LOG-CONCAVITY OF THE ALEXANDER POLYNOMIAL

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Abstract. The central question of knot theory is that of distinguishing links up to isotopy. The first polynomial invariant of links devised to help answer this question was the Alexander polynomial (1928). Almost a century after its introduction, it still presents us with tantalizing questions, such as Fox's conjecture (1962) that the absolute values of the coefficients of the Alexander polynomial $\Delta_L(t)$ of an alternating link L are unimodal. Fox's conjecture remains open in general with special cases settled by Hartley (1979) for two-bridge knots, by Murasugi (1985) for a family of alternating algebraic links, and by Ozsváth and Szabó (2003) for the case of genus 2 alternating knots, among others.

We settle Fox's conjecture for special alternating links. We do so by proving that a certain multivariate generalization of the Alexander polynomial of special alternating links is Lorentzian. As a consequence, we obtain that the absolute values of the coefficients of $\Delta_L(t)$, where L is a special alternating link, form a log-concave sequence with no internal zeros. In particular, they are unimodal.

1. Introduction

The central question of knot theory is that of distinguishing links up to isotopy. Knot invariants are devised for this purpose. The Alexander polynomial $\Delta_L(t)$, associated to an oriented link L, was the first polynomial knot invariant, discovered in the 1920s [Ale28]. The key property of the Alexander polynomial is that if oriented links L_1 and L_2 are isotopic, then $\Delta_{L_1}(t) = \Delta_{L_2}(t)$ up to multiplication by $\pm t^k$ for some integer k.

The coefficients of $\Delta_L(t)$ for an arbitrary link L are palindromic. In 1962, Fox [Fox62] conjectured that for alternating links, the absolute values of the coefficients of Alexander polynomials are unimodal. For alternating links L, [Cro59, Mur58a, Mur58b] show that the Alexander polynomial can be normalized so that $\Delta_L(-t) \in \mathbb{Z}_{\geq 0}[t]$ and that its sequence of coefficients contains no internal zeros. With this normalization, we can write Fox's conjecture as:

Conjecture 1.1. [Fox62] *Let L be an alternating link. Then the coefficients of $\Delta_L(-t)$ form a unimodal sequence.*

The conjecture remains open in general, although some special cases have been settled by Hartley [Har79] for two-bridge knots, Murasugi [Mur85] for a family of alternating algebraic links, and Ozsváth and Szabó [OSz03] for the case of genus 2 alternating knots, among others. That Fox's conjecture holds for genus 2 alternating knots was also confirmed by Jong [Jon09]. At the 2018 ICM, June Huh highlighted this sequence as one of *"the most interesting sequences that are conjectured to be* **log-concave***"* [Huh18]. Huh was referring to Stoimenow's [Sto05] strengthening of Fox's conjecture from unimodality to log-concavity.

In this paper, we show:

Theorem 1.2. *The coefficients of the Alexander polynomial $\Delta_L(-t)$ of a special alternating link L form a log-concave sequence with no internal zeros. In particular, they are unimodal, proving Fox's conjecture in this case.*

Inspired by Crowell's combinatorial model for the Alexander polynomial of alternating links [Cro59], we study a homogeneous multivariate polynomial which we term the M-polynomial (because its support is M-convex). This polynomial previously appeared in the works of Kálmán [Ká13] and Juhász, Kálmán, and Rasmussen [JKR12]. We discovered the M-polynomial via Crowell's construction. We prove that the M-polynomial specializes to $\Delta_L(-t)$ for special alternating links L. We also prove that the M-polynomial is denormalized Lorentzian, opening the door to use the powerful theory of Lorentzian polynomials developed by Brändén and Huh [BH20]. Lorentzian polynomials were independently developed by Anari, Liu, Oveis

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Gharan and Vinzant [AGV21, ALGV19, ALGV18] under the name of completely log-concave polynomials. Relying on the theory of Lorentzian polynomials, we prove that the coefficients of the Alexander polynomial $\Delta_L(-t)$ of a special alternating link L form a log-concave sequence with no internal zeros.

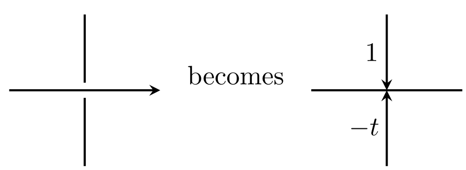
Roadmap of the paper. In Section 2, we review the necessary background for the paper. In Section 3, we show that the M-polynomial arises naturally from Crowell's model of the Alexander polynomial of alternating links. In Section 4, we prove Theorem 1.2.

2. Background

In this section, we collect the most important results and notions used in our paper: (1) Crowell's model for the Alexander polynomial of alternating links; (2) the construction of special alternating links from planar bipartite graphs; (3) background on M-convex sets; (4) background on the theory of Lorentzian polynomials.

2.1. **Crowell's model.** We will use the following combinatorial model for the Alexander polynomial of alternating links due to Crowell [Cro59]. Recall that a link is alternating if it has an alternating diagram.

Let $\mathcal{G}(L)$ be the planar graph obtained by flattening the crossings of an alternating diagram of L; the crossings of L are the vertices of $\mathcal{G}(L)$ while the arcs between the crossings are the edges of $\mathcal{G}(L)$. Note that $\mathcal{G}(L)$ is a planar 4-regular 2-face colorable graph. Next, we assign directions to the edges of $\mathcal{G}(L)$ – but not those coming from the orientation of the link – as well as weights in the following way:



On the left, we see the orientation of the link L in an overcrossing, and on the right, we see how the edges of $\mathcal{G}(L)$ are directed and weighted. Denote by $\overrightarrow{\mathcal{G}(L)}$ the oriented weighted graph obtained from $\mathcal{G}(L)$ in this fashion. Let $var(e)$ be the weight $-t$ or 1 assigned to the edge $e \in E(\overrightarrow{\mathcal{G}(L)})$. See Figure 1 for a full example.

Theorem 2.1 ([Cro59] Theorem 2.12)**.** *Given an alternating diagram of the link L, fix an arbitrary vertex $r \in V(\overrightarrow{\mathcal{G}(L)})$. Denote by $\mathcal{A}(L,r)$ the set of arborescences of $\overrightarrow{\mathcal{G}(L)}$ rooted at r. The Alexander polynomial of L is:*

$$\Delta_L(t) = \sum_{T \in \mathcal{A}(L,r)} \prod_{e \in E(T)} var(e).$$

Recall that an **arborescence** rooted at r is a spanning tree in which there is a unique directed path to any vertex from the root r.

2.2. **Special alternating links.** We follow the construction presented by Juhász, Kálmán, Rasmussen [JKR12] and Kálmán, Murakami [KM17], associating a positive special alternating link L_G to a planar bipartite graph G. Let $M(G)$ be the medial graph of G: the vertices of $M(G)$ are the edges of G, and two vertices of $M(G)$ are connected by an edge if the edges of G that they come from are consecutive in the boundary of a face of G. We think of a particular planar drawing of $M(G)$ here: the midpoints of the edges of the planar drawing of G are the vertices of $M(G)$. Thinking of $M(G)$ as a flattening of a link, there are two ways to choose under and overcrossings at each vertex of $M(G)$ to make it into an alternating link L_G. We select the over and undercrossings and orient L_G so that each crossing is positive. This procedure yields a positive special alternating link. Moreover, any positive special alternating link arises from such a construction. Figure 1 shows an example of this construction.

If the link associated to the planar bipartite graph G as above is instead oriented so that every crossing is negative, we obtain a negative special alternating link and denote it L_G^{neg}. All special alternating links are either positive or negative.

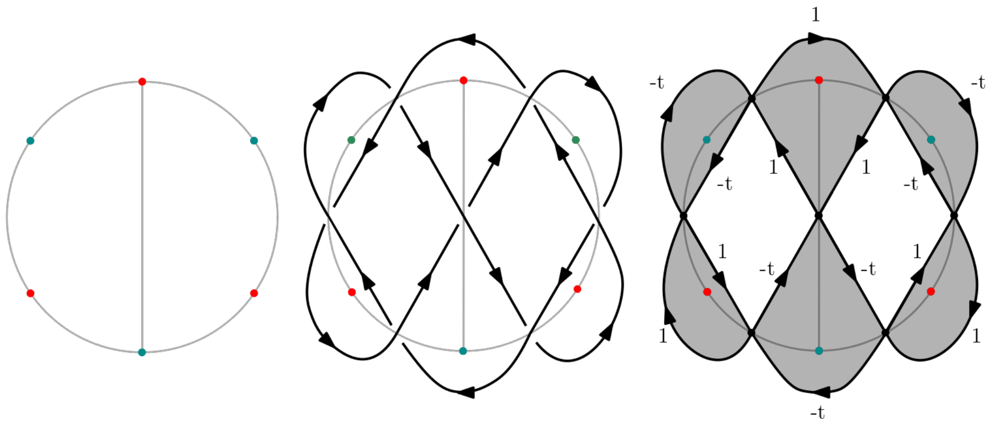


FIGURE 1. A planar bipartite graph along with its associated positive special alternating link L_G. On the right, we show $\overrightarrow{\mathcal{G}(L_G)}$, oriented and weighted as in Crowell's model.

2.3. **M-convex sets.** Let $\mathbb{N} = \{0, 1, 2, \ldots\}$, and denote by e_i the ith standard basis vector of $\mathbb{N}^n$. A subset $J \subseteq \mathbb{N}^n$ is called M**-convex** if for any index i and any $\alpha, \beta \in J$ whose ith coordinates satisfy $\alpha_i > \beta_i$, there is an index j satisfying

$$\alpha_j < \beta_j, \quad \alpha - e_i + e_j \in J, \quad \text{and} \quad \beta - e_j + e_i \in J.$$

A wealth of information on M-convex sets can be found in [Mur03]. We emphasize here that the sum of the coordinates of the integer points in an M-convex set J is a fixed constant. We also provide below a point of view through generalized permutahedra.

Convex hulls of M-convex sets are well-studied integer polytopes called generalized permutahedra. The name generalized permutahedron was coined by Postnikov [Pos09] since, as we explain below, we can see these polytopes as generalizing the standard permutahedron Π_n. Generalized permutahedra have been studied since the 1970s in the combinatorial optimization literature under various names such as base polytopes and polymatroids (see [Fra11, Sch03]). The standard permutahedron $\Pi_n \subset \mathbb{R}^n$ is the convex hull of all permutations of the vector $(1, 2, \ldots, n)$. Note that Π_n lies in the hyperplane $x_1 + x_2 + \cdots + x_n = \binom{n+1}{2}$. The edge directions of Π_n are all in root directions $e_i - e_j$ for $i, j \in [n]$. A generalized permutahedron can be defined as a polytope with all edge directions parallel to $e_i - e_j$ for $i, j \in [n]$. All integer points of an integer generalized permutahedron in $\mathbb{R}^n$, which are exactly the points in an M-convex set (possibly translated in order to lie in $\mathbb{N}^n$), lie in a hyperplane $x_1 + x_2 + \cdots + x_n = c$ for some constant $c \in \mathbb{Z}$.

M-convex sets, or equivalently, integer points of integer generalized permutahedra, play a fundamental role in the theory of Lorentzian polynomials. The latter theory is the first comprehensive tool for proving log-concavity results, introduced in the seminal work of Brändén and Huh [BH20]. All Lorentzian polynomials have supports that are M-convex. Recall that the **support** of a polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ is the set $\operatorname{supp}(f) \subseteq \mathbb{N}^n$ of all tuples $(\alpha_1, \ldots, \alpha_n)$ such that the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ has nonzero coefficient in f.

We define Lorentzian polynomials in Section 2.4. Here we consider a particular M-convex set that will be relevant for us in Section 4.1.

A matroid is a discrete structure inspired by the properties of bases in linear algebra. The M in M-convex comes from the word matroid. We do not need to know the definition of matroid here, and for those interested, we defer to [Oxl11, Sch03]. We will, however, introduce the base polytope of a graphic matroid. The latter polytope is also often referred to as a graphic matroid polytope and is a generalized permutahedron. Given a connected graph $G = (V, E)$, the base polytope of the graphic matroid of G is the convex hull of the indicator vectors χ_T of spanning trees T of G. An indicator vector χ_T of T is a $0, 1$ vector in $\mathbb{R}^E$ such that its e^{th} coordinate is 1 if the edge e is in T and 0 otherwise. Thus, we see that all integer points of the base polytope of the graphic matroid of G are vertices and $0, 1$ vectors.

The integer point enumerator of a polytope $P \subset \mathbb{R}^n$ is the n-variable polynomial $\displaystyle\sum_{(\alpha_1, \ldots, \alpha_n) \in P \cap \mathbb{Z}^n} {x_1}^{\alpha_1} \cdots {x_n}^{\alpha_n}$.

We will be working with the integer point enumerator of the base polytope of the graphic matroid in Section 4.1. Theorem 2.3 below, when applied to graphic matroids, could be stated as: the normalization of an integer point enumerator of a base polytope of graphic matroids is Lorentzian. Since the integer points of the base polytopes of graphic matroids are all $0, 1$, the normalization becomes trivial, and so we can say that the integer point enumerator of a base polytope of a graphic matroid is Lorentzian. We re-explain this and exploit it further in Section 4.1.

2.4. **Lorentzian polynomials.** Let H_n^d be the space of degree d homogeneous polynomials with real coefficients in the n variables $x_1, \dots, x_n$. Denote by $\frac{\partial}{\partial x_i} f$ the partial derivative of f relative to x_i. The **Hessian** of a homogeneous quadratic polynomial $f \in \mathrm{H}_n^2$ is the symmetric $n \times n$ matrix $H = (H_{ij})_{i,j \in [n]}$ defined by $H_{ij} = \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} f$. The set L_n^d of **Lorentzian polynomials** with degree d in n variables is defined as follows. Set $\mathrm{L}_n^1 \subseteq \mathrm{H}_n^1$ to be the set of all linear polynomials with nonnegative coefficients. Let $\mathrm{L}_n^2 \subseteq \mathrm{H}_n^2$ be the subset of quadratic polynomials with nonnegative coefficients whose Hessians have at most one positive eigenvalue and which have M-convex support. For $d > 2$, define $\mathrm{L}_n^d \subseteq \mathrm{H}_n^d$ recursively by

$$\mathrm{L}_n^d = \left\{ f \in \mathrm{M}_n^d \colon \frac{\partial}{\partial x_i} f \in \mathrm{L}_n^{d-1} \text{ for all } i \right\}$$

where $\mathrm{M}_n^d \subseteq \mathrm{H}_n^d$ is the set of polynomials with nonnegative coefficients whose supports are M-convex.

Since $f \in \mathrm{M}_n^d$ implies $\frac{\partial}{\partial x_i} f \in \mathrm{M}_n^{d-1}$, we have

$$\mathrm{L}_n^d = \left\{ f \in \mathrm{M}_n^d \colon \frac{\partial}{\partial x_{i_1}} \frac{\partial}{\partial x_{i_2}} \cdots \frac{\partial}{\partial x_{i_{d-2}}} f \in \mathrm{L}_n^2 \text{ for all } i_1, i_2, \dots, i_{d-2} \in [n] \right\}.$$

The **normalization operator** N on $\mathbb{R}[x_1, \dots, x_n]$ is defined by:

$$N(\mathbf{x}^\alpha) = \frac{\mathbf{x}^\alpha}{\alpha!},$$

where for a vector $\alpha = (\alpha_1, \dots, \alpha_n)$ of nonnegative integers, we write $\alpha!$ to mean $\prod_{i=1}^n \alpha_i!$.

A polynomial $f \in \mathrm{H}_n^d$ is a **denormalized Lorentzian polynomial** in n variables if $N(f) \in \mathrm{L}_n^d$.

We collect here four results that we will utilize in this paper:

Theorem 2.2 ([BH20, Theorem 2.10])**.** *If $f \in \mathrm{L}_n^d$ is a Lorentzian polynomial in n variables and A is an $n \times m$ matrix with nonnegative entries, then $f(A\mathbf{v}) \in \mathrm{L}_m^d$ is a Lorentzian polynomial in the m variables $\mathbf{v} = (v_1, \dots, v_m)$.*

Theorem 2.3 ([BH20, Theorem 3.10])**.** *Let J be an M-convex set. Then the polynomial $f_J = N(\sum_{\alpha \in J} \mathbf{x}^\alpha)$ is a Lorentzian polynomial.*

Proposition 2.4 ([BH20, Proposition 4.4])**.** *If $f(\mathbf{x}) = \sum_\alpha c_\alpha \mathbf{x}^\alpha$ is a homogeneous polynomial on n variables so that $N(f)$ is Lorentzian, then for any $\alpha \in \mathbb{N}^n$ and any $i, j \in [n]$, the inequality*

$$c_\alpha^2 \geq c_{\alpha+e_i-e_j} c_{\alpha-e_i+e_j}$$

holds.

Lemma 2.5. [BLP22, Lemma 4.8] *If $f(x_1, x_2, x_3, \dots, x_n) \in \mathbb{R}_{\geq 0}[x_1, \dots, x_n]$ is a denormalized Lorentzian polynomial, then $f(x_1, x_1, x_3, x_4, \dots, x_n)$, generated by specializing to $x_2 = x_1$, is also a denormalized Lorentzian polynomial.*

3. A multivariate generalization of Crowell's Alexander polynomial

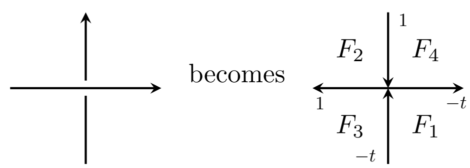
Theorem 2.1 reveals the possibility of a multivariate generalization of the Alexander polynomial: instead of assigning weights $-t$ and 1 to the edges, we can assign a different weight/variable to each of the edges of $\overrightarrow{\mathcal{G}(L)}$. Our goal is to make a Lorentzian generalization of the Alexander polynomial in such a way that the (denormalized) Lorentzian property carries over to the homogenized Alexander polynomial $\Delta_{L_G}(-t)$ for any planar bipartite graph G. This, in turn, would imply the log-concavity of the coefficients of $\Delta_{L_G}(-t)$.

Assigning all different variables to the edges of $\overrightarrow{\mathcal{G}(L)}$ does not give us an M-convex support, needed for the Lorentzian property; however, a different approach does. We note that for special alternating links L, the oriented graph $\overrightarrow{\mathcal{G}(L)}$ is an **alternating dimap**: a planar Eulerian digraph oriented so that the edges around each vertex are directed alternately into and out of that vertex. Moreover, we prove a particularly special distribution of the weights $-t$ and 1 with respect to the regions of $\overrightarrow{\mathcal{G}(L)}$ in Lemma 3.1 below.

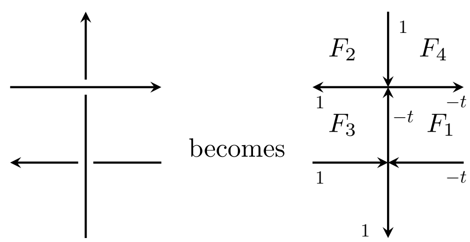
Recall that any alternating dimap D is two face colorable. The edges surrounding faces in one color class are clockwise oriented cycles, and the edges surrounding the other faces are counterclockwise oriented cycles. Throughout this paper, we will use the terms "faces" and "regions" of a dimap D interchangeably.

Lemma 3.1. *Let G be a planar bipartite graph. Recall that $\overrightarrow{\mathcal{G}(L_G)}$ is the alternating dimap obtained by flattening the crossings of L_G with the orientation and edge labeling given by Crowell as in Theorem 2.1. Suppose R is the set of all regions of $\overrightarrow{\mathcal{G}(L_G)}$ whose boundaries are clockwise oriented cycles. Then, R is either precisely the set of regions associated to vertices of G or the set of regions associated to vertices in its planar dual G^*. Furthermore, the boundary of every face in R is either labeled with a 1 on every edge or with a $-t$ on every edge.*

Proof. Using the fact that L_G is alternating and has only positive crossings, the edges incident to any fixed vertex v of $\overrightarrow{\mathcal{G}(L_G)}$ will be oriented and labeled as shown.



Following along the bottom strand, the next crossing (and the corresponding vertex in $\overrightarrow{\mathcal{G}(L_G)}$) will have the following form.



Repeating this for the crossing to the right of the previous one and all the other crossings on the boundary of the region F_1, we see that F_1 is bounded by a clockwise oriented cycle in which every edge is labeled with $-t$. Similarly, we can show that F_2 is bounded by a clockwise oriented cycle in which every edge is labeled with 1.

In particular, observe that $F_1, F_2 \in R$ and $F_3, F_4 \notin R$. Since this pattern is the same at any vertex of $\overrightarrow{\mathcal{G}(L_G)}$, all regions in R will have either the labeling of F_1 (with all $-t$'s) or the labeling of F_2 (with all 1's).

Furthermore, R and R', the set of faces not in R, will form a proper 2-coloring of the faces of $\overrightarrow{\mathcal{G}(L_G)}$. By construction, one of the sets R or R' will be the regions corresponding to the vertices of G, and the other will be the regions corresponding to the vertices of G^*. □

See Figure 2 for a full example of the edge labeling on a positive special alternating link.

Remark 3.2. *If L_G^{neg} is any negative special oriented link, then Lemma 3.1 holds with "clockwise" replaced by "counterclockwise."*

In light of Lemma 3.1, we consider the following generalization of the Alexander polynomial $\Delta_{L_G}(-t)$. We define a multivariate polynomial for all alternating dimaps D and show that when we take $D = \overrightarrow{\mathcal{G}(L_G)}$ and specialize this polynomial, we get $\Delta_{L_G}(-t)$.

Definition 3.3. *Denote the set of clockwise oriented regions of the alternating dimap D by $R(D)$. Let $R(D) = \{R_1, \ldots, R_k\}$. Each edge $e \in E(D)$ belongs to the boundary of exactly one region in $R(D)$. Assign a variable $var(e) = x_i$ to each edge e where e is in the boundary of R_i, $i \in [k]$. See Figure 3 for an example.*

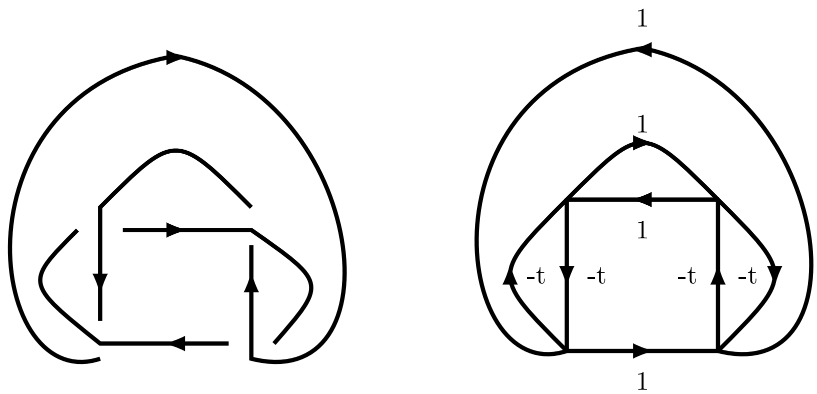


FIGURE 2. A positive special alternating link L on the left and the associated dimap $\overrightarrow{\mathcal{G}(L)}$ with orientations and labeling from Crowell's model on the right. Note that in this example, the set R of clockwise oriented regions includes the exterior of $\overrightarrow{\mathcal{G}(L)}$.

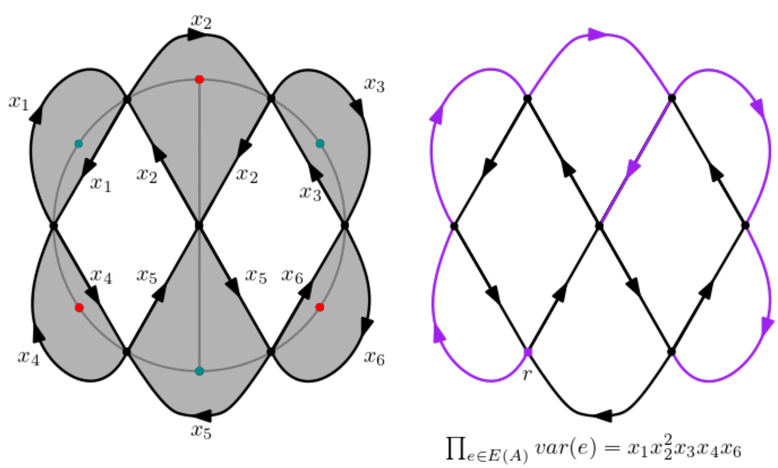


FIGURE 3. The left shows the graph $\overrightarrow{\mathcal{G}(L_G)}$ for the positive special alternating link constructed in Figure 1 with edge variables as in Definition 3.3. The right shows an arborescence rooted at r and the monomial associated to it.

Definition 3.4. *Let D be an alternating dimap, and define $var(e)$, $e \in E(D)$, as in Definition 3.3. Fix a vertex $r \in V(D)$. Define*

$$M_{D,r}(x_1, \dots, x_k) = \sum_A \prod_{e\in E(A)} var(e) \tag{1}$$

where the sum is over all arborescences A of D rooted at r.

We call the multivariate polynomial $M_{D,r}$ the **M-polynomial** of the dimap D, as we devised it so that its support would be M-convex. We will prove this and other important properties in Section 4. We will also see that $M_{D,r}(x_1, \dots, x_k)$ does not depend on the choice of root r but only on the dimap D (Theorem 4.1), and for this reason, we denote it simply by $M_D(\mathbf{x})$ for the rest of this section. The polynomial we term the M-polynomial appeared as a determinant in works by Juhász, Kálmán, and Rasmussen [JKR12] and by Kálmán [Ká13] with a different, but closely related, prelude.

Theorem 3.5. *Let G be a planar bipartite graph, and assign $\overrightarrow{\mathcal{G}(L_G)}$ the orientation and labeling from Crowell's model as described in Section 2. Let $R(\overrightarrow{\mathcal{G}(L_G)})_1 = \{R_1, \dots, R_l\}$ and $R(\overrightarrow{\mathcal{G}(L_G)})_2 = \{R_{l+1}, \dots, R_k\}$ be the clockwise oriented regions of $\overrightarrow{\mathcal{G}(L_G)}$ labeled with $-t$'s and 1's respectively. Then,*

$$\Delta_{L_G}(-t) = M_{\overrightarrow{\mathcal{G}(L_G)}}(t, \ldots, t, 1, \ldots, 1) \tag{2}$$

where we set $x_1 = \cdots = x_l = t$ *and* $x_{l+1} = \cdots = x_k = 1$ *in the* M*-polynomial* $M_{\overrightarrow{\mathcal{G}(L_G)}}$.

Similarly,

$$\mathrm{Homog}_q(\Delta_{L_G}(-t)) = M_{\overrightarrow{\mathcal{G}(L_G)}}(t, \ldots, t, q, \ldots, q), \tag{3}$$

where $\mathrm{Homog}_q(\Delta_{L_G}(-t))$ *denotes the* q*-homogenization of* $\Delta_{L_G}(-t)$ *and we set* $x_1 = \cdots = x_l = t$ *and* $x_{l+1} = \cdots = x_k = q$ *in the* M*-polynomial* $M_{\overrightarrow{\mathcal{G}(L_G)}}$.

Proof. This is an immediate corollary of Theorem 2.1, Lemma 3.1, and Definition 3.4. □

Remark 3.6. *For any digraph* D*, let the* ***transpose*** *of* D*, denoted* D^T*, be the digraph obtained from* D *by reversing the orientation of each edge. By Remark 3.2,* $M_{\overrightarrow{\mathcal{G}(L_G^{neg})}^T}(x_1, \ldots, x_k)$ *specializes to* $\Delta_{L_G^{neg}}(-t)$ *for any negative special alternating link* L_G^{neg} *in the same way as above.*

The specialization taken on the right-hand side of (3) has a discrete geometric meaning. Given any polynomial $M(x_1, \ldots, x_k) = \sum_{(p_1,\ldots,p_k)\in\mathbb{Z}^k} c_{p_1,\ldots,p_k} \prod_{i=1}^k x_i^{p_i} \in \mathbb{Z}[x_1, \ldots, x_k]$, we can equivalently represent it as a **labeled lattice** $\mathbb{Z}^k$ of the polynomial $M(x_1, \ldots, x_k)$: for each point $(p_1, \ldots, p_k) \in \mathbb{Z}^k$ label it by the coefficient $c_{p_1,\ldots,p_k}$ of $\prod_{i=1}^k x_i^{p_i}$ in $M(x_1, \ldots, x_k)$. All but finitely many points in the labeled lattice $\mathbb{Z}^k$ of $M(x_1, \ldots, x_k)$ are labeled by 0, and the convex hull of the integer points with nonzero labels is called the **Newton polytope** of $M(x_1, \ldots, x_k)$.

Let

$$\widetilde{M}(t, q) = M(t, \ldots, t, q, \ldots, q) \tag{4}$$

where we set $x_1 = \cdots = x_l = t$ and $x_{l+1} = \cdots = x_k = q$ in $M(x_1, \ldots, x_k)$. In particular, $\mathrm{Homog}_q(\Delta_{L_G}(-t)) = \widetilde{M}_{\overrightarrow{\mathcal{G}(L_G)}}(t, q)$ by (3). We can readily interpret the coefficients of $\widetilde{M}(t, q)$ as follows.

Lemma 3.7. *The coefficient of* $t^m q^n$ *in* $\widetilde{M}(t, q)$*, for any fixed* $m, n \in \mathbb{Z}_{\geq 0}$*, is equal to the sum of the labels – in the labeled lattice* $\mathbb{Z}^k$ *of* $M(x_1, \ldots, x_k)$ *– of the integer points in the intersection of the Newton polytope of* $M(x_1, \ldots, x_k)$ *with the hyperplanes* $p_1 + \cdots + p_l = m$ *and* $p_{l+1} + \cdots + p_k = n$.

Proof. By definition, the coefficient of $t^m q^n$ in $\widetilde{M}(t, q)$, for any fixed $m, n \in \mathbb{Z}_{\geq 0}$, is equal to the sum of the coefficients of the monomials $\prod_{i=1}^k x_i^{p_i}$ of $M(x_1, \ldots, x_k)$ whose exponents lie in the hyperplanes $p_1 + \cdots + p_l = m$ and $p_{l+1} + \cdots + p_k = n$. □

Corollary 3.8. *If the support of* $M(x_1, \ldots, x_k)$ *is an* M*-convex set, then the sequence of coefficients of* $\widetilde{M}(t, q)$ *is equal to the sum of the labels – in the labeled lattice* $\mathbb{Z}^k$ *of* $M(x_1, \ldots, x_k)$ *– of the integer points in the intersection of the Newton polytope of* $M(x_1, \ldots, x_k)$ *with parallel hyperplanes of the form* $p_1 + \cdots + p_l = c$*,* $c \in \mathbb{Z}_{\geq 0}$.

Proof. Since the support of $M(x_1, \ldots, x_k)$ is M-convex, the sum $p_1 + \cdots + p_k$ is constant on the support of $M(x_1, \ldots, x_k)$ (the latter is true for any M-convex set as noted in Section 2.3). As such, the statement follows from Lemma 3.7. □

In the next section, we show that the M-polynomial of any alternating dimap D is denormalized Lorentzian. Using Theorem 3.5, this will imply that the sequence of coefficients of $\Delta_L(-t)$ is log-concave with no internal zeros when L is a special alternating link. Corollary 3.8 will afford us a geometric viewpoint as well.

4. The M-polynomial and the homogenized Alexander polynomial $\Delta_{L_G}(-t)$ are denormalized Lorentzian

The goal of this section is to prove that the M-polynomial of any alternating dimap D is denormalized Lorentzian:

Theorem 4.1. *For any alternating dimap* D*, the polynomial* $M_D(\mathbf{x}) = M_{D,r}(\mathbf{x})$ *is independent of the choice of root* $r \in D$*. Moreover,* $M_D(\mathbf{x})$ *is denormalized Lorentzian.*

In the special case of $\overrightarrow{\mathcal{G}(L)}$ where L is a special alternating link, Theorems 3.5 and 4.1 and Lemma 2.5 will imply that the homogenized Alexander polynomial $\mathrm{Homog}_q(\Delta_{L_G}(-t))$ is also denormalized Lorentzian.

We now study the support and coefficients of the M-polynomial in order to prove Theorem 4.1. Lemmas 4.3 and 4.6 were first discovered by Kálmán [Ká13] and presented as part of his beautiful proof of [Ká13, Theorem 10.1]. We include their proofs here for completeness. For Lemma 4.3, we present our proof which is closely related to Kálmán's. For Lemma 4.6, we present his original proof, adapted to our conventions.

4.1. **The support of the M-polynomial.** We relate the support of the M-polynomial to the integer points of the base polytope of a graphic matroid as defined in Section 2.3; we recall the necessary definitions here. Let $\mathcal{T}(D)$ be the set of all spanning trees of the alternating dimap D (spanning trees here are considered without orientation). We will let g_D be the integer point enumerator, defined below, of the base polytope of the graphic matroid of D considered without orientation. If we let $R_1, \dots, R_k$ of D be the regions bounded by the clockwise oriented cycles $C_1, \dots, C_k$ and denote the edges of C_i by $e_{i,1}, \dots, e_{i,|C_i|}$, $i \in [k]$, then:

$$g_D(x_{1,1}, \dots, x_{n,|C_k|}) = \sum_{T \in \mathcal{T}(D)} \prod_{e_{i,j} \in E(T)} x_{i,j}.$$

Theorem 2.3 implies that $N(g_D(\mathbf{x}))$ is Lorentzian since the integer points of a matroid base polytope form an M-convex set. Moreover, since all integer points of a matroid base polytope are $0, 1$, we have that $N(g_D(\mathbf{x})) = g_D(\mathbf{x})$. Thus, $g_D(\mathbf{x})$ itself is Lorentzian. Next, we specialize $g_D(\mathbf{x})$ in a way that preserves the Lorentzian property:

$$f_D(x_1, \dots, x_k) = \sum_{T \in \mathcal{T}(D)} \prod_{i=1}^{k} x_i^{a_i(T)},$$

where $a_i(T)$ is the number of edges of T belonging to the cycle C_i, $i \in [k]$.

Lemma 4.2. *Given an alternating dimap D, the polynomial f_D is Lorentzian. In particular, f_D has M-convex support.*

Proof. The polynomial g_D is the exponential generating function of the graphic matroid of D (where we consider D without its orientation). Thus, by Theorem 2.3, the polynomial g_D is Lorentzian. The polynomial f_D is obtained from g_D by a nonnegative linear change of variable, so f_D is also Lorentzian by Theorem 2.2. Since f_D is Lorentzian, it must have M-convex support. □

Next, we show that for any $r \in D$, $\mathrm{supp}(f_D(x_1, \dots, x_k)) = \mathrm{supp}(M_{D,r}(x_1, \dots, x_k))$. To do this, we need the following auxiliary lemma:

Lemma 4.3. [Ká13, see proof of Theorem 10.1]. *Let D be an alternating dimap. Denote the cycles surrounding the clockwise oriented regions by $C_1, \dots, C_k$. Let T be any spanning tree in D, and fix any $r \in V(D)$. Let $a_i(T)$ be the number of edges of T in the cycle C_i. Then, there exists an arborescence A, rooted at r, such that $a_i(A) = a_i(T)$ for all $i \in [k]$.*

Proof. Define a subset $V'(T) \subset V(T) = V(G)$ as follows. Let $r \in V'(T)$ if and only if T contains no edges with final vertex r. For any vertex $v \neq r$, there is a unique (undirected) path from r to v in T. Let e_v be the unique edge on this path with v as an endpoint. Then $v \in V'(T)$ if and only if all other vertices along this path from r to v in T are in $V'(T)$ and e_v is the only edge in T with final vertex v. We will refer to $V'(T)$ as the *good* vertices of T.

If $V'(T)$ is nonempty, fix a vertex v_1 such that $v_1 \notin V'(T)$ but all other vertices on the unique path between v_1 and r in T are good. By construction, there exists some edge $e_1 \neq e_{v_1}$ in T with final vertex v_1. If $V'(T)$ is empty, define $v_1 = r$, and fix an edge e_1 with final vertex r.

Note that e_1 is in the cycle C_i for a unique i. Let $V_1 \subset V(T)$ be the set of all vertices such that their unique path to r in T passes through e_1.

Observe that both $V_1 \cap V(C_i)$ and $(V(T) - V_1) \cap V(C_i)$ are non-empty (since v_1 is not in V_1 but the initial vertex of e_1 is). Since C_i is an oriented cycle, we can find an edge e_2 of C_i with final vertex in V_1 and initial vertex not in V_1. By construction, e_2 is not in the tree T. Set $T_1 = (V(T), (E(T) - \{e_1\}) \cup \{e_2\})$.
By construction, T_1 has the following properties:

(1) T_1 is a spanning tree of G

(2) $a_i(T_1) = a_i(T)$ for all $i \in [n]$
(3) $V'(T_1) \supset V'(T)$

If $v_1 \notin V'(T_1)$, then T_1 must still contain an edge other than e_{v_1} with final vertex v_1, and we can perform the same procedure as above to remove and replace this edge. Otherwise, we select a new vertex $v_2 \notin V'(T_1)$ such that all other vertices along its unique path to r are good and repeat the process. This procedure will terminate when all the vertices of our tree are good, i.e. when we have an arborescence. □

Corollary 4.4. *For any $r \in V$, $\text{supp}(f_D(x_1, \ldots, x_k)) = \text{supp}(M_{D,r}(x_1, \ldots, x_k))$. In particular, $\text{supp}(M_{D,r}(x_1, \ldots, x_k))$ is M-convex.*

Proof. This is an immediate consequence of Lemmas 4.2 and 4.3 and the definitions of f_D and $M_{D,r}$. □

Corollary 4.5. *The support of $M_{D,r}(x_1, \ldots, x_n)$ is M-convex and independent of the choice of root r.*

Proof. Follows from Corollary 4.4. □

4.2. **The coefficients of the M-polynomial.** In this section, we see that for any choice of root r in an alternating dimap D, the polynomial $M_{D,r}(\mathbf{x})$ has all 0 and 1 coefficients. In particular, since the support of $M_{D,r}(\mathbf{x})$ is independent of the choice of root r by Corollary 4.5, the polynomial $M_{D,r}(\mathbf{x})$ is also independent of the root r, and we are justified in denoting it by $M_D(\mathbf{x})$.

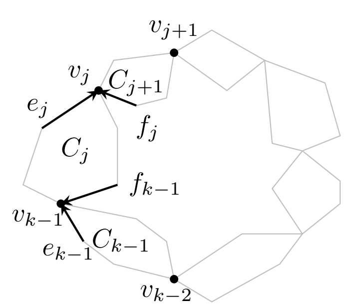
Lemma 4.6. [Ká13, see proof of Theorem 10.1]. *Let D be an alternating dimap. Denote the cycles surrounding the clockwise oriented regions by $C_1, \ldots, C_n$. For any spanning tree T of D, let $a_i(T)$ denote the number of edges of T in the cycle C_i. Then, for a fixed vertex r of D and a fixed sequence $\{s_1, \ldots, s_n\}$, there exists at most one arboresccence T rooted at r such that $a_i(T) = s_i$ for all i.*

Proof. Suppose we have two such arborescences T_1 and T_2. Since these trees are distinct, there must exist some edge $e_1 = (v_0, v_1)$ which is in T_1 but not T_2. Without loss of generality, suppose C_1 is the cycle containing e_1. Observe that since v_1 has an edge pointing towards it in T_1, it cannot be equal to the root r. This implies that T_2 must also have some unique edge $f_1 \neq e_1$ which is directed into v_1 and that f_1 is not in T_1. Let C_2 be the cycle containing f_1. Since $a_2(T_1) = a_2(T_2)$, there must exist some edge e_2 in C_2 which is in T_1 but not T_2. We will denote the final vertex of e_2 by v_2.

Repeating this process, we obtain a sequence of edges $e_1, f_1, e_2, f_2, \ldots$ and a sequence of cycles $C_1, C_2, \ldots$ such that for all i,

(1) e_i is in T_1 but not T_2
(2) f_i is in T_2 but not T_1
(3) e_i and f_{i-1} are in C_i
(4) e_i and f_i share the same final vertex v_i

Let k be the smallest index such that $C_k = C_j$ for some $j < k$. The interiors of the cycles $C_j, \ldots, C_{k-1}$ now form a cycle, as shown in the example below.



In particular, D can be divided into two subgraphs, the portion inside this cycle and the portion outside it, which share only the vertices $v_j, \ldots, v_{k-1}$. Let D_1 and D_2 respectively denote these subgraphs. Since $C_j, \ldots, C_{k-1}$ are all clockwise oriented cycles, all of $e_j, \ldots, e_{k-1}$ will be directed out of one subgraph and all of $f_j, \ldots, f_{k-1}$ out of the other. Suppose without loss of generality that the edges $e_j, \ldots, e_{k-1}$ point into $v_j, \ldots, v_{k-1}$ from D_1.

Each of D_1 and D_2 must contain at least one vertex besides $v_j, \ldots, v_{k-1}$ (otherwise, $\{e_j, \ldots, e_{k-1}\}$ or

$\{f_j, \ldots, f_{k-1}\}$ would form a cycle, contradicting the fact that they are edges of trees). Since $v_j, \ldots, v_{k-1}$ are all the final vertices of some edges of T_1 and T_2, none of them will equal the root r. We can, thus, conclude that r is in precisely one of D_1 or D_2. There is, however, no directed path in T_1 into a vertex in $V(D_1) - \{v_j, \ldots, v_{k-1}\}$ from a vertex in $V(D_2) - \{v_j, \ldots, v_{k-1}\}$ because the only edges of T_1 which are directed into $\{v_j, \ldots, v_{k-1}\}$ are $\{e_j, \ldots, e_{k-1}\}$, all of which point away from D_1. Similarly, there is no directed path in T_2 into a vertex in $V(D_2) - \{v_j, \ldots, v_{k-1}\}$ from a vertex in $V(D_1) - \{v_j, \ldots, v_{k-1}\}$. This gives a contradiction. □

Next, we prove Theorem 4.1.

Proof of Theorem 4.1. By Corollary 4.5, the support of $M_{D,r}(\mathbf{x})$ is M-convex and independent of the choice of root r. It follows from Lemma 4.6 that all coefficients of $M_{D,r}(\mathbf{x})$ are 1 on its support. Thus, $M_{D,r}(\mathbf{x})$ is independent of the choice of r, and we may denote it by $M_D(\mathbf{x})$. By Theorem 2.3, we conclude that $M_D(\mathbf{x})$ is denormalized Lorentzian. □

Corollary 4.7. *The sequence of coefficients of $\widetilde{M_D}(t,q)$ is the number of integer points in the intersection of the Newton polytope of $M_D(x_1, \ldots, x_k)$ with parallel hyperplanes of the form $p_1 + \cdots + p_l = c$, $c \in \mathbb{Z}_{\geq 0}$.*

Proof. By the proof of Theorem 4.1, all coefficients of M_D are 1 and its support is M-convex. Thus, applying Corollary 3.8 yields the result. □

It follows from Theorem 3.5 and Corollary 4.7 that in the special case where $D = \overrightarrow{\mathcal{G}(L_G)}$, we can interpret the coefficients of the Alexander polynomial of $\Delta_L(-t)$ for a special alternating link L as the integer point counts in a series of parallel hyperplanes intersecting a generalized permutahedron. Such an interpretation is closely related to the work of Li and Postnikov on slicing zonotopes [LP13]. Moreover, Kálmán's work [Ká13] implies that the support of M_D is a trimmed generalized permutahedron [Pos09, Definition 11.2] dependent on the dimap D.

4.3. **Log-concavity of the coefficients of $\Delta_{L_G}(-t)$.** From Theorems 3.5 and 4.1 and Lemma 2.5, we readily obtain:

Theorem 1.2. *The coefficients of the Alexander polynomial $\Delta_L(-t)$ of a special alternating link L form a log-concave sequence with no internal zeros. In particular, they are unimodal, proving Fox's conjecture in this case.*

Proof. If L is a positive special alternating link, then by Theorems 3.5 and 4.1 and Lemma 2.5, we conclude that $\mathrm{Homog}_q(\Delta_L(-t))$ is denormalized Lorentzian. Therefore, $N(\mathrm{Homog}_q(\Delta_L(-t)))$ is Lorentzian. By Proposition 2.4, this implies that the coefficients of $\mathrm{Homog}_q(\Delta_L(-t))$, which are the same as those of $\Delta_L(-t)$, are log-concave with no internal zeros.

The case where L is a negative special alternating link follows from Remarks 3.2 and 3.6. □

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