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Subspace configurations and low degree points on curves

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ABSTRACT

This paper is devoted to understanding curves X over a number field k that possess infinitely many solutions in extensions of k of degree at most d; such solutions are the titular low degree points. For $d = 2, 3$ it is known ([9], [2]) that such curves, after a base change to $\overline{k}$, admit a map of degree at most d onto $\mathbb{P}^1$ or an elliptic curve. For $d \geqslant 4$ the analogous statement was shown to be false [3]. We prove that once the genus of X is high enough, the low degree points still have geometric origin: they can be obtained as pullbacks of low degree points from a lower genus curve. We introduce a discrete-geometric invariant attached to such curves: a family of subspace configurations, with many interesting properties. This structure gives a natural alternative construction of curves from [3]. As an application of our methods, we obtain a classification of such curves over k for $d = 2, 3$, and a classification over $\overline{k}$ for $d = 4, 5$.

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1. Introduction

Suppose k is a number field and X/k is a nice curve (smooth, projective, and geometrically integral). The density degree set $\delta(X/k)$ of X/k is the set of integers d for which the collection of closed points of degree d on X are Zariski dense. Since X is a curve, this is equivalent to asking that the degree d points be infinite, yet the definition in terms of Zariski density is natural for a variety of any dimension.

In this paper we are concerned with the most basic such piece of information: the minimum density degree[1] $\min(\delta(X/k))$ is the smallest positive integer in $\delta(X/k)$. There is also a geometric version of the minimal density degree that is stable under finite extensions of the ground field. Let $\wp(X/k)$ be the union of $\delta(X/L)$ as L ranges over all finite extensions of k. The minimum potential density degree is $\min(\wp(X/k))$. For more on the structure of $\delta(X/k)$ and $\wp(X/k)$ see [20]. Our motivating problem is:

Main Problem 1.1. Classify curves X with $\min(\delta(X/k)) = d$, and those with $\min(\wp(X/k)) = d$.

Faltings' theorem classifies curves X with $\min(\wp(X/k)) = 1$. Main Problem 1.1 can therefore be viewed as a generalization of this fundamental problem.

There are two natural geometric sources of Zariski dense degree d points: if X is a degree d cover of $\mathbb{P}^1$ or an elliptic curve E of positive rank, then pulling back rational points on $\mathbb{P}^1$ or E gives an infinite family of degree d (or less) points on X. Previous work on Main Problem 1.1 has focused on the geometric invariant $\min(\wp)$. Harris–Silverman (for $d = 2$) and Abramovich–Harris (for $d = 3$) showed that the above two natural geometric sources of low degree points characterize when $\min(\wp)$ takes the value 2 or 3. More precisely, $\min(\wp(X/k)) = 2$ or 3 if and only if $X_{\bar{k}}$ is a degree 2 or 3 cover of $\mathbb{P}^1$ or an elliptic curve. Based on this evidence, Abramovich–Harris conjectured that the same should hold for all values of d. However, Debarre and Fahlaoui [3] showed that more obscure constructions of infinite families of degree $d \geqslant 4$ points exist by cleverly constructing certain curves on the symmetric square of an elliptic curve. A full classification for any larger values of $d \geqslant 4$ has remained stubbornly out of reach, and there have been essentially no classification results for the arithmetic invariant $\min(\delta)$.

In the present paper we refocus on Main Problem 1.1, including the thornier arithmetic classification. As a result, we obtain the following new classification.

Theorem 1.2. *Suppose X/k is a nice curve. Then the following statements hold:*

(1) If $\min(\delta(X/k)) = 2$, then X is a double cover of $\mathbb{P}^1$ or an elliptic curve of positive rank;

[1] In previous work [17], the second author called this invariant the arithmetic degree of irrationality. We prefer to switch to terminology that better generalizes to higher dimensional X.

(2) If $\min(\delta(X/k)) = 3$*, then one of the following three cases holds:*

(a) X *is a triple cover of* $\mathbb{P}^1$ *or an elliptic curve of positive rank;*
(b) X *is a smooth plane quartic with no rational points, positive rank Jacobian, and at least one cubic point;*
(c) X *is a genus* 4 *Debarre–Fahlaoui curve (see Section 5 for a precise definition);*

(3) If $\min(\wp(X/k)) = d \leqslant 3$*, then* $X_{\bar{k}}$ *is a degree* d *cover of* $\mathbb{P}^1$ *or an elliptic curve;*
(4) If $\min(\wp(X/k)) = d = 4, 5$*, then either* $X_{\bar{k}}$ *is a Debarre-Fahlaoui curve, or* $X_{\bar{k}}$ *is a degree* d *cover of* $\mathbb{P}^1$ *or an elliptic curve.*

Surprisingly, the seemingly clever construction by Debarre–Fahlaoui of counterexamples to the conjecture of Abramovich–Harris arises perfectly naturally from our perspective. The next open case is to classify curves of genus 11 with $\min(\wp(X/k)) = 6$, see Section 7.1. As can be seen from Case 2b of Theorem 1.2, there are certain arithmetic subtleties involved in the classification; some open questions concerning these subtleties are described in Section 7.2.

The classification in Theorem 1.2 is obtained from a systematic study of the possible infinite collections of degree d points. Our guiding philosophy is that when d is small compared to the genus of X, such infinite collections *still* occur for good geometric reasons. The first step in our analysis is thus to make this precise with the following genus bound, which reduces Main Problem 1.1 to finitely many genera for each value of d.

Theorem 1.3. *Suppose* X/k *is a nice curve of genus* g *and* $\min(\delta(X/k)) = d$*. Let* $m := \lceil d/2 \rceil - 1$ *and let* $\varepsilon := 3d - 1 - 6m < 6$*. Then one of the following holds:*

(1) There exists a nonconstant morphism of curves $\phi\colon X \to Y$ *of degree at least* 2 *such that* $d = \min(\delta(Y/k)) \cdot \deg \phi$*;*
(2) The genus of X *is bounded*

$$g \leqslant \max\left(\frac{d(d-1)}{2} + 1,\ 3m(m-1) + m\varepsilon\right).$$

In case (1) there is a clear source of low degree points on X: they can be obtained as pullbacks of low degree point on Y under ϕ. There is a long history of genus bounds in problems related to Main Problem 1.1; see [18], [19], [2], [16]. Theorem 1.3 is indirectly claimed in [2] by combining [2, Lemma 3] with [2, Theorem 2]; however, the statement of [2, Lemma 3] has an error, and the proof of [2, Theorem 2] contains a gap. See the discussion in Section 1.1.

Following the ideas introduced by Abramovich and Harris, we study the geometry of curves with $\min(\delta(X/k)) = d$ by studying the geometry of linear systems of the form $|nD|$

for degree d points D. The Mordell–Lang conjecture ensures that these linear systems have positive dimension. An important step in proving Theorem 1.3 is Theorem 3.5, which states that unless case (1) holds, the linear systems $|nD|$ are birational for most D and $n \geqslant 2$.

With birationality proved, we can investigate finer questions concerning the geometry of the linear systems $|nD|$. We do so by equipping these linear systems with a discrete-geometric structure: there is an infinite family of multisecant planes within each of the projective spaces $|nD|$, which form an combinatorially interesting configuration. The presence of this additional structure allows us to prove the following finer classification of curves X with $\min(\delta(X/k)) = d$. To formally state this classification we require a notion of a sufficiently general degree d point D; this is rigorously defined in Section 2.

Theorem 1.4. *Suppose X is a curve with $\min(\delta(X/k)) = d$. Let $m := \lceil d/2 \rceil - 1$ and let $\varepsilon := 3d - 1 - 6m < 6$. Then for a sufficiently general degree d point D one of the following holds:*

(1) $\dim |2D| = 1$*, and X is a degree d cover of an elliptic curve of positive rank;*
(2) $\dim |2D| \geqslant 2$*, the associated map $X \to \mathbb{P}^{|2D|}$ is not birational onto its image, and there exists a covering of curves $\phi \colon X \to Y$ of degree at least 2 such that $d = \min(\delta(Y/k)) \cdot \deg \phi$;*
(3) $\dim |2D| = 2$*, the associated map $X \to \mathbb{P}^2$ is birational onto its image, and X is one of the Debarre-Fahlaoui curves (see Section 5 for the precise definition);*
(4) $\dim |2D| > 2$*, the associated map $X \to \mathbb{P}^{|2D|}$ is birational onto its image, and the genus g of X satisfies*

$$g \leqslant \max\left(\frac{(d-1)(d-2)}{2} + 2,\ 3m(m-1) + m\varepsilon\right)$$

The proof of Theorem 1.4 involves a detailed analysis of the configuration geometry in $|3D|$; it shows how the geometry of the linear systems $|nD|$ naturally gives rise to the Debarre-Fahlaoui examples.

Remark 1.5. The results of [2] imply that the gonality of a curve with $\min(\delta(X/k)) = d$ is at most $2d$; this fact was also independently observed by Frey [6]. One corollary of Theorem 1.4 is that the geometric gonality can equal $2d$ only for (geometric) degree d covers of elliptic curves; and can equal $2d - 1$ only for (geometrically) Debarre-Fahlaoui curves. See Remarks 3.4 and 5.9.

The method of examining multisecant configurations can be applied to study the low degree points on special families of curves. As a demonstration, in Section 6.1 we prove that projective curves of large genus have finitely many sufficiently low degree points. The statement of this estimate uses the Castelnuovo function $\pi(d, r)$; we recall its definition in Section 6.1. The number $\pi(d, r)$ is an upper bound for the genus of a nondegenerate degree d curve in $\mathbb{P}^r$. When r is fixed and d is growing, $\pi(d, r) \sim d^2/(2r-2)$.

Theorem 1.6. *Suppose $X \subset \mathbb{P}^r$ is an irreducible (possibly singular) curve of degree e and genus g. Suppose X has infinitely many points of degree d not contained in hyperplanes of $\mathbb{P}^r$. Then*

$$g \leqslant \pi(e + 2d, 2r + 1).$$

There are many open questions concerning the geometry of curves with abundant low degree points, both of arithmetic and purely geometric nature. We survey these questions in Section 7.

1.1. Relation to previous work

The first results on low degree points were obtained by Hindry [8], who studied quadratic points on modular curves $X_0(p)$ and asked if in general a curve with infinitely many quadratic points is either hyperelliptic or bielliptic. Later, Faltings [5], and Vojta [18] used diophantine approximation techniques to describe low degree points on curves of small gonality. The strongest of these results was obtained by Vojta, who showed that a degree s cover of $\mathbb{P}^1$ with infinitely many degree d points not contracted by the map to $\mathbb{P}^1$ has genus at most $s(d-1)+1$ [19].

The resulting genus bound is sharp: if E is an elliptic curve of positive rank, then a (d, s)-curve on $E \times \mathbb{P}^1$ satisfies the conditions of the theorem and has genus $g = s(d-1)+1$. The general question of describing curves with minimum potential density degree d was first addressed in [9] in the case $d = 2$ and in [2] for $d = 3$. Based on these results, Abramovich and Harris proposed the following conjecture, which was soon disproved by Debarre and Fahlaoui.

Conjecture 1.7 *([2]; proved for $d = 2$ [9]; proved for $d = 3$ [2]; disproved for all $d \geqslant 4$ [3]).* Suppose $\min(\wp(X/k)) = d$. Then $X_{\bar{k}}$ has a degree d map to $\mathbb{P}^1$ or an elliptic curve.

The presence of counterexamples makes it hard to analyze the minimum density degree for arbitrary curves; however, the methods used in [2] can still be applied to certain classes of special curves. For example, Debarre and Klassen [4] showed that a smooth plane curve X of degree $d \geqslant 8$ has minimum density degree d or $d-1$ corresponding to the cases $X(k) = \emptyset$ and $X(k) \neq \emptyset$ respectively. For a generalization to curves on other surfaces see [17].

A word of warning is warranted concerning the work of Abramovich and Harris [2]: as detailed in [3], the paper contains several errors (including in Lemma 3, Lemma 6, Lemma 8, and Corollary 1), which, while not severe enough to make the main results false, can be misleading. Some corrections are described in [3], however one of the main results – [2, Theorem 2] – should be considered unproved. For this reason, we give full proofs of several simple lemmas appearing in [2] when we need them.

Other related work includes the study of integral points of low degree [13], generalization of the work of Vojta to covers of curves [16], results on low degree points for curves on product surfaces [12].

1.2. Structure of the paper

In Section 2 we describe how the Mordell–Lang conjecture gives geometric restrictions on the curves with $\min(\delta(X/k)) = d$; this observation was also used in [9], [2].

In Section 3 we prove a key technical result: the birationality of the linear systems $|nD|$ for $n \geqslant 2$ and for sufficiently general degree d points D on X. More precisely, we show that these linear systems are birational if there does not exist a cover $\phi\colon X \to Y$ of degree at least 2 with $d = \min(\delta(Y/k)) \cdot \deg \phi$.

In Section 4 we enrich the linear systems $|nD|$ with additional discrete-geometric structure of a configuration of multisecant subspaces to X. The main properties of this structure are summarized in Section 4.2. As an application, we prove Theorem 1.3.

In Section 5 we use subspace configurations to relate geometry of special linear systems to the construction of Debarre–Fahlaoui; in particular we prove most of Theorem 1.4.

In Section 6 we collect geometric corollaries of the results obtained so far. We prove Theorem 1.6, finish the proof of Theorem 1.4, and summarize what we know about curves with minimum density degree at most 5 to prove Theorem 1.2.

In Section 7 we collect some open questions on the geometry of curves with $\min(\delta(X/k)) = d$.

1.3. Notation

Throughout the paper k will denote a fixed number field and X/k will denote a nice curve. Let $\bar{k}$ be an algebraic closure of k. We write $\bar{X} = X_{\bar{k}}$ for the base-change of X to $\bar{k}$. By a degree d point on X we mean a closed point with residue field a degree d extension of k. Write $\operatorname{Sym}^d X = X^d /\!\!/ S_d$ for the dth symmetric power of the curve X, and Pic^d_X for the degree d component of the Picard scheme of X/k. We write $\operatorname{Pic}^d X$ for the group of isomorphism classes of degree d line bundles on X/k. There is an inclusion $\operatorname{Pic}^d X \subset \operatorname{Pic}^d_X(k) = \operatorname{Pic}^d_X(\bar{k})^{\operatorname{Gal}(\bar{k}/k)}$, which need not be an equality if $X(k) = \emptyset$.

We write $W_d X = W^0_d X$ for the image of the Abel–Jacobi map $\operatorname{Sym}^d X \to \operatorname{Pic}^d_X$ sending an effective divisor of degree d to its linear equivalence class. This is a Brill–Noether locus of Pic^d_X (see [1, Chapter IV §3] for more details on Brill–Noether loci). The fiber of the Abel–Jacobi map over a line bundle $L \in \operatorname{Pic}^d X$ is isomorphic to the complete linear system $|L| \simeq \mathbb{P}H^0(X, L)$ of L. When $\dim |L| = r$, such a linear system is called a g^r_d. The minimal value of d for which X has a (k-rational) g^1_d is called the gonality of X, denoted $\operatorname{gon}(X)$.

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2. Corollaries of Mordell–Lang

In this section we collect some corollaries of the Mordell–Lang conjecture, now a theorem of Faltings [5, Theorem 1], for the structure of rational points on subvarieties of the Picard scheme of a curve over k.

Suppose that $\min(\delta(X/k)) = d$ and that the gonality $\operatorname{gon}(X)$ is strictly greater than d. Since $\min(\delta(X/k)) = d$, the set $\operatorname{Sym}^d X(k)$ is infinite. Since $\operatorname{gon}(X) > d$, the Abel–Jacobi map $\operatorname{Sym}^d X(k) \to \operatorname{Pic}^d_X(k)$ is injective. Consequently, the rational points of the image $W_d X(k)$ are an infinite set. Since $W_d X$ is a subvariety of a torsor under an abelian variety, the Mordell Lang conjecture [5, Theorem 1] implies that there exists a translate $A \subset W_d X$ of a positive-dimensional abelian subvariety $A^0 \subset \operatorname{Pic}^0_X$, such that $A(k)$ is Zariski dense in A.

By the semicontinuity theorem, there is an open dense locus in A consisting of points for which the corresponding (isomorphism class of) line bundle $[L]$ has the minimal achieved value of $h^0(X, L)$. (By virtue of the fact that $A \subset W^0_d X$, this minimal value is positive.) The fibers of $\operatorname{Sym}^d X \to \operatorname{Pic}^d_X$ are Severi–Brauer varieties of dimension $h^0(X, L) - 1$. Since $\operatorname{gon}(X) > d$, any fiber of $\operatorname{Sym}^d X \to \operatorname{Pic}^d_X$ having a rational point is necessarily a $\mathbb{P}^0_k$. Since $\operatorname{Sym}^d X(k) \neq \emptyset$, the minimal achieved value of $h^0(X, L)$ must be 1. Hence there is an open dense locus $U \subset A$ over which $\operatorname{Sym}^d X \to \operatorname{Pic}^d_X$ is an isomorphism. In the case $\dim A = 1$, applying the curve-to-projective extension theorem to the map $U \to \operatorname{Sym}^d X$, we see that there is a canonical effective divisor associated to every point of A (and in particular $U(k) = A(k)$). Moreover, even when $\dim A > 1$, resolving the indeterminacy of the inverse $U \to \operatorname{Sym}^d X$ (in a similar manner to the proof the Lang–Nishimura lemma as in [15, Proposition A.6]) proves that $U(k) = A(k)$.

Since we will focus on this setup for the majority of the paper, we codify it in the following:

Setup 2.1. Let X be a nice curve with $\min(\delta(X/k)) = d$ and $\operatorname{gon}(X) > d$. Write $A \subset W_d X$ for an associated positive-dimensional abelian translate and $U \subset A$ for the open dense locus over which $\operatorname{Sym}^d X \to \operatorname{Pic}^d_X$ is an isomorphism.

Since the rational points of A are Zariski dense, any nonempty open in A also has Zariski dense rational points. By a general rational point of A, we mean a rational point in an open dense subvariety of A. By a general effective divisor D in $A(k) = U(k)$, we mean a rational point in an open dense subvariety of $U \subset A$ where there exists a unique effective divisor representing each isomorphism class of line bundle. A general effective divisor D in $A(k)$ is a degree d point on X.

Consider the incidence correspondence

$$I := \{(p, [D]) \in X \times U : p \in D\}. \tag{1}$$

Then I necessarily dominates X via the first projection, and is a degree d cover of U via the second projection. The dimension of I is therefore equal to the dimension of A. The (arithmetic) monodromy of this degree d cover is transitive, since a general effective divisor in U corresponds to a degree d point on X. In particular, we have that

$$\text{a general pair of effective divisors in } U \text{ have disjoint support.} \tag{2}$$

The following lemma shows that at the same time, there are many pairs of divisors in U that share points on X.

Lemma 2.2. *Suppose we are in Setup 2.1 and that if* $\dim A = 1$*, then X is not a degree d cover of A. Then for any open subset $V \subset U$ and a general point $P \in X(\bar{k})$ there exists a pair of distinct divisors $D_1, D_2 \in V$ containing P.*

Proof. Suppose to the contrary, that for some open $V \subset U$ a general point $P \in X$ is contained in a unique divisor from V. The rational map $\phi : X \dashrightarrow V$ that sends a point $P \in X$ to the unique divisor in V that contains P is dominant, and so $\dim A = 1$. In this case, the map ϕ extends to a degree d covering $X \to A$ (by (2)). □

2.1. The abelian translate property

The abelian translate A is a torsor under an abelian variety $A^0 \subset \operatorname{Pic}^0_X$. The group law on A^0 has the following consequence, which we term the abelian translate property: for any three points L_1, L_2 and L_3 of A, the line bundle $L_1 \otimes L_2 \otimes L_3^{-1}$ is again a point of A. Moreover, on rational points, for effective divisors D_1, D_2 and D_3 in $U(k)$, the divisor $D_1 + D_2 - D_3$ is again in $U(k)$.

Fix an effective divisor D in $U(k)$. The abelian translate property implies that for any $n-1$ divisors $D_1, \ldots, D_{n-1}$ in $U(k)$, there exists a nth divisor D_n in $U(k)$ that is linearly equivalent to

$$nD - D_1 - \cdots - D_{n-1}.$$

In the next section, we will prove that Setup 2.1 implies that the linear systems $|\mathcal{O}(nD)|$ for D a general effective divisor in $A(k)$ are birational unless there is a natural geometric source of degree d points on X. We will then interpret the abelian translate property geometrically in terms of spans of divisors in $|\mathcal{O}(nD)|$.

3. Birationality

The main result of this section is Theorem 3.5 which shows that in Setup 2.1, unless an infinite collection of degree d points on X is obtained by pullback from a lower genus curve, the linear system $|2D|$ is birational for a general $D \in A(k)$. In particular, this immediately implies that the genera of such curves are bounded by $(d-1)(2d-1)$. These results are closely related to [2]. Our main Theorem 3.5 is similar to [2, Lemma 3]; note, however, that the statement of [2, Lemma 3] has an error (the last formula of the statement is false), and more importantly the proof does not go through for curves which are degree d covers of pointless conics – the case that requires most work in our Theorem 3.5.

We will use the following version of de Franchis theorem due to Kani.

Theorem 3.1. *Suppose k is a field of characteristic zero and X/k is a nice curve. Then*

(1) There exist at most finitely many surjective morphisms $X \to Y$ to curves Y/k of genus at least 2;
(2) For any integer d there exists at most finitely many surjective morphisms $X \to C$ of degree less than d to curves of genus 1, up to translations on the target.

Proof. See [10] Theorem 3 and [10] Corollary after Theorem 4. □

One way of obtaining infinitely many degree d points on X is via pullback from an elliptic curve of positive rank. The following lemma describes a situation in which this is the case.

Lemma 3.2. *Assume that we are in Setup 2.1 and $\dim A = 1$. Let $D \in A$ be a general divisor. For every effective divisor $E \in A$, the divisor $E' := 2D - E$ belongs to A by the abelian translate property. Suppose that there exists a map*

$$\pi \colon X \to \mathbb{P}^1$$

of degree $2d$ such that all $E \in A$ are contracted to a point $\psi(E)$ by π and further $\psi(E) = \psi(E')$. Then there exists a degree d map $\pi' \colon X \to A$ factoring the map $\pi = \psi \circ \pi'$.

Proof. The morphism $\psi\colon A \to \mathbb{P}^1$ that sends a divisor $E \in A$ to the point $\pi(E)$ evidently factors through the quotient by the involution sending E to $2D - E$. By computing the ramification of ψ, we will show that ψ has degree 2 and is hence equal to this quotient map. As a result, the original map π factors $X \xrightarrow{d:1} A \xrightarrow{\psi} \mathbb{P}^1$, and X is a degree d cover of the elliptic curve A.

Since $\dim A = 1$, we can extend an inverse of the Abel–Jacobi map to a regular map $A \to \mathrm{Sym}^d X$. Since the effective divisor corresponding to a general point of A is reduced, the union of the supports of all nonreduced divisors from A is finite. In particular, we may assume that D, and all of its translates D' by the finitely many 2-torsion points of A^0, are disjoint from this finite set. The points of D' are ramification points of π, and in particular we have the equality of sets $\pi^{-1}(\psi(D')) = \operatorname{supp} D'$. Since no nonreduced divisor intersects $\operatorname{supp} D' = \pi^{-1}(\psi(D'))$, any divisor from A supported on the fiber $\pi^{-1}(\psi(D'))$ is equal to D'. Therefore the map $\psi\colon A \to \mathbb{P}^1$ is totally ramified over the 4 points of the form $\psi(D')$ satisfying $2D' = 2D$. Since A has genus 1 and ψ is totally ramified over at least 4 points, the Riemann–Hurwitz formula gives

$$0 = -2\deg\psi + \sum_P (e_P - 1) \geqslant -2\deg\psi + 4(\deg\psi - 1) = 2\deg\psi - 4.$$

Therefore $\deg\psi = 2$. This means that there are exactly two divisors from A supported on a general fiber of π. Recall the incidence correspondence $I \subset X \times U$ given by formula (1) that represents the relation "point belongs to a divisor". Since we just saw that a general point of X belongs to a unique divisor from U, the correspondence I is a graph of a rational map $\pi' : X \to A$. The map π' represents the association $P \mapsto$ (unique $D \in A$ with $P \in D$) defined on an open dense subset of X, and gives the desired factorization $\pi = \psi \circ \pi'$. □

Proposition 3.3. *Suppose we are in Setup 2.1 and additionally that X is not a degree d cover of an elliptic curve. Then for any divisor class $D \in A(k)$ the linear system $|2D|$ is basepoint free and* $\dim |2D| \geqslant \max(2, \dim A)$.

Proof. By [2, Lemma 1], $|2D|$ is basepoint free and $\dim |2D| \geqslant \dim A$. (Indeed, by the abelian translate property, for all $E \in A$, the class $2D - E$ is effective. The association $E \mapsto E + |2D - E|$ defines a ($\geqslant \dim A$)-dimensional family of effective divisors in class $2D$. Since the divisors $E \in U$ don't have a shared point, the family of divisors of the form $E \cup [2D - E]$ do not have any common points either, and so $|2D|$is base point free.) This completes the proof when $\dim A > 1$, so we assume for the remainder of the proof that $\dim A = 1$.

Since $\dim |2D|$ is upper-semicontinuous, it suffices to prove the result for a general $D \in A$. Suppose that $\dim |2D| = 1$, and let $\phi : X \to \mathbb{P}^1$ be the associated map. For every effective divisor $E \in A$, the divisor $2D - E$ belongs to A by the abelian translate property, and therefore is effective. Hence every divisor $E \in A$ is supported on a fiber of ϕ and both E and $2D - E$ are supported on the same fiber. Therefore the assumptions of Lemma 3.2 are satisfied, which is a contradiction since we assumed that X is not a degree d cover of an elliptic curve. □

Remark 3.4. Abramovich and Harris, and, independently, Frey [6], observed that the gonality of a curve with $\min(\delta(X/k)) = d$ is at most $2d$. An immediate corollary of Proposition 3.3 is that the geometric gonality of a curve with $\min(\delta(X/k)) = d$ that is not a degree d cover of an elliptic curve is at most $2d - 1$.

We now prove the main result of this section.

Theorem 3.5. *Suppose that we are in Setup 2.1 and $D \in U(k)$ is a general divisor. Then one of the following holds:*

(1) there exists a covering of curves $\phi\colon X \to Y$ of degree at least 2 with $\min(\delta(Y/k)) = d/\deg\phi$;
(2) the associated map $X \to \mathbb{P}^{\dim|2D|}$ is birational. (The basepoint free line bundle $|2D|$ is birationally very ample).

Proof. By Theorem 3.1, X has only finitely many nonconstant maps $f_1, ..., f_N$ of degree at most d to curves of genus $\geqslant 1$ up to automorphisms of the base. Since D is general, we can assume that D does not intersect the preimage of the branch locus of any of the f_i.

Suppose that case (2) does not hold, i.e., that the morphism $X \to \mathbb{P}^{|2D|}$ factors as $X \xrightarrow{f} Y \hookrightarrow \mathbb{P}^{|2D|}$ with $m := \deg f \geqslant 2$. Then we will show that case (1) holds. Note that it suffices to show that $\min(\delta(Y/k)) \leqslant d/\deg\phi$, since X has finitely many points of degree less than d. Write $\widetilde{f} : X \to \widetilde{Y}$ for the map to the normalization of Y. Since the nondegenerate curve $Y \subset \mathbb{P}^{|2D|}$ has degree at least $\dim|2D| \geqslant 2$, the degree of $\widetilde{f}$ is at most d.

First suppose that the genus of $\widetilde{Y}$ is at least 1. By assumption, the divisor D has trivial intersection with the preimage of the branch locus of the map $\widetilde{f}$. Observe that for any curve C and any effective divisor Δ the following property holds: if for some positive k the linear system $k\Delta$ is base-point free and $\Phi : C \to \mathbb{P}^N$ is the associated morphism, then we have the equality of sets $\Phi^{-1}(\Phi(\Delta)) = \Delta$. Applying this to $\Delta = D$ and $k = 2$ gives $\widetilde{f}^{-1}\left(\widetilde{f}(D)\right) = D$. Since D does not intersect the preimage of the branch locus, we have

$$d = \#D = \#\widetilde{f}^{-1}\left(\widetilde{f}(D)\right) = m\#\widetilde{f}(D).$$

Hence the image of D in $\widetilde{Y}$ is a point of degree equal to d/m. Since there are infinitely many choices of D and only finitely many choices for the morphism $\widetilde{f}$, by Theorem 3.1, there exists a map $\widetilde{f} : X \to \widetilde{Y}$ of degree m such that for infinitely many $D \in \operatorname{Sym}^d X(k)$ the image $\widetilde{f}(D)$ has degree d/m, in which case (1) holds.

Now consider the case that $\widetilde{Y}$ has genus 0. Since a genus 0 curve has infinitely many quadratic points either (1) holds, or $m = \deg \widetilde{f} > d/2$, and so $\deg Y = 2d/m < 4$. If $\deg Y = 3$ and the genus of $\widetilde{Y}$ is zero, then Y has odd degree points, and thus $Y = \mathbb{P}^1$; this implies that case (1) holds.

It therefore remains to consider the case $\deg Y = 2$, $\dim |2D| = 2$, Y is a smooth pointless conic and f is a covering of degree d. In this way, a general divisor $D \in U$ defines a map to a rational curve $f_D : X \to Y_D$; let $U' \subset U$ be the open subset of divisors that define such maps. For every pair $(D_1, D_2) \in U' \times U'$ we get a map $\phi_{D_1,D_2} = (f_{D_1}, f_{D_2}) : X \to Y_{D_1} \times Y_{D_2}$. Let Z_{D_1,D_2} denote its image. By semicontinuity, Z_{D_1,D_2} will have a constant bidegree (e_1, e_2) on an open dense subset of $U' \times U'$, and comparing the degrees of Z_{D_1,D_2} and Z_{D_2,D_1} we see that the bidegree is necessarily symmetric: $e_1 = e_2 = e$. To simplify notation, fix a general pair $(D_1, D_2) \in U' \times U'$ and write $Y_1 = Y_{D_1}$, $Y_2 = Y_{D_2}$, $f_1 = f_{D_1}$, $f_2 = f_{D_2}$, $\phi = \phi_{D_1,D_2}$, and $Z = Z_{D_1,D_2}$. In what follows, we will show that $e > 1$, so that the map $\phi\colon X \to Z$ has degree $d/e \leqslant d/2$, and that the images $\phi(E)$ of divisors $E \in U(k)$ have low enough degree to force us to be in case (1).

Since the line bundles $\mathcal{O}_X(D_1) = f_1^*\mathcal{O}_{\mathbb{P}^1}(1)$ and $\mathcal{O}_X(D_2) = f_2^*\mathcal{O}_{\mathbb{P}^1}(1)$ are distinct, there does not exist an automorphism of $\mathbb{P}^1$ bringing one to the other and thus Z cannot be a $(1,1)$-divisor. Hence, the degree e of the projection from Z to Y_1 and Y_2 is at least 2. For a general divisor $E \in U(k)$, the divisors $2D_i - E$ are effective by the abelian translate property. Therefore $f_i(E)$ is contained in the hyperplane section of a (pointless) conic and hence $\deg f_1(E) = \deg f_2(E) = 2$, and so $\deg \phi(E) \leqslant 4$. Since the map f_1 factors through ϕ and $\deg f_1(E) = 2$, the degree of $\phi(E)$ is either 2 or 4.

Case 1: $\deg \phi(E) = 2$ for infinitely many $E \in U(k)$. Since $\deg(\phi) \leqslant d/2$, we are in evidently in case (1), unless $d = 2$ and the map ϕ is birational onto its image. However an integral $(2,2)$-curve (which necessarily has geometric genus 0 or 1) with infinitely many degree 2 points is always a degree 2 cover of $\mathbb{P}^1$, so we are again in case (1).

Case 2: $\deg \phi(E) = 4$ for general $E \in U$. In this case, we will show that $e \geqslant 4$, and hence we are in case (1) unless $d = 4$ and the map ϕ is birational onto its image. Consider a general $E \in A$ and the divisors $f_2(E)$ and $f_2(2D_1 - E)$ on Y_2. Each one is a degree 2 point on Y_2, and as D_1 varies the point $f_2(2D_1 - E)$ will vary as well. Since D_1, D_2 are a general pair, we can assume that the divisors $f_2(E)$ and $f_2(2D_1 - E)$ are disjoint for general $E \in A$, and hence the divisors $\phi(E), \phi(2D_1 - E) \in Z$ are necessarily disjoint degree 4 divisors. Since $f_1(E) = f_1(2D_1 - E)$, the projection of the degree 8 divisor $\phi(E) + \phi(2D_1 - E)$ to Y_1 is supported on a degree 2 point $f_1(E)$. Therefore the degree e of the projection $Z \to Y_1$ is at least 4, and hence the degree of $\phi : X \to Z$ is at most $\deg f_1/4 = d/4$. If ϕ is not birational onto its image, then since $\deg \phi \leqslant d/4$ and Z has infinitely many degree 4 points, case (1) holds.

It remains to consider the case when $d = 4$ and X is birational to a $(4,4)$ curve Z on $Y_1 \times Y_2$. If Z is smooth, then $X = Z$ and the projections onto Y_1 and Y_2 give the only two degree 4 maps from $\bar{X}$ to $\mathbb{P}^1$ [1, Chapter IV, Exercise F-2]. This is a contradiction, since the infinite family of D_i define distinct maps. Therefore Z is singular. Since Y_1 is pointless, the singular locus of Z has to have cardinality 2 or $\geqslant 4$, for otherwise the projection of the singular locus would be a zero-cycle of odd degree on Y_1. Therefore the genus of X is either 7 or at most 5. Since $\dim|2D| = 2$, the Riemann–Roch theorem implies that $g(X) = 7$. Consider now the geometric curve $\bar{X}$. By Mumford's extension of Marten's theorem (see [1, Chapter IV, Theorem 5.2]), since the curve $\bar{X}$ has a positive-dimensional family of g^1_4's, it is either hyperelliptic, trigonal, bielliptic, or a smooth plane quintic. Since $g(X) = 7$, it is not isomorphic to a smooth plane quintic. If $\bar{X}$ is hyperelliptic, then X is a degree 2 cover of a conic, which has infinitely many degree 2 points, and we are in case (1). If $\bar{X}$ is trigonal, then the associated $(3, 4)$ map onto $\mathbb{P}^1 \times Y_2$ has to be birational onto its image (since 4 and 3 are relatively prime), contradicting $g(X) = 7$. Therefore $\bar{X}$ is bielliptic, so there is a degree 2 covering $\phi : \bar{X} \to C$, for an elliptic curve C. Consider the composite map $\bar{X} \to C \times Y_2$. Since X has genus 7, it cannot be birational to a $(2, 4)$ curve (genus would be at most 5) on $C \times Y_2$, therefore the morphism $\bar{X} \to Y_2$ factors through ϕ. Similarly, the morphism $\bar{X} \to Y_1$ factors through ϕ, contradicting the birationality of $X \to Z$. □

Motivated by Theorem 3.5 we make the following definition.

Definition 3.6. Suppose X/k is a curve with $\min(\delta(X/k)) = d$. We say that X is d-minimal if there does not exist a covering of curves $\pi : X \to Y$ of degree at least 2 such that $\min(\delta(Y/k)) \deg \pi = d$.

The problem of understanding the minimum density degree is reduced, by Theorem 3.5, to analyzing the geometry of d-minimal curves. Note that Theorem 3.5 already gives us some control over this geometry: since a d-minimal curve X has a birational embedding of degree $2d$, the genus of X is bounded by $(d - 1)(2d - 1)$. We will prove a stronger genus bound in Theorem 4.10 below.

In our analysis of d-minimal curves X we occasionally need to use hyperbolicity of X; the following lemma allows us to do so.

Lemma 3.7. *Suppose X is a d-minimal curve with $d \geqslant 2$. Then the genus of X is at least 3.*

Proof. If X has genus zero, then it is isomorphic to a plane conic. If $X(k) = \emptyset$, then projection from a rational point on the plane defines a degree 2 map $X \to \mathbb{P}^1$ and $\min(\delta(X/k)) = 2$. If $X(k) \neq \emptyset$, then $X = \mathbb{P}^1$ and X is 1-minimal.

If X has genus 1 and infinitely many rational points, then X is 1-minimal. Otherwise, by Riemann–Roch, if D is a rational degree $d \geqslant 2$ divisor on X, then $\dim H^0(X, \mathcal{O}(D)) \geqslant 2$, and so X is a degree d cover of $\mathbb{P}^1$.

If X has genus 2, then $\min(\delta(X/k)) \geq 2$ by Faltings' Theorem. On the other hand, the canonical linear series exhibits that X is a degree 2 cover of $\mathbb{P}^1_k$, for which $\min(\delta(\mathbb{P}^1/k)) = 1$. □

The birationality of $|2D|$ on a d-minimal curve implies a classification of 2-minimal curves. The resulting theorem is an arithmetic strengthening of [9, Corollary 3], which proves that any curve with $\min(\delta(C/k)) = 2$ is a degree 2 cover of a genus 0 or 1 curve.

Theorem 3.8. *There are no* 2*-minimal curves over any number field.*

Proof. Suppose to the contrary that we are in Setup 2.1 and X is a 2-minimal curve. By Theorem 3.5, for a general divisor $D \in A$ the linear system $|2D|$ is birationally very ample. A nondegenerate degree 4 curve in $\mathbb{P}^n$ for $n \geqslant 3$ has genus at most 1. By Lemma 3.7 we can assume that for a general D the linear system $2D$ realizes X as a plane quartic $Y_D \subset \mathbb{P}^2$.

If Y_D is smooth, then $X = Y_D$ is a canonical genus 3 curve. In particular $2D = K_X$. Since D was general, we can assume $2D \neq K_X$. If Y_D is singular, its geometric genus (the genus of X) is at most 2, so X cannot be d-minimal for any $d \geq 2$ by Lemma 3.7. □

4. Subspace configurations

We will analyze the geometry of d-minimal curves by studying structures ("subspace configurations") associated to the birational linear systems $|nD|$, where $n \geqslant 2$ is an integer and D is a degree d point on X. We first establish notation for and basic properties of these objects, building to a proof of Theorem 1.3. We summarize the discrete-geometric structure of these subspace configurations in Section 4.2.

From now on we use notation of Setup 2.1 and assume additionally that X is d-minimal.

4.1. Geometric considerations

Given an abelian translate $A \hookrightarrow W_d X$, the tensor product map on line bundles gives a map

$$\underbrace{A \times A \times \cdots \times A}_{n} \to W_{nd}X,$$

whose image $A^{(n)}$ is (noncanonically) isomorphic to A (since we assumed A is a trivial torsor). Every divisor in $A^{(n)}$ is (geometrically) of the form nD for some $D \in A$. By Theorem 3.5 the linear system $|nD|$, for $n \geqslant 2$, is birationally very ample. By upper-semicontinuity of dimensions of global sections, there is an open subset of D in A with the same (minimal) value of $\dim |nD|$; we denote this minimal value by $r(n)$ (so in fact $A^{(n)} \subset W_{nd}^{r(n)} X$.)

Given any divisor D' on X, there is an evaluation map

$$H^0(X, nD) \xrightarrow{\mathrm{ev}_{D'}} \mathcal{O}(nD)_{D'},$$

whose kernel is identified with the space of sections vanishing along D'. If we let D' vary among the divisors parameterized by A, the dimension of kernel is an upper-semicontinuous and achieves a generic value on an open subset of A. We write $s(n)$ for $r(n)$ minus this generic dimension of $h^0(X, nD - D')$ as D' varies over the divisors parameterized by A.

The number $s(n)$ has a geometric interpretation in terms of the map to projective space given by $|nD|$. We will write $\operatorname{Span}_{|nD|}(D')$ for the linear span of the images of the points of D' under the map $|nD|$. Then $\operatorname{Span}_{|nD|}(D')$ is a projective space of dimension at most $s(n)$. For $D' \in A$ general, $\operatorname{Span}_{|nD|}(D')$ has dimension exactly $s(n)$. (When the linear system nD is unambiguous, we will implicitly write $\operatorname{Span}(D')$.) The abelian translate property from Section 2.1 in this geometric language says that for any collection of $n-1$ divisors $D_1, \ldots, D_{n-1}$ from A, there exists a divisor D_n such that their spans $\operatorname{Span}_{|nD|}(D_1), \ldots, \operatorname{Span}_{|nD|}(D_n)$ in $|nD|$ are contained in a common hyperplane.

Lemma 4.1. *Let X be d-minimal. Suppose that D_1 and D_2 are general divisors from A and that D is an independently general divisor from A.*

- *If $n \geqslant 3$, then $X \cap \operatorname{Span}_{|nD|} D_1 = D_1$.*
- *If $n = 2$, then $X \cap \operatorname{Span}_{|2D|} D_1 = D_1 \sqcup (2D - D_1)$.*

In particular, $D_2 \cap \operatorname{Span}_{|nD|} D_1 = D_1 \cap D_2$.

Proof. First suppose that $n \geqslant 3$. Since D is general, the line bundle $nD - D_1$ is basepoint free by Proposition 3.3. On the other hand, any point of $(\operatorname{Span}_{|nD|} D_1) \cap D_2$ that is not a point of D_1 would be a basepoint of $nD - D_1$.

Now suppose $n = 2$. Since D_1, D are a general pair, the space $\operatorname{Span}_{|2D|} D_1$ is a hyperplane, for otherwise the projection from a codimension 2 space containing $\operatorname{Span}_{|2D|} D_1$ is a degree d (or less) map from X to $\mathbb{P}^1$. Since $D_1 + (2D - D_1) = 2D$, the hyperplane section $X \cap \operatorname{Span}_{|2D|} D_1$ equals $D_1 \sqcup (2D - D_1)$. □

By definition, $r(n) - r(n-1) = s(n) + 1$. The difference $s(n) - s(n-1)$ also has a geometric interpretation.

Lemma 4.2. *We have $s(n) - s(n-1) = \lambda + 1$, where $\lambda = \dim(\operatorname{Span}_{|nD|}(D_1) \cap \operatorname{Span}_{|nD|}(D_2))$, for general $D_1, D_2 \in A$.*

Proof. Since D_1 and D_2 are general, we have $D_1 \cap D_2 = \emptyset$ by (2), and so Lemma 4.1 guarantees that the projection of $\operatorname{Span}_{|nD|} D_2$ from $\operatorname{Span}_{|nD|} D_1$ is $\operatorname{Span}_{|nD-D_1|} D_2$. Since D, D_1, D_2 are general, we have $s(n) = \dim \operatorname{Span}_{|nD|} D_2$ and $s(n-1) = \dim \operatorname{Span}_{|nD-D_1|} D_2$. □

We want to analyze the geometry of the configuration of $\operatorname{Span} D'$ in $|nD|$ for various $n \geqslant 2$. It will be convenient to project from a maximal subspace that is common to $\operatorname{Span} D'$ for almost all D'; to formalize this we make the following definition.

Definition 4.3 *(Definition/Notation).* For a dense open subset $W \subset A$, let

$$V_{|nD|,W} := \bigcap_{D' \in W} \operatorname{Span}_{|nD|} D'.$$

Let $V_{|nD|}$ be the maximal subspace of the form $V_{|nD|,W}$ as W varies over dense opens in A. Explicitly, $V_{|nD|} = V_{|nD|,W}$ for W the locus of D' where $\operatorname{Span} D'$ has the maximal dimension $s(n)$.

Lemma 4.4. *Suppose X is a d-minimal curve and $D \in A(k)$ is a general divisor. Then the codimension of $V = V_{|2D|}$ in $|2D|$ is at least* 3.

Proof. Suppose that to the contrary the codimension of V is equal to 2. The projection from V defines a morphism $\pi_V : X \to \mathbb{P}^1$ of degree at most $2d$. Since for a general divisor D', $\operatorname{Span}_{|2D|} D'$ is contained in a hyperplane and contains V, a general divisor $D' \in A$ is contracted to a point by π_V. In particular the divisors from A vary in a one-dimensional family, and so $\dim A = 1$. Moreover, since D' and $2D - D'$ belong to the same hyperplane, $\pi_V(D') = \pi_V(2D - D')$. For general $D' \in A$, the divisors D' and $2D - D'$ don't share points by (2), and so the degree of π_V equals $2d$. By Lemma 3.2 this implies that X is a degree d cover of the elliptic curve A, which contradicts our assumption of d-minimality. □

In Lemma 2.2 we showed that when X is d-minimal, a general point P on X is contained in at least two distinct divisors from A. It will be convenient for us to consider separately the case when a general point P in X is the intersection of exactly two divisors from A. We refer to this property as condition (†), formalized as follows:

$$\text{For a general point } P \in X \text{ there exists a pair of divisors } F, F' \in A \text{ such that } F \cap F' = P \qquad (\dagger)$$

We do not know of examples in which condition (†) fails for a d-minimal curve X. We have the following sufficient condition for (†):

Lemma 4.5. *Suppose X is d-minimal and $r(2) = 2$. Suppose $D \in A$ is general, $x \in D$ is a point, and D' is a general divisor containing x. Then $D \cap D' = \{x\}$. In particular, condition (†) holds.*

Proof. Choose a general divisor E disjoint from D and D'. The pair $(x, D+E)$ is a general point of $X \times A^{(2)}$. In particular, x is a smooth point on the image of $X \subset |D+E| \simeq \mathbb{P}^2$. The span of D in $|D+E|$ is a line ℓ that intersects the curve in $D \cup E$. The span of D'

is a line ℓ' that does not equal ℓ since $D' \not\subset D \cup E$. A pair of distinct lines shares exactly one point, and so $D \cap D' \subset \ell \cap \ell' = \{x\}$. □

Under condition (†) the linear configuration of $\operatorname{Span} D'$ in $|nD|$ has interesting incidence structure, as we show in Proposition 4.7. We first need to prove the following linear nondegeneracy property of $D \subset \operatorname{Span} D$.

Lemma 4.6. *Suppose $n \geqslant 2$ is an integer such that $s(n) \leqslant d-2$. Let $D \in A$ be a general divisor. Then for a general divisor $D' \in A$ and any point $x \in D'$ we have $\operatorname{Span}_{|nD|}(D' \setminus \{x\}) = \operatorname{Span}_{|nD|} D'$.*

Proof. Suppose that for a general divisor D' there is a point $x \in D'$ such that the set $D' \setminus \{x\}$ is contained in a hyperplane inside $\operatorname{Span} D'$. Choose a divisor $D' \in U(k)$ such that the Galois group G_k acts transitively on D' and the complement of a point $x \in D'$ belongs to a hyperplane $H \subset \operatorname{Span} D'$. By transitivity of the Galois action, for every $x \in D'$ there exists a hyperplane H_x that contains $D' \setminus \{x\}$. Choose points $x_1, \dots, x_{s(n)+1} \in D'$ that span $\operatorname{Span} D'$. Since $s(n)+1 \leqslant d-1$ there is a point $x \in D'$ such that $x \neq x_i$. Then H_x would contain the points $x_1, \dots, x_{s(n)+1}$, so H_x contains their span $\operatorname{Span} D'$. This is a contradiction. □

Proposition 4.7. *Suppose X is a d-minimal curve, condition (†) holds, and $n \geqslant 2$ is an integer. Suppose that for a general $D' \in A$, we have $\dim \operatorname{Span}_{|nD|} D' \leqslant d-2$. Then for a general pair of divisors $D_1, D_2 \in A$ we have $\operatorname{Span}_{|nD|} D_1 \cap \operatorname{Span}_{|nD|} D_2 \neq V_{|nD|}$.*

Proof. If $n = 2$, $D' \in A$ is general, and $\operatorname{Span}_{|2D|} D'$ is not a hyperplane, then projection from $\operatorname{Span}_{|2D|} D'$ defines a morphism from X to a positive-dimensional projective space of degree at most d. Therefore $\operatorname{Span}_{|2D|} D'$ is a hyperplane, and so for a general pair D_1, D_2, we have that $\operatorname{Span}_{|2D|} D_1 \cap \operatorname{Span}_{|2D|} D_2$ has codimension 2. Since $V_{|2D|}$ has codimension at least 3 by Lemma 4.4, the conclusion holds.

Assume for the remainder of the proof that $n \geqslant 3$. Consider the linear system $(n-1)E$ for a general $E \in A$ and another general divisor $F \in A$. Choose a point $x \in F$ and a divisor F' such that $F \cap F' = \{x\}$; this is possible since (†) holds. Since F was general, F' is general as well (although the pair F, F' is not general). In particular $\dim \operatorname{Span}_{|(n-1)E|} F' = s(n-1)$. Consider the linear system $|(n-1)E + F|$. Since E and F are general, $|(n-1)E + F| = |nD|$ for a general D and F, D form a general pair. The points of F do not belong to $V_{|nD|}$ (for example, by Lemma 4.1). Therefore, both $\operatorname{Span}_{|nD|} F$ and $\operatorname{Span}_{|nD|} F'$ contain the point x, which is outside of $V_{|nD|}$, so

$$\operatorname{Span}_{|nD|} F \cap \operatorname{Span}_{|nD|} F' \neq V_{|nD|}.$$

We have $F \cap F' = x$ by construction. Considering the projection π from $\operatorname{Span}_{|nD|} F$, we have

$$\begin{aligned}\dim \pi(\operatorname{Span}_{|nD|} F') = &\dim \operatorname{Span}_{|nD|} F' \\ &- \dim(\operatorname{Span}_{|nD|} F \cap \operatorname{Span}_{|nD|} F') - 1 \\ < &s(n) - \dim V_{|nD|} - 1. \end{aligned} \tag{3}$$

On the other hand,

$$\begin{aligned}\pi(\operatorname{Span}_{|nD|} F') &= \pi(\operatorname{Span}_{|nD|}(F' \setminus \{x\})) && \text{(Lemma 4.6)} \\ &= \operatorname{Span}_{|nD-F|}(F' \setminus \{x\}) && \text{(Lemma 4.1)} \\ &= \operatorname{Span}_{|(n-1)E|}(F' \setminus \{x\}) \\ &= \operatorname{Span}_{|(n-1)E|} F' && \text{(Lemma 4.6).}\end{aligned} \tag{4}$$

Combining (3) and (4) we see

$$s(n-1) = \dim \operatorname{Span}_{|(n-1)E|} F' < s(n) - \dim V_{|nD|} - 1.$$

Therefore, by Lemma 4.2, the intersection of a general pair of divisor spans is larger than $V_{|nD|}$. □

We are now in the position to analyze the geometry of linear systems obtained by projecting $|nD|$ from the subspace $V_{|nD|}$. To do so we introduce the following definition.

Definition 4.8. Suppose $n \geqslant 2$ is an integer. We denote by $|nD|'$ the linear system obtained from $|nD|$ by projection from $V_{|nD|}$. Similarly, let $r'(n) = \dim |nD|'$ and $s'(n) = \dim_{|nD|'} \operatorname{Span} D'$ for general $D, D' \in A$.

Proposition 4.7 immediately implies:

Corollary 4.9. *Suppose (†) holds. Then we have $s'(n) \geqslant \min(s(n-1)+1, d-1)$.*

Proof. Suppose $s'(n) \leqslant d-2$. By Proposition 4.7, for general $D_1, D_2 \in A$ the spaces $\operatorname{Span}_{|nD|'} D_1$, $\operatorname{Span}_{|nD|'} D_2$ share a point. Since D_1, D_2 are general, no point of D_2 belongs to $\operatorname{Span}_{|nD|'} D_1$ by Lemma 4.1. Projecting from $\operatorname{Span}_{|nD|'} D_1$ we get

$$\dim \operatorname{Span}_{|nD|'} D_2 \geqslant \dim \operatorname{Span}_{|nD-D_1|} D_2 + 1.$$

Therefore $s'(n) \geqslant s(n-1)+1$. □

Theorem 4.10. *Suppose that we are in Setup 2.1 and X is d-minimal. Suppose (†) holds. Then for a general divisor $D \in A(k)$ and every number $n \leqslant d$ we have*

$$\dim |nD| \geqslant \dim |nD|' \geqslant \frac{n(n+1)}{2} - 1.$$

Proof. By Lemma 4.4, we have $r'(2) \geqslant 2$ and $s'(2) \geqslant 1$. Combining $s'(2) \geqslant 1$ with Corollary 4.9, we have $s(n) \geqslant s'(n) \geqslant \min(d-1, n-1)$. Therefore for $2 < n \leqslant d$,

$$\begin{aligned} r'(n) &= (s'(n)+1) + r(n-1) \\ &= (s'(n)+1) + (s(n-1)+1) + \cdots + (s(3)+1) + r(2) \\ &\geqslant \frac{n(n+1)}{2} - 1. \quad \square \end{aligned}$$

Theorem 4.10 can be used to bound the genus of curves for which condition (†) holds. When (†) does not hold, we use the following lemma instead.

Lemma 4.11. *Suppose we are in Setup 2.1 and X is d-minimal. Suppose $r(2) \geqslant 3$ and $d \geqslant 4$. Then $r'(3) \geqslant 7$.*

Proof. Because X does not admit a degree d map to $\mathbb{P}^1$, we have $s(2) = r(2) - 1 \geqslant 2$. We also have $s'(3) \geqslant s(2)$, and $r'(3) = s'(3) + r(2) + 1$. Therefore the only case in which $r'(3) = 6$ is $r(2) = 3$, $s'(3) = s(2) = 2$.

By Lemma 2.2, for a general point $P \in X$ there exists a pair of divisors $D_1, D_2 \in U$, $D_1 \neq D_2$ both containing P. We can assume that both D_1, D_2 are general divisors. If condition (†) holds, then $s'(3) > s(2)$ by Corollary 4.9, contradicting our calculation that $s'(3) = s(2) = 2$ above. We may therefore assume that $D_1 \cap D_2$ contains at least 2 points, but that there exists some point $y \in D_2 \smallsetminus D_1$. Choose a general divisor $D \in U$. By Lemma 4.1, the point y is in $\operatorname{Span}_{|2D|} D_2$ but not in $\operatorname{Span}_{|2D|} D_1$. Therefore the planes $\operatorname{Span}_{|2D|} D_1$ and $\operatorname{Span}_{|2D|} D_2$ intersect along a line.

By Lemma 4.6 the intersection $D_1 \cap D_2$ contains at most $d-2$ points. Consider now the embedding of X into $\mathbb{P}^6$ given by the linear system $|2D + D_2|'$. Since D was general, the spans of D_1 and D_2 in this linear system have dimension $s'(3) = 2$. However $\operatorname{Span}_{|2D+D_2|'} D_1$ and $\operatorname{Span}_{|2D+D_2|'} D_2$ share at least 2 points, and therefore intersect along a line, again applying Lemma 4.1. Therefore the projection from $\operatorname{Span}_{|2D+D_2|'} D_2$ maps all points of $D_1 \smallsetminus D_2$ onto a single point in $|2D|$. Since $D_1 \smallsetminus D_2$ contains at least 2 points, this is a contradiction. $\square$

A similar argument can be used to improve the estimate for the value of $r(4)$; it will be useful in our considerations of low values of d.

Lemma 4.12. *Suppose we are in Setup 2.1 and X is d-minimal, and $d \geqslant 5$ is odd. Suppose (†) does not hold. Then $r'(4) \geqslant 12$.*

Proof. Since (†) does not hold, we have $r(2) \geqslant 3$ by Lemma 4.5. Because X does not admit a degree d map to $\mathbb{P}^1$, we have $s(2) = r(2) - 1$. Furthermore, by considering projections from divisor spans, we see that $s'(n) \geqslant s(2)$ for all $n > 2$, and that $r'(4) \geqslant r(2) + 2s(2) + 2$. Combining these, if $r(2) \geqslant 4$, then $r'(4) \geqslant 3r(2) \geqslant 12$. It therefore suffices to consider the case $r(2) = 3$. In this case $r'(3) \geqslant 7$ by Lemma 4.11, and so $s(4) \geqslant s(3) \geqslant 3$. Again considering projection, we have

$$r'(4) = s'(4) + 1 + r(3) \geqslant s(3) + 1 + r'(3) \geqslant 3 + 1 + 7 = 11.$$

Therefore the only way to have $r'(4) < 12$ is to have equality everywhere, and hence $r(2) = 3$, $r(3) = r'(3) = 7$, $s(3) = s'(4) = 3$, and $r'(4) = 11$.

Consider a general pair of divisors $D, D_1 \in A$. Suppose $D_2 \in A$ is a divisor that shares points with D_1. Note that D_2 is a general divisor in A since D_1 is general, and moreover, since (†) does not hold, we can assume that $D_1 \cap D_2$ contains at least 2 points.

Consider the linear system $|3D| = |3D|'$ and the subspaces $\operatorname{Span}_{|3D|} D_1$ and $\operatorname{Span}_{|3D|} D_2$. By assumption these are both 3-dimensional. Since $\operatorname{Span}_{|3D|} D_1$ and $\operatorname{Span}_{|3D|} D_2$ have nontrivial intersection but do not coincide by Lemma 4.1, Lemma 4.6 implies that $D_1 \setminus D_2$ contains at least 2 points. Suppose $\dim \operatorname{Span}_{|3D|} D_1 \cap \operatorname{Span}_{|3D|} D_2 = 2$. Projecting from $\operatorname{Span}_{|3D|} D_2$ we see that in the linear system $|3D - D_2|$, the points of $D_1 \setminus D_2$ map to a single point. Since D_1 and $|3D - D_2|$ is a general pair of divisors, this is a contradiction. Therefore $\dim \operatorname{Span}_{|3D|} D_1 \cap \operatorname{Span}_{|3D|} D_2 = 1$, and so all points of $D_1 \cap D_2$ in $|3D|$ belong to a single line.

Consider the linear system $|3D + D_2|'$. Since D is general, $3D + D_2$ is a general point of $A^{(4)}$. By assumption the subspaces $\operatorname{Span}_{|3D+D_2|'} D_1$ and $\operatorname{Span}_{|3D+D_2|'} D_2$ are 3-dimensional, distinct, and meet in at least 2 points. Therefore the projection of $D_1 \setminus D_2$ from $\operatorname{Span}_{|3D+D_2|'} D_2$ maps to a space of dimension at most 1 in $|3D|$. Thus for a general divisor D, the image of D_1 in $|3D|$ is contained in a pair of skew lines each containing at least 2 points (since $D_1 \setminus D_2$ and $D_1 \cap D_2$ both contain at least 2 points). A nondegenerate set S of $d \geqslant 5$ distinct points in $\mathbb{P}^3$ is contained in at most one pair of skew lines with each line containing at least 2 points. Therefore the pair of lines $\operatorname{Span}_{|3D|}(D_1 \setminus D_2)$ and $\operatorname{Span}_{|3D|}(D_1 \cap D_2)$ are preserved by the Galois action on D_1, and, in particular, each line has to contain the same number of points of D_1. This contradicts the assumption that d is odd. □

We now prove Theorem 1.3 from the introduction.

Theorem 4.13. *Given an integer d, let $m := \lceil d/2 \rceil - 1$ and let $\varepsilon := 3d - 1 - 6m < 6$. Suppose X is a d-minimal curve. If (†) holds, then the genus of X is bounded by*

$$d(d-1)/2 + 1.$$

If (†) does not hold, then the genus is at most

$$3m(m-1) + m\varepsilon.$$

Proof. If (†) holds, this follows from Theorem 4.10 for $n = d$ and Castelnuovo's genus bound (see [1, Chapter III, page 116] for the proof of the bound, and Section 6 Equation

(6) for the statement). Alternatively, since any nondegenerate special curve in $\mathbb{P}^r$ has degree at least $2r$, the linear system $|dD|$ on X is nonspecial for $d > 2$ and the genus bound follows from Riemann–Roch.

If (†) does not hold, this is Castelnuovo's bound for a degree $3d$ curve in $\mathbb{P}^7$, which applies by Lemma 4.11. □

An immediate corollary of this bound is the theorem of Abramovich–Harris on degree 3 points on curves.

Corollary 4.14. *Suppose* $\min(\delta(X/k)) = 3$. *Then* $\bar{X}$ *is a degree* 3 *cover of* $\mathbb{P}^1$ *or an elliptic curve.*

Proof. If X is not 3-minimal the conclusion holds, so we can assume X is 3-minimal. Then by Theorem 4.13 the genus of X is at most 4. The geometric gonality of a curve of genus $g \leqslant 4$ is at most 3. □

4.2. Summary of setup and notation

We give a brief summary of the basic structures and properties introduced in the previous section. We fix a d-minimal curve X, and let $A \subset W_d X$ be a corresponding abelian variety with dense k-points. For every $D \in A$ and every integer $n \geqslant 2$ we consider the linear system $|nD|$ and the corresponding projective embedding of X. Within the projective space $\mathbb{P}^{|nD|}$, we look at the linear spaces of the form $\operatorname{Span}_{|nD|} E$ for all divisors $E \in A$. The resulting system of subspace configurations enjoys a number of unusual properties. We use $V = V_{|nD|}$ to denote the maximal subspace shared by all spaces $\operatorname{Span}_{|nD|} E$ for a Zariski open family of $E \in A$. Projecting from V defines the linear system $|nD|'$ on X equipped with a similar family of linear spaces $\operatorname{Span}_{|nD|'} E$. The basic properties of these structures are the following:

(1) The dimensions of $|nD|$ and $|nD|'$ have fixed values $r(n), r'(n)$ for a generic choice of $D \in A$;
(2) For general D, E the dimensions of $\operatorname{Span}_{|nD|} E$ and $\operatorname{Span}_{|nD'|} E$ have constant values $s(n), s'(n)$;
(3) If (†) holds, and n is such that $s(n) \leqslant d-2$, then for a general D and a general pair E_1, E_2 the subspaces $\operatorname{Span}_{|nD|'} E_1$ and $\operatorname{Span}_{|nD|'} E_2$ have nonempty intersection;
(4) The intersection of all subspaces $\operatorname{Span}_{|nD|'} E$ as E varies over any Zariski open subset in A is empty.
(5) (The abelian translate property) For any $D \in A$ and any divisors $E_1, \ldots, E_{n-1} \in A$ there exists a divisor $E_n \in A$ such that the subspaces $\operatorname{Span}_{|nD|'} E_i$ all belong to the same hyperplane;

(6) For general $D, E \in A$ the projection of X in $|nD|$ from $\operatorname{Span}_{|nD|} E$ is equivalent to the embedding given by $|nD - E|$; in particular we have the identities $r(n) - s(n) = r'(n) - s'(n) = r(n-1) + 1$.

The presence of these linear configurations allows us to give various restrictions on the geometry of the curve X. In the next section we will use this structure to identify the curves with $r(2) = 2$ with the curves constructed by Debarre and Fahlaoui [3].

5. Debarre–Fahlaoui curves

Let A be a positive rank elliptic curve over k. For all $d \geqslant 4$, Debarre and Fahlaoui give examples of d-minimal curves lying on the smooth surface $\operatorname{Sym}^2 A$. We first recall their construction, and then we show that any d-minimal curve with $r(2) = 2$ naturally arises in this way. This shows that the simplest class of d-minimal curves is the one provided by the Debarre–Fahlaoui construction.

We begin by recalling the setup from [3, Section 4.1]. The addition law on A induces a natural map $\pi \colon \operatorname{Sym}^2 A \to A$. Let $o \in A(k)$ be the origin, and let $\mathscr{E}$ be the unique nonsplit extension

$$0 \to \mathcal{O}_A \to \mathscr{E} \to \mathcal{O}_A(o) \to 0.$$

Then the fibration $\operatorname{Sym}^2 A \xrightarrow{\pi} A$ is isomorphic to $\mathbb{P}\mathscr{E} \to A$. Let H denote the relative $\mathcal{O}_{\mathbb{P}\mathscr{E}}(1)$. Then we have

$$\operatorname{Pic}(\operatorname{Sym}^2 A) \simeq \pi^* \operatorname{Pic}(A) \oplus \mathbb{Z}H.$$

We will write F_x for the divisor $\pi^*\mathcal{O}_A(x)$; in terms of the moduli description of $\operatorname{Sym}^2 A$, this consists of all degree 2 effective divisors on A that sum to x under the group law. The divisors F_x for all $x \in A$ are numerically equivalent, and we simply write F for this numerical class. Another natural divisor on $\operatorname{Sym}^2 A$, which we denote H_x consists of all effective divisors of degree 2 on A that contain x. The rational equivalence class of this divisor is $H_x = H - F_o + F_x$ [3, Section 4.1 (ii)]. (In particular, $H_o = H$.) The numerical classes of divisors are spanned by H and F, with the following intersection relations:

$$H^2 = 1, \qquad H \cdot F = 1, \qquad F^2 = 0.$$

The canonical class K on $\operatorname{Sym}^2 A$ has numerical class $K = -2H + F$. The Nef and effective cones both consist of all classes $aH + bF$ where $a \geqslant 0$ and $a + 2b \geqslant 0$ [7, Chapter V, Proposition 2.21].

Definition 5.1. A Debarre–Fahlaoui curve is a geometrically integral curve on $\operatorname{Sym}^2 A$ in numerical class

$$(d+m)H - mF,$$

for some $1 \leqslant m \leqslant d$.

This terminology comes from the fact that Debarre and Fahlaoui consider the case $m = 1$ in [3] to give counterexamples to the conjecture of Abramovich–Harris [2, page 229]. Let us recall this construction.

Let X be a Debarre–Fahlaoui curve. The family of divisors H_x on $\operatorname{Sym}^2 A$ restricts to a family of degree d effective divisors on X parameterized by A, since $H \cdot ((d+m)H - mF) = d$. This family gives rise to an embedding

$$\psi\colon A \hookrightarrow W_d X.$$

This family of degree d divisors is *not* induced by a map $X \to A$: if $H_x \cdot X$ contains the degree 2 effective divisor $[x + x']$, then so does $H_{x'} \cdot X$; as such these cannot be (the necessarily disjoint) fibers of a map.

Proposition 5.2 *([3, Propositions 5.7 and 5.14]). Let $d \geqslant 4$ and $1 \leqslant m \leqslant d$ be integers. Consider the numerical class $(d+m)H - mF$.*

(1) If $m < d/2$, then for any nice curve X in this class, we have $\operatorname{gon} \bar{X} > d$.
(2) If the class of X is very ample (e.g., if $m = 1$), then a general curve in this class admits no nontrivial maps of degree at most d to a non-isomorphic curve of genus at least 1.

In particular, under both of these assumptions, such a curve X is d-minimal.

We now turn to d-minimal curves with $r(2) = 2$. We will analyze the geometry of such curves by looking at the induced subspace configurations in $|3D|'$. We begin by showing, in Lemma 5.5, that this structure is a configuration of 2-planes in a 5-space; in other words $r'(3) = 5$ (which forces $s'(3) = 2$).

Lemma 5.3. *Let $V_i \subset \mathbb{P}^n, i \in I$ be a collection of codimension 2 subspaces of $\mathbb{P}^n$ spanning all of $\mathbb{P}^n$. Suppose that for any $i, j \in I$, the subspaces V_i, V_j belong to a common hyperplane. Then there is a codimension 3 subspace Λ that belongs to V_i for all $i \in I$.*

Proof. Since V_i have codimension 2, if V_i, V_j are two distinct subspaces, then $\dim \operatorname{Span}(V_i, V_j) = n - 1$, and so $\dim V_i \cap V_j = n - 3$. Choose a subspace V_k such that V_k does not belong to $\operatorname{Span}(V_i, V_j)$. Then $\dim V_k \cap \operatorname{Span}(V_i, V_j) = n - 3$, and on the other hand V_k intersects each of V_i and V_j in a subspace of dimension $n - 3$. Therefore V_k contains $V_i \cap V_j$. Finally take any subspace V_ℓ, $\ell \neq i, j, k$. Then V_ℓ does not belong to $\operatorname{Span}(V_u, V_w)$ for some $u, w \in \{i, j, k\}$ and by the previous argument V_ℓ contains $V_u \cap V_w = V_i \cap V_j$. Thus $\Lambda = V_i \cap V_j$ satisfies the conclusion of the lemma. □

Lemma 5.4. *Let X be a d-minimal curve with $r(2) = 2$. Suppose that $P \in X$ is a general point, and $D \in A$ is general. Then there exists a pair of distinct divisors D_1, D_2 from A through P such that* $\operatorname{Span}_{|3D|'} D_1$ *and* $\operatorname{Span}_{|3D|'} D_2$ *span a hyperplane in* $|3D|'$.

Proof. By Lemma 2.2 there exists a pair of distinct divisors D_1, D_2 through a general point of X, and moreover each D_i individually is general in A. Choose such a pair. Recall that since $r(2) = 2$ and $r'(3) - r(2) = s'(3) + 1$, the spaces $\operatorname{Span}_{|3D|'} D_i$ have codimension 3 in $|3D|'$. By Lemma 4.5, we may assume that D_1 and D_2 meet only at P.

Consider the divisor $E \in A$ such that $3D - D_1 = 2E$; such E is a general point of A, and in particular $|2E|$ is an embedding along each D_i. Note that $\operatorname{Span}_{|3D|'}(D_1 + D_2)$ has codimension either 1 or 2. If $\operatorname{Span}_{|3D|'}(D_1 + D_2)$ has codimension 2, then the projection π from $\operatorname{Span}_{|3D|'} D_1$ sends the points of $D_2 \setminus D_1$ to the same point in $|2E|$. The set $D_2 \setminus D_1$ contains at least $d - 1$ points, contradicting the assumption that $|2E|$ is an embedding along D_2. □

Lemma 5.5. *Suppose $r(2) = 2$. Then $r'(3) = 5$.*

Proof. We will consider the configuration of divisor spans in $|3D|'$. Since $r(2) = 2$, we have $s'(3) = r'(3) - r(2) - 1 = r'(3) - 3$. Since two general divisor spans span a hyperplane by Lemma 5.4, their intersection has codimension 5. Furthermore, $r'(3) \geqslant 5$ by Lemma 4.5 and Theorem 4.10. Suppose $r'(3) \geqslant 6$.

Let E be a general divisor in A. Consider a general collection of divisors $D_1, \dots, D_N$ in A. The spaces $W_i := \operatorname{Span} D_i \cap \operatorname{Span} E$ form a collection of distinct codimension 2 subspaces of $\operatorname{Span} E$, any two of which span a hyperplane $\operatorname{Span}(D_i + D_j) \cap \operatorname{Span} E$. For N large enough, the intersection of all W_i is empty (since we are working in $|3D|'$). Thus by Lemma 5.3, there is a hyperplane Λ in $\operatorname{Span} E$ containing all W_i. Since N was arbitrarily large, for a general $D' \in A$ we have $\operatorname{Span} D' \cap \operatorname{Span} E \subset \Lambda$. Being contained in Λ is a closed condition; therefore every divisor D' for which the codimension of $\operatorname{Span} D' \cap \operatorname{Span} E$ in $\operatorname{Span} E$ is 2 satisfies $\operatorname{Span} D' \cap \operatorname{Span} E \subset \Lambda$. This contradicts Lemma 5.4 applied to D, $D_1 = E$ and a point P of E outside Λ. □

We will relate curves with $r(2) = 2$ to Debarre–Fahlaoui curves as follows. If $r'(3) = 5$, then the resulting configuration of divisor spans in $\mathbb{P}^5$ is a family of 2-planes, parametrized by A, pairwise sharing points. This will naturally gives rise to a rational map $\psi\colon \operatorname{Sym}^2 A \to \mathbb{P}^5$ sending a pair of divisors to the intersection of their spans. Since there are at least two divisors from A through every point on X we expect X to be in $\psi(\operatorname{Sym}^2 A)$. In this way X "wants to be" a curve on $\operatorname{Sym}^2 A$.

To realize this idea, we first need to reduce to the case $\dim A = 1$ (in Lemma 5.6) and establish a nondegeneracy property of our configuration (Lemma 5.7).

Lemma 5.6. *Suppose that X is d-minimal and $\dim A > 1$. Then $r(2) > 2$.*

Proof. By Proposition 3.3 we have $r(2) \geqslant \dim A$. Thus it suffices to show that the case $\dim A = r(2) = 2$ does not occur. We consider the cases $d = 3$ and $d \geqslant 4$ separately.

Suppose $d \geqslant 4$. Choose a general divisor $D \in A$ and consider the rational map $\phi : A \to (\mathbb{P}^2)^\vee$ that sends a divisor class E to the line $\operatorname{Span}_{|2D|} E$.

The set of $2d$ points on a general linear section of $X \subset \mathbb{P}^2$ is thereby equipped with a nonempty collection of d-element subsets coming from A. But the monodromy of the linear section is the symmetric group (see, for example, [1, Lemma, Chapter III, page 111]), and so every d-element subset of a general linear section of X is a divisor from A. Therefore, for a general divisor $E \in A$, there exists another divisor $E' \in A$ such that both $E \cap E'$ and $E' \setminus E$ consist of at least two points. Choose a general E and consider the linear system $|2D + E|'$. By Lemma 5.5, we have $\dim |2D + E|' = 5$ and $\dim \operatorname{Span}_{|2D+E|'} E = \dim \operatorname{Span}_{|2D+E|'} E' = r'(3) - r(2) - 1 = 2$. Since E and E' share at least two points, the 2-planes $\operatorname{Span}_{|2D+E|'} E$ and $\operatorname{Span}_{|2D+E|'} E'$ share a line ℓ. Therefore the projection of $\operatorname{Span}_{|2D+E|'} E'$ from E is a single point. Therefore all the points of $E' \setminus E$ are mapped to the same point under $2D$, which is a contradiction.

Suppose now that $d = 3$. Consider the linear system $|3D|'$ and the associated embedding of X in $\mathbb{P}^5$. Consider a general point $P \in X$ and a general pair of divisors D_P, D'_P through P. Such a pair does not share any points on X except for P by Lemma 4.5. By Lemma 5.4 we have

$$\dim \operatorname{Span}_{|3D|'} D_P \cap \operatorname{Span}_{|3D|'} D'_P = 0.$$

Choose a pair of general points $P, Q \in X$. Since the pair is general, projection from the line $\ell = PQ$ realizes X as a degree 7 curve in $\mathbb{P}^3$. The projection from ℓ maps the divisor spans that contain P or Q to lines in $\mathbb{P}^3$. Since P, Q are a general pair of points, a general divisor D_P containing P and a general divisor D_Q containing Q form a general pair of divisors (as P, Q vary), and so $\operatorname{Span}_{|3D|'} D_P$ and $\operatorname{Span}_{|3D|'} D_Q$ intersect at a point. By the above description of generic intersections, we can choose an infinite collection of divisors $D_P^1, D_P^2, \ldots$ and $D_Q^1, D_Q^2, \ldots$ such that the projections $\ell_i = \pi_\ell(\operatorname{Span}_{|3D|'} D_P^i)$ and $\ell'_i = \pi_\ell(\operatorname{Span}_{|3D|'} D_Q^i)$ form two families of lines with lines in each family pairwise skew, and $\ell_i \cap \ell'_j \neq \emptyset$ for all i, j. Such a pair of families is always contained in a smooth quadric. Since every line ℓ_i, ℓ'_i contains points from $\pi_\ell(X)$, the curve $\pi_\ell(X)$ shares infinitely many points with the quadric, and so belongs to the quadric. Therefore projection from PQ realizes X as a degree 7, (e_1, e_2)-curve on the quadric. As (P, Q) varies, the value of (e_1, e_2) achieves a generic value on an open subset of $\operatorname{Sym}^2 X$; by monodromy, for this generic value $e_1 = e_2$. However, X has degree $7 = e_1 + e_2$, contradiction. □

Lemma 5.7. *Suppose we are in Setup 2.1, X is d-minimal, and $r(2) = 2$. Consider a general triple of divisors $D, D_1, D_2 \in A$ and let $D_3 = 3D - D_1 - D_2$. Then*

$$\bigcap_i \operatorname{Span}_{|3D|'} D_i = \emptyset.$$

Proof. Suppose the spaces $\mathrm{Span}_{|3D|'} D_i$ share a common point P. Consider a divisor E such that D, D_1, D_2, E is a general quadruple; then $D_1, D_2, D_3, 3D - E$ is a general quadruple as well and P does not belong to $\mathrm{Span}_{|3D|'} E$. Consider the projection π_E from $\mathrm{Span}_{|3D|'} E$. We have

$$\pi_E(P) \in \pi_E(\mathrm{Span}_{|3D|'} D_i) = \mathrm{Span}_{|3D-E|} D_i.$$

By the generality of the quadruple $D_1, D_2, D_3, 3D - E$, the lines $\mathrm{Span}_{|3D-E|} D_i$ do not share a point, contradiction. □

We now prove the main result of this section.

Theorem 5.8. *Suppose we are in Setup 2.1 and X is d-minimal. If $r(2) = 2$, then X is birational to a Debarre–Fahlaoui curve.*

Proof. By Lemma 5.6, we have $\dim A = 1$. By Lemma 5.5, we have $r'(3) = 5$. Let D be a point of $A(k)$ achieving these generic values, so that the linear systems $|nD|$ are basepoint-free for $n \geqslant 2$, $\dim |2D| = 2$, and $\dim |3D|' = 5$.

Write $\varphi\colon X \to \mathbb{P}^5$ for the morphism associated to $|3D|'$. We will now define a rational map from $\mathrm{Sym}^2 A$ to $\mathbb{P}^5$, whose image contains X in its closure. Since $s'(3) = r'(3) - r(2) - 1 = 2$, a general divisor $[D_1] \in A$ has 2-dimensional span.

Given a general pair of divisors $[D_1], [D_2] \in A$, the spans $\mathrm{Span}_{|3D|'} D_1$, $\mathrm{Span}_{|3D|'} D_2$ belong to a common hyperplane (by the abelian translate property (5)). This means that for a general pair of divisors $[D_1], [D_2] \in A$, we must have $\dim \mathrm{Span}_{|3D|'} D_1 \cap \mathrm{Span}_{|3D|'} D_2 \geqslant 0$. Since by Lemma 5.4 there exists a pair of divisors with zero-dimensional intersection of spans, by semicontinuity we have in general $\dim \mathrm{Span}_{|3D|'} D_1 \cap \mathrm{Span}_{|3D|'} D_2 = 0$.

This yields a rational map

$$\begin{aligned} \psi\colon \mathrm{Sym}^2 A &\dashrightarrow \mathbb{P}^5 \\ (D_1, D_2) &\mapsto \mathrm{Span}\, D_1 \cap \mathrm{Span}\, D_2. \end{aligned}$$

If $\mathrm{Span}\, D_1 \cap \mathrm{Span}\, D_2$ has dimension 1, then projecting from $\mathrm{Span}(D_1 + D_2)$ yields a (geometric) degree d map $X \to \mathbb{P}^1$; in particular, by Setup 2.1, the divisor $[3D - D_1 - D_2] \in A$ lies in a proper Zariski closed (dimension at most 0) locus. From this we observe

For general D_1, there exist finitely many lines Σ in $\mathrm{Span}\, D_1$, such that for any $P \notin \Sigma$,

$$\text{if } D_2 \neq D_1 \text{ and } P \in \mathrm{Span}\, D_1 \cap \mathrm{Span}\, D_2, \text{ then } P = \mathrm{Span}\, D_1 \cap \mathrm{Span}\, D_2. \quad (5)$$

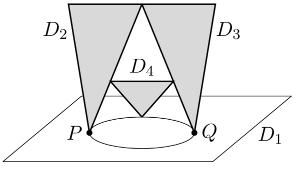
If the divisors D_1 and D_2 share a point, then the intersection of $\mathrm{Span}\, D_1$ and $\mathrm{Span}\, D_2$ necessarily contains that point. The closure of the image of ψ contains the image of X

under φ since by Lemma 5.4 for a general point P of $X(\bar{k})$, there exist divisors D_1 and D_2 such that

$$\operatorname{Span} D_1 \cap \operatorname{Span} D_2 = P.$$

Our goal is to show that a general point $P \in X$ is contained in *exactly* two divisors. Once we do so, we will have a natural map $X \to \operatorname{Sym}^2 A$, and we will then show via a simple argument that the image is indeed a Debarre–Fahlaoui curve. Fix a general divisor class D_1. We first analyze the image of the morphism $\eta : A \to \operatorname{Span} D_1$ that sends a divisor D_2 to $\operatorname{Span} D_2 \cap \operatorname{Span} D_1$. This map is nondegenerate by Lemma 5.4. We will show that it must be the inclusion of A as a plane cubic curve by considering several cases based on the possible degrees of the image of η.

Case 1: $\deg \eta(A) = 2$. Consider divisors D_2, D_3 such that D_1, D_2, D_3 form a general triple. Let $P := \operatorname{Span} D_2 \cap \operatorname{Span} D_1$ and $Q := \operatorname{Span} D_3 \cap \operatorname{Span} D_1$. Consider the divisor $D_4 := 3D - D_2 - D_3$; since D_1, D_2, D_3 are a general triple we can assume that D_4 does not pass through P or Q.



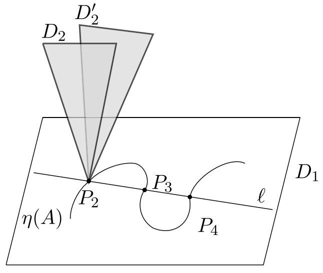
On the other hand, $\operatorname{Span}(D_2, D_3)$ is a hyperplane that contains $\operatorname{Span} D_4$ and intersects $\operatorname{Span} D_1$ in the line $\overline{PQ}$. Since $\overline{PQ}$ meets $\eta(A)$ only at P and Q, this is a contradiction.

Case 2: $\deg \eta(A) \geqslant 3$ and η multiple-to-one onto its image.

Choose a general line ℓ in $\operatorname{Span} D_1$ that intersects the image of η in at least three smooth points P_2, P_3, P_4. Through each of the points P_2, P_3, P_4 passes at least two divisor-spans $\operatorname{Span} D_i, \operatorname{Span} D_i'$, $i = 2, 3, 4$. Since the line ℓ is general, the pair D_1, D_2 is a general pair of divisors in $A \times A$. Hence, using the fact that η is nondegenerate, we can assume that the point where they meet is not in the finite set Σ guaranteed by (5) on either $\operatorname{Span} D_1$ or $\operatorname{Span} D_2$. In particular

$$\operatorname{Span} D_1 \cap \operatorname{Span} D_2 = \operatorname{Span} D_1 \cap \operatorname{Span} D_2' = \operatorname{Span} D_2 \cap \operatorname{Span} D_2' = P_2.$$

By symmetry the same holds for $i = 3$ and $i = 4$.

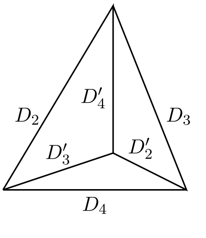


By Lemma 5.7 applied to the triple D, D_1, D_i we may choose $D_i' \neq 3D - D_1 - D_i$. Since D_1, D_2 is general, for every possible D_2' through P_2, the divisor $D_2 - D_2'$ is not a 2-torsion point on A^0, as we now explain. Equivalently, for a general D_2, and each of the finitely many possible 2-torsion points T on A^0 giving rise to $D_2' = D_2 - T$, a general divisor span $\operatorname{Span} D_1$ does not meet $\operatorname{Span} D_2 \cap \operatorname{Span} D_2'$, which is clear since the map η associated to D_2 is nondegenerate.

We may further assume that the 15 pairwise intersection points of $\operatorname{Span} D_i, \operatorname{Span} D_j'$ do not belong to X (if $i \neq j$ then D_i, D_j' are a general pair of divisors; if $i = j$, then P_i is a general point on $\eta(A)$ which contains only finitely many points of X).

Projection from ℓ maps each D_i and D_i' for $i = 2, 3, 4$ to a line. Since $D_i' \neq 3D - D_1 - D_i$, we have that $\operatorname{Span}(D_i + D_i')$, for $i = 2, 3, 4$, does not contain D_1. Further, since ℓ is a general line in $\operatorname{Span}(D_1)$ through P_i, we also have that $\operatorname{Span}(D_i + D_i')$ does not contain ℓ. Since in addition $\operatorname{Span} D_i$ and $\operatorname{Span} D_i'$ meet only at P_i, their images under projection from ℓ are skew. Moreover, since $\operatorname{Span} D_i \cap \operatorname{Span} D_j'$ is a point off of ℓ for $i \neq j$, any two such lines meet.

The only such configuration of a triple of pairs of skew lines $\pi_\ell(\operatorname{Span} D_i), \pi_\ell(\operatorname{Span} D_i')$ is the configuration of edges of a tetrahedron.



The triples of divisors corresponding to faces of the tetrahedron sum to $3D$, since they are coplanar. Summing the faces containing a shared edge and subtracting the two faces containing the opposite edge, we get $2(D_i - D_i') = 0$ for all i, contradicting the generality assumption $D_2 - D_2' \notin A[2]$.

Case 3: $\deg \eta(A) \geqslant 4$ and η birational onto its image. We proceed as before by choosing a general line $\ell \in \operatorname{Span} D_1$ and analyzing the projection from ℓ. Let $P_2, ..., P_n$, $n \geqslant 5$ be the points of $\ell \cap \eta(A)$. Since ℓ is general there is a unique divisor-span $\operatorname{Span} D_i$ through P_i. Projection from ℓ maps the $\operatorname{Span}(D_i)$ into a collection of lines in $\mathbb{P}^3$ pairwise sharing points. Since they cannot all belong to the same plane, by Lemma 5.3 they have to share a point P.

Consider two divisors D_i, D_j for $i, j > 2$ and let $D_{ij} := 3D - D_i - D_j$ be the remaining divisor contained in the hyperplane $\operatorname{Span}(D_i, D_j)$. Since (D_1, D_i, D_j) is general, $D_{ij} \neq D_1$; equivalently, D_1 is not contained in $\operatorname{Span}(D_i, D_j)$. In particular $\operatorname{Span}(D_i, D_j) \cap \operatorname{Span} D_1 = \ell$, and so $\operatorname{Span} D_{ij} \cap \operatorname{Span}(D_1) \in \ell$. Thus $D_{ij} = D_k$ for some index k. Projecting the configuration of lines $\pi_\ell(\operatorname{Span}(D_i))$ from P gives a configuration of $n-1 \geqslant 4$ points $Q_2, ..., Q_n \in \mathbb{P}^2$ with the following two properties:

(1) for any two distinct points Q_i, Q_j there exists a point Q_k, $k \neq i, j$ collinear with Q_i, Q_j;
(2) no more than 3 of Q_i are collinear.

Configurations of points satisfying Property (1) are known as Sylvester–Gallai configurations; see [11, Theorems 3.1 – 3.6] for classification results for small values of n. In particular, either $n - 1 = 9$ and the configuration is Hesse configuration (and the points are a base locus for a pencil of cubic curves) or $n - 1 \geqslant 12$.

We first show that for general choices of D_1 and ℓ, the 2-plane $\Lambda := \pi_\ell^{-1}(P)$ does not meet X. Indeed, suppose to the contrary that Λ meets X in λ points for a general choice of D_1 and ℓ. Fixing D_1 and varying ℓ, we obtain a map

$$(\mathbb{P}^2)^\vee \dashrightarrow \operatorname{Sym}^\lambda X.$$

If this map is nonconstant, then $\lambda \geqslant d + 1$ since X is d-minimal. Since $\operatorname{Span} D_1$ meets X in d points, none of which are on $\ell = \Lambda \cap \operatorname{Span} D_1$, we must have that $\operatorname{Span}(D_1, \Lambda)$ meets X in at least $2d + 1$ points. If D_1 and ℓ are defined over the ground field, then so is the unique point P, and hence so is the 3-plane $\operatorname{Span}(D_1, \Lambda)$. Projection from this plane defines a degree at most $d - 1$ map to $\mathbb{P}^1$, contradicting d-minimality.

We may therefore assume that the plane Λ meets X in points independent of ℓ. If it meets X in at least 2 distinct points in $\mathbb{P}^5$, then their span meets $\operatorname{Span} D_1$ in a unique point, which does not lie on a general line ℓ. Hence Λ meets the image of X in $\mathbb{P}^5$ in a unique point (possibly with multiplicity). Varying D_1, and noting that a d-minimal curve cannot have genus at most 1, we see that this unique point must also be independent of D_1. This is a contradiction, since a general pair of divisors D_1, D_2 can be chosen so that their span $\operatorname{Span}(D_1, D_2)$ misses any specific point.

Consider the projection $\pi_\Lambda \colon X \to \mathbb{P}^2$, and suppose that it factors as $X \to Y \to Y' \subset \mathbb{P}^2$, where $X \to Y$ is a finite degree s morphism of smooth curves and $Y \to Y'$ is birational. The plane curve Y' has degree $3d/s$ and every one of the points $Q_2, ..., Q_n$ is a singular point of multiplicity at least d/s. Moreover, the point $Q_1 := \pi_\Lambda(\operatorname{Span} D_1))$ is also a singular point of Y' of multiplicity at least d/s.

Suppose $n - 1 = 9$. Then the configuration of points $Q_2, ..., Q_{10}$ is a Hesse configuration, in particular there is a pencil of cubics through $Q_2, ..., Q_{10}$. We may therefore choose Q to be a cubic though $Q_2, ..., Q_{10}$ and Q_1. The curve Y' is not a cubic, since it has at least 10 singular points, and so $Y' \cap Q$ is a finite scheme. However Q intersects Y' in at least 10 points of multiplicity d/s, thus the total multiplicity of the intersection is at least $10d/s > 9d/s = \deg Q \deg Y'$, contradiction.

Suppose $n - 1 \geqslant 12$. Then the geometric genus of Y' is at most

$$g_{Y'} \leqslant \frac{(3d/s - 1)(3d/s - 2)}{2} - 13\frac{d/s(d/s - 1)}{2} = 1 + 2\frac{d}{s} - 2\frac{d^2}{s^2}.$$

Since Y' has at least 13 singular points, the degree of Y' is at least 5, and so $g_{Y'} < 0$, contradiction.

Thus the map $\eta\colon A \to \operatorname{Span} D_1$ is an isomorphism onto a plane cubic curve. This means that for every point $P \in D_1$ there is exactly one divisor $D_2 \neq D_1$, with $[D_2] \in A$ that contains P. Since D_1 was an arbitrary general divisor, we conclude that for a general point $P \in X$ there exist exactly two divisors D_1, D_2 from A that contain P. Therefore we can define a birational morphism $\mu\colon X \to \operatorname{Sym}^2 A$ that sends a point $P \in X$ to the unique pair of divisors $(D_1, D_2) \in \operatorname{Sym}^2 A$ that contain P. We claim that μ is the birational equivalence of X with a Debarre–Fahlaoui curve that we seek. To do this we need to identify the numerical class of $\mu(X)$ on $\operatorname{Sym}^2 A$.

For $x \in A$, recall that H_x is the set of all pairs of divisors in $\operatorname{Sym}^2 A$ that contain x. Hence the intersection $H_x \cap \mu(X)$ is supported on the points $\mu(\operatorname{supp}(x))$. Since a general divisor $x \in A$ is a degree d point (and hence a single monodromy orbit), $H_x \cap \mu(X)$ is a multiple of $\mu(\operatorname{supp}(x))$. Consider a general point (x, x') on $\mu(X)$. Since the divisors H_x, and $H_{x'}$ for $x \neq x'$ intersect transversely at $(x, x') \in \operatorname{Sym}^2 A$, the intersections $H_x \cap \mu(X)$ and $H_{x'} \cap \mu(X)$ cannot both be nontransverse at (x, x'). Hence the generic intersection $H_x \cap \mu(X)$ cannot consist of multiple points. In other words, for a general $x \in A$ the intersection $H_x \cap \mu(X) \subset \operatorname{Sym}^2 A$ is smooth and $H_x \cap \mu(X) = \mu(\operatorname{supp}(x))$, so $[\mu(X)] \cdot H = d$. Therefore, numerically, $[\mu(X)] = aH + (d - a)F$. Since $[\mu(X)]$ is effective, by the description of the effective cone we have $a \geqslant 0$ and $2d - a \geqslant 0$. The fibers of the addition map $\operatorname{Sym}^2 A \to A$ have numerical class F, and since X does not admit maps of degree less than d to A, we conclude that $d < [\mu(X)] \cdot F = a$. Thus $[\mu(X)] = (d + m)H - mF$ for some m between 1 and d as claimed. □

Remark 5.9. As observed in Remark 3.4, the geometric gonality of a curve with $\min(\delta(X/k)) = d$ which is not a degree d cover of an elliptic curve is at most $2d - 1$. Theorem 5.8 implies that if, in addition, such curves are not Debarre–Fahlaoui curves, then their geometric gonality is at most $2d - 2$.

6. Applications and extensions

6.1. Low degree points on projective curves

Our main strategy can be applied to study low degree points on "special" curves. The geometry of configurations of divisor spans as summarized in Section 4.2 can be used to estimate the dimensions of various linear systems from below. Combining this with Castelnuovo's bound yields a bound on the genus of the curve. We now recall Castelnuovo's theorem [1, Chapter III, page 116]. Given positive integers δ, n, write

$$\delta - 1 = m(n-1) + \varepsilon,$$

for integers m and $0 \leqslant \varepsilon < n-1$. Then the genus of a nondegenerate curve of degree δ in $\mathbb{P}^n$ is bounded by

$$\pi(\delta, n) = \frac{m(m-1)}{2}(n-1) + m\varepsilon. \tag{6}$$

For fixed n and large δ, the genus bound $\pi(\delta, n)$ is roughly $\delta^2/(2n-2)$.

Theorem 6.1. *Suppose $X \subset \mathbb{P}^r$ is an irreducible (possibly singular) curve of degree e and genus g. Suppose X has infinitely many points of degree d not contained in hyperplanes of $\mathbb{P}^r$. Then*

$$g \leqslant \pi(e + 2d, 2r + 1).$$

Proof. By the Mordell–Lang Conjecture, as explained in Section 2, for all but finitely many degree d points D on X, either D moves in a pencil, or the class of D in $W_d X$ belongs to a translate of an abelian subvariety in Pic^d_X. In either of those cases, the class $2D$ is basepoint-free.

Let $[H]$ denote the divisor class corresponding to the embedding $X \subset \mathbb{P}^r$, choose a degree d point D for which $2D$ is basepoint-free and such that D is not contained in divisors from $|H|$. Consider the linear system $|2D + H|$. Since $2D$ is basepoint-free, for a divisor $H' \in |H|$ we have $X \cap \mathrm{Span}_{|2D+H|} H' = H'$. Suppose the dimension of $\mathrm{Span}_{|2D+H|} D$ is equal to s and choose a set $S \subset D$ of size $s+1$ such that $\mathrm{Span}_{|2D+H|} S = \mathrm{Span}_{|2D+H|} D$. If $s < r$, then there exists a divisor $H' \in |H|$ that contains S. Then $\mathrm{Span}_{|2D+H|} H' \supset \mathrm{Span}_{|2D+H|} S = \mathrm{Span}_{|2D+H|} D$, and so the points of D are contained in $X \cap \mathrm{Span}_{|2H+D|} H' = H'$, contradicting our assumption. Therefore $s \geqslant r$. Since the projection from $\mathrm{Span}_{|2D+H|} D$ maps to a space of dimension at least r, we have $\dim |2D + H| \geqslant 2r + 1$. By Castelnuovo's theorem applied to the embedding $|2D + H|$, the genus of X is at most $\pi(e + 2d, 2r + 1)$. □

6.2. Genus estimate for non-Debarre–Fahlaoui curves

The argument of Theorem 4.13 with the added assumption $r(2) > 2$ gives a better genus bound for curves that satisfy (†). Together Theorems 4.13, 5.8, 3.5 and 6.2 below yield Theorem 1.4 from the introduction.

Theorem 6.2. *Suppose X is a d-minimal curve with $r(2) > 2$. Suppose that condition (†) holds. Then the genus g of X satisfies*

$$g \leqslant \frac{(d-1)(d-2)}{2} + 2.$$

Proof. The argument is identical to the proof of Theorem 4.13. We have $r(2) \geqslant 3$, $s(2) \geqslant 2$ by assumption and $s(n) \geqslant \min(d-1, s(n-1)+1)$ by Proposition 4.7. Therefore for $n \leqslant d-1$ we have

$$r(n) \geqslant \frac{(n+1)(n+2)}{2} - 3.$$

By Castelnuovo's theorem applied to the linear system $|(d-1)D|$ we get the desired genus bound. □

6.3. Classification results for low values of d

We now summarize what the main results say about curves with small minimum density degree.

Proposition 6.3. *The following Table 1 summarizes the classification of curves X of genus g with small values of $\min(\delta(X/k))$ or $\min(\wp(X/k))$. We use the following shorthand:*

- *"covers": a degree d cover of $\mathbb{P}^1$ or a (positive rank) elliptic curve*
- *"DF": a normalization of a Debarre–Fahaloui curve*

Proof. If X is not d-minimal, then it is a cover of an s-minimal curve for some $s \mid d$. Since there are no 2-minimal curves by Theorem 3.8, and in our cases $d \leqslant 5$, we can assume that X is either d-minimal or a cover of a 1-minimal curve (i.e., $\mathbb{P}^1$ or an elliptic curve of positive rank). From now on, we assume that X is d-minimal. If X is not a Debarre–Fahlaoui curve, but satisfies the condition (†), then by Theorem 6.2, the genus of X is at most $(d-1)(d-2)/2 + 2$. If (†) does not hold, then the genus bound from Theorem 4.13 applies. Finally, when $d = 5$ the genus of X is bounded by the Castelnuovo function $\pi(20, 12) = 8$ by Lemma 4.12.

Any curve of genus g has geometric gonality at most $\lfloor (g+3)/2 \rfloor$ and gonality at most $2g-2$. Combining this with the genus bounds described above gives the result. □

Table 1
The classification of curves with small minimum density degree.

d	2	3	4	5
$\min(\wp) = d$	covers	covers	covers + DF	covers + DF
$\min(\delta) = d$	covers	covers + DF + $g = 3$	covers + DF + $g = 4, 5$	covers + DF + $g = 5, 6, 7, 8$

In the case $d = 3$ and $g = 3$, a 3-minimal curve X cannot be hyperelliptic, since any conic with a degree 3 point is isomorphic to $\mathbb{P}^1$. Hence a 3-minimal curve X of genus 3 is isomorphic to a smooth plane quartic. Since the gonality of a plane quartic is 3 if and only if it has a rational point, we see that X must be pointless. In this case, $\operatorname{Sym}^3 X(k) = \operatorname{Pic}^3_X(k)$, and so the Jacobian of X must have positive rank. Conversely, any such curve with a single degree 3 point which is not a degree 3 cover of an elliptic curve (i.e., with simple Jacobian) is 3-minimal. Combining this with Proposition 6.3 proves Theorem 1.2.

7. Questions and problems

7.1. Geometric questions

All of the questions we consider have a geometric analogue, that applies to curves X over any field k, and concerns the existence of abelian translates in $W_d\bar{X}$. The resulting geometric questions are usually slightly easier then the arithmetic ones.

Given a curve X over any field k, the union of all positive-dimensional abelian translates in $W_d\bar{X}$ is the Kawamata–Ueno locus $\operatorname{Ueno}(W_d\bar{X})$. When k is a number field, the Mordell–Lang conjecture implies that

$$\min(\wp(X/k)) = \min(\operatorname{gon}(\bar{X}), \min(d : \operatorname{Ueno}(W_d\bar{X}) \neq \emptyset)).$$

However, this more general definition makes sense for complex curves. It is thus natural to define the locus $Z_d(g)$ of curves $[X] \in M_g(\mathbb{C})$ with $\min(\operatorname{gon}(\bar{X}), \min(d : \operatorname{Ueno}(W_d\bar{X}) \neq \emptyset)) = d$. For $d \geqslant \lfloor (g+3)/2 \rfloor$, the locus $Z_d(g)$ is all of $M_g(\mathbb{C})$. The general theory of Kawamata–Ueno loci [14, Theorem 1.2] implies that $Z_d(g)$ is the set of complex points a subvariety of M_g. For lower values of d it is interesting to study the ubiquity of curves in $Z_d(g)$.

Question 7.1. For $d < \lfloor (g+3)/2 \rfloor$, what is the codimension of $Z_d(g)$ in M_g?

A geometrically d-minimal curve X is a curve in $Z_d(g(X))$ for which there does not exist a degree $s \geqslant 2$ covering of curves $X \to Y$ with $Y \in Z_{d/s}(g(Y))$. The next specific case in which we don't know the classification of geometrically d-minimal curves is $d = 6$ and $g = 11$.

Question 7.2. Do there exist geometrically 6-minimal curves of genus 11?

Abramovich and Harris claimed [2][Theorem 2] that the genus of a geometrically d-minimal curve is bounded by $d(d-1)/2$; however the proof presented there is incomplete. The key to obtaining this bound is the inequality $s(n) \geqslant \min(s(n-1)+1, d-1)$, which we only prove in Corollary 4.9 under the assumption (†). It seems likely that the genus bound holds without additional assumptions.

Question 7.3. Do there exist geometrically d-minimal curves with genus larger than $d(d-1)/2$?

In the geometric situation, it is natural to treat $a = \dim A$ as an extra parameter together with d and g. The key to the proof of the Main Theorem is Proposition 4.7 concerning the difference between the dimensions $s(n)$ and $s(n-1)$ of divisor spans in $|nD|$ and $|(n-1)D|$. It seems likely that such a bound can be strengthened to depend on a.

Question 7.4. If $\dim A = a$, is it true that when $s(n) \leqslant d-2$, we have $s(n)-s(n-1) \geqslant a$ for $n \geqslant 3$?

If Question 7.4 has a positive answer, then one can obtained significant improvements on the genus bound for d-minimal curves with $a > 1$. In particular it would imply that the $a = 2$-family constructed by Debarre and Fahlaoui [3] achieves the largest possible genus. Such an estimate is claimed in [2], but the proof has a gap (as remarked in [3]).

The problem of classifying curves in $Z_d(g)$ is interesting over any field k. The results of this paper use the assumption $\operatorname{char} k = 0$ in a few places. For instance, Kani's version of de Franchis theorem (Theorem 3.1) requires a separability assumption in positive characteristic, the classification of small Sylvester–Gallai configurations used in the proof of Theorem 5.8 is more complicated when $\operatorname{char} k = 2, 3$, and the monodromy argument in Lemma 5.6 only works in characteristic zero.

Question 7.5. Do the geometric analogues of our main Theorems 4.13, 5.8 hold in positive characteristic?

7.2. Arithmetic questions

The smallest cases for which some questions are still left open concern d-minimal curves with $d = 3$ and $g = 3, 4$. In both of these cases we expect that 3-minimal curves exist. The case $d = g = 3$ is discussed in Problem 7.7. In the case $d = 3, g = 4$, Proposition 6.3 implies that such a curve is a Debarre–Fahlaoui curve of class $4H - F$. However, a genus 4 curve X over $\bar{k}$ generically admits two maps of degree 3 to $\mathbb{P}^1$: the canonical embedding realizes X as a complete intersection of a quadric and a cubic, and projections from rulings on the quadric give a pair of g^1_3's. We expect that there exist Debarre–Fahlaoui curves in class $4H - F$ for which these two maps are Galois conjugate, and that such Debarre–Fahlaoui curves are 3-minimal.

It is in principle possible to verify this claim (if true) by exhibiting a specific curve on $\operatorname{Sym}^2 A$ and checking that the unique quadric containing the canonical curve is nonsplit (and independently verifying that it is not a triple cover of an elliptic curve), but such a computation is nontrivial in practice. We thus leave this question as a problem.

Problem 7.6. Show that a general curve in numerical class $4H - F$ on $\operatorname{Sym}^2 A$ is 3-minimal.

There is another natural source of low degree points on curves of genus g, as we now describe. Consider a general (in a non-technical sense) genus g curve X over a number field equipped with a degree g point. The abelian variety Pic^g_X may have positive rank, at the same time it appears that there is no clear reason for X to have maps to other curves or low gonality. If this is the case, then $\operatorname{Sym}^g X$, which is birational to Pic^g_X, will have an infinite family of rational points. While we expect such curves to be abundant, we do not know if examples can be proved to exist, and thus leave this as a problem.

Problem 7.7. Show that for every $d \geqslant 3$ there exists a d-minimal curve of genus d.

The problem for $d = 3$ (i.e., smooth plane quartics) is already interesting and should be computationally feasible.

Theorem 4.13 shows that, under condition (†), the genus of a d-minimal curve is bounded by $\frac{d(d-1)}{2} + 1$; curiously, this number is exactly the (maximal) genus of a Debarre-Fahlaoui curve, and in Theorem 5.8 we explained this coincidence by identifying curves with $r(2) = 2$ with Debarre–Fahlaoui curves. Question 7.4 predicts a similar situation for $\dim A = 2$: the maximal genus for such a curve is $d^2/4 + 1$, which is exactly the genus of $\dim A = 2$ examples constructed in [3]. We hope that this can also be explained in terms of geometry of configurations.

Question 7.8. Is it true that a d-minimal curve with $\dim A = 2$ and $r(2) = 3$ is birational to one of the curves constructed in [3]?

It would be very interesting to obtain better results for special curves; in particular, we do not expect Theorem 6.1 to be close to optimal. One way to test optimality of such a genus bound is to compare it to known results in low dimension.

Problem 7.9. Suppose X is a curve equipped with a g^r_e linear system. Show that for a certain function $g(r, e, d)$ the following holds: if the genus of X is larger than $g(r, e, d)$, and X has infinitely many points of degree d, then all but finitely many of those points are contained in the divisors from g^r_e. Can the function $g(r, e, d)$ be such that the value of $g(1, e, d)$ implies Vojta's estimate [19] for low degree covers of $\mathbb{P}^1$, and the value of $g(2, e, d)$ implies the Debarre–Klassen theorem [4] on smooth plane curves?

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