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VORTICITY CONVERGENCE FROM BOLTZMANN TO 2D INCOMPRESSIBLE EULER EQUATIONS BELOW YUDOVICH CLASS*

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Abstract. It is challenging to perform a multiscale analysis of mesoscopic systems exhibiting singularities at the macroscopic scale. In this paper, we study the hydrodynamic limit of the Boltzmann equation $\mathrm{St}\partial_t F + v\cdot\nabla_x F = \frac{1}{\mathrm{Kn}}Q(F,F)$ toward the singular solutions of 2D incompressible Euler equations whose vorticity is unbounded: $\partial_t u + u\cdot\nabla_x u + \nabla_x p = 0$, $\operatorname{div} u = 0$. We obtain a microscopic description of the singularity through the so-called kinetic vorticity and understand its behavior in the vicinity of the macroscopic singularity. As a consequence of our new analysis, we settle affirmatively an open problem of convergence toward Lagrangian solutions of the 2D incompressible Euler equation whose vorticity is unbounded ($\omega \in L^{\mathfrak{p}}$ for any fixed $1 \le \mathfrak{p} < \infty$). Moreover, we prove the convergence of kinetic vorticities toward the vorticity of the Lagrangian solution of the Euler equation. In particular, we obtain the rate of convergence when the vorticity blows up moderately in $L^{\mathfrak{p}}$ as $\mathfrak{p} \to \infty$ (a localized Yudovich class).

Key words. hydrodynamic limit, Euler equation, Boltzmann equation

MSC codes. 35Q20, 35Q35

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1. Introduction.

One of the fundamental questions in the area of partial differential equations is Hilbert's sixth problem, seeking a unified theory of the gas dynamics including different levels of descriptions from a mathematical standpoint by connecting the mesoscopic Boltzmann equations to the macroscopic fluid models that arise in formal limits. The Boltzmann equation is a fundamental model of kinetic theory for dilute collections of gas particles, which undergo elastic binary collisions. The dimensionless form of the equation is given as an integro-differential equation, where $F(t,x,v) \ge 0$ is a density distribution of particles on the phase space. Here, the *Strouhal number* and *Knudsen number* are denoted by St and Kn, which are a ratio of the characteristic length to the characteristic time and a ratio of mean free path to the characteristic length, respectively.

The effect of binary collision between particles is described by $Q(F,F)$, which takes various forms of the nonlocal-in-velocity operator depending on the nature of particles and its intermolecular interaction [11]. An intrinsic equilibrium, satisfying $Q(\cdot,\cdot) = 0$, is given by the so-called local Maxwellian associated with $(R,U,\Theta) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}_+$:

$$M_{R,U,\Theta}(v) := \frac{R}{(2\pi\Theta)^{3/2}} \exp\left\{-\frac{|v-U|^2}{2\Theta}\right\}. \tag{1.1}$$

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The collision operator enjoys the so-called collision invariance: $\int Q(F,G)\begin{bmatrix}1 & v & |v|^2\end{bmatrix} \mathrm{d}v = 0$ for arbitrary F, G. The celebrated Boltzmann's H-theorem (entropy $H = \int F \ln F \mathrm{d}v$) reveals the entropy dissipation: $\int Q(F,F)\ln F \mathrm{d}v \leq 0$. In this paper, we consider the most basic hard-sphere collision cross section:

$$Q(F,G)(v) = \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |(v-v_*)\cdot\sigma|\{F(v')G(v'_*) + G(v')F(v'_*) - F(v)G(v_*) - G(v)F(v_*)\}\mathrm{d}\sigma \mathrm{d}v_*, \tag{1.2}$$

where postcollision velocities are denoted by $v' = v - ((v-v_*)\cdot\sigma)\sigma$ and $v'_* = v_* + ((v-v_*)\cdot\sigma)\sigma$.

Besides St and Kn, we introduce the *Mach number* Ma as a size of fluctuations of F around the global Maxwellian $M_{1,0,1}(v)$ of the reference state $(1,0,1)$. Relations between St, Kn, and Ma are important. Naturally, Ma is bounded above by St$/c$, where c is denoted by the speed of sound. On the other hand, the famous *Reynolds number* Re appears as a ratio between Kn and Ma through the von Karman relation: $1/\mathrm{Re} = \mathrm{Kn}/\mathrm{Ma}$. By passing Kn to zero and choosing different St(Kn) and Ma(Kn) as functions of Kn, we can formally derive various PDEs of macroscopic variables. Formally, the incompressible Euler limit can be realized in the following scaling of the large Re limit:

$$\mathrm{St} = \varepsilon = \mathrm{Ma} \quad \text{and} \quad \mathrm{Kn} = \kappa\varepsilon \quad \text{with } \kappa = \kappa(\varepsilon) \to 0 \text{ as } \varepsilon \to 0. \tag{1.3}$$

In the diffusive scaling, the same scaling of (1.3) with $\kappa = 1$, the corresponding macroscopic PDE is the incompressible Navier–Stokes–Fourier system. This scaling problem is better understood as a singular perturbation in being *milder* than our case (1.3) (see [23, 31, 25, 26] and references therein). In [23], Esposito et al. establish a uniform bound of a perturbation f in $F = M_{1,0,1} + \varepsilon f\sqrt{M_{1,0,1}}$ without a priori information of the fluid solutions, and hence, they derive (actually construct) a strong solution of the incompressible Navier–Stokes–Fourier system for both steady and unsteady cases in the presence of a boundary. One of the key ingredients is to obtain an L^6_x ($\hookleftarrow H^1_x$ in three dimensions) control of f by realizing a hidden elliptic equation of the bulk velocity part of f in

$$v\cdot\nabla_x f \sim \frac{1}{\varepsilon}Lf \qquad (\textit{macro-micro scale balance}) \tag{1.4}$$

for a linearized operator L of Q. Unfortunately, a uniform bound of f in the Euler scaling seems not feasible even in two dimensions without a priori information of solutions of the incompressible Euler equations, due to an additional singularity in both macro-micro scale balance and nonlinear perturbation, which are major obstacles in our analysis.

The regularity of fluid solutions plays a crucial rule in the multiscale analysis in the Euler scaling (1.3), which has been revealed differently in a modulated entropy inequality by Saint-Raymond [42] and an asymptotic expansion by Jang and Kim [35]. This effect appears as an growth in the microscopic scale (see (1.15)), which resembles the famous Beale, Kato, and Majda result [6]. For a spatially Lipschitz continuous velocity field, Saint-Raymond proves in [42] a hydrodynamic limit toward such solutions of the incompressible Euler equations. It has been an open problem to study the hydrodynamic limit toward solutions of the Euler equations that are not spatially Lipschitz continuous, such as vortex sheet solutions. Due to the transport feature of 2D Euler equations, such singular solutions have been well understood. For compactly supported initial vorticities in $L^{\mathfrak{p}}$ for $1 < \mathfrak{p} < \infty$, global existence theory was first proved by DiPerna and Majda in [21]. Using the so-called concentration-cancellation, the result was extended for a finite measure with distinguished sign by Delort in [18], and L^1 vorticities by Vecchi and Wu in [43]. Recently, Bohun, Bouchut, and Crippa constructed Lagrangian solutions of $\omega \in L^1$ in [7] using a stability estimate of [8].

A. Main theorems. We recall the main object of this paper: the scaled Boltzmann equation of the scaling (1.3)

$$\varepsilon \partial_t F^\varepsilon + v \cdot \nabla_x F^\varepsilon = \frac{1}{\kappa \varepsilon} Q(F^\varepsilon, F^\varepsilon) \quad \text{in } [0,T] \times \mathbb{T}^2 \times \mathbb{R}^3. \tag{1.5}$$

In this paper, we set that the spatial variables and velocity variables belong to the 2D periodic domain and 3D whole space, respectively:

$$x = (x_1, x_2) \in \mathbb{T}^2 := \left[-\frac{1}{2}, \frac{1}{2}\right] \times \left[-\frac{1}{2}, \frac{1}{2}\right] \text{ with the periodic boundary} \tag{1.6}$$

$$v = (\underline{v}, v_3) := (v_1, v_2, v_3) \in \mathbb{R}^3. \tag{1.7}$$

The existence and uniqueness of the Boltzmann equation with fixed scaling have been extensively studied in [28, 29, 30]; the initial-boundary value problem in [32, 38, 39]; the singularity formation in [37]; the boundary regularity estimate in [33, 10]; and nonequilibrium steady states in [22]. For the weak solution contents, we refer to [19, 26] and the references therein.

As the main quantities in the hydrodynamic limit, we are interested in the following observables and their convergence toward the counterparts in fluid.

Definition 1 (Boltzmann's macroscopic velocity and vorticity).

$$\begin{aligned} u_B^\varepsilon(t,x) &= \frac{1}{\varepsilon} \int_{\mathbb{R}^3} (F^\varepsilon(t,x,v) - M_{1,0,1}(v)) \underline{v} \mathrm{d}v, \\ \omega_B^\varepsilon(t,x) &:= \nabla^\perp \cdot u_B^\varepsilon(t,x) = \Big(-\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1} \Big) \cdot u_B^\varepsilon(t,x). \end{aligned} \tag{1.8}$$

In two dimensions, the incompressible Euler equation has the vorticity formulation

$$\partial_t \omega + u \cdot \nabla \omega = 0 \quad \text{in } [0,T] \times \mathbb{T}^2, \tag{1.9}$$

$$u = -\nabla^\perp (-\Delta)^{-1} \omega \quad \text{in } [0,T] \times \mathbb{T}^2, \tag{1.10}$$

$$\omega|_{t=0} = \omega_0 \quad \text{in } \mathbb{T}^2. \tag{1.11}$$

We will present the Biot–Savart formula of (1.10) in the periodic box $\mathbb{T}^2$ at (3.19). When a velocity field is Lipschitz continuous, there exists a Lagrangian flow $X(s;t,x)$ solving

$$\frac{d}{ds} X(s;t,x) = u(s, X(s;t,x)), \quad X(s;t,x)|_{s=t} = x. \tag{1.12}$$

Then, a smooth solution of the vorticity equation (1.9), (1.10), and (1.11) is given by

$$\omega(t,x) = \omega_0(X(0;t,x)), \quad u(t,x) = -\nabla^\perp (-\Delta)^{-1} \omega(t,x). \tag{1.13}$$

Out of the smooth context, a general notion of Lagrangian flow has been introduced.

Definition 2 ([20, 15]). *Let* $u \in L^1([0,T]\times\mathbb{T}^2;\mathbb{R}^2)$. *A map* $X:[0,T]\times\mathbb{T}^2\to\mathbb{T}^2$ *is a regular Lagrangian flow of* (1.12) *if and only if, for almost every* $x\in\mathbb{T}^2$ *and for any* $t\in[0,T]$, *the map* $s\in[0,t]\mapsto X(s;t,x)\in\mathbb{T}^2$ *is an absolutely continuous integral solution of* (1.12) *and there exists a constant* $\mathfrak{C}>0$ *such that, for all* $(s,t)\in[0,t]\times[0,T]$, *there holds*

$$\int_{\mathbb{T}^2}\phi(X(s;t,x))\mathrm{d}x \le \mathfrak{C}\int_{\mathbb{T}^2}\phi(x)\mathrm{d}x \tag{1.14}$$

for every measurable function $\phi:\mathbb{T}^2\to[0,\infty]$.

For a given regular Lagrangian flow to (1.12), we can define the *Lagrangian solution* (u,ω) along with the regular Lagrangian flow as in (1.13). In fact, the existence and uniqueness (for a given u) of the regular Lagrangian flow is proved in [20, 15, 8] as long as (1.10) holds while $\omega\in L^{\mathfrak{p}}$ for $\mathfrak{p}\ge 1$.

Our first theorem is about the convergence of ω_B^ε to the Lagrangian solution ω when vorticities belong to $L^{\mathfrak{p}}(\mathbb{T}^2)$ when $\mathfrak{p}<\infty$.

Theorem 1 (informal statement of Theorem 8: strong convergence). *Suppose that* $\varepsilon,\kappa=\kappa(\varepsilon),\beta=\beta(\varepsilon)$ *satisfy* (2.3). *Let arbitrary* $T>0$ *and* $(u_0,\omega_0)\in L^2(\mathbb{T}^2)\times L^{\mathfrak{p}}(\mathbb{T}^2)$ *for* $\mathfrak{p}\ge 1$. *Let* $(u,\omega)\in L^\infty((0,T);L^2(\mathbb{T}^2)\times L^{\mathfrak{p}}(\mathbb{T}^2))$ *be a Lagrangian solution of* 2*D incompressible Euler equations* (1.9), (1.10), *and* (1.11) *with initial data* (u_0,ω_0). *Then, we construct a family of solutions to the Boltzmann equation* (1.5) *whose macroscopic velocity and vorticity* $(u_B^\varepsilon,\omega_B^\varepsilon)$ *of* (1.8) *converge to the Lagrangian solution. Moreover, we have*

$$\omega_B^\varepsilon\to\omega \quad \textit{strongly in } [0,T]\times\mathbb{T}^2.$$

Remark 1. The uniqueness of the incompressible Euler equations in two dimensions is only known for vorticities with moderate growth of $L^{\mathfrak{p}}$ norm as $\mathfrak{p}\to\infty$ by Yudovich [36, 45]. In some sense, we can view the theorem as a "selection principle" of a Lagrangian solution of the incompressible Euler equations from the Boltzmann equation.

Remark 2. Our proof does not rely on a result of the inviscid limit of the nonlinear Navier–Stokes equations (cf. [35]) nor the higher-order Hilbert expansion (cf. the results by Guo [31] and de Masi, Esposito, and Lebowitz [17]). A direct approach we develop in this paper is based on stability analysis for both the Lagrangian solutions of the inviscid fluid and the Boltzmann solutions with a new corrector.

Our second theorem is about the quantitative rate of convergence/stability of ω_B^ε to ω when the uniqueness of the fluid is guaranteed. In [45], Yudovich extends his uniqness result for bounded vorticities [36] to the so-called localized Yudovich class; namely, $\omega_0\in Y_{\mathrm{ul}}^{\Theta}(\Omega)$ with a certain modulus of continuity for its velocity u. Here,

$$\|\omega\|_{Y_{\mathrm{ul}}^{\Theta}(\mathbb{T}^2)}:=\sup_{1\le\mathfrak{p}<\infty}\frac{\|\omega\|_{L^{\mathfrak{p}}(\mathbb{T}^2)}}{\Theta(\mathfrak{p})} \quad \text{for some } \Theta(\mathfrak{p})\to\infty \text{ as } \mathfrak{p}\to\infty.$$

Here, we specify $\Theta:\mathbb{R}_+\to\mathbb{R}_+$; there exists $m\in\mathbb{Z}_+$ such that $\Theta(\mathfrak{p})=\prod_{k=1}^m\log_k\mathfrak{p}$ for large $\mathfrak{p}>1$, where $\log_k\mathfrak{p}$ is defined inductively by $\log_0\mathfrak{p}=1$, $\log_1\mathfrak{p}=\log\mathfrak{p}$, and $\log_{k+1}\mathfrak{p}=\log\log_k\mathfrak{p}$. Also, we denote the inverse function of $\log_m(\mathfrak{p})$ (defined for large $\mathfrak{p}$) by e_m. Finally, we note that $\int_{e_m(1)}^\infty\frac{1}{\mathfrak{p}\Theta(\mathfrak{p})}=\infty$, which turns out to be important in the uniqueness of the solution.

THEOREM 2 (informal statement of Theorem 9: rate of convergence). *Suppose that* $\varepsilon, \kappa = \kappa(\varepsilon), \beta = \beta(\varepsilon)$ *satisfy* (2.3). *If we further assume* $\omega_0 \in Y^{\Theta}_{\mathrm{ul}}(\mathbb{T}^2)$ *in addition to Theorem* 1, *then*

$$\omega_B^{\varepsilon} \to \omega \quad \textit{strongly in } [0,T] \times \mathbb{T}^2 \textit{ with an explicit rate,}$$

where the explicit rates are defined as in (4.35) *and* (4.45).

B. Novelties, difficulties, and idea. The major novelty of this paper is to establish the incompressible Euler limit in the level of vorticity without using the inviscid limit of the Navier–Stokes equations in the vicinity of the *macroscopic singularity* ($\omega \notin L^{\infty}(\mathbb{T}^2)$). We study the convergence of Boltzmann's macroscopic vorticity toward Euler's vorticity because interesting singular behavior, e.g., interfaces in vortex patches, can be observed only in a stronger topology of velocities. We believe this new approach will shed light on the validity of Euler equations in a more direct fashion. A possible application would be a direct validity proof of Euler solutions from the kinetic theory without relying on the inviscid limit results. In addition, we are able to allow quite far-from-equilibrium initial data (see (1.16)).

There are two major difficulties in the proof. First, the macroscopic solutions are singular, and their singularity appears as growth at the microscopic level ([35]):

$$\exp\left(\int_0^t \|\nabla_x u(s)\|_{L^\infty_x} \mathrm{d}s\right). \tag{1.15}$$

This factor becomes significantly difficult to control when we study the Boltzmann solutions close to the solution of Euler equations instead of Navier–Stokes equations. The diffusion in the bulk velocity has a considerable magnitude and causes a singular term due to the growth of (1.15). Second, the macro-micro scale balance is singular in the Euler scaling. Because the transport effect is weaker, this results in the lack of a scale factor of the hydrodynamic bound in the dissipation. In fact, an integrability gain in L^p ($\hookleftarrow H^1_x$ in 2D) of [23] or velocity average lemma [24] are not useful to control the singular nonlinearity. In addition, the perturbation equations suffer a loss of scale due to the commutator of spatial derivatives and the linearized operator around a local Maxwellian associated with macroscopic solutions.

To overcome the difficulties, we devise a novel *viscosity-canceling correction* in an asymptotic expansion of the scaled Boltzmann equations. To handle the low regularity of fluid velocity fields, we regularize the initial data with scale β and expand the Boltzmann equations around the local Maxwellian $M_{1,\varepsilon u^\beta,1}$ associated with the Euler solution u^β starting from u_0^β. In the first place, one may try a form of the standard Hilbert expansion

$$M_{1,\varepsilon u^\beta,1} + \varepsilon^2 p^\beta M_{1,\varepsilon u^\beta,1} - \varepsilon^2\kappa(\nabla_x u^\beta) : \mathfrak{A}\sqrt{M_{1,\varepsilon u^\beta,1}} + \varepsilon f_R\sqrt{M_{1,\varepsilon u^\beta,1}}, \tag{1.16}$$

where $\mathfrak{A}$ is defined as (2.13), by matching to cancel most singular terms. The Euler equation is in the hierarchy of $O(\varepsilon^2)$; it comes from $\varepsilon\partial_t M_{1,\varepsilon u^\beta,1}$ and correctors. However, the third term of order $\varepsilon^2\kappa$ introduces the viscosity contribution $-\varepsilon^2\kappa\eta_0\Delta_x u^\beta \cdot (v - \varepsilon u^\beta) M_{1,\varepsilon u^\beta,1}$, and, comparing it to $\varepsilon f_R\sqrt{M_{1,\varepsilon u^\beta,1}}$, we see that, if this term is not canceled, then it will drive the remainder to order $O(\varepsilon\kappa)$. However, once the remainder f_R grows to $O(\varepsilon\kappa)$ size, the effect of nonlinearity $\frac{1}{\varepsilon\kappa}\Gamma(f_R, f_R)$ becomes $O(\varepsilon\kappa)$ as well (see (2.37)). As a consequence, we cannot close the bootstrap argument; we need to keep the remainder f_R to the size $o(\varepsilon\kappa)$. Note that this term is hydrodynamic, so we cannot rely on coercivity provided by L; it provides additional $\varepsilon\sqrt{\kappa}$ smallness only for nonhydrodynamic terms.

A simple but useful observation is that this term is still in a lower hierarchy than that of the Euler equation. When $\kappa=\varepsilon$, this observation leads to an introduction of next-order expansion, which absorbs the viscosity contribution to the hydrodynamic equation of smaller-scale fluctuation. However, in our setting, κ is not an integer power of ε, and therefore, choosing the right next-level corrector is a nontrivial problem. One key observation in this work is that a corrector can be found in a very similar fashion as the $\kappa=\varepsilon$ case; by introducing an additional corrector in $\varepsilon\kappa$ level, we were able to cancel out the viscosity contribution. Of course, one needs to be careful because we introduce an $\varepsilon\kappa$-size term to cancel out an $\varepsilon^2\kappa$-size term! However, by carefully choosing the form of the $\varepsilon\kappa$-size corrector

$$F^\varepsilon = (1.16) + \varepsilon\kappa\tilde{u}^\beta \cdot (v-\varepsilon u^\beta) M_{1,\varepsilon u^\beta,1} + \varepsilon^2\kappa\tilde{p}^\beta M_{1,\varepsilon u^\beta,1}, \tag{1.17}$$

we can actually fulfill our goal.

1. $\varepsilon\kappa\tilde{u}\cdot(v-\varepsilon u^\beta)M_{1,\varepsilon u^\beta,1}$ is fully hydrodynamic, and therefore, the most singular term coming from collision with the local Maxwellian vanishes. Then, the largest term coming from collision is the collision of this corrector with itself, which is of size $\varepsilon\kappa$ but nonhydrodynamic. Thus, it is in fact small (due to $\varepsilon\sqrt{\kappa}$ gain for a nonhydrodynamic term, nonhydrodynamic source terms of $\varepsilon\sqrt{\kappa}$ drive the remainder to order $O(\varepsilon\kappa)$.)
2. By imposing $\nabla_x\cdot\tilde{u}=0$, we can cancel out the hydrodynamic part for $v\cdot\nabla_x(\tilde{u}\cdot(v-\varepsilon u^\beta)M_{1,\varepsilon u^\beta,1})$, which is of order $\varepsilon\kappa$. Also, by introducing an additional corrector at $\varepsilon^2\kappa$ level, one can cancel out all hydrodynamic terms of $\varepsilon^2\kappa$ level by the evolution equation for $\tilde{u}$, including $\Delta_x u$. Therefore, the remaining hydrodynamic terms are of order $o(\varepsilon^2\kappa)$ and nonhydrodynamic terms are of order $O(\varepsilon\kappa)$, and both are small.
3. The interaction of this corrector and the remainder also turns out to be innocuous as well.

It is worth remarking that, in this corrector-based Hilbert expansion, we do not need to set up $\varepsilon=\kappa$ as in the usual Hilbert expansion [17]; we only need $\varepsilon/\kappa^2\to 0$. This is satisfactory, in the sense that a regime that is close to the Navier–Stokes regime (whose κ vanishes slowly) should be more tractable in philosophy, and indeed, for such a regime, we can allow a larger deviation from the equilibrium. In addition, we note that this expansion in fact allows even more general data than (1.16); we have additional freedom in choosing $\tilde{u}_0$, so, in principle, a remainder with a certain part of size $\varepsilon\kappa$ is in fact admissible, while in (1.16), all parts of the remainder should be of size $o(\varepsilon\kappa)$. We believe that this new idea of correction will have many applications.

The search for an additional corrector in (1.17) has been largely indebted to our $H^2_xL^2_v$ framework. The framework was introduced for two reasons: First, our goal was to obtain the hydrodynamic limit in a stronger topology. Second, this framework gives better control of the nonlinearity. To elaborate, one can start from the observation that, when measuring the $L^2_{x,v}$ norm of the nonlinearity $\Gamma(f_R,f_R)$, one only lacks L^2-integrability of x (see Lemma 1) since Γ is also an integral operator whose kernel decays rapidly in v. Therefore, to control the nonlinearity, we may first establish $H^2_xL^2_v$ control of the remainder and use interpolation inequalities (see Lemma 4). It turns out that this framework gives a sharper scaling than previously considered methods, which is reasonable in the sense that this method relies on the coercivity of the linearized collision operator, while other methods do not use it and treat the linearized collision operator, which is the most singular term in (2.37), as a forcing.

In addition to the aforementioned advantages, the $H^2_xL^2_v$ framework greatly simplifies the searching process for correctors. One can almost close the estimate for f_R in the $H^2_xL^2_v$ framework, except for the term coming from the momentum-stream ($f_R(\frac{(\partial_t+\frac{v}{\varepsilon}\cdot\nabla_x)\sqrt{\mu}}{\sqrt{\mu}})$ in (2.37)), which can be closed using an L^∞ estimate with a very small prefactor. Therefore, to check if a proposed corrector works, one only needs to measure the size of its effect in the energy space.

The paper is organized as follows. In section 2, we introduce our new expansion, derive the equation for the remainder, and conclude that the size of the remainder can be controlled in the $H^2_xL^2_v\cap L^\infty_{xv}$ space. In section 3, we present the derivative bounds for Euler equations with smoothly approximated initial data. In section 4, we show the stability of class of solutions of Euler equations we consider (Yudovich/localized Yudovich/Diperna–Majda) under smooth approximation of initial data. In particular, for the localized Yudovich class, we find explicit convergence rate for both velocity and vorticity. Finally, in section 5, we prove our main hydrodynamic limit theorem.

Notations. For the sake of readers' convenience, we list notations used often in this paper.

$$\partial \ : \partial f = \ \partial_{x_1} f \text{ or } \partial_{x_2} f \tag{1.18}$$

$$\partial^s \ : \partial^s f = \sum_{\alpha_1+\alpha_2\le s} \partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2} f \tag{1.19}$$

$$f*g \ : f*g(x) := \int_{\mathbb{T}^2} f(x-y)g(y)\mathrm{d}y \tag{1.20}$$

$$f*_{\mathbb{R}^2} g \ : f*_{\mathbb{R}^2} g(x) = \int_{\mathbb{R}^2} f(x-y)g(y)\mathrm{d}y \tag{1.21}$$

$$(\ \cdot\ )_+ \ : (a)_+ = \max\{a,0\} \tag{1.22}$$

$$\log_+ \ : \ \log_+ a = \max\{\log a, 0\} \tag{1.23}$$

$$\lesssim : \text{there exists } C>0 \text{ such that } a\lesssim b \text{ implies } a\le Cb \tag{1.24}$$

$$a\simeq b \ : a \text{ consists of an appropriate linear combination of the terms in } b \tag{1.25}$$

$$[\![\cdot,\cdot]\!] \ : [\![A,B]\!]g := A(Bg)-B(Ag) \quad \text{(commutator)} \tag{1.26}$$

$$\|\cdot\|_{L^p_\square} \ : \|f\|_{L^p_t} = \|f\|_{L^p(0,T)},\ \|f\|_{L^p_x} = \|f\|_{L^p(\mathbb{T}^2)},\ \|f\|_{L^p_v} = \|f\|_{L^p(\mathbb{R}^3)} \tag{1.27}$$

$$\|\cdot\|_{L^p_xL^2_v} \ : \|f\|_{L^p_xL^2_v} := \|f\|_{L^p(\mathbb{T}^2;L^2(\mathbb{R}^3))} = \big\|\|f(x,v)\|_{L^2(\mathbb{R}^3_v)}\big\|_{L^p(\mathbb{T}^2_x)} \tag{1.28}$$

$$d_{\mathbb{T}^2}(x,y) \ : \text{geodesic distance between } x \text{ and } y \text{ in } \mathbb{T}^2, \text{ often abused as } |x-y| \tag{1.29}$$

2. Hilbert-type expansion with viscosity-canceling corrector.

2.1. Formulation around a local Maxwellian.

We denote a local Maxwellian corresponding to $(1,\varepsilon u^\beta,1)$ by

$$\mu := M_{1,\varepsilon u^\beta,1}. \tag{2.1}$$

We try to construct a family of solutions F^ε in the form of

$$F^\varepsilon = \mu + \varepsilon^2 p^\beta\mu - \varepsilon^2\kappa(\nabla_x u^\beta):\mathfrak{A}\sqrt{\mu} + \{\varepsilon\kappa\tilde{u}^\beta\cdot(v-\varepsilon u^\beta)+\varepsilon^2\kappa\tilde{p}^\beta\}\mu + \varepsilon f_R\sqrt{\mu}, \tag{2.2}$$

where p^β, $\tilde{u}^\beta$, and $\tilde{p}^\beta$ satisfy (3.9) and (3.10) and $\mathfrak{A}$ will be defined in (2.13).

Also, we assume the following assumption on the relative magnitudes on $\varepsilon, \kappa = \kappa(\varepsilon), \beta = \beta(\varepsilon)$:

$$\begin{aligned} \lim_{\varepsilon\to 0} \frac{\varepsilon}{\kappa^2} &= 0, \\ \lim_{\varepsilon\to 0} \kappa^{\frac{1}{4}} V(\beta) &= 0, \\ \lim_{\varepsilon\to 0} \kappa^{\frac{1}{2}} e^{2\mathbf{C}_0 T \|\nabla_x u^\beta\|^2_{L^\infty((0,T)\times\mathbb{T}^2)}} &= 0, \end{aligned} \tag{2.3}$$

where $\mathbf{C}_0$ is specified in section 2.5.

We define

$$Lf = \frac{-2}{\sqrt{\mu}} Q(\mu, \sqrt{\mu} f), \quad \Gamma(f,g) = \frac{1}{\sqrt{\mu}} Q(\sqrt{\mu} f, \sqrt{\mu} g). \tag{2.4}$$

From the collision invariance, a null space of L, denoted by $\mathcal{N}$, has five orthonormal bases $\{\varphi_i \sqrt{\mu}\}_{i=1}^5$ with

$$\begin{aligned} &\varphi = (\varphi_0, \varphi_1, \varphi_2, \varphi_3, \varphi_4), \\ &\varphi_0 := 1, \quad \varphi_i := v_i - \varepsilon u_i^\beta \ \text{ for } i = 1,2,3, \quad \varphi_4 := \frac{|v - \varepsilon u^\beta|^2 - 3}{\sqrt{6}}. \end{aligned} \tag{2.5}$$

We define $\mathbf{P}$, an L^2_v-projection on $\mathcal{N}$, as

$$\begin{aligned} &Pg := (P_0 g, P_1 g, P_2 g, P_3 g, P_4 g), \quad P_j g := \int_{\mathbb{R}^3} g \varphi_j \sqrt{\mu} \mathrm{d}v \ \text{ for } j = 0,1,\dots,4, \\ &\mathbf{P} g := \sum_{j=0}^4 (P_j g) \varphi_j \sqrt{\mu} = Pg \cdot \varphi \sqrt{\mu}. \end{aligned} \tag{2.6}$$

We record the exact form of L and Γ for the later purpose; the calculation is due to Grad [27], and one can also read [24] for details of derivations. Also, the exact form of the formulae were excerpted from [35]: For certain positive constants c_1, c_2, c_3,

$$\begin{aligned} Lf(v) &= \nu f(v) - Kf(v) = \nu(v) f(v) - \int_{\mathbb{R}^3} \mathbf{k}(v, v_*) f(v_*) \mathrm{d}v_*, \\ \nu(v) &= c_1 \left(\left(2|v - \varepsilon u^\beta| + \frac{1}{|v - \varepsilon u^\beta|} \right) \int_0^{|v-\varepsilon u^\beta|} e^{-\frac{z^2}{2}} \mathrm{d}z + e^{-\frac{|v-\varepsilon u^\beta|^2}{2}} \right), \\ \mathbf{k}(v, v_*) &= c_2 |v - v_*| e^{-\frac{|v-\varepsilon u^\beta|^2 + |v_* - \varepsilon u^\beta|^2}{4}} - \frac{c_3}{|v - v_*|} e^{-\frac{1}{8}|v-v_*|^2 - \frac{1}{8}\frac{(|v-\varepsilon u^\beta|^2 - |v_* - \varepsilon u^\beta|^2)^2}{|v-v_*|^2}}, \\ \Gamma(f,g)(v) &= \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |(v - v_*) \cdot \omega| \sqrt{\mu(v_*)} (f(v') g(v'_*) + g(v') f(v'_*)) \mathrm{d}\omega \mathrm{d}v_* \\ &\quad - \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |(v - v_*) \cdot \omega| \sqrt{\mu(v_*)} (f(v) g(v_*) + g(v) f(v_*)) \mathrm{d}\omega \mathrm{d}v_*, \end{aligned} \tag{2.7}$$

where $v' = v - ((v - v_*) \cdot \omega)\omega$, $v'_* = v_* + ((v - v_*) \cdot \omega)\omega$. Here, all $\nu, \mathbf{k}, \Gamma$ also depend on x and t in a straightforward manner; that is, $Lf(x,t,v)$ and $\Gamma(f,g)(x,t,v)$ depend on $f(x,t,\cdot)$, $g(x,t,\cdot)$, and $u^\beta(x,t)$. We omitted them for the sake of simplicity.

Also, we define ${}_{\partial^s}L$ and ${}_{\partial^s}\Gamma$ for $s \geq 1$:

(2.8)
$$
\begin{aligned}
{}_{\partial^s}Lf(v) &= \partial^s(\nu)(v)f(v) - \int_{\mathbb{R}^3} \partial^s(\mathbf{k})(v,v_*)f(v_*)\mathrm{d}v_*,\\
{}_{\partial^s}\Gamma(f,g)(v) &= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |(v-v_*)\cdot\omega|\partial^s(\sqrt{\mu(v_*)})(f(v')g(v'_*)+g(v')f(v'_*))\mathrm{d}\omega\mathrm{d}v_*\\
&\quad - \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |(v-v_*)\cdot\omega|\partial^s(\sqrt{\mu(v_*)})(f(v)g(v_*)+g(v)f(v_*))\mathrm{d}\omega\mathrm{d}v_*.
\end{aligned}
$$

We list standard results that will be used later in this section for the sake of readers' convenience. First, we note that

$$
Q(\mu,\mu) = 0 = \mathbf{P}L = L\mathbf{P} = \mathbf{P}\Gamma \tag{2.9}
$$

from the collision invariance.

Lemma 1 ([23, 31, 29]). *Suppose that* (2.3) *holds. Then,*

(2.10)
$$
\begin{aligned}
&\|\nu^{-1/2}Lf\|_{L^2(\mathbb{T}^2\times\mathbb{R}^3)} \lesssim \|\sqrt{\nu}(\mathbf{I}-\mathbf{P})f\|_{L^2(\mathbb{T}^2\times\mathbb{R}^3)},\\
&\|\nu^{\frac12}(\mathbf{I}-\mathbf{P})f\|^2_{L^2_v} \lesssim \left|\int Lf(v)f(v)\mathrm{d}v\right|,\\
&\left|\int {}_{\partial^s}Lf(v)g(v)\mathrm{d}v\right|\\
&\lesssim \varepsilon\|\partial^s u^\beta\|_{L^\infty_{t,x}}\left(\|\mathbf{P}f\|_{L^2_v} + \|\nu^{\frac12}(\mathbf{I}-\mathbf{P})f\|_{L^2_v}\right)\left(\|\mathbf{P}g\|_{L^2_v} + \|\nu^{\frac12}(\mathbf{I}-\mathbf{P})g\|_{L^2_v}\right),
\end{aligned}
$$

(2.11)
$$
\begin{aligned}
&\left|\int \Gamma(f,g)h\mathrm{d}v\mathrm{d}x\mathrm{d}t\right|\\
&\lesssim \int\left[\left(\|\mathbf{P}f\|_{L^2_v} + \|\nu^{\frac12}(\mathbf{I}-\mathbf{P})f\|_{L^2_v}\right)\|g\|_{L^2_v}\right.\\
&\qquad\left. + \left(\|\mathbf{P}g\|_{L^2_v} + \|\nu^{\frac12}(\mathbf{I}-\mathbf{P})g\|_{L^2_v}\right)\|f\|_{L^2_v}\right]\|\nu^{\frac12}(\mathbf{I}-\mathbf{P})h\|_{L^2_v}\mathrm{d}x\mathrm{d}t,\\
&\left|\int {}_{\partial^s}\Gamma(f,g)h\mathrm{d}v\mathrm{d}x\mathrm{d}t\right|\\
&\lesssim \varepsilon\|\partial^s u\|_{L^\infty_{t,x}}\int\left[\left(\|\mathbf{P}f\|_{L^2_v} + \|\nu^{\frac12}(\mathbf{I}-\mathbf{P})f\|_{L^2_v}\right)\|g\|_{L^2_v}\right.\\
&\qquad\left. + \left(\|\mathbf{P}g\|_{L^2_v} + \|\nu^{\frac12}(\mathbf{I}-\mathbf{P})g\|_{L^2_v}\right)\|f\|_{L^2_v}\right]\\
&\quad\times\left(\|\mathbf{P}h\|_{L^2_v} + \|\nu^{\frac12}(\mathbf{I}-\mathbf{P})h\|_{L^2_v}\right)\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Next, we introduce a lemma illustrating the structure of higher derivatives of Lf. Recall the notation $[\![\cdot,\cdot]\!]$ for the commutator (1.26).

Lemma 2. *For $s \geq 1$, $[\![\partial^s, L]\!]f$ is a linear combination, whose coefficient depends only on s, of the terms having one of the following forms:*

1. ${}_{\partial^j}L(\mathbf{I}-\mathbf{P})\partial^{s-j}f$*, where* $1 \leq j \leq s$;
2. $L\partial\cdots[\![\mathbf{P},\partial]\!]\cdots\partial f$*, where* $\partial\cdots[\![\mathbf{P},\partial]\!]\cdots\partial f$ *is an application of* $s-1$ ∂ *and one* $[\![\mathbf{P},\partial]\!]$ *at* j*th order to* f $(0 \leq j \leq s)$*; or*
3. ${}_{\partial^j}L\partial\cdots[\![\mathbf{P},\partial]\!]\cdots\partial f$*, where* $1 \leq j \leq s-1$ *and* $\partial\cdots[\![\mathbf{P},\partial]\!]\cdots\partial f$ *is an application of* $s-j-1$ ∂ *and one* $[\![\mathbf{P},\partial]\!]$ *at* i*th order to* f $(0 \leq i \leq s-j)$.

Proof. We proceed by the induction on s; first, we note that

$$\begin{aligned}\partial(Lf) &= \partial L(\mathbf{I}-\mathbf{P})f = {}_{\partial}L(\mathbf{I}-\mathbf{P})f + L\partial(\mathbf{I}-\mathbf{P})f\\ &= {}_{\partial}L(\mathbf{I}-\mathbf{P})f + L[\![\mathbf{P},\partial]\!]f + L(\mathbf{I}-\mathbf{P})\partial f,\\ [\![\partial, L]\!]f &= {}_{\partial}L(\mathbf{I}-\mathbf{P})f + L[\![\mathbf{P},\partial]\!]f,\end{aligned}$$

which proves the claim for $s=1$. Next, for $s\geq 1$, we have

$$[\![\partial^{s+1}, L]\!]f = \partial^{s+1}Lf - L\partial^{s+1}f = \partial[\![\partial^s, L]\!]f + [\![\partial, L]\!]\partial^s f,$$

and, by the first step, $[\![\partial, L]\!]\partial^s f$ consists of terms in the lemma. Also, application of ∂ to the terms of the second and third form of the lemma produces terms of the second and third form again, while application of ∂ to the first form produces

$$\begin{aligned}\partial_{\partial^j}L(\mathbf{I}-\mathbf{P})\partial^{s-j}f &= {}_{\partial^{j+1}}L(\mathbf{I}-\mathbf{P})\partial^{s-j}f + {}_{\partial^j}L\partial(\mathbf{I}-\mathbf{P})\partial^{s-j}f\\ &= {}_{\partial^{j+1}}L(\mathbf{I}-\mathbf{P})\partial^{s-j}f + {}_{\partial^j}L[\![\mathbf{P},\partial]\!]\partial^{s-j}f + \cdots + {}_{\partial^j}L\partial^{s-j}[\![\partial^{s-j}[\![\mathbf{P},\partial]\!]f\\ &\quad + {}_{\partial^j}L(\mathbf{I}-\mathbf{P})\partial^{s-j+1}f,\end{aligned}$$

which proves the claim. ◻

Also, we have the following straightforward estimate for $[\![\mathbf{P},\partial]\!]f$.

LEMMA 3. *Suppose that* (2.3) *holds. For* $s_1+s_2\leq 1$*, the following holds:*

$$\begin{aligned}[\![\mathbf{P},\partial]\!]f &= -\sum_{i=0}^{4}\langle f, \varphi_i\sqrt{\mu}\rangle_{L^2_v}\partial(\varphi_i\sqrt{\mu}),\\ \|[\![\mathbf{P},\partial]\!]f\|_{L^2_v} &\lesssim \varepsilon\|\nabla_x u^\beta\|_{L^\infty_{t,x}}\|f\|_{L^2_v},\\ \|\partial^{s_1}[\![\mathbf{P},\partial]\!]\partial^{s_2}f\|_{L^2_v} &\lesssim \varepsilon V(\beta)\|\partial^{s_1+s_2}f\|_{L^2_v}.\end{aligned}$$

Next, we introduce anisotropic spaces; this will be key to our analysis. For $p\in[1,\infty]$, we recall the space $L^p(\mathbb{T}^2;L^2(\mathbb{R}^3))$ by the norm $\|f\|_{L^p(\mathbb{T}^2;L^2(\mathbb{R}^3))}$ in (1.28). For $p,q\in[1,\infty]$, $L^q([0,T];L^p(\mathbb{T}^2;L^2(\mathbb{R}^3)))$ is defined similarly. We have the following anisotropic interpolations.

LEMMA 4. *We have the following:*
1. *(anisotropic Ladyzhenskaya)* $\|f\|_{L^4_xL^2_v}\lesssim\|f\|^{\frac12}_{L^2_xL^2_v}\|\partial f\|^{\frac12}_{L^2_xL^2_v}$ *and*
2. *(anisotropic Agmon)* $\|f\|_{L^\infty_xL^2_v}\lesssim\|f\|^{\frac12}_{L^2_xL^2_v}\|\partial^2 f\|^{\frac12}_{L^2_xL^2_v}$.

Proof. We only prove the former; the latter is derived in a similar manner.

$$\begin{aligned}\|f\|_{L^4_xL^2_v} &= \left(\int_{\mathbb{T}^2}\left(\int_{\mathbb{R}^3}|f(x,v)|^2\mathrm{d}v\right)^{\frac42}\mathrm{d}x\right)^{\frac{1}{2\cdot 2}} \leq \left(\int_{\mathbb{R}^3}\left(\int_{\mathbb{T}^2}|f(x,v)|^4\mathrm{d}x\right)^{\frac12}\mathrm{d}v\right)^{\frac12}\\ &= \left(\int_{\mathbb{R}^3}\|f(\cdot,v)\|^2_{L^4_x}\mathrm{d}v\right)^{\frac12} \lesssim \left(\int_{\mathbb{R}^3}\|f(\cdot,v)\|_{L^2_x}\|\partial f(\cdot,v)\|_{L^2_x}\mathrm{d}v\right)^{\frac12}\\ &\leq \left(\int_{\mathbb{R}^3}\int_{\mathbb{T}^2}|f(x,v)|^2\mathrm{d}x\mathrm{d}v\right)^{\frac{1}{2\cdot 2}}\left(\int_{\mathbb{R}^3}\int_{\mathbb{T}^2}|\partial f(x,v)|^2\mathrm{d}x\mathrm{d}v\right)^{\frac{1}{2\cdot 2}}\\ &= \|f\|^{\frac12}_{L^2_xL^2_v}\|\partial f\|^{\frac12}_{L^2_xL^2_v},\end{aligned}$$

where we applied Minkowski for the first, the usual Ladyzhenskaya for the second, and Hölder for the last inequalities. ◻

From Lemma 4, we have the following.

LEMMA 5.

$$\begin{aligned}
&\|\nu^{\frac12}(\mathbf{I}-\mathbf{P})f\|_{L^4_xL^2_v}\\
&\lesssim \varepsilon^{\frac12}\|\nu^{\frac12}(\mathbf{I}-\mathbf{P})f\|^{\frac12}_{L^2_xL^2_v}\\
&\quad\times\left(\|\partial u^\beta\|_{L^\infty_x}\|f\|_{L^2_xL^2_v}+\|\varepsilon^{-1}\nu^{\frac12}(\mathbf{I}-\mathbf{P})\partial f\|_{L^2_xL^2_v}+V(\beta)\|\varepsilon^{-1}\nu^{\frac12}(\mathbf{I}-\mathbf{P})f\|_{L^2_xL^2_v}\right)^{\frac12},\\
&\|\nu^{\frac12}(\mathbf{I}-\mathbf{P})f\|_{L^\infty_xL^2_v}\\
&\lesssim \varepsilon^{\frac12}\|\nu^{\frac12}(\mathbf{I}-\mathbf{P})f\|^{\frac12}_{L^2_xL^2_v}\\
&\quad\times\Big[\|\partial u^\beta\|_{L^\infty_x}\|\partial f\|_{L^2_xL^2_v}+\|\varepsilon^{-1}\nu^{\frac12}(\mathbf{I}-\mathbf{P})\partial^2 f\|_{L^2_xL^2_v}\\
&\quad+V(\beta)\left(\|\varepsilon^{-1}\nu^{\frac12}(\mathbf{I}-\mathbf{P})f\|_{L^2_xL^2_v}+\|\varepsilon^{-1}\nu^{\frac12}(\mathbf{I}-\mathbf{P})\partial f\|_{L^2_xL^2_v}+\|f\|_{L^2_xL^2_v}\right)\Big]^{\frac12}.
\end{aligned}$$

Proof. We only give the proof for the first inequality; the second inequality can be proved by a similar argument. By Lemma 4, it suffices to control $\partial(\nu^{\frac12}(\mathbf{I}-\mathbf{P})f)$; we have

$$\partial(\nu^{\frac12}(\mathbf{I}-\mathbf{P})f)=\frac12\nu^{-1}\partial(\nu)\nu^{\frac12}(\mathbf{I}-\mathbf{P})f+\nu^{\frac12}[\![\mathbf{P},\partial]\!]f+\nu^{\frac12}(\mathbf{I}-\mathbf{P})\partial f.$$

One can easily check that $\sup_{x,v}|\nu^{-1}\partial(\nu)|\lesssim\varepsilon\|\partial u^\beta\|_{L^\infty_x}$, and thus, the inequality follows. ∎

LEMMA 6 ([11, 31]). *$L|_{\mathcal{N}^\perp}:\mathcal{N}^\perp\to\mathcal{N}^\perp$ is a bijection, and thus, $L^{-1}:\mathcal{N}^\perp\to\mathcal{N}^\perp$ is well defined. Also, L^{-1} is symmetric under any orthonormal transformation. In particular, if $f\in\mathcal{N}^\perp$ is an even (resp., odd) function, then so is $L^{-1}f$.*

Proof. The proof follows the Fredholm alternative and rotational invariance of Q. We refer to [11, 31] for the proof. ∎

The term $(v-\varepsilon u^\beta)\otimes(v-\varepsilon u^\beta)\sqrt{\mu}$ and its image over L^{-1} turns out to play an important role in the Hilbert expansion. Note that

(2.12)
$$(\mathbf{I}-\mathbf{P})\left((v-\varepsilon u^\beta)\otimes(v-\varepsilon u^\beta)\sqrt{\mu}\right)=\left((v-\varepsilon u^\beta)\otimes(v-\varepsilon u^\beta)-\frac13|v-\varepsilon u^\beta|^2\mathbf{I}_3\right)\sqrt{\mu}.$$

Thus, we define $\mathfrak{A}:=\mathfrak{A}(t,x)\in\mathbb{M}_{3\times3}(\mathbb{R})$ by (see [5])

$$\mathfrak{A}_{ij}=L^{-1}\left(\left((v-\varepsilon u^\beta)_i(v-\varepsilon u^\beta)_j-\frac{|v-\varepsilon u^\beta|^2}{3}\delta_{ij}\right)\sqrt{\mu}\right). \tag{2.13}$$

Regarding $\mathfrak{A}$, we have the following useful lemma.

LEMMA 7 ([5, 4]). $\langle L\mathfrak{A}_{\ell k},\mathfrak{A}_{ij}\rangle=\eta_0(\delta_{ik}\delta_{j\ell}+\delta_{i\ell}\delta_{jk})-\frac23\eta_0\delta_{ij}\delta_{k\ell}$.

Proof. We refer to [5, 4] for the proof. ∎

From explicit calculation, we can also establish the following result.

LEMMA 8. *For $i,j,k\in\{1,2,3\}$,*

$$\mathbf{P}(\varphi_i\varphi_j\varphi_k\sqrt{\mu})=\sum_{\ell=1}^3(\delta_{ij}\delta_{k\ell}+\delta_{ik}\delta_{j\ell}+\delta_{jk}\delta_{i\ell})\varphi_\ell\sqrt{\mu}.$$

We also have the following useful pointwise estimates. First, we have the following pointwise estimates on $\partial^s(f\frac{(\partial_t+\frac{v}{\varepsilon}\cdot\nabla_x)\sqrt{\mu}}{\sqrt{\mu}})$.

LEMMA 9. *Suppose that* (2.3) *holds. Then, for* $s\leq 2$, *we have*

$$\begin{aligned}\partial^s\left(f\frac{(\partial_t+\frac{v}{\varepsilon}\cdot\nabla_x)\sqrt{\mu}}{\sqrt{\mu}}\right)&=\partial^s f\left(\frac{(\partial_t+\frac{v}{\varepsilon}\cdot\nabla_x)\sqrt{\mu}}{\sqrt{\mu}}\right)\\&\quad+\sum_{s'<s}(\partial^{s'}f)\frac{1}{2}\sum_{i,j}(\partial^{s-s'}\partial_{x_i}u_j^\beta)\varphi_i\varphi_j+R,\end{aligned}\tag{2.14}$$

where $|R|\lesssim\varepsilon V(\beta)\nu(v)\sum_{s'<s}|\partial^{s'}f|$.

Proof. It suffices to notice that

$$\frac{(\partial_t+\frac{v}{\varepsilon}\cdot\nabla_x)\sqrt{\mu}}{\sqrt{\mu}}=\frac{1}{2}\sum_{i,j}\partial_{x_i}u_j^\beta\varphi_i\varphi_j+\frac{1}{2}\varepsilon\sum_i(\partial_t u^\beta+u^\beta\cdot\nabla_x u^\beta)_i\varphi_i$$

and that the first two terms of the right-hand side of (2.14) correspond to the terms where all ∂ are applied to either f or $\partial_{x_i}u_j^\beta$ and R are all others. ∎

Next, we present pointwise estimates on A and its derivatives ([35]).

LEMMA 10 (Lemma 3 of [35]). *Suppose that* (2.3) *holds. For* $\varrho\in(0,1/4)$,

$$\begin{aligned}|\mathfrak{A}_{ij}(v)|&\lesssim e^{-\varrho|v-\varepsilon u^\beta|^2},\\ \sum_{s\leq 2,D\in\{\partial_t,\partial\}}\left|\partial^s\left((1+(u^\beta,\tilde{u}^\beta))D\mathfrak{A}_{ij}(v)\right)\right|&\lesssim\varepsilon V(\beta)e^{-\varrho|v-\varepsilon u^\beta|^2}.\end{aligned}$$

Next, we have the following pointwise estimates on Γ and L.

LEMMA 11 (Lemma 4 of [35]). *Suppose that* $\varepsilon|u^\beta(x,t)|\lesssim 1$. *For* $0<\varrho<1/4$, $C\in\mathbb{R}^3$, *and* $s\leq 2$, *we have*

$$\begin{aligned}|\Gamma(f,g)(v)|&\lesssim\|e^{\varrho|v|^2+C\cdot v}f(v)\|_{L^\infty_v}\|e^{\varrho|v|^2+C\cdot v}g(v)\|_{L^\infty_v}\frac{\nu(v)}{e^{\varrho|v|^2+C\cdot v}},\\|_{\partial^s}\Gamma(f,g)(v)|&\lesssim\varepsilon V(\beta)\|e^{\varrho|v|^2+C\cdot v}f(v)\|_{L^\infty_v}\|e^{\varrho|v|^2+C\cdot v}g(v)\|_{L^\infty_v}\frac{\nu(v)}{e^{\varrho|v|^2+C\cdot v}},\\|_{\partial^s}Lf(v)|&\lesssim\varepsilon V(\beta)\|e^{\varrho|v|^2+C\cdot v}f(v)\|_{L^\infty_v}\frac{\nu(v)^2}{e^{\varrho|v|^2+C\cdot v}}.\end{aligned}$$

Here, we can choose the constant for the bound uniformly for $\{|C|\leq 1\}$.

Finally, we present pointwise estimates regarding projections $\mathbf{P}$ and $\mathbf{I}-\mathbf{P}$.

LEMMA 12. *Suppose that* $f(t,x,v)\in L^2_v$ *satisfies* $|f(t,x,v)|\leq C(t,x)\exp(-\varrho|v-\varepsilon u^\beta(t,x)|^2)$ *for some constant* $C(t,x)$ *independent of* v *and* $\varrho\in(0,1/4)$. *Then,*

$$\begin{aligned}|\mathbf{P}f(t,x,v)|&\lesssim C(t,x)\exp\left(-\varrho|v-\varepsilon u^\beta(t,x)|^2\right),\\|(\mathbf{I}-\mathbf{P})f(t,x,v)|&\lesssim C(t,x)\exp\left(-\varrho|v-\varepsilon u^\beta(t,x)|^2\right),\end{aligned}\tag{2.15}$$

where the constants for inequalities are independent of t,x,v *but depend on* ϱ.

Proof. It suffices to show (2.15) only; the other follows from $|(\mathbf{I}-\mathbf{P})f(t,x,v)| \le |\mathbf{P}f(t,x,v)| + |f(t,x,v)|$. Note that, from (2.6),

$$\begin{aligned} &|\mathbf{P}f(t,x,v)| \\ &\quad \le \sum_{\ell=1}^{5} C(t,x) \int \langle v-\varepsilon u^\beta\rangle^2 \exp\left(-\left(\varrho+\frac{1}{4}\right)|v-\varepsilon u^\beta(t,x)|^2\right) dv \langle v-\varepsilon u^\beta\rangle^2 \sqrt{\mu} \\ &\quad \le C(t,x) C_\varrho \exp\left(-\varrho |v-\varepsilon u^\beta(t,x)|^2\right). \end{aligned}$$

∎

2.2. New Hilbert-type expansion. We recall an explicit form of derivatives of μ^k:

$$\begin{aligned} \left[\partial_t + u^\beta\cdot\nabla_x\right]\mu^k &= \varepsilon k(\partial_t u^\beta + u^\beta\cdot\nabla_x u^\beta)\cdot(v-\varepsilon u^\beta)\mu^k, \\ (v-\varepsilon u^\beta)\cdot\nabla_x \mu^k &= \varepsilon k(\nabla_x u^\beta) : ((v-\varepsilon u^\beta)\otimes(v-\varepsilon u^\beta))\mu^k, \end{aligned} \tag{2.16}$$

where $k>0$ and $A:B = \mathrm{tr}(AB) = \sum_{i,j=1}^3 A_{ij}B_{ji}$ for arbitrary rank 2 tensors A, B.

Now, we derive an equation of f_R. First, we plug (2.2) into (1.5) to obtain

$$(\underline{v}-\varepsilon u^\beta)\cdot\nabla_x\left(\mu + \varepsilon^2 p^\beta\mu - \varepsilon^2\kappa(\nabla_x u^\beta):\mathfrak{A}\sqrt{\mu} + \varepsilon\kappa\tilde{u}^\beta\cdot(v-\varepsilon u^\beta)\mu + \varepsilon^2\kappa\tilde{p}^\beta\mu\right) \tag{2.17}$$

$$+\,\varepsilon(\partial_t + u^\beta\cdot\nabla_x)\left(\mu + \varepsilon^2 p^\beta\mu - \varepsilon^2\kappa(\nabla_x u^\beta):\mathfrak{A}\sqrt{\mu} + \varepsilon\kappa\tilde{u}^\beta\cdot(v-\varepsilon u^\beta)\mu + \varepsilon^2\kappa\tilde{p}^\beta\mu\right) \tag{2.18}$$

$$-\frac{1}{\kappa\varepsilon}Q(\mu + \varepsilon^2 p^\beta\mu - \varepsilon^2\kappa(\nabla_x u^\beta):\mathfrak{A}\sqrt{\mu} + \varepsilon\kappa\tilde{u}^\beta\cdot(v-\varepsilon u^\beta)\mu + \varepsilon^2\kappa\tilde{p}^\beta\mu) \tag{2.19}$$

$$+\,\varepsilon^2\left\{\partial_t(f_R\sqrt{\mu}) + \frac{v}{\varepsilon}\cdot\nabla_x(f_R\sqrt{\mu}) - \frac{1}{\varepsilon\kappa}Q(f_R\sqrt{\mu}, f_R\sqrt{\mu})\right\} \tag{2.20}$$

$$-\frac{2}{\kappa}Q(\mu + \varepsilon^2 p^\beta\mu - \varepsilon^2\kappa(\nabla_x u^\beta):\mathfrak{A}\sqrt{\mu} + \varepsilon\kappa\tilde{u}^\beta\cdot(v-\varepsilon u^\beta)\mu + \varepsilon^2\kappa\tilde{p}^\beta\mu, f_R\sqrt{\mu}) = 0, \tag{2.21}$$

where we have used an abbreviation $Q(g) = Q(g,g)$ in (2.19).

We group the source terms (2.17), (2.18), and (2.19) with corresponding orders of magnitude; it is good to keep in mind that, in our method, all hydrodynamic terms of order of magnitude less than $\varepsilon^2\kappa$ are considered small and all nonhydrodynamic terms of order of magnitude less than $\varepsilon\sqrt{\kappa}$ are considered small. In the end, we will group all small terms altogether.

Terms that are greater than ε. Among terms that are independent of f_R, there are no terms whose magnitude is greater than ε. For terms in (2.17) and (2.18), this is obvious; the largest term comes from $(\underline{v}-\varepsilon u^\beta)\cdot\nabla_x\mu$, which is of order ε. For terms in (2.19), we note that, since $(v-\varepsilon u^\beta)\sqrt{\mu}, \sqrt{\mu}\in\mathcal{N}$, in fact, (2.19) can be rewritten as

$$\begin{aligned} &2\varepsilon Q(\mu(1+\varepsilon^2 p^\beta + \varepsilon^2\kappa\tilde{p}^\beta), (\nabla_x u^\beta):\mathfrak{A}\sqrt{\mu}) - \kappa\varepsilon Q(\tilde{u}^\beta\cdot(v-\varepsilon u^\beta)\mu, \tilde{u}^\beta\cdot(v-\varepsilon u^\beta)\mu) \\ &\quad + 2\varepsilon^2\kappa Q(\tilde{u}^\beta\cdot(v-\varepsilon u^\beta)\mu, (\nabla_x u^\beta):\mathfrak{A}\sqrt{\mu}) - \varepsilon^3\kappa Q((\nabla_x u^\beta):\mathfrak{A}\sqrt{\mu}, (\nabla_x u^\beta):\mathfrak{A}\sqrt{\mu}), \end{aligned} \tag{2.22}$$

whose leading order is ε.

2.2.1. Order ε. Among terms that are independent of f_R, there are two terms of order ε:

$$\begin{aligned}&(\underline{v}-\varepsilon u^\beta)\cdot\nabla_x\mu+\frac{2}{\kappa\varepsilon}Q(\mu,\varepsilon^2\kappa(\nabla_x u^\beta):\mathfrak{A}\sqrt{\mu})\\&=\varepsilon\nabla_x u^\beta:(v-\varepsilon u^\beta)\otimes(v-\varepsilon u^\beta)\mu-\varepsilon(\nabla_x u^\beta):L\mathfrak{A}\sqrt{\mu}=0\end{aligned}$$

because $\nabla_x\cdot u^\beta=0$.

2.2.2. Order $\varepsilon\kappa$. Among terms that are independent of f_R, there are two terms of order $\varepsilon\kappa$.

$$\begin{aligned}&\varepsilon\kappa(\underline{v}-\varepsilon u^\beta)\cdot\nabla_x(\tilde{u}^\beta\cdot(v-\varepsilon u^\beta)\mu)-\varepsilon\kappa Q(\tilde{u}^\beta\cdot(v-\varepsilon u^\beta)\mu,\tilde{u}^\beta\cdot(v-\varepsilon u^\beta)\mu)\\&=\varepsilon\kappa\left((\nabla_x\tilde{u}^\beta):L\mathfrak{A}-\Gamma\left(\tilde{u}^\beta\cdot(\underline{v}-\varepsilon u^\beta)\sqrt{\mu},\tilde{u}^\beta\cdot(\underline{v}-\varepsilon u^\beta)\sqrt{\mu}\right)\right)\sqrt{\mu}\end{aligned}\tag{2.23}$$

$$+\varepsilon^2\kappa\left(\sum_{i,j}(\underline{v}-\varepsilon u^\beta)_i\tilde{u}^\beta_j\left(-\partial_{x_i}u^\beta_j+(\underline{v}-\varepsilon u^\beta)_j(\underline{v}-\varepsilon u^\beta)_k\partial_{x_i}u^\beta_k\right)\right)\mu\tag{2.24}$$

because $\nabla_x\cdot\tilde{u}^\beta=0$. Note that terms of order $\varepsilon\kappa$ are nonhydrodynamic: $\frac{1}{\sqrt{\mu}}(2.23)\in\mathcal{N}^\perp$.

2.2.3. Order ε^2. The following terms are of order ε^2:

$$\begin{aligned}&\varepsilon(\partial_t+u^\beta\cdot\nabla_x)\mu+\varepsilon^2(\underline{v}-\varepsilon u^\beta)\cdot\nabla_x(p^\beta\mu)\\&=\varepsilon^2\left((\partial_t+u^\beta\cdot\nabla_x)u^\beta+\nabla_x p^\beta\right)\cdot(\underline{v}-\varepsilon u^\beta)\mu\\&\quad+\varepsilon^3p^\beta\nabla_{x_i}u^\beta_j(\underline{v}-\varepsilon u^\beta)_i(\underline{v}-\varepsilon u^\beta)_j\mu=\varepsilon^3p^\beta\nabla_{x_i}u^\beta_j(\underline{v}-\varepsilon u^\beta)_i(\underline{v}-\varepsilon u^\beta)_j\mu\end{aligned}\tag{2.25}$$

since $(\partial_t+u^\beta\cdot\nabla_x)u^\beta+\nabla_x p^\beta=0$.

2.2.4. Order $\varepsilon^2\kappa$. The key reason to introduce correctors $\varepsilon\kappa\tilde{u}^\beta\cdot(v-\varepsilon u^\beta)\mu$ and $\varepsilon^2\kappa\tilde{p}^\beta\mu$ is to get rid of hydrodynamic terms of order $\varepsilon^2\kappa$; as payback, we obtained terms of order $\varepsilon\kappa$, which is larger, but all of them are nonhydrodynamic, so they are small in our scale. The following is the collection of all terms of order $\varepsilon^2\kappa$:

$$\begin{aligned}&-\varepsilon^2\kappa(\underline{v}-\varepsilon u^\beta)\cdot\nabla_x((\nabla_x u^\beta):\mathfrak{A}\sqrt{\mu})+\varepsilon^2\kappa(\underline{v}-\varepsilon u^\beta)\cdot\nabla_x(\tilde{p}^\beta\mu)+(2.24)\\&+\varepsilon^2\kappa(\partial_t+u^\beta\cdot\nabla_x)(\tilde{u}^\beta\cdot(\underline{v}-\varepsilon u^\beta)\mu)+2\varepsilon^2\kappa\Gamma(\tilde{u}^\beta\cdot(\underline{v}-\varepsilon u^\beta)\sqrt{\mu},(\nabla_x u^\beta):\mathfrak{A})\sqrt{\mu}\end{aligned}$$

(2.26)

$$=\varepsilon^2\kappa\{-\eta_0\Delta_x u^\beta+\nabla_x\tilde{p}^\beta+(\partial_t+u^\beta\cdot\nabla_x)\tilde{u}^\beta\}\cdot(\underline{v}-\varepsilon u^\beta)\mu$$

(2.27)

$$+\varepsilon^2\kappa\left(-\sum_{i,j}\tilde{u}^\beta_j\partial_{x_i}u^\beta_j(\underline{v}-\varepsilon u^\beta)_i+\sum_{i,j,k,\ell}\tilde{u}^\beta_j\partial_{x_i}u^\beta_k(\delta_{ij}\delta_{k\ell}+\delta_{ik}\delta_{j\ell}+\delta_{jk}\delta_{i\ell})(\underline{v}-\varepsilon u^\beta)_\ell\right)\mu$$

$$+\varepsilon^2\kappa\left(2\Gamma(\tilde{u}^\beta\cdot(\underline{v}-\varepsilon u^\beta)\sqrt{\mu},(\nabla_x u^\beta):\mathfrak{A})-(\nabla_x^2u^\beta):(\mathbf{I}-\mathbf{P})(\underline{v}-\varepsilon u^\beta)\mathfrak{A}\right)\sqrt{\mu}\tag{2.28}$$

$$+\varepsilon^2\kappa\sum_{i,j,k}\tilde{u}^\beta_j\partial_{x_i}u^\beta_k(\mathbf{I}-\mathbf{P})\left((\underline{v}-\varepsilon u^\beta)_i(\underline{v}-\varepsilon u^\beta)_j(\underline{v}-\varepsilon u^\beta)_k\sqrt{\mu}\right)\sqrt{\mu}\tag{2.29}$$

$$\begin{aligned}&+\varepsilon^2\kappa\Big(-(\nabla_x u^\beta):(\underline{v}-\varepsilon u^\beta)\cdot\nabla_x(\mathfrak{A}\sqrt{\mu})+\tilde{p}^\beta(\underline{v}-\varepsilon u^\beta)\cdot\nabla_x\mu+\tilde{u}^\beta\\&\quad\cdot(\partial_t+u^\beta\cdot\nabla_x)((\underline{v}-\varepsilon u^\beta)\mu)\Big)=(2.28)+(2.29)+(2.30).\end{aligned}\tag{2.30}$$

Here, we have used Lemma 8 and that (2.26) and (2.27) can be gathered to form

$$(2.26)+(2.27)=\varepsilon^2\kappa((\partial_t+u^\beta\cdot\nabla_x)\tilde{u}^\beta+\tilde{u}^\beta\cdot\nabla_x u^\beta-\eta_0\Delta_x u^\beta+\nabla_x\tilde{p}^\beta)\cdot(\underline{v}-\varepsilon u^\beta)\mu=0.$$

Note that $\frac{1}{\sqrt{\mu}}((2.28)+(2.29))\in\mathcal{N}^\perp$; that is, it is nonhydrodynamic, so small in our scales, and (2.30) is small—in fact, it is of order $\varepsilon^3\kappa$.

2.2.5. Small, nonnecessarily nonhydrodynamic remainders. The remaining terms are small in our scales; the following gathers all remaining terms.

$$\begin{aligned}\varepsilon^3\sqrt{\mu}\mathfrak{R}_1=\;&(2.25)+(2.30)+\varepsilon^3(\partial_t+u^\beta\cdot\nabla_x)(p^\beta\mu)\\&+\varepsilon^3\kappa(\partial_t+u^\beta\cdot\nabla_x)(-(\nabla_x u^\beta):\mathfrak{A}\sqrt{\mu}+\tilde{p}^\beta\mu)\\&-\varepsilon^3p^\beta(L(\nabla_x u^\beta):\mathfrak{A})\sqrt{\mu}-\varepsilon^3\kappa\tilde{p}^\beta(L(\nabla_x u^\beta):\mathfrak{A})\sqrt{\mu}\\&-\varepsilon^3\kappa\Gamma((\nabla_x u^\beta):\mathfrak{A},(\nabla_x u^\beta):\mathfrak{A})\sqrt{\mu}.\end{aligned}\tag{2.31}$$

One can easily observe the following.

Proposition 1. *Suppose that* (2.3) *holds.* $\mathfrak{R}_1$ *consists of a linear combination of the terms in the following tensor product:*

$$\begin{pmatrix}1\\\kappa\\\varepsilon\\\varepsilon\kappa\end{pmatrix}\otimes\begin{pmatrix}1\\p^\beta\\\nabla_x u^\beta\\\tilde{p}^\beta\\\tilde{u}^\beta\\\tilde{u}^\beta\otimes u^\beta\\u^\beta\end{pmatrix}\otimes D\begin{pmatrix}p^\beta\\u^\beta\\\nabla_x u^\beta\\\tilde{p}^\beta\end{pmatrix}\otimes\mathfrak{P}^{\le 2}((\underline{v}-\varepsilon u^\beta))\begin{pmatrix}\sqrt{\mu}\\\frac{1}{\varepsilon}\partial_t\mathfrak{A}\\\frac{1}{\varepsilon}\partial\mathfrak{A}\\L\mathfrak{A}\\\Gamma(\mathfrak{A},\mathfrak{A})\end{pmatrix},$$

where D is either ∂_t or ∂, which is applied to $p^\beta,u^\beta,\nabla_x u^\beta,\tilde{p}^\beta$, and $\mathfrak{P}^{\le 2}$ is a polynomial of degree ≤ 2 of its arguments. In particular, for $\varrho\in(0,\frac14)$ and $s\le 2$, we have the following pointwise estimate:

$$|\partial^s\mathfrak{R}_1|\lesssim V(\beta)e^{-\varrho|v-\varepsilon u^\beta|^2}.\tag{2.32}$$

2.2.6. Small nonhydrodynamic remainders. Equations (2.23), (2.28), and (2.29) are nonhydrodynamic remainders. We group them to obtain the following proposition.

Proposition 2. *Suppose that* (2.3) *holds. Let* $\mathfrak{R}_2$ *be defined by*

$$\varepsilon\kappa\sqrt{\mu}\mathfrak{R}_2=(2.23)+(2.28)+(2.29).\tag{2.33}$$

Then, $\mathfrak{R}_2$ *consists of a linear combination of the terms in the following tensor product:*

$$\begin{pmatrix}1\\\varepsilon\end{pmatrix}\otimes\begin{pmatrix}\nabla_x\tilde{u}^\beta\\\tilde{u}^\beta\otimes\tilde{u}^\beta\\\tilde{u}^\beta\otimes\nabla_x u^\beta\\\nabla_x^2u^\beta\end{pmatrix}\otimes\begin{pmatrix}L\mathfrak{A}\\\Gamma((\underline{v}-\varepsilon u^\beta)\sqrt{\mu},(\underline{v}-\varepsilon u^\beta)\sqrt{\mu})\\\Gamma((\underline{v}-\varepsilon u^\beta)\sqrt{\mu},\mathfrak{A})\\(\mathbf{I}-\mathbf{P})(\underline{v}-\varepsilon u^\beta)\mathfrak{A}\\(\mathbf{I}-\mathbf{P})(\underline{v}-\varepsilon u^\beta)^{\otimes 3}\sqrt{\mu}.\end{pmatrix}.$$

In particular, $\mathfrak{R}_2\in\mathcal{N}^\perp$, and, for $\varrho\in(0,\frac14)$ and $s\le 2$, we have the following pointwise estimate:

$$\begin{aligned}|(\mathbf{I}-\mathbf{P})\partial^s\mathfrak{R}_2|&\lesssim V(\beta)e^{-\varrho|v-\varepsilon u^\beta|^2},\\|\mathbf{P}\partial^s\mathfrak{R}_2|&\lesssim\varepsilon V(\beta)e^{-\varrho|v-\varepsilon u^\beta|^2}.\end{aligned}\tag{2.34}$$

Proof. It suffices to show (2.34); we see that, if all ∂^s are applied to macroscopic quantities $\nabla_x \tilde{u}^\beta, \ldots, \nabla_x^2 u^\beta$, then the resulting term is still nonhydrodynamic. In that case, the first inequality of (2.34) applies. On the other hand, if some of ∂ are applied to microscopic quantities $g = L\mathfrak{A}, \ldots, (\mathbf{I}-\mathbf{P})(\underline{v}-\varepsilon u^\beta)^{\otimes^3}\sqrt{\mu}$, we note that

$$\partial^{s'} g = \partial^{s'}(\mathbf{I}-\mathbf{P})g = (\mathbf{I}-\mathbf{P})\partial^{s'} g + [\![\mathbf{P}, \partial^{s'}]\!] g.$$

The first term belongs to $\mathcal{N}^\perp$, and the second term belongs to $\mathcal{N}$ and is bounded by $\varepsilon(1+\sum_{s''\le s'} \|\partial^{s''} u\|_{L^\infty})\|\partial^{s'-1} g\|_{L^\infty_x L^2_v} e^{-\varrho|v-\varepsilon u^\beta|^2}$. In both cases, (2.34) is valid. ☐

Also, we can collect terms in (2.21) except for μ and f_R by $\mathfrak{R}_3$.

PROPOSITION 3. *Suppose that* (2.3) *holds. Let* $\mathfrak{R}_3$ *be defined by*

$$\varepsilon\kappa\sqrt{\mu}\mathfrak{R}_3 = 2\varepsilon\kappa\tilde{u}^\beta \cdot (\underline{v}-\varepsilon u^\beta)\mu + \varepsilon^2 p^\beta \mu - \varepsilon^2\kappa(\nabla_x u^\beta) : \mathfrak{A}\sqrt{\mu} + \varepsilon^2\kappa\tilde{p}^\beta\mu. \tag{2.35}$$

Then, for $\varrho \in (0, \frac{1}{4})$ *and* $s \le 2$, *we have the following pointwise estimate:*

$$|\partial^s \mathfrak{R}_3| \lesssim V(\beta) e^{-\varrho|v-\varepsilon u^\beta|^2}. \tag{2.36}$$

2.3. Remainder equation and its derivatives. We have simplified (2.17), (2.18), (2.19), (2.20), and (2.21) so far. Finally, by dividing (2.17), (2.18), (2.19), (2.20), and (2.21) by $\varepsilon^2\sqrt{\mu}$, we obtain

$$\begin{aligned} &\partial_t f_R + \frac{v}{\varepsilon}\cdot\nabla_x f_R + f_R\left(\frac{(\partial_t + \frac{v}{\varepsilon}\cdot\nabla_x)\sqrt{\mu}}{\sqrt{\mu}}\right) + \frac{1}{\varepsilon^2\kappa} L f_R \\ &\quad = \frac{1}{\varepsilon\kappa}\Gamma(f_R, f_R) + \frac{1}{\varepsilon}\Gamma(\mathfrak{R}_3, f_R) - \varepsilon\mathfrak{R}_1 - \frac{\kappa}{\varepsilon}\mathfrak{R}_2, \end{aligned} \tag{2.37}$$

where $\mathfrak{R}_1, \mathfrak{R}_2$, and $\mathfrak{R}_3$ are defined by (2.31), (2.33), and (2.35), respectively.

Also, we have the equation for $\partial^s f$, for $s \le 2$; by Lemma 9,

$$\begin{aligned} &\partial_t\partial^s f_R + \frac{v}{\varepsilon}\cdot\nabla_x\partial^s f_R + \partial^s f_R\left(\frac{(\partial_t + \frac{v}{\varepsilon}\cdot\nabla_x)\sqrt{\mu}}{\sqrt{\mu}}\right) + \frac{1}{\varepsilon^2\kappa} L\partial^s f_R \\ &\quad = -\sum_{s'<s}\partial^{s'} f_R \frac{1}{2}\sum_{i,j}(\partial^{s-s'}\partial_{x_i} u_j^\beta)\varphi_i\varphi_j + R_s \\ &\qquad + \frac{1}{\varepsilon^2\kappa}[\![\partial^s, L]\!] f_R + \frac{1}{\varepsilon\kappa}\partial^s\Gamma(f_R, f_R) + \frac{1}{\varepsilon}\partial^s\Gamma(\mathfrak{R}_3, f_R) \\ &\qquad - \varepsilon\partial^s\mathfrak{R}_1 - \frac{\kappa}{\varepsilon}(\mathbf{I}-\mathbf{P})\partial^s\mathfrak{R}_2 - \frac{\kappa}{\varepsilon}\mathbf{P}\partial^s\mathfrak{R}_2, \end{aligned} \tag{2.38}$$

where $|R_s| \lesssim \varepsilon V(\beta)\nu(v)\sum_{s'<s}|\partial^{s'} f|$.

2.4. Scaled L^∞-estimate. In this section, we prove a pointwise estimate (with a weight (2.43)) of an L^p solution of the linear Boltzmann equation with a force term. We consider the following transport equation with (2.40) term:

$$[\partial_t + \varepsilon^{-1}\underline{v}\cdot\nabla_x] f + \frac{1}{\varepsilon^2\kappa} L f - \frac{\left(\partial_t + \frac{1}{\varepsilon}\underline{v}\cdot\nabla_x\right)\sqrt{\mu}}{\sqrt{\mu}} f_R = \tilde{H} \text{ in } [0,T]\times\mathbb{T}^2\times\mathbb{R}^3. \tag{2.39}$$

Also, we have an issue of momentum stream; the remainder equation (2.37) in our case contains the term

$$\frac{\left(\partial_t + \frac{1}{\varepsilon}\underline{v}\cdot\nabla_x\right)\sqrt{\mu}}{\sqrt{\mu}} f_R, \tag{2.40}$$

which cannot be controlled by f_R for large v. This term precisely comes from the one that we expand around the local Maxwellian, not the global one. In [35], a weight function of the form

$$w(x,v) := \exp(\vartheta|v|^2 - Z(x)\cdot v), \tag{2.41}$$

where $Z(x)$ is a suitable vector field, was introduced to bound the (2.40) term in the expansion around the local Maxwellian

$$w\left(\partial_t + \frac{1}{\varepsilon}\underline{v}\cdot\nabla_x\right) f_R = \left(\partial_t + \frac{1}{\varepsilon}\underline{v}\cdot\nabla_x\right)(wf_R) + \frac{1}{\varepsilon}\left(\underline{v}\cdot\nabla_x Z(x)\cdot\underline{v}\right) wf_R, \tag{2.42}$$

and if $Z(x)$ is chosen so that $\underline{v}\cdot\nabla_x Z(x)\cdot\underline{v} > 0$ for any v ($Z(x) = z(x)x$ for a suitably chosen function $z(x)$ works), one may control the most problematic term in (2.40): $(\nabla_x u^\beta : \underline{v}\otimes\underline{v})wf_R$.

Inspired by this, we introduce a suitable weight function, which is appropriate for the periodic domain. Unlike the whole Euclidean space, the existence of such $Z(x)$ in $\mathbb{T}^2$ is less obvious; in fact, if $Z = (Z_1, Z_2)$ is smooth, then, since $\int_{\mathbb{T}^1}\partial_1 Z_1(x_1,x_2)dx_1 = 0$, $\partial_1 Z_1$ will have a mixed sign along the circle $\mathbb{T}^1\times\{x_2\}$ for each $x_2\in\mathbb{T}^1$ unless it is 0 over whole circle. Thus, $\nabla_x Z + (\nabla_x Z)^T$ is neither positive definite nor negative definite over the whole domain $\mathbb{T}^2$.

To overcome this difficulty, we introduce a weight function that cancels the most problematic term of (2.40) instead of controlling it; we introduce

$$w(t,x,v) := \exp\left(\vartheta|v|^2 - \frac{1}{2}\varepsilon u^\beta(t,x)\cdot\underline{v}\right), \tag{2.43}$$

where $\vartheta\in(0,\frac{1}{4})$, under the assumption

$$\varepsilon|u^\beta(t,x)| = o(1). \tag{2.44}$$

In our scale regime, (2.3) and (2.44) hold.

Proposition 4. *For an arbitrary $T>0$, suppose that $f(t,x,v)$ is a distribution solution to* (2.39)*. Also, suppose that* (2.3) *holds.*

Then, for $w = e^{\vartheta|v|^2 - \frac{1}{2}\varepsilon u^\beta(t,x)\cdot\underline{v}}$ with $\vartheta\in(0,\frac{1}{4})$ in (2.43),

$$\begin{aligned}&\varepsilon\kappa\sup_{t\in[0,T]}\|wf(t)\|_{L^\infty(\mathbb{T}^2\times\mathbb{R}^3)}\\ &\quad\lesssim \varepsilon\kappa\|wf_0\|_{L^\infty(\mathbb{T}^2\times\mathbb{R}^3)} + \sup_{t\in[0,T]}\|f(s)\|_{L^2(\mathbb{T}^2\times\mathbb{R}^3)} + \varepsilon^3\kappa^2\sup_{t\in[0,T]}\|\nu^{-1}w\tilde{H}\|_{L^\infty(\mathbb{T}^2\times\mathbb{R}^3)}.\end{aligned} \tag{2.45}$$

The proof is based on the Duhamel formula (2.56) along the trajectory with scaled variables and the L^p-L^∞ interpolation argument based on the change of variable.

Let, with w of (2.43),

$$h = wf. \tag{2.46}$$

From (2.39), we can write the evolution equation of h:

$$
\begin{aligned}
\left[\partial_t + \frac{1}{\varepsilon}\underline{v}\cdot\nabla_x\right] h &= w\left[\partial_t + \frac{1}{\varepsilon}\underline{v}\cdot\nabla_x\right] f + f\left[\partial_t + \frac{1}{\varepsilon}\underline{v}\cdot\nabla_x\right] w \\
&= -\frac{1}{\varepsilon^2\kappa} wLf + \frac{\left[\partial_t + \frac{1}{\varepsilon}\underline{v}\cdot\nabla_x\right]\sqrt{\mu}}{\sqrt{\mu}} h + w\tilde{H} + h\left[\partial_t - \frac{1}{\varepsilon}\underline{v}\cdot\nabla_x\right]\frac{1}{2}\varepsilon u^\beta\cdot\underline{v} \\
&= -\frac{1}{\varepsilon^2\kappa} wLf + w\tilde{H} \\
&\quad + h\left(-\frac{1}{2}(\underline{v}-\varepsilon u^\beta)\cdot[\partial_t + \frac{1}{\varepsilon}\underline{v}\cdot\nabla_x](-\varepsilon u^\beta) - \frac{1}{2}\left[\partial_t + \frac{1}{\varepsilon}\underline{v}\cdot\nabla_x\right]\varepsilon u^\beta\cdot\underline{v}\right) \\
&= -\frac{1}{\varepsilon^2\kappa} wL\left(\frac{h}{w}\right) - h\left(\frac{\varepsilon^2}{2}u^\beta\cdot\partial_t u^\beta + \frac{\varepsilon}{2}\underline{v}\cdot(\nabla_x u^\beta)\cdot u^\beta\right) + w\tilde{H}.
\end{aligned} \tag{2.47}
$$

Next, we recall that $Lf = \nu f - Kf$ from (2.7). From the explicit form of ν in (2.7), we have a positive constant $\nu_0 > 0$ such that

$$\nu_0(|v-\varepsilon u^\beta|+1) \le \nu(v) \le 2\nu_0(|v-\varepsilon u^\beta|+1). \tag{2.48}$$

In particular, (2.48) and (2.3) imply that

$$\tilde{\nu}(t,x,v) := \nu(t,x,v) + \frac{\varepsilon^4\kappa}{2}u^\beta\cdot\partial_t u^\beta + \frac{\varepsilon^3\kappa}{2}\underline{v}\cdot(\nabla_x u^\beta)\cdot u^\beta \tag{2.49}$$

satisfies

$$\frac{1}{2}\nu_0(|v|+1) \le \tilde{\nu}(t,x,v) \le \frac{5}{2}\nu_0(|v|+1). \tag{2.50}$$

With $\tilde{\nu}$, we can write the evolution equation for h:

$$\left(\partial_t + \frac{1}{\varepsilon}\underline{v}\cdot\nabla_x\right) h + \frac{1}{\varepsilon^2\kappa}\tilde{\nu} h = \frac{1}{\varepsilon^2\kappa} wK\frac{h}{w} + w\tilde{H}. \tag{2.51}$$

Let $K_w h(v) = \int_{\mathbb{R}^3}\mathbf{k}_w(v,v_*)h(v_*)\mathrm{d}v_*$ with $\mathbf{k}_w(v,v_*) := \mathbf{k}(v,v_*)\frac{w(v)}{w(v_*)}$. Then,

$$w(v)K\frac{h}{w}(v) = \int_{\mathbb{R}^3}\mathbf{k}(v,v_*)\frac{w(v)}{w(v_*)}h(v_*)\mathrm{d}v_* = K_w h(v). \tag{2.52}$$

We will need the following estimate for $\mathbf{k}_w$.

Lemma 13 (Lemma 2 of [35]; also [23]). *Suppose that* (2.44) *holds. For* $w = e^{\vartheta|v|^2 - \frac{1}{2}\varepsilon u^\beta\cdot v}$ *with* $\vartheta\in(0,\frac{1}{4})$, *there exists* $C_\vartheta > 0$ *such that*

$$\mathbf{k}_w(v,v_*) \lesssim \frac{1}{|v-v_*|}e^{-C_\vartheta\frac{|v-v_*|^2}{2}} =: \mathbf{k}^\vartheta(v-v_*), \tag{2.53}$$

$$\int_{\mathbb{R}^3}(1+|v-v_*|)\mathbf{k}_w(v,v_*)\mathrm{d}v_* \lesssim \frac{1}{\nu(v)} \lesssim \frac{1}{1+|v|}, \tag{2.54}$$

$$\int_{\mathbb{R}^3}\frac{1}{|v-v_*|}\mathbf{k}_w(v,v_*)\mathrm{d}v_* \lesssim \frac{1}{\nu(v)} \lesssim 1. \tag{2.55}$$

Note that $\mathbf{k}^\vartheta \in L^1(\mathbb{R}^3)$.

We solve (2.51) along the characteristics

(2.56)

$$
\begin{aligned}
h(t,x,v) &= h_0(Y(0;t,x,\underline{v}),v)\exp\left(-\int_0^t \frac{\tilde{\nu}(\tau,Y(\tau;t,x,\underline{v}),v)}{\varepsilon^2\kappa}\mathrm{d}\tau\right)\\
&+\int_0^t \frac{e^{-\int_s^t \frac{\tilde{\nu}(\tau,Y(\tau;t,x,\underline{v}),v)\mathrm{d}\tau}{\varepsilon^2\kappa}}}{\varepsilon^2\kappa}\int_{\mathbb{R}^3}\mathbf{k}_w(s,Y(s;t,x,\underline{v}),v,v_*)h(s,Y(s;t,x,\underline{v}),v_*)\mathrm{d}v_*\mathrm{d}s\\
&+\int_0^t e^{-\int_s^t \frac{\tilde{\nu}(\tau,Y(\tau;t,x,\underline{v}),v)\mathrm{d}\tau}{\varepsilon^2\kappa}}(w\tilde{H})(s,Y(s;t,x,\underline{v}),v)\mathrm{d}s.
\end{aligned}
$$

Proof of Proposition 4. We again apply (2.56) to the second term on the right-hand side of (2.56):

$$
\begin{aligned}
h(t,x,v) =\ & h_0(Y(0;t,x,\underline{v}),v)\exp\left(-\int_0^t \frac{\tilde{\nu}(\tau,Y(\tau;t,x,\underline{v}),v)}{\varepsilon^2\kappa}\mathrm{d}\tau\right)\\
&+\int_0^t e^{-\int_s^t \frac{\tilde{\nu}(\tau,Y(\tau;t,x,\underline{v}),v)\mathrm{d}\tau}{\varepsilon^2\kappa}}(wH)(s,Y(s;t,x,\underline{v}),v)\mathrm{d}s\\
&+\int_0^t \frac{e^{-\int_s^t \frac{\tilde{\nu}(\tau,Y(\tau;t,x,\underline{v}),v)\mathrm{d}\tau}{\varepsilon^2\kappa}}}{\varepsilon^2\kappa}\int_{\mathbb{R}^3}\mathbf{k}_w(s,Y(s;t,x,\underline{v}),v,v_*)\\
&\times h_0(Y(0;s,Y(s;t,x,\underline{v}),\underline{v}_*),v_*)e^{-\int_0^s \frac{\tilde{\nu}(\tau',Y(\tau';s,Y(s;t,x,\underline{v}),\underline{v}_*),v_*)}{\varepsilon^2\kappa}\mathrm{d}\tau'}\mathrm{d}v_*\mathrm{d}s\\
&+\int_0^t \frac{e^{-\int_s^t \frac{\tilde{\nu}(\tau,Y(\tau;t,x,\underline{v}),v)\mathrm{d}\tau}{\varepsilon^2\kappa}}}{\varepsilon^2\kappa}\int_{\mathbb{R}^3}\mathbf{k}_w(s,Y(s;t,x,\underline{v}),v,v_*)\\
&\times\int_0^s e^{-\int_\tau^s \frac{\tilde{\nu}(\tau',Y(\tau';s,Y(s;t,x,\underline{v}),\underline{v}_*),v_*)\mathrm{d}\tau'}{\varepsilon^2\kappa}}(wH)(\tau,Y(\tau;s,Y(s;t,x,\underline{v}),\underline{v}_*),v_*)\mathrm{d}\tau\mathrm{d}v_*\mathrm{d}s\\
&+\int_0^t \frac{e^{-\int_s^t \frac{\tilde{\nu}(\tau,Y(\tau;t,x,\underline{v}),v)\mathrm{d}\tau}{\varepsilon^2\kappa}}}{\varepsilon^2\kappa}\int_{\mathbb{R}^3}\mathbf{k}_w(s,Y(s;t,x,\underline{v}),v,v_*)\\
&\times\int_0^s \frac{e^{-\int_\tau^s \frac{\tilde{\nu}(\tau',Y(\tau';s,Y(s;t,x,\underline{v}),\underline{v}_*),v_*)\mathrm{d}\tau'}{\varepsilon^2\kappa}}}{\varepsilon^2\kappa}\int_{\mathbb{R}^3}\mathbf{k}_w(\tau,Y(\tau;s,Y(s;t,x,\underline{v}),\underline{v}_*),v_*,v_{**})\\
&\times h(\tau,Y(\tau;s,Y(s;t,x,\underline{v}),\underline{v}_*),v_{**})\mathrm{d}v_{**}\mathrm{d}\tau\mathrm{d}v_*\mathrm{d}s\\
=:\ & I_1+I_2+I_3+I_4+I_K.
\end{aligned}
$$

First, we control $I_0 := I_1 + I_3$, the contribution from the initial data. We easily notice from (2.50) and (2.54) that

$$
\begin{aligned}
|I_1| &\le \|h_0\|_{L^\infty(\mathbb{T}^2\times\mathbb{R}^3)}e^{-\frac{\nu_0(|v|+1)t}{2\varepsilon^2\kappa}} \le \|h_0\|_{L^\infty(\mathbb{T}^2\times\mathbb{R}^3)},\\
|I_3| &\le \int_0^t \frac{e^{-\frac{\nu_0(|v|+1)(t-s)}{2\varepsilon^2\kappa}}}{\varepsilon^2\kappa}\int_{\mathbb{R}^3}\mathbf{k}_w(v,v_*)e^{-\frac{\nu_0(|v_*|+1)s}{2\varepsilon^2\kappa}}\|h_0\|_{L^\infty(\mathbb{T}^2\times\mathbb{R}^3)}\mathrm{d}v_*\mathrm{d}s \lesssim \|h_0\|_{L^\infty(\mathbb{T}^2\times\mathbb{R}^3)}.
\end{aligned}
$$

In the second inequality, the dependence of $\mathbf{k}_w$ on the t,x variables is omitted because the bound is uniform on them.

Next, we control $I_H := I_2 + I_4$, the contribution from source H. Again, from (2.50) and (2.54), we have

$$
\begin{aligned}
|I_2| &\le \int_0^t e^{-\frac{\nu_0(|v|+1)(t-s)}{2\varepsilon^2\kappa}}|wH(v)|\mathrm{d}s \lesssim \varepsilon^2\kappa\|\nu^{-1}wH\|_{L^\infty([0,T]\times\mathbb{T}^2\times\mathbb{R}^3)},\\
|I_4| &\le \int_0^t \frac{e^{-\frac{\nu_0(|v|+1)(t-s)}{2\varepsilon^2\kappa}}}{\varepsilon^2\kappa}\int_{\mathbb{R}^3}\mathbf{k}_w(v,v_*)\int_0^s e^{-\frac{\nu_0(|v_*|+1)(s-\tau)}{2\varepsilon^2\kappa}}|wH(v_*)|\mathrm{d}\tau\mathrm{d}v_*\mathrm{d}s\\
&\lesssim \varepsilon^2\kappa\|\nu^{-1}wH\|_{L^\infty([0,T]\times\mathbb{T}^2\times\mathbb{R}^3)}.
\end{aligned}
$$

Finally, we control I_K. The idea is the following: We decompose the time interval $[0,s]$ into $[0, s-\varepsilon^2\kappa o(1)]$ and $[s-\varepsilon^2\kappa o(1), s]$; the first integral is controlled using the change of variables $\underline{v}_* \to Y(\tau; s, Y(s;t,x,\underline{v}), \underline{v}_*)$, and thus, we can rewrite the integral of h with respect to v_*, v_{**} variables into the space-time integral of f. For that reason, we plugged (2.56) into itself. Also, the splitting of the time gives control of the Jacobian factor obtained from change of variables. On the other hand, the second term is controlled by the fact that it is a short time integral; this gives smallness, and thus, we can bound the integral with $o(1)\|h\|_{L^\infty([0,T]\times\mathbb{T}^2\times\mathbb{R}^3)}$.

For this purpose, we introduce a small positive number $\eta > 0$, which is to be determined. Using (2.50) and (2.54), we have the following:

$$\begin{aligned}
|I_K| &\le \int_0^t \frac{e^{-\frac{\nu_0(|v|+1)(t-s)}{2\varepsilon^2\kappa}}}{\varepsilon^2\kappa}\int_0^s \frac{e^{-\frac{\nu_0(|v_*|+1)(s-\tau)}{2\varepsilon^2\kappa}}}{\varepsilon^2\kappa}\\
&\qquad\times\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\mathbf{k}^\vartheta(v-v_*)\mathbf{k}^\vartheta(v_*-v_{**})|h(\tau, Y(\tau;s,Y(s;t,x,\underline{v}),\underline{v}_*),v_{**})|\mathrm{d}v_{**}\mathrm{d}v_*\mathrm{d}\tau\mathrm{d}s\\
&= \int_0^t \frac{e^{-\frac{\nu_0(|v|+1)(t-s)}{2\varepsilon^2\kappa}}}{\varepsilon^2\kappa}\int_0^{s-\varepsilon^2\kappa\eta} \frac{e^{-\frac{\nu_0(|v_*|+1)(s-\tau)}{2\varepsilon^2\kappa}}}{\varepsilon^2\kappa}\\
&\qquad\times\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\mathbf{k}^\vartheta(v-v_*)\mathbf{k}^\vartheta(v_*-v_{**})|h(\tau, Y(\tau;s,Y(s;t,x,\underline{v}),\underline{v}_*),v_{**})|\mathrm{d}v_{**}\mathrm{d}v_*\mathrm{d}\tau\mathrm{d}s\\
&\quad+\int_0^t \frac{e^{-\frac{\nu_0(|v|+1)(t-s)}{2\varepsilon^2\kappa}}}{\varepsilon^2\kappa}\int_{s-\varepsilon^2\kappa\eta}^{s} \frac{e^{-\frac{\nu_0(|v_*|+1)(s-\tau)}{2\varepsilon^2\kappa}}}{\varepsilon^2\kappa}\\
&\qquad\times\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\mathbf{k}^\vartheta(v-v_*)\mathbf{k}^\vartheta(v_*-v_{**})|h(\tau, Y(\tau;s,Y(s;t,x,\underline{v}),\underline{v}_*),v_{**})|\mathrm{d}v_{**}\mathrm{d}v_*\mathrm{d}\tau\mathrm{d}s\\
&=: I_{5,1}+I_{5,2}.
\end{aligned}$$

We first bound $I_{5,2}$. From the integrability of $\mathbf{k}^\vartheta$, we have

$$I_{5,2} \le \int_0^t \frac{e^{-\frac{\nu_0(|v|+1)(t-s)}{2\varepsilon^2\kappa}}}{\varepsilon^2\kappa}\mathrm{d}s\frac{\varepsilon^2\kappa\eta}{\varepsilon^2\kappa}\|\mathbf{k}^\vartheta\|^2_{L^1(\mathbb{R}^3)}\|h\|_{L^\infty([0,T]\times\mathbb{T}^2\times\mathbb{R}^3)} \lesssim \eta\|h\|_{L^\infty([0,T]\times\mathbb{T}^2\times\mathbb{R}^3)}.$$

Next, to treat $I_{5,1}$, we introduce the following decomposition of $\mathbf{k}^\vartheta(v-v_*)$; for a given $N>0$,

$$\begin{aligned}
&\mathbf{k}^\vartheta(v-v_*) = \mathbf{k}^\vartheta_N(v,v_*) + \mathbf{k}^\vartheta_R(v,v_*), \text{ where}\\
&\mathbf{k}^\vartheta_N(v,v_*) = \mathbf{k}^\vartheta(v-v_*)\mathbf{1}_{B_N(0)\setminus B_{\frac{1}{N}}(0)}(v-v_*)\mathbf{1}_{B_N(0)}(v_*) \text{ and}\\
&\mathbf{k}^\vartheta_R(v,v_*) = \mathbf{k}^\vartheta(v-v_*) - \mathbf{k}^\vartheta_N(v,v_*).
\end{aligned}$$

With this decomposition, we can split $I_{5,1}$ by

$$\begin{aligned}
I_{5,1} &= \int_0^t \frac{e^{-\frac{\nu_0(|v|+1)(t-s)}{2\varepsilon^2\kappa}}}{\varepsilon^2\kappa}\int_0^{s-\varepsilon^2\kappa\eta} \frac{e^{-\frac{\nu_0(|v_*|+1)(s-\tau)}{2\varepsilon^2\kappa}}}{\varepsilon^2\kappa}\\
&\quad\times\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\mathbf{k}^\vartheta_N(v,v_*)\mathbf{k}^\vartheta_N(v_*,v_{**})|h(\tau, Y(\tau;s,Y(s;t,x,\underline{v}),\underline{v}_*),v_{**})|\mathrm{d}v_{**}\mathrm{d}v_*\mathrm{d}\tau\mathrm{d}s\\
&\quad+\int_0^t \frac{e^{-\frac{\nu_0(|v|+1)(t-s)}{2\varepsilon^2\kappa}}}{\varepsilon^2\kappa}\int_0^{s-\varepsilon^2\kappa\eta} \frac{e^{-\frac{\nu_0(|v_*|+1)(s-\tau)}{2\varepsilon^2\kappa}}}{\varepsilon^2\kappa}
\end{aligned}$$

$$
\begin{aligned}
&\times \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathbf{k}_N^{\vartheta}(v,v_*)\mathbf{k}_R^{\vartheta}(v_*,v_{**})|h(\tau, Y(\tau; s, Y(s;t,x,\underline{v}),\underline{v}_*),v_{**})|\mathrm{d}v_{**}\mathrm{d}v_*\mathrm{d}\tau\mathrm{d}s\\
&+\int_0^t \frac{e^{-\frac{\nu_0(|v|+1)(t-s)}{2\varepsilon^2\kappa}}}{\varepsilon^2\kappa}\int_0^{s-\varepsilon^2\kappa\eta}\frac{e^{-\frac{\nu_0(|v_*|+1)(s-\tau)}{2\varepsilon^2\kappa}}}{\varepsilon^2\kappa}\\
&\times \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathbf{k}_R^{\vartheta}(v,v_*)\mathbf{k}_N^{\vartheta}(v_*,v_{**})|h(\tau, Y(\tau; s, Y(s;t,x,\underline{v}),\underline{v}_*),v_{**})|\mathrm{d}v_{**}\mathrm{d}v_*\mathrm{d}\tau\mathrm{d}s\\
&+\int_0^t \frac{e^{-\frac{\nu_0(|v|+1)(t-s)}{2\varepsilon^2\kappa}}}{\varepsilon^2\kappa}\int_0^{s-\varepsilon^2\kappa\eta}\frac{e^{-\frac{\nu_0(|v_*|+1)(s-\tau)}{2\varepsilon^2\kappa}}}{\varepsilon^2\kappa}\\
&\times \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathbf{k}_R^{\vartheta}(v,v_*)\mathbf{k}_R^{\vartheta}(v_*,v_{**})|h(\tau, Y(\tau; s, Y(s;t,x,\underline{v}),\underline{v}_*),v_{**})|\mathrm{d}v_{**}\mathrm{d}v_*\mathrm{d}\tau\mathrm{d}s\\
&=: I_{5,1}^{NN}+I_{5,1}^{NR}+I_{5,1}^{RN}+I_{5,1}^{RR}.
\end{aligned}
$$

Since $\int_{\mathbb{R}^3}\mathbf{k}_N^{\vartheta}(v,v_*)\mathrm{d}v_* \uparrow \|\mathbf{k}^{\vartheta}\|_{L^1(\mathbb{R}^3)}$ as $N\to\infty$, and thus, $A_N := \int_{\mathbb{R}^3}\mathbf{k}_R^{\vartheta}(v,v_*)\mathrm{d}v_* \to 0$ as $N\to\infty$ by the monotone convergence theorem, we have

$$
\begin{aligned}
I_{5,1}^{NR} &\lesssim A_N\|\mathbf{k}^{\vartheta}\|_{L^1(\mathbb{R}^3)}\|h\|_{L^\infty([0,T]\times\mathbb{T}^2\times\mathbb{R}^3)},\\
I_{5,1}^{RN} &\lesssim A_N\|\mathbf{k}^{\vartheta}\|_{L^1(\mathbb{R}^3)}\|h\|_{L^\infty([0,T]\times\mathbb{T}^2\times\mathbb{R}^3)},\\
I_{5,1}^{RR} &\lesssim A_N^2\|h\|_{L^\infty([0,T]\times\mathbb{T}^2\times\mathbb{R}^3)}.
\end{aligned}
$$

Finally, we estimate $I_{5,1}^{NN}$. First, we recall that $\mathbf{k}_N^{\vartheta}(v,v_*)$ is supported on $\{\frac{1}{N} < |v-v_*| < N\}$ and therefore is bounded by some constant C_N. Thus, we have

$$
\begin{aligned}
\mathbf{k}_N^{\vartheta}(v,v_*) &\le C_N\mathbf{1}_{B_N(0)}(v_*),\\
\mathbf{k}_N^{\vartheta}(v_*,v_{**}) &\le C_N\mathbf{1}_{B_N(0)}(v_{**}).
\end{aligned}
$$

Next, we expand $|h(\tau, Y(\tau; s, Y(s;t,x,\underline{v}),\underline{v}_*),v_{**})|$; in support of $\mathbf{k}_N^{\vartheta}(v,v_*)\mathbf{k}_N^{\vartheta}(v_*,v_{**})$, $|v_*|, |v_{**}| < N$. Note that this implies that $|\underline{v}_*| < N$ and $|(v_*)_3| < N$, where $v_* = (\underline{v}_*, (v_*)_3)$. Together with (2.44), we have

$$
\begin{aligned}
|h(\tau, Y(\tau; s, Y(s;t,x,\underline{v}),\underline{v}_*),v_{**})| &= |wf(\tau, Y(\tau; s, Y(s;t,x,\underline{v}),\underline{v}_*),v_{**})|\\
&\le e^{\vartheta|v_{**}|^2+\frac12\varepsilon\|u^\beta\|_{L^\infty([0,T]\times\mathbb{T}^2)}|\underline{v_{**}}|}|f(\tau, Y(\tau; s, Y(s;t,x,\underline{v}),\underline{v}_*),v_{**})|\\
&\le C_N|f(\tau, Y(\tau; s, Y(s;t,x,\underline{v}),\underline{v}_*),v_{**})|.
\end{aligned}
$$

Also, we rewrite $Y(\tau; s, Y(s;t,x,\underline{v}),\underline{v}_*)$; we have

$$
Y(\tau; s, Y(s;t,x,\underline{v}),\underline{v}_*) = x - \frac{t-s}{\varepsilon}\underline{v} - \frac{s-\tau}{\varepsilon}\underline{v}_*/\mathbb{Z}^2 \in \mathbb{T}^2.
$$

Finally, we remark that, since $\tau\in[0, s-\varepsilon^2\kappa\eta]$, we have $\frac{s-\tau}{\varepsilon} > \varepsilon\kappa\eta$. Combining these all together, for $\tau\in[0,s-\varepsilon^2\kappa\eta]$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\mathbf{k}_N^{\vartheta}(v,v_*)\mathbf{k}_N^{\vartheta}(v_*,v_{**})|h(\tau, Y(\tau; s, Y(s;t,x,\underline{v}),\underline{v}_*),v_{**})|\mathrm{d}v_{**}\mathrm{d}v_*\\
&\lesssim_N \int_{|(v_*)_3|<N}\int_{|\underline{v}_*|<N}\int_{|v_{**}|<N}\left|f_R\left(\tau, x-\frac{t-s}{\varepsilon}\underline{v}-\frac{s-\tau}{\varepsilon}\underline{v}_*/\mathbb{Z}^2, v_{**}\right)\right|\mathrm{d}v_{**}\mathrm{d}v_*\\
&\lesssim_N \left(\int_{|\underline{v}_*|<N}\int_{\mathbb{R}^3}\left|f_R\left(\tau, x-\frac{t-s}{\varepsilon}\underline{v}-\frac{s-\tau}{\varepsilon}\underline{v}_*/\mathbb{Z}^2, v_{**}\right)\right|^2\mathrm{d}v_{**}\mathrm{d}v_*\right)^{\frac12},
\end{aligned}
$$

where we have used that the integrand is independent of $(v_*)_3$ and $\|\mathbf{1}_{\{|\underline{v}_*|<N\}\times\{|v_{**}|<N\}}\|_{L^2(\mathbb{R}^2\times\mathbb{R}^3)} \lesssim_N 1$.

Next, we apply the change of variables $\underline{v}_* \to y = x - \frac{t-s}{\varepsilon}\underline{v} - \frac{s-\tau}{\varepsilon}\underline{v}_* \in \mathbb{R}^2$. This map is one-to-one and maps $\underline{v}_* \in B_N(0)$ onto $y \in B_{\frac{s-\tau}{\varepsilon}N}(x - \frac{t-s}{\varepsilon}\underline{v})$ with $\mathrm{d}y = \left(\frac{s-\tau}{\varepsilon}\right)^2 \mathrm{d}\underline{v}_*$. Therefore, we have

(2.57)

$$\begin{aligned}
&\left(\int_{|\underline{v}_*|<N}\int_{\mathbb{R}^3}\left|f_R(\tau, x - \frac{t-s}{\varepsilon}\underline{v} - \frac{s-\tau}{\varepsilon}\underline{v}_*/\mathbb{Z}^2, v_{**})\right|^2 \mathrm{d}v_{**}\mathrm{d}v_*\right)^{\frac{1}{2}} \\
&\quad = \left(\int_{y\in B_{\frac{s-\tau}{\varepsilon}N}(x-\frac{t-s}{\varepsilon}\underline{v})}\int_{\mathbb{R}^3}\left|f_R(\tau, y/\mathbb{Z}^2, v_{**})\right|^2\left(\frac{\varepsilon}{s-\tau}\right)^2 \mathrm{d}v_{**}\mathrm{d}y\right)^{\frac{1}{2}} \\
&\quad = \left(\sum_{k\in\mathbb{Z}^2}\int_{y\in([-\frac{1}{2},\frac{1}{2}]^2+k)\cap B_{\frac{s-\tau}{\varepsilon}N}(x-\frac{t-s}{\varepsilon}\underline{v})}\int_{\mathbb{R}^3}|f_R(\tau, y-k, v_{**})|^2\left(\frac{\varepsilon}{s-\tau}\right)^2 \mathrm{d}v_{**}\mathrm{d}y\right)^{\frac{1}{2}} \\
&\quad = \left(\sum_{k\in\mathbb{Z}^2}\int_{z\in[-\frac{1}{2},\frac{1}{2}]^2\cap B_{\frac{s-\tau}{\varepsilon}N}(x-\frac{t-s}{\varepsilon}\underline{v}-k)}\int_{\mathbb{R}^3}|f_R(\tau, z, v_{**})|^2\left(\frac{\varepsilon}{s-\tau}\right)^2 \mathrm{d}v_{**}\mathrm{d}z\right)^{\frac{1}{2}},
\end{aligned}$$

where $z = y - k$ in each integral. Next, we count the number of $k \in \mathbb{Z}^2$ such that $\left[-\frac{1}{2}, \frac{1}{2}\right]^2 \cap B_{\frac{s-\tau}{\varepsilon}N}(x - \frac{t-s}{\varepsilon}\underline{v} - k) \neq \emptyset$. There are two cases: If $N\frac{s-\tau}{\varepsilon} \leq 1$, there are $O(1)$ such $k \in \mathbb{Z}^2$. If $N\frac{s-\tau}{\varepsilon} > 1$, there are $O((N\frac{s-\tau}{\varepsilon})^2)$ such $k \in \mathbb{Z}^2$. Therefore, we have

$$\begin{aligned}
(2.57) &\lesssim \left(\max\left(\left(\frac{\varepsilon}{s-\tau}\right)^2, N^2\right)\int_{\mathbb{T}^2}\int_{\mathbb{R}^3}|f_R(\tau, z, v_{**})|^2 \mathrm{d}v_{**}\mathrm{d}z\right)^{\frac{1}{2}} \\
&\lesssim_{N,\eta} \frac{1}{\varepsilon\kappa}\sup_{\tau\in[0,t]}\|f_R(\tau)\|_{L^2(\mathbb{T}^2\times\mathbb{R}^3)}.
\end{aligned}$$

Choosing N large enough and η small enough so that we can bury $I_{5,2}$, $I_{5,1}^{NR}, I_{5,1}^{RN}$, and $I_{5,1}^{RR}$ gives

$$\begin{aligned}
\|h\|_{L^\infty([0,T]\times\mathbb{T}^2\times\mathbb{R}^3)} &\lesssim \|h_0\|_{L^\infty(\mathbb{T}^2\times\mathbb{R}^3)} + \varepsilon^2\kappa\|\nu^{-1}wH\|_{L^\infty([0,T]\times\mathbb{T}^2\times\mathbb{R}^3)} \\
&\quad + \frac{1}{\varepsilon\kappa}\|f_R\|_{L^\infty([0,T];L^2(\mathbb{T}^2\times\mathbb{R}^3))},
\end{aligned}$$

which is the desired conclusion. ∎

2.5. Remainder estimate. To admit far-from-equilibrium initial data, we need to keep the characteristic size of remainder as large as possible. A heuristic calculation suggests that the size $o(\varepsilon\kappa)$ for the remainder is the threshold; if the remainder becomes of the size $O(\varepsilon\kappa)$, we lose control of the nonlinearity of the remainder equation. Thus, we aim to keep our characteristic size of the remainder slightly smaller than $\varepsilon\kappa$.

There is only very slight room for this; the only possible gain is the coercivity of the linearized Boltzmann operator L. However, many conventional techniques (averaging lemma, L^∞-estimates) do not rely on it; up to the authors' knowledge, the coercivity of L is exploited only in L^2_v estimates. If we rely on other techniques at too early a stage, we enormously lose the scale and fail to achieve the goal.

As a consequence, we need to push the L^2_v estimates as far as possible. The important observation made in [29] is that, even for the nonlinear term, we have control by the L^2-in-v integral of remainders since the nonlinear term is also expressed in terms of an integral with nicely decaying kernel; what is lacking is L^2 integrability in x. This observation naturally leads us to pursue the L^2_v-estimate for derivatives of the remainder and then rely on interpolation—H^2_x, but L^2_v estimate.

It turns out that this idea gives a sharper scaling than many conventional techniques; the commutator $[\![\partial^s, L]\!]$ between spatial derivatives and L forces us to lose $\sqrt{\kappa}$ scale for each derivative, but we do not lose scale in nonlinearity for the 2D domain. Thus, by setting the initial data decaying to 0 at an arbitrary slow rate as $\varepsilon \to 0$, we can keep the $L^2_x L^2_v$ norms of the remainder and its derivatives small provided that the source terms are also small, which is the main point of the next idea.

Furthermore, we note that $H^2_x L^2_v$ fits very well with our goal to see convergence in a stronger topology; because we can control up to second derivatives of the remainder small, we can keep our Boltzmann solution close to the local Maxwellian $M_{1,\varepsilon u^\beta,1}$. Its zeroth and first derivatives may converge; they correspond to the velocity and vorticity. Its second derivatives may blow in general, which represents the formation of a singular object, e.g., interfaces.

Now, we are ready to prove the compactness of f_R in a suitable topology, thereby proving convergence. For a fixed $T > 0$ and $t \in (0,T)$, we use the following scaled energy and its dissipation:

$$\begin{aligned} \mathcal{E}(t) &:= \sum_{s\le 2} \sup_{t'\in(0,t)} \|\kappa^{-1+\frac{s}{2}} \partial^s f_R(t')\|^2_{L^2_x L^2_v}, \\ \mathcal{D}(t) &:= \sum_{s\le 2} \|\varepsilon^{-1}\kappa^{-\frac{3}{2}+\frac{s}{2}} \nu^{\frac{1}{2}} (\mathbf{I}-\mathbf{P})\partial^s f_R\|^2_{L^2((0,t);L^2_x L^2_v))}. \end{aligned} \tag{2.58}$$

We also need the following auxilliary norm:

$$\mathcal{F}(t) := \varepsilon \sup_{t'\in(0,t)} \|f_R(t')\|_{L^\infty(\mathbb{T}^2\times\mathbb{R}^3)}. \tag{2.59}$$

Also, we will frequently use the following basic inequality:

$$\sum_{s\le 2} \|\kappa^{-1+\frac{s}{2}} \partial^s f_R\|^2_{L^2((0,t);L^2_x L^2_v)} = \int_0^t \mathcal{E} \lesssim_T \mathcal{E}(t).$$

The main theorem of this section is the following.

THEOREM 3. *Let $T > 0$. Suppose that $\delta_s = \delta_s(\varepsilon)$, $s = 0,1,2$ satisfy the following:*

$$\lim_{\varepsilon\to 0} \delta_0(\varepsilon)^2 \left(\|\nabla_x u^\beta\|^2_{L^\infty((0,T)\times\mathbb{T}^2)} + 2 \right) \exp\left(2\mathbf{C}_0 \left(\|\nabla_x u^\beta\|^2_{L^\infty((0,T)\times\mathbb{T}^2)} + 2 \right) T \right) = 0, \tag{2.60}$$

$$\delta_s(\varepsilon) < (\varepsilon^{-1}\kappa^{-1/2})^s, s = 1,2. \tag{2.61}$$

Suppose that $f_R(0)$ satisfies

$$\sqrt{\mathcal{E}(0)},\ \mathcal{F}(0) < \delta_0(\varepsilon),\ \|w_s \partial^s f_{R0}\|_{L^\infty(\mathbb{T}^2\times\mathbb{R}^3)} < \delta_s(\varepsilon), s = 1,2.$$

Then, (2.37) *with initial data $f_R(0)$ and $\tilde{u}^\beta(0) \equiv 0$ has a solution $f_R(t)$, $t \in (0,T)$ such that*

$$
\begin{aligned}
&\mathcal{E}(t)+\mathcal{D}(t)\\
&\quad\le(\delta_0^2+\kappa)(1+T)C(\mathbf{C}_0)\\
&\qquad\times\left(2\mathbf{C}_0\left(\|\nabla_x u^\beta\|^2_{L^\infty((0,T)\times\mathbb{T}^2)}+2\right)\exp\left(2\mathbf{C}_0\left(\|\nabla_x u^\beta\|^2_{L^\infty((0,T)\times\mathbb{T}^2)}+2\right)T\right)\right)
\end{aligned}
$$

and $\lim_{\varepsilon\to0}\sup_{t\in(0,T)}(\mathcal{E}(t)+\mathcal{F}(t))=0$.

2.5.1. Energy estimate. By taking the L^2 norm for (2.37) and (2.38) for $s\le2$ and integrating over time, we have

$$
\begin{aligned}
&\mathcal{E}(t)+\mathcal{D}(t)\lesssim\mathcal{E}(0)\\
&\quad+\|\nabla_x u^\beta\|_{L^\infty_{t,x}}\sum_{s\le2}\int_{(0,t)\times\mathbb{T}^2\times\mathbb{R}^3}\left|\frac{\partial^s f_R}{\kappa^{1-\frac{s}{2}}}\right|^2\nu^2\mathrm{d}v\mathrm{d}x\mathrm{d}t' \qquad (2.62)\\
&\quad+\sum_{s'<s}\kappa^{\frac{s-s'}{2}}V(\beta)\int_{(0,t)\times\mathbb{T}^2\times\mathbb{R}^3}\left|\frac{\partial^{s'} f_R}{\kappa^{1-\frac{s'}{2}}}\right|\left|\frac{\partial^s f_R}{\kappa^{1-\frac{s}{2}}}\right|\nu^2\mathrm{d}v\mathrm{d}x\mathrm{d}t'+\varepsilon V(\beta)(\mathcal{E}(t)+\mathcal{D}(t)) \qquad (2.63)\\
&\quad+\sum_{s\le2}\int_{(0,t)\times\mathbb{T}^2\times\mathbb{R}^3}\varepsilon^{-2}\kappa^{-2+\frac{s}{2}}[\![\partial^s,L]\!]f_R\frac{\partial^s f_R}{\kappa^{1-\frac{s}{2}}}\mathrm{d}v\mathrm{d}x\mathrm{d}t' \qquad (2.64)\\
&\quad+\sum_{s\le2}\int_{(0,t)\times\mathbb{T}^2\times\mathbb{R}^3}\varepsilon^{-1}\kappa^{-2+\frac{s}{2}}\partial^s\Gamma(f_R,f_R)\frac{\partial^s f_R}{\kappa^{1-\frac{s}{2}}}\mathrm{d}v\mathrm{d}x\mathrm{d}t' \qquad (2.65)\\
&\quad+\sum_{s\le2}\int_{(0,t)\times\mathbb{T}^2\times\mathbb{R}^3}\varepsilon^{-1}\kappa^{-1+\frac{s}{2}}\partial^s\Gamma(\mathfrak{R}_3,f_R)\frac{\partial^s f_R}{\kappa^{1-\frac{s}{2}}}\mathrm{d}v\mathrm{d}x\mathrm{d}t' \qquad (2.66)\\
&\quad-\sum_{s\le2}\int_{(0,t)\times\mathbb{T}^2\times\mathbb{R}^3}\kappa^{-1+\frac{s}{2}}\left(\varepsilon\partial^s\mathfrak{R}_1+\frac{\kappa}{\varepsilon}(\mathbf{I}-\mathbf{P})\partial^s\mathfrak{R}_2+\frac{\kappa}{\varepsilon}\partial^s\mathfrak{R}_2\right)\frac{\partial^s f_R}{\kappa^{1-\frac{s}{2}}}\mathrm{d}v\mathrm{d}x\mathrm{d}t'. \qquad (2.67)
\end{aligned}
$$

Step 1. *Control of* (2.67). From (2.32) and (2.34), we have

$$
\begin{aligned}
(2.67)&\lesssim\sum_{s\le2}\left(\varepsilon\kappa^{-1+\frac{s}{2}}V(\beta)\sqrt{\mathcal{E}(t)}+\kappa^{\frac{3}{2}}V(\beta)\sqrt{\mathcal{D}(t)}+\kappa V(\beta)\sqrt{\mathcal{E}(t)}\right)\\
&\lesssim\kappa^{\frac{1}{2}}\left(\sqrt{\mathcal{E}(t)}+\sqrt{\mathcal{D}(t)}\right)
\end{aligned}
\qquad (2.68)
$$

by (2.3).

Step 2. *Control of* (2.66). We note that

$$\partial^s\Gamma(\mathfrak{R}_3,f_R)=\sum_{s_1+s_2+s_3=s}\partial^{s_1}\Gamma(\partial^{s_2}\mathfrak{R}_3,\partial^{s_3}f_R).$$

There are two cases. First, if $s_1=0$, then

$$
\begin{aligned}
&\sum_{s_2+s_3=s}\int_{(0,t)\times\mathbb{T}^2\times\mathbb{R}^3}\varepsilon^{-1}\kappa^{-1+\frac{s}{2}}\Gamma(\partial^{s_2}\mathfrak{R}_3,\partial^{s_3}f_R)\frac{\partial^s f_R}{\kappa^{1-\frac{s}{2}}}\mathrm{d}v\mathrm{d}x\mathrm{d}t'\\
&\qquad\lesssim\sum_{s_3\le s}\kappa^{-\frac{1}{2}+\frac{s}{2}}V(\beta)\|\partial^{s_3}f_R\|_{L^2((0,t);L^2_xL^2_v)}\sqrt{\mathcal{D}(t)}\lesssim\kappa^{\frac{1}{2}}V(\beta)\sqrt{\mathcal{E}(t)}\sqrt{\mathcal{D}(t)}.
\end{aligned}
$$

If $s_1 \geq 1$, then, by Lemma 1, we have

$$\sum_{s_1+s_2+s_3=s} \int_{(0,t)\times\mathbb{T}^2\times\mathbb{R}^3} \varepsilon^{-1}\kappa^{-1+\frac{s}{2}} {}_{\partial^{s_1}}\Gamma(\partial^{s_2}\mathfrak{R}_3, \partial^{s_3} f_R)\frac{\partial^s f_R}{\kappa^{1-\frac{s}{2}}} \mathrm{d}v\mathrm{d}x\mathrm{d}t'$$
$$\lesssim \sum_{s_3<s} V(\beta)\kappa^{-1+\frac{s}{2}} \left(\|\partial^{s_3} f_R\|_{L^2((0,t);L^2_xL^2_v)} + \|\nu^{\frac12}(\mathbf{I}-\mathbf{P})\partial^{s_3} f_R\|_{L^2((0,T);L^2_xL^2_v)} \right)$$
$$\times \left(\sqrt{\mathcal{E}(t)} + \varepsilon\kappa^{\frac12}\sqrt{\mathcal{D}(t)}\right) \lesssim \kappa^{\frac12} V(\beta)(\mathcal{E}(t) + \varepsilon^2\kappa\mathcal{D}(t))$$

since $s_3 < s$. In conclusion, we have

$$(2.66) \lesssim \kappa^{\frac12} V(\beta)(\mathcal{E}(t) + \mathcal{D}(t)). \tag{2.69}$$

Step 3. *Control of* (2.64). For $s = 0$, $[\![\partial^s, L]\!] = 0$. When $s = 1$, $[\![\partial^s, L]\!] f_R$ consists of type 1 and type 2 terms in Lemma 2. When $s = 2$, there is exactly one term in $[\![\partial^s, L]\!] f_R$ that is of type 3 in Lemma 2: ${}_{\partial}L[\![\mathbf{P},\partial]\!] f_R$. For a given $s \leq 2$ and type 1 term in Lemma 2, we have an upper bound

$$\left(\|\nabla_x u^\beta\|_{L^\infty_{t,x}} + \kappa^{\frac12}V(\beta)\right)\sqrt{\mathcal{D}(t)}\left(\sqrt{\int_0^t \mathcal{E}} + \varepsilon\kappa^{\frac12}\sqrt{\mathcal{D}(t)}\right) \tag{2.70}$$
$$\lesssim (\|\nabla_x u^\beta\|^2_{L^\infty_{t,x}} + 1)\int_0^t \mathcal{E} + o(1)\mathcal{D}(t),$$

where the first $\|\nabla_x u^\beta\|_{L^\infty_{t,x}}$ term corresponds to ${}_{\partial^{s_1}}L(\mathbf{I}-\mathbf{P})\partial^{s_2} f_R$ and the second $\kappa^{\frac12}V(\beta)$ term corresponds to ${}_{\partial^2}L(\mathbf{I}-\mathbf{P})f_R$. For example, for $s = 2$ with ${}_{\partial}L(\mathbf{I}-\mathbf{P})\partial f_R$ term, we have

$$\int_{(0,t)\times\mathbb{T}^2\times\mathbb{R}^3} \varepsilon^{-2}\kappa^{-1} {}_{\partial}L(\mathbf{I}-\mathbf{P})\partial f_R \partial^2 f_R \mathrm{d}v\mathrm{d}x\mathrm{d}t'$$
$$\lesssim \|\nabla_x u^\beta\|_{L^\infty_{t,x}} \|\varepsilon^{-1}\kappa^{-1}\nu^{\frac12}(\mathbf{I}-\mathbf{P})\partial f_R\|_{L^2((0,t);L^2_xL^2_v)}$$
$$\times \left(\|\partial^2 f_R\|_{L^2((0,t);L^2_xL^2_v)} + \varepsilon\kappa^{\frac12}\|\varepsilon^{-1}\kappa^{-\frac12}\nu^{\frac12}(\mathbf{I}-\mathbf{P})\partial^2 f_R\|_{L^2((0,t);L^2_xL^2_v)}\right),$$

which is bounded by the right-hand side of (2.70).

For a given $s \leq 2$ and type 2 term in Lemma 2, we have a similar upper bound

$$\sum \varepsilon^{-1}\kappa^{-\frac32+\frac{s}{2}} \|\partial\cdots[\![\mathbf{P},\partial]\!]\cdots\partial f_R\|_{L^2((0,t);L^2_xL^2_v)}\sqrt{\mathcal{D}(t)},$$

where summation is over possible combinations of $\partial\cdots[\![\mathbf{P},\partial]\!]\cdots\partial$, consisting of $s-1$ ∂ and one $[\![\mathbf{P},\partial]\!]$. We note that

$$\|\partial\cdots[\![\mathbf{P},\partial]\!]\cdots\partial f_R\|_{L^2((0,t);L^2_xL^2_v)}$$
$$\lesssim \varepsilon\left(\|\nabla_x u^\beta\|_{L^\infty_{t,x}} \|\partial^{s-1} f_R\|_{L^2((0,t);L^2_xL^2_v)} + V(\beta)\sum_{s'<s-1}\|\partial^{s'} f_R\|_{L^2((0,t);L^2_xL^2_v)}\right),$$

where the former term corresponds to the case that all $s-1$ derivatives ∂ are applied to f_R and the latter corresponds to the others. Thus, again, we have a bound

$$\left(\|\nabla_x u^\beta\|_{L^\infty_{t,x}} + \kappa^{\frac12}V(\beta)\right)\sqrt{\int_0^t \mathcal{E}}\sqrt{\mathcal{D}(t)} \lesssim (\|\nabla_x u^\beta\|^2_{L^\infty_{t,x}} + 1)\int_0^t \mathcal{E} + o(1)\mathcal{D}(t).$$

Finally, for a type 3 term in Lemma 2 (which immediately implies that $s=2$), we have

$$\|\nabla_x u^\beta\|^2_{L^\infty_{t,x}} \left\| \frac{f_R}{\kappa} \right\|_{L^2((0,t);L^2_x L^2_v)} \sqrt{\int_0^t \mathcal{E}} \lesssim \|\nabla_x u^\beta\|^2_{L^\infty_{t,x}} \int_0^t \mathcal{E}.$$

To summarize, we have

$$(2.64) \lesssim (\|\nabla_x u^\beta\|^2_{L^\infty_{t,x}} + 1) \int_0^t \mathcal{E} + o(1)\mathcal{D}(t). \tag{2.71}$$

Step 4. *Control of* (2.62) and (2.63). We use the following standard estimate: Let $0 < \vartheta_2 < \vartheta_1 < \vartheta_0 < \frac{1}{4}$, and let

$$w_j = e^{\vartheta_j |v|^2 - \frac{1}{2}\varepsilon u^\beta \cdot \underline{v}}, j = 0, 1, 2. \tag{2.72}$$

For $s \le 2$, we have

$$\begin{aligned}
\int_{(0,t)\times\mathbb{T}^2\times\mathbb{R}^3} \left| \frac{\partial^s f_R}{\kappa^{1-\frac{s}{2}}} \right|^2 \nu^2 \mathrm{d}v\mathrm{d}x\mathrm{d}t' &\lesssim \left\| \frac{\mathbf{P}\partial^s f_R}{\kappa^{1-\frac{s}{2}}} \right\|^2_{L^2((0,t);L^2_x L^2_v)} \\
&\quad + \left\| \nu^1 \frac{(\mathbf{I}-\mathbf{P})\partial^s f_R}{\kappa^{1-\frac{s}{2}}} \right\|^2_{L^2((0,t);L^2_x L^2_v)} \\
&\lesssim \int_0^t \mathcal{E} + \left\| \mathbf{1}_{\{|v-\varepsilon u^\beta| > (\varepsilon\sqrt{\kappa})^{-o(1)}\}} \nu^1 \frac{(\mathbf{I}-\mathbf{P})\partial^s f_R}{\kappa^{1-\frac{s}{2}}} \right\|^2_{L^2((0,t);L^2_x L^2_v)} \\
&\quad + \left\| \mathbf{1}_{\{|v-\varepsilon u^\beta| \le (\varepsilon\sqrt{\kappa})^{-o(1)}\}} \nu^1 \frac{(\mathbf{I}-\mathbf{P})\partial^s f_R}{\kappa^{1-\frac{s}{2}}} \right\|^2_{L^2((0,t);L^2_x L^2_v)} \\
&\lesssim \int_0^t \mathcal{E} + \|\mathbf{1}_{\{|v-\varepsilon u^\beta| > (\varepsilon\sqrt{\kappa})^{-o(1)}\}} \nu^1 w_s^{-1}\|^2_{L^2((0,t);L^2_x L^2_v)} \|w_s \partial^s f\|^2_{L^\infty((0,t);L^\infty(\mathbb{T}^2\times\mathbb{R}^3))} \\
&\quad + \left(\varepsilon\sqrt{\kappa}\right)^{1-o(1)} \mathcal{D}(t) \\
&\lesssim \int_0^t \mathcal{E} + \left(\varepsilon\sqrt{\kappa}\right)^{1-o(1)} \mathcal{D}(t) + e^{-\frac{1}{(\varepsilon\sqrt{\kappa})^{o(1)}}} \|w_s \partial^s f\|^2_{L^\infty((0,t);L^\infty(\mathbb{T}^2\times\mathbb{R}^3))}.
\end{aligned}$$

Similar calculation for (2.63) gives the following bound:

$$\begin{aligned}
(2.62) + (2.63) &\lesssim (1 + \|\nabla_x u^\beta\|_{L^\infty_{t,x}}) \int_0^t \mathcal{E} + o(1)\mathcal{D}(t) \\
&\quad + e^{-\frac{1}{(\varepsilon\sqrt{\kappa})^{o(1)}}} \sum_{s\le 2} \|w_s \partial^s f\|^2_{L^\infty((0,t);L^\infty(\mathbb{T}^2\times\mathbb{R}^3))}.
\end{aligned} \tag{2.73}$$

Step 5. *Control of* (2.65). Finally, we control the nonlinear contribution (2.65); here, we use the anisotropic interpolation result (Lemma 4) and Lemma 1. First, from Lemma 5 and (2.3), we remark that

$$\begin{aligned}
&\left\| \nu^{\frac{1}{2}} (\mathbf{I}-\mathbf{P}) \frac{f}{\kappa} \right\|_{L^2((0,t);L^4_x L^2_v)} + \left\| \nu^{\frac{1}{2}} (\mathbf{I}-\mathbf{P}) \frac{\partial f}{\sqrt{\kappa}} \right\|_{L^2((0,t);L^4_x L^2_v)} + \left\| \nu^{\frac{1}{2}} (\mathbf{I}-\mathbf{P}) \frac{f}{\kappa} \right\|_{L^2((0,t);L^\infty_x L^2_v)} \\
&\quad \lesssim \varepsilon^{\frac{1}{2}} \left(\sqrt{\mathcal{D}(t)} + \sqrt{\int_0^t \mathcal{E}} \right).
\end{aligned}$$

Next, we estimate the following integrals. First, we estimate

$$\begin{aligned}
&\int_{(0,t)\times\mathbb{T}^2\times\mathbb{R}^3} \frac{1}{\varepsilon\kappa^2}\Gamma(f_R,f_R)\frac{f_R}{\kappa}\mathrm{d}v\mathrm{d}x\mathrm{d}t' \\
&\quad\lesssim \left(\left\|\frac{f_R}{\kappa}\right\|_{L^2_{tvx}} + \left\|\nu^{\frac12}(\mathbf{I}-\mathbf{P})\frac{f_R}{\kappa}\right\|_{L^2_{tvx}}\right)\sqrt{\kappa}^{-1}\|f_R\|_{L^\infty_{tx}L^2_v}\sqrt{\mathcal{D}(t)} \\
&\quad\lesssim \left(\sqrt{\int_0^t\mathcal{E}}+\varepsilon\sqrt{\kappa}\sqrt{D(t)}\right)\sqrt{D(t)}\left\|\frac{f_R}{\kappa}\right\|^{\frac12}_{L^\infty_tL^2_{xv}}\|\partial^2 f_R\|^{\frac12}_{L^\infty_tL^2_{xv}} \lesssim \left(\int_0^t\mathcal{E}+\mathcal{D}(t)\right)\sqrt{\mathcal{E}(t)}.
\end{aligned}$$

In a similar fashion, we see

$$\begin{aligned}
&\int_{(0,t)\times\mathbb{T}^2\times\mathbb{R}^3} \frac{1}{\varepsilon\kappa^{2-\frac{s}{2}}}\Gamma(\partial^s f_R,f_R)\frac{\partial^s f_R}{\kappa^{1-\frac{s}{2}}}\mathrm{d}v\mathrm{d}x\mathrm{d}t' \\
&\quad\lesssim \sqrt{\mathcal{D}(t)}\frac{1}{\kappa^{\frac{3-s}{2}}}\Big[\left(\|\partial^s f_R\|_{L^2_{txv}}+\|\nu^{\frac12}(\mathbf{I}-\mathbf{P})\partial^s f_R\|_{L^2_{txv}}\right)\|f_R\|_{L^\infty_{tx}L^2_v} \\
&\qquad+\left(\|f_R\|_{L^2_tL^\infty_xL^2_v}+\|\nu^{\frac12}(\mathbf{I}-\mathbf{P})f_R\|_{L^2_tL^\infty_xL^2_v}\right)\|\partial^s f_R\|_{L^\infty_tL^2_{xv}}\Big] \\
&\quad\lesssim \sqrt{\mathcal{D}(t)}\left[\left(\sqrt{\int_0^t\mathcal{E}}+\varepsilon\sqrt{\kappa}\sqrt{\mathcal{D}(t)}\right)\frac{1}{\sqrt{\kappa}}\|f_R\|_{L^\infty_{tx}L^2_v}\right. \\
&\qquad\left.+\sqrt{\mathcal{E}(t)}\left(\frac{1}{\sqrt{\kappa}}\|f_R\|_{L^2_tL^\infty_xL^2_v}+\varepsilon^{\frac12}\left(\sqrt{\mathcal{D}(t)}+\sqrt{\int_0^t\mathcal{E}}\right)\right)\right] \lesssim \left(\int_0^t\mathcal{E}+\mathcal{D}(t)\right)\sqrt{\mathcal{E}(t)},\ s\le 2,
\end{aligned}$$

$$\begin{aligned}
&\int_{(0,t)\times\mathbb{T}^2\times\mathbb{R}^3} \frac{1}{\varepsilon\kappa}\Gamma(\partial f_R,\partial f_R)\partial^2 f_R\mathrm{d}v\mathrm{d}x\mathrm{d}t' \\
&\quad\lesssim \sqrt{\mathcal{D}(t)}\frac{1}{\sqrt{\kappa}}\left(\|\partial f_R\|_{L^2_tL^4_xL^2_v}+\|\nu^{\frac12}(\mathbf{I}-\mathbf{P})\partial f_R\|_{L^2_tL^4_xL^2_v}\right)\|\partial f_R\|_{L^\infty_tL^4_xL^2_v} \\
&\quad\lesssim \left(\int_0^t\mathcal{E}+\mathcal{D}(t)\right)\sqrt{\mathcal{E}(t)}.
\end{aligned}$$

Here, we have used Lemma 1 to first bound terms with L^2_v norms with mixed L^p_x norms and then Lemma 4 to turn back to the L^2_x norm. In a similar manner, we have, for $s\le 2$, $s_1+s_2=s$, and $s_1\ge 1$,

$$\begin{aligned}
&\int_{(0,t)\times\mathbb{T}^2\times\mathbb{R}^3} \frac{1}{\varepsilon\kappa^{2-\frac{s}{2}}}\partial^{s_1}\Gamma(\partial^{s_2} f_R,f_R)\frac{\partial^s f_R}{\kappa^{1-\frac{s}{2}}}\mathrm{d}v\mathrm{d}x\mathrm{d}t' \\
&\quad\lesssim \|\partial^{s_1}u^\beta\|_{L^\infty_{t,x}}\sqrt{\int_0^t\mathcal{E}}\frac{1}{\kappa^{2-\frac{s}{2}}}\Big[\left(\|\partial^{s_2} f_R\|_{L^2_{txv}}+\|\nu^{\frac12}(\mathbf{I}-\mathbf{P})\partial^{s_2} f_R\|_{L^2_{txv}}\right)\|f_R\|_{L^\infty_{tx}L^2_v} \\
&\qquad+\left(\|f_R\|_{L^2_tL^\infty_xL^2_v}+\|\nu^{\frac12}(\mathbf{I}-\mathbf{P})f_R\|_{L^2_tL^\infty_xL^2_v}\right)\|\partial^{s_2} f_R\|_{L^\infty_tL^2_{xv}}\Big] \\
&\quad\lesssim \|\partial^{s_1}u^\beta\|_{L^\infty_{t,x}}\sqrt{\int_0^t\mathcal{E}}\frac{1}{\kappa^{\frac12-\frac{s-s_2}{2}}}\left[\left(\sqrt{\int_0^t\mathcal{E}}+\varepsilon\sqrt{\kappa}\sqrt{\mathcal{D}(t)}\right)\sqrt{\mathcal{E}(t)}\right. \\
&\qquad\left.+\left(\sqrt{\int_0^t\mathcal{E}}+\varepsilon^{\frac12}\left(\sqrt{\mathcal{D}(t)}+\sqrt{\int_0^t\mathcal{E}}\right)\right)\sqrt{\mathcal{E}(t)}\right] \\
&\quad\lesssim (\|\nabla_x u^\beta\|_{L^\infty_{tx}}+V(\beta)\sqrt{\kappa})\sqrt{\mathcal{E}(t)}\left(\int_0^t\mathcal{E}+\varepsilon\mathcal{D}(t)\right) \lesssim \sqrt{\mathcal{E}(t)}\left(\|\nabla_x u^\beta\|_{L^\infty_{tx}}\int_0^t\mathcal{E}+\mathcal{D}(t)\right),
\end{aligned}$$

where the first factor $\|\nabla_x u^\beta\|_{L^\infty_{tx}}$ comes from the case $s_1 = 1$ and the second factor $V(\beta)\sqrt{\kappa}$ comes from the case $s_2 = 2, s_2 = 0$. Also, we have used (2.3) to bury the contribution of $\|\nabla_x u^\beta\|_{L^\infty_{t,x}}$ in $\mathcal{D}(t)$.

Therefore, we have

$$(2.65) \lesssim \left((\|\nabla_x u^\beta\|_{L^\infty_{tx}} + 1) \int_0^t \mathcal{E} + \mathcal{D}(t) \right) \sqrt{\mathcal{E}(t)}. \tag{2.74}$$

Summing up (2.68), (2.69), (2.71), (2.73), and (2.74), we have

$$\begin{aligned} \mathcal{E}(t) + \mathcal{D}(t) \lesssim \mathcal{E}(0) + (\|\nabla_x u^\beta\|^2_{L^\infty_{t,x}} + 1 + \sqrt{\mathcal{E}(t)}) \int_0^t \mathcal{E} + \kappa + \sqrt{\mathcal{E}(t)}\mathcal{D}(t) \\ + e^{-\frac{1}{(\varepsilon\sqrt{\kappa})^{o(1)}}} \sum_{s\le 2} \|w_s \partial^s f\|^2_{L^\infty((0,t);L^\infty(\mathbb{T}^2\times\mathbb{R}^3))}. \end{aligned} \tag{2.75}$$

2.5.2. L^∞ control. From Proposition 4 and (2.37), we obtain the following:

$$\begin{aligned} &\|w_0 f_R\|_{L^\infty((0,t);L^\infty(\mathbb{T}^2\times\mathbb{R}^3))} \\ &\lesssim \|w_0 f_{R0}\|_{L^\infty(\mathbb{T}^2\times\mathbb{R}^3)} + \frac{1}{\varepsilon}\sqrt{\mathcal{E}(t)} + \varepsilon \|w_0 f_R\|^2_{L^\infty((0,t);L^\infty(\mathbb{T}^2\times\mathbb{R}^3))} \\ &\quad + \varepsilon\kappa V(\beta) \|w_0 f_R\|_{L^\infty((0,t);L^\infty(\mathbb{T}^2\times\mathbb{R}^3))} + \varepsilon^3\kappa V(\beta) + \varepsilon\kappa^2 V(\beta). \end{aligned} \tag{2.76}$$

Here, we have used Lemma 11 to bound the right-hand side of (2.37). Proceeding similar argument to (2.38), for $1 \le s \le 2$, we obtain

$$\begin{aligned} &\|w_s \partial^s f_R\|_{L^\infty((0,t);L^\infty(\mathbb{T}^2\times\mathbb{R}^3))} \lesssim \|w_s \partial^s f_{R0}\|_{L^\infty(\mathbb{T}^2\times\mathbb{R}^3)} + \frac{1}{\varepsilon\kappa^{\frac{s}{2}}}\mathcal{E}(t) \\ &\quad + \varepsilon \|w_s \partial^s f_R\|_{L^\infty((0,t);L^\infty(\mathbb{T}^2\times\mathbb{R}^3))} \|w_0 f_R\|_{L^\infty((0,t);L^\infty(\mathbb{T}^2\times\mathbb{R}^3))} \\ &\quad + \varepsilon\kappa V(\beta) \|w_s \partial^s f_R\|_{L^\infty((0,t);L^\infty(\mathbb{T}^2\times\mathbb{R}^3))} \\ &\quad + \varepsilon V(\beta) \sum_{s'<s} \|w_{s'} \partial^{s'} f_R\|_{L^\infty((0,t);L^\infty(\mathbb{T}^2\times\mathbb{R}^3))} \\ &\quad + \varepsilon V(\beta) \sum_{s_1+s_2\le s, s_1, s_2 < s} \|w_{s_1} \partial^{s_1} f_R\|_{L^\infty((0,t);L^\infty(\mathbb{T}^2\times\mathbb{R}^3))} \|w_{s_2} \partial^{s_2} f_R\|_{L^\infty((0,t);L^\infty(\mathbb{T}^2\times\mathbb{R}^3))} \\ &\quad + \varepsilon^3\kappa V(\beta) + \varepsilon\kappa^2 V(\beta). \end{aligned}$$

Here, we have used a pointwise bound $w_0 > \nu^2 w_1 > \nu^4 w_2$ for the third line. Therefore, we have

$$\begin{aligned} &\mathcal{F}(t) \lesssim \mathcal{F}(0) + \varepsilon^2 + \sqrt{\mathcal{E}(t)} + \mathcal{F}(t)^2, \\ &\|w_1 \partial^1 f_R\|_{L^\infty((0,t);L^\infty(\mathbb{T}^2\times\mathbb{R}^3))} \lesssim \|w_1 \partial^1 f_{R0}\|_{L^\infty(\mathbb{T}^2\times\mathbb{R}^3)} + \mathcal{F}(t) \|w_1 \partial^1 f_R\|_{L^\infty((0,t);L^\infty(\mathbb{T}^2\times\mathbb{R}^3))} \\ &\qquad + \frac{1}{\varepsilon\sqrt{\kappa}}\sqrt{\mathcal{E}(t)} + \varepsilon V(\beta) \left(1 + \frac{\mathcal{F}(t)}{\varepsilon}\right)^2, \\ &\|w_2 \partial^2 f_R\|_{L^\infty((0,t);L^\infty(\mathbb{T}^2\times\mathbb{R}^3))} \lesssim \|w_2 \partial^2 f_{R0}\|_{L^\infty(\mathbb{T}^2\times\mathbb{R}^3)} + \mathcal{F}(t) \|w_2 \partial^2 f_R\|_{L^\infty((0,t);L^\infty(\mathbb{T}^2\times\mathbb{R}^3))} \\ &\qquad + \frac{1}{\varepsilon\kappa}\sqrt{\mathcal{E}(t)} + \varepsilon V(\beta) \left(1 + \frac{\mathcal{F}(t)}{\varepsilon} + \|w_1 \partial^1 f_R\|_{L^\infty((0,t);L^\infty(\mathbb{T}^2\times\mathbb{R}^3))}\right)^2. \end{aligned} \tag{2.77}$$

In particular, giving explicit constants for (2.75) and (2.77), we obtain

$$\begin{aligned}\mathcal{E}(t)+\mathcal{D}(t)\le \mathbf{C}_0\Bigg(&\mathcal{E}(0)+\Big(\|\nabla_x u^\beta\|^2_{L^\infty((0,T)\times\mathbb{T}^2)}+1+\sqrt{\mathcal{E}(t)}\Big)\int_0^t\mathcal{E}+\kappa+\sqrt{\mathcal{E}(t)}\mathcal{D}(t)\\&e^{-\frac{1}{(\varepsilon\sqrt{\kappa})^{o(1)}}}\sum_{s\le 2}\|w_s\partial^s f\|^2_{L^\infty((0,t);L^\infty(\mathbb{T}^2\times\mathbb{R}^3))}\cdot\Bigg),\\ \mathcal{F}(t)\le{}& \mathbf{C}_0\left(\mathcal{F}(0)+\varepsilon^2+\sqrt{\mathcal{E}(t)}+\mathcal{F}(t)^2\right),\\ \|w_s\partial^s f_R\|_{L^\infty((0,t);L^\infty(\mathbb{T}^2\times\mathbb{R}^3))}\le{}&\mathbf{C}_0\Big(\|w_s\partial^s f_{R0}\|_{L^\infty(\mathbb{T}^2\times\mathbb{R}^3)}\\&+\mathcal{F}(t)\|w_s\partial^s f_R\|_{L^\infty((0,t);L^\infty(\mathbb{T}^2\times\mathbb{R}^3))}+\varepsilon^{-1}\kappa^{-\frac{s}{2}}\sqrt{\mathcal{E}(t)}\\&+\varepsilon V(\beta)\Big(1+\varepsilon^{-1}\mathcal{F}(t)+\sum_{1\le s'<s}\|w_{s'}\partial^{s'}f_R\|_{L^\infty((0,t);L^\infty(\mathbb{T}^2\times\mathbb{R}^3))}\Big)^2\Big)\end{aligned}$$

for some constant $\mathbf{C}_0>1$.

2.6. Proof of Theorem 3. For given any arbitrary positive time $T>0$, choose $T_*\in[0,T]$ such that

$$T_*=\sup\Big\{t>0:\sqrt{\mathcal{E}(t)}<\frac{1}{10\mathbf{C}_0},\mathcal{F}(t)<\frac{1}{10\mathbf{C}_0}\Big\}. \tag{2.78}$$

Then, for $t\in[0,T_*]$,

$$\begin{aligned}\mathcal{E}(t)+\mathcal{D}(t)\le{}&2\mathbf{C}_0\mathcal{E}(0)+2\mathbf{C}_0\left(\|\nabla_x u^\beta\|^2_{L^\infty((0,T)\times\mathbb{T}^2)}+2\right)\int_0^t\mathcal{E}+2\mathbf{C}_0\kappa\\&+2\mathbf{C}_0e^{-\frac{1}{(\varepsilon\sqrt{\kappa})^{o(1)}}}\sum_{s\le2}\|w_s\partial^s f\|^2_{L^\infty((0,t);L^\infty(\mathbb{T}^2\times\mathbb{R}^3))},\\ \mathcal{F}(t)\le{}&2\mathbf{C}_0\mathcal{F}(0)+2\mathbf{C}_0\varepsilon^2+2\mathbf{C}_0\sqrt{\mathcal{E}(t)},\end{aligned}$$

and for $1\le s\le 2$,

$$\begin{aligned}\|w_s\partial^s f_R\|_{L^\infty((0,t);L^\infty(\mathbb{T}^2\times\mathbb{R}^3))}\le{}&2\mathbf{C}_0\Bigg(\|w_s\partial^s f_{R0}\|_{L^\infty(\mathbb{T}^2\times\mathbb{R}^3)}+\varepsilon^{-1}\kappa^{-\frac{s}{2}}\\&+\varepsilon V(\beta)\Bigg(1+\varepsilon^{-1}/2+\sum_{1\le s'<s}\|w_{s'}\partial^{s'}f_R\|_{L^\infty((0,t);L^\infty(\mathbb{T}^2\times\mathbb{R}^3))}\Bigg)^2\Bigg),\\ \|w_1\partial^1 f_R\|_{L^\infty((0,t);L^\infty(\mathbb{T}^2\times\mathbb{R}^3))}\le{}&2\mathbf{C}_0\|w_1\partial^1 f_{R0}\|_{L^\infty(\mathbb{T}^2\times\mathbb{R}^3)}+4\mathbf{C}_0\varepsilon^{-1}\kappa^{-\frac12},\\ \|w_2\partial^2 f_R\|_{L^\infty((0,t);L^\infty(\mathbb{T}^2\times\mathbb{R}^3))}\le{}&2\mathbf{C}_0\|w_2\partial^2 f_{R0}\|_{L^\infty(\mathbb{T}^2\times\mathbb{R}^3)}\\&+C(\mathbf{C}_0)\left(\|w_1\partial^1 f_{R0}\|^2_{L^\infty(\mathbb{T}^2\times\mathbb{R}^3)}+\varepsilon^{-1}\kappa^{-1}V(\beta)\right).\end{aligned}$$

Since $\sqrt{\mathcal{E}(0)},\mathcal{F}(0)<\delta_0=\delta_0(\varepsilon)$, and $\|w_s\partial^s f_{R0}\|_{L^\infty(\mathbb{T}^2\times\mathbb{R}^3)}<\delta_s=\delta_s(\varepsilon)$ satisfy (2.61) for $s=1,2$, we have

$$\begin{aligned}&\|w_s\partial^s f_R\|_{L^\infty((0,t);L^\infty(\mathbb{T}^2\times\mathbb{R}^3))}\le C(\mathbf{C}_0)(\varepsilon^{-1}\kappa^{-1/2})^s,\\ &\qquad\mathcal{E}(t)+\mathcal{D}(t)\le C(\mathbf{C}_0)(\delta_0^2+\kappa)+2\mathbf{C}_0\left(\|\nabla_x u^\beta\|^2_{L^\infty((0,T)\times\mathbb{T}^2)}+2\right)\int_0^t\mathcal{E}\end{aligned}$$

since $e^{-\frac{1}{(\varepsilon\sqrt{\kappa})^{o(1)}}}$ factor decays faster than any algebraic blowups. By Gronwall's lemma, we have

$$\begin{aligned}\mathcal{E}(t)+\mathcal{D}(t) &\le C(\mathbf{C}_0)(\delta_0^2+\kappa)(1+T)\\ &\quad\times\left(2\mathbf{C}_0\left(\|\nabla_x u^\beta\|^2_{L^\infty((0,T)\times\mathbb{T}^2)}+2\right)\exp\left(2\mathbf{C}_0\left(\|\nabla_x u^\beta\|^2_{L^\infty((0,T)\times\mathbb{T}^2)}+2\right)T\right)\right).\end{aligned}$$

Since δ_0 satisfies (2.60), we see that, for sufficiently small ε, $\sqrt{\mathcal{E}(T_*)}, \mathcal{F}(T_*)$ satisfies (2.78). Therefore, $T_*=T$, and we proved the claim.

3. Approximation of the Lagrangian solutions of the Euler equations.

As discussed in the introduction, we would like to obtain a limit to weak solutions that do not have enough regularity in the framework of the standard Hilbert expansion in general. Moreover, we want a convergent sequence in a stronger topology than $L^{\mathfrak{p}}$ for velocity because interesting singular behavior can be observed only in a stronger topology. However, control in a stronger topology requires more regularity for the velocity field as well. A straightforward remedy for low regularity of the fluid velocity field is to regularize the initial data; therefore, instead of choosing the initial data as a perturbation around the local Maxwellian $M_{1,\varepsilon u_0,1}$, we choose the initial data as a perturbation around the local Maxwellian $M_{1,\varepsilon u_0^\beta,1}$, where u_0^β is the initial data regularization of u_0 with scale β. Then, if one can prove the stability of the Euler solution under the perturbation of the initial data, as well as control of the remaining small terms, we can construct a sequence of Boltzmann solutions whose bulk velocity converges to the Euler solution.

It turns out that this simple idea works well; in the class of solutions of the Euler equation we consider, we have a certain stability, so we can prove that the solution u^β starting from u_0^β converges to the solution u from u_0. Also, for the estimate of the remainders, the introduction of the regularization scale β gives an additional freedom in our analysis; by sacrificing the speed of regularization convergence, we can control the size of higher derivatives appearing in the remainder equation. In addition, many weak solutions of fluid equations are interpreted as a limit of smooth solutions. In that regard, this initial data regularization approach is quite natural.

3.1. Regularization.

In our proof of the hydrodynamic limit from the Boltzmann equations, it is important to regularize the Largangian solutions of the Euler equation (1.9). We achieve this by regularizing the initial data using the standard mollifier. Let $\varphi\in C_c^\infty(\mathbb{R}^2)$ be a smooth nonnegative function with $\int_{\mathbb{R}^2}\varphi(x)dx=1$ and $\varphi(x)=0$ for $|x-(0,0)|\ge\frac{1}{4}$. For $\beta\in(0,1)$, we define

$$\varphi^\beta(x):=\frac{1}{\beta^2}\varphi\left(\frac{x}{\beta}\right)\quad\text{for}\quad x\in\left[-\frac{1}{2},\frac{1}{2}\right]^2. \tag{3.1}$$

Note that φ^β can be extended periodically so that $\varphi^\beta\in C^\infty(\mathbb{T}^2)$ and $\int_{\mathbb{T}^2}\varphi^\beta(x)dx=1$ as well. Also, φ^β is supported on $B_{\frac{\beta}{4}}(0)$. Note that $\{\varphi^\beta\}_\beta$ are approximate identities; thus, for $1\le p<\infty$ and $\psi\in L^p(\mathbb{T}^2)$, we have

$$\lim_{\beta\to 0}\|\varphi^\beta*\psi-\psi\|_{L^p(\mathbb{T}^2)}=0. \tag{3.2}$$

Note that we cannot expect a universal rate of convergence, which is independent of ψ if ψ is merely in $L^p(\mathbb{T}^2)$ or $p=\infty$. However, if we have a certain regularity for ψ, we have the rate of convergence; for example, if $\psi\in W^{1,2}(\mathbb{T}^2)$, we have

$$
\begin{aligned}
\|\varphi^\beta * \psi - \psi\|_{L^2(\mathbb{T}^2)} &= \left(\int_{\mathbb{T}^2} \left| \int_{\mathbb{T}^2} \varphi^\beta(y)(\psi(x-y) - \psi(x)) \mathrm{d}y \right|^2 \mathrm{d}x \right)^{\frac{1}{2}} \\
&\le \int_{\mathbb{T}^2} |\varphi^\beta(y)| \left(\int_{\mathbb{T}^2} |\psi(x-y) - \psi(x)|^2 dx \right)^{\frac{1}{2}} dy \\
&\le C \int_{\mathbb{T}^2} |y||\varphi^\beta(y)| dy \|\psi\|_{W^{1,2}(\mathbb{T}^2)} \le C\beta \|\psi\|_{W^{1,2}(\mathbb{T}^2)}.
\end{aligned} \tag{3.3}
$$

We consider approximation solutions (u^β, ω^β) for the mollified initial data:

$$
\partial_t \omega^\beta + u^\beta \cdot \nabla \omega^\beta = 0 \quad \text{in } [0,T] \times \mathbb{T}^2, \tag{3.4}
$$

$$
u^\beta = -\nabla^\perp (-\Delta)^{-1} \omega^\beta \quad \text{in } [0,T] \times \mathbb{T}^2, \tag{3.5}
$$

$$
\omega^\beta|_{t=0} = \omega_0^\beta := \varphi^\beta * \omega_0 \quad \text{in } \mathbb{T}^2. \tag{3.6}
$$

Note that, for each $\beta \in (0,1)$ this problem (3.4), (3.5), and (3.6) has a smooth (therefore unique) solution, which is the Lagrangian solution:

$$
\omega^\beta(t,x) = \omega_0^\beta(X^\beta(0;t,x)), \tag{3.7}
$$

$$
\frac{d}{ds} X^\beta(s;t,x) = u^\beta(s, X^\beta(s;t,x)), \quad X^\beta(s;t,x)|_{s=t} = x. \tag{3.8}
$$

Remark 3. If u^β is obtained from (1.10) with $\omega^\beta \in C^\infty(\mathbb{T})$, u^β is incompressible, and thus, the associated flow X^β by (1.12) satisfies (1.14) with an equality and $\mathfrak{C} = 1$ (measure-preserving).

We define a pressure as a unique solution of $-\Delta p^\beta = \mathrm{div}(\mathrm{div}(u^\beta \otimes u^\beta))$ with $⨍_{\mathbb{T}^2} p^\beta = 0$. Then, we have

$$
\begin{aligned}
(\partial_t + u^\beta \cdot \nabla_x) u^\beta + \nabla_x p^\beta &= 0 \quad \text{in } [0,T] \times \mathbb{T}^2, \\
\nabla_x \cdot u^\beta &= 0 \quad \text{in } [0,T] \times \mathbb{T}^2, \\
u^\beta(x,0) &= u_0^\beta(x) \quad \text{in } \mathbb{T}^2.
\end{aligned} \tag{3.9}
$$

Also, we will consider the following auxiliary linear equation:

$$
\begin{aligned}
(\partial_t + u^\beta \cdot \nabla_x)\tilde{u}^\beta + \tilde{u}^\beta \cdot \nabla_x u^\beta + \nabla_x \tilde{p}^\beta - \eta_0 \Delta_x u^\beta &= 0 \quad \text{in } [0,T] \times \mathbb{T}^2, \\
\nabla_x \cdot \tilde{u}^\beta &= 0 \quad \text{in } [0,T] \times \mathbb{T}^2, \\
\tilde{u}^\beta(0,x) &= \tilde{u}_0(x) \quad \text{in } \mathbb{T}^2.
\end{aligned} \tag{3.10}
$$

Here, η_0 is given by Lemma 7.

3.2. Biot–Savart law in a periodic domain. In this part, we discuss the asymptotic form of the kernel for the Biot–Savart law, which gives u from ω, and the singular integral, which gives $\nabla_x u$ from ω in our setting, the periodic domain $\mathbb{T}^2 = \left[-\frac{1}{2}, \frac{1}{2}\right]^2$. This is important since the compactness results we have used, in particular [8], have the $\mathbb{R}^N$ setting; in particular, the key estimate, the weak L^1 estimate for $\nabla_x u$, relies on the form of the Calderon–Zygmund kernel of a Riesz transform. Therefore, we need an asymptotic form of Biot–Savart kernels and Riesz transforms.

We start from [12].

PROPOSITION 5 ([12], Lemma 1). *The function G—defined on $\mathbb{R}^2 \simeq \mathbb{C}$ by*

$$G(z) := \operatorname{Im}\left(\frac{|z|^2 - z^2}{-4i} - \frac{z}{2} + \frac{i}{12}\right) - \frac{1}{2\pi}\log\left|(1-e(z)) \times \prod_{n=1}^{\infty}(1-e(ni+z))(1-e(ni-z))\right|, \tag{3.11}$$

where $e(z) = e^{2\pi i z}$—is $\mathbb{Z}^2$-periodic and is the Green's function with mass on $\mathbb{Z}^2$; that is,

$$-\Delta_x G(x) = \sum_{\zeta\in\mathbb{Z}^2} \delta(x-\zeta) - 1 \text{ for } x \in \mathbb{R}^2, \quad \int G(x)dx = 0. \tag{3.12}$$

In particular, the infinite product inside converges absolutely, and G is of the form

$$G(z) = \frac{|z|^2}{4} - \frac{1}{2\pi}\log|\mathfrak{h}(z)|, \tag{3.13}$$

where $\mathfrak{h}$ is a holomorphic function with simple zeros exactly on $\mathbb{Z}^2$.

For the sake of completeness, we briefly reason (3.13). We recall the following result from complex analysis.

PROPOSITION 6 (Theorem 15.5 of [41]). *Suppose that $\{g_n\}$ is a sequence of nonzero holomorphic functions on $\mathbb{C}$ such that*

$$\sum_{n=1}^{\infty} |1 - g_n(z)| \tag{3.14}$$

converges uniformly on compact subsets of $\mathbb{C}$. Then, the product

$$g(z) = \prod_{n=1}^{\infty} g_n(z) \tag{3.15}$$

converges uniformly on compact subsets of $\mathbb{C}$, and thus, g is holomorphic on $\mathbb{C}$. Furthermore, the multiplicity of g at z_0 (i.e., the smallest nonnegative integer k such that $\lim_{z\to z_0} \frac{g(z)}{(z-z_0)^k} \neq 0$) is the sum of multiplicities of g_n at z_0.

Now, we see that $\mathfrak{h}(z)$ is the product of $1-e(z) = 1-e^{2\pi i z}$, $1-e(ni+z) = 1-e^{-2\pi n+2\pi i z}$, and $1-e(ni-z) = 1-e^{-2\pi n-2\pi i z}$. Note that $|1-(1-e(ni+z))| = |1-(1-e(ni-z))| = e^{-2\pi n}$ so that the premise of the proposition is satisfied. Thus, $\mathfrak{h}(z)$ is holomorphic. Furthermore, the zeros of $\mathfrak{h}$ are exactly the union of zeros of $1-e(z)$, which are $\{mi|m\in\mathbb{Z}\}$; zeros of $1-e(ni+z)$, which are $\{m-ni|m\in\mathbb{Z}\}$; and zeros of $1-e(ni-z)$, which are $\{m+ni|m\in\mathbb{Z}\}$, for each integer $n\geq 1$. The union is exactly $\mathbb{Z}^2$. Moreover, the multiplicity of each point in $\mathbb{Z}^2$ is 1; in other words, all roots are simple.

Thus, on $\mathbb{R}^2\setminus\mathbb{Z}^2$, G is infinitely differentiable. Furthermore, let $\zeta\in\mathbb{Z}^2$. Then, there exists an $\mathfrak{r}_\zeta > 0$ such that

$$\mathfrak{h}(z) = (z-\zeta)\mathfrak{H}(z), \tag{3.16}$$

where $\mathfrak{H}(z) = \frac{\mathfrak{h}(z)}{z-\zeta}$ is an holomorphic function on $B_{\mathfrak{r}_\zeta}(\zeta)$ and $\inf_{z\in B_{\mathfrak{r}_\zeta}(\zeta)} |\mathfrak{H}(z)| \geq c_\zeta > 0$. Therefore, we can rewrite (3.13) in the following form and differentiate; for $z \in B_{\mathfrak{r}_\zeta}(\zeta)$,

$$
\begin{aligned}
G(z) &= -\frac{1}{2\pi}\log|z-\zeta| + \mathfrak{B}_\zeta(z),\\
\nabla G(z) &= -\frac{1}{2\pi}\frac{z-\zeta}{|z-\zeta|^2} + \nabla \mathfrak{B}_\zeta(z),\\
\nabla^2 G(z) &= \frac{1}{4\pi}\frac{(z-\zeta)\otimes(z-\zeta) - \frac{1}{2}|z-\zeta|^2\mathbb{I}_2}{|z-\zeta|^4} + \nabla^2\mathfrak{B}_\zeta(z),
\end{aligned} \tag{3.17}
$$

where $z=x+iy$ is identified with (x,y), $\nabla=(\partial_x,\partial_y)$, and

$$
\mathfrak{B}_\zeta(z) = \frac{|z|^2}{4} - \frac{1}{2\pi}\log|\mathfrak{H}(z)| \tag{3.18}
$$

is a smooth function (in x,y) whose all derivatives are bounded. In particular, taking $\zeta=0=(0,0)$ and taking $\mathfrak{r}=\mathfrak{r}_0$, we have the following.

Proposition 7. *Let G be defined by* (3.11) *so that the solution to the Poisson equation $-\Delta_x q = h - \fint_{\mathbb{T}^2} h$ is given by $q=G*h$ and the Biot–Savart law by*

$$
u(x) = \mathbf{b}*w = \left(\frac{1}{2\pi}\frac{x^\perp}{|x|^2} + \nabla_x^\perp\mathfrak{B}\right)*\omega. \tag{3.19}
$$

Then, there exists a $\mathfrak{r}>0$ such that $G,\nabla_x G,\nabla_x^2 G$ are smooth and bounded in $\mathbb{T}^2\setminus B_\mathfrak{r}(0) = \left[-\frac{1}{2},\frac{1}{2}\right]^2\setminus B_\mathfrak{r}(0)$, and in $B_\mathfrak{r}(0)$, we have

$$
\begin{aligned}
G(x) &= -\frac{1}{2\pi}\log|x| + \mathfrak{B}(x), x\in B_\mathfrak{r}(0),\\
\nabla_x G(x) &= -\frac{1}{2\pi}\frac{x}{|x|^2} + \nabla_x\mathfrak{B}(x), x\in B_\mathfrak{r}(0),\\
\nabla_x^2 G(x) &= \frac{1}{4\pi}\frac{x\otimes x - \frac{1}{2}|x|^2\mathbb{I}_2}{|x|^4} + \nabla_x^2\mathfrak{B}(x), x\in B_\mathfrak{r}(0),
\end{aligned} \tag{3.20}
$$

where $\nabla_x^k\mathfrak{B}$ are bounded in $B_\mathfrak{r}(0)$ for all $k\geq 0$.

3.3. Higher regularity of the approximations (u^β,ω^β). In this section, we establish the regularity estimate of (u^β,ω^β) solving (3.9) and (3.4), (3.5), and (3.6) and $(\tilde{u}^\beta,\tilde{p}^\beta)$ solving (3.10).

First, we prove that, for $1\leq r,p\leq\infty$,

$$
\|\omega_0^\beta\|_{L^r(\mathbb{T}^2)} \lesssim \beta^{-2\left(\frac{1}{p}-\frac{1}{r}\right)_+}\|\omega_0\|_{L^p}, \tag{3.21}
$$

$$
\|\nabla^k\omega_0^\beta\|_{L^r(\mathbb{T}^2)} \lesssim \beta^{-k-2\left(\frac{1}{p}-\frac{1}{r}\right)_+}\|\omega_0\|_{L^p}. \tag{3.22}
$$

From Young's inequality, for $1+1/r=1/p+1/q$ and $r,p,q\in[1,\infty]$,

$$
\|\omega_0^\beta\|_{L^r(\mathbb{T}^2)} \leq \|\varphi^\beta\|_{L^q(\mathbb{T}^2)}\|\omega_0\|_{L^p(\mathbb{T}^2)} \lesssim \beta^{-2\left(\frac{1}{p}-\frac{1}{r}\right)}\|\omega_0\|_{L^p} \quad \text{for } r\geq p,
$$

$$
\|\nabla^k\omega_0^\beta\|_{L^r(\mathbb{T}^2)} \leq \|\nabla^k\varphi^\beta\|_{L^q(\mathbb{T}^2)}\|\omega_0\|_{L^p(\mathbb{T}^2)} \leq \beta^{-k-2\left(\frac{1}{p}-\frac{1}{r}\right)}\|\omega_0\|_{L^p} \quad \text{for } r\geq p.
$$

For both, we have used

$$
\left(\int_{\mathbb{T}^2}|\nabla_x^k\varphi^\beta|^q \mathrm{d}x\right)^{1/q} = \left(\frac{\beta^2}{\beta^{q(2+k)}}\int_{\mathbb{T}^2}|\nabla^k\varphi(\frac{x}{\beta})|^q\mathrm{d}\frac{x_1}{\beta}\mathrm{d}\frac{x_2}{\beta}\right)^{1/q} = \beta^{-k-\frac{2(q-1)}{q}}\|\nabla^k\varphi\|_{L^q(\mathbb{T}^2)}.
$$

Using $|\mathbb{T}^2|=1$, we have

$$\|\omega_0^\beta\|_{L^r(\mathbb{T}^2)} \le \|\omega_0^\beta\|_{L^p(\mathbb{T}^2)} \lesssim \|\omega_0\|_{L^p(\mathbb{T}^2)} \quad \text{for } p \ge r,$$
$$\|\nabla^k\omega_0^\beta\|_{L^r(\mathbb{T}^2)} \le \|\nabla^k\omega_0^\beta\|_{L^p(\mathbb{T}^2)} \lesssim \beta^{-k}\|\omega_0\|_{L^p(\mathbb{T}^2)} \quad \text{for } p \ge r.$$

Collecting the bounds, we conclude (3.21) and (3.22).

3.3.1. Bounds for $\|\nabla_x u^\beta(t)\|_{L^\infty(\mathbb{T}^2)}$.

THEOREM 4. *Let* (u^β, ω^β) *be the Lagrangian solution of* (3.7) *supplemented with* (3.8) *and* (3.5). *For* $\mathfrak{p} \in [1,\infty]$ *and* $\beta \ll \|\omega_0\|_{L^\mathfrak{p}}$, *we have the following estimate for all* $t \ge 0$*:*

(3.23)
$$\|\nabla u^\beta(t,\cdot)\|_{L^\infty} \lesssim \mathfrak{Lip}(\beta,\mathfrak{p}) := \left(\beta^{-\frac{2}{\mathfrak{p}}}\log_+\frac{1}{\beta}\right)\|\omega_0\|_{L^\mathfrak{p}} e^{tC\beta^{-\frac{2}{\mathfrak{p}}}\|\omega_0\|_{L^\mathfrak{p}}} \quad \textit{for some } C>1.$$

We will estimate $\nabla_x X$ by applying Gronwall's inequality to the differentiation of (3.8):

$$\frac{d}{ds}\nabla_x X^\beta(s;t,x) = \nabla_x X(s;t,x)\cdot(\nabla_x u)(s,X(s;t,x)). \tag{3.24}$$

The initial condition for each purely spatial derivative can be driven from (1.12):

$$\nabla_x X(s;t,x)|_{s=t} = id. \tag{3.25}$$

We use the following version of Gronwall's inequality.

LEMMA 14 ([3], Lemma 3.3). *Let* q *and* z *be two* C^0 *(resp.,* C^1*) nonnegative functions on* $[t_0,T]$. *Let* $\mathcal{G}$ *be a continuous function on* $[t_0,T]$. *Suppose that, for* $t\in[t_0,T]$,

$$\frac{d}{dt}z(t) \le \mathcal{G}(t)z(t) + q(t). \tag{3.26}$$

For any time $t\in[t_0,T]$, *we have*

$$z(t) \le z(t_0)\exp\left(\int_{t_0}^t \mathcal{G}(\tau)d\tau\right) + \int_{t_0}^t q(\tau)\exp\left(\int_\tau^t \mathcal{G}(\tau')d\tau'\right)d\tau. \tag{3.27}$$

LEMMA 15. *For any* $r\in[1,\infty]$ *and* $0\le s\le t$,

$$\|\nabla_x X^\beta(s;t,\cdot)\|_{L^r(\mathbb{T}^2)} \le e^{\int_s^t \|\nabla_x u(t')\|_{L^\infty_x} \mathrm{d}t'}. \tag{3.28}$$

Proof. The proof is immediate from Gronwall's inequality to (3.24) and the initial condition $\|\nabla X^\beta(t;t,x)\|_{L^r(\mathbb{T}^2)} = \|\nabla x\|_{L^r(\mathbb{T}^2)} = \|id\|_{L^r(\mathbb{T}^2)} = 1$ from (3.25).

Next, using Morrey's inequality

$$W^{1,r}(\mathbb{T}^2) \subset C^{0,1-\frac{2}{r}}(\mathbb{T}^2) \quad \text{for} \quad r>2, \tag{3.29}$$

we estimate the Hölder seminorm of ω^β.

LEMMA 16. *For* $r\in(2,\infty)$,

$$[\omega^\beta(t,\cdot)]_{C^{0,1-\frac{2}{r}}(\mathbb{T}^2)} \lesssim \beta^{-1-2\left(\frac{1}{\mathfrak{p}}-\frac{1}{r}\right)_+}\|\omega_0\|_{L^\mathfrak{p}(\mathbb{T}^2)} e^{\left(1-\frac{2}{r}\right)\int_0^t \|\nabla_x u^\beta(t')\|_{L^\infty_x}\mathrm{d}t'}. \tag{3.30}$$

Proof. We note that

$$[\omega^\beta(t,\cdot)]_{C^{0,1-\frac{2}{r}}(\mathbb{T}^2)} = \sup_{x\neq y\in\mathbb{T}^2} \frac{|\omega_0^\beta(X^\beta(0;t,x)) - \omega_0^\beta(X^\beta(0;t,y))|}{|x-y|^{\left(1-\frac{2}{r}\right)}} \le [\omega_0^\beta]_{C^{0,1-\frac{2}{r}}(\mathbb{T})} \|\nabla_x X^\beta(0;t,\cdot)\|_{L^\infty_x}^{\left(1-\frac{2}{r}\right)}, \tag{3.31}$$

where we slightly abused the notation by

$$|x-y| = \mathrm{dist}_{\mathbb{T}^2}(x,y). \tag{3.32}$$

Applying Morrey's inequality (3.29) for $[\omega_0^\beta]_{C^{0,1-\frac{2}{r}}(\mathbb{T}^2)}$ and applying (3.28) gives the result. ∎

The following standard estimate is important in the proof.

Lemma 17. *Let* (u^β, ω^β) *satisfy* (3.5). *Then, for any* $\gamma > 0$,

$$\|\nabla_x u\|_{L^\infty(\mathbb{T}^2)} \lesssim 1 + \|\omega\|_{L^1(\mathbb{T}^2)} + \|\omega\|_{L^\infty(\mathbb{T}^2)} \log_+([\omega]_{C^{0,\gamma}(\mathbb{T}^2)}). \tag{3.33}$$

Proof. The result is well known from the potential theory (e.g., [40]), so we just briefly sketch the proof. Assume that $\omega \in L^1(\mathbb{T}^2) \cap C^{0,\gamma}(\mathbb{T}^2)$. From (1.10) and (3.19), for $R \ge d > 0$, there exists $C_2 > 0$ only depending on the spatial dimension (2 in our case)

$$\partial_{x_j} u_i(x) = \int_{|x-y|\ge R} \partial_j \mathbf{b}_i(x-y)\omega(y)\mathrm{d}y + \int_{d\le|x-y|\le R} \partial_j \mathbf{b}_i(x-y)\omega(y)\mathrm{d}y + \int_{|x-y|\le d} \partial_j \mathbf{b}_i(x-y)[\omega(y)-\omega(x)]\mathrm{d}y + C_2\delta_{i+1,j}\omega(x) \tag{3.34}$$

for

$$\partial_j \mathbf{b}(x-y) := \frac{1}{2\pi}\left(\frac{2(x_{i+1}-y_{i+1})(x_j-y_j)}{|x-y|^4} - \frac{\delta_{i+1,j}}{|x-y|^2}\right) + \partial_j \mathfrak{B}(x-y). \tag{3.35}$$

Here, the index $i+1$ should be understood on a modulus of 2, and $\delta_{i+1,j} = 1$ if $i+1 = j$ mod 2 and $\delta_{i+1,j} = 0$ if $i+1 \neq j$ mod 2. We bound (3.34) as

$$\begin{aligned} |(3.34)| &\le \int_{|x-y|\ge R} \frac{4}{|x-y|^2}|\omega(y)|\mathrm{d}y + \int_{d\le|x-y|\le R} \frac{4}{|x-y|^2}|\omega(y)|\mathrm{d}y \\ &\quad + [\omega]_{C^{0,\gamma}(\mathbb{T}^2)} \int_{|x-y|\le d} \frac{4}{|x-y|^{2-\gamma}}\mathrm{d}y + C_2|\omega(x)| \\ &\lesssim R^{-1/2}\|\omega\|_{L^1(\mathbb{T}^2)} + \ln\left(\frac{R}{d}\right)\|\omega\|_{L^\infty(\mathbb{T}^2)} + d^\gamma[\omega]_{C^{0,\gamma}(\mathbb{T}^2)} + \|\omega\|_{L^\infty(\mathbb{T}^2)}. \end{aligned} \tag{3.36}$$

We finalize the proof by choosing $R = 1$ and $d = \max\left(1, [\omega]^{1/\gamma}_{C^{0,\gamma}(\mathbb{T}^2)}\right)$. ∎

Proof of Theorem 4. To prove (3.23), we apply $(3.21)|_{r=1,\infty}$ and $(3.30)|_{r>2}$ to (3.33)| to conclude that

$$\begin{aligned} &\|\nabla u^\beta(t,\cdot)\|_{L^\infty}/\|\omega_0\|_{L^\mathfrak{p}} \\ &\lesssim 1 + \beta^{-\frac{2}{\mathfrak{p}}} \log_+\left(\beta^{-1-2(\frac{1}{\mathfrak{p}}-\frac{1}{r})_+}\|\omega_0\|_{L^\mathfrak{p}} e^{\int_0^t \|\nabla_x u^\beta(s)\|_{L^\infty_x}\mathrm{d}s}\right) \\ &\lesssim 1 + \beta^{-\frac{2}{\mathfrak{p}}}\left\{\log_+\frac{1}{\beta} + \log_+\|\omega_0\|_{L^\mathfrak{p}} + \int_0^t \|\nabla u^\beta(s,\cdot)\|_{L^\infty}\mathrm{d}s\right\}. \end{aligned} \tag{3.37}$$

Applying Gronwall's inequality gives the result. ∎

3.3.2. Bounds for $V(\beta)$. We introduce the growth-of-estimate function for $(u^\beta, p^\beta, \tilde{u}^\beta, \tilde{p}^\beta)$, which is a function of β:

$$V(\beta) := \sum_{s_1+s_2\le 2, D\in\{\partial_t,\partial\}} \|\partial^{s_1} D(u^\beta, \partial u^\beta, p^\beta, \tilde{u}^\beta, \tilde{p}^\beta)\|_{L^\infty_{t,x}} \times \left(1 + \|\partial^{s_2}(\tilde{u}^\beta, u^\beta)\|_{L^\infty_{t,x}}\right)\left(1+\sum_{j\le 2}\|\partial^j u^\beta\|_{L^\infty_{t,x}}\right)^2. \tag{3.38}$$

This is a pointwise bound for all derivatives of $(u^\beta, p^\beta, \tilde{u}^\beta, \tilde{p}^\beta)$ appearing in the remainder estimates in section 2.5.

We have the following explicit bound for $V(\beta)$.

Theorem 5. *Suppose that $\omega_0 \in L^{\mathfrak{p}}(\mathbb{T}^2)$. Then,*

$$V(\beta) \lesssim \left(\|\tilde{u}_0\|_{H^6(\mathbb{T}^2)} + TU(\beta,\mathfrak{p})e^{TU(\beta,\mathfrak{p})} + U(\beta,\mathfrak{p})\right)^6,$$

where $U(\beta)$ is as defined in (3.39).

Proof. By Sobolev embedding and the formula for p^β, $\tilde{p}^\beta$, $\partial_t u^\beta$, and $\partial_t\tilde{u}^\beta$, we have a bound

$$V(\beta) \lesssim \left(\|u^\beta\|_{L^\infty((0,T);H^8(\mathbb{T}^2))} + \|\tilde{u}^\beta\|_{L^\infty((0,T);H^6(\mathbb{T}^2))}\right)^6.$$

We invoke the standard energy, commutator estimate, and algebra property of $H^s(\mathbb{T}^2)$, $s>1$:

$$\begin{aligned}
&\frac{\mathrm{d}}{2\mathrm{d}t}\|\partial^8 u^\beta(t)\|^2_{L^2(\mathbb{T}^2)}\\
&\le \|\partial^8 u^\beta(t)\|_{L^2(\mathbb{T}^2)}\|[\![\partial^8, u^\beta\cdot\nabla_x]\!]u\|_{L^2(\mathbb{T}^2)} \lesssim \|\nabla_x u^\beta u^\beta\|_{L^\infty(\mathbb{T}^2)}\|\partial^8 u^\beta(t)\|^2_{L^2(\mathbb{T}^2)},\\
&\frac{\mathrm{d}}{2\mathrm{d}t}\|\partial^6 \tilde{u}^\beta(t)\|^2_{L^2(\mathbb{T}^2)}\\
&\lesssim \|\partial^6\tilde{u}^\beta(t)\|_{L^2(\mathbb{T}^2)}\\
&\quad\times\left(\|[\![\partial^6, u^\beta\cdot\nabla_x]\!]\tilde{u}(t)\|_{L^2(\mathbb{T}^2)} + \|\partial^6\tilde{u}^\beta(t)\|_{L^2(\mathbb{T}^2)}\|\partial^7 u^\beta(t)\|_{L^2(\mathbb{T}^2)} + \|\partial^8 u^\beta(t)\|_{L^2(\mathbb{T}^2)}\right)\\
&\lesssim \|\partial^8 u^\beta(t)\|_{L^2(\mathbb{T}^2)}\|\partial^6\tilde{u}^\beta(t)\|^2_{L^2(\mathbb{T}^2)} + \|\partial^8\tilde{u}^\beta(t)\|^2_{L^2(\mathbb{T}^2)}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\|u^\beta\|_{L^\infty((0,T);H^8(\mathbb{T}^2))} &\lesssim e^{\|\nabla_x u^\beta\|_{L^\infty((0,T)\times\mathbb{T}^2)}}\|u^\beta(0)\|_{L^\infty((0,T);H^8(\mathbb{T}^2))}\\
&\lesssim e^{\mathfrak{Lip}(\beta,\mathfrak{p})}\beta^{-8-2\left(\frac{1}{\mathfrak{p}}-\frac{1}{2}\right)_+}\|\omega_0\|_{L^{\mathfrak{p}}} =: U(\beta,\mathfrak{p}),\\
\|\tilde{u}^\beta\|_{L^\infty((0,T);H^6(\mathbb{T}^2))} &\lesssim e^{\|u^\beta\|_{L^\infty((0,T);H^8(\mathbb{T}^2))}T}\left(\|\tilde{u}_0\|_{H^6(\mathbb{T}^2)} + T\|u\|_{L^\infty((0,T);H^8(\mathbb{T}^2))}\right)\\
&\lesssim (\|\tilde{u}_0\|_{H^6(\mathbb{T}^2)} + TU(\beta,\mathfrak{p}))e^{TU(\beta,\mathfrak{p})}.
\end{aligned} \tag{3.39}$$

∎

4. Vorticity convergence of the approximate solutions of Euler.

4.1. Stability of the regular Lagrangian flow when the vorticity is unbounded.

To study the stability of the regular Lagrangian flow when the vorticities do not belong to L^∞, we adopt the functional used in [2, 15, 8]; for $(u^{\beta_i}, X^{\beta_i})$ solving (3.8),

$$\Lambda(s;t) = \Lambda^{\beta_1,\beta_2}(s;t) := \int_{\mathbb{T}^2} \log\left(1 + \frac{|X^{\beta_1}(s;t,x) - X^{\beta_2}(s;t,x)|}{\lambda}\right) \mathrm{d}x, \tag{4.1}$$

where we again abused the notation

$$|X^{\beta_1}(s;t,x) - X^{\beta_2}(s;t,x)| = \mathrm{dist}_{\mathbb{T}^2}(X^{\beta_1}(s;t,x), X^{\beta_2}(s;t,x)), \tag{4.2}$$

that is, the geodesic distance between $X^{\beta_1}(s;t,x)$ and $X^{\beta_2}(s;t,x)$. We note that

$$\Lambda(t;t) = 0 \tag{4.3}$$

due to the last condition in both (1.12) and (3.8). From (1.12) and (3.8), a direct computation yields that

$$\begin{aligned}
|\dot{\Lambda}(s;t)| &\le \int_{\mathbb{T}^2} \frac{|\dot{X}^{\beta_1}(s) - \dot{X}^{\beta_2}(s)|}{\lambda + |X^{\beta_1}(s) - X^{\beta_2}(s)|} \mathrm{d}x \\
&\le \int_{\mathbb{T}^2} \frac{|u^{\beta_1}(s, X^{\beta_1}(s)) - u^{\beta_2}(s, X^{\beta_2}(s)|}{\lambda + |X^{\beta_1}(s) - X^{\beta_2}(s)|} \mathrm{d}x \\
&\le \int_{\mathbb{T}^2} \frac{|u^{\beta_1}(s, X^{\beta_1}(s)) - u^{\beta_1}(s, X^{\beta_2}(s)|}{\lambda + |X^{\beta_1}(s) - X^{\beta_2}(s)|} \mathrm{d}x \qquad (4.4)\\
&\quad + \int_{\mathbb{T}^2} \frac{|u^{\beta_1}(s, X^{\beta_2}(s)) - u^{\beta_2}(s, X^{\beta_2}(s)|}{\lambda + |X^{\beta_1}(s) - X^{\beta_2}(s)|} \mathrm{d}x. \qquad (4.5)
\end{aligned}$$

PROPOSITION 8 ([15, 8]). *Let $(u^{\beta_i}, \omega^{\beta_i})$ satisfy (3.6), (3.5), and (3.7), and let X^{β_i} be the regular Lagrangian flow of (3.8) for $i = 1, 2$. Suppose that $\|u^{\beta_1} - u^{\beta_2}\|_{L^1((0,T);L^1(\mathbb{T}^2))} \ll 1$. Then,*

$$\begin{aligned}
&\|X^{\beta_1}(s;t,\cdot) - X^{\beta_2}(s;t,\cdot)\|_{L^1(\mathbb{T}^2)} \\
&\lesssim \frac{1 + \|\nabla u^{\beta_1}\|_{L^1((0,T);L^p(\mathbb{T}^2))}}{|\log \|u^{\beta_1} - u^{\beta_2}\|_{L^1((0,T);L^1(\mathbb{T}^2))}|} \quad \textit{for} \ \ p > 1.
\end{aligned} \tag{4.6}$$

For $p = 1$, for every $\delta > 0$, there exists $C_\delta > 0$ such that, for every $\gamma > 0$,

$$\begin{aligned}
&\mathscr{L}^2(\{x \in \mathbb{T}^2 : |X^{\beta_1}(s;t,x) - X^{\beta_2}(s;t,x)| > \gamma\}) \\
&\le \frac{e^{\frac{4C_\delta}{\delta}}}{\frac{4C_\delta}{\delta}} \frac{\|u^{\beta_1} - u^{\beta_2}\|_{L^1((0,T);L^1(\mathbb{T}^2))}}{\gamma} + \varepsilon
\end{aligned} \tag{4.7}$$

holds.

For the convenience of the reader, we provide a sketch of the argument. The argument follows the line of [15] for $p > 1$ and that of [8] for $p = 1$.

Proof. For (4.5), using (1.14), we have

$$\begin{aligned}
(4.5) &\le \frac{1}{\lambda} \int_{\mathbb{T}^2} |u^{\beta_1}(s, X^{\beta_2}(s;t,x)) - u^{\beta_2}(s, X^{\beta_2}(s;t,x)| \mathrm{d}x \\
&\le \frac{\mathfrak{C}}{\lambda} \|u^{\beta_1}(s,\cdot) - u^{\beta_2}(s,\cdot)\|_{L^1(\mathbb{T}^2)}
\end{aligned} \tag{4.8}$$

with the common compressibility bound $\mathfrak{C}=1$. In the rest of the proof, we estimate (4.4).

Step 1. *The case of* $p>1$. Recall that the maximal function of u is given by

$$Mu(x)=\sup_{\varepsilon>0}\fint_{B_\varepsilon(x)}|u(y)|\mathrm{d}y=\sup_{\varepsilon>0}\frac{1}{\mathscr{L}^2(B_\varepsilon(x))}\int_{B_\varepsilon(x)}|u(y)|\mathrm{d}y. \tag{4.9}$$

We have the following (e.g., [34], section 2):

$$|u(x)-u(y)|\lesssim|x-y|\{(M\nabla u)(x)+(M\nabla u)(y)\}\quad \text{a.e. } x,y\in\mathbb{T}^2, \tag{4.10}$$

$$\|Mw\|_{L^p(\mathbb{T}^2)}\lesssim\|w\|_{L^p(\mathbb{T}^2)}\quad\text{for}\quad p\in(1,\infty]. \tag{4.11}$$

Now, we bound (4.4) for $p>1$, using (4.10) and (4.11), as

$$\begin{aligned}(4.4)&\le\int_{\mathbb{T}^2}\{M\nabla u^{\beta_1}(s,X^{\beta_1}(s;t,x))+M\nabla u^{\beta_1}(s,X^{\beta_2}(s;t,x))\}\mathrm{d}x\\&\lesssim\|\nabla u^{\beta_1}\|_{L^p(\mathbb{T}^2)}\quad\text{for}\quad p\in(1,\infty].\end{aligned} \tag{4.12}$$

Using the above (4.12) and (4.8), together with (4.3), we derive that

$$\begin{aligned}\Lambda(s;t)&\lesssim\|\nabla u^{\beta_1}\|_{L^1((0,T);L^p(\mathbb{T}^2))}\\&\qquad+\frac{1}{\lambda}\|u^{\beta_1}-u^{\beta_2}\|_{L^1((0,T);L^1(\mathbb{T}^2))}\quad\text{for all}\quad(s,t)\in[0,t]\times[0,T].\end{aligned} \tag{4.13}$$

On the other hand, for any $(s,t)\in[0,t]\times[0,T]$,

$$\mathbf{1}_{|X^{\beta_1}(s;t,x)-X^{\beta_2}(s;t,x)|\ge\gamma}\log\left(1+\frac{|X^{\beta_1}(s;t,x)-X^{\beta_2}(s;t,x)|}{\lambda}\right)\ge\log\left(1+\frac{\gamma}{\lambda}\right). \tag{4.14}$$

Then, (4.14) with $\gamma=\sqrt{\lambda}$, together with the definition (4.1), implies that

$$\mathscr{L}^2(\{x\in\mathbb{T}^2:|X^{\beta_1}(s;t,x)-X^{\beta_2}(s;t,x)|\ge\sqrt{\lambda}\})\le\frac{1}{|\log\sqrt{\lambda}|}\Lambda(s;t). \tag{4.15}$$

Therefore, by applying (4.13) to (4.15), together with $\mathscr{L}^2(\mathbb{T}^2)=1$ and $|x-y|\le\sqrt{2}$ for $x,y\in\mathbb{T}^2$, we establish the stability:

$$\begin{aligned}&\|X^{\beta_1}(s;t,\cdot)-X^{\beta_2}(s;t,\cdot)\|_{L^1(\mathbb{T}^2)}=\int_{\mathbb{T}^2}|X^{\beta_1}(s;t,x)-X^{\beta_2}(s;t,x)|\mathrm{d}x\\&\quad=\int_{|X^{\beta_1}(s;t,\cdot)-X^{\beta_2}(s;t,\cdot)|\le\sqrt{\lambda}}+\int_{|X^{\beta_1}(s;t,\cdot)-X^{\beta_2}(s;t,\cdot)|\ge\sqrt{\lambda}}\\&\quad\le\sqrt{\lambda}+\frac{\sqrt{2}}{|\log\sqrt{\lambda}|}\Lambda(s;t)\\&\quad\lesssim\sqrt{\lambda}+\frac{1}{|\log\sqrt{\lambda}|}\Big\{\|\nabla u^{\beta_1}\|_{L^1((0,T);L^p(\mathbb{T}^2))}+\frac{1}{\lambda}\|u^{\beta_1}-u^{\beta_2}\|_{L^1((0,T);L^1(\mathbb{T}^2))}\Big\}.\end{aligned}$$

Choosing

$$\lambda=\|u^{\beta_1}-u^{\beta_2}\|_{L^1((0,T);L^1(\mathbb{T}^2))}, \tag{4.16}$$

we have that

$$
\begin{aligned}
&\|X^{\beta_1}(s;t,\cdot) - X^{\beta_2}(s;t,\cdot)\|_{L^1(\mathbb{T}^2)} \\
&\lesssim \|u^{\beta_1} - u^{\beta_2}\|^{1/2}_{L^1((0,T);L^1(\mathbb{T}^2))} + \frac{\|\nabla u^{\beta_1}\|_{L^1((0,T);L^p(\mathbb{T}^2))}}{|\log \|u^{\beta_1} - u^{\beta_2}\|_{L^1((0,T);L^1(\mathbb{T}^2))}|}.
\end{aligned} \tag{4.17}
$$

For $\|u^{\beta_1} - u^{\beta_2}\|_{L^1((0,T);L^1(\mathbb{T}^2))} \ll 1$, we prove (4.6).

Step 2. *The case of* $p = 1$. Note that $p = 1$ fails (4.11), but $\|Mu\|_{L^{1,\infty}(\mathbb{T}^2)} \lesssim \|u\|_{L^1(\mathbb{T}^2)}$ only holds instead of (4.11). Here, we recall the quasi-norm of the Lorentz space $L^{p,q}$:

$$
\begin{aligned}
\|u\|_{L^{p,q}(\mathbb{T}^2,\mathrm{m})} &:= p^{1/q} \|\lambda \mathscr{L}^2(\{x\in\mathbb{T}^2 : |u(x)| > \lambda\})^{1/p}\|_{L^q(\mathbb{R}_+,\frac{\mathrm{d}\lambda}{\lambda})}, \\
\|u\|^p_{L^{p,\infty}(\mathbb{T}^2)} = \|u\|^p_{L^{p,\infty}(\mathbb{T}^2,\mathscr{L}^2)} &= \sup_{\lambda>0}\{\lambda^p \mathscr{L}^2(\{x\in\mathbb{T}^2 : |u(x)| > \lambda\})\}.
\end{aligned} \tag{4.18}
$$

For $p = 1$, there exists a map $\tilde{M}$, defined as in Definition 3.1 of [8] with choice of functions in Proposition 4.2 of [8], such that (Theorem 3.3 of [8])

$$
\tilde{M} : \omega \mapsto \tilde{M}\nabla(\nabla^\perp(\Delta)^{-1}\omega) \geq 0 \text{ is bounded in } L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2) \text{ and } L^1(\mathbb{T}^2) \to L^{1,\infty}(\mathbb{T}^2). \tag{4.19}
$$

Note that, if (u, ω) satisfy (1.10) in the sense of distributions, then $\tilde{M}\nabla(B * \omega) = \tilde{M}\nabla u$. The argument follows the line of [8], with translation to the periodic domain by Proposition 7.

PROPOSITION 9. *There exists an operator* $\omega \to U(\omega)$*, which will be denoted by* $\tilde{M}\nabla u$*, defined either on* $L^1(\mathbb{T}^2)$ *or* $L^2(\mathbb{T}^2)$*, satisfying*

$$
\begin{aligned}
&U(\omega)(x) \geq 0, \\
&\|U(\omega)\|_{L^{1,\infty}(\mathbb{T}^2)} \lesssim \|\omega\|_{L^1(\mathbb{T}^2)}, \\
&\|U(\omega)\|_{L^2(\mathbb{T}^2)} \lesssim \|\omega\|_{L^2(\mathbb{T}^2)}.
\end{aligned}
$$

Also, if $\omega \in L^1(\mathbb{T}^2)$ *and* $u = B * \omega$*, then there is a Lebesgue measure 0 set* $\mathcal{N}$ *such that*

$$
|u(x) - u(y)| \leq |x - y|(U(x) + U(y)), x, y \in \mathbb{T}^2 \setminus \mathcal{N}.
$$

Proof. We first identify $x \in \mathbb{T}^2$ with $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]^2 \subset \mathbb{R}^2$, denote $K(y) := \nabla_y^2 G(y) \chi_{\left[-\frac{1}{2},\frac{1}{2}\right]^2}(y), y \in \mathbb{R}^2$, and define

$$
K_0(y) = \frac{1}{4\pi} \frac{y \otimes y - \frac{1}{2}|y|^2 \mathbb{I}_2}{|y|^4}, y \in \mathbb{R}^2.
$$

Also, we regard ω and u as a $\mathbb{Z}^2$-periodic function in $\mathbb{R}^2$: $\omega(x+m) = \omega(x)$, $u(x+m) = u(x)$ for $m \in \mathbb{Z}^2$. Now, for $x \in \left[-\frac{5}{2}, \frac{5}{2}\right]^2 \subset \mathbb{R}^2$, $\int_{\mathbb{R}^2} K(y)\omega(x - y)dy$ is well defined because it is exactly $(\nabla^2 G *_{\mathbb{T}^2} \omega)(x - m)$ for some $m \in \mathbb{Z}^2$ so that $x - m \in \left[-\frac{1}{2}, \frac{1}{2}\right]^2$. Then, we see that $D(x)$, defined by

$$
\begin{aligned}
D(x) &:= \int_{\mathbb{R}^2} K(y)\omega(x - y) - K_0(y)\omega(x - y)\chi_{B_{100}(0)}(x - y)dy \\
&= \int_{\mathbb{R}^2} (K(y)\chi_{\left[-\frac{1}{2},\frac{1}{2}\right]^2}(y) - K_0(y)\chi_{B_{100}(x)}(y))\omega(x - y)dy
\end{aligned}
$$

for $x \in \left[-\frac{5}{2}, \frac{5}{2}\right]$ and $D(x) := 0$ for $x \notin \left[-\frac{5}{2}, \frac{5}{2}\right]$, is bounded. First, since $B_{\mathfrak{r}}(0) \subset B_{100}(x) \cap \left[-\frac{1}{2}, \frac{1}{2}\right]^2$, $(K(y)\chi_{\left[-\frac{1}{2},\frac{1}{2}\right]^2}(y) - K_0(y)\chi_{B_{100}(x)}(y))$ is thus bounded for $y \in$

$B_{\mathfrak{r}}(0)$. For $y \notin B_{\mathfrak{r}}(0)$, $(K(y)\chi_{[-\frac{1}{2},\frac{1}{2}]^2}(y) - K_0(y)\chi_{B_{100}(x)}(y))$ is bounded as well. Finally, $(K(y)\chi_{[-\frac{1}{2},\frac{1}{2}]^2}(y) - K_0(y)\chi_{B_{100}(x)}(y))$ is supported on $B_{100}(x)$; thus, we have

$$|D(x)| \lesssim \int_{\mathbb{R}^2} \chi_{B_{100}(x)}(y)|\omega(x-y)|dy \lesssim C\|\omega\|_{L^1(\mathbb{T}^2)}.$$

Furthermore, this implies that $|D| \le C\|\omega\|_{L^1(\mathbb{T}^2)}\chi_{[-\frac{5}{2},\frac{5}{2}]^2}$, so, in fact, $D \in L^1$ as well. Therefore, we have

$$\nabla_x u(x) = D(x) + K_0 \star_{\mathbb{R}^2} (\omega\chi_{B_{100}(0)})(x), x \in \left[-\frac{5}{2}, \frac{5}{2}\right]^2. \tag{4.20}$$

Next, we closely follow the argument of Proposition 4.2 of [8]. Let $\bar{h}$ be a smooth, nonnegative function supported on $B_{\frac{1}{100}}(0)$ with $\int_{\mathbb{R}^2} \bar{h}(y)dy = 1$. Also, we denote $\bar{h}_r(x) = \frac{1}{r^2}\bar{h}\left(\frac{x}{r}\right)$ for $x \in \mathbb{R}^2$ and $r > 0$. Finally, for $\xi \in \mathbb{S}^1$ and $j = 1, 2$ we define

$$\mathfrak{T}^{\xi,j}(w) := h\left(\frac{\xi}{2} - w\right) w_j,$$

and $\mathfrak{T}^{\xi,j}_r$ is similarly defined for $r > 0$. Now, let $x, y \in \mathbb{T}^2 = \left[-\frac{1}{2}, \frac{1}{2}\right]^2$. Then, there exists $\tilde{y} \in \left[-\frac{3}{2}, \frac{3}{2}\right]$, $\tilde{y} - y \in \mathbb{Z}^2$ such that the projection of the line segment of $\tilde{y}$ and x in $\mathbb{R}^2$ is the geodesic connecting x, y in $\mathbb{T}^2$. Then, we have

$$\begin{aligned} u(x) - u(y) &= u(x) - u(\tilde{y}) \\ &= \int_{\mathbb{R}^2} \bar{h}_{|x-\tilde{y}|}\left(z - \frac{x+\tilde{y}}{2}\right)(u(x) - u(z))dz + \int_{\mathbb{R}^2} \bar{h}_{|x-\tilde{y}|}\left(z - \frac{x+\tilde{y}}{2}\right)(u(z) - u(y))dz. \end{aligned}$$

We focus on the first term; the other gives a similar contribution. Following the argument of Proposition 4.2 of [8], we have

$$\int_{\mathbb{R}^2} \bar{h}_{|x-\tilde{y}|}\left(z - \frac{x+\tilde{y}}{2}\right)(u(x) - u(z))dz = |x-\tilde{y}|\sum_{j=1}^{2}\int_0^1\int_{\mathbb{R}^2} \mathfrak{T}^{\frac{x-y}{|x-y|},j}_{s|x-\tilde{y}|}(w)(\partial_j u)(x-w)dwds.$$

Note that $\mathfrak{T}^{\frac{x-\tilde{y}}{|x-\tilde{y}|},j}_{s|x-\tilde{y}|}$ is supported on $B_{\frac{1}{100}s|x-\tilde{y}|}\left(\frac{x-y}{2|x-\tilde{y}|}\right)$ and $|x-\tilde{y}| \le \frac{\sqrt{2}}{2}$, so, if $w \in B_{\frac{1}{100}s|x-\tilde{y}|}\left(\frac{x-\tilde{y}}{2|x-\tilde{y}|}\right)$, $|w| \le \frac{2}{3}$, and thus, $x - w \in \left[-\frac{5}{2}, \frac{5}{2}\right]^2$, which implies that (4.20) is satisfied at $x - w$. (Similar consideration shows that at $\tilde{y} - w$, (4.20) is satisfied.) Therefore,

$$\begin{aligned} \left|\int_{\mathbb{R}^2} \bar{h}_{|x-\tilde{y}|}\left(z - \frac{x+\tilde{y}}{2}\right)(u(x) - u(z))dz\right| &\le |x-\tilde{y}|\sum_{j=1}^{2}\int_0^1\left|\left(\mathfrak{T}^{\frac{x-y}{|x-y|},j}_{s|x-\tilde{y}|} \star_{\mathbb{R}^2} D\right)(x)\right|ds \\ &\quad + |x-\tilde{y}|\sum_{j=1}^{2}\int_0^1\left|\left(\mathfrak{T}^{\frac{x-y}{|x-y|},j}_{s|x-\tilde{y}|} \star_{\mathbb{R}^2} (K_0 \star_{\mathbb{R}^2} \omega\chi_{B_{100}})\right)(x)\right|ds \\ &\le |x-\tilde{y}|\sum_{j=1}^{2}\left(M_{\{\mathfrak{T}^{\xi,j}|\xi\in\mathbb{S}^1\}}(D)(x) + M_{\{\mathfrak{T}^{\xi,j}|\xi\in\mathbb{S}^1\}}(K_0 \star_{\mathbb{R}^2} \omega\chi_{B_{100}})(x)\right), \end{aligned}$$

where

$$M_{\{\mathfrak{T}^{\xi,j}|\xi\in\mathbb{S}^1\}}(g)(x) = \sup_{\{\mathfrak{T}^{\xi,j}|\xi\in\mathbb{S}^1\}}\sup_{r>0}\left|\left(\mathfrak{T}^{\xi,j}_r \star_{\mathbb{R}^2} g\right)(x)\right|, x \in \mathbb{R}^2.$$

By Theorem 3.3 of [8], we have

$$\|M_{\{\mathfrak{T}^{\xi,j}|\xi\in\mathbb{S}^1\}}(K_0 \star_{\mathbb{R}^2} \omega\chi_{B_{100}})\|_{L^{1,\infty}(\mathbb{R}^2)} \le C\|\omega\chi_{B_{100}}\|_{L^1(\mathbb{R}^2)} \le C\|\omega\|_{L^1(\mathbb{T}^2)}.$$

Also, by Young's inequality, we have

$$\|M_{\{\mathfrak{T}^{\xi,j}|\xi\in\mathbb{S}^1\}}(D)\|_{L^{1,\infty}(\mathbb{R}^2)} \le \|M_{\{\mathfrak{T}^{\xi,j}|\xi\in\mathbb{S}^1\}}(D)\|_{L^{\infty}(\mathbb{R}^2)} \le C\|D\|_{L^\infty(\mathbb{R}^2)} \le C\|\omega\|_{L^1(\mathbb{T}^2)}.$$

Finally, for $x\in\mathbb{T}^2$ identified with $\left[\frac{-1}{2},\frac{1}{2}\right]^2$, we define

$$U(x) := \sum_{\tilde{x}\in\left[-\frac{3}{2},\frac{3}{2}\right], x-\tilde{x}\in\mathbb{Z}^2} \sum_{j=1}^{2} \left(M_{\{\mathfrak{T}^{\xi,j}|\xi\in\mathbb{S}^1\}}(D)(\tilde{x}) + M_{\{\mathfrak{T}^{\xi,j}|\xi\in\mathbb{S}^1\}}(K_0 \star_{\mathbb{R}^2} \omega\chi_{B_{100}})(\tilde{x})\right).$$

Then, obviously, for $x,y\in\mathbb{T}^2$,

$$|u(x)-u(y)| \le d_{\mathbb{T}^2}(x,y)(U(x)+U(y)),$$

and if $U(x)>\lambda$, then, for $\tilde{x}_1,\ldots,\tilde{x}_9\in\left[-\frac{3}{2},\frac{3}{2}\right]^2$ such that $\tilde{x}_j - x\in\mathbb{Z}^2$, at least one of $\tilde{x}_i$ satisfies

$$\sum_{j=1}^{2} \left(M_{\{\mathfrak{T}^{\xi,j}|\xi\in\mathbb{S}^1\}}(D)(\tilde{x}_i) + M_{\{\mathfrak{T}^{\xi,j}|\xi\in\mathbb{S}^1\}}(K_0 \star_{\mathbb{R}^2} \omega\chi_{B_{100}})(\tilde{x}_i)\right) > \frac{\lambda}{9};$$

therefore,

$$\begin{aligned}
&\left\{x\in\left[-\frac{1}{2},\frac{1}{2}\right]^2 | U(x)>\lambda\right\} \\
&\quad\subset \bigcup_{m=(a,b),a,b\in\{-1,0,1\}} \Bigg\{ y\in\left[-\frac{1}{2},\frac{1}{2}\right]^2 + m \Big| M_{\{\mathfrak{T}^{\xi,j}|\xi\in\mathbb{S}^1\}}(D)(y) \\
&\qquad\qquad + M_{\{\mathfrak{T}^{\xi,j}|\xi\in\mathbb{S}^1\}}(K_0 \star_{\mathbb{R}^2} \omega\chi_{B_{100}})(y) > \frac{\lambda}{9}\Bigg\} \\
&\quad\subset \left\{ y\in\mathbb{R}^2 | M_{\{\mathfrak{T}^{\xi,j}|\xi\in\mathbb{S}^1\}}(D)(y) + M_{\{\mathfrak{T}^{\xi,j}|\xi\in\mathbb{S}^1\}}(K_0 \star_{\mathbb{R}^2} \omega\chi_{B_{100}})(y) > \frac{\lambda}{9}\right\}.
\end{aligned}$$

Therefore, we see that

$$\|U\|_{L^{1,\infty}(\mathbb{T}^2)} \le C\|\omega\|_{L^1(\mathbb{T}^2)}.$$

Also, if $\omega\in L^2(\mathbb{T}^2)$, we see that

$$\begin{aligned}
\|U\|_{L^2(\mathbb{T}^2)} &\le C(\|M_{\{\mathfrak{T}^{\xi,j}|\xi\in\mathbb{S}^1\}}(K_0 \star_{\mathbb{R}^2} \omega\chi_{B_{100}})\|_{L^2(\mathbb{R}^2)} + \|M_{\{\mathfrak{T}^{\xi,j}|\xi\in\mathbb{S}^1\}}(D)\|_{L^2(\mathbb{R}^2)} \\
&\le C(\|\omega\chi_{B_{100}}\|_{L^2(\mathbb{R}^2)} + \|D\|_{L^2(\mathbb{R}^2)} \le C\|\omega\|_{L^2(\mathbb{T}^2)}
\end{aligned}$$

by, again, Theorem 3.3 of [8] and Young's inequality. ∎

We return to the proof of (4.7). We have (Proposition 4.2 in [8])

$$|u(x)-u(y)| \le |x-y|\{\tilde{M}\nabla u(x) + \tilde{M}\nabla u(y)\} \quad \text{a.e. } x,y\in\mathbb{T}^2. \tag{4.21}$$

Now, we check that $\{\omega^\beta\}$ of (3.7) with (3.6) is equi-integrable (in the sense of (4.25)). Fix any $\varepsilon > 0$. We choose $\delta > 0$ such that

$$\text{if } \mathscr{L}^2(E') < \delta, \text{ then } \int_{E'} |\omega_0(x)|\mathrm{d}x < \frac{\varepsilon}{2\mathfrak{C}}. \tag{4.22}$$

From (3.6) and (1.14), for any Borel set $E \subset \mathbb{T}^2$ with $\mathscr{L}^2(E) < \delta/\mathfrak{C}$,

$$\begin{aligned} \|\omega^\beta(t,x)\|_{L^1(E)} &= \|\omega_0^\beta(X^\beta(0;t,x))\|_{L^1(\{x\in E\})} \\ &\le \mathfrak{C}\int_{X^\beta(t;0,x)\in E} |\omega_0^\beta(x)|\mathrm{d}x \\ &\le \mathfrak{C}\int_{\mathbb{R}^2}\left(\int_{\mathbb{T}^2} \mathbf{1}_{X^\beta(t;0,x)\in E}|\omega_0(x-y)|\mathrm{d}x\right)\varphi^\beta(y)\mathrm{d}y, \end{aligned} \tag{4.23}$$

where ω_0 is regarded as a $\mathbb{Z}^2$-periodic function. For $y \in \mathbb{R}^2$, we define

$$\tilde{E}_y := \{\tilde{x} \in \mathbb{R}^2 : X^\beta(t;0,\tilde{x}+y) \in E + \mathbb{Z}^2\}/\mathbb{Z}^2 \subset \mathbb{T}^2.$$

From (1.14) and the fact that $x \mapsto x - y$ is measure-preserving for fixed y, we have

$$\mathscr{L}^2(\{\tilde{x} \in \tilde{E}_y\}) = \mathscr{L}^2(\{x \in \mathbb{T}^2 : X^\beta(t;0,x) \in E\}) \le \mathfrak{C}\mathscr{L}^2(E) < \delta. \tag{4.24}$$

Therefore, applying (4.24) to (4.22), we have that, from (4.23),

$$\text{if } \mathscr{L}^2(E) < \delta/\mathfrak{C}, \text{ then } \|\omega^\beta(t,\cdot)\|_{L^1(E)} \le \|\varphi^\beta\|_{L^1(\mathbb{R}^2)} \sup_{y\in\mathbb{R}^2} \mathfrak{C}\int_{\tilde{x}\in\tilde{E}_y} |\omega_0(\tilde{x})|\mathrm{d}\tilde{x} < \varepsilon. \tag{4.25}$$

Since ω^β is equi-integrable, for every $\delta > 0$, there exists $C_\delta > 0$ and a Borel set $A_\delta \subset \mathbb{T}^2$ such that $\omega^\beta = \omega_1^\beta + \omega_2^\beta$ such that $\|\omega_1^\beta\|_{L^1} \le \delta$ and $\mathrm{supp}(\omega_2^\beta) \subset A_\delta$, $\|\omega_2^\beta\|_{L^2} \le C_\delta$ (Lemma 5.8 of [8], whose proof can be established by noting that equi-integrability with $\sup_\beta \|\omega^\beta\|_{L^1} < \infty$ is equivalent to $\lim_{K\to\infty}\sup_\beta \int_{\{|\omega^\beta|>K\}\cap\mathbb{T}^2} |\omega^\beta|dx = 0$). Now, apply (4.21) to (4.4), and use the decomposition of $u^\beta = u_1^\beta + u_2^\beta$ with $u_i^\beta = \nabla^\perp(-\Delta)^{-1}$ ω_i^β to derive that

$$\begin{aligned} &(4.4) \le \int_{\mathbb{T}^2} U_1^\lambda(s;t,x)\mathrm{d}x + \int_{\mathbb{T}^2} U_2^\lambda(s;t,x)\mathrm{d}x, \\ &U_i^\lambda(s;t,x) := \min\left\{\frac{|u_i^{\beta_1}(s,X^{\beta_1}(s;t,x))|}{\lambda} + \frac{|u_i^{\beta_1}(s,X^{\beta_2}(s;t,x))|}{\lambda},\right. \\ &\qquad\left. \tilde{M}\nabla u_i^{\beta_1}(s,X^{\beta_1}(s;t,x)) + \tilde{M}\nabla u_i^{\beta_1}(s,X^{\beta_2}(s;t,x))\right\} \ge 0. \end{aligned} \tag{4.26}$$

For U_2^λ, we use (1.14) and (4.19) and simply derive that

$$\|U_2^\lambda(s;t,\cdot)\|_{L^2(\mathbb{T}^2)} \le \mathfrak{C}\min\left\{\frac{2\|u_2^{\beta_1}(s)\|_{L^2(\mathbb{T}^2)}}{\lambda}, \|\omega_2^{\beta_1}\|_{L^2}\right\} \le \mathfrak{C}C_\delta. \tag{4.27}$$

For U_1^λ, using (4.19),

$$\|U_1^\lambda(s;t,\cdot)\|_{L^{1,\infty}} \lesssim \min\left\{\frac{\|u_1^\beta(s)\|_{L^{1,\infty}}}{\lambda}, \|\omega_1\|_{L^1(\mathbb{T}^2)}\right\} \le \|\omega_1\|_{L^1(\mathbb{T}^2)} \le \delta,$$

$$\|U_\lambda^1(s;t,\cdot)\|_{L^{p,\infty}} \lesssim \|U_\lambda^1(s;t,\cdot)\|_{L^p} \lesssim \min\left\{\frac{\|u_1^\beta(s)\|_{L^p}}{\lambda}, \|\omega_1\|_{L^p(\mathbb{T}^2)}\right\} \lesssim \frac{\|u_1^\beta(s)\|_{L^p}}{\lambda} \lesssim \frac{\delta}{\lambda}$$

for some $p \in (1,2)$, using fractional integration.

Using the interpolation $\|g\|_{L^1(\mathbb{T}^2)} \lesssim \|g\|_{L^{1,\infty}}\{1+\log(\frac{\|g\|_{L^{p,\infty}}}{\|g\|_{L^{1,\infty}}})\}$ (Lemma 2.2 of [8]), we end up with

$$\|U^1_\lambda(s;t,\cdot)\|_{L^1} \lesssim \|U^1_\lambda(s,\cdot)\|_{L^{1,\infty}}\left\{1+\log_+\left(\frac{\|U^1_\lambda(s,\cdot)\|_{L^{p,\infty}}}{\|U^1_\lambda(s,\cdot)\|_{L^{1,\infty}}}\right)\right\} \lesssim \delta+\delta|\log\lambda|, \tag{4.28}$$

where we have used that the map $z\to z(1+\log_+(K/z))$ is nondecreasing for $z\in[0,\infty)$.

Together with (4.8), (4.27), and (4.28), we conclude that

$$\begin{aligned}\Lambda(s;t) &\le \int_s^t |\dot\Lambda(\tau;t)|\mathrm{d}\tau \le \int_0^t \{(4.4)+(4.5)\}\mathrm{d}s\\ &\le \int_0^t \left\{\|U^1_\lambda(s;t,\cdot)\|_{L^1(\mathbb{T}^2)}+\|U^2_\lambda(s;t,\cdot)\|_{L^2(\mathbb{T}^2)}+\frac{\mathfrak{C}}{\lambda}\|u^{\beta_1}(s,\cdot)-u^{\beta_2}(s,\cdot)\|_{L^1(\mathbb{T}^2)}\right\}\mathrm{d}s\\ &\le \mathfrak{C}C_\delta T+\delta\{1+|\log\lambda|\}T+\frac{\mathfrak{C}}{\lambda}\|u^{\beta_1}-u^{\beta_2}\|_{L^1((0,T);L^1(\mathbb{T}^2))}.\end{aligned}$$

From this inequality and (4.14) and (4.3), we derive that

$$\begin{aligned}&\mathscr{L}^2(\{x\in\mathbb{T}^2 : |X^{\beta_1}(s;t,x)-X^{\beta_2}(s;t,x)|>\gamma\})\\ &\le \frac{\Lambda(s;t)}{\log\left(1+\frac{\gamma}{\lambda}\right)} \lesssim \frac{\|u^{\beta_1}-u^{\beta_2}\|_{L^1L^1}}{\lambda\log(1+\frac{\gamma}{\lambda})|}+\frac{C_\delta}{|\log(1+\frac{\gamma}{\lambda})|}+\delta\end{aligned} \tag{4.29}$$

for $\lambda,\gamma\in(0,1/e)$. Here, for the last term, we have used that, for $0<\lambda<1/e$ and $0<\gamma<1/e$,

$$\frac{\delta|\log\lambda|}{|\log(1+\frac{\gamma}{\lambda})|}=\delta\frac{|\log\lambda|}{-\log\lambda+\log(\lambda+\gamma)}=\delta\frac{|\log\lambda|}{|\log\lambda|-|\log(\lambda+\gamma)|}\le\delta\frac{|\log\lambda|}{|\log\lambda|}\le\delta.$$

Choose

$$\lambda=\lambda_{\delta,\gamma}=(e^{\frac{4C_\delta}{\delta}}-1)^{-1}\gamma. \tag{4.30}$$

Note that $\log(1+\frac{\gamma}{\lambda_{\varepsilon,\gamma}})=\log(e^{\frac{4C_\delta}{\delta}})=\frac{4C_\delta}{\delta}$. Then, (4.29) yields (4.7). ∎

4.2. Convergence of the velocity field u^β.

LEMMA 18. *Let $T>0$. Assume that* (3.5) *holds and that*

$$\sup_\beta\|\omega^\beta\|_{L^\infty((0,T);L^1(\mathbb{T}^2))}<\infty,\quad \sup_\beta\|u^\beta\|_{L^\infty((0,T);L^2(\mathbb{T}^2))}<\infty.$$

Then, there exists a subsequence $\{\beta'\}\subset\{\beta\}$ such that $u^{\beta'}$ is Cauchy in $L^1((0,T);L^1(\mathbb{T}^2))$.

Proof. The proof is due to the elliptic regularity; the Frechet–Kolmogorov theorem, which states that $W^{s,p}(\mathbb{T}^2)\hookrightarrow\hookrightarrow L^q(\mathbb{T}^2)$ for $s>0$ and $1\le q\le p<\infty$; and the Aubin–Lions lemma, which states that, for reflexive Banach spaces X,Y,Z such that $Y\hookrightarrow\hookrightarrow X\hookrightarrow Z$,

$$W^{1,r}((0,T);Z)\cap L^1((0,T);Y)\hookrightarrow\hookrightarrow L^1((0,T);X) \text{ for } r>1. \tag{4.31}$$

Note that, from $L^1(\mathbb{T}^2)\hookrightarrow H^s(\mathbb{T}^2)$, for any $s<-1$,

$$\omega^\beta\in C^0([0,T];H^s(\mathbb{T}^2)) \text{ uniformly-in-}\beta \text{ for any } s<-1.$$

On the other hand, we have $-\Delta p^\beta = \mathrm{div}(\mathrm{div}(u^\beta \otimes u^\beta))$ with $⨍_{\mathbb{T}^2} p^\beta = 0$. Since $u^\beta \in L^\infty((0,T);L^2)$ uniformly-in-β, $u^\beta \otimes u^\beta \in L^\infty((0,T);L^1(\mathbb{T}^2))$ uniformly-in-β. Using $L^1(\mathbb{T}^2) \hookrightarrow H^s(\mathbb{T}^2)$ for $s < -1$, an elliptic regularity says that $L^\infty((0,T);H^{s-1}(\mathbb{T}^2)) \ni \mathrm{div}(u^\beta \otimes u^\beta) \mapsto \nabla p^\beta \in L^\infty((0,T);H^{s-1}(\mathbb{T}^2))$ uniformly-in-β. Therefore, from $\partial_t u^\beta = -\mathrm{div}(u^\beta \otimes u^\beta) - \nabla p^\beta$, we derive that $\partial_t u^\beta \in L^\infty((0,T);H^{s-1})$ uniformly-in-β for any $s < -1$. Therefore, we conclude that

$$u^\beta \in W^{1,\infty}((0,T);H^{-5/2}(\mathbb{T}^2)) \text{ uniformly-in-}\beta. \tag{4.32}$$

Next, we note that $L^1(\mathbb{T}^2) \hookrightarrow W^{-\frac{3}{4},3}(\mathbb{T}^2)$. This is a consequence of an embedding $W^{\frac{3}{4},3}(\mathbb{T}^2) \hookrightarrow L^\infty(\mathbb{T}^2)$ (note $\frac{3}{4} > \frac{2}{3}$) and the duality argument $L^1(\mathbb{T}^2) \hookrightarrow (L^\infty(\mathbb{T}^2))^* \hookrightarrow (W^{\frac{3}{4},3}(\mathbb{T}^2))^* = W^{-\frac{3}{4},\frac{3}{2}}(\mathbb{T}^2)$. Therefore, we derive that $\omega^\beta \in L^\infty((0,T);W^{-\frac{3}{4},\frac{3}{2}}(\mathbb{T}^2))$. Now, applying the elliptic regularity theory to (3.5), we derive that

$$u^\beta \in L^\infty((0,T);W^{\frac{1}{4},\frac{3}{2}}(\mathbb{T}^2)) \;\; \text{uniformly-in-}\beta. \tag{4.33}$$

Now, we set $Y = W^{\frac{1}{4},\frac{3}{2}}(\mathbb{T}^2), X = L^1(\mathbb{T}^2), Z = H^{-\frac{5}{2}}(\mathbb{T}^2)$. Using the Frechet–Kolmogorov theorem, we have $Y = W^{\frac{1}{4},\frac{3}{2}}(\mathbb{T}^2) \hookrightarrow\hookrightarrow X = L^1(\mathbb{T}^2) \hookrightarrow Z = H^{-\frac{5}{2}}(\mathbb{T}^2)$. Finally, we prove Lemma 18 using the Aubin–Lions lemma (4.31). ∎

4.3. Rate of convergence of u^β: Localized Yudovich solutions. We use the following version of the theorem, presented in [16]. The theorem in [16] provides the modulus of continuity for u that we will use and explicitly states that the unique solution is regular Lagrangian.

We begin with introducing the localized Yudovich class of vorticity. Intuitively, the localized Yudovich class consists of vorticities with moderate growth of L^p norm as $p \to \infty$. The existence and uniqueness results of the Yudovich class of vorticity extend to the localized Yudovich class. We refer to [16] and references therein for further details.

$$\|\omega\|_{Y^\Theta_{\mathrm{ul}}(\mathbb{T}^2)} := \sup_{1 \le \mathfrak{p} < \infty} \frac{\|\omega\|_{L^\mathfrak{p}(\mathbb{T}^2)}}{\Theta(\mathfrak{p})}.$$

In this paper, we focus on the growth function with the following condition, which gives quantitative bounds on the behavior of velocity field u; it would be interesting to see if one can generalize the presented results to arbitrary admissible growth functions. We assume that $\Theta : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ satisfies the following: There exists $m \in \mathbb{Z}_{>0}$ such that

$$\Theta(\mathfrak{p}) = \prod_{k=1}^{m} \log_k \mathfrak{p} \tag{4.34}$$

for large $\mathfrak{p} > 1$, where $\log_k \mathfrak{p}$ is defined inductively by $\log_1 \mathfrak{p} = \log \mathfrak{p}$ and

$$\log_{k+1} \mathfrak{p} = \log \log_k \mathfrak{p}.$$

Also, we adopt the convention that $\log_0 \mathfrak{p} = 1$. We remark that we are only interested in the behavior of Θ for large $\mathfrak{p}$. Also, we denote the inverse function of $\log_m(\mathfrak{p})$ (defined for large $\mathfrak{p}$) by e_m. Finally, we note that

$$\int_{e_m(1)}^{\infty} \frac{1}{\mathfrak{p}\Theta(\mathfrak{p})} = \infty,$$

which turns out to be important in the uniqueness of the solution.

THEOREM 6 ([16]). *If $\omega_0 \in Y^{\Theta}_{\mathrm{ul}}(\mathbb{T}^2)$, for every $T > 0$, there exists a unique weak solution $\omega \in L^\infty([0,T]; Y^{\Theta}_{\mathrm{ul}}(\mathbb{T}^2))$ with $u \in L^\infty([0,T]; C_b^{0,\varphi_\Theta}(\mathbb{T}^2, \mathbb{R}^2))$, which is a regular Lagrangian. Here, the function space $C_b^{0,\varphi_\Theta}(\mathbb{T}^2,\mathbb{R}^2)$ is defined by*

$$C_b^{0,\varphi_\Theta}(\mathbb{T}^2,\mathbb{R}^2) = \left\{ v \in L^\infty(\mathbb{T}^2,\mathbb{R}^2) | \sup_{x\neq y} \frac{|v(x)-v(y)|}{\varphi_\Theta(d(x,y)) < \infty} \right\},$$

where $d(x,y)$ is the geodesic distance on the torus $\mathbb{T}^2 = \mathbb{T}^2$ and φ_Θ is defined by

$$\varphi_\Theta(r) = \begin{cases} 0, r = 0, \\ r(1-\log r)\Theta(1-\log r), r \in (0, e^{-2}), \\ e^{-2}3\Theta(3), r \geq e^{-2}. \end{cases}$$

Also, $\|\omega\|_{L^\infty([0,T];Y^{\Theta}_{\mathrm{ul}}(\mathbb{T}^2))}$ and $\|u\|_{C_b^{0,\varphi_\Theta}(\mathbb{T}^2,\mathbb{R}^2)}$ depend only on $\|\omega_0\|_{Y^{\Theta}_{\mathrm{ul}}(\mathbb{T}^2)}$ and T. The dependence is nondecreasing in both $\|\omega_0\|_{Y^{\Theta}_{\mathrm{ul}}(\mathbb{T}^2)}$ and T.

In this subsection, we prove the following proposition.

PROPOSITION 10. *Let $\omega_0 \in Y^{\Theta}_{\mathrm{ul}}(\mathbb{T}^2)$. There exist constants M, depending only on m and $\sup_{t\in[0,T]} \|u(t)\|_{L^\infty}$ (and therefore, $\|\omega_0\|_{L^3}$) (and dimension $d = 2$), and C ($C = 2e$ works), which is universal, such that*

(4.35)
$$\sup_{0\leq t\leq T} \|u^\beta(t) - u(t)\|^2_{L^2(\mathbb{T}^2)} \leq \frac{M}{e_m\left(\left(\log_m\left(\frac{M}{\beta^2\|\omega_0\|^2_{L^2(\mathbb{T}^2)}}\right)\right)^{e^{-C\|\omega_0\|_{Y^{\Theta}_{\mathrm{ul}}}T}}\right)} =: \mathrm{Rate}(\omega_0;\beta).$$

Note that $\lim_{\beta\to 0^+} \mathrm{Rate}(\omega_0;\beta) = 0$.

In particular, the case $m = 0$ corresponds to the Yudovich class with $\mathrm{Rate}(\beta) = \beta^{2e^{-C\|\omega_0\|_{Y^{\Theta}_{\mathrm{ul}}}T}}$.

Proof. We follow the proof of [45]. By letting $v = u^\beta - u$, we have

$$\partial_t v + u^\beta \cdot \nabla_x v - v \cdot \nabla_x u + \nabla_x(p^\beta - p) = 0.$$

Noting that v is incompressible and taking the L^2 norm of v, we obtain

$$\frac{d}{2dt}\|v\|^2_{L^2(\mathbb{T}^2)} \leq \int_{\mathbb{T}^2} v \cdot \nabla_x u \cdot v \mathrm{d}x$$

or

$$\|v(t)\|^2_{L^2(\mathbb{T}^2)} \leq \|v(0)\|_{L^2(\mathbb{T}^2)} + 2\int_0^t \int_{\mathbb{T}^2} |\nabla_x u||v|^2 \mathrm{d}x.$$

Next, we note that, by Sobolev embedding,

$$\|v\|^2_{L^\infty(\mathbb{T}^2)} \leq 2(\|u\|^2_{L^\infty(\mathbb{T}^2)} + \|u^\beta\|^2_{L^\infty(\mathbb{T}^2)}) \leq 2C\|\omega_0\|^2_{L^3(\mathbb{T}^2)},$$

while energy conservation gives

$$\|v(t)\|^2_{L^2(\mathbb{T}^2)} \leq 2(\|u^\beta(t)\|^2_{L^2(\mathbb{T}^2)} + \|u(t)\|^2_{L^2(\mathbb{T}^2)}) \leq 4\|u_0\|^2_{L^2(\mathbb{T}^2)}.$$

Therefore, there exists a constant M, explicitly given by

$$M := 1 + 4\|u_0\|^2_{L^2(\mathbb{T}^2)} e_m(1) + 2C\|\omega_0\|^2_{L^3},$$

satisfying

$$\frac{M}{\|v(t)\|^2_{L^2(\mathbb{T}^2)}} > e_m(1), \|v(t)\|^2_{L^\infty(\mathbb{T}^2)} \leq M.$$

Then, by the definition of Y^Θ_{ul} and the Calderon–Zygmund inequality

$$\|\nabla_x u\|_{L^{\mathfrak{p}}(\mathbb{T}^2)} \leq C\mathfrak{p}\|\omega\|_{L^{\mathfrak{p}}(\mathbb{T}^2)}$$

for $p \in (1, \infty)$, we have

$$\|\nabla_x u\|_{L^{\mathfrak{p}}(\mathbb{T}^2)} \leq \|\omega_0\|_{Y^\Theta_{\mathrm{ul}}} \mathfrak{p}\Theta(\mathfrak{p}) := \|\omega_0\|_{Y^\Theta_{\mathrm{ul}}} \phi(\mathfrak{p}),$$

where we have used the conservation of $\|\omega\|_{L^{\mathfrak{p}}(\mathbb{T}^2)}$ for every $1 \leq \mathfrak{p} < \infty$. We first treat the case of $m \geq 1$. By Hölder's inequality, for each $\epsilon \in (0, \frac{1}{e_{m-1}(1)})$ ($\frac{1}{e_{m-1}(1)} \leq 1$), we have

$$\begin{aligned}
\int_{\mathbb{T}^2} |\nabla_x u||v|^2 dx &\leq \|v\|^{2\epsilon}_{L^\infty(\mathbb{T}^2)} \int |v|^{2(1-\epsilon)}|\nabla_x u| \mathrm{d}x \\
&\leq M^\epsilon \left(\int_{\mathbb{T}^2} |v|^2 \mathrm{d}x\right)^{1-\epsilon} \left(\int_{\mathbb{T}^2} |\nabla_x u|^{\frac{1}{\epsilon}} \mathrm{d}x\right)^\epsilon \\
&\leq M^\epsilon \left(\|v\|^2_{L^2(\mathbb{T}^2)}\right)^{1-\epsilon} \|\omega_0\|_{Y^\Theta_{\mathrm{ul}}} \phi\left(\frac{1}{\epsilon}\right) \\
&= \|\omega_0\|_{Y^\Theta_{\mathrm{ul}}} \|v\|^2_{L^2(\mathbb{T}^2)} \left(\frac{M}{\|v\|^2_{L^2(\mathbb{T}^2)}}\right)^\epsilon \phi\left(\frac{1}{\epsilon}\right).
\end{aligned}$$

Now, choose

$$\epsilon^* = \frac{1}{\log \frac{M}{\|v(t)\|^2_{L^2(\mathbb{T}^2)}}}.$$

Then, since $\frac{M}{\|v(t)\|^2_{L^2(\mathbb{T}^2)}} > e_m(1)$, $\log(\frac{M}{\|v(t)\|^2_{L^2(\mathbb{T}^2)}}) > \log(e_m(1)) = e_{m-1}(1)$ so that $\epsilon^* \in (0, \frac{1}{e_{m-1}(1)})$. There, we have

$$\begin{aligned}
&\left(\frac{M}{\|v\|^2_{L^2(\mathbb{T}^2)}}\right)^{\epsilon^*} \phi\left(\frac{1}{\epsilon^*}\right) \\
&\quad = e \log\left(\frac{M}{\|v(t)\|^2_{L^2(\mathbb{T}^2)}}\right) \log\left(\log\left(\frac{M}{\|v(t)\|^2_{L^2(\mathbb{T}^2)}}\right)\right) \cdots \log_m\left(\log\left(\frac{M}{\|v(t)\|^2_{L^2(\mathbb{T}^2)}}\right)\right) \\
&\quad = e\Theta\left(\frac{M}{\|v(t)\|^2_{L^2(\mathbb{T}^2)}}\right).
\end{aligned}$$

For $m = 0$ (the Yudovich case), $\epsilon \to \left(\frac{M}{\|v\|^2_{L^2(\mathbb{T}^2)}}\right)^\epsilon \phi(\frac{1}{\epsilon}) = \left(\frac{M}{\|v\|^2_{L^2(\mathbb{T}^2)}}\right)^\epsilon \frac{1}{\epsilon}$ attains its minimum at $\epsilon^* = \frac{1}{\log\left(\frac{M}{\|v\|^2_{L^2(\mathbb{T}^2)}}\right)}$, so we choose M such that $\epsilon^* < 1$.

Therefore, we have

$$\int_{\mathbb{T}}^{2} |\nabla_x u||v|^2 \mathrm{d}x \leq e\|\omega_0\|_{Y_{\mathrm{ul}}^{\Theta}} \|v\|_{L^2(\mathbb{T}^2)}^2 \Theta\left(\frac{M}{\|v\|_{L^2(\mathbb{T}^2)}^2}\right).$$

To sum up, we have

$$\|v(t)\|_{L^2(\mathbb{T}^2)}^2 \leq \|v_0\|_{L^2(\mathbb{T}^2)}^2 + \int_0^t 2e\|\omega_0\|_{Y_{\mathrm{ul}}^{\Theta}} \Psi(\|v(s)\|_{L^2(\mathbb{T}^2)}^2)\mathrm{d}s,$$

where

$$\Psi(r) = r\Theta\left(\frac{M}{r}\right).$$

Then, by Osgood's lemma, we have

$$-\mathcal{M}(\|v(t)\|_{L^2(\mathbb{T}^2)}^2) + \mathcal{M}(\|v_0\|_{L^2(\mathbb{T}^2)}^2) \leq 2e\|\omega_0\|_{Y_{\mathrm{ul}}^{\Theta}} t,$$

where

$$\begin{aligned}\mathcal{M}(x) &= \int_x^a \frac{\mathrm{d}r}{\Psi(r)} = \int_x^a \frac{\mathrm{d}r}{r\prod_{k=1}^m \log_k\left(\frac{M}{r}\right)} \\ &= \int_{\frac{M}{a}}^{\frac{M}{x}} \frac{\mathrm{d}z}{z\prod_{k=1}^m \log_k(z)} = \int_{\log_m\left(\frac{M}{a}\right)}^{\log_m\left(\frac{M}{x}\right)} \frac{\mathrm{d}y}{y} = \log_{m+1}\left(\frac{M}{x}\right) - \log_{m+1}\left(\frac{M}{a}\right),\end{aligned}$$

where $a = 2\|u_0\|_{L^2(\mathbb{T}^2)}^2$ and we have used the substitution $z = \frac{M}{r}$ for the third identity and $y = \log_m(z)$ with

$$\frac{\mathrm{d}y}{\mathrm{d}z} = \frac{1}{z\prod_{k=1}^{m-1}\log_k(z)}$$

for the fourth identity. In particular, we have

$$\begin{aligned}&\log_{m+1}\left(\frac{M}{\|v(t)\|_{L^2(\mathbb{T}^2)}^2}\right) \\ &\quad \geq \log_{m+1}\left(\frac{M}{\|v_0\|_{L^2(\mathbb{T}^2)}^2}\right) - C\|\omega_0\|_{Y_{\mathrm{ul}}^{\Theta}} t = \log\left(\log_m\left(\frac{M}{\|v_0\|_{L^2(\mathbb{T}^2)}^2}\right) e^{-Ct\|\omega_0\|_{Y_{\mathrm{ul}}^{\Theta}}}\right),\end{aligned}$$

and taking e_{m+1} and reciprocal gives the desired conclusion. Certainly, Rate(β) is a continuous function of β, and it converges to 0 as $\beta \to 0$ as $\mathcal{M}(0) = \infty$. ∎

4.4. Convergence of ω^{β}.

PROPOSITION 11. *For any fixed* $\mathfrak{p} \in [1,\infty]$, *suppose that* $\omega_0 \in L^{\mathfrak{p}}(\mathbb{T}^2)$. *Recall the regularization of the initial data* ω_0^{β} *in* (3.6). *Let* $(u^{\beta},\omega^{\beta})$ *and* (u,ω) *be Lagrangian solutions of* (3.4) *and* (3.5) *and* (1.9) *and* (1.10), *respectively. For any* $T > 0$ *and the subsequence* $\{\beta'\} \subset \{\beta\}$ *in Lemma* 18, *we have*

$$\sup_{t\in[0,T]} \|\omega^{\beta'}(t,\cdot) - \omega(t,\cdot)\|_{L^{\mathfrak{p}}(\mathbb{T}^2)} \to 0 \quad \text{as} \quad \beta' \to \infty. \tag{4.36}$$

Proof. For the subsequence $\{\beta'\} \subset \{\beta\}$ in Lemma 18,

$$\begin{aligned} &|\omega(t,x) - \omega^{\beta'}(t,x)| \\ &\quad = |\omega_0(X(0;t,x)) - \omega_0^{\beta'}(X^{\beta'}(0;t,x))| \\ &\quad \le |\omega_0(X(0;t,x)) - \omega_0^{\ell}(X(0;t,x))| + |\omega_0^{\ell}(X^{\beta'}(0;t,x)) - \omega_0^{\beta'}(X^{\beta'}(0;t,x))| \qquad (4.37) \\ &\quad\quad + |\omega_0^{\ell}(X(0;t,x)) - \omega_0^{\ell}(X^{\beta'}(0;t,x))|. \qquad (4.38) \end{aligned}$$

Using the compressibility (1.14), we derive that, for $\mathfrak{p} \in [1,\infty]$,

$$\|(4.37)\|_{L^p} \le 2\mathfrak{C}\|\omega_0 - \omega_0^{\ell}\|_{L^p}. \tag{4.39}$$

For the last term, we need a stability of the Lagrangian flows:

$$\begin{aligned} \|(4.38)\|_{L^{\mathfrak{p}}(\mathbb{T}^2)} &\le \|\nabla \omega_0^{\ell}\|_{L^\infty} \|X(0;t,\cdot) - X^{\beta'}(0;t,\cdot)\|_{L^{\mathfrak{p}}(\mathbb{T}^2)} \\ &\le \|\nabla \varphi^{\ell}\|_{L^\infty} \|\omega_0\|_{L^1} \|X(0;t,\cdot) - X^{\beta'}(0;t,\cdot)\|_{L^{\mathfrak{p}}(\mathbb{T}^2)} \\ &\le \frac{1}{\ell^3} \|\nabla \varphi\|_{L^\infty(\mathbb{T}^2)} \|\omega_0\|_{L^1} \|X(0;t,\cdot) - X^{\beta'}(0;t,\cdot)\|_{L^{\mathfrak{p}}(\mathbb{T}^2)}, \end{aligned} \tag{4.40}$$

where we have used (3.22).

For $\mathfrak{p} > 1$, we use (4.6) in Proposition 8 and Lemma 18 to have

$$(4.40) \lesssim \frac{1}{\ell^3} \frac{1 + \|\nabla u^{\beta'}\|_{L^1((0,T);L^{\mathfrak{p}}(\mathbb{T}^2))}}{|\log \|u - u^{\beta'}\|_{L^1((0,T);L^1(\mathbb{T}^2))}|}. \tag{4.41}$$

Now, we choose

$$\ell = \ell(\beta') \sim |\log \|u - u^{\beta'}\|_{L^1((0,T);L^1(\mathbb{T}^2))}|^{-\frac{1}{10}} \quad \text{for each } \beta' \tag{4.42}$$

such that

$$\begin{aligned} \ell = \ell(\beta') \downarrow 0 \quad &\text{as } \beta' \downarrow 0, \\ \ell^3 |\log \|u - u^{\beta'}\|_{L^1((0,T);L^1(\mathbb{T}^2))}| \to \infty \quad &\text{as } \beta' \downarrow 0. \end{aligned}$$

Therefore, for $\mathfrak{p} > 1$, we prove (4.41) $\to 0$ as $\beta' \downarrow 0$. Combining this with (4.39), we conclude (4.36) for $\mathfrak{p} > 1$.

For $p = 1$, there exists $C_\varepsilon > 0$ for any $\varepsilon > 0$ such that

$$\begin{aligned} &\mathscr{L}^2(\{x \in \mathbb{T}^2 : |X^{\beta_1}(s;t,x) - X^{\beta_2}(s;t,x)| > \gamma\}) \\ &\le \frac{e^{\frac{4C_\varepsilon}{\varepsilon}}}{\frac{4C_\varepsilon}{\varepsilon}} \frac{\|u^{\beta_1} - u^{\beta_2}\|_{L^1((0,T);L^1(\mathbb{T}^2))}}{\gamma} + \varepsilon \quad \text{for any } \gamma > 0. \end{aligned} \tag{4.43}$$

For $\mathfrak{p} = 1$, using (4.7), we have

$$\begin{aligned} &\|X(0;t,\cdot) - X^{\beta'}(0;t,\cdot)\|_{L^1(\mathbb{T}^2)} \\ &\quad \le \int_{|X(0;t,\cdot) - X^{\beta'}(0;t,\cdot)| \le \gamma} |X(0;t,x) - X^{\beta'}(0;t,x)| \mathrm{d}x \\ &\quad\quad + \int_{|X(0;t,\cdot) - X^{\beta'}(0;t,\cdot)| \ge \gamma} |X(0;t,x) - X^{\beta'}(0;t,x)| \mathrm{d}x \\ &\quad \le \gamma + \frac{e^{\frac{4C_\varepsilon}{\varepsilon}}}{\frac{4C_\varepsilon}{\varepsilon}} \frac{\|u - u^{\beta'}\|_{L^1((0,T);L^1(\mathbb{T}^2))}}{\gamma} + \varepsilon, \end{aligned}$$

and hence,

$$(4.40) \lesssim \frac{1}{\ell^3}\left\{\gamma + \frac{e^{\frac{4C_\varepsilon}{\varepsilon}}}{\frac{4C_\varepsilon}{\varepsilon}} \frac{\|u-u^{\beta'}\|_{L^1((0,T);L^1(\mathbb{T}^2))}}{\gamma} + \varepsilon\right\}. \tag{4.44}$$

For each $\varepsilon > 0$, we choose $\gamma = \varepsilon$, $\ell = \varepsilon^{\frac{1}{10}}$, and $\beta' \gg_\varepsilon 1$ such that $\frac{e^{\frac{4C_\varepsilon}{\varepsilon}}}{\frac{4C_\varepsilon}{\varepsilon}} \frac{1}{\varepsilon^{\frac{13}{10}}}\|u - u^{\beta'}\|_{L^1((0,T);L^1(\mathbb{T}^2))} \to 0$. Combining with (4.39), we conclude (4.36) for $\mathfrak{p}=1$. ∎

4.4.1. When $\boldsymbol{\omega_0}$ has no regularity. If $\omega_0 \in Y^\Theta_{\mathrm{ul}}(\mathbb{T}^2)$ and no additional regularity is assumed, one cannot expect a convergence rate that is uniform over ω_0; the rate crucially depends on how fast ω_0^β converges to ω_0. Suppose that $\omega(t)$ is the Lagrangian solution with initial data ω_0. Then, we have

$$\begin{aligned}|\omega(t,x) - \omega^\beta(t,x)| &= |\omega_0(X(0;t,x)) - \omega_0^\beta(X^\beta(0;t,x))|\\ &\le |\omega_0(X(0;t,x)) - \omega_0^\ell(X(0;t,x))| + |\omega_0^\ell(X^\beta(0;t,x)) - \omega_0^\beta(X^\beta(0;t,x))|\\ &+ |\omega_0^\ell(X(0;t,x)) - \omega_0^\ell(X^\beta(0;t,x))|,\end{aligned}$$

where ω_0^ℓ is the initial data regularization of ω_0 with parameter ℓ. Therefore, by the compression property, we have

$$\begin{aligned}\|\omega(t) - \omega^\beta(t)\|_{L^\mathfrak{p}(\mathbb{T}^2)} &\le \mathfrak{C}\|\omega_0 - \omega_0^\ell\|_{L^\mathfrak{p}(\mathbb{T}^2)} + \|\omega_0^\ell - \omega_0^\beta\|_{L^\mathfrak{p}(\mathbb{T}^2)}\\ &+ \|\omega_0^\ell(X(0,t;\cdot) - \omega_0^\ell(X^\beta(0;t,\cdot))\|_{L^\mathfrak{p}(\mathbb{T}^2)}.\end{aligned}$$

Using (4.40), we can estimate the first two terms:

$$\mathfrak{C}\|\omega_0 - \omega_0^\ell\|_{L^\mathfrak{p}(\mathbb{T}^2)} + (\|\omega_0^\ell - \omega_0^\beta\|_{L^\mathfrak{p}(\mathbb{T}^2)} \le (\mathfrak{C}+1)\|\omega_0 - \omega_0^\ell\|_{L^\mathfrak{p}(\mathbb{T}^2)} + \|\omega_0^\beta - \omega_0\|_{L^\mathfrak{p}(\mathbb{T}^2)}.$$

The last term is estimated by (4.40) and (4.41):

$$\|\omega_0^\ell(X(0,t;\cdot)) - \omega_0^\ell(X^\beta(0;t,\cdot))\|_{L^\mathfrak{p}(\mathbb{T}^2)} \le \frac{C(1+\mathfrak{p}\|\omega_0\|_{L^\mathfrak{p}(\mathbb{T}^2)}t)}{\ell^3|\log \mathrm{Rate}(\omega_0;\beta)|}.$$

Choosing $\ell = |\log \mathrm{Rate}(\beta)|^{-\frac{1}{4}}$ gives that, for $t\in[0,T]$,

$$\begin{aligned}\|\omega(t) - \omega^\beta(t)\|_{L^\mathfrak{p}(\mathbb{T}^2)} &\lesssim \|\omega_0^\beta - \omega_0\|_{L^\mathfrak{p}(\mathbb{T}^2)} + \|\omega_0 - \omega_0^{|\log \mathrm{Rate}(\omega_0;\beta)|^{-\frac{1}{4}}}\|_{L^\mathfrak{p}(\mathbb{T}^2)}\\ &\quad + \frac{1+\mathfrak{p}\|\omega_0\|_{L^\mathfrak{p}(\mathbb{T}^2)}T}{|\log \mathrm{Rate}(\omega_0;\beta)|^{\frac{1}{4}}}\\ &=: \mathrm{Rate}_\omega(\omega_0;\beta).\end{aligned} \tag{4.45}$$

Since there is no explicit rate for the convergence of $\|\omega_0^\beta - \omega_0\|_{L^\mathfrak{p}(\mathbb{T}^2)}$, the first two terms dominate the rate of convergence in general.

4.4.2. When $\boldsymbol{\omega_0}$ has some regularity. An important class of localized Yudovich vorticity functions belong to the Besov space of positive regularity index; for example, $f(x) = \log(\log|x|)\varphi(x) \in Y^\Theta_{\mathrm{ul}}$ with $\Theta(\mathfrak{p}) = \log \mathfrak{p}$, where $\varphi(x)$ is a smooth cutoff function, belongs to $W^{1,r}(\mathbb{T}^2)$, where $r<2$ and thus in Besov space $B^s_{2,\infty}$ with $s<1$. Of course, vortex patches χ_D with box-counting dimension of the boundary $d_F(\partial D) < 2$ belong to $B_{p,\infty}^{\frac{2-d_F(\partial D)}{p}}$ for $1 \le p < \infty$ [14], and thus, the vortex patch with a mild singularity in the interior of D also belongs to a certain Besov space with positive regularity.

In this subsection, we provide the rate of convergence of vorticity when $\omega_0 \in Y^{\Theta}_{\mathrm{ul}}(\mathbb{T}^2) \cap B^s_{2,\infty}(\mathbb{T}^2)$ or $\omega_0 \in L^\infty(\mathbb{T}^2) \cap B^s_{2,\infty}(\mathbb{T}^2)$. Unlike the Yudovich $\omega_0 \in L^\infty(\mathbb{T}^2)$ case, if ω_0 is in the localized Yudovich class $Y^{\Theta}_{\mathrm{ul}}(\mathbb{T}^2)$, even if initial vorticity has additional Besov regularity—that is, $\omega_0 \in Y^{\Theta}_{\mathrm{ul}}(\mathbb{T}^2) \cap B^s_{2,\infty}(\mathbb{T}^2)$ for some $s > 0$—the Besov regularity of vorticity $\omega(t)$ may not propagate, even in the losing manner. The key obstruction is failure of generalization of propagation of regularity result. We will explain this after proving the result, following the argument of [13], [3], and [44].

Proposition 12. *If $\omega_0 \in Y^{\Theta}_{\mathrm{ul}}(\mathbb{T}^2) \cap B^s_{2,\infty}(\mathbb{T}^2)$ for some $s > 0$, then we have*

$$\begin{aligned} &\|\omega_0 - \omega_0^\beta\|_{L^2(\mathbb{T}^2)} \\ &\le C(T, \|\omega_0\|_{L^2(\mathbb{T}^2)}, \|\omega_0\|_{B^s_{2,\infty}(\mathbb{T}^2)}) \left(\beta^{\frac{s'}{1+s'}} + \left(\frac{1}{|\log \mathrm{Rate}(\omega_0;\beta)|} \right)^{\frac{s'}{3+4s'}} \right) \\ &=: \mathrm{Rate}_{\omega,s,loc-Y}(\beta) \end{aligned} \tag{4.46}$$

for any $s' \in (0,s)$. Moreover, if $\omega_0 \in L^\infty(\mathbb{T}^2) \cap B^s_{2,\infty}(\mathbb{T}^2)$,

$$\|\omega^\beta(t) - \omega(t)\|_{L^2(\mathbb{T}^2)} \le C(s,T,\|\omega_0\|_{B^s_{2,\infty}(\mathbb{T}^2)}) \beta^{C(s) e^{-C(\|\omega_0\|_{L^\infty(\mathbb{T}^2)})T}} =: \mathrm{Rate}_{\omega,s,Y}(\beta). \tag{4.47}$$

In particular, if ω_0 is Yudovich with some Besov regularity, the vorticity converges with an algebraic rate β^α.

Proof. First, we prove the rate for $\omega_0 \in Y^{\Theta}_{\mathrm{ul}}(\mathbb{T}^2) \cap B^s_{2,\infty}(\mathbb{T}^2)$. We rely on the above rate:

$$\|\omega(t) - \omega^\beta(t)\|_{L^2(\mathbb{T}^2)} \le C(\|\omega_0 - \omega_0^\ell\|_{L^2(\mathbb{T}^2)} + \|\omega_0 - \omega_0^\beta\|_{L^2(\mathbb{T}^2)}) + \frac{C(1 + T\|\omega_0\|_{L^2(\mathbb{T}^2)})}{\ell^3 |\log \mathrm{Rate}(\beta)|}.$$

Since $\omega_0 \in B^s_{2,\infty}(\mathbb{T}^2)$, we may use the following interpolation:

$$\begin{aligned} &\|\omega_0 - \omega_0^\beta\|_{L^2(\mathbb{T}^2)} \\ &\le \|\omega_0 - \omega_0^\beta\|^{\frac{s'}{1+s'}}_{H^{-1}(\mathbb{T}^2)} \|\omega_0 - \omega_0^\beta\|^{\frac{1}{1+s'}}_{H^{s'}(\mathbb{T}^2)} \le \|\omega_0 - \omega_0^\beta\|^{\frac{s'}{1+s'}}_{H^{-1}(\mathbb{T}^2)} \|\omega_0 - \omega_0^\beta\|^{\frac{1}{1+s'}}_{B^s_{2,\infty}(\mathbb{T}^2)} \end{aligned}$$

for arbitrary $s' \in (0,s)$, where we have used that $H^s = B^s_{2,2}$ and $B^s_{p,q}(\mathbb{T}^2) \subset B^{s'}_{p,q'}(\mathbb{T}^2)$ for $s' < s$ and arbitrary q, q'. (The proof for the whole space, which is standard, can be easily translated to periodic domain $\mathbb{T}^2$.) Since

$$\|\omega_0 - \omega_0^\beta\|_{H^{-1}(\mathbb{T}^2)} \le \|u_0 - u_0^\beta\|_{L^2(\mathbb{T}^2)} \le C\beta \|\omega_0\|_{L^2(\mathbb{T}^2)},$$

we have

$$\|\omega_0 - \omega_0^\beta\|_{L^2(\mathbb{T}^2)} \le C\beta^{\frac{s'}{1+s'}} \|\omega_0\|^{\frac{s'}{1+s'}}_{L^2(\mathbb{T}^2)} \|\omega_0\|^{\frac{1}{1+s'}}_{B^s_{2,\infty}(\mathbb{T}^2)},$$

and similarly,

$$\|\omega_0 - \omega_0^\ell\|_{L^2(\mathbb{T}^2)} \le C\ell^{\frac{s'}{1+s'}} \|\omega_0\|^{\frac{s'}{1+s'}}_{L^2(\mathbb{T}^2)} \|\omega_0\|^{\frac{1}{1+s'}}_{B^s_{2,\infty}(\mathbb{T}^2)}.$$

Finally, we match ℓ and β to find a rate of convergence; we match ℓ so that

$$\frac{1}{\ell^3 |\log \mathrm{Rate}(\beta)|} = \ell^{\frac{s'}{1+s'}}.$$

Then, we have

$$\ell^{\frac{s'}{1+s'}} = \frac{1}{\ell^3|\log \mathrm{Rate}(\beta)|} = \left(\frac{1}{|\log \mathrm{Rate}(\beta)|}\right)^{\frac{s'}{3+4s'}} \to 0$$

as $\beta \to 0$. To summarize, we have

$$\|\omega_0 - \omega_0^\beta\|_{L^2(\mathbb{T}^2)} \le C(T, \|\omega_0\|_{L^2(\mathbb{T}^2)}, \|\omega_0\|_{B^s_{2,\infty}(\mathbb{T}^2)}) \left(\beta^{\frac{s'}{1+s'}} + \left(\frac{1}{|\log \mathrm{Rate}(\beta)|}\right)^{\frac{s'}{3+4s'}}\right),$$

as desired. Note that, in the Yudovich class, $\mathrm{Rate}(\beta) = \beta^C$, and thus, this rate is dominated by $\frac{1}{|\log \beta|^\alpha}$, which is much slower than algebraic rate β^α.

Next, we prove the improved rate for the Yudovich initial data $\omega_0 \in L^\infty(\mathbb{T}^2)$. First, we calculate the rate of distance $d(X^\beta(0;t,x), X^\beta(0;t,y))$ with respect to $d(x,y)$, which is uniform in β. For the later purpose, we calculate the rate for localized Yudovich class as well; $m=0$ corresponds to $\omega_0 \in L^\infty(\mathbb{T}^2)$.

If $\omega_0 \in Y^\Theta_{\mathrm{ul}}(\mathbb{T}^2) \cap B^s_{2,\infty}(\mathbb{T}^2)$, then so is $\omega_0^\beta \in Y^\Theta_{\mathrm{ul}}(\mathbb{T}^2) \cap B^s_{2,\infty}(\mathbb{T}^2)$ with

$$\sup_\beta (\|\omega_0^\beta\|_{Y^\Theta_{\mathrm{ul}}(\mathbb{T}^2)} + \|\omega_0^\beta\|_{B^s_{2,\infty}(\mathbb{T}^2)}) \le (\|\omega_0\|_{Y^\Theta_{\mathrm{ul}}(\mathbb{T}^2)} + \|\omega_0\|_{B^s_{2,\infty}(\mathbb{T}^2)}).$$

We first estimate the modulus of continuity for u^β with $\omega_0 \in Y^\Theta_{\mathrm{ul}}(\mathbb{T}^2)$, given by Theorem 6.

$$\varphi_\Theta(r) \le \begin{cases} 0, r=0, \\ r(1-\log r)\prod_{k=1}^m \log_k(1-\log r), 0 < r < \frac{1}{e^{e_m(1)-1}}, \\ C(\Theta), r \ge \frac{1}{e^{e_m(1)-1}}, \end{cases}$$

where $C(\Theta)$ is a constant depending on Θ.

We have

$$\begin{aligned} |X^\beta(0;t,x) - X^\beta(0;t,y)| &\le |x-y| + \int_0^t \left|\frac{d}{ds}X^\beta(s;t,x) - \frac{d}{ds}X^\beta(s;t,y)\right| \mathrm{d}s \\ &= |x-y| + \int_0^t |u(X^\beta(s;t,x),s) - u(X^\beta(s;t,y),s)| \mathrm{d}s \\ &\le |x-y| + \int_0^t \varphi_\Theta(|X^\beta(s;t,x) - X^\beta(s;t,y)|) B \mathrm{d}s. \end{aligned}$$

Here, by Theorem 6, C is uniform in β. Then, by Osgood's lemma, we have

$$-\mathcal{M}(\left|X^\beta(0;t,x), X^\beta(0;t,y)\right|) + \mathcal{M}(|x-y|) \le Bt,$$

where

$$\mathcal{M}(x) = \int_x^1 \frac{\mathrm{d}r}{\varphi_\Theta(r)} = \begin{cases} \int_x^{\exp(\frac{1}{e_m(1)-1})} \frac{1}{r(1-\log r)\prod_{k=1}^m \log_k(1-\log r)} \mathrm{d}r \\ + \int_{\frac{1}{e^{e_m(1)-1}}}^1 \frac{\mathrm{d}r}{\varphi^\Theta(r)}, x < \exp(\frac{1}{e^{e_m(1)-1}}), \\ \int_x^1 \frac{\mathrm{d}r}{\varphi_\Theta(r)}, x \ge \exp(\frac{1}{e^{e_m(1)-1}}) \end{cases}$$

and B is an upper bound for $\|u^\beta\|_{L^\infty([0,T];C_b^{0,\varphi_\Theta}(\mathbb{T}^2,\mathbb{R}^2))}$. For future purposes, we take B so that $e^{BT} > e_m(1)$. Thus, if $x \ge \exp(\frac{1}{e^{e_m(1)-1}})$, $\mathcal{M}(x) \le C_0$ for some positive constant C_0. If $x < \exp(\frac{1}{e^{e_m(1)-1}})$, then

$$\int_x^{\exp(\frac{1}{e_m(1)-1})} \frac{1}{r(1-\log r)\prod_{k=1}^m \log_k(1-\log r)} \mathrm{d}r = \log_{m+1}(1-\log x)$$

using the substitution $y = \log_m(1-\log r)$, and thus,

$$\mathcal{M}(x) \in [\log_{m+1}(1-\log x), \log_{m+1}(1-\log x) + C_0]$$

for a (possibly larger) positive constant C_0. Therefore, if $|x-y|$ is sufficiently small such that $\log_{m+1}(1-\log|x-y|) - BT > C_0$, then, since

$$\mathcal{M}(|X^\beta(0;t,x) - X^\beta(0;t,y)|) \geq \mathcal{M}(|x-y|) - Bt \geq \log_{m+1}(1-\log|x-y|) - BT,$$

$|X^\beta(0;t,x) - X^\beta(0;t,y)| < \exp(\frac{1}{e^{e_m(1)-1}})$, and therefore, we have

$$\log_{m+1}(1-\log(|X^\beta(0;t,x) - X^\beta(0;t,y)|)) \geq \log_{m+1}(1-\log|x-y|) - BT - C_0,$$

which gives

$$1 - \log(|X^\beta(0;t,x) - X^\beta(0;t,y)|) \geq e_{m+1}(\log_{m+1}(1-\log|x-y|) - BT - C_0)$$

or

$$|X^\beta(0;t,x) - X^\beta(0;t,y)| \leq e\exp\left(-\left(e_{m+1}\left(\log_{m+2}\left(\frac{e}{|x-y|}\right) - BT - C_0\right)\right)\right),$$

which is uniform in β.

From now on, we assume $m = 0$. We closely follow the proof of [13] (and [3]). We rewrite the above as

$$|X^\beta(0;t,x) - X^\beta(0;t,y)| \leq e\left(\frac{|x-y|}{e}\right)^{e^{-(BT+C_0)}} =: C(T)(|x-y|)^{\alpha(T)},$$

where $\alpha(T) = \exp(-(BT+C_0))$, which is deteriorating in time, and $C(T) = \exp(1 - e^{-(BT+C_0)})$, which increases in time.

Next, we introduce the space $F^s_{\mathfrak{p}}(\mathbb{T}^2)$, which belongs to the family of Triebel–Lizorkin spaces $F^s_{\mathfrak{p}} = F^s_{\mathfrak{p},\infty}$ for $\mathfrak{p} > 1$:

$$\begin{aligned} F^s_{\mathfrak{p}}(\mathbb{T}^2) = \{f \in L^{\mathfrak{p}}(\mathbb{T}^2)|\ &\text{there exists } g \in L^{\mathfrak{p}}(\mathbb{T}^2) \text{ such that, for every } x,y \in \mathbb{T}^2, \\ &\frac{|f(x)-f(y)|}{|x-y|^s} \leq g(x) + g(y)\}, \end{aligned} \tag{4.48}$$

and its seminorm $[\cdot]_{F^s_{\mathfrak{p}}}$ is defined by

$$[f]_{F^s_{\mathfrak{p}}} := \inf_{g \in L^{\mathfrak{p}}(\mathbb{T}^2)} \{\|g\|_{L^{\mathfrak{p}}(\mathbb{T}^2)} | |f(x)-f(y)| \leq (|x-y|)^s (g(x)+g(y)) \text{ for every } x,y \in \mathbb{T}^2\}.$$

The norm on $F^s_{\mathfrak{p}}(\mathbb{T}^2)$ is naturally defined by $\|\cdot\|_{L^{\mathfrak{p}}(\mathbb{T}^2)} + [\cdot]_{F^s_{\mathfrak{p}}}$.

Now, we argue that a solution in the Yudovich class propagates Besov regularity. First, we use the following embeddings: For $s_3 > s_2 > s_1$, we have continuous embeddings (the proof for the whole space, which is standard, can be easily translated to the periodic domain $\mathbb{T}^2$).

$$B^{s_3}_{\mathfrak{p},\infty}(\mathbb{T}^2) \subset B^{s_2}_{\mathfrak{p},1}(\mathbb{T}^2) \subset W^{s_2,\mathfrak{p}}(\mathbb{T}^2) \subset F^{s_1}_{\mathfrak{p}}(\mathbb{T}^2) \subset B^{s_1}_{\mathfrak{p},\infty}(\mathbb{T}^2). \tag{4.49}$$

Therefore, since $\omega_0 \in B^s_{2,\infty}(\mathbb{T}^2)$ for some $s>0$, we have $\omega_0 \in F_2^{s_1}$ for some $s_1 \in (0,s)$, and thus, so are ω_0^βs with uniform bounds on the $F_2^{s_1}$ norm. Then, for any $\beta \geq 0$ (we introduce the convention that $X^0 = X$ and $\omega^0 = \omega$), we have

$$\begin{aligned}\frac{|\omega^\beta(x,t)-\omega^\beta(y,t)|}{(|x-y|)^{s_1\alpha(T)}} &= \frac{|\omega_0^\beta(X^\beta(0;t,x))-\omega_0^\beta(X^\beta(0;t,y))|}{(|x-y|)^{s_1\alpha(T)}}\\ &= \frac{|\omega_0^\beta(X^\beta(0;t,x))-\omega_0^\beta(X^\beta(0;t,y))|}{d(X^\beta(0;t,x),X^\beta(0;t,y))^{s_1}}\frac{(|X^\beta(0;t,x)-X^\beta(0;t,y)|)^{s_1}}{(|x-y|)^{s_1\alpha(T)}}\\ &\leq \big(g(X^\beta(0;t,x))+g(X^\beta(0;t,y))\big)C(T)\end{aligned}$$

for any $g \in L^2(\mathbb{T}^2)$ satisfying (4.48). Therefore, $C(T)g\circ X^\beta(0;t,\cdot)$ satisfies the defining condition for (4.48), and thus, $\omega^\beta(t) \in F_2^{s_1\alpha(T)}$ with

$$\|\omega^\beta(t)\|_{F_2^{s_1\alpha(T)}} \leq C(T)\|\omega_0\|_{F_2^{s_1}}.$$

Therefore, using (4.49), we have

$$\|\omega^\beta(t)\|_{B_{2,\infty}^{s_1\alpha(T)}(\mathbb{T}^2)} \leq C\|\omega^\beta(t)\|_{F_2^{s_1\alpha(T)}(\mathbb{T}^2)} \leq C(T)\|\omega_0\|_{F_2^{s_1}(\mathbb{T}^2)} \leq C(T)\|\omega_0\|_{B^s_{2,\infty}(\mathbb{T}^2)}.$$

Now, we use the interpolation inequality

$$\|\omega^\beta(t)-\omega(t)\|_{L^2(\mathbb{T}^2)} \leq \|\omega^\beta(t)-\omega(t)\|_{H^{-1}(\mathbb{T}^2)}^{\frac{s_0}{1+s_0}}\|\omega^\beta(t)-\omega(t)\|_{B_{2,\infty}^{s_1\alpha(T)}}^{\frac{1}{1+s_0}}$$

for some $s_0 < s_1\alpha(T)$. Therefore, we have

$$\begin{aligned}\|\omega^\beta(t)-\omega(t)\|_{L^2(\mathbb{T}^2)} &\leq \|u^\beta(t)-u(t)\|_{L^2(\mathbb{T}^2)}^{\frac{s_0}{1+s_0}}C(T,\|\omega_0\|_{B^s_{2,\infty}(\mathbb{T}^2)})\\ &\leq C(T,\|\omega_0\|_{B^s_{2,\infty}(\mathbb{T}^2)})\beta^{Ce^{-C(\|\omega_0\|_{L^\infty(\mathbb{T}^2)})T}}\end{aligned}$$

by noting that the rate function for Yudovich case is algebraic; that is, $\mathrm{Rate}(\beta) = \beta^{2e^{-C(\|\omega_0\|_{L^\infty(\mathbb{T}^2)})T}}$. ∎

Remark 4. One may naturally ask if one can obtain a faster rate than (4.46), analogous to (4.47). It seems that the argument we presented for (4.47) does not extend to the localized Yudovich space.

First, if $m>0$ for the modulus of continuity given by

$$\mu(|x-y|,T) = \exp\left(-e_{m+1}\left(\log_{m+2}\frac{e}{|x-y|}-(BT+C_0)\right)\right), \tag{4.50}$$

it cannot be bounded by any Hölder exponent $|x-y|^\alpha$ for any $\alpha \in (0,1)$. Thus, we cannot continue the argument from there. To see this, suppose that there exists a $\alpha>0$ and $C>0$ such that

$$\mu(r,T) \leq Cr^\alpha$$

for any $r<1$ very small. This amounts to saying that

$$\log_{m+2}\frac{e}{r} - \log_{m+2}\frac{1}{Cr^\alpha} \geq BT + C_0.$$

Taking the exponential, we have

$$\frac{\log_{m+1}\frac{e}{r}}{\log_{m+1}\frac{1}{Cr^\alpha}} \geq e^{BT+C_0}.$$

Since both denominator and numerator diverge as $r \to 0^+$, we may apply L'Hôpital's rule:

$$\frac{d}{dr}\log_{m+1}\frac{e}{r} = \frac{1}{\prod_{k=1}^m \log_k \frac{e}{r}}\left(-\frac{1}{r}\right),$$
$$\frac{d}{dr}\log_{m+1}\frac{1}{Cr^\alpha} = \frac{1}{\prod_{k=1}^m \log_k \frac{1}{Cr^\alpha}}\left(-\frac{\alpha}{r}\right).$$

Inductively, we have

$$\lim_{r\to 0^+}\frac{\log_{0+1}\frac{e}{r}}{\log_{0+1}\frac{1}{Cr^\alpha}} = \frac{1}{\alpha},$$
$$\lim_{r\to 0^+}\frac{\log_{1+1}\frac{e}{r}}{\log_{1+1}\frac{1}{Cr^\alpha}} = \lim_{r\to 0^+}\frac{\log_1\frac{1}{Cr^\alpha}}{\log_1\frac{e}{r}}\frac{1}{\alpha} = \frac{\alpha}{\alpha} = 1,$$
$$\dots$$
$$\lim_{r\to 0^+}\frac{\log_{m+1}\frac{e}{r}}{\log_{m+1}\frac{1}{Cr^\alpha}} = \prod_{k=1}^m \frac{\log_k\frac{1}{Cr^\alpha}}{\log_k\frac{e}{r}}\frac{1}{\alpha} = 1.$$

Therefore, except for $m=0$, where the limit is given by $\frac{1}{\alpha}$, for any $\alpha > 0$ and $C > 0$, there exists small $r > 0$ such that $\mu(r,T) > Cr^\alpha$. Thus, control of vorticity in Triebel–Lizorkin space $F_p^{s(t)}$ is not available.

There are other methods for propagation of regularity (in a losing manner), but it seems that they also suffer from similar issue; flows generated by the localized Yudovich class do not propagate enough regularity.

The argument of [3] does not extend to the localized Yudovich class as well; when ω_0 is locally Yudovich, the modulus of continuity for u is weaker than log-Lipschitz. It is known that the norm defined by

$$\|u\|_{LL'} = \|u\|_{L^\infty} + \sup_{j\geq 0}\frac{\|\nabla S_j u\|_{L^\infty}}{(j+1)},$$

where $S_j u = \sum_{k=-1}^j \Delta_k u$, is equivalent to the norm of the log-Lipschitz space (Proposition 2.111 of [3], which is for the whole case but can be adopted to the periodic domain easily). However, if $\omega_0 \in Y_{\mathrm{ul}}^\Theta$, then u has the modulus of continuity φ_Θ, and the norm for $C_b^{0,\varphi_\Theta}(\mathbb{T}^2)$ is equivalent to

$$\|u\|_{L^\infty} + \sup_{j\geq 0}\frac{\|\nabla S_j u\|_{L^\infty}}{\prod_{k=1}^{m+1}\log_k(e2^j)},$$

which is less than $\|u\|_{LL'}$. However, the critical growth rate for the denominator in applying the linear loss of regularity result (for example, Theorem 3.28 of [3]) is $j+1$, which is the rate of the log-Lipschitz norm. Therefore, we cannot rely on the argument of [3] to conclude that $\omega(t)$ has certain Besov regularity.

Finally, a borderline Besov space B_Γ, introduced by Vishik in [44], has a certain regularity (in the sense that B_Γ restricts the rate of growth of frequency components) propagates, but it is not clear how to use this to obtain convergence rate for vorticity. For simplicity, we focus on one particular form of growth function: Let

$$\Gamma(r) = (r+2)\frac{\log(r+3)}{\log 2}, \Gamma_1(r) = \frac{\log(r+3)}{\log 2}$$

for $r \geq -1$ and $\Gamma(r) = \Gamma_1(r) = 1$ for $r \leq -1$. We define the space B_Γ by

$$B_\Gamma = \left\{ f \| \|f\|_\Gamma := \sup_{N \geq -1} \frac{\sum_{j=-1}^N \|\Delta_j f\|_{L^\infty}}{\Gamma(N)} < \infty \right\},$$

and we define B_{Γ_1} in a similar manner. In [44], the following was proved.

THEOREM 7 ([44]). *If $\omega_0 \in L^{\mathfrak{p}_0} \cap L^{\mathfrak{p}_1} \cap B_{\Gamma_1}$ for $1 < \mathfrak{p}_0 < 2 < \mathfrak{p}_1 < \infty$, then, for any $T > 0$, there uniquely exists a weak solution $\omega(t)$ of the Euler equation satisfying*

$$\|\omega(t)\|_\Gamma \leq \lambda(t),$$

where $\lambda(t)$ depends only on the bounds on $\|\omega_0\|_{L^{\mathfrak{p}_0} \cap L^{\mathfrak{p}_1} \cap B_{\Gamma_1}}$.

Therefore, one can prove the uniform boundedness of vorticity in B_Γ space. However, it is not clear how one can interpolate B_Γ space and the velocity space (where we have rate of convergence) to obtain the rate for the L^p norm of the vorticity.

Indeed, it was recently shown that if the velocity field is worse than Lipschitz ($u \in W^{1,p}$ for $p < \infty$), then it is possible for smooth data to lose all Sobolev regularity instantaneously from the transport by u ([1]). Instead, only a logarithm of a derivative can be preserved (see, e.g., [9]), and this loss of regularity prohibits faster convergence.

5. Proof of the main theorems.

LEMMA 19.

$$\left\| \frac{F^\varepsilon(t) - M_{1,\varepsilon u(t),1}}{\varepsilon \sqrt{M_{1,0,1}}} \right\|_{L^p_x L^2_v} \lesssim e^{\frac{\varepsilon^2}{4}\|u^\beta\|_\infty^2} \left\{ \|u^\beta(t) - u(t)\|_{L^p_x} e^{\varepsilon^2 \|u-u^\beta\|_\infty^2} + \kappa^{\min\{1, \frac{p+2}{2p}\}} \sqrt{\mathcal{E}(t)} + \varepsilon\kappa V(\beta) \right\}. \tag{5.1}$$

$$\begin{aligned} &\left\| \frac{\nabla_x (F^\varepsilon - M_{1,\varepsilon u(t),1})}{\varepsilon(1+|v|)\sqrt{M_{1,0,1}}} \right\|_{L^p_x L^2_v} \\ &\lesssim \{ \|\nabla_x u^\beta - \nabla_x u\|_{L^p_x} + \varepsilon \|\nabla_x u\|_{L^p_x} + \varepsilon \|\nabla_x u^\beta\|_{L^p_x} \} e^{\varepsilon^2 \|u-u^\beta\|_\infty^2} e^{\varepsilon^2 \|u^\beta\|_\infty^2} \\ &\quad + e^{\frac{\varepsilon^2 \|u^\beta\|_{L^\infty(\mathbb{T}^2)}^2}{4}} \{ \kappa^{\min\{\frac{1}{p}, \frac{1}{2}\}} \sqrt{\mathcal{E}(t)} + \varepsilon\kappa V(\beta) \}. \end{aligned} \tag{5.2}$$

Proof. We only prove (5.2) because the proof of (5.1) is similar and simpler. We decompose

$$\begin{aligned} &\left\| \frac{\nabla_x (F^\varepsilon - M_{1,\varepsilon u,1})}{\varepsilon(1+|v|)\sqrt{M_{1,0,1}}} \right\|_{L^p_x L^2_v} \\ &\leq \left\| \frac{\nabla_x (M_{1,\varepsilon u^\beta,1} - M_{1,\varepsilon u,1})}{\varepsilon(1+|v|)\sqrt{M_{1,0,1}}} \right\|_{L^p_x L^2_v} + \left\| \sqrt{\frac{M_{1,\varepsilon u^\beta,1}^{1+o(1)}}{M_{1,0,1}}} \right\|_{L^\infty_{x,v}} \left\| \frac{\nabla_x (F^\varepsilon - M_{1,\varepsilon u^\beta,1})}{\varepsilon(1+|v|)\sqrt{M_{1,\varepsilon u^\beta,1}^{1+o(1)}}} \right\|_{L^p_x L^2_v} \\ &= (5.3)_1 + (5.3)_2 (5.3)_3. \end{aligned} \tag{5.3}$$

The bound of $(5.3)_1$ raises the need for consideration of $\sqrt{M_{1,A,1}^{1+o(1)}/M_{1,0,1}}$ for $A \in \mathbb{R}^3$:

$$\sqrt{M_{1,A,1}^{1+o(1)}/M_{1,0,1}} \lesssim e^{\frac{-(1+o(1))|v-A|^2+|v|^2}{4}} \leq e^{\frac{|A|^2}{4}}. \tag{5.4}$$

Using (5.4) and the Taylor expansion, we derive that

$$\begin{aligned}
&\frac{|\nabla_x (M_{1,\varepsilon u^\beta,1} - M_{1,\varepsilon u,1})|}{\varepsilon\sqrt{M_{1,0,1}}} \\
&= \frac{1}{\varepsilon}\Big|\int_0^\varepsilon \nabla_x \Big(\big((v-\varepsilon u) + a(u-u^\beta)\big)\cdot (u^\beta - u)\frac{M_{1,\varepsilon u - a(u-u^\beta),1}}{\sqrt{M_{1,0,1}}}\Big)\mathrm{d}a\Big| \\
&\lesssim \{|\nabla_x u^\beta - \nabla_x u| + \varepsilon|\nabla_x u| + \varepsilon|\nabla_x u^\beta|\} e^{\varepsilon^2|u-u^\beta|^2} e^{\varepsilon^2|u^\beta|^2}\frac{1}{\varepsilon}\int_0^\varepsilon |M_{1,\varepsilon u - a(u-u^\beta),1}(v)|^{\frac14}\mathrm{d}a,
\end{aligned} \tag{5.5}$$

where we have used $|(v-\varepsilon u)+a(u-u^\beta)||M_{1,\varepsilon u-a(u-u^\beta),1}|^{\frac12 - o(1)/2} \lesssim |M_{1,\varepsilon u - a(u-u^\beta),1}|^{\frac14}$ and $|\varepsilon u - a(u-u^\beta)| = |(\varepsilon - a)u - (\varepsilon - a)u^\beta + \varepsilon u^\beta| \leq |\varepsilon - a||u-u^\beta| + \varepsilon|u^\beta| \leq \varepsilon\{|u-u^\beta| + |u^\beta|\}$.

Now, taking an $L^p_x L^2_v$-norm to (5.5), we conclude that

$$(5.3)_1 \lesssim \big\{\|\nabla_x u^\beta - \nabla_x u\|_{L^p_x} + \varepsilon\|\nabla_x u\|_{L^p_x} + \varepsilon\|\nabla_x u^\beta\|_{L^p_x}\big\} e^{\varepsilon^2\|u-u^\beta\|_\infty^2} e^{\varepsilon^2\|u^\beta\|_\infty^2}. \tag{5.6}$$

From (5.4), clearly we have

$$(5.3)_2 \lesssim e^{\frac{\varepsilon^2\|u^\beta\|^2_{L^\infty(\mathbb{T}^2)}}{4}}. \tag{5.7}$$

Using the expansion (2.2), we can bound $(5.3)_3$:

$$\begin{aligned}
(5.3)_3 &\lesssim \|\nabla_x f^\varepsilon\|_{L^p_x L^2_x} + \varepsilon\|u^\beta\|_\infty \|f^\varepsilon\|_{L^p_x L^2_x} + \varepsilon\kappa V(\beta) \\
&\lesssim \|\nabla_x^2 f_R\|_{L^2_{x,v}}^{\frac{p-2}{p}} \|\nabla_x f_R\|_{L^2_{x,v}}^{\frac{2}{p}} + \varepsilon\|u^\beta\|_\infty \|\nabla_x f_R\|_{L^2_{x,v}}^{\frac{p-2}{p}} \|f_R\|_{L^2_{x,v}}^{\frac{2}{p}} + \varepsilon\kappa V(\beta) \\
&\lesssim \kappa^{\min\{\frac1p,\frac12\}}\sqrt{\mathcal{E}(t)} + \varepsilon\kappa V(\beta).
\end{aligned} \tag{5.8}$$

We finish the proof by applying (5.6), (5.7), and (5.8) to (5.3). ∎

We claim the following.

Lemma 20.

$$\|\omega_B^\varepsilon(t) - \omega(t)\|_{L^p(\mathbb{T}^2)} \lesssim \|\omega^\beta(t) - \omega(t)\|_{L^p(\mathbb{T}^2)} + \kappa^{\min\{\frac12,\frac1p\}}\sqrt{\mathcal{E}(t)} + \varepsilon\kappa V(\beta). \tag{5.9}$$

Proof. Recall F^ε in (2.2). Note that

$$\begin{aligned}
\omega_B^\varepsilon(t,x) - \omega(t,x) &= \nabla^\perp \cdot u_B^\varepsilon(t,x) - \nabla^\perp\cdot u(t,x) \\
&= \frac{1}{\varepsilon}\int_{\mathbb{R}^3} \underline{v}\cdot\nabla^\perp (F^\varepsilon(t,x,v) - M_{1,\varepsilon u,1}(v))\mathrm{d}v \\
&= \frac{1}{\varepsilon}\int_{\mathbb{R}^3} \underline{v}\cdot\nabla^\perp (M_{1,\varepsilon u^\beta,1}(v) - M_{1,\varepsilon u,1}(v))\mathrm{d}v \quad (= \omega^\beta - \omega) &(5.10)\\
&\quad + \int_{\mathbb{R}^3} \nabla^\perp f_R(t,x,v)\cdot \underline{v}\sqrt{\mu}\mathrm{d}v &(5.11)\\
&\quad + \nabla^\perp\cdot\int_{\mathbb{R}^3}\{\varepsilon^2 p^\beta\mu - \varepsilon^2\kappa(\nabla_x u^\beta):\mathfrak{A}\sqrt{\mu} + \varepsilon\kappa\tilde{u}^\beta\cdot(v-\varepsilon u^\beta)\mu + \varepsilon^2\kappa\tilde{p}^\beta\mu\}\mathrm{d}v. &(5.12)
\end{aligned}$$

Clearly,

$$\|(5.10)\|_{L^{\mathfrak{p}}(\mathbb{T}^2)} = \|\omega^{\beta}(t) - \omega(t)\|_{L^{\mathfrak{p}}(\mathbb{T}^2)}.$$

From Theorem 3, we conclude that

$$\|(5.11)\|_{L^{\mathfrak{p}}(\mathbb{T}^2)} \lesssim \begin{cases} \|\nabla_x f_R(t)\|_{L^2(\mathbb{T}^2\times\mathbb{R}^3)} \lesssim \sqrt{\kappa}\sqrt{\mathcal{E}(t)} \text{ for } \mathfrak{p}\in[1,2], \\ \|\nabla_x^2 f_R(t)\|_{L^2(\mathbb{T}^2\times\mathbb{R}^3)}^{\frac{\mathfrak{p}-2}{\mathfrak{p}}} \|\nabla_x f_R(t)\|_{L^2(\mathbb{T}^2\times\mathbb{R}^3)}^{\frac{2}{\mathfrak{p}}} \lesssim \kappa^{\frac{1}{\mathfrak{p}}}\sqrt{\mathcal{E}(t)} \text{ for } \mathfrak{p}\in(2,\infty), \end{cases}$$

where we have used (anisotropic) Gagliardo–Nirenberg interpolation for the second, whose proof is analogous to Lemma 4.

Using Theorem 5, we get that $\|(5.12)\|_{L^{\mathfrak{p}}(\mathbb{T}^2)} \lesssim \varepsilon\kappa V(\beta)$. ☐

Equipped with Proposition 4, Proposition 11, and Proposition 3, we are ready to prove the main theorem of this paper.

THEOREM 8. *Suppose that* $\varepsilon, \kappa = \kappa(\varepsilon), \beta = \beta(\varepsilon)$ *satisfy* (2.3). *Choose an arbitrary* $T \in (0,\infty)$. *Suppose that* $(u_0,\omega_0) \in L^2(\mathbb{T}^2) \times L^{\mathfrak{p}}(\mathbb{T}^2)$ *for* $\mathfrak{p} \in [1,\infty)$ *and* (u,ω) *be a Lagrangian solution of* (1.9), (1.10), *and* (1.11). *Assume that the initial data* F_0 *to* (1.5) *satisfy conditions in Theorem* 3. *Then, there exists a family of Boltzmann solutions* $F^{\varepsilon}(t,x,v)$ *to* (1.5) *in* $[0,T]$ *such that*

$$\sup_{t\in[0,T]} \left\| \frac{F^{\varepsilon}(t) - M_{1,\varepsilon u(t),1}}{\varepsilon\sqrt{M_{1,0,1}}} \right\|_{L^2(\mathbb{T}^2\times\mathbb{R}^3)} \to 0. \tag{5.13}$$

Moreover, the Boltzmann vorticity converges to the Lagrangian solution ω*:*

$$\sup_{0\le t\le T} \|\omega_B^{\varepsilon}(t,\cdot) - \omega(t,\cdot)\|_{L^{\mathfrak{p}}(\mathbb{T}^2)} \to 0. \tag{5.14}$$

THEOREM 9. *Suppose that* $\varepsilon, \kappa = \kappa(\varepsilon), \beta = \beta(\varepsilon)$ *satisfy* (2.3). *Choose an arbitrary* $T \in (0,\infty)$. *Suppose that* $\omega_0 \in Y_{\mathrm{ul}}^{\Theta}(\mathbb{T}^2)$ *for some* Θ *in* (4.34) *with* $m \in \mathbb{Z}_{\ge 0}$, *and let* (u,ω) *be the unique weak solution of* (1.9), (1.10), *and* (1.11). *Assume that the initial data* F_0 *to* (1.5) *satisfy conditions in Theorem* 3. *Then, there exists a family of Boltzmann solutions* $F^{\varepsilon}(t,x,v)$ *to* (1.5) *in* $[0,T]$ *such that*

$$\sup_{t\in[0,T]} \left\| \frac{F^{\varepsilon}(t) - M_{1,\varepsilon u(t),1}}{\varepsilon\sqrt{M_{1,0,1}}} \right\|_{L^2(\mathbb{T}^2\times\mathbb{R}^3)} \to 0. \tag{5.15}$$

Moreover, the Boltzmann velocity and vorticity converge to the solution ω *with an explicit rate* $\mathrm{Rate}(\beta(\varepsilon))$, $\mathrm{Rate}_{\omega}(\beta(\varepsilon))$ *as defined in* (4.35) *and* (4.45)*:*

$$\begin{aligned} &\sup_{0\le t\le T} \|u_B^{\varepsilon}(t,\cdot) - u(t,\cdot)\|_{L^2(\mathbb{T}^2)} \lesssim \mathrm{Rate}(\beta(\varepsilon)), \\ &\sup_{0\le t\le T} \|\omega_B^{\varepsilon}(t,\cdot) - \omega(t,\cdot)\|_{L^{\mathfrak{p}}(\mathbb{T}^2)} \lesssim \mathrm{Rate}_{\omega}(\beta(\varepsilon)). \end{aligned} \tag{5.16}$$

Furthermore, if $\omega_0 \in Y_{\mathrm{ul}}^{\Theta}(\mathbb{T}^2) \cap B_{2,\infty}^{s}(\mathbb{T}^2)$ *for some* $s > 0$, *Boltzmann vorticity converges to the solution* ω *with a rate that is uniform in* ω_0 *as in* (4.46) *and* (4.47)*:*

(5.17)

$$\begin{aligned} &\sup_{0\le t\le T} \|\omega_B^{\varepsilon}(t,\cdot) - \omega(t,\cdot)\|_{L^{\mathfrak{p}}(\mathbb{T}^2)} \lesssim \mathrm{Rate}_{\omega,s,loc-Y}(\beta), m > 0 \textit{ (localized Yudovich)}, \\ &\sup_{0\le t\le T} \|\omega_B^{\varepsilon}(t,\cdot) - \omega(t,\cdot)\|_{L^{\mathfrak{p}}(\mathbb{T}^2)} \lesssim \mathrm{Rate}_{\omega,s,Y}(\beta), m = 0 \textit{ (Yudovich)}. \end{aligned}$$

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