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ORIGINAL RESEARCH ARTICLE

Boundary Effect Under 2D Newtonian Gravity Potential in the Phase Space

Jiaxin Jin[1] · **Chanwoo Kim**[2]

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Abstract
We study linear two-and-a-half-dimensional Vlasov equations under the logarithmic gravity potential in the half-space of diffuse reflection boundary. We prove decay-in-time of the exponential moments with a polynomial rate, which depends on the base logarithm.

Keywords Vlasov equation · Diffusive reflection boundary · Logarithmic gravity potential

1 Introduction

In this paper, we consider a free molecules without intermolecular interaction which are contained in a horizontally periodic three- dimensional half-space $\Omega = \mathbb{T}^2 \times \mathbb{R}_+$ and subjected to the gravity field. A governing kinetic model of the system is the Vlasov equations:

$$\partial_t F + v \cdot \nabla_x F - \nabla\Phi(x) \cdot \nabla_v F = 0, \text{ for } (t, x, v) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^3. \tag{1.1}$$

Here, $\Phi(x)$ is a given external field (gravity), which will be specified later in (1.6).

At the bottom of domain, the phase boundary $\gamma := \{(x, v) \in \partial\Omega \times \mathbb{R}^3\}$ is decomposed into the outgoing boundary and incoming boundary $\gamma_\pm := \{(x, v) \in \partial\Omega \times \mathbb{R}^3, n(x)\cdot v \gtrless 0\}$ with the outward normal $n(x)$ at $x \in \partial\Omega$. It is clear that $|\partial\Omega| = 1$. Further, we

Jiaxin Jin and Chanwoo Kim have contributed equally to this work.

✉ Chanwoo Kim
chanwookim.math@gmail.com

Jiaxin Jin
jin.1307@osu.edu

1 Department of Mathematics, The Ohio State University, Columbus, OH 43210, USA

2 Department of Mathematics, University of Wisconsin-Madison, Madison, WI 53706, USA

consider the molecules interact with the boundary thermodynamically via a *diffusive reflection* boundary condition

$$F(\cdot, x, v) = \mu(x, v) \int_{n(x)\cdot v^1 > 0} F(\cdot, x, v^1)\{n(x)\cdot v^1\}\mathrm{d}v^1 \ \text{ for } \ (x, v) \in \gamma_- := \{x \in \partial\Omega \text{ and } v_3 > 0\}, \tag{1.2}$$

such that an outgoing distribution is proportional to the thermal equilibrium of the unit boundary temperature:

$$\mu(x, v) = \frac{1}{2\pi} e^{-\frac{|v|^2}{2}} \qquad \text{(wall Maxwellian).} \tag{1.3}$$

where $\int_{n(x)\cdot v^1 > 0} \mu(x, v^1)\{n(x)\cdot v^1\}\mathrm{d}v^1 = 1$, and we have a null flux at the boundary and enjoy the conservation of total mass:

$$\iint_{\Omega\times\mathbb{R}^3} F(t, x, v)\mathrm{d}x\mathrm{d}v = \iint_{\Omega\times\mathbb{R}^3} F(0, x, v)\mathrm{d}x\mathrm{d}v = \mathfrak{m} > 0.$$

Throughout this paper, we always assume that the total mass equals $\mathfrak{m}$.

If the boundary temperature varies with the position on the boundary, then stationary solutions to (1.2) are neither given by explicit formulas nor are equilibria (local Maxwellian) in general, if they exist (see [5] for the construction of steady solutions). This is because any explicit solution can be obtained by backtracking along the characteristics until the boundary. Under the non-isothermal case when $\mu_\theta(x, v) = \frac{1}{2\pi} e^{-\frac{|v|^2}{2\theta(x)}}$ and $\theta(x)$ varies with x, local Maxwellian doesn't satisfy the diffusive boundary condition in general.

In this paper, we only focus on the asymptotic stability of simpler isothermal boundary for the sake of simplicity. In this case of the isothermal boundary (1.3), a stationary solution has an explicit form: for some $c_{\mathfrak{m}} > 0$

$$\tilde{\mu}(x, v) := \frac{c_{\mathfrak{m}}}{2\pi} e^{-\left(\frac{|v|^2}{2} + \Phi(x)\right)}.$$

The uniqueness of stationary problem can be easily proved as the problem is linear (see [5] for the details).

The main interest in this paper is to study stabilizing effect of the diffusive reflection boundary to the Vlasov equations under the logarithmic potential

$$\Phi(x) = \log_a(1 + x_3). \tag{1.4}$$

This potential is physically relevant in the 2D universe. Indeed the logarithmic potential (1.4) corresponds to the Newtonian potential in the 2-dimensional universe. A relevant model is the two-and-a-half-dimensional Vlasov equation:

$$\partial_t F + \sum_{i=1,3} v_i \partial_{x_i} F - \partial_{x_3}\Phi(x)\partial_{v_3} F = 0, \tag{1.5}$$

where the spatial domain is $\mathbb{T}\times\mathbb{R}_+ := \{(x_1,x_3)\in\mathbb{T}\times\mathbb{R} : x_3 > 0\}$.

Our full 3-dimensional problem (1.1) can directly apply to this two-and-a-half-dimensional model (1.5) by setting data homogeneous in x_2-direction, that is, $F = F(t,x_1,x_3,v)$ and $F_0 = F_0(x_1,x_3,v)$ in the spatial domain $\Omega = \mathbb{T}\times\mathbb{R}_+$ and the domain of the velocities is still $\mathbb{R}^3$.

Notations. Here we clarify some notations: $\mathbb{N}^+$ represents the set of all positive natural numbers; $A \lesssim B$ if $A \le CB$ for a constant $C > 0$ which is independent on A, B; $A \lesssim_\theta B$ if $A \le CB$ for a constant $C = C(\theta) > 0$ which depends on θ but is independent on A, B; $\|\cdot\|_{L^1_{x,v}}$ for the norm of $L^1(\Omega\times\mathbb{R}^3)$; $\|\cdot\|_{L^\infty_{x,v}}$ or $\|\cdot\|_\infty$ for the norm of $L^\infty(\bar\Omega\times\mathbb{R}^3)$; $|g|_{L^1_{\gamma_\pm}} = \int_{\gamma_\pm}|g(x,v)||n(x)\cdot v|\mathrm{d}S_x\mathrm{d}v$ where $\mathrm{d}S_x = \mathrm{d}x_1\mathrm{d}x_2$ represents the measure on the boundary $\partial\Omega$ and $n(x)$ is the outward normal at $x\in\partial\Omega$; an integration $\int_Y f(y)\mathrm{d}y$ is often abbreviated to $\int_Y f$, if it is not ambiguous. We remark that n represents an integer without $x\in\partial\Omega$ (e.g.,Proposition 10). Finally, when we write $(A.1) \le C$, we mean that C is an upper bound of the most right-hand side of the equation $(A.1)$.

Main Theorems. The main interest in this work is to study a long-time behavior of solutions to the Vlasov equations for the field as follows:

$$\Phi(x) = \log_a(1+x_3), \quad \text{and} \quad \mathcal{A} = \left[\frac{1}{\ln(a)}\right] \ge 8, \tag{1.6}$$

where $[m]$ represents the biggest integer less than or equal to m. Here we set $\mathcal{A}$ as the integer part of $1/\ln(a)$ for the convenience of decay rates in main results (see Theorem 1 and Theorem 3).

The gravitational potential in the logarithm form plays an important role to the convergence speed which turns out a polynomial rate depends on the base of the logarithm.

We express the perturbation form as

$$F(t,x,v) = \tilde\mu(x,v) + f(t,x,v), \tag{1.7}$$

and the initial data $F_0(x,v) = \tilde\mu(x,v) + f_0(x,v)$.

Theorem 1 shows L^1-estimates on every fluctuation which is of zero initial mass.

Theorem 1 *Consider the initial data $F_0(x,v) = \tilde\mu(x,v) + f_0(x,v) \ge 0$, such that*

$$\iint_{\Omega\times\mathbb{R}^3} f_0(x,v)\mathrm{d}x\mathrm{d}v = 0, \quad \|e^{\frac12|v|^2+\Phi(x)} f_0\|_{L^\infty_{x,v}} < \infty. \tag{1.8}$$

There exists a unique global-in-time solution

$$F(t,x,v)=\tilde{\mu}(x,v)+f(t,x,v)\geq 0 \tag{1.9}$$

to (1.1) and the boundary condition (1.2) with the initial condition $F(t,x,v)|_{t=0}=F_0(x,v)$ *in* $\Omega\times\mathbb{R}^3$*, such that*

$$\iint_{\Omega\times\mathbb{R}^3} f(t,x,v)\mathrm{d}x\mathrm{d}v=0,\ \ \textit{for all } t\geq 0. \tag{1.10}$$

Moreover, we have

$$\|f(t)\|_{L^1_{x,v}}\leq C(\ln\langle t\rangle)^{\mathcal{A}-6-\frac{\delta}{2}}\langle t\rangle^{-(\mathcal{A}-6)}\times\|e^{\frac{1}{2}|v|^2+\Phi(x)}f_0\|_{L^\infty_{x,v}}, \tag{1.11}$$

where $C=C(\Omega)$ *only depends on the domain* Ω*,* $0<\delta<1$ *and* $\mathcal{A}$ *is given as in (1.6).*

Remark 2 To prove Theorem 1, we introduce and compute the norms of $f(t,x,v)$ at time $t=kT_0$ with $k\in\mathbb{N}$ (see (3.38)). Further, the time interval T_0 depends only on the domain Ω (see Propositions 18 and 21). Therefore, the constant C only depends on the domain Ω.

Theorem 3 proves the decay of the exponential moment on the fluctuation.

Theorem 3 *Assume all conditions in Theorem 1. For all* $t\geq 0$ *and* $0\leq 2\theta<\theta'=\frac{1}{2}$*,*

$$\sup_{t\geq 0}\|e^{\theta'(|v|^2+2\Phi(x))}f(t)\|_{L^\infty_{x,v}}\lesssim\|e^{\theta'(|v|^2+2\Phi(x))}f_0\|_{L^\infty_{x,v}}. \tag{1.12}$$

$$\sup_{x\in\bar{\Omega}}\int_{\mathbb{R}^3}e^{\theta(|v|^2+2\Phi(x))}|f(t,x,v)|\mathrm{d}v\lesssim_\theta\langle t\rangle^{7-\mathcal{A}}. \tag{1.13}$$

Remark 4 The decay rate and the potential have a close relation. When the gravity is constant (for example: $\Phi(x)=gx_3$), then the system has an exponential decay [5, 6]. On the other hand, when the domain is bounded and the potential is zero, the decay rate is polynomial depending on the spatial dimension. This is due to the fact that low velocities stay in the system for a long time. About this direction, we refer to [1, 2, 4, 7] and the references therein.

Difficulties and Ideas. Throughout this paper, we use the fundamental idea where for each velocity obtained from the diffusive reflection boundary condition, we compute how the velocity transfers through space under the kinetic operator. This idea is realized by *the stochastic cycles*.

The characteristics of (1.1) are determined by the Hamilton ODEs

$$\begin{cases}\frac{\mathrm{d}}{\mathrm{d}s}X(s;t,x,v)=V(s;t,x,v),\\ \frac{\mathrm{d}}{\mathrm{d}s}V(s;t,x,v)=-\nabla\Phi(X(s;t,x,v)),\end{cases} \tag{1.14}$$

for $-\infty < s, t < \infty$ with $(X(t; t, x, v), V(t; t, x, v)) = (x, v)$.

Definition 5 (Stochastic Cycles)
Consider (X, V) solving (1.14), which is the characteristics of the Vlasov equations (1.1). Define the backward exit time $t_{\mathbf{b}}$ and the forward exit time $t_{\mathbf{f}}$,

$$
\begin{aligned}
&t_{\mathbf{b}}(x, v) := \sup\{s \geq 0 : X(t-\tau; t, x, v) \in \Omega,\ \forall \tau \in [0, s)\},\ x_{\mathbf{b}}(x, v) := X(t - t_{\mathbf{b}}(x, v); t, x, v),\\
&t_{\mathbf{f}}(x, v) := \sup\{s \geq 0 : X(t+\tau; t, x, v) \in \Omega,\ \forall \tau \in [0, s)\},\ x_{\mathbf{f}}(x, v) := X(t + t_{\mathbf{f}}(x, v); t, x, v).
\end{aligned}
\tag{1.15}
$$

We define the stochastic cycles:

$$
\begin{aligned}
t^1(t, x, v) &= t - t_{\mathbf{b}}(x, v),\ x^1(x, v) = x_{\mathbf{b}}(x, v) = X(t^1, t, x, v),\ v_{\mathbf{b}}(x, v) = V(t^1, t, x, v),\\
&t^k(t, x, v, v^1, ..., v^{k-1}) = t^{k-1} - t_{\mathbf{b}}(x^{k-1}, v^{k-1}),\ t^k_{\mathbf{b}} = t^{k+1}_{\mathbf{f}} = t^k - t^{k+1},\\
&x^k(t, x, v, v^1, ..., v^{k-1}) = X(t^k; t^{k-1}, x^{k-1}, v^{k-1}),\ v^k_{\mathbf{b}} = V(t^{k+1}; t^k, x^k, v^k),
\end{aligned}
\tag{1.16}
$$

where we define $v^j \in \mathcal{V}_j := \{v^j \in \mathbb{R}^3 : n(x^j) \cdot v^j > 0\}$ with the measure $\mathrm{d}\sigma_j = \mathrm{d}\sigma_j(x^j)$ on $\mathcal{V}_j$ which is given by

$$
\mathrm{d}\sigma_j := \mu(x^{j+1}, v^j_{\mathbf{b}})\{n(x^j) \cdot v^j\}\mathrm{d}v^j. \tag{1.17}
$$

Here, $n(x)$ is the outward normal at $x \in \partial\Omega$. To clarify the notation, in the rest of this paper,we let the superscript of $x, v, t, v_{\mathbf{b}}, x_{\mathbf{b}}, t_{\mathbf{b}}$ (e.g.,x^i, $t^i_{\mathbf{b}}$) denote the notation in the stochastic cycles; we include absolute brackets or parentheses or angle brackets to denote the power of these terms (e.g., $(t_{\mathbf{b}})^2$, $\langle t^i_{\mathbf{b}}\rangle^4$).

Given $(t, x, v) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^3$, suppose that $(X(s; t, x, v), V(s; t, x, v))$ solves (1.14), the backward exit time $t_{\mathbf{b}}$ stands for the longest backward time, for which the characteristic $X(s; t, x, v)$ stays in the domain Ω. And $x_{\mathbf{b}} = X(t - t_{\mathbf{b}}; t, x, v)$ is the boundary position when $s = t - t_{\mathbf{b}}$. Similarly, the forward exit time $t_{\mathbf{f}}$ is the longest forward time, for which the characteristic $X(s; t, x, v)$ stays in the domain Ω, and $x_{\mathbf{f}} = X(t + t_{\mathbf{f}}; t, x, v)$ is the boundary position when $s = t + t_{\mathbf{f}}$. Moreover, since the field $\Phi(x)$ is timely independent, this leads that both $t_{\mathbf{b}}$ and $t_{\mathbf{f}}$ are also timely independent.

Now we explain a major difficulty in the presence of logarithmic potential. Compared to the constant potential considered in [5], the backward exit time $t_{\mathbf{b}}$ and the forward exit time $t_{\mathbf{f}}$ have much weaker control. Indeed,we can derive that, for any $(x, v) \in \gamma_-$,

$$
a^{\frac{1}{2}|v_3|^2}\sqrt{1 - a^{-\frac{1}{2}|v_3|^2}} \lesssim t_{\mathbf{b}}(x, v) \lesssim a^{\frac{1}{2}|v_3|^2},
$$

using the conservation of mass on the characteristic line crucially. This control shows that the backward exit time $t_{\mathbf{b}}$ is comparable to $a^{\frac{1}{2}|v_3|^2}$ when $n(x) \cdot v \gg 1$. The crucial observation is that the Maxwellian $\mu(x,v)=\frac{1}{2\pi}e^{-\frac{|v|^2}{2}}$ has a polynomial control on $t_{\mathbf{b}}$ (or $t_{\mathbf{f}}$ for $(x,v)\in\gamma_+$) depending on $\mathcal{A}=[\frac{1}{\ln(a)}]$. Therefore, we are able to control the sum of infinite Maxwellian terms produced by the periodic domain (see Lemma 13).

The proof of dynamical stability on the fluctuations $f(t,x,v)$, which solves (1.1), (1.2), and (1.8), is based on a lower bound with the unreachable defect (see Proposition 17) as follows:

$$\begin{aligned} f(NT_0,x,v) \ &\geq \mathfrak{m}(x,v)\Big\{\iint_{\Omega\times\mathbb{R}^3} f((N-1)T_0,x,v)\mathrm{d}v\mathrm{d}x \\ &\quad -\iint_{\Omega\times\mathbb{R}^3}\mathbf{1}_{t_{\mathbf{f}}(x,v)\geq\frac{T_0}{4}} f((N-1)T_0,x,v)\mathrm{d}v\mathrm{d}x\Big\}, \end{aligned}$$

where $\mathfrak{m}(x,v)$ is defined in (3.32). This is also considered as the Doeblin condition where $f(t,x,v)$ is bounded below by the part of the mass of molecules in previous stochastic cycles. We refer to [3], which includes a systematic exposition of Doeblin-type arguments.

Next we control the unreachable defect (see Lemma 15). Since the forward exit time under Vlasov operator can be controlled as follows:

$$\frac{\partial}{\partial t}t_{\mathbf{f}}(t,x,v)+v\cdot\frac{\partial}{\partial x}t_{\mathbf{f}}(t,x,v)-\nabla\Phi(x)\cdot\frac{\partial}{\partial v}t_{\mathbf{f}}(t,x,v)=-1,$$

any weight function $\varphi(t_{\mathbf{f}})$ satisfies $(v\cdot\nabla_x-\nabla\Phi(x)\cdot\nabla_v)\varphi(t_{\mathbf{f}})=-\varphi'(t_{\mathbf{f}})$. Moreover, we consider the weight function $\varphi:[0,\infty)\to\mathbb{R}$ to satisfy that for any $\tau\geq 0$, $\varphi(\tau)\geq 0$, $\varphi'\geq 0$, and

$$\int_1^\infty \tau^{3-\mathcal{A}}\varphi(\tau)\mathrm{d}\tau<\infty. \tag{1.18}$$

It is worth to compare to the constant gravity case [5] when we allow $\int_1^\infty e^{-\frac{1}{2}\tau^2}\varphi(\tau)\mathrm{d}\tau<\infty$ and then the system has an exponential decay. This weaker weight in τ restricts the range of φ and consequently deduces a polynomial decay.

Suppose f solves (1.1) and (1.2), there exists $C>0$ independent of t_*, t, such that for all $0\leq t_*\leq t$,

$$\begin{aligned} &\|\varphi(t_{\mathbf{f}})f(t)\|_{L^1_{x,v}}+\int_{t_*}^t\|\varphi'(t_{\mathbf{f}})f\|_{L^1_{x,v}}\mathrm{d}s+\int_{t_*}^t|\varphi(t_{\mathbf{f}})f|_{L^1_{\gamma_+}}\mathrm{d}s \\ &\quad\leq\|\varphi(t_{\mathbf{f}})f(t_*)\|_{L^1_{x,v}}+C(t-t_*+1)\|f(t_*)\|_{L^1_{x,v}}+\frac{1}{4}\int_{t_*}^t|f|_{L^1_{\gamma_+}}\mathrm{d}s. \end{aligned}$$

We remark that the exponent $3-\mathcal{A}$ in (1.18) is determined from the initial condition $\|e^{\frac{1}{2}|v|^2+\Phi(x)}f_0\|_{L^\infty_{x,v}}<\infty$ and polynomial control between $\mu(x,v)$ and $t_{\mathbf{f}}$ for $(x,v)\in\gamma_+$. Furthermore, this exponent will restrict the decay rate of Theorem 3. Then we introduce two norms $|||\cdot|||_2$ and $|||\cdot|||_4$ as

$$|||f|||_i := \|f\|_{L^1_{x,v}} + \frac{4\mathfrak{m}_{T_0}}{\varphi_{i-1}\left(\frac{3T_0}{4}\right)}\|\varphi_{i-1}(t_{\mathbf{f}})f\|_{L^1_{x,v}} + \frac{4e\mathfrak{m}_{T_0}}{T_0\varphi_{i-1}\left(\frac{3T_0}{4}\right)}\|\varphi_i(t_{\mathbf{f}})f\|_{L^1_{x,v}},$$

where four polynomial weights $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ are defined in (3.39). We derive the polynomial decay in L^1 after using an energy estimate on these norms.

At last, to conclude a pointwise bound on the exponential moment, we introduce several weight functions $\varrho(t)$ and $w'(x, v)$. Then we control the bound on $\varrho(t)w'(x, v)f(t, x, v)$ via stochastic cycles expansions and polynomial decay on the fluctuations proved before. This allows us to conclude the decay of the exponential moment.

Structural of the paper. For the rest of the paper, we collect some basic preliminaries in Sect. 2. Then in Sect. 3, we study the weighted L^1-estimates and prove Theorem 1. Finally in Sect. 4, we show an L^∞-estimate of moments in Theorem 3.

2 Background

We first list some properties for (1.14), the characteristics of (1.1).

Lemma 6 [5] *For any $g(t, x, v)$ and (X, V) solving (1.14), we have*

$$\int_{\gamma_+}\int_0^{t_-} g(t, X(t, t+s, x, v), V(t, t+s, x, v))|n(x)\cdot v|\mathrm{d}s\mathrm{d}v\mathrm{d}S_x = \iint_{\Omega\times\mathbb{R}^3} g(t, y, v)\mathrm{d}y\mathrm{d}v, \quad (2.1)$$

$$\int_{\gamma_-}\int_0^{t_+} g(t, X(t, t-s, x, v), V(t, t-s, x, v))|n(x)\cdot v|\mathrm{d}s\mathrm{d}v\mathrm{d}S_x = \iint_{\Omega\times\mathbb{R}^3} g(t, y, v)\mathrm{d}y\mathrm{d}v, \quad (2.2)$$

$$\int_{\gamma_\pm} g(t, x_\mp(x, v), v_\mp(x, v))|n(x)\cdot v|\mathrm{d}v\mathrm{d}S_x = \int_{\gamma_\mp} g(t, y, v)|n(y)\cdot v|\mathrm{d}v\mathrm{d}S_y. \quad (2.3)$$

Here, for the sake of simplicity, we have abused the notations temporarily: $t_- = t_{\mathbf{b}}, x_- = x_{\mathbf{b}}$ and $t_+ = t_{\mathbf{f}}, x_+ = x_{\mathbf{f}}$.

The following Lemma will let us derive the stochastic cycles.

Lemma 7 [5]

Suppose $F(x, v)$ solves (1.1) and (1.2). Consider (X, V) solving (1.14) with $0 \le t_* \le t$, then for $k \ge 1$,

$$\begin{aligned}F(x, v) = \mathbf{1}_{t^1 < t_*} F(X(t_*; t, x, v), V(t_*; t, x, v)) \\ +\mu(x^1, v_{\mathbf{b}})\sum_{i=1}^{k-1}\int_{\prod_{j=1}^{i}\mathcal{V}_j} \\ \left\{\mathbf{1}_{t^{i+1} < t_* \le t^i} F(X(t_*; t^i, x^i, v^i), V(t_*; t^i, x^i, v^i))\right\}\mathrm{d}\Sigma_i\end{aligned}$$

$$+\mu(x^1, v_{\mathbf{b}}) \int_{\prod_{j=1}^{k} \mathcal{V}_j} \mathbf{1}_{t^k \geq t_*} F(x^k, v^k) \mathrm{d}\Sigma_k, \tag{2.4}$$

where $\mathrm{d}\Sigma_i := \frac{\mathrm{d}\sigma_i}{\mu(x^{i+1}, v_{\mathbf{b}}^i)} \mathrm{d}\sigma_{i-1} \cdots \mathrm{d}\sigma_1$, *with* $\mathrm{d}\sigma_j = \mu(x^{j+1}, v_{\mathbf{b}}^j)\{n(x^j) \cdot v^j\}\mathrm{d}v^j$ *in (1.17), and* $v_{\mathbf{b}}^j = v_{\mathbf{b}}(x^j, v^j)$ *defined in (1.16).*

Proof The proof follows from a similar argument, Lemma 2 in [5]. □

Lemma 8 *Consider* (X, V) *solving (1.14), then for* $x \in \partial\Omega$ *and* $v \in \mathcal{V} := \{v \in \mathbb{R}^3 : n(x) \cdot v > 0\}$,

$$|v_{\mathbf{b}}| = |v|.$$

where $v_{\mathbf{b}} = v_{\mathbf{b}}(x, v)$ *defined in (1.16).*

Proof The proof follows from a similar argument, Lemma 3 in [5]. Since (X, V) solves (1.14) with $v \in \mathcal{V}$, we compute the following derivative:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}s}\Big(\frac{|V(s;t,x,v)|^2}{2} + \Phi(X(s;t,x,v))\Big) \\ &= V(s;t,x,v) \cdot \frac{\mathrm{d}V}{\mathrm{d}s} + \nabla\Phi \cdot \frac{\mathrm{d}X}{\mathrm{d}s} = -V(s;t,x,v) \cdot \nabla\Phi(X(s;t,x,v)) \\ &\quad + \nabla\Phi \cdot V(s;t,x,v) = 0. \end{aligned} \tag{2.5}$$

Recall that $(X(t;t,x,v), V(t;t,x,v)) = (x, v)$, $(X(t - t_{\mathbf{b}}; t, x, v), V(t - t_{\mathbf{b}}; t, x, v)) = (x_{\mathbf{b}}, v_{\mathbf{b}})$. By taking $s = t - t_{\mathbf{b}}$ and $s = t$, we obtain

$$|v|^2/2 + \Phi(x) = |v_{\mathbf{b}}|^2/2 + \Phi(x_{\mathbf{b}}).$$

Since $\Phi(x)|_{x_3=0} = \log_a(1 + x_3)|_{x_3=0} \equiv 0$ and $x, x_{\mathbf{b}} \in \partial\Omega$, we have

$$\Phi(x) = \Phi(x_{\mathbf{b}}) = 0,$$

which implies $|v_{\mathbf{b}}| = |v|$. □

Remark 9 We compute that $\mu(x, v) = \frac{1}{2\pi} e^{-\frac{|v|^2}{2}}$ is radial, that is, $\mu(x, v_1) = \mu(x, v_2)$ if $|v_1| = |v_2|$. From the Lemma 8, we obtain $\mu(x^{j+1}, v_{\mathbf{b}}^j) = \mu(x^{j+1}, v^j)$ where $v_{\mathbf{b}}^j = v_{\mathbf{b}}(x^j, v^j)$. Therefore, in the rest of the paper, we write $\mathrm{d}\sigma_j$ as

$$\mathrm{d}\sigma_j = \mu(x^{j+1}, v^j)\{n(x^j) \cdot v^j\}\mathrm{d}v^j,$$

where $v^j \in \mathcal{V}_j := \{v^j \in \mathbb{R}^3 : n(x^j) \cdot v^j > 0\}$ and (X, V) solves (1.14).

Now we consider the change of variables $v \in \{v \in \mathbb{R}^3 : n(x) \cdot v > 0\} \mapsto (x_{\mathbf{b}}(x, v), t_{\mathbf{b}}(x, v)) \in \partial\Omega \times \mathbb{R}_+$. Since the domain is periodic, this is a local bijective mapping. For fixed x, $t_{\mathbf{b}}$ and $x_{\mathbf{b}}$, we introduce the set of velocities $\{v^{m,n}\}$ with $m, n \in \mathbb{Z}$ such that

$$\begin{aligned} v^{m,n} &\in \{v^{m,n} \in \mathbb{R}^3 : n(x) \cdot v^{m,n} > 0\} \mapsto (x_{\mathbf{b}}, t_{\mathbf{b}}) := (x_{\mathbf{b}} + (m, n, 0), t_{\mathbf{b}}) \\ &= (x_{\mathbf{b}}, t_{\mathbf{b}}) \in \partial\Omega \times \mathbb{R}_+. \end{aligned} \tag{2.6}$$

Proposition 10 *Consider (X, V) solving (1.14),*

- *For fixed $x \in \partial\Omega$, and $m, n \in \mathbb{Z}$, we introduce the following map:*

$$\begin{aligned} v \in \{v \in \mathbb{R}^3 : n(x) \cdot v > 0\} &\mapsto (x_{\mathbf{b}}, t_{\mathbf{b}}) \\ &:= (x_{\mathbf{b}}(x, v) + (m, n, 0), t_{\mathbf{b}}(x, v)) \in \partial\Omega \times \mathbb{R}_+. \end{aligned} \tag{2.7}$$

Then the map (2.7) is locally bijective and has the change of variable formula as

$$(t_{\mathbf{b}})^{-2}(1 + |v_3|t_{\mathbf{b}})^{-1} \mathrm{d}t_{\mathbf{b}} \mathrm{d}S_{x_{\mathbf{b}}} \lesssim \mathrm{d}v \lesssim (t_{\mathbf{b}})^{-2} \mathrm{d}t_{\mathbf{b}} \mathrm{d}S_{x_{\mathbf{b}}}. \tag{2.8}$$

- *Similarly we have a locally bijective map:*

$$v \in \{v \in \mathbb{R}^3 : n(x) \cdot v < 0\} \mapsto (x_{\mathbf{f}}, t_{\mathbf{f}}) := (x_{\mathbf{f}}(x, v) + (m, n, 0), t_{\mathbf{f}}(x, v)) \in \partial\Omega \times \mathbb{R}_+,$$

with

$$(t_{\mathbf{f}})^{-2}(1 + |v_3|t_{\mathbf{f}})^{-1} \mathrm{d}t_{\mathbf{f}} \mathrm{d}S_{x_{\mathbf{f}}} \lesssim \mathrm{d}v \lesssim (t_{\mathbf{f}})^{-2} \mathrm{d}t_{\mathbf{f}} \mathrm{d}S_{x_{\mathbf{f}}}. \tag{2.9}$$

Proof We just need to show (2.8), since (2.9) can be deduced after changing the backward variables into forward variables. For the sake of simplicity, we have abused the notations temporarily:

$$\begin{aligned} &x_{\mathbf{b}} := x^1 = (x_1^1, x_2^1, x_3^1) = (x_{\parallel}^1, x_3^1), \quad v = (v_1, v_2, v_3) = (v_{\parallel}, v_3), \\ &v_{\mathbf{b}} = (v_{\mathbf{b},1}, v_{\mathbf{b},2}, v_{\mathbf{b},3}), \quad t_{\mathbf{b}} = t_{\mathbf{b}}(x, v). \end{aligned}$$

Recall $\Phi(x) = \log_a(1 + x_3)$, then we get $\nabla\Phi = (0, 0, \frac{1}{(1+x_3)\ln(a)})$ with $\frac{1}{\ln(a)} > 1$.

Now we compute the determinant of the Jacobian matrix. Fixing x, t and following the characteristics trajectory, we deduce

$$x^1 + \int_{t-t_{\mathbf{b}}}^{t} V(s; t, x, v)\mathrm{d}s + (m, n, 0) = x, \tag{2.10}$$

$$v_{\mathbf{b}} + \int_{t-t_{\mathbf{b}}}^{t} -\nabla\Phi(X(s; t, x, v))\mathrm{d}s = v. \tag{2.11}$$

Inputting (2.11) into (2.10), we have

$$\begin{aligned} x_1^1 + t_{\mathbf{b}} v_1 + m &= x_1, \\ x_2^1 + t_{\mathbf{b}} v_2 + n &= x_2. \end{aligned} \tag{2.12}$$

From (2.11), we obtain $t_{\mathbf{b}} = t_{\mathbf{b}}(v_3)$ and

$$\frac{\partial t_{\mathbf{b}}}{\partial v} = (0, 0, \frac{\partial t_{\mathbf{b}}}{\partial v_3}), \text{ and } \frac{\partial x_{\parallel}^1}{\partial v} = \begin{pmatrix} -t_{\mathbf{b}} & 0 & -v_1 \frac{\partial t_{\mathbf{b}}}{\partial v_3} \\ 0 & -t_{\mathbf{b}} & -v_2 \frac{\partial t_{\mathbf{b}}}{\partial v_3} \end{pmatrix}.$$

Therefore, we get

$$\det\left(\frac{\partial x_{\parallel}^1}{\partial v}, \frac{\partial t_{\mathbf{b}}}{\partial v}\right) = (t_{\mathbf{b}})^2 \times \frac{\partial t_{\mathbf{b}}}{\partial v}. \tag{2.13}$$

Now recall (1.14),

$$\frac{\mathrm{d}}{\mathrm{d}s} V_3(s; t, x, v) = -\frac{1}{(1 + X_3(s; t, x, v)) \ln(a)}. \tag{2.14}$$

Thus, we obtain

$$1 + X_3(s; t, x, v) = a^{\frac{1}{2}(v_3^2 - V_3^2(s;t,x,v))}, \tag{2.15}$$

and

$$|v_{\mathbf{b},3}| = |v_3|. \tag{2.16}$$

Inputting (2.15) into (2.14), we derive

$$\frac{\mathrm{d}}{\mathrm{d}s} V_3(s; t, x, v) = -\frac{1}{a^{-\frac{1}{2}(v_3^2 - V_3^2(s;t,x,v))} \ln(a)},$$

and thus

$$a^{-\frac{1}{2} V_3^2(s;t,x,v)} \mathrm{d}V_3(s; t, x, v) = -\frac{a^{-\frac{1}{2} v_3^2}}{\ln(a)} \mathrm{d}s. \tag{2.17}$$

Note that $v_3 = V_3(t; t, x, v)$ and $v_{\mathbf{b},3} = V_3(t - t_{\mathbf{b}}; t, x, v)$. Taking the integration toward time $s \in [t - t_{\mathbf{b}}, t]$ on (2.17), we get

$$\int_{v_{\mathbf{b},3}}^{v_3} a^{-\frac{1}{2} V_3^2(s;t,x,v)} \mathrm{d}V_3(s; t, x, v) = \int_{t-t_{\mathbf{b}}}^{t} -\frac{a^{-\frac{1}{2} v_3^2}}{\ln(a)} \mathrm{d}s = -\frac{a^{-\frac{1}{2} v_3^2}}{\ln(a)} t_{\mathbf{b}}.$$

From Lemma 8, $v_{\mathbf{b},3} = -v_3 > 0$. Further, since $a^{-\frac{1}{2} V^2}$ is an even function, we have

$$\int_{v_{\mathbf{b},3}}^{v_3} a^{-\frac{1}{2}V_3^2(s;t,x,v)}\mathrm{d}V_3(s;t,x,v) = -2\int_0^{|v_3|} a^{-\frac{1}{2}V_3^2(s;t,x,v)}\mathrm{d}V_3(s;t,x,v) = -\frac{a^{-\frac{1}{2}v_3^2}}{\ln(a)}t_{\mathbf{b}}. \tag{2.18}$$

We estimate the following integration:

$$\int_0^{|v_3|} a^{-\frac{1}{2}y^2}\mathrm{d}y \leq \int_0^{\infty} a^{-\frac{1}{2}y^2}\mathrm{d}y \lesssim \sqrt{\frac{1}{\ln(a)}}.$$

On the other hand,

$$\int_0^{|v_3|} a^{-\frac{1}{2}y^2}\mathrm{d}y \gtrsim \sqrt{\int_0^{\frac{\pi}{2}}\int_0^{v_3} a^{-\frac{1}{2}r^2} r\mathrm{d}r\mathrm{d}\theta} \gtrsim \sqrt{\frac{1-a^{-\frac{1}{2}v_3^2}}{\ln(a)}}.$$

From (2.18), we get

$$t_{\mathbf{b}} = 2\ln(a)a^{\frac{1}{2}v_3^2}\int_0^{|v_3|} a^{-\frac{1}{2}V_3^2(s;t,x,v)}\mathrm{d}V_3(s;t,x,v). \tag{2.19}$$

Then

$$\frac{2\ln(a)}{\sqrt{\ln(a)}}a^{\frac{1}{2}v_3^2}\sqrt{1-a^{-\frac{1}{2}v_3^2}} \lesssim t_{\mathbf{b}} \lesssim \frac{2\ln(a)}{\sqrt{\ln(a)}}a^{\frac{1}{2}v_3^2}.$$

Since a is fixed, for simplicity we rewrite the above as

$$a^{\frac{1}{2}v_3^2}\sqrt{1-a^{-\frac{1}{2}v_3^2}} \lesssim t_{\mathbf{b}} \lesssim a^{\frac{1}{2}v_3^2}. \tag{2.20}$$

Note that for $0 \leq |v_3| \ll 1$, we use the Taylor expansion on $a^{-\frac{1}{2}v_3^2}$, and obtain

$$t_{\mathbf{b}} \gtrsim a^{\frac{1}{2}v_3^2}\sqrt{1-a^{-\frac{1}{2}v_3^2}} \gtrsim \sqrt{1-a^{-\frac{1}{2}v_3^2}} \gtrsim |v_3|. \tag{2.21}$$

Next, we take the derivative $\frac{\mathrm{d}}{\mathrm{d}v_3}$ on (2.19) and write $\frac{\mathrm{d}t_{\mathbf{b}}}{\mathrm{d}v_3}$ as

$$\frac{\mathrm{d}t_{\mathbf{b}}}{\mathrm{d}v_3} = -2\ln(a) + v_3 t_{\mathbf{b}} < 0.$$

Thus, we derive that

$$1+|v_3|a^{\frac{1}{2}v_3^2}\sqrt{1-a^{-\frac{1}{2}v_3^2}} \lesssim \left|\frac{\mathrm{d}t_{\mathbf{b}}}{\mathrm{d}v_3}\right| \lesssim 1+|v_3|a^{\frac{1}{2}v_3^2}. \tag{2.22}$$

Since a is a fixed constant, we can write

$$1+|v_3|a^{\frac{1}{2}v_3^2}\sqrt{1-a^{-\frac{1}{2}v_3^2}} \lesssim \left|\frac{\mathrm{d}t_{\mathbf{b}}}{\mathrm{d}v_3}\right| \lesssim 1+|v_3|t_{\mathbf{b}}.$$

Inputting (2.20), (2.22) into (2.13), we get the following:

$$\left|\det\left(\frac{\partial x_{\parallel}^1}{\partial v},\frac{\partial t_{\mathbf{b}}}{\partial v}\right)\right| = (t_{\mathbf{b}})^2\times\left|\frac{\partial t_{\mathbf{b}}}{\partial v}\right| \gtrsim (t_{\mathbf{b}})^2\times\left(1+|v_3|a^{\frac{1}{2}v_3^2}\sqrt{1-a^{-\frac{1}{2}v_3^2}}\right) \gtrsim (t_{\mathbf{b}})^2,$$

and

$$\left|\det\left(\frac{\partial x_{\parallel}^1}{\partial v},\frac{\partial t_{\mathbf{b}}}{\partial v_3}\right)\right| = (t_{\mathbf{b}})^2\times\left|\frac{\partial t_{\mathbf{b}}}{\partial v}\right| \lesssim (t_{\mathbf{b}})^2\times\left(1+|v_3|t_{\mathbf{b}}\right)$$

Therefore, we conclude

$$\frac{1}{(t_{\mathbf{b}})^2(1+|v_3|t_{\mathbf{b}})} \lesssim \left|\det\left(\frac{\partial x_{\parallel}^1}{\partial v},\frac{\partial t_{\mathbf{b}}}{\partial v}\right)\right|^{-1} \lesssim \frac{1}{(t_{\mathbf{b}})^2},$$

and we conclude (2.8). □

The following lemma is a consequence of Proposition 10.

Lemma 11 *Consider* (X,V) *solving (1.14),*

- *For* $x\in\partial\Omega$ *and* $v\in\mathcal{V}:=\{v\in\mathbb{R}^3 : n(x)\cdot v>0\}$, *we consider the map (2.7) with* $m,n\in\mathbb{Z}$, *then*

$$|v_3| = |v_{\mathbf{b},3}| \lesssim t_{\mathbf{b}}(x,v). \tag{2.23}$$

- *Similarly for* $x\in\partial\Omega$ *and* $v\in\{v\in\mathbb{R}^3 : n(x)\cdot v<0\}$, *we consider the map (2.9) with* $m,n\in\mathbb{Z}$, *then*

$$|v_3| = |v_{\mathbf{f},3}| \lesssim t_{\mathbf{f}}(x,v). \tag{2.24}$$

Proof We just need to show (2.23), since (2.24) can be deduced after changing the backward variables into forward variables. Similar to Proposition 10, we have abused the notations temporarily:

$$v=(v_1,v_2,v_3),\quad v_{\mathbf{b}}=(v_{\mathbf{b},1},v_{\mathbf{b},2},v_{\mathbf{b},3}),\quad t_{\mathbf{b}}=t_{\mathbf{b}}(x,v).$$

The first equality $|v_3|=|v_{\mathbf{b},3}|$ follows from (2.16).

Next, from (2.20) we have

$$a^{\frac{1}{2}v_3^2}\sqrt{1-a^{-\frac{1}{2}v_3^2}} \lesssim t_{\mathbf{b}}.$$

Since $a > 1$ is fixed, then for $|v_3| > 0$,

$$\frac{t_{\mathbf{b}}}{|v_3|} \gtrsim \frac{a^{\frac{1}{2}v_3^2}\sqrt{1 - a^{-\frac{1}{2}v_3^2}}}{|v_3|} > 0. \tag{2.25}$$

Moreover, there exists sufficiently large $n > 1$ such that for any $|v_3| \geq n$,

$$a^{-\frac{1}{2}v_3^2} \leq \frac{1}{2}. \tag{2.26}$$

Using the Taylor expansion on $a^{\frac{1}{2}v_3^2}$, we obtain

$$a^{\frac{1}{2}v_3^2} \gtrsim v_3^2 \geq |v_3|. \tag{2.27}$$

From (2.26), (2.27), we derive that for any $|v_3| \geq n$,

$$t_{\mathbf{b}} \gtrsim a^{\frac{1}{2}v_3^2}\sqrt{1 - a^{-\frac{1}{2}v_3^2}} \gtrsim \frac{1}{\sqrt{2}}|v_3|. \tag{2.28}$$

On the other hand, from (2.21) for $0 \leq |v_3| \ll 1$, we have

$$t_{\mathbf{b}} \gtrsim |v_3|. \tag{2.29}$$

Together with (2.25), (2.28) and (2.29), we conclude (2.23). □

Remark 12 We can apply Proposition 10 on $v^j \in \mathcal{V}_j \mapsto (x^{j+1}, t^j_{\mathbf{b}}) := (x_{\mathbf{b}}(x^j, v^j), t_{\mathbf{b}}(x^j, v^j))$, and this is also a local bijective mapping. For fixed $t^j_{\mathbf{b}}$ and x^{j+1}, we introduce the set of velocities $\{v_j^{m,n}\}$ with $m, n \in \mathbb{Z}$ such that

$$v_j^{m,n} \in \mathcal{V}_j \mapsto (x^{j+1} + (m, n, 0), t^j_{\mathbf{b}}) = (x^{j+1}, t^j_{\mathbf{b}}) \in \partial\Omega \times [0, t^j], \tag{2.30}$$

with the change of variable formula as

$$\mathrm{d}v_j^{m,n} \lesssim |t^j_{\mathbf{b}}|^{-2}\mathrm{d}t^j_{\mathbf{b}}\mathrm{d}S_{x^{j+1}}. \tag{2.31}$$

Because of the periodic domain, we will gain an infinite sum of Maxwellian terms as the integrand after the change of variable in Remark 12. In the following lemma, we do an estimate on this infinite sum.

Lemma 13 *Consider (X, V) solving (1.14) with $x^{i-1} \in \partial\Omega$, for $t^{i-2}_{\mathbf{b}} \geq 1$,*

$$\sum_{m,n\in\mathbb{Z}} \mu(x^{i-1}, v^{m,n}_{i-2,\mathbf{b}}) \lesssim |t^{i-2}_{\mathbf{b}}|^{4-\mathcal{A}}. \tag{2.32}$$

For $0 \le t_{\mathbf{b}}^{i-2} < 1$,

$$\sum_{m,n\in\mathbb{Z}} \mu(x^{i-1}, v_{i-2,\mathbf{b}}^{m,n}) \lesssim \sum_{|m|<2,|n|<2} \mu(x^{i-1}, v_{i-2,\mathbf{b}}^{m,n}) + e^{-\frac{1}{2(t_{\mathbf{b}}^{i-2})^2}}, \tag{2.33}$$

where $v_{i-2,\mathbf{b}}^{m,n} = v_{\mathbf{b}}(x^{i-1}, v_{i-2}^{m,n})$*, which was defined in (1.16) and (2.6).*

Proof Here, for the sake of simplicity, we have abused the notations temporarily:

$$x^i = (x_1^i, x_2^i, x_3^i) = (x_{\parallel}^i, x_3^i),\ \ v_{i,\mathbf{b}}^{m,n} = (v_{i,\mathbf{b}_1}^{m,n}, v_{i,\mathbf{b}_2}^{m,n}, v_{i,\mathbf{b}_3}^{m,n}) = (v_{i,\mathbf{b}_{\parallel}}^{m,n}, v_{i,\mathbf{b}_3}^{m,n}).$$

To estimate $v_{i-2,\mathbf{b}_{\parallel}}^{m,n}$, we recall (2.12) and get

$$|v_{i-2,\mathbf{b}_1}^{m,n}| = \frac{|x_1^{i-1} + m - x_1^{i-2}|}{t_{\mathbf{b}}^{i-2}},\ \ |v_{i-2,\mathbf{b}_2}^{m,n}| = \frac{|x_2^{i-1} + n - x_2^{i-2}|}{t_{\mathbf{b}}^{i-2}}. \tag{2.34}$$

Now we split the length of $t_{\mathbf{b}}^{i-2}$ into two cases:

Case 1: $t_{\mathbf{b}}^{i-2} \ge 1$. From (2.34), for $|m| \ge (t_{\mathbf{b}}^{i-2})^2$, we bound

$$\frac{|x_1^{i-1} + m - x_1^{i-2}|}{t_{\mathbf{b}}^{i-2}} \gtrsim \frac{|m|}{2t_{\mathbf{b}}^{i-2}}.$$

Similarly, for $|n| \ge (t_{\mathbf{b}}^{i-2})^2$, we bound

$$\frac{|x_2^{i-1} + n - x_2^{i-2}|}{t_{\mathbf{b}}^{i-2}} \gtrsim \frac{|n|}{2t_{\mathbf{b}}^{i-2}}.$$

For $|m| < (t_{\mathbf{b}}^{i-2})^2$, we bound $|v_{i-2,\mathbf{b}_1}^{m,n}| \ge 0$, and for $|n| < (t_{\mathbf{b}}^{i-2})^2$, we bound $|v_{i-2,\mathbf{b}_2}^{m,n}| \ge 0$. In order to derive (2.32), we divide $\{v_{i-2,\mathbf{b}}^{m,n}\}_{m,n\in\mathbb{Z}}$ into four parts.

(a) For $|m| < (t_{\mathbf{b}}^{i-2})^2$ and $|n| < (t_{\mathbf{b}}^{i-2})^2$, we bound $a^{\frac{1}{2}|v_{i-2,\mathbf{b}_3}^{m,n}|^2} \gtrsim t_{\mathbf{b}}^{i-2}$ in (2.20). Therefore, we have

$$\begin{aligned}\sum_{|m|<(t_{\mathbf{b}}^{i-2})^2,|n|<(t_{\mathbf{b}}^{i-2})^2} \mu(x^{i-1}, v_{i-2,\mathbf{b}}^{m,n}) &\lesssim (t_{\mathbf{b}}^{i-2})^4 e^{-\frac{1}{2}|v_{i-2,\mathbf{b}}^{m,n}|^2} \lesssim (t_{\mathbf{b}}^{i-2})^4 (t_{\mathbf{b}}^{i-2})^{-\mathcal{A}} \\ &= |t_{\mathbf{b}}^{i-2}|^{4-\mathcal{A}}.\end{aligned} \tag{2.35}$$

(b) For $|m| < (t_{\mathbf{b}}^{i-2})^2$ and $|n| \geq (t_{\mathbf{b}}^{i-2})^2$, we bound $|v_{i-2,\mathbf{b}_2}^{m,n}| \gtrsim \frac{|n|}{2t_{\mathbf{b}}^{i-2}}$. Thus, we have

$$\begin{aligned}
\sum_{|m|<(t_{\mathbf{b}}^{i-2})^2, |n|\geq (t_{\mathbf{b}}^{i-2})^2} \mu(x^{i-1}, v_{i-2,\mathbf{b}}^{m,n}) &\lesssim \sum_{|m|<(t_{\mathbf{b}}^{i-2})^2} \mu(x^{i-1}, (0, \frac{|n|}{2t_{\mathbf{b}}^{i-2}}, v_{i-2,\mathbf{b}_3}^{m,n})) \\
&\lesssim (t_{\mathbf{b}}^{i-2})^2 (t_{\mathbf{b}}^{i-2})^{-\mathcal{A}} \sum_{n=0}^{\infty} \mu(x^{i-1}, \frac{|n|}{2t_{\mathbf{b}}^{i-2}}) \\
&\leq (t_{\mathbf{b}}^{i-2})^2 (t_{\mathbf{b}}^{i-2})^{-\mathcal{A}} (1 - e^{-\frac{1}{8(t_{\mathbf{b}}^{i-2})^2}})^{-1} \\
&\lesssim (t_{\mathbf{b}}^{i-2})^4 (t_{\mathbf{b}}^{i-2})^{-\mathcal{A}} = |t_{\mathbf{b}}^{i-2}|^{4-\mathcal{A}},
\end{aligned} \tag{2.36}$$

where the last inequality holds from the Taylor expansion.

(c) For $|m| \geq (t_{\mathbf{b}}^{i-2})^2$ and $|n| < (t_{\mathbf{b}}^{i-2})^2$ case, we bound $|v_{i-2,\mathbf{b}_1}^{m,n}| \gtrsim \frac{|m|}{2t_{\mathbf{b}}^{i-2}}$. Similar as in (2.36), we get

$$\sum_{|m|\geq (t_{\mathbf{b}}^{i-2})^2, |n|<(t_{\mathbf{b}}^{i-2})^2} \mu(x^{i-1}, v_{i-2,\mathbf{b}}^{m,n}) \lesssim |t_{\mathbf{b}}^{i-2}|^{4-\mathcal{A}}. \tag{2.37}$$

(d) For $|m| \geq (t_{\mathbf{b}}^{i-2})^2$ and $|n| \geq (t_{\mathbf{b}}^{i-2})^2$, we use two lower bounds $|v_{i-2,\mathbf{b}_2}^{m,n}| \gtrsim \frac{|n|}{2t_{\mathbf{b}}^{i-2}}$, $|v_{i-2,\mathbf{b}_1}^{m,n}| \gtrsim \frac{|m|}{2t_{\mathbf{b}}^{i-2}}$. Then, we derive that

$$\begin{aligned}
&\sum_{|m|\geq (t_{\mathbf{b}}^{i-2})^2, |n|\geq (t_{\mathbf{b}}^{i-2})^2} \mu(x^{i-1}, v_{i-2,\mathbf{b}}^{m,n}) \\
&\lesssim \sum_{m,n=0}^{\infty} \mu\left(x^{i-1}, \left(\frac{|m|}{2t_{\mathbf{b}}^{i-2}}, \frac{|n|}{2t_{\mathbf{b}}^{i-2}}, v_{i-2,\mathbf{b}_3}^{m,n}\right)\right) \\
&\lesssim (t_{\mathbf{b}}^{i-2})^{-\mathcal{A}} \sum_{n=0}^{\infty} \mu(x^{i-1}, \frac{|n|}{2t_{\mathbf{b}}^{i-2}})(1 - e^{-\frac{1}{8(t_{\mathbf{b}}^{i-2})^2}})^{-1} \\
&\lesssim (t_{\mathbf{b}}^{i-2})^{-\mathcal{A}} (1 - e^{-\frac{1}{8(t_{\mathbf{b}}^{i-2})^2}})^{-2} \lesssim (t_{\mathbf{b}}^{i-2})^4 (t_{\mathbf{b}}^{i-2})^{-\mathcal{A}} = |t_{\mathbf{b}}^{i-2}|^{4-\mathcal{A}}.
\end{aligned} \tag{2.38}$$

From (2.35), (2.36), (2.37) and (2.38), we conclude that for $t_{\mathbf{b}}^{i-2} \geq 1$,

$$\sum_{m,n\in\mathbb{Z}} \mu(x^{i-1}, v_{i-2,\mathbf{b}}^{m,n}) \lesssim |t_{\mathbf{b}}^{i-2}|^{4-\mathcal{A}}.$$

Case 2: $0 \le t_{\mathbf{b}}^{i-2} < 1$. In this case $t_{\mathbf{b}}^{i-2}$ is small, for $|m| \ge 2$ and $|n| \ge 2$, we bound (2.34) as

$$\frac{|x_1^{i-1} + m - x_1^{i-2}|}{t_{\mathbf{b}}^{i-2}} \gtrsim \frac{|m|}{2t_{\mathbf{b}}^{i-2}}, \quad \frac{|x_2^{i-1} + n - x_2^{i-2}|}{t_{\mathbf{b}}^{i-2}} \gtrsim \frac{|n|}{2t_{\mathbf{b}}^{i-2}}.$$

For $|m| < 2$, we bound $|v_{i-2,\mathbf{b}_1}^{m,n}| \ge 0$, and for $|n| < 2$, we bound $|v_{i-2,\mathbf{b}_2}^{m,n}| \ge 0$. To obtain (2.33), we again divide $\{v_{i-2,\mathbf{b}}^{m,n}\}_{m,n\in\mathbb{Z}}$ into four parts.

(a) For $|m| < 2$ and $|n| < 2$, we keep the following five terms summation:

$$\sum_{|m|<2,|n|<2} \mu(x^{i-1}, v_{i-2,\mathbf{b}}^{m,n}). \tag{2.39}$$

(b) For $|m| < 2$ and $|n| \ge 2$, we bound $|v_{i-2,\mathbf{b}_2}^{m,n}| \gtrsim \frac{|n|}{2t_{\mathbf{b}}^{i-2}}$. Thus, we have

$$\begin{aligned}\sum_{|m|<2,|n|\ge2} \mu(x^{i-1}, v_{i-2,\mathbf{b}}^{m,n}) &\lesssim \sum_{|m|<2,|n|\ge2} \mu(x^{i-1}, (0, \frac{|n|}{2t_{\mathbf{b}}^{i-2}}, v_{i-2,\mathbf{b}_3}^{m,n})) \\ &\lesssim \sum_{n=2}^{\infty} \mu(x^{i-1}, \frac{|n|}{2t_{\mathbf{b}}^{i-2}}) \lesssim \sum_{n=2}^{\infty} e^{-\frac{n^2}{8(t_{\mathbf{b}}^{i-2})^2}} \\ &\le e^{-\frac{1}{2(t_{\mathbf{b}}^{i-2})^2}} (1 - e^{-\frac{1}{8(t_{\mathbf{b}}^{i-2})^2}})^{-1} \lesssim e^{-\frac{1}{2(t_{\mathbf{b}}^{i-2})^2}},\end{aligned} \tag{2.40}$$

where the last inequality holds from $0 \le t_{\mathbf{b}}^{i-2} < 1$.

(c) For $|m| \ge 2$ and $|n| < 2$ case, we bound $|v_{i-2,\mathbf{b}_1}^{m,n}| \gtrsim \frac{|m|}{2t_{\mathbf{b}}^{i-2}}$. Similar as in (2.40), we get

$$\sum_{|m|\ge2,|n|<2} \mu(x^{i-1}, v_{i-2,\mathbf{b}}^{m,n}) \lesssim e^{-\frac{1}{2(t_{\mathbf{b}}^{i-2})^2}}. \tag{2.41}$$

(d) For $|m| \ge 2$ and $|n| \ge 2$, we bound $|v_{i-2,\mathbf{b}_2}^{m,n}| \gtrsim \frac{|n|}{2t_{\mathbf{b}}^{i-2}}$, $|v_{i-2,\mathbf{b}_1}^{m,n}| \gtrsim \frac{|m|}{2t_{\mathbf{b}}^{i-2}}$ and we derive

$$\begin{aligned}\sum_{|m|\ge2,|n|\ge2} \mu(x^{i-1}, v_{i-2,\mathbf{b}}^{m,n}) &\lesssim \sum_{m,n=2}^{\infty} \mu\left(x^{i-1}, \left(\frac{|m|}{2t_{\mathbf{b}}^{i-2}}, \frac{|n|}{2t_{\mathbf{b}}^{i-2}}, v_{i-2,\mathbf{b}_3}^{m,n}\right)\right) \\ &\lesssim \sum_{n=2}^{\infty} \mu(x^{i-1}, \frac{|n|}{2t_{\mathbf{b}}^{i-2}}) e^{-\frac{1}{2(t_{\mathbf{b}}^{i-2})^2}} \lesssim e^{-\frac{1}{2(t_{\mathbf{b}}^{i-2})^2}}.\end{aligned} \tag{2.42}$$

From (2.39), (2.40), (2.41) and (2.42), we conclude that for $0 \le t_{\mathbf{b}}^{i-2} < 1$,

$$\sum_{m,n\in\mathbb{Z}} \mu(x^{i-1}, v_{i-2,\mathbf{b}}^{m,n}) \lesssim \sum_{|m|<2,|n|<2} \mu(x^{i-1}, v_{i-2,\mathbf{b}}^{m,n}) + e^{-\frac{1}{2(t_{\mathbf{b}}^{i-2})^2}},$$

so we prove (2.32) and (2.33). □

3 Weighted L^1-Estimates

The main purpose of this section is to prove Theorem 1, in which we do L^1-estimates on fluctuations. Then we show the existence and uniqueness of the stationary solution.

3.1 $f(t, x, v)$ via Stochastic Cycles

The main purpose of this section is to show Lemma 15, where we control $\|\varphi(t_{\mathbf{f}}) f(t)\|_{L^1_{x,v}}$ under some weight function s φ. To prove Lemma 15, we first express $f(t, x, v)$ with the stochastic cycles in Lemma 14, then we do some energy estimates in Lemma 16.

Lemma 14 *For any integer $k \ge 2$, suppose $f(t, x, v)$ solves (1.1) and (1.2), and $t_* \le t$, then we have*

$$f(t, x, v) = \mathbf{1}_{t^1 < t_*} f(t_*, X(t_*; t, x, v), V(t_*; t, x, v)) \tag{3.1}$$

$$+ \mu(x^1, v_{\mathbf{b}}) \sum_{i=1}^{k-1} \int_{\prod_{j=1}^{i} \mathcal{V}_j} \Big\{ \mathbf{1}_{t^{i+1} < t_* \le t^i} f(t_*, X(t_*; t^i, x^i, v^i), V(t_*; t^i, x^i, v^i)) \Big\} \mathrm{d}\Sigma_i \tag{3.2}$$

$$+ \mu(x^1, v_{\mathbf{b}}) \int_{\prod_{j=1}^{k} \mathcal{V}_j} \mathbf{1}_{t^k \ge t_*} f(t^k, x^k, v^k) \mathrm{d}\Sigma_k, \tag{3.3}$$

where $\mathrm{d}\Sigma_i := \frac{\mathrm{d}\sigma_i}{\mu(x^{i+1}, v^i)} \mathrm{d}\sigma_{i-1} \cdots \mathrm{d}\sigma_1$, *with* $\mathrm{d}\sigma_j = \mu(x^{j+1}, v^j)\{n(x^j)\cdot v^j\}\mathrm{d}v^j$ *in (1.17). Here, (X, V) solves (1.14).*

Proof We can obtain this Lemma by following Lemma 7 and Remark 9. □

Lemma 15 *Given a function $\varphi : [0, \infty) \to \mathbb{R}$, suppose φ satisfies that for any $\tau \ge 0$, $\varphi(\tau) \ge 0$, $\varphi' \ge 0$, and*

$$\int_1^\infty \tau^{3-\mathcal{A}} \varphi(\tau) \mathrm{d}\tau < \infty. \tag{3.4}$$

Suppose f solves (1.1) and (1.2), there exists $C > 0$ independent of t_*, t, such that for all $0 \le t_* \le t$,

$$\begin{aligned}&\|\varphi(t_{\mathbf{f}})f(t)\|_{L^1_{x,v}} + \int_{t_*}^t \|\varphi'(t_{\mathbf{f}})f\|_{L^1_{x,v}}\mathrm{d}s + \int_{t_*}^t |\varphi(t_{\mathbf{f}})f|_{L^1_{\gamma+}}\mathrm{d}s \\ &\le \|\varphi(t_{\mathbf{f}})f(t_*)\|_{L^1_{x,v}} + C(t-t_*+1)\|f(t_*)\|_{L^1_{x,v}} + \frac{1}{4}\int_{t_*}^t |f|_{L^1_{\gamma+}}\mathrm{d}s.\end{aligned} \tag{3.5}$$

For the proof of Lemma 15, we shall start it from the energy estimate, Lemma 16.

Lemma 16 [5] *Suppose f solves (1.1) and (1.2), then for $0 \le t_* \le t$ with $0 < \delta < \min(1, t-t_*)$,*

$$\|f(t)\|_{L^1_{x,v}} \le \|f(t_*)\|_{L^1_{x,v}}, \tag{3.6}$$

$$\int_{t_*}^t |f(s)|_{L^1_{\gamma+}}\mathrm{d}s \le \left\lceil \frac{t-t_*}{\delta}\right\rceil \|f(t_*)\|_{L^1_{x,v}} + O(\delta^2)\int_{t_*}^t |f(s)|_{L^1_{\gamma+}}\mathrm{d}s, \tag{3.7}$$

and if f_0 is non-negative, so is $f(t,x,v)$ for all $(t,x,v) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^3$.

Proof Since $f(t,x,v)$ solves (1.1) and (1.2) in the L^1 sense, according to [2, Lemma 2], $|f(t,x,v)|$ is also a solution to (1.1) and (1.2).

From (1.1) and (1.2), taking integration on $|f(t,x,v)|$ over $(t_*,t) \times \Omega \times \mathbb{R}^3$, we derive that

$$\|f(t)\|_{L^1_{x,v}} + \int_{t*}^t \int_{\gamma_+} |f|\mathrm{d}s - \int_{t*}^t\int_{\gamma_-} |f|\mathrm{d}s = \|f(t_*)\|_{L^1_{x,v}}.$$

Due to the choice of $\mu(x,v)$ in (1.2), for $\forall t \ge 0$,

$$\int_{\gamma_-} |f(t,x,v)||n(x)\cdot v|\mathrm{d}S_x\mathrm{d}v = \left|\int_{\gamma_+} f(t,x,v)\{n(x)\cdot v\}\mathrm{d}S_x\mathrm{d}v\right|.$$

Therefore, we have

$$\begin{aligned}&\int_{t*}^t\int_{\gamma_+}|f|\mathrm{d}s - \int_{t*}^t\int_{\gamma_-}|f|\mathrm{d}s \\ &= \int_{t*}^t\int_{\gamma_+}|f|\mathrm{d}s - \int_{t*}^t\left|\int_{\gamma_+} f\right|\mathrm{d}s \ge \int_{t*}^t\int_{\gamma_+}|f|\mathrm{d}s - \int_{t*}^t\int_{\gamma_+}|f|\mathrm{d}s = 0,\end{aligned}$$

therefore we prove (3.6).

Next we work on (3.7). For $\delta \in (0, t-t_*)$ and $(x,v) \in \gamma_+$, we split the time interval $[t^*, t]$ into some subintervals as follows:

$$[t^*, t^*+\delta], [t^*+\delta, t^*+2\delta], \ldots, [t^* + \lceil\frac{t-t_*-\delta}{\delta}\rceil\delta, t].$$

Since f is invariant along the characteristic, we backward $f(s,x,v)$ into a new time depending on s and $t_{\mathbf{b}}(x,v)$. Then we do estimates on different cases.

$$\begin{aligned}
&|f(s,x,v)| \\
&\leq \underbrace{\sum_{k=1}^{\lceil \frac{t-t_*-\delta}{\delta}\rceil} \mathbf{1}_{\{t_*+k\delta\leq s\leq t_*+(k+1)\delta,\ \delta<t_{\mathbf{b}}(x,v)\}}|f(t_*+k\delta, X(t_*+k\delta,s,x,v), V(t_*+k\delta,s,x,v))|}_{(3.8)_1} \\
&\quad + \underbrace{\mathbf{1}_{\{t_*+\delta\leq s,\ \delta\geq t_{\mathbf{b}}(x,v)\}}|f(s-t_{\mathbf{b}}(x,v), x_{\mathbf{b}}(x,v), v_{\mathbf{b}}(x,v))|}_{(3.8)_2} \\
&\quad + \underbrace{\mathbf{1}_{\{s-t_*<\delta,\ s-t_*<t_{\mathbf{b}}(x,v)\}}|f(t_*, X(t_*,s,x,v), V(t_*,s,x,v))|}_{(3.8)_3} \\
&\quad + \underbrace{\mathbf{1}_{\{t_{\mathbf{b}}(x,v)\leq s-t_*<\delta\}}|f(s-t_{\mathbf{b}}(x,v), x_{\mathbf{b}}(x,v), v_{\mathbf{b}}(x,v))|}_{(3.8)_4},
\end{aligned} \tag{3.8}$$

where $s\in[t_*,t]$.

First we do estimate on $(3.8)_1$. From (2.1), (3.6) and $t_*+\delta\leq s\leq t$ with $\delta<t_{\mathbf{b}}(x,v)$,

$$\begin{aligned}
&\int_{t_*}^{t}\int_{\gamma_+}(3.8)_1\mathrm{d}s \\
&\leq \sum_{k=1}^{\lceil \frac{t-t_*-\delta}{\delta}\rceil}\int_{\gamma_+} \\
&\quad \int_{t_*+k\delta}^{t_*+k\delta+t_{\mathbf{b}}(x,v)} |f(t_*+k\delta, X(t_*+k\delta,s,x,v), V(t_*+k\delta,s,x,v))|\mathrm{d}s\{n(x)\cdot v\}\mathrm{d}S_x\mathrm{d}v \\
&\leq \sum_{k=1}^{\lceil \frac{t-t_*-\delta}{\delta}\rceil}\|f(t_*+k\delta)\|_{L^1_{x,v}} \leq \left\lceil \frac{t-t_*-\delta}{\delta}\right\rceil \times \|f(t_*)\|_{L^1_{x,v}}.
\end{aligned} \tag{3.9}$$

Now we consider $(3.8)_2$. For $y=x_{\mathbf{b}}(x,v)$ and $s\in[t_*,t]$, we have

$$\mathbf{1}_{\delta\geq t_{\mathbf{b}}(x,v)} = \mathbf{1}_{\delta\geq t_{\mathbf{f}}(y,v_{\mathbf{b}})}. \tag{3.10}$$

From Lemma 11, for $(x,v)\in\gamma_+$ and $v_3\lesssim t_{\mathbf{b}}(x,v)<\delta<1$, we get

$$\mathbf{1}_{|v_3|\lesssim\delta} \geq \mathbf{1}_{\{\delta>t_{\mathbf{b}}(x,v)}.$$

Thus, we compute that

$$\begin{aligned}&\int_{n(x)\cdot v>0} \mathbf{1}_{\delta>t_{\mathbf{b}}(x,v)}\mu(x_{\mathbf{b}},v)|n(x)\cdot v|\mathrm{d}v\\ &\lesssim \int_{|v_3|\lesssim\delta}\mu(x_{\mathbf{b}},v)|n(x)\cdot v|\mathrm{d}v \\ &\le \int_{|v_3|\lesssim\delta} e^{-\frac{v_3^2}{2}}|v_3|\mathrm{d}v_3 \lesssim C\delta^2.\end{aligned} \tag{3.11}$$

From (2.3), (3.10) and using the Fubini's theorem, we derive

$$\begin{aligned}\int_{t_*}^t\int_{\gamma_+}(3.8)_2\mathrm{d}s &= \int_{\gamma_+}\int_{t_*+\delta}^t \mathbf{1}_{\delta\ge t_{\mathbf{b}}(x,v)}|f(s-t_{\mathbf{b}}(x,v),x_{\mathbf{b}},v_{\mathbf{b}})|\mathrm{d}s\{n(x)\cdot v\}\mathrm{d}S_x\mathrm{d}v\\ &\le \int_{\partial\Omega}\int_{n(y)\cdot v<0}\mathbf{1}_{\delta>t_{\mathbf{f}}(y,v)}\int_{t_*}^t |f(s,y,v)|\mathrm{d}s|n(y)\cdot v|\mathrm{d}S_y\mathrm{d}v\\ &\le \int_{\partial\Omega}\underbrace{\left(\int_{n(y)\cdot v<0}\mathbf{1}_{\delta>t_{\mathbf{f}}(y,v)}\mu(y,v)|n(y)\cdot v|\mathrm{d}v\right)}_{(3.12)_*}\\ &\quad\int_{t_*}^t\int_{n(y)\cdot v^1>0}|f(s,y,v^1)|\{n(y)\cdot v^1\}\mathrm{d}v^1\mathrm{d}s\mathrm{d}S_y.\end{aligned} \tag{3.12}$$

From (3.11), we derive

$$(3.12)\le O(\delta^2)\int_{t_*}^t\int_{\gamma_+}|f|\mathrm{d}s.$$

From (2.1) and $s<t_*+t_{\mathbf{b}}(x,v)$, we have

$$\int_{t_*}^t\int_{\gamma_+}(3.8)_3\mathrm{d}s\le\int_{t_*}^{t_*+t_{\mathbf{b}}(x,v)}\int_{\gamma_+}(3.8)_3\mathrm{d}s\le\|f(t_*)\|_{L^1_{x,v}}.$$

Again setting $y=x_{\mathbf{b}}(x,v)$ and $s\in[t_*,t]$, we have

$$\mathbf{1}_{\{t_{\mathbf{b}}(x,v)\le s-t_*<\delta\}}\le\mathbf{1}_{\delta\ge t_{\mathbf{f}}(y,v_{\mathbf{b}})}. \tag{3.13}$$

From (2.3), (3.13) and using the Fubini's theorem, we derive

$$\begin{aligned}\int_{t_*}^t\int_{\gamma_+}(3.8)_4\mathrm{d}s &= \int_{\gamma_+}\int_{t_*+t_{\mathbf{b}}(x,v)}^{t_*+\delta}\mathbf{1}_{\delta\ge t_{\mathbf{b}}(x,v)}|f(s-t_{\mathbf{b}}(x,v),x_{\mathbf{b}},v_{\mathbf{b}})|\mathrm{d}s\{n(x)\cdot v\}\mathrm{d}S_x\mathrm{d}v\\ &\le\int_{\partial\Omega}\int_{n(y)\cdot v<0}\mathbf{1}_{\delta>t_{\mathbf{f}}(y,v)}\int_{t_*}^{t_*+\delta}|f(s,y,v)|\mathrm{d}s|n(y)\cdot v|\mathrm{d}S_y\mathrm{d}v\end{aligned}$$

$$\leq \int_{\partial\Omega} \underbrace{\Big(\int_{n(y)\cdot v<0} \mathbf{1}_{\delta > t_{\mathbf{f}}(y,v)} \mu(y,v)|n(y)\cdot v| \mathrm{d}v \Big)}_{(3.14)_*} \int_{t_*}^{t_*+\delta} \int_{\gamma_+} |f| \mathrm{d}s. \tag{3.14}$$

Then we conclude (3.14) $\leq O(\delta^2) \int_{t_*}^{t} \int_{\gamma_+} |f| \mathrm{d}s$, therefore we prove (3.7).

To prove the positivity property, we write

$$f_- = \frac{|f| - f}{2}.$$

Since both $f(t,x,v)$ and $|f(t,x,v)|$ are solutions to (1.1) and (1.2), it is clear that f_- also solves (1.1) and (1.2). From (3.6) and the assumption $f_0 \geq 0$, we have

$$\|f_-(t)\|_{L^1_{x,v}} = \Big\| \frac{|f|(t) - f(t)}{2} \Big\|_{L^1_{x,v}} = \Big\| \frac{\big(|f| - f\big)(t)}{2} \Big\|_{L^1_{x,v}} \leq \Big\| \frac{|f_0| - f_0}{2} \Big\|_{L^1_{x,v}} = 0, \tag{3.15}$$

then we conclude $f_-(t,x,v) = 0$ on $\Omega \times \mathbb{R}^3$. □

Now we are ready to prove Lemma 15, which will be used frequently in this paper.

Proof of Lemma 15 Consider $\big(s, X(s;t,x,v), V(s;t,x,v)\big)$ solving (1.14), we now compute the forward exit time $t_{\mathbf{f}}(x,v)$ under this characteristics. Recall that $t_{\mathbf{f}}$ is timely independent because of the timely independent field $\Phi(x)$.

$$\begin{aligned} -1 &= \frac{d}{ds} t_{\mathbf{f}}(X(s;t,x,v), V(s;t,x,v)) \\ &= \frac{\partial}{\partial X} t_{\mathbf{f}}(X,V) \cdot \frac{d}{ds} X + \frac{\partial}{\partial V} t_{\mathbf{f}}(X,V) \cdot \frac{d}{ds} V, \end{aligned}$$

By setting $s = t$, we have

$$\frac{\partial}{\partial x} t_{\mathbf{f}}(x,v) \cdot v + \frac{\partial}{\partial v} t_{\mathbf{f}}(x,v) \cdot -\nabla\Phi(x) = -1.$$

On the other hand, since $f(t,x,v)$ solves (1.1) and (1.2), then $|f(t,x,v)|$ also solves (1.1) and (1.2), that is, $[\partial_t + v\cdot\nabla_x - \nabla\Phi\cdot\nabla_v]|f| = 0$. Then, in the sense of distribution

$$\begin{aligned} [\partial_t + v\cdot\nabla_x - \nabla\Phi\cdot\nabla_v]\big(\varphi(t_{\mathbf{f}})|f|\big) &= \varphi'(t_{\mathbf{f}})[\partial_t + v\cdot\nabla_x \\ &\quad -\nabla\Phi\cdot\nabla_v](t_{\mathbf{f}})|f| = -\varphi'(t_{\mathbf{f}})|f|. \end{aligned} \tag{3.16}$$

From (1.1), (3.16), $\varphi(\tau) \geq 0$, $\varphi' \geq 0$ and taking integration over $(t_*, t) \times \Omega \times \mathbb{R}^3$, we derive

$$\|\varphi(t_{\mathbf{f}}) f(t)\|_{L^1_{x,v}} + \int_{t_*}^{t} \|\varphi'(t_{\mathbf{f}}) f(s)\|_{L^1_{x,v}} \mathrm{d}s + \int_{t_*}^{t} \int_{\gamma_+} \varphi(t_{\mathbf{f}})|f| \mathrm{d}s \tag{3.17}$$

$$
\begin{aligned}
&\leq \|\varphi(t_{\mathbf{f}}) f(t_*)\|_{L^1_{x,v}} + \int_{t_*}^t \int_{\partial\Omega} \int_{n(x)\cdot v<0} \varphi(t_{\mathbf{f}})|f||n(x)\cdot v|\mathrm{d}v\mathrm{d}S_x\mathrm{d}s \\
&= \|\varphi(t_{\mathbf{f}}) f(t_*)\|_{L^1_{x,v}} + \int_{t_*}^t \int_{\partial\Omega} \int_{n(x)\cdot v<0} \varphi(t_{\mathbf{f}})\mu(x,v)|n(x)\cdot v|\mathrm{d}v\mathrm{d}S_x\mathrm{d}s \\
&\quad \times \int_{n(x)\cdot v^1>0} |f(s,x,v^1)|\{n(x)\cdot v^1\}\mathrm{d}v^1. \qquad (3.18)
\end{aligned}
$$

We remark that from Definition 5, $t_{\mathbf{f}}(x,v)=0$ for any $(x,v)\in\gamma_+$. Thus, the third integration in (3.17) follows

$$
\int_{t_*}^t \int_{\gamma_+} \varphi(t_{\mathbf{f}})|f|\mathrm{d}s = \int_{t_*}^t \int_{\gamma_+} \varphi(0)|f|\mathrm{d}s. \qquad (3.19)
$$

Now we prove the following claim: If (3.4) holds, then

$$
\sup_{x\in\partial\Omega} \int_{n(x)\cdot v<0} \varphi(t_{\mathbf{f}})(x,v)\mu(x,v)|n(x)\cdot v|\mathrm{d}v \lesssim 1. \qquad (3.20)
$$

We split $\int_{n(x)\cdot v<0} \varphi(t_{\mathbf{f}})(x,v)\mu(x,v)|n(x)\cdot v|\mathrm{d}v$ into two parts:

$$
\int_{n(x)\cdot v<0} \mathbf{1}_{t_{\mathbf{f}}\leq 1}\, \varphi(t_{\mathbf{f}})\mu(x,v)|n(x)\cdot v|\mathrm{d}v \ \text{ and } \int_{n(x)\cdot v<0} \mathbf{1}_{t_{\mathbf{f}}>1}\, \varphi(t_{\mathbf{f}})\mu(x,v)|n(x)\cdot v|\mathrm{d}v.
$$

For $t_{\mathbf{f}} \leq 1$, since $x\in\partial\Omega$ and $n(x)\cdot v<0$, we consider $x_{\mathbf{f}}(x,v)$, $v_{\mathbf{f}}(x,v)$ and get $t_{\mathbf{b}}(x_{\mathbf{f}},v_{\mathbf{f}}) = t_{\mathbf{f}}(x,v)$. Using Lemma 11, we have

$$
|v_3| = |v_{\mathbf{f},3}| \lesssim t_{\mathbf{b}}(x_{\mathbf{f}},v_{\mathbf{f}}),
$$

and thus

$$
|v_3| \lesssim t_{\mathbf{f}}(x,v) \leq 1.
$$

Combining with $\varphi' \geq 0$, we bound

$$
\int_{n(x)\cdot v<0} \mathbf{1}_{t_{\mathbf{f}}\leq 1}\, \varphi(t_{\mathbf{f}})\mu(x,v)|n(x)\cdot v|\mathrm{d}v \lesssim \varphi(1)\int_{\mathbb{R}^3} e^{-|v|^2/2}\mathrm{d}v \lesssim 1. \qquad (3.21)
$$

For $t_{\mathbf{f}} > 1$, applying (2.9) in Proposition 10 and Lemma 13, together with $|n(x)\cdot v| \lesssim t_{\mathbf{f}}(x,v)$ from Lemma 11, we obtain

$$
\int_{n(x)\cdot v<0} \varphi(t_{\mathbf{f}})\mu(x,v)|n(x)\cdot v|\mathrm{d}v \lesssim \int_{\partial\Omega}\int_1^\infty \varphi(t_{\mathbf{f}}) \sum_{m,n\in\mathbb{Z}} \mu(x_{\mathbf{f}},v_{\mathbf{f}})\frac{|t_{\mathbf{f}}|}{|t_{\mathbf{f}}|^2}\mathrm{d}t_{\mathbf{f}}\mathrm{d}S_{x_{\mathbf{f}}}. \qquad (3.22)
$$

From Lemma 13 and (3.4), we derive that

$$(3.22) \lesssim \int_1^\infty \frac{\varphi(t_{\mathbf{f}})}{|t_{\mathbf{f}}|}|t_{\mathbf{f}}|^{4-\mathcal{A}}\mathrm{d}t_{\mathbf{f}} \lesssim \int_1^\infty \varphi(t_{\mathbf{f}})|t_{\mathbf{f}}|^{3-\mathcal{A}}\mathrm{d}t_{\mathbf{f}} \lesssim 1.$$

Combining the above bound with (3.21), we prove (3.20). Then picking sufficiently small δ in (3.7) and using (3.20), we conclude (3.5), through, for $C > 1$,

$$\begin{aligned}(3.18) &\lesssim \int_{t_*}^t \int_{\gamma_+} |f(s,x,v^1)|\{n(x^1)\cdot v^1\}\mathrm{d}v^1\mathrm{d}S_x\mathrm{d}s \\ &\le C(t-t_*+1)\|f(t_*)\|_{L^1_{x,v}} + \frac{1}{4}\int_{t_*}^t |f(s)|_{L^1(\gamma_+)}.\end{aligned}$$

□

3.2 Lower Bound with the Unreachable Defect

In this section, we prove Proposition 17 to obtain a lower bound with the unreachable defect. It is the key to control the fluctuations.

Proposition 17 *Suppose f solves (1.1) and (1.2). Assume $f_0(x,v) \ge 0$. For any $T_0 \gg 1$ and $N \in \mathbb{N}^+$, there exists $\mathfrak{m}(x,v) \ge 0$, which only depends on Ω and T_0 (see (3.32) for the precise form), such that*

$$\begin{aligned} f(NT_0,x,v) \ge \mathfrak{m}(x,v)\Big\{ &\iint_{\Omega\times\mathbb{R}^3} f((N-1)T_0,x,v)\mathrm{d}v\mathrm{d}x \\ &- \iint_{\Omega\times\mathbb{R}^3} \mathbf{1}_{t_{\mathbf{f}}(x,v)\ge \frac{T_0}{4}} f((N-1)T_0,x,v)\mathrm{d}v\mathrm{d}x\Big\}. \end{aligned} \tag{3.23}$$

Proof **Step 1.** From (3.15) the assumption $f_0(x,v) \ge 0$, we have $f(t,x,v) \ge 0$. From (3.1)–(3.3) and setting $t = NT_0$, $t_* = (N-1)T_0$, $k = 2$, we can derive that

$$\begin{aligned} f(NT_0,x,v) &\ge \mathbf{1}_{t_{\mathbf{b}}(x,v)\le\frac{T_0}{4}}\mu(x^1,v_{\mathbf{b}}) \\ \int_{\mathcal{V}_1}\int_{\mathcal{V}_2} &\mathbf{1}_{t^2\ge(N-1)T_0} f(t^2,x^2,v^2)\{n(x^2)\cdot v^2\}\mathrm{d}v^2\mathrm{d}\sigma_1. \end{aligned} \tag{3.24}$$

Now we apply Proposition 10 on $v^1 \in \mathcal{V}_1$ with (2.7) and (2.8). In order to have the bijective mapping with (2.7), we restrict the range of $v^1_{\mathbf{b}}$ as

$$\mathcal{V}_{1,b} := \{v^1_{\mathbf{b}} \in \mathbb{R}^3 : x^2 + \int_{t-t_{\mathbf{b}}}^t \big(v_{\mathbf{b}} + \int_{t-t_{\mathbf{b}}}^s -\nabla\Phi(X(\alpha;t^1,x^1,v^1))\mathrm{d}\alpha\big)\mathrm{d}s = x^1\}. \tag{3.25}$$

This implies all characteristic trajectories $X(\alpha;t^1,x^1,v^1)$ between x^1 and x^2 under $v^1_{\mathbf{b}} \in \mathcal{V}_{1,b}$ don't cross the periodic boundary.

Therefore, using the change of variables $v^1 \in \mathcal{V}_1 \mapsto (x^2, t^1_{\mathbf{b}}) \in \partial\Omega \times \mathbb{R}_+$ for $v^1_{\mathbf{b}} \in \mathcal{V}_{1,b}$ in (2.8), together with Fubini's theorem, we derive

$$(3.24) \gtrsim \mathbf{1}_{t_{\mathbf{b}}(x,v) \le \frac{T_0}{4}} \mu(x^1, v_{\mathbf{b}}) \int_0^{T_0 - t_{\mathbf{b}}(x,v)} \int_{\partial\Omega} \underbrace{\frac{n(x^1)\cdot v^1}{(t^1_{\mathbf{b}})^2(1+|v^1_3|t^1_{\mathbf{b}})} \mu(x^2, v^1_{\mathbf{b}})}_{(3.26)_*} \times \int_{n(x^2)\cdot v^2 > 0} f(t^2, x^2, v^2)\{n(x^2)\cdot v^2\} \mathrm{d}v^2 \mathrm{d}S_{x^2} \mathrm{d}t^1_{\mathbf{b}}, \tag{3.26}$$

where $t^2 = NT_0 - t_{\mathbf{b}}(x,v) - t^1_{\mathbf{b}}$ and $n(x^1)\cdot v^1 = v^1_3$.

Step 2. In order to bound the integrand of the first line in (3.26), we will further restrict integration regimes. Note that $x^1 = x_{\mathbf{b}}(x,v)$ is given, x^2 is free variable and $t^2 \ge (N-1)T_0$.

Now we restrict the integral regimes of the variable $t^1_{\mathbf{b}}$ as

$$\mathfrak{T}^{T_0} := \left\{ t^1_{\mathbf{b}} \in [0,\infty) : T_0 - t_{\mathbf{b}}(x,v) - \min\left(t_{\mathbf{b}}(x^2,v^2), \frac{T_0}{4}\right) \le t^1_{\mathbf{b}} \le T_0 - t_{\mathbf{b}}(x,v) \right\}. \tag{3.27}$$

As a consequence of (3.27) and $t_{\mathbf{b}}(x,v) \le \frac{T_0}{4}$ in (3.26), we will derive (3.28) and (3.29),

$$\frac{T_0}{2} \le T_0 - t_{\mathbf{b}}(x,v) - \frac{T_0}{4} \le t^1_{\mathbf{b}} \le T_0. \tag{3.28}$$

Secondly, we prove (3.29). Note that if $t^1_{\mathbf{b}} \in \mathfrak{T}^{T_0}$, we have

$$(N-1)T_0 \le t^2 = NT_0 - t_{\mathbf{b}}(x,v) - t^1_{\mathbf{b}} \le (N-1)T_0 + \min\{t_{\mathbf{b}}(x^2,v^2), \frac{T_0}{4}\}.$$

This implies that, for $y_* = X((N-1)T_0; t^2, x^2, v^2)$ and $v_* = V((N-1)T_0; t^2, x^2, v^2)$, we have

$$t_{\mathbf{f}}(y_*, v_*) = t^2 - (N-1)T_0 = T_0 - t_{\mathbf{b}}(x,v) - t^1_{\mathbf{b}} \in \left[0, \frac{T_0}{4}\right], \tag{3.29}$$

where we use $t_{\mathbf{f}}(y_*, v_*) \le t_{\mathbf{b}}(x^2, v^2)$ since $x^2 = x_{\mathbf{f}}(y_*, v_*)$.

Step 3. For (3.26), we apply the restriction of integral regimes in (3.25) and (3.27). Note that

$$\frac{n(x^1)\cdot v^1}{(t^1_{\mathbf{b}})^2(1+|v^1_3|t^1_{\mathbf{b}})} = \frac{|v^1_3|}{(t^1_{\mathbf{b}})^2 + |v^1_3|(t^1_{\mathbf{b}})^3} \ge \frac{1}{|t^1_{\mathbf{b}}|^3}.$$

Using $\frac{T_0}{2} \leq t^1_{\mathbf{b}} \leq T_0$ in (3.28), we have

$$
\begin{aligned}
(3.26)_* \gtrsim \frac{1}{|t^1_{\mathbf{b}}|^3}\mu(x^2, v^1_{\mathbf{b}}) &= \frac{1}{|t^1_{\mathbf{b}}|^3}\mu\big(|n(x^2)\cdot v^1_{\mathbf{b}}|\big)\mu\Big(\frac{|x^2-x^1|}{|t^1_{\mathbf{b}}|}\Big) \\
&\gtrsim \frac{1}{|t^1_{\mathbf{b}}|^3}\mu(|v^1_3|)\mu\Big(\frac{\sqrt{2}}{|t^1_{\mathbf{b}}|}\Big) \gtrsim \frac{1}{(T_0)^3}e^{-\frac{1}{2}|v^1_3|^2}e^{-\frac{4}{T_0^2}} \\
&\gtrsim \frac{1}{(T_0)^3}(t^1_{\mathbf{b}})^{-(\mathcal{A}+1)} \geq (T_0)^{-4-\mathcal{A}},
\end{aligned}
$$

where the second last inequality follows from $T_0 \gg 1$, (2.20) and $\mathcal{A} \leq \frac{1}{\ln(a)} < \mathcal{A}+1$.

Finally, we get

$$
\begin{aligned}
(3.26) \geq\ & \mathbf{1}_{t_{\mathbf{b}}(x,v)\leq \frac{T_0}{4}}(T_0)^{-4-\mathcal{A}}\mu(x^1, v_{\mathbf{b}})\int_{\partial\Omega}\mathrm{d}S_{x^2}\int_{n(x^2)\cdot v^2>0}\mathrm{d}v^2\{n(x^2)\cdot v^2\} \\
&\times\int_{\mathfrak{T}^{T_0}}\mathrm{d}t^1_{\mathbf{b}}\, f(NT_0 - t_{\mathbf{b}}(x,v) - t^1_{\mathbf{b}}, x^2, v^2) \\
\gtrsim\ & \mathbf{1}_{t_{\mathbf{b}}(x,v)\leq \frac{T_0}{4}}(T_0)^{-4-\mathcal{A}}\mu(x^1, v_{\mathbf{b}})\int_{\partial\Omega}\mathrm{d}S_{x^2}\int_{n(x^2)\cdot v^2>0}\mathrm{d}v^2\{n(x^2)\cdot v^2\} \\
&\times\int_{T_0 - t_{\mathbf{b}}(x,v)-\min\left(t_{\mathbf{b}}(x^2,v^2),\frac{T_0}{4}\right)}^{T_0 - t_{\mathbf{b}}(x,v)}\mathrm{d}t^1_{\mathbf{b}}\, f(NT_0 - t_{\mathbf{b}}(x,v) - t^1_{\mathbf{b}}, x^2, v^2). \quad (3.30)
\end{aligned}
$$

Now we focus on the integrand of (3.30). Recall (3.27), we have

$$
(NT_0 - t_{\mathbf{b}}(x,v) - t^1_{\mathbf{b}}) - (N-1)T_0 = T_0 - t_{\mathbf{b}}(x,v) - t^1_{\mathbf{b}} \in \Big[0, \min\Big(t_{\mathbf{b}}(x^2,v^2), \frac{T_0}{4}\Big)\Big].
$$

Now setting $y_* = X((N-1)T_0; t^2, x^2, v^2)$, $v_* = V((N-1)T_0; t^2, x^2, v^2)$ and $\alpha = T_0 - t_{\mathbf{b}}(x,v) - t^1_{\mathbf{b}}$, we have

$$
(3.30) = \int_0^{\min\left(t_{\mathbf{b}}(x^2,v^2),\frac{T_0}{4}\right)} f\big((N-1)T_0, y_*, v_*\big)\mathrm{d}\alpha. \quad (3.31)
$$

From (3.29), we have $t_{\mathbf{f}}(y_*, v_*) \in \big[0, \frac{T_0}{4}\big]$. Now applying (2.1), we conclude that

$$
(3.26) \geq \mathbf{1}_{t_{\mathbf{b}}(x,v)\leq \frac{T_0}{4}}(T_0)^{-4-\mathcal{A}}\mu(x^1, v_{\mathbf{b}})\iint_{\Omega\times\mathbb{R}^3}\mathbf{1}_{t_{\mathbf{f}}(y,v)\in[0,\frac{T_0}{4}]}f((N-1)T_0, y, v)\mathrm{d}v\mathrm{d}y.
$$

We conclude (3.23) by setting

$$
\mathfrak{m}(x,v) := \mathbf{1}_{t_{\mathbf{b}}(x,v)\leq \frac{T_0}{4}}(T_0)^{-4-\mathcal{A}}\mu(x^1, v_{\mathbf{b}}). \quad (3.32)
$$

□

An immediate consequence of Proposition 17. follows.

Proposition 18 *Suppose f solves (1.1), (1.2) and satisfies (1.10). Then for all $T_0 \gg 1$ and $N \in \mathbb{N}^+$,*

$$\|f(NT_0)\|_{L^1_{x,v}} \leq (1 - \|\mathfrak{m}\|_{L^1_{x,v}})\|f((N-1)T_0)\|_{L^1_{x,v}} + 2\|\mathfrak{m}\|_{L^1_{x,v}}\|\mathbf{1}_{t_{\mathbf{f}} \geq \frac{T_0}{4}} f((N-1)T_0)\|_{L^1_{x,v}}. \tag{3.33}$$

Moreover, there exists $T_0 = T_0(\Omega)$, such that

$$\|\mathfrak{m}\|_{L^1_{x,v}} := \mathfrak{m}_{T_0} \lesssim (T_0)^{-3-\mathcal{A}}|\partial\Omega| < 1. \tag{3.34}$$

Proof We decompose

$$f((N-1)T_0, x, v) = f_{N-1,+}(x, v) - f_{N-1,-}(x, v),$$

where

$$f_{N-1,+}(x, v) = \mathbf{1}_{f((N-1)T_0,x,v)\geq 0} f\big((N-1)T_0, x, v\big),$$
$$f_{N-1,-}(x, v) = \mathbf{1}_{f((N-1)T_0,x,v)< 0} |f\big((N-1)T_0, x, v\big)|.$$

Let $f_\pm(s, x, v)$ solve (1.1) for $s \in [(N-1)T_0, NT_0]$ with the initial data $f_{N-1,+}$ and $f_{N-1,-}$ at $s = (N-1)T_0$, respectively. Now we apply Proposition 17 on $f_\pm(t, x, v)$ and conclude (3.23) for $f = f_+$ and $f = f_-$,respectively. We also note that

$$\iint_{\Omega\times\mathbb{R}^3} f((N-1)T_0, x, v)\mathrm{d}x\mathrm{d}v = \iint_{\Omega\times\mathbb{R}^3} f_{N-1,+}(x, v)\mathrm{d}x\mathrm{d}v - \iint_{\Omega\times\mathbb{R}^3} f_{N-1,-}(x, v)\mathrm{d}x\mathrm{d}v = 0.$$

This implies

$$\iint_{\Omega\times\mathbb{R}^3} f_{N-1,\pm}(x, v)\mathrm{d}x\mathrm{d}v = \frac{1}{2}\iint_{\Omega\times\mathbb{R}^3} |f((N-1)T_0, x, v)|\mathrm{d}x\mathrm{d}v. \tag{3.35}$$

From (3.23),

$$f_{N-1,\pm}(x, v) \geq \mathfrak{m}(x, v)\iint f_{N-1,\pm}(x, v)\mathrm{d}x\mathrm{d}v - \mathfrak{m}(x, v)\iint_{\Omega\times\mathbb{R}^3} \mathbf{1}_{t_{\mathbf{f}}(x,v)\geq\frac{T_0}{4}} f_{N-1,\pm}(x, v)\mathrm{d}x\mathrm{d}v$$

Using (3.35), we have

$$f_{N-1,\pm}(x, v) \geq \underbrace{\mathfrak{m}(x, v)\Big(\frac{1}{2}\|f((N-1)T_0)\|_{L^1_{x,v}} - \|\mathbf{1}_{t_{\mathbf{f}}\geq\frac{T_0}{4}} f((N-1)T_0)\|_{L^1_{x,v}}\Big)}_{\mathfrak{l}(x,v)}.$$

(3.36)

Then we deduce

$$
\begin{aligned}
|f(NT_0,x,v)| &= |f_{N-1,+}(x,v) - f_{N-1,-}(x,v) + \mathfrak{l}(x,v) - \mathfrak{l}(x,v)| \\
&\le |f_{N-1,+}(x,v) - \mathfrak{l}(x,v)| + |f_{N-1,-}(x,v) - \mathfrak{l}(x,v)|.
\end{aligned}
$$

From (3.36),

$$
|f(NT_0,x,v)| \le f_{N-1,+}(x,v) + f_{N-1,-}(x,v) - 2\mathfrak{l}(x,v). \tag{3.37}
$$

Note that $f_{N-1,+}(NT_0,x,v)+f_{N-1,-}(NT_0,x,v)$ solves (1.1) with the initial datum

$$
f_{N-1,+} + f_{N-1,-} = \big|f\big((N-1)T_0,x,v\big)\big|.
$$

Using (3.35), (3.37) and taking the integration on (3.36) over $\Omega \times \mathbb{R}^3$, we derive

$$
\begin{aligned}
\|f(NT_0)\|_{L^1_{x,v}} \le &\iint_{\Omega\times\mathbb{R}^3} f_{N-1,+}(x,v)\mathrm{d}x\mathrm{d}v \\
&+ \iint_{\Omega\times\mathbb{R}^3} f_{N-1,-}(x,v)\mathrm{d}x\mathrm{d}v - \iint_{\Omega\times\mathbb{R}^3} 2\mathfrak{l}(x,v)\mathrm{d}x\mathrm{d}v \\
&= (1-\|\mathfrak{m}\|_{L^1_{x,v}})\|f((N-1)T_0)\|_{L^1_{x,v}} \\
&+ 2\|\mathfrak{m}\|_{L^1_{x,v}}\|\mathbf{1}_{t_{\mathbf{f}}\ge\frac{T_0}{4}} f((N-1)T_0)\|_{L^1_{x,v}},
\end{aligned}
$$

therefore we prove (3.33).

To derive (3.34), it suffices to bound $\|\mathbf{1}_{t_{\mathbf{b}}(x,v)\le\frac{T_0}{4}}\mu(x^1,v_{\mathbf{b}})\|_{L^1_{x,v}}$. From Lemma 6, Lemma 8, and $t_{\mathbf{b}}(X(t-s,t,x,v),V(t-s,t,x,v)) = t_{\mathbf{b}}(x,v)-s$, we have

$$
\begin{aligned}
&\|\mathbf{1}_{t_{\mathbf{b}}(x,v)\le\frac{T_0}{4}}\mu(x^1,v_{\mathbf{b}})\|_{L^1_{x,v}} \\
&= \int_{\gamma_+}\int_{\max\{0,t_{\mathbf{b}}(x,v)-\frac{T_0}{4}\}}^{t_{\mathbf{b}}(x,v)} \mu(x^1,v)\{n(x)\cdot v\}\mathrm{d}s\mathrm{d}v\mathrm{d}S_x \\
&\lesssim \int_{\gamma_+}\Big(\mathbf{1}_{t_{\mathbf{b}}(x,v)\le\frac{T_0}{4}}\int_0^{t_{\mathbf{b}}(x,v)}\mathrm{d}s + \mathbf{1}_{t_{\mathbf{b}}(x,v)\ge\frac{T_0}{4}}\int_{t_{\mathbf{b}}(x,v)-\frac{T_0}{4}}^{t_{\mathbf{b}}(x,v)}\mathrm{d}s\Big)e^{-\frac{1}{2}|v|^2}\{n(x)\cdot v\}\mathrm{d}v\mathrm{d}S_x \\
&\le \frac{T_0}{4}\int_{\partial\Omega}\mathrm{d}S_x\int_{n(x)\cdot v>0} e^{-\frac{1}{2}|v|^2}\{n(x)\cdot v\}\mathrm{d}v \lesssim T_0|\partial\Omega|.
\end{aligned}
$$

Combining the above bound with (3.32), we conclude (3.34). □

Remark 19 Throughout this paper, we consider $\Omega = \mathbb{T}^2 \times \mathbb{R}_+$ and $|\partial\Omega| = 1$. Thus, any $T_0 > 1$ satisfies (3.34). In general, T_0 depends heavily on $|\partial\Omega|$, otherwise $\mathfrak{m}_{T_0} > 1$ and it leads to a negative estimate for L^1 in (3.33).

3.3 Proof of Weighted L^1-Estimates

In this section, we prove Theorem 1. We start with establishing the uniform estimates of the following energies:

$$|||f|||_i:=\|f\|_{L^1_{x,v}}+\frac{4\mathfrak{m}_{T_0}}{\varphi_{i-1}\left(\frac{3T_0}{4}\right)}\|\varphi_{i-1}(t_{\mathbf{f}})f\|_{L^1_{x,v}}+\frac{4e\mathfrak{m}_{T_0}}{T_0\varphi_{i-1}\left(\frac{3T_0}{4}\right)}\|\varphi_i(t_{\mathbf{f}})f\|_{L^1_{x,v}}, \tag{3.38}$$

where $\|\mathfrak{m}\|_{L^1_{x,v}}:=\mathfrak{m}_{T_0}$ (see (3.34)) and φ_i's defined in (3.39) with $i=2,4$.

Here we first introduce the weight functions φ_i's.

Definition 20 For $0<\delta<1$, we set

$$\begin{aligned}
&\varphi_1(\tau):=(e\ln(e+1))^{-1}(e+\tau)\ln(e+\ln(e+\tau)),\\
&\varphi_2(\tau):=(e^2\ln(e+1))^{-1}(e+\tau)^2\ln(e+\ln(e+\tau)),\\
&\varphi_3(\tau):=e^{5-\mathcal{A}}(\tau+e)^{\mathcal{A}-5}\big(\ln(\tau+e)\big)^{-(1+\delta)},\\
&\varphi_4(\tau):=e^{4-\mathcal{A}}(\tau+e)^{\mathcal{A}-4}\big(\ln(\tau+e)\big)^{-(1+\delta)}.
\end{aligned} \tag{3.39}$$

First, φ_i satisfies (3.4) for $i=1,2,3,4$: for example, for $i=4$,

$$\begin{aligned}
\int_1^\infty \tau^{3-\mathcal{A}}\varphi_4(\tau)\mathrm{d}\tau &= \int_1^\infty \tau^{3-\mathcal{A}}e^{4-\mathcal{A}}(\tau+e)^{\mathcal{A}-4}\big(\ln(\tau+e)\big)^{-(1+\delta)}\mathrm{d}\tau\\
&\lesssim \int_1^\infty \frac{1}{(\tau+e)(\ln(\tau+e))^{1+\delta}}\mathrm{d}s<\infty.
\end{aligned}$$

Second, φ_i satisfies

$$\varphi_i(0)=1, \quad \text{for } i=1,2,3,4. \tag{3.40}$$

Finally, we have

$$\begin{aligned}
&\varphi_2'(\tau)\geq(e^2\ln(e+1))^{-1}2(e+\tau)\ln(e+\ln(e+\tau))\geq 2e^{-1}\varphi_1(\tau),\quad \varphi_1'(\tau)\geq 0,\\
&\varphi_4'(\tau)=\big(\mathcal{A}-4-\frac{1+\delta}{\ln(\tau+e)}\big)e^{4-\mathcal{A}}(\tau+e)^{\mathcal{A}-5}\big(\ln(\tau+e)\big)^{-(1+\delta)}\geq\varphi_3(\tau),\quad \varphi_3'(\tau)\geq 0.
\end{aligned} \tag{3.41}$$

Proposition 21 *Choose $T_0>20$, such that for the constant C in (3.5),*

$$4C(e+3T_0)\Big(\varphi_i\big(\frac{3T_0}{4}\big)\Big)^{-1}\leq\frac{1}{2},\ \textit{for } i=1,3. \tag{3.42}$$

For any $N \in \mathbb{N}^+$, and $i = 2, 4$,

$$\begin{aligned}
&\|f(NT_0)\|_{L^1_{x,v}} + \frac{4\mathfrak{m}_{T_0}}{\varphi_{i-1}\left(\frac{3T_0}{4}\right)}\Big\{2\|\varphi_{i-1}(t_{\mathbf{f}})f(NT_0)\|_{L^1_{x,v}} + \frac{e}{T_0}\|\varphi_i(t_{\mathbf{f}})f(NT_0)\|_{L^1_{x,v}}\Big\} \\
&\quad\leq \ (3.43)_* \times \|f((N-1)T_0)\|_{L^1_{x,v}} \qquad (3.43)\\
&\qquad + \frac{4\mathfrak{m}_{T_0}}{\varphi_{i-1}\left(\frac{3T_0}{4}\right)}\Big\{\frac{3}{4}\|\varphi_{i-1}(t_{\mathbf{f}})f((N-1)T_0)\|_{L^1_{x,v}} + \frac{e}{T_0}\|\varphi_i(t_{\mathbf{f}})f((N-1)T_0)\|_{L^1_{x,v}}\Big\},
\end{aligned}$$

where (3.43)$_*$:=$1 - \mathfrak{m}_{T_0}\left\{1 - \frac{4C(e+3T_0)}{\varphi_{i-1}\left(\frac{3T_0}{4}\right)}\right\}$, with $\mathfrak{m}_{T_0}$ defined in (3.34).

Proof As key steps, we apply Lemma 15 on $f(t, x, v)$ solving (1.1) and (1.2) with φ_i's in (3.39). Using $\varphi_i(0) = 1$ for $i = 1, 2, 3, 4$ in (3.40), together with (3.19), we get for $i = 1, 2, 3, 4$,

$$\int_{t_*}^{t}\int_{\gamma_+}\varphi_i(t_{\mathbf{f}})|f|\mathrm{d}s = \int_{t_*}^{t}\int_{\gamma_+}|f|\mathrm{d}s.$$

Thus, we derive that, for $(N-1)T_0 \leq t_* \leq NT_0$ and $i = 2, 4$,

$$\begin{aligned}
&\|\varphi_{i-1}(t_{\mathbf{f}})f(NT_0)\|_{L^1_{x,v}} + \frac{3}{4}\int_{t_*}^{NT_0}|f|_{L^1_{\gamma_+}}\mathrm{d}s \leq \|\varphi_{i-1}(t_{\mathbf{f}})f(t_*)\|_{L^1_{x,v}} \\
&\quad + CT_0\|f(t_*)\|_{L^1_{x,v}}, \qquad (3.44)
\end{aligned}$$

and

$$\begin{aligned}
&\|\varphi_i(t_{\mathbf{f}})f(NT_0)\|_{L^1_{x,v}} + \int_{(N-1)T_0}^{NT_0}\{\|\varphi_i'(t_{\mathbf{f}})f\|_{L^1_{x,v}} + \frac{3}{4}|f|_{L^1_{\gamma_+}}\}\mathrm{d}s \\
&\quad \leq \|\varphi_i(t_{\mathbf{f}})f((N-1)T_0)\|_{L^1_{x,v}} + CT_0\|f((N-1)T_0)\|_{L^1_{x,v}}, \qquad (3.45)
\end{aligned}$$

where we set $t_* = (N-1)T_0$ in (3.45).

From (3.6), (3.41) and (3.44), we derive that, for $i = 2, 4$,

$$\begin{aligned}
\int_{(N-1)T_0}^{NT_0}\|\varphi_i'(t_{\mathbf{f}})f\|_{L^1_{x,v}} &\geq \int_{(N-1)T_0}^{NT_0}2e^{-1}\|\varphi_{i-1}(t_{\mathbf{f}})f(t_*)\|_{L^1_{x,v}}\mathrm{d}t_* \\
&\geq 2e^{-1}T_0\|\varphi_{i-1}(t_{\mathbf{f}})f(NT_0)\|_{L^1_{x,v}} \qquad (3.46)\\
&\quad - 2e^{-1}C(T_0)^2\|f((N-1)T_0)\|_{L^1_{x,v}}.
\end{aligned}$$

Applying (3.46) on (3.45), we conclude that, for $i = 2, 4$,

$$\begin{aligned}&\|\varphi_i(t_{\mathbf{f}})f(NT_0)\|_{L^1_{x,v}} + 2e^{-1}T_0\|\varphi_{i-1}(t_{\mathbf{f}})f(NT_0)\|_{L^1_{x,v}} + \frac{3}{4}\int_{(N-1)T_0}^{NT_0}|f|_{L^1_{\gamma+}}\\&\leq \|\varphi_i(t_{\mathbf{f}})f((N-1)T_0)\|_{L^1_{x,v}} + CT_0(1+2e^{-1}T_0)\|f((N-1)T_0)\|_{L^1_{x,v}}.\end{aligned} \tag{3.47}$$

Note that from (3.41), we have, for $i = 2, 4$,

$$\mathbf{1}_{t_{\mathbf{f}}\geq\frac{3T_0}{4}} \leq \left(\varphi_{i-1}\left(\frac{3T_0}{4}\right)\right)^{-1}\varphi_{i-1}(t_{\mathbf{f}}), \tag{3.48}$$

Now we combine (3.33) with (3.44)–(3.48) and $\mathfrak{m}_{T_0}$ in (3.34) with $|\partial\Omega| = 1$, and obtain

$$\begin{aligned}\|f(NT_0)\|_{L^1_{x,v}} &\leq (1-\mathfrak{m}_{T_0})\|f((N-1)T_0)\|_{L^1_{x,v}}\\&\quad + \frac{2\mathfrak{m}_{T_0}}{\varphi_{i-1}\left(\frac{3T_0}{4}\right)}\|\varphi_{i-1}(t_{\mathbf{f}})f((N-1)T_0)\|_{L^1_{x,v}}.\end{aligned} \tag{3.49}$$

For $i = 2, 4$ and $T_0 \gg 1$ in (3.42), considering (3.49) $+\frac{4\mathfrak{m}_{T_0}}{\varphi_{i-1}\left(\frac{3T_0}{4}\right)}\Big\{\frac{1}{4}$ (3.44) $|_{t_*=(N-1)T_0} + \frac{e}{T_0}$ (3.47) $\Big\}$, then we conclude (3.43). □

Now we are well equipped to prove Theorem 1.

Proof of Theorem 1 Fix T_0 in (3.42) and recall norms of $|||\cdot|||_2$ and $|||\cdot|||_4$ in (3.38). From (3.43), for $i = 2, 4$,

$$|||f(NT_0)|||_i \leq |||f((N-1)T_0)|||_i \leq \cdots \leq |||f(0)|||_i, \quad \text{for all } N \in \mathbb{N}^+. \tag{3.50}$$

Step 1. Under direct computation, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\varphi_2(\tau)}{\varphi_4(\tau)}\right) \lesssim \frac{(1+\delta)\ln(e+\ln(e+\tau)) - (\mathcal{A}-6)\ln(e+\ln(e+\tau))(\ln(\tau+e))^{\delta}}{(\ln(\tau+e))^{-1}(\tau+e)^{\mathcal{A}-5}}, \tag{3.51}$$

which shows that the function $\varphi_2(\tau)/\varphi_4(\tau)$ is decreasing when $\tau \gg 1$. Thus, we can choose $M \gg 1$ satisfying (3.55) and (3.60), such that

$$\begin{aligned}\varphi_2(t_{\mathbf{f}}) &= \mathbf{1}_{t_{\mathbf{f}}\geq M}\varphi_2(t_{\mathbf{f}}) + \mathbf{1}_{t_{\mathbf{f}}<M}\varphi_2(t_{\mathbf{f}})\\&\leq \mathbf{1}_{t_{\mathbf{f}}\geq M}\frac{\varphi_2(M)}{\varphi_4(M)}\varphi_4(t_{\mathbf{f}}) + \mathbf{1}_{t_{\mathbf{f}}<M}M\varphi_1(t_{\mathbf{f}}),\end{aligned} \tag{3.52}$$

where we use $\varphi_2(\tau) = \frac{e+\tau}{\tau}\varphi_1(\tau)$ and $\frac{e+\tau}{\tau} < M$ for $M \gg 1$.

Applying (3.50) for $i = 4$ and (3.52) with $M \gg 1$, we obtain for $1 \leq N \in \mathbb{N}^+$,

$$\begin{aligned}
&\frac{1}{M}\|\varphi_2(t_{\mathbf{f}})f((N-1)T_0)\|_{L^1_{x,v}} \\
&\leq \frac{1}{M}\frac{\varphi_2(M)}{\varphi_4(M)}\|\varphi_4(t_{\mathbf{f}})f((N-1)T_0)\|_{L^1_{x,v}} + \|\varphi_1(t_{\mathbf{f}})f((N-1)T_0)\|_{L^1_{x,v}} \\
&\leq \frac{1}{M}\frac{\varphi_2(M)}{\varphi_4(M)}\frac{T_0\varphi_3\left(\frac{3T_0}{4}\right)}{4e\mathfrak{m}_{T_0}}|||f(0)|||_4 + \|\varphi_1(t_{\mathbf{f}})f((N-1)T_0)\|_{L^1_{x,v}}.
\end{aligned} \tag{3.53}$$

After inputting (3.53) into (3.43) for $i = 2$, we derive that

$$|||f(NT_0)|||_2 \leq (3.54)_* \times |||f((N-1)T_0)|||_2 + \frac{1}{M}\frac{\varphi_2(M)}{\varphi_4(M)}\frac{\varphi_3\left(\frac{3T_0}{4}\right)}{\varphi_1\left(\frac{3T_0}{4}\right)}|||f(0)|||_4, \tag{3.54}$$

with $(3.54)_* := \max\left\{(1-\mathfrak{m}_{T_0}\{1-\frac{4C(e+3T_0)}{\varphi_1\left(\frac{3T_0}{4}\right)}\}),\ (\frac{3}{4}+\frac{e}{T_0}),\ (1-\frac{1}{M})\right\}$.

Step 2. Using $T_0 > 20$ in (3.42), we have $\frac{3}{4}+\frac{e}{T_0} < 1$. Thus, tentatively we make an assumption, which will be justified later behind (3.60),

$$\left(1+\frac{1}{M}\right)^{-1} \geq \max\left\{(1-\mathfrak{m}_{T_0}\{1-\frac{4C(e+3T_0)}{\varphi_1\left(\frac{3T_0}{4}\right)}\}),\ (\frac{3}{4}+\frac{e}{T_0}),\ (1-\frac{1}{M})\right\}. \tag{3.55}$$

For any $t \geq 0$, we choose $N_* \in \mathbb{N}$ such that $t \in [N_*T_0, (N_*+1)T_0]$. From (3.54) and (3.55), we derive, for all $1 \leq N \leq N_*+1$,

$$|||f(NT_0)|||_2 \leq \left(1+\frac{1}{M}\right)^{-1}|||f((N-1)T_0)|||_2 + \mathfrak{R}, \tag{3.56}$$

where $\mathfrak{R} := \frac{1}{M}\frac{\varphi_2(M)}{\varphi_4(M)}\frac{\varphi_3\left(\frac{3T_0}{4}\right)}{\varphi_1\left(\frac{3T_0}{4}\right)}|||f(0)|||_4$.

From (3.5) and $0 \leq N_*T_0 \leq t$, there exists a constant $C > 0$, such that

$$\|\varphi(t_{\mathbf{f}})f(t)\|_{L^1_{x,v}} \leq \|\varphi(t_{\mathbf{f}})f(N_*T_0)\|_{L^1_{x,v}} + CT_0\|f(N_*T_0)\|_{L^1_{x,v}}. \tag{3.57}$$

Now applying (3.57) first and using (3.56) successively, we conclude that

$$\begin{aligned}
|||f(t)|||_2 \lesssim_{T_0} |||f(N_*T_0)|||_2 &\leq \left(1+\frac{1}{M}\right)^{-1}|||f((N_*-1)T_0)|||_2 + \mathfrak{R} \\
&\leq \left(1+\frac{1}{M}\right)^{-2}|||f((N_*-2)T_0)|||_2 + \left(1+\frac{1}{M}\right)^{-1}\mathfrak{R} + \mathfrak{R}
\end{aligned}$$

$$\leq \cdots \leq \left(1+\frac{1}{M}\right)^{-N_*} \|\| f(0) \|\|_2 + (1+M)\mathfrak{R}. \tag{3.58}$$

From $N_* T_0 \leq t \leq (N_*+1)T_0$ and $1 \leq \frac{1+M}{M} \leq 2$, we get

$$\begin{aligned}
&\big(1+\frac{1}{M}\big)^{-N_*} \lesssim \big((1+\frac{1}{M})^{-M}\big)^{\frac{N_*+1}{M}} \lesssim e^{-\frac{N_*+1}{2M}} \leq e^{-\frac{t}{2T_0 M}}, \\
&(1+M)\mathfrak{R} \leq 2\frac{\varphi_2(M)}{\varphi_4(M)} \frac{\varphi_3\left(\frac{3T_0}{4}\right)}{\varphi_1\left(\frac{3T_0}{4}\right)} \|\| f(0) \|\|_4.
\end{aligned}$$

Then we have

$$\begin{aligned}
\|f\|_{L^1_{x,v}} \leq \|\| f(t) \|\|_2 \leq (3.58) \lesssim \max\left\{e^{-\frac{t}{2T_0 M}},\ \varphi_2(M)/\varphi_4(M)\right\} \\
\times\left\{\|\| f(0) \|\|_2 + \|\| f(0) \|\|_4\right\}.
\end{aligned} \tag{3.59}$$

Step 3. To make $|e^{-\frac{t}{2T_0 M}} - \varphi_2(M)/\varphi_4(M)| \ll 1$ as $t \to \infty$, we set M as follows:

$$M = t\big[2T_0 \ln(10 + t^{\mathcal{A}-6})\big]^{-1}, \tag{3.60}$$

so that

$$\max\left\{e^{-\frac{t}{2T_0 M}},\ \varphi_2(M)/\varphi_4(M)\right\} \lesssim_{T_0} (\ln\langle t\rangle)^{\mathcal{A}-6-\frac{\delta}{2}} \langle t\rangle^{6-\mathcal{A}}. \tag{3.61}$$

Clearly such a choice assures our precondition (3.55) for $t \gg 1$.

Now we claim that

$$\|\| f(0) \|\|_2 + \|\| f(0) \|\|_4 \lesssim \|e^{\frac{1}{2}|v|^2+\Phi(x)} f_0\|_{L^\infty_{x,v}}. \tag{3.62}$$

Note that it suffices to check that $\|\varphi_4(t_{\mathbf{f}}) f_0\|_{L^1_{x,v}} \lesssim \|e^{\frac{1}{2}|v|^2+\Phi(x)} f_0\|_{L^\infty_{x,v}}$.

Assume $\|e^{\frac{1}{2}|v|^2+\Phi(x)} f_0\|_{L^\infty_{x,v}} < \infty$, from (2.1), (2.20), $\Phi(x)|_{x\in\partial\Omega} = 0$ and $\mathcal{A} = [\frac{1}{\ln(a)}]$, then we derive

$$\begin{aligned}
\iint_{\Omega\times\mathbb{R}^3} |\varphi_4(t_{\mathbf{f}}) f_0(y,w)| \mathrm{d}y\mathrm{d}w &\lesssim \int_{\gamma_+}\int_0^{t_-} (t_{\mathbf{b}}(x,v))^{\mathcal{A}-4} e^{-\frac{1}{2}|v|^2-\Phi(x)} |n(x)\cdot v| \mathrm{d}s\mathrm{d}v\mathrm{d}S_x \\
&= \int_{\gamma_+}\int_0^{t_-} (t_{\mathbf{b}}(x,v))^{\mathcal{A}-4} e^{-\frac{|v|^2}{2}} |n(x)\cdot v| \mathrm{d}s\mathrm{d}v\mathrm{d}S_x \\
&\lesssim \int_{\gamma_+} (a^{\frac{1}{2}} v_3^2)^{\mathcal{A}-3} e^{-\frac{|v|^2}{2}} |v_3| \mathrm{d}v\mathrm{d}S_x \\
&\lesssim \int_{v_3<0} a^{-|v_3|^2} \mathrm{d}v_3 < \infty,
\end{aligned} \tag{3.63}$$

and this concludes the claim. Finally, together with (3.59), (3.61) and (3.62), we prove (1.11). □

4 Estimates on Exponential Moments

Now we are able to show the asymptotic behavior of the exponential moments. The main purpose of this section is to prove Theorem 3.

4.1 Some Preparation on Exponential Moments

To estimate the exponential moments, we include two weight functions: (i) a time-dependent weight function $\varrho(t)$, and (ii) a time-independent weight function $w'(x, v)$, which is constant along the characteristic trajectory (1.14). Then we consider the stochastic cycle representation of $\varrho(t)w'(x, v)f(t, x, v)$.

Lemma 22 *Suppose $f(t, x, v)$ solves (1.1) and (1.2) with $0 = t_* \le t$. Consider a time-dependent function $\varrho(t)$ and a time-independent function $w'(x, v)$, which is constant along the characteristic (1.14). Then for $k \ge 1$,*

$$\varrho(t)w'(x,v)f(t,x,v) = \mathbf{1}_{t^1<0}\,\varrho(t)w'(X(0;t,x,v),V(0;t,x,v))f(0,X,V) \tag{4.1}$$

$$+\, w'\mu(x^1,v_{\mathbf{b}})\sum_{i=1}^{k-1}\int_{\prod_{j=1}^{i}\mathcal{V}_j}\Big\{\mathbf{1}_{t^{i+1}<0\le t^i}\varrho(0)w'(X(0;t^i,x^i,v^i),V(0;t^i,x^i,v^i))$$

$$f(0,X,V)\Big\}\mathrm{d}\tilde{\Sigma}_i \tag{4.2}$$

$$+\, w'\mu(x^1,v_{\mathbf{b}})\sum_{i=1}^{k-1}\int_{\prod_{j=1}^{i}\mathcal{V}_j}\mathbf{1}_{0\le t^i}\Big\{\int_{\max(0,t^{i+1})}^{t^i}\varrho'(s)w'(X(s;t^i,x^i,v^i),V(s;t^i,x^i,v^i))$$

$$\times\, f(s,X(s;t^i,x^i,v^i),V(s;t^i,x^i,v^i))\mathrm{d}s\Big\}\mathrm{d}\tilde{\Sigma}_i \tag{4.3}$$

$$+\, w'\mu(x^1,v_{\mathbf{b}})\int_{\prod_{j=1}^{k}\mathcal{V}_j}\mathbf{1}_{t^k\ge 0}\,\varrho(t^k)w'f(t^k,x^k,v^k)\mathrm{d}\tilde{\Sigma}_k, \tag{4.4}$$

where $\mathrm{d}\tilde{\Sigma}_i := \frac{\mathrm{d}\sigma_i}{\mu(x^{i+1},v^i)w'(x^i,v^i)}\mathrm{d}\sigma_{i-1}\cdots\mathrm{d}\sigma_1$, with $\mathrm{d}\sigma_j = \mu(x^{j+1},v^j)\{n(x^j)\cdot v^j\}\mathrm{d}v^j$. Here, (X, V) solves (1.14).

Proof Following Lemma 7 and Remark 9, we obtain this Lemma. □

We start with a simple case when $w'(x, v) \equiv 1$. Applying Lemma 22, we derive the stochastic cycle representation of $\varrho(t)f(t, x, v)$ as follows.

$$\varrho(t)f(t,x,v) = \mathbf{1}_{t^1<0}\;\varrho(t)f(0,X(0;t,x,v),V(0;t,x,v)) \tag{4.5}$$

$$+\,\mu(x^1,v_{\mathbf{b}})\sum_{i=1}^{k-1}\int_{\prod_{j=1}^{i}\mathcal{V}_j}\Big\{\mathbf{1}_{t^{i+1}<0\le t^i}\varrho(0)f(0,X(0;t^i,x^i,v^i),V(0;t^i,x^i,v^i))\Big\}\mathrm{d}\tilde{\Sigma}_i \tag{4.6}$$

$$+\,\mu(x^1,v_{\mathbf{b}})\sum_{i=1}^{k-1}\int_{\prod_{j=1}^{i}\mathcal{V}_j}\mathbf{1}_{0\le t^i}\Big\{\int_{\max(0,t^{i+1})}^{t^i}\varrho'(s)f(s,X(s;t^i,x^i,v^i),$$

$$V(s;t^i,x^i,v^i))\mathrm{d}s\Big\}\mathrm{d}\tilde{\Sigma}_i \tag{4.7}$$

$$+\,\mu(x^1,v_{\mathbf{b}})\int_{\prod_{j=1}^{k}\mathcal{V}_j}\mathbf{1}_{t^k\geq 0}\,\varrho(t^k)f(t^k,x^k,v^k)\mathrm{d}\tilde{\Sigma}_k, \tag{4.8}$$

where $\mathrm{d}\tilde{\Sigma}_i=\frac{\mathrm{d}\sigma_i}{\mu(x^{i+1},v^i)}\mathrm{d}\sigma_{i-1}\cdots\mathrm{d}\sigma_1$, with $\mathrm{d}\sigma_j=\mu(x^{j+1},v^j)\{n(x^j)\cdot v^j\}\mathrm{d}v^j$, and (X,V) solves (1.14).

Here, we put emphasis on (4.7) and (4.8) since (4.5) and (4.6) can be controlled by Theorem 1 and initial condition, which will be shown in the proof of Theorem 3.

To estimate (4.7), for $j=i-2,i-1$, we apply Proposition 10 on $\mathcal{V}_j$, together with $t_{\mathbf{b}}^j\gtrsim|n(x^j)\cdot v^j|$ in Lemma 11, then we obtain for $j=i-2,i-1$,

$$\begin{aligned}\mathrm{d}\sigma_j&=\mu(x^{j+1},v^j)\{n(x^j)\cdot v^j\}\mathrm{d}v^j\\&\lesssim\mu(x^{j+1},v^j)\{n(x^j)\cdot v^j\}(t_{\mathbf{b}}^j)^{-2}\mathrm{d}t_{\mathbf{b}}\mathrm{d}S_{x_{\mathbf{b}}}\lesssim\mu(x^{j+1},v^j)\frac{1}{t_{\mathbf{b}}^j}\mathrm{d}t_{\mathbf{b}}^j\mathrm{d}S_{x_{\mathbf{b}}}.\end{aligned}$$

Then we derive

$$\begin{aligned}&\int_{\mathcal{V}_{i-2}}\mathrm{d}\sigma_{i-2}\int_{\mathcal{V}_{i-1}}\mathrm{d}\sigma_{i-1}\int_{\mathcal{V}_i}\mathbf{1}_{t^{i+1}<0\leq t^i}\\&\int_0^{t^i}\varrho'(s)f(s,X(s;t^i,x^i,v^i),V(s;t^i,x^i,v^i))\{n(x^i)\cdot v^i\}\mathrm{d}s\mathrm{d}v^i\\&\quad\lesssim\int_0^{t^{i-2}}\mathrm{d}t_{\mathbf{b}}^{i-1}\int_{\partial\Omega}\frac{\mathrm{d}S_{x^{i-1}}}{t_{\mathbf{b}}^{i-2}}\sum_{m,n\in\mathbb{Z}}\mu(x^{i-1},v_{i-2,\mathbf{b}}^{m,n})\\&\int_0^{t^{i-2}-t_{\mathbf{b}}^{i-1}}\mathrm{d}t_{\mathbf{b}}^{i-2}\int_{\partial\Omega}\frac{\mathrm{d}S_{x^i}}{t_{\mathbf{b}}^{i-1}}\sum_{m,n\in\mathbb{Z}}\mu(x^i,v_{i-1,\mathbf{b}}^{m,n})\\&\qquad\times\underbrace{\int_{\mathcal{V}_i}\mathbf{1}_{t^{i+1}<0\leq t^i}\int_0^{t^i}\varrho'(s)|f(s,X(s;t^i,x^i,v^i),V(s;t^i,x^i,v^i))|\mathrm{d}s\{n(x^i)\cdot v^i\}\mathrm{d}v^i}_{(4.9)^*},\end{aligned} \tag{4.9}$$

with $t^{i-1}=t^{i-2}-t_{\mathbf{b}}^{i-2}$, $t^i=t^{i-1}-t_{\mathbf{b}}^{i-1}$ and $v_{i-1,\mathbf{b}}^{m,n}=v_{\mathbf{b}}(x^i,v_{i-1}^{m,n})$, $v_{i-2,\mathbf{b}}^{m,n}=v_{\mathbf{b}}(x^{i-1},v_{i-2}^{m,n})$.

Now we can control (4.7) via the following lemma:

Lemma 23 *Suppose $f(t,x,v)$ solves (1.1), (1.2) and (X,V) solves (1.14), for $0\leq t^i\leq t$ and $i=3,\cdots,k-1$,*

$$\begin{aligned}&\int_{\prod_{j=1}^{i}\mathcal{V}_j}\mathbf{1}_{t^{i+1}<0\leq t^i}\int_0^{t^i}\varrho'(s)f(s,X(s;t^i,x^i,v^i),\\&\qquad V(s;t^i,x^i,v^i))\mathrm{d}s\mathrm{d}\tilde{\Sigma}_i\lesssim\int_0^t\|\varrho'(s)f(s)\|_{L^1_{x,v}}\mathrm{d}s.\end{aligned} \tag{4.10}$$

where $\mathrm{d}\tilde{\Sigma}_i = \frac{\mathrm{d}\sigma_i}{\mu(x^{i+1},v^i)}\mathrm{d}\sigma_{i-1}\cdots\mathrm{d}\sigma_1$, *with* $\mathrm{d}\sigma_j = \mu(x^{j+1},v^j)\{n(x^j)\cdot v^j\}\mathrm{d}v^j$, *and* (X,V) *solves (1.14).*

Proof **Step 1.** For (4.10), it suffices to prove this upper bound for $i = 2, ..., k-1$,

$$\int_{\prod_{j=1}^{i-1}\mathcal{V}_j}\int_{\mathcal{V}_i}\mathbf{1}_{t^{i+1}<0\le t^i}\int_0^{t^i}\varrho'(s)|f(s,X(s;t^i,x^i,v^i), V(s;t^i,x^i,v^i))|\mathrm{d}s\{n(x^i)\cdot v^i\}\mathrm{d}v^i\mathrm{d}\sigma_{i-1}\cdots\mathrm{d}\sigma_1. \tag{4.11}$$

Applying Prop osition 10 as in (4.9), we bound the above integration as

$$(4.11) \lesssim \int_{\mathcal{V}_1}\mathrm{d}\sigma_1\cdots\int_{\mathcal{V}_{i-3}}\mathrm{d}\sigma_{i-3}\int_0^{t^{i-2}}\mathrm{d}t_{\mathbf{b}}^{i-1}\int_0^{t^{i-2}-t_{\mathbf{b}}^{i-1}}\mathrm{d}t_{\mathbf{b}}^{i-2}\int_{\partial\Omega}\mathrm{d}S_{x^i} \times \underbrace{\left(\int_{\partial\Omega}\frac{\sum_{m,n\in\mathbb{Z}}\mu(x^{i-1},v_{i-2,\mathbf{b}}^{m,n})}{|t_{\mathbf{b}}^{i-2}|}\times\frac{\sum_{m,n\in\mathbb{Z}}\mu(x^{i},v_{i-1,\mathbf{b}}^{m,n})}{|t_{\mathbf{b}}^{i-1}|}\mathrm{d}S_{x^{i-1}}\right)}_{(4.12)_*}\times(4.9)^*. \tag{4.12}$$

Step 2. We claim that

$$(4.12)_* \lesssim \mathbf{1}_{t_{\mathbf{b}}^{i-1}\le t_{\mathbf{b}}^{i-2}}\langle t_{\mathbf{b}}^{i-2}\rangle^{4-\mathcal{A}} + \mathbf{1}_{t_{\mathbf{b}}^{i-1}\ge t_{\mathbf{b}}^{i-2}}\langle t_{\mathbf{b}}^{i-1}\rangle^{4-\mathcal{A}}. \tag{4.13}$$

In order to prove this claim, we split into the following two cases:

Case 1: $t_{\mathbf{b}}^{i-1}\le t_{\mathbf{b}}^{i-2}$. Using (2.32) and (2.33) in Lemma 13, we bound

$$\frac{\sum_{m,n\in\mathbb{Z}}\mu(x^{i-1},v_{i-2,\mathbf{b}}^{m,n})}{|t_{\mathbf{b}}^{i-2}|} \lesssim \mathbf{1}_{t_{\mathbf{b}}^{i-2}\le 1}\frac{1}{|t_{\mathbf{b}}^{i-2}|} + \mathbf{1}_{t_{\mathbf{b}}^{i-2}\ge 1}\frac{1}{|t_{\mathbf{b}}^{i-2}|^{\mathcal{A}-4}}, \tag{4.14}$$

Replacing i with $i+1$ in (2.32) and (2.33), we bound

$$\frac{\sum_{m,n\in\mathbb{Z}}\mu(x^{i},v_{i-1,\mathbf{b}}^{m,n})}{|t_{\mathbf{b}}^{i-1}|} \lesssim \underbrace{\sum_{|a|<2,|b|<2}\mathbf{1}_{t_{\mathbf{b}}^{i-1}\le 1}\frac{1}{|t_{\mathbf{b}}^{i-1}|}\mu\Big(x^i,\frac{|x^i+(a,b)-x^{i-1}|}{|t_{\mathbf{b}}^{i-1}|}\Big)}_{(4.15)_1} + \underbrace{\mathbf{1}_{t_{\mathbf{b}}^{i-1}\le 1}\frac{1}{|t_{\mathbf{b}}^{i-1}|}e^{-\frac{1}{2(t_{\mathbf{b}}^{i-1})^2}}}_{(4.15)_2} + \underbrace{\mathbf{1}_{t_{\mathbf{b}}^{i-1}\ge 1}|t_{\mathbf{b}}^{i-1}|^{4-\mathcal{A}}}_{(4.15)_3}. \tag{4.15}$$

For (4.15)$_1$, we employ a change of variables, for $x^i \in \partial\Omega$, $|a| < 2$, $|b| < 2$ and $0 < t_{\mathbf{b}}^{i-1} \le 1$,

$$x^{i-1} \in \partial\Omega \mapsto z := \frac{1}{t_{\mathbf{b}}^{i-1}}(x^{i-1} + (a,b) - x^i) \in \mathfrak{S}^{a,b}_{x^i, t_{\mathbf{b}}^{i-1}},$$

where the image $\mathfrak{S}^{a,b}_{x^i, t_{\mathbf{b}}^{i-1}}$ of the map is a two-dimensional smooth plane. Using the local chart of $\partial\Omega$, we have $\mathrm{d}S_{x^{i-1}} \lesssim |t_{\mathbf{b}}^{i-1}|^2 \mathrm{d}S_z$. From this change of variables and (4.14), we conclude that

$$\begin{aligned}
&\mathbf{1}_{t_{\mathbf{b}}^{i-1} \le t_{\mathbf{b}}^{i-2}} \int_{\partial\Omega} \frac{\sum_{m,n\in\mathbb{Z}} \mu(x^{i-1}, v^{m,n}_{i-2,\mathbf{b}})}{|t_{\mathbf{b}}^{i-2}|} \times (4.15)_1 \, \mathrm{d}S_{x^{i-1}} \\
&\lesssim \mathbf{1}_{t_{\mathbf{b}}^{i-1} \le t_{\mathbf{b}}^{i-2}} (4.14) \times \sum_{|a|<2, |b|<2} \int_{\mathfrak{S}^{a,b}_{x^i, t_{\mathbf{b}}^{i-1}}} \mathbf{1}_{t_{\mathbf{b}}^{i-1} \le 1} e^{-\frac{1}{2}|z|^2} |t_{\mathbf{b}}^{i-1}| \mathrm{d}S_z \\
&\lesssim \mathbf{1}_{t_{\mathbf{b}}^{i-1} \le t_{\mathbf{b}}^{i-2}} \left\{ \mathbf{1}_{t_{\mathbf{b}}^{i-2} \le 1} \frac{1}{|t_{\mathbf{b}}^{i-2}|} + \mathbf{1}_{t_{\mathbf{b}}^{i-2} \ge 1} \frac{1}{|t_{\mathbf{b}}^{i-2}|^{\mathcal{A}-4}} \right\} \mathbf{1}_{t_{\mathbf{b}}^{i-1} \le 1} |t_{\mathbf{b}}^{i-1}| \\
&\lesssim \mathbf{1}_{t_{\mathbf{b}}^{i-1} \le t_{\mathbf{b}}^{i-2}} \left\{ \mathbf{1}_{t_{\mathbf{b}}^{i-2} \le 1} \frac{|t_{\mathbf{b}}^{i-1}|}{|t_{\mathbf{b}}^{i-2}|} + \mathbf{1}_{t_{\mathbf{b}}^{i-2} \ge 1} \frac{1}{|t_{\mathbf{b}}^{i-2}|^{\mathcal{A}-4}} \right\} \lesssim \mathbf{1}_{t_{\mathbf{b}}^{i-2} \le 1} + \mathbf{1}_{t_{\mathbf{b}}^{i-2} \ge 1} \frac{1}{|t_{\mathbf{b}}^{i-2}|^{\mathcal{A}-4}}.
\end{aligned} \tag{4.16}$$

For (4.15)$_2$, since $e^{-\frac{1}{2t^2}} \lesssim t^2$ for $0 < t \le 1$, then we have

$$\begin{aligned}
&\mathbf{1}_{t_{\mathbf{b}}^{i-1} \le t_{\mathbf{b}}^{i-2}} \int_{\partial\Omega} \frac{\sum_{m,n\in\mathbb{Z}} \mu(x^{i-1}, v^{m,n}_{i-2,\mathbf{b}})}{|t_{\mathbf{b}}^{i-2}|} \times (4.15)_2 \, \mathrm{d}S_{x^{i-1}} \\
&\lesssim \mathbf{1}_{t_{\mathbf{b}}^{i-1} \le t_{\mathbf{b}}^{i-2}} (4.14) \times \mathbf{1}_{t_{\mathbf{b}}^{i-1} \le 1} \frac{1}{|t_{\mathbf{b}}^{i-1}|} e^{-\frac{1}{2(t_{\mathbf{b}}^{i-1})^2}} \int_{\partial\Omega} \mathrm{d}S_{x^{i-1}} \\
&\lesssim \mathbf{1}_{t_{\mathbf{b}}^{i-1} \le t_{\mathbf{b}}^{i-2}} \left\{ \mathbf{1}_{t_{\mathbf{b}}^{i-2} \le 1} \frac{1}{|t_{\mathbf{b}}^{i-2}|} + \mathbf{1}_{t_{\mathbf{b}}^{i-2} \ge 1} \frac{1}{|t_{\mathbf{b}}^{i-2}|^{\mathcal{A}-4}} \right\} \mathbf{1}_{t_{\mathbf{b}}^{i-1} \le 1} |t_{\mathbf{b}}^{i-1}| \\
&\le \mathbf{1}_{t_{\mathbf{b}}^{i-1} \le t_{\mathbf{b}}^{i-2}} \frac{|t_{\mathbf{b}}^{i-1}|}{|t_{\mathbf{b}}^{i-2}|} \mathbf{1}_{t_{\mathbf{b}}^{i-2} \le 1} + \mathbf{1}_{t_{\mathbf{b}}^{i-2} \ge 1} \frac{1}{|t_{\mathbf{b}}^{i-2}|^{\mathcal{A}-4}} \\
&\le \mathbf{1}_{t_{\mathbf{b}}^{i-2} \le 1} + \mathbf{1}_{t_{\mathbf{b}}^{i-2} \ge 1} \frac{1}{|t_{\mathbf{b}}^{i-2}|^{\mathcal{A}-4}}.
\end{aligned} \tag{4.17}$$

For (4.15)$_3$, from $\mathbf{1}_{t_{\mathbf{b}}^{i-1}\geq 1}|t_{\mathbf{b}}^{i-1}|^{4-\mathcal{A}} \lesssim \mathbf{1}_{t_{\mathbf{b}}^{i-1}\geq 1}$, we derive

$$\begin{aligned}
&\mathbf{1}_{t_{\mathbf{b}}^{i-1}\leq t_{\mathbf{b}}^{i-2}} \int_{\partial\Omega} \frac{\sum_{m,n\in\mathbb{Z}} \mu(x^{i-1}, v_{i-2,\mathbf{b}}^{m,n})}{|t_{\mathbf{b}}^{i-2}|} \times (4.15)_3 \, \mathrm{d}S_{x^{i-1}} \\
&\lesssim \mathbf{1}_{t_{\mathbf{b}}^{i-1}\leq t_{\mathbf{b}}^{i-2}} (4.14) \times \mathbf{1}_{t_{\mathbf{b}}^{i-1}>1} |t_{\mathbf{b}}^{i-1}|^{4-\mathcal{A}} \int_{\partial\Omega} \mathrm{d}S_{x^{i-1}} \\
&\lesssim \mathbf{1}_{t_{\mathbf{b}}^{i-1}\leq t_{\mathbf{b}}^{i-2}} \left\{ \mathbf{1}_{t_{\mathbf{b}}^{i-2}\leq 1} \frac{1}{|t_{\mathbf{b}}^{i-2}|} + \mathbf{1}_{t_{\mathbf{b}}^{i-2}\geq 1} \frac{1}{|t_{\mathbf{b}}^{i-2}|^{\mathcal{A}-4}} \right\} \mathbf{1}_{t_{\mathbf{b}}^{i-1}\geq 1} \\
&\lesssim \mathbf{1}_{t_{\mathbf{b}}^{i-2}\geq 1} \frac{1}{|t_{\mathbf{b}}^{i-2}|^{\mathcal{A}-4}}.
\end{aligned} \tag{4.18}$$

Collecting estimate from (4.16)–(4.18), we deduce that

$$\mathbf{1}_{t_{\mathbf{b}}^{i-1}\leq t_{\mathbf{b}}^{i-2}} (4.12)_* \lesssim \mathbf{1}_{t_{\mathbf{b}}^{i-2}\leq 1} + \mathbf{1}_{t_{\mathbf{b}}^{i-2}\geq 1} |t_{\mathbf{b}}^{i-2}|^{4-\mathcal{A}}. \tag{4.19}$$

Case 2: $t_{\mathbf{b}}^{i-1} \geq t_{\mathbf{b}}^{i-2}$. We change the role of $i-1$ and $i-2$ and follow the argument of the previous case. We employ a change of variables, for $x^{i-1} \in \partial\Omega$ $|a| < 2$, $|b| < 2$ and $0 < t_{\mathbf{b}}^{i-2} \leq 1$,

$$x^{i-2} \in \partial\Omega \mapsto z := \frac{1}{t_{\mathbf{b}}^{i-2}} (x^{i-2} - x^{i-1}) \in \mathfrak{S}_{x^{i-1}, t_{\mathbf{b}}^{i-2}}, \tag{4.20}$$

with $\mathrm{d}S_{x^{i-2}}^{a,b} \lesssim |t_{\mathbf{b}}^{i-2}|^2 \mathrm{d}S_z$. Then we can conclude that

$$\mathbf{1}_{t_{\mathbf{b}}^{i-1}\geq t_{\mathbf{b}}^{i-2}} (4.12)_* \lesssim \mathbf{1}_{t_{\mathbf{b}}^{i-1}\leq 1} + \mathbf{1}_{t_{\mathbf{b}}^{i-1}\geq 1} |t_{\mathbf{b}}^{i-1}|^{4-\mathcal{A}}. \tag{4.21}$$

Therefore, we show (4.13).

Step 3. Now we apply (4.13) on (4.12). Then we have

$$\begin{aligned}
(4.11) \lesssim & \int_{\mathcal{V}_1} \mathrm{d}\sigma_1 \cdots \int_{\mathcal{V}_{i-3}} \mathrm{d}\sigma_{i-3} \int_0^{t^{i-2}} \frac{\mathrm{d}t_{\mathbf{b}}^{i-1}}{\langle t_{\mathbf{b}}^{i-1}\rangle^{\mathcal{A}-4}} \int_0^{\min\{t^{i-2}-t_{\mathbf{b}}^{i-1}, t_{\mathbf{b}}^{i-1}\}} \mathrm{d}t_{\mathbf{b}}^{i-2} \\
& \int_{\partial\Omega} \mathrm{d}S_{x^i} \times (4.9)^* \qquad (4.22) \\
& + \int_{\mathcal{V}_1} \mathrm{d}\sigma_1 \cdots \int_{\mathcal{V}_{i-3}} \mathrm{d}\sigma_{i-3} \int_0^{t^{i-2}} \frac{\mathrm{d}t_{\mathbf{b}}^{i-2}}{\langle t_{\mathbf{b}}^{i-2}\rangle^{\mathcal{A}-4}} \int_0^{\min\{t^{i-2}-t_{\mathbf{b}}^{i-2}, t_{\mathbf{b}}^{i-2}\}} \mathrm{d}t_{\mathbf{b}}^{i-1} \\
& \int_{\partial\Omega} \mathrm{d}S_{x^i} \times (4.9)^*. \qquad (4.23)
\end{aligned}$$

For (4.22), we employ the change of variables

$$(x^i, t_{\mathbf{b}}^{i-2}, v^i) \mapsto (y, w) = (X(s; t^{i-2} - t_{\mathbf{b}}^{i-2} - t_{\mathbf{b}}^{i-1}, x^i, v^i),$$

$$V(s; t^{i-2} - t_{\mathbf{b}}^{i-2} - t_{\mathbf{b}}^{i-1}, x^i, v^i)) \in \Omega \times \mathbb{R}^3,$$

and we have $|n(x^i) \cdot v^i| \mathrm{d}S_{x^i} \mathrm{d}t_{\mathbf{b}}^{i-2} \mathrm{d}v^i \lesssim \mathrm{d}y \mathrm{d}w$ from (2.1). From $0 \le t^i \le t$ and $\mathcal{A} \ge 8$, we bound (4.22) as

$$\begin{aligned}(4.22) \le & \int_{\mathcal{V}_1} \mathrm{d}\sigma_1 \cdots \int_{\mathcal{V}_{i-3}} \mathrm{d}\sigma_{i-3} \int_0^{t^{i-2}} \mathrm{d}t_{\mathbf{b}}^{i-1} \langle t_{\mathbf{b}}^{i-1} \rangle^{4-\mathcal{A}} \\ & \times \int_0^{t^i} \varrho'(s) \iint_{\Omega \times \mathbb{R}^3} |f(s, y, w)| \mathrm{d}y \mathrm{d}w \mathrm{d}s \\ \lesssim & \int_0^t \|\varrho'(s) f(s)\|_{L^1_{x,v}} \mathrm{d}s.\end{aligned}$$

A bound of (4.23) can be derived similarly, by using the change of variables

$$\begin{aligned}(x^i, t_{\mathbf{b}}^{i-1}, v^i) \mapsto (y, w) = (X(s; t^{i-2} - t_{\mathbf{b}}^{i-2} - t_{\mathbf{b}}^{i-1}, x^i, v^i), \\ V(s; t_{i-2} - t_{\mathbf{b}}^{i-2} - t_{\mathbf{b}}^{i-1}, x^i, v^i)) \in \Omega \times \mathbb{R}^3,\end{aligned}$$

with $|n(x^i) \cdot v^i| \mathrm{d}S_{x^i} \mathrm{d}t_{\mathbf{b}}^{i-1} \mathrm{d}v^i \lesssim \mathrm{d}y \mathrm{d}w$. □

Next, we control (4.8) by establishing the following estimate:

Lemma 24 *Consider (X, V) solving (1.14), there exists $\mathfrak{C} = \mathfrak{C}(\Omega) > 0$ (see (4.28) for the precise choice), such that*

$$\text{if } k \ge \mathfrak{C}t, \text{ then } \sup_{(x,v) \in \bar{\Omega} \times \mathbb{R}^3} \Big(\int_{\prod_{j=1}^{k-1} \mathcal{V}_j} \mathbf{1}_{t^k(t,x,v,v^1,\cdots,v^{k-1}) \ge 0} \, \mathrm{d}\sigma_1 \cdots \mathrm{d}\sigma_{k-1} \Big) \lesssim e^{-t}, \tag{4.24}$$

where $\mathrm{d}\sigma_j = \mu(x^{j+1}, v^j)\{n(x^j) \cdot v^j\} \mathrm{d}v^j$ in (1.17).

Proof From (3.11), we have

$$\int_{n(x) \cdot v > 0} \mathbf{1}_{\delta > t_{\mathbf{b}}(x,v)} \mu(x_{\mathbf{b}}, v) |n(x) \cdot v| \mathrm{d}v \lesssim C\delta^2.$$

Thus we define $\mathcal{V}_i^\delta := \{v^i \in \mathcal{V}_i : |n(x^i) \cdot v^i| < \delta\}$ and derive that

$$\int_{\mathcal{V}_j^\delta} \mathrm{d}\sigma_j \le C\delta^2.$$

On the other hand, since $t_{\mathbf{b}}(x^i, v^i) \gtrsim |n(x^i) \cdot v^i|$, we derive that for $v^i \in \mathcal{V}_i \backslash \mathcal{V}_i^\delta$,

$$t_{\mathbf{b}}(x^i, v^i) \ge C_\Omega \delta.$$

If $t_k(t,x,v^1,\cdots,v^{k-1})\geq 0$, we conclude such $v^i\in\mathcal{V}_i\backslash\mathcal{V}_i^\delta$ can exist at most $[\frac{t}{C_\Omega\delta}]+1$ times. Denote the combination $\binom{M}{N}=\frac{M(M-1)\cdots(M-N+1)}{N(N-1)\cdots 1}=\frac{M!}{N!(M-N)!}$ for $M,N\in\mathbb{N}$ and $M\geq N$. From $0<\delta\ll 1$, we have

$$\begin{aligned}&\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\mathbf{1}_{t_k(t,x,v^1,\cdots,v^{k-1})\geq 0}\,\mathrm{d}\sigma_{k-1}\cdots\mathrm{d}\sigma_1\\&\leq\sum_{m=0}^{[\frac{t}{C_\Omega\delta}]+1}\binom{k}{m}\Big(\int_{\mathcal{V}_i^\delta}\mathrm{d}\sigma_i\Big)^{k-m}\leq (C\delta^2)^{k-[\frac{t}{C_\Omega\delta}]}\underbrace{\sum_{m=0}^{[\frac{t}{C_\Omega\delta}]+1}\binom{k}{m}}_{(4.25)_*}.\end{aligned}\tag{4.25}$$

Recall the Stirling's formula,

$$\sqrt{2\pi}k^{k+\frac12}e^{-k}\leq k!\leq k^{k+\frac12}e^{-k+1}.\tag{4.26}$$

Using $(1+\frac{1}{\mathfrak{a}-1})^{\mathfrak{a}-1}\leq e$ and (4.26), we have for $\mathfrak{a}\in\mathbb{N}^+$ and $\mathfrak{a}\geq 2$,

$$\begin{aligned}\binom{k}{\frac{k}{\mathfrak{a}}}=\frac{k!}{(k-\frac{k}{\mathfrak{a}})!\frac{k}{\mathfrak{a}}!}&\leq\Big(\frac{\mathfrak{a}}{\mathfrak{a}-1}\Big)^{\frac{\mathfrak{a}}{\mathfrak{a}-1}k}\mathfrak{a}^{\frac{k}{\mathfrak{a}}}\sqrt{\frac{\mathfrak{a}^2}{k(\mathfrak{a}-1)}}\\&=\frac{1}{\sqrt{k}}\Big(\mathfrak{a}^{\frac{1}{\mathfrak{a}}}\big(\frac{\mathfrak{a}}{\mathfrak{a}-1}\big)^{\frac{\mathfrak{a}}{\mathfrak{a}-1}}\Big)^k\sqrt{\frac{\mathfrak{a}^2}{\mathfrak{a}-1}}\leq\frac{1}{\sqrt{k}}(e\mathfrak{a})^{4\frac{k}{\mathfrak{a}}}\sqrt{\frac{\mathfrak{a}^2}{\mathfrak{a}-1}},\end{aligned}$$

where the last inequality follows from $\frac{\mathfrak{a}}{\mathfrak{a}-1}\leq 2$. Hence, we derive that

$$\sum_{i=1}^{[\frac{k}{\mathfrak{a}}]}\binom{k}{i}\leq\frac{k}{\mathfrak{a}}\binom{k}{\frac{k}{\mathfrak{a}}}\leq\frac{e}{2\pi}\sqrt{\frac{k}{\mathfrak{a}}}(e\mathfrak{a})^{4\frac{k}{\mathfrak{a}}}.\tag{4.27}$$

Now we estimate $(4.25)_*$. For fixed $0<\delta\ll 1$ which is independent of t, we choose

$$\mathfrak{a}\in\mathbb{N}^+\ \text{ such that }(\delta^{2\mathfrak{a}}e\mathfrak{a})^{\frac{1}{C_\Omega\delta}}\leq e^{-2},\ \text{ and set }k{:=}\frac{\mathfrak{a}}{4}\Big([\frac{t}{C_\Omega\delta}]+1\Big).\tag{4.28}$$

Using (4.27), we have

$$(4.25)_*\lesssim\sqrt{[\frac{t}{C_\Omega\delta}]+1}\Big(e\frac{k}{[\frac{t}{C_\Omega\delta}]+1}\Big)^{[\frac{t}{C_\Omega\delta}]+1}\lesssim\sqrt{[\frac{t}{C_\Omega\delta}]+1}(e\mathfrak{a})^{[\frac{t}{C_\Omega\delta}]+1}.$$

Hence, we bound (4.25) by

$$(\delta^{2\mathfrak{a}}e\mathfrak{a})^{[\frac{t}{C_\Omega\delta}]+1}\sqrt{[\frac{t}{C_\Omega\delta}]+1}\lesssim e^{-t}.\qquad\square$$

4.2 Estimates on Exponential Moments

Now we are ready to prove Theorem 3. First, we set

$$w(x,v):=e^{\theta(|v|^2+2\Phi(x))} \ \text{ and } \ w'(x,v):=e^{\theta'(|v|^2+2\Phi(x))}, \tag{4.29}$$

where $0 \le 2\theta < \theta' = \frac{1}{2}$. Suppose (X, V) solves (1.14). From (2.5), we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\big(|V(s;t,x,v)|^2/2 + \Phi(X(s;t,x,v))\big) = 0.$$

This indicates that both $w(x, v)$ and $w'(x, v)$ are constant along the the characteristic (1.14).

Proof of Theorem 3 We start to prove (1.12), and pick $\varrho(t) = t + 1$ to utilize the L^1-decay of Theorem 1. Then we work on the stochastic cycle representation of $\varrho(t)w'(x,v)f(t,x,v)$ in (4.1)–(4.4).

For the contribution of (4.1), note that $t^1 < 0$, and both w' and f are constant along the characteristic trajectory. Thus, we deduce that

$$\begin{aligned} &w'(x,v)f(t,x,v) = w'(X(0;t,x,v),V(0;t,x,v))f(0,X(0;t,x,v),\\ &V(0;t,x,v)) \le \|w'f(0)\|_{L^\infty_{x,v}}. \end{aligned} \tag{4.30}$$

Now we bound the contribution of (4.2). Since $|n(x)\cdot v| \lesssim w'(x,v) = \frac{1}{2\pi}\mu^{-1}(x,v)$, we derive

$$\begin{aligned} \frac{1}{\varrho(t)}|(4.2)| &\lesssim \frac{k}{\varrho(t)}\left(\sup_i \int_{\prod_{j=1}^k \mathcal{V}_j} \mathbf{1}_{t^{i+1}<0\le t^i}\,\mathrm{d}\tilde{\Sigma}_i\right)\varrho(0)\|w'f(0)\|_{L^\infty_{x,v}} \\ &\lesssim \frac{k}{\varrho(t)}\left(\int_{n(x^j)\cdot v^j>0} \frac{|n(x^j)\cdot v^j|}{w'(x^j,v^j)}\mathrm{d}v^j\right)\|w'f(0)\|_{L^\infty_{x,v}} \\ &\lesssim \frac{k}{\varrho(t)}\|w'f(0)\|_{L^\infty_{x,v}}. \end{aligned} \tag{4.31}$$

Applying Lemma 23 and Theorem 1, we bound the contribution of (4.3). Since $|n(x)\cdot v| \lesssim w'(x,v) = \frac{1}{2\pi}\mu^{-1}(x,v)$ and $\varrho' = 1$, together with $w'(x^i,v^i) = w'(X(s;t^i,x^i,v^i),V(s;t^i,x^i,v^i))$ for $\max(0,t^{i+1}) \le s \le t^i$ and

$\mathrm{d}\tilde{\Sigma}_i := \frac{\mathrm{d}\sigma_i}{\mu(x^{i+1}, v^i) w'(x^i, v^i)} \mathrm{d}\sigma_{i-1} \cdots \mathrm{d}\sigma_1$, we have

$$\begin{aligned}
\frac{1}{\varrho(t)} |(4.3)| &\lesssim \frac{k}{\varrho(t)} \sup_i \int_{\prod_{j=1}^{i} \mathcal{V}_j} \mathbf{1}_{0 \le t^i} \int_{\max(0, t^{i+1})}^{t^i} w'(X(s; t^i, x^i, v^i), V(s; t^i, x^i, v^i)) \\
&\quad \times f(s, X(s; t^i, x^i, v^i), V(s; t^i, x^i, v^i)) \mathrm{d}s \mathrm{d}\tilde{\Sigma}_i \\
&= \frac{k}{\varrho(t)} \sup_i \int_{\prod_{j=1}^{i} \mathcal{V}_j} \mathbf{1}_{0 \le t^i} \\
&\quad \times \int_{\max(0, t^{i+1})}^{t^i} f(s, X(s; t^i, x^i, v^i), V(s; t^i, x^i, v^i)) \mathrm{d}s \\
&\quad \frac{\mathrm{d}\sigma_i}{\mu(x^{i+1}, v^i)} \mathrm{d}\sigma_{i-1} \cdots \mathrm{d}\sigma_1 \\
&\lesssim \frac{k}{\varrho(t)} \int_0^t \| f(s) \|_{L^1_{x,v}} \mathrm{d}s \lesssim \frac{k}{\varrho(t)} \times \| w' f(0) \|_{L^\infty_{x,v}}.
\end{aligned} \tag{4.32}$$

Lastly we bound the contribution of (4.4). From Lemma 24, we get

$$\begin{aligned}
\frac{1}{\varrho(t)} |(4.4)| &\lesssim \frac{\varrho(t^k)}{\varrho(t)} \sup_{(x,v) \in \bar{\Omega} \times \mathbb{R}^3} \Big(\int_{\prod_{j=1}^{k-1} \mathcal{V}_j} \mathbf{1}_{t_k(t,x,v,v^1,\cdots,v^{k-1}) \ge 0} \mathrm{d}\sigma_1 \cdots \mathrm{d}\sigma_{k-1} \Big) \\
&\quad \| w' f(t_k) \|_{L^\infty_{x,v}} \\
&\lesssim e^{-t} \sup_{t \ge s \ge 0} \| w' f(s) \|_{L^\infty_{x,v}}.
\end{aligned} \tag{4.33}$$

Collecting estimates from (4.30)–(4.33) and using $k \lesssim t$, we derive

$$(1 - e^{-t}) \sup_{t \ge 0} \| w' f(t) \|_{L^\infty_{x,v}} \lesssim (1 + \frac{k}{\varrho(t)}) \times \| w' f(0) \|_{L^\infty_{x,v}}. \tag{4.34}$$

Therefore, we prove (1.12).

Next, we prove (1.13). To show the decay of exponential moments and again utilize the L^1-decay, we set a new weight function

$$\varrho(t) := (\ln \langle t \rangle)^{6-\mathcal{A}} \langle t \rangle^{\mathcal{A}-5}. \tag{4.35}$$

Clearly we have $\varrho'(t) \lesssim (\ln \langle t \rangle)^{6-\mathcal{A}} \langle t \rangle^{\mathcal{A}-6}$ for $t \gg 1$.

Step 1. From Lemma 14, we derive the form of $\int_{\mathbb{R}^3} w(x,v) |f(t,x,v)| \mathrm{d}v$. First we split $|v| \ge t/2$ and $t_1 \le 3t/4$ case to get (4.36) and (4.37). Next, for $t_1 \ge 3t/4$ case, we follow along the stochastic cycles twice with $k = 2$, $t_* = t/2$, and obtain (4.38) and (4.39).

$$\int_{\mathbb{R}^3} w(x,v) |f(t,x,v)| \mathrm{d}v \le \int_{|v| \ge t/2} w(x,v) |f(t,x,v)| \mathrm{d}v \tag{4.36}$$

$$+\int_{|v|\le t/2}\mathbf{1}_{t^1\le 3t/4}w(x,v)|f(3t/4,X(3t/4;t,x,v),V(3t/4;t,x,v))|\mathrm{d}v \tag{4.37}$$

$$+\int_{\mathbb{R}^3}\mathbf{1}_{t^1\ge 3t/4}w(x,v)\mu(x^1,v_{\mathbf{b}})\int_{\prod_{j=1}^2\mathcal{V}_j}\mathbf{1}_{t^2<t/2<t^1}w(x^1,v^1)|f(t^1,x^1,v^1)|\mathrm{d}\Sigma_1^2\mathrm{d}v \tag{4.38}$$

$$+\int_{\mathbb{R}^3}\mathbf{1}_{t^1\ge 3t/4}w(x,v)\mu(x^1,v_{\mathbf{b}})\left|\int_{\prod_{j=1}^2\mathcal{V}_j}\mathbf{1}_{t^2\ge t/2}w(x^2,v^2)f(t^2,x^2,v^2)\mathrm{d}\Sigma_2^2\right|\mathrm{d}v, \tag{4.39}$$

where $\mathrm{d}\Sigma_1^2=\mathrm{d}\sigma_2\frac{\mathrm{d}\sigma_1}{\mu(x^2,v^1)w(x^1,v^1)}$ and $\mathrm{d}\Sigma_2^2=\frac{\mathrm{d}\sigma_2}{\mu(x^3,v^2)w(x^2,v^2)}\mathrm{d}\sigma_1$.

For (4.36), from the L^∞-boundedness, $\Phi(x)|_{x\in\bar{\Omega}}\ge 0$, and $0<w<w'$, we derive that

$$\begin{aligned}\int_{|v|\ge t/2}w(x,v)|f(t,x,v)|\mathrm{d}v &\le \int_{|v|\ge t/2}\frac{w(x,v)}{w'(x,v)}\mathrm{d}v\|w'f(0)\|_{L^\infty_{x,v}}\\ &\le \int_{|v|\ge t/2}e^{-(\theta'-\theta)|v|^2}\mathrm{d}v\|w'f(0)\|_{L^\infty_{x,v}}\\ &\lesssim \frac{1}{(\theta'-\theta)^{3/2}}e^{-\frac{(\theta'-\theta)t^2}{4}}\|w'f(0)\|_{L^\infty_{x,v}}.\end{aligned} \tag{4.40}$$

For (4.37), from $t^1\le 3t/4$ and Lemma 8, we have for $t^1\le s\le t$,

$$t_{\mathbf{b}}(x,v)=t-t^1\ge t/4,\quad \frac{|V(s;t,x,v)|^2}{2}+\Phi(X(s;t,x,v))=\frac{|v_{\mathbf{b}}|^2}{2}. \tag{4.41}$$

On the other hand, using (2.20), we get

$$t_{\mathbf{b}}(x,v)\lesssim a^{\frac{1}{2}v_{\mathbf{b},3}^2(x,v)}. \tag{4.42}$$

Then, from the L^∞-boundedness, (4.41) and (4.42), we deduce that

$$\begin{aligned}(4.37) &\lesssim \int_{|v|\le t/2}\frac{w(x,v)}{w'(X(3t/4;t,x,v),V(3t/4;t,x,v))}\mathrm{d}v\|w'f(0)\|_{L^\infty_{x,v}}\\ &\le \int_{|v|\le t/2}e^{(\theta-\theta')|v_{\mathbf{b}}(x,v)|^2}\mathrm{d}v\|w'f(0)\|_{L^\infty_{x,v}}\\ &\le \int_{|v|\le t/2}e^{-\frac{1}{4}|v_{\mathbf{b}}(x,v)|^2}\mathrm{d}v\|w'f(0)\|_{L^\infty_{x,v}}\\ &\le \int_{|v_3|\le t/2}|t_{\mathbf{b}}(x,v)|^{-\frac{\mathcal{A}}{2}}\mathrm{d}v_3\|w'f(0)\|_{L^\infty_{x,v}}\lesssim \langle t\rangle^{1-\frac{\mathcal{A}}{2}}\|w'f(0)\|_{L^\infty_{x,v}}.\end{aligned} \tag{4.43}$$

Next, we bound $\int_{\mathbb{R}^3}w(x,v)\mu(x^1,v_{\mathbf{b}})\mathrm{d}v$ shown in (4.38) and (4.39). Note that from (4.41), we have

$$\mu^{-1}(x^1,v_{\mathbf{b}})=2\pi w'(x,v).$$

Thus, we derive

$$\int_{\mathbb{R}^3} w(x,v)\mu(x^1,v_{\mathbf{b}})\mathrm{d}v = \int_{\mathbb{R}^3} \frac{w(x,v)}{2\pi w'(x,v)}\mathrm{d}v \lesssim_\theta 1. \tag{4.44}$$

For (4.38), since $\int_{\mathcal{V}_2} \mathrm{d}\sigma_2$ is bounded and from (4.44), we have

$$(4.38) \lesssim \int_{\mathcal{V}_1} \mathbf{1}_{\{t^2<t/2<t^1\}} |f(t^1,x^1,v^1)| \{n(x^1)\cdot v^1\}\mathrm{d}v^1. \tag{4.45}$$

From $t^1 \geq 3t/4$, $t^2 < t/2$ and (2.20), we have

$$t_{\mathbf{b}}(x^1,v^1) = t^1 - t^2 \geq t/4, \quad a^{\frac{1}{2}(v_3^1)^2} \gtrsim t_{\mathbf{b}}(x^1,v^1). \tag{4.46}$$

Then, from the L^∞-boundedness and $0 < n(x^1)\cdot v^1 \lesssim e^{\varepsilon|v^1|^2} < w'(x^1,v^1)$ for $0<\varepsilon \ll 1/2$, we derive

$$\begin{aligned}
(4.45) &\lesssim \int_{\mathcal{V}_1} \frac{n(x^1)\cdot v^1}{w'(x^1,v^1)}\mathrm{d}v^1 \|w' f(0)\|_{L^\infty_{x,v}} \\
&\lesssim \int_{v_3^1\leq 0} e^{(\varepsilon-\theta')|v_3^1|^2}\mathrm{d}v_3^1 \|w' f(0)\|_{L^\infty_{x,v}} \\
&\lesssim \int_{v_3^1\leq 0} e^{-\frac{\theta'}{2}|v_3^1|^2} e^{(\varepsilon-\frac{\theta'}{2})|v_3^1|^2}\mathrm{d}v_3^1 \|w' f(0)\|_{L^\infty_{x,v}} \\
&\lesssim \int_{v_3^1\leq 0} \left(t_{\mathbf{b}}(x^1,v^1)\right)^{-\frac{\mathcal{A}}{2}} e^{(\varepsilon-\frac{\theta'}{2})|v_3^1|^2}\mathrm{d}v_3^1 \|w' f(0)\|_{L^\infty_{x,v}} \\
&\lesssim \langle t\rangle^{-\frac{\mathcal{A}}{2}} \int_{v_3^1\leq 0} e^{(\varepsilon-\frac{\theta'}{2})|v_3^1|^2}\mathrm{d}v_3^1 \|w' f(0)\|_{L^\infty_{x,v}} \lesssim \langle t\rangle^{-\frac{\mathcal{A}}{2}} \|w' f(0)\|_{L^\infty_{x,v}}.
\end{aligned} \tag{4.47}$$

Step 2. Now we only need to bound (4.39). Since $\int_{\mathbb{R}^3} w(x,v)\mu(x^1,v_{\mathbf{b}})\mathrm{d}v \lesssim_\theta 1$, and $\int_{\mathcal{V}_1} \mathrm{d}\sigma_1$ is bounded, it suffices to prove the decay of

$$\sup_{v\in\mathbb{R}^3, v^1\in\mathcal{V}_1} \left| \int_{\mathcal{V}_2} \mathbf{1}_{t^2\geq t/2} f(t^2,x^2,v^2)\{n(x^2)\cdot v^2\}\mathrm{d}v^2 \right|. \tag{4.48}$$

Here we define $g(t,x,v) := \varrho(t) w(x,v) f(t,x,v)$ and note that

$$\begin{aligned}
&\frac{1}{\varrho(t^2)} \int_{\mathcal{V}_2} \mathbf{1}_{t^2\geq t/2} \frac{|n(x^2)\cdot v^2|}{w(x^2,v^2)} g(t^2,x^2,v^2)\mathrm{d}v^2 \\
&\quad = \int_{\mathcal{V}_2} \mathbf{1}_{t^2\geq t/2} f(t^2,x^2,v^2)\{n(x^2)\cdot v^2\}\mathrm{d}v^2.
\end{aligned}$$

Therefore, it suffices to show the decay of $\left|\frac{1}{\varrho(t^2)} \int_{\mathcal{V}_2} \mathbf{1}_{t^2\geq t/2} \frac{|n(x^2)\cdot v^2|}{w(x^2,v^2)} g(t^2,x^2,v^2)\mathrm{d}v^2\right|$.

Applying Lemma 22 with $w(x,v)=e^{\theta(|v|^2+2\Phi(x))}$ and $\varrho(t)$ in (4.35), and choosing $k\geq \mathfrak{C}t$ as in Lemma 24, we obtain the following stochastic cycle representation of $g(t^2,x^2,v^2)=\varrho(t^2)w(x^2,v^2)f(t^2,x^2,v^2)$:

$$
\begin{aligned}
&g(t^2,x^2,v^2)\\
&\quad=\mathbf{1}_{t^3<0}\varrho(0)w(x^2,v^2)f(0,X(0;t^2,x^2,v^2),V(0;t^2,x^2,v^2)) && (4.49)\\
&\qquad+w(x^2,v^2)\int_{\max(0,t^3)}^{t^2}\varrho'(s)f(s,X(s;t^2,x^2,v^2),V(s;t^2,x^2,v^2))\mathrm{d}s && (4.50)\\
&\qquad+w\mu(x^3,v^2_{\mathbf{b}})\sum_{i=3}^{k-1}\\
&\qquad\int_{\prod_{j=3}^{i}\mathcal{V}_j}\Big\{\mathbf{1}_{t^{i+1}<0\leq t^i}\varrho(0)w(x^i,v^i)\\
&\qquad\times f(0,X(0;t^i,x^i,v^i),V(0;t^i,x^i,v^i))\Big\}\mathrm{d}\tilde{\Sigma}_i && (4.51)\\
&\qquad+w\mu(x^3,v^2_{\mathbf{b}})\sum_{i=3}^{k-1}\\
&\qquad\int_{\prod_{j=3}^{i}\mathcal{V}_j}\mathbf{1}_{0\leq t^i}\Big\{\int_{\max(0,t^{i+1})}^{t^i}\varrho'(s)w(x^i,v^i)\\
&\qquad\times f(s,X(s;t^i,x^i,v^i),V(s;t^i,x^i,v^i))\mathrm{d}s\Big\}\mathrm{d}\tilde{\Sigma}_i && (4.52)\\
&\qquad+w\mu(x^3,v^2_{\mathbf{b}})\int_{\prod_{j=3}^{k}\mathcal{V}_j}\mathbf{1}_{t^k\geq 0}\,g(t^k,x^k,v^k)\mathrm{d}\tilde{\Sigma}_k, && (4.53)
\end{aligned}
$$

where $\mathrm{d}\tilde{\Sigma}_i:=\frac{\mathrm{d}\sigma_i}{\mu(x^{i+1},v^i)w(x^i,v^i)}\mathrm{d}\sigma_{i-1}\cdots\mathrm{d}\sigma_3$ with $3\leq i\leq k$. Here, we regard t^2,x^2,v^2 as free parameters and from Lemma 8, we have $\mu(x^3,v^2_{\mathbf{b}})=\mu(x^3,v^2)$.

Step 3. Next we estimate the contribution of (4.49)–(4.53) in $\frac{1}{\varrho(t^2)}\int_{\mathcal{V}_2}\frac{|n(x^2)\cdot v^2|}{w(x^2,v^2)}$ $g(t^2,x^2,v^2)\mathrm{d}v^2$ term by term.

We start with the contribution of (4.49). From $t^2\geq t/2$ and $t^3\leq 0$, we have

$$\|w(x^2,v^2)f(t^2,x^2,v^2)\|_{L^\infty_{x,v}}\leq \|w(x,v)f(0,x,v)\|_{L^\infty_{x,v}}.$$

From the L^∞-boundedness and $0<n(x^2)\cdot v^2\lesssim w(x^2,v^2)<w'(x^2,v^2)$, we deduce that

$$
\begin{aligned}
\frac{1}{\varrho(t^2)}\int_{\mathcal{V}_2}\frac{|n(x^2)\cdot v^2|}{w(x^2,v^2)}|(4.49)|\mathrm{d}v^2 &\lesssim \frac{1}{\varrho(t^2)}\int_{\mathcal{V}_2}\frac{|n(x^2)\cdot v^2|}{w(x^2,v^2)}\varrho(0)\mathrm{d}v^2\times\|wf(0)\|_{L^\infty_{x,v}}\\
&\lesssim\frac{1}{\varrho(t)}\varrho(0)\|wf(0)\|_{L^\infty_{x,v}}\lesssim\frac{1}{\varrho(t)}\|w'f(0)\|_{L^\infty_{x,v}}.
\end{aligned}
\tag{4.54}
$$

Now we bound the contribution of (4.50). Recall Theorem 1 with $\varrho'(t) \lesssim (\ln\langle t\rangle)^{6-\mathcal{A}}\langle t\rangle^{\mathcal{A}-6}$ for $t \gg 1$, and Lemma 23, we get

$$\begin{aligned}\frac{1}{\varrho(t^2)}\int_{\mathcal{V}_2}\frac{|n(x^2)\cdot v^2|}{w(x^2,v^2)}|(4.50)|\mathrm{d}v^2 &\lesssim \frac{1}{\varrho(t)}\int_0^t \|\varrho'(s)f(s)\|_{L^1_{x,v}}\mathrm{d}s\\ &\lesssim \frac{1}{\varrho(t)}\int_0^t \|(\ln\langle s\rangle)^{6-\mathcal{A}}\langle s\rangle^{\mathcal{A}-6}f(s)\|_{L^1_{x,v}}\mathrm{d}s\\ &\lesssim \frac{t}{\varrho(t)}\times\|w'f(0)\|_{L^\infty_{x,v}}.\end{aligned} \tag{4.55}$$

Next, we bound the contribution of (4.51). From $t^{i+1} < 0 \le t^i$, we have

$$\|w(x^i,v^i)f(t^i,x^i,v^i)\|_{L^\infty_{x,v}} \le \|w(x,v)f(0,x,v)\|_{L^\infty_{x,v}}.$$

From the L^∞-boundedness and $0 < n(x^2)\cdot v^2 \lesssim w(x^2,v^2) < w'(x^2,v^2) = \frac{1}{2\pi}\mu^{-1}(x^3,v^2)$, we derive

$$\begin{aligned}&\frac{1}{\varrho(t^2)}\int_{\mathbb{R}^3}\frac{|n(x^2)\cdot v^2|}{w(x^2,v^2)}|(4.51)|\mathrm{d}v^2\\ &\lesssim \frac{k}{\varrho(t)}\left(\sup_i\int_{\prod_{j=3}^i\mathcal{V}_j}\mathbf{1}_{t^{i+1}<0\le t^i}\mathrm{d}\tilde{\Sigma}_i\right)\varrho(0)\|wf(0)\|_{L^\infty_{x,v}}\\ &\lesssim \frac{k}{\varrho(t)}\left(\int_{n(x^i)\cdot v^i>0}\frac{|n(x^i)\cdot v^i|}{w(x^i,v^i)}\mathrm{d}v^i\right)\|wf(0)\|_{L^\infty_{x,v}}\\ &\lesssim \frac{k}{\varrho(t)}\|w'f(0)\|_{L^\infty_{x,v}}.\end{aligned} \tag{4.56}$$

Again using Lemma 23 and Theorem 1, we bound the contribution of (4.52). From $0 < n(x^2)\cdot v^2 \lesssim w(x^2,v^2) \le \mu^{-1}(x^3,v^2)$ and $\varrho'(t) \lesssim (\ln\langle t\rangle)^{6-\mathcal{A}}\langle t\rangle^{\mathcal{A}-6}$ for $t \gg 1$, we have

$$\begin{aligned}&\frac{1}{\varrho(t^2)}\int_{\mathbb{R}^3}\frac{|n(x^2)\cdot v^2|}{w(x^2,v^2)}|(4.52)|\mathrm{d}v^2\\ &\lesssim \frac{k}{\varrho(t)}\times\sup_i\int_{\prod_{j=3}^i\mathcal{V}_j}\mathbf{1}_{0\le t^i}\int_{\max(0,t^{i+1})}^{t^i}w(x^i,v^i)\varrho'(s)\\ &f(s,X(s;t^i,x^i,v^i),V(s;t^i,x^i,v^i))\mathrm{d}s\mathrm{d}\tilde{\Sigma}_i\\ &\lesssim \frac{k}{\varrho(t)}\int_0^t\|\varrho'(s)f(s)\|_{L^1_{x,v}}\mathrm{d}s \lesssim \frac{k}{\varrho(t)}\int_0^t\|(\ln\langle s\rangle)^{6-\mathcal{A}}\langle s\rangle^{\mathcal{A}-6}f(s)\|_{L^1_{x,v}}\mathrm{d}s\\ &\lesssim \frac{kt}{\varrho(t)}\times\|w'f(0)\|_{L^\infty_{x,v}}.\end{aligned} \tag{4.57}$$

Lastly we bound the contribution of (4.53). Applying Lemma 24 with $k \geq \mathfrak{C}t$, we get

$$\begin{aligned}
&\frac{1}{\varrho(t^2)} \int_{\mathbb{R}^3} \frac{|n(x^2)\cdot v^2|}{w(x^2,v^2)} |(4.53)| \mathrm{d}v^2 \\
&\lesssim \frac{\varrho(t^k)}{\varrho(t^2)} \sup_{(x,v)\in\bar{\Omega}\times\mathbb{R}^3} \Big(\int_{\prod_{j=3}^{k-1}\mathcal{V}_j} \mathbf{1}_{t^k(t^2,x^2,v^2,\cdots,v^{k-1})\geq 0} \mathrm{d}\sigma_3\cdots \mathrm{d}\sigma_{k-1} \Big) \sup_{t_k\geq 0} \|wf(t^k)\|_{L^\infty_{x,v}} \\
&\lesssim e^{-t} \sup_{t_k\geq 0} \|w' f(t^k)\|_{L^\infty_{x,v}} \lesssim e^{-t} \|w' f(0)\|_{L^\infty_{x,v}}.
\end{aligned} \tag{4.58}$$

Collecting estimates from (4.54)–(4.58) and using $k \lesssim t$, we derive

$$\begin{aligned}
&\Big|\frac{1}{\varrho(t^2)} \int_{\mathcal{V}_2} \mathbf{1}_{t^2\geq t/2} \frac{|n(x^2)\cdot v^2|}{w(x^2,v^2)} g(t^2,x^2,v^2)\mathrm{d}v^2\Big| \\
&\lesssim \max\{\frac{1}{\varrho(t)}, \frac{(k+1)t}{\varrho(t)}, e^{-t}\} \times \|w' f(0)\|_{L^\infty_{x,v}} \lesssim \frac{\langle t\rangle^2}{\varrho(t)} \times \|w' f(0)\|_{L^\infty_{x,v}}.
\end{aligned} \tag{4.59}$$

Using $\varrho(t) = (\ln\langle t\rangle)^{6-\mathcal{A}}\langle t\rangle^{\mathcal{A}-5}$, $0 < w(x,v) < \mu^{-1}(x,v)$ and (4.59), we conclude

$$(4.39) \lesssim \frac{\langle t\rangle^2}{\varrho(t)} \lesssim \langle t\rangle^{\mathcal{A}-7}.$$

From the above estimate, together with (4.40), (4.43) and (4.47), we prove (1.13). □

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Data availability Data sharing does not apply to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest The authors have no competing interests to declare that are relevant to the content of this article.

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