\|\mathrm{Rm}\|_q \le \beta$ for $q = \frac{n}{2} + 1$, where $\beta = c(n)R_0$ for some $c(n) > 0$, on the interval $\left[\frac{\delta(n)R_0^{-1}}{2}, \delta(n)R_0^{-1}\right]$ along the Ricci flow starting from (M^n, g_0). We also have the uniform Sobolev inequality of Lemma 3.2 in this interval, so that in the notation of Theorem 2.3, $A = \frac{C(n)}{R_0}$ and $B = C(n)$.

We now apply Theorem 2.3 to estimate $|Z|$ on the time interval $\left[\frac{\delta(n)R_0^{-1}}{2}, \delta(n)R_0^{-1}\right]$ with $q = \frac{n}{2} + 1$ and $p_0 = \frac{n}{2}$. Note that the factor $\left[\frac{B}{A} + A^{\frac{n}{2q-n}} \beta^{\frac{2q}{2q-n}} + \frac{1}{t}\right]^{\frac{n+2}{2p_0}}$ appearing in the right-hand side of (2.10) involves a sum of three different terms, and this complication may appear to make troubles for our scheme. However, applying our earlier estimates established on this particular time interval we can readily bound each of these terms by a constant multiple of R_0. Putting everything together we obtain exactly the correct power of R_0 on the right needed to apply pointwise pinching results:

$$
\begin{aligned}
|Z| &\lesssim R_0^{\frac{2}{n}} \left(\int_0^t \int |Z|^{\frac{n}{2}} \, dV_t \, dt \right)^{\frac{2}{n}} \\
&\lesssim R_0^{\frac{2}{n}} \left(\Lambda(n) R_0^{-1+\frac{n}{2}} \right)^{\frac{2}{n}} \\
&\lesssim \Lambda(n)^{\frac{2}{n}} R_0.
\end{aligned}
$$

If we assume that $\Lambda(n)$ is sufficiently small, then since the scalar curvature of (M, g_t) is everywhere greater than R_0, we have at the time $t = \delta(n) R_0^{-1}$ the pinching

$$
|Z| < \sqrt{\frac{1}{2n(n-1)(n-2)}} \, R. \tag{3.11}
$$

By Theorem 2.4, we may therefore conclude that the normalized Ricci flow starting from (M^n, g_0) exists for all times and converges to a spherical space form. □

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