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The core of monomial ideals

Louiza Fouli, Jonathan Montaño, Claudia Polini and Bernd Ulrich

The core of an ideal is defined as the intersection of all of its reductions. We provide an explicit description for the core of a monomial ideal I satisfying certain residual conditions, showing that $\operatorname{core}(I)$ coincides with the largest monomial ideal contained in a general reduction of I. We prove that the class of lex-segment ideals satisfies these residual conditions and study the core of lex-segment ideals generated in one degree. For monomial ideals that do not necessarily satisfy the residual conditions and that are generated in one degree, we conjecture an explicit formula for the core, and make progress towards this conjecture.

1. Introduction

The *core* of an ideal I in a Noetherian ring is the intersection of all reductions of I, i.e., all ideals over which I is integral. Since reductions, even minimal ones, are highly nonunique, one uses the core to encode information about all of them. The core appears naturally in the context of Briançon–Skoda theorems that compare the integral closure filtration with the adic filtration of an ideal [Lipman and Sathaye 1981; Hochster and Huneke 1990; Lipman 1994; Lazarsfeld 2004a; 2004b]. It is also related to adjoints and multiplier ideals [Lipman 1994; Huneke and Swanson 1995], to Kawamata's conjecture on the nonvanishing of sections of certain line bundles [Hyry and Smith 2003; 2004], and to the Cayley–Bacharach property of finite sets of points in projective space [Fouli et al. 2010]. Knowing the core, say of a zero-dimensional ideal in a local Cohen–Macaulay ring, can be helpful in proofs via reduction to the Artinian case; for the elements of $I \setminus \operatorname{core}(I)$ are exactly those elements in I that remain nonzero when reducing modulo some general system of parameters inside I; see for instance [Engheta 2009; Huneke et al. 2015].

Being an a priori infinite intersection of reductions, the core is difficult to compute. Explicit formulas for the core have been found, but they require strong hypotheses [Huneke and Swanson 1995; Corso et al. 2002; Hyry and Smith 2003; Polini and Ulrich 2005; Huneke and Trung 2005; Polini et al. 2007; Wang 2008; Fouli et al. 2008; 2010; Smith 2011; Fouli and Morey 2012; Kohlhaas 2014; 2016; Cumming 2018; Okuma et al. 2018]. Without such hypotheses, the best one could hope for is that the core is a finite intersection of *general* reductions. This was proved in the local case assuming fairly weak *residual conditions* [Corso et al. 2001]; see Section 2 for definitions. The first main theorem, Theorem 3.9, in the

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current article generalizes this result to the nonlocal setting, a nontrivial generalization as the core is not known to be compatible with localization. In fact, our result shows a posteriori that the core does localize in the setting of the theorem; see Corollary 3.10. If in addition I is generated by homogeneous polynomials of the same degree, we also prove that the core coincides with the *graded core*, the intersection of all homogeneous reductions of I; see Corollary 3.12. The question of when this equality holds was also considered by Hyry and Smith [2003] in connection with their work on Kawamata's conjecture. Without a result as in Theorem 3.9, the core is essentially uncomputable as one does not know how to identify the *special* reductions needed in the intersection. In this paper we propose a method for finding such reductions in the case of monomial ideals; see Section 5.

With the same weak residual conditions as in Theorem 3.9 we come close to proving a formula for the core in the monomial case, by expressing the core of a monomial ideal in terms of a single general reduction. This result is based on the fact that the core of a monomial ideal I is again monomial, and hence contained in the largest monomial ideal $\mathrm{mono}(K)$ contained in any reduction K. When the reduction K is general, it is highly nonmonomial and hence $\mathrm{mono}(K)$ is as close to the core as possible. In Theorem 4.7 we prove that in fact

$$\mathrm{core}(I) = \mathrm{mono}(K)$$

if the aforementioned residual conditions are satisfied. This generalizes a result from [Polini et al. 2007] for the case of zero-dimensional monomial ideals. The mono of any ideal can be computed using an algorithm by Saito, Sturmfels, and Takayama [Saito et al. 2000], and this is implemented in Macaulay2 and can be accessed with the command `monomialSubideal`.

Examples show that the results described above do not hold without any residual conditions; see Examples 4.8 and 4.9. In Section 5 we treat the graded core of monomial ideals that are generated in a single degree but do not satisfy any further assumptions. Whereas the graded core is always contained in $\mathrm{mono}(K)$ for K a general reduction, in Theorem 5.4 we come up with a monomial ideal $\mathfrak{A}$ contained in the graded core, and we conjecture that in fact $\mathrm{gradedcore}(I) = \mathfrak{A}$; see Conjecture 5.5. We also propose a way to find the special reductions required in the intersection that gives the graded core; see Discussion 5.8. These results use, in an essential way, the ideal J generated by d linear combinations of the monomial generators of $I \subset R = \Bbbk[x_1, \dots, x_d]$ with new variables $\underline{z} = z_{ij}$ as coefficients. Considering $\Bbbk[\underline{z}][x_1, \dots, x_d]$ as a polynomial ring in the variables $x_1, \dots, x_d$, we form the ideal $\mathrm{mono}(J)$, which is generated by monomials $m \in \Bbbk[x_1, \dots, x_d]$ times ideals $C_m \subset \Bbbk[\underline{z}]$. Due to the variation of the ideals C_m, the ideal $\mathrm{mono}(J)$ carries considerably more information than $\mathrm{mono}(K)$ for a general reduction $K \subset R$, which only records the monomials m and does not suffice to determine $\mathrm{gradedcore}(I)$. As an application we prove that if the ideals C_m are constant up to radical then the graded core is the mono of a general minimal reduction without any residual conditions; see Theorem 5.10.

In the last section of the article we focus on the special class of lex-segment ideals. We first show that these ideals satisfy the residual conditions as in Theorem 3.9. We conjecture that the core of a lex-segment ideal I generated in a single degree is equal to I times a certain power of the maximal homogeneous ideal;

see Conjecture 6.1. We prove one inclusion in full generality and establish the conjecture for a large number of cases; see Theorem 6.9 and Remark 6.2. We also show that the core of I is contained in the adjoint of I^g, where $g = \mathrm{ht}(I)$. The connection between cores and adjoints is particularly attractive in the context of monomial ideals, since there is an explicit combinatorial description for adjoints in terms of Newton polyhedra [Howald 2001], a description that is lacking for cores, even in the zero-dimensional case.

2. Background

In this section we provide some background information and fix notations needed in the rest of the article, including the residual conditions mentioned in the Introduction. For further information we refer to [Ulrich 1994; Chardin et al. 2001; Huneke and Swanson 2006].

Let R be a Cohen–Macaulay ring and I an ideal. A subideal $J \subseteq I$ is a *reduction* of I if I and J have the same integral closure, or equivalently, if

$$I^{n+1} = JI^n \quad \text{for } n \gg 0. \tag{1}$$

The *reduction number of I with respect to J*, denoted by $r_J(I)$, is the smallest nonnegative integer n for which (1) holds true.

Suppose either R is local with maximal ideal $\mathfrak{m}$ and residue field $\Bbbk$, or R is positively graded over a field $\Bbbk$ with maximal homogeneous ideal $\mathfrak{m}$ and I is homogeneous generated in a single degree. We denote by $\ell(I)$ the *analytic spread* of I, i.e., the dimension of the *special fiber ring* $\mathscr{F}(I) = \bigoplus_{n\geqslant 0} I^n/\mathfrak{m}I^n$. If $\Bbbk$ is an infinite field, then $\ell(I)$ is equal to the minimal number of generators $\mu(J)$ of any minimal reduction J of I. Recall that a minimal reduction is a reduction that is minimal with respect to inclusion. The *reduction number of I* is $r(I) = \min\{r_J(I) \mid J \text{ is a minimal reduction of } I\}$.

Artin and Nagata [1972] defined the notion of s-residual intersection that generalizes the notion of linkage when the linked ideals may not have the same height. To be precise, an R-ideal K in an arbitrary Cohen–Macaulay ring R is an *s-residual intersection of I* if $K = \mathfrak{a} : I$ for some s generated ideal $\mathfrak{a} \subsetneq I$ such that $\mathrm{ht}(K) \geqslant s$. We say K is a *geometric s-residual intersection of I* if in addition we have $\mathrm{ht}(I + K) > s$. The ideal I is said to be *weakly s-residually S_2* if the ring R/K satisfies Serre's condition S_2 for every $0 \leqslant i \leqslant s$ and for every geometric i-residual intersection K of I. We say that I satisfies G_s if $\mu(I_\mathfrak{p}) \leqslant \mathrm{ht}(\mathfrak{p})$ for every $\mathfrak{p} \in V(I)$ such that $\mathrm{ht}(\mathfrak{p}) \leqslant s - 1$.

In this article we deal with ideals that satisfy the residual conditions G_d and are weakly $(d-2)$-residually S_2, where $d = \dim(R)$. Classes of ideals that satisfy these two conditions include ideals of dimension 1 that are generically complete intersections. Moreover, if an ideal I satisfies G_d, then I is weakly $(d-2)$-residually S_2 if it is strongly Cohen–Macaulay or, more generally, if after localizing it has the sliding depth property; see [Huneke 1983, Theorem 3.1] and [Herzog et al. 1985, Theorem 3.3]. Examples of strongly Cohen–Macaulay ideals are Cohen–Macaulay almost complete intersections, Cohen–Macaulay ideals in a Gorenstein ring generated by $\mathrm{ht}(I) + 2$ elements [Avramov and Herzog 1980, page 259], and ideals in the linkage class of a complete intersection [Huneke 1982, Theorem 1.11], such as perfect ideals of height 2 [Apéry 1945; Gaeta 1952] and perfect Gorenstein ideals of height 3 [Watanabe 1973]. In this article we add to this list by proving that lex-segment ideals satisfy both residual conditions; see Proposition 6.5.

3. The core and general reductions

In this section we define a notion of *general reductions* for homogeneous ideals that are not necessarily generated in one degree, and we use this new notion to show a homogeneous version of [Corso et al. 2001, Theorem 4.5] for ideals of maximal analytic spread (see Theorem 3.9). We begin by setting up some notation.

Notation 3.1. Let $\Bbbk$ be an infinite field, R a Noetherian $\Bbbk$-algebra, and $I \subset R$ an ideal. Fix a positive integer s and $\underline{f} = f_1, \dots, f_u$ a generating sequence for I. Consider an $s \times u$ matrix of variables $[\underline{z}] = [z_{i,j}]$ with $1 \leqslant i \leqslant s$ and $1 \leqslant j \leqslant u$. An ideal $J_{s,\underline{z}} := J_{s,\underline{z}}(\underline{f})$ of $R[\underline{z}]$ is said to be generated by s *generic elements of* I (*with respect to* $\underline{f}$), if it is generated by the entries of the column vector $[\underline{z}][\underline{f}]^T$.

For every $\underline{\lambda} = (\lambda_{ij}) \in \mathbb{A}_{\Bbbk}^{su}$ we define $\pi_{\underline{\lambda}} : R[\underline{z}] \to R$ to be the evaluation map given by $z_{i,j} \mapsto \lambda_{i,j}$. For a positive integer n we say that the R-ideals $J_1, \dots, J_n$ are generated by s *general elements* (*with respect to* $\underline{f}$) of I, if $J_i = \pi_{\underline{\lambda_i}}(J_{s,\underline{z}})$ and $(\underline{\lambda_1}, \dots, \underline{\lambda_n})$ ranges over a Zariski dense open subset of $\mathbb{A}_{\Bbbk}^{nsu}$.

Remark 3.2. Using Notation 3.1, let $A \in \mathrm{GL}_u(R)$ and consider the R-automorphism ϕ of $R[\underline{z}]$ that sends the matrix $[\underline{z}]$ to $[\underline{z}]A$. If $[\underline{g}]^T := A[\underline{f}]^T$, then $\phi(J_{s,\underline{z}}(\underline{f})) = J_{s,\underline{z}}(\underline{g})$.

The following lemma shows that saturating the ideal $J_{s,\underline{z}}$ with respect to I is the same as saturating it with respect to any nonzero element $f \in I$.

Lemma 3.3. *We use Notation 3.1. Let $f \in I$ be a nonzero element. If R is a domain, then*

$$J_{s,\underline{z}} :_{R[\underline{z}]} I^\infty = J_{s,\underline{z}} :_{R[\underline{z}]} f^\infty$$

and this is a prime ideal of height s.

Proof. We clearly have $J_{s,\underline{z}} : I^\infty \subseteq J_{s,\underline{z}} : f^\infty$. To prove the equality it suffices to show that the ideal on the left is a prime ideal of height s and the one on the right has height at most s. Notice that $\mathrm{ht}(J_{s,\underline{z}} : f^\infty) \leqslant \mathrm{ht}((J_{s,\underline{z}} : f^\infty)_f) = \mathrm{ht}((J_{s,\underline{z}})_f) \leqslant s$. The last inequality follows by Krull's altitude theorem; notice that $(J_{s,\underline{z}})_f$ is a proper ideal as $f \notin \sqrt{J_{s,\underline{z}}}$.

Since I contains a nonzerodivisor modulo $J_{s,\underline{z}} : I^\infty$, a general $\Bbbk$-linear combination of $f_1, \dots, f_u$ is a nonzerodivisor modulo $J_{s,\underline{z}} : I^\infty$. Hence there exists $A \in \mathrm{GL}_u(\Bbbk)$ such that $[g_1 \cdots g_u]^T = A[f_1 \cdots f_u]^T$ with g_1 a nonzerodivisor modulo $J_{s,\underline{z}} : I^\infty$. By Remark 3.2 we may replace $f_1, \dots, f_u$ by $g_1, \dots, g_u$ to assume f_1 is a nonzerodivisor modulo $J_{s,\underline{z}} : I^\infty$. Notice that

$$(J_{s,\underline{z}} : I^\infty)_{f_1} = (J_{s,\underline{z}})_{f_1} = \left(\left\{z_{i,1} + f_1^{-1} \sum_{j=2}^{u} z_{i,j} f_j \;\middle|\; 1 \leqslant i \leqslant s\right\}\right) R[\underline{z}]_{f_1},$$

which is a prime ideal of height s. Since f_1 is a nonzerodivisor modulo $J_{s,\underline{z}} : I^\infty$, it follows that $J_{s,\underline{z}} : I^\infty$ is a prime ideal of height s. □

The following lemma is needed in the proof of Proposition 3.5, which in turn provides a way to construct general reductions of ideals.

Lemma 3.4. *Let $\Bbbk$ be an infinite field, R a finitely generated $\Bbbk$-algebra, and I an ideal. If J is an ideal generated by s general elements of I, then*

$$\dim(R/(J :_R I^\infty)) \leqslant \dim(R) - s$$

and

$$\dim(R/((J :_R I^\infty) + I)) \leqslant \dim(R) - s - 1.$$

Proof. The first inequality is [Fouli et al. 2008, Lemma 2.2]. The second inequality follows from the first because I contains a nonzerodivisor modulo $J : I^\infty$. □

If either the ambient ring R is local or the ideal I is generated by forms of the same degree and R is a positively graded $\Bbbk$-algebra, then d general elements of I generate a reduction, where $d = \dim(R)$. The following proposition gives a method to construct finite sets of general reductions for arbitrary ideals in any Noetherian $\Bbbk$-algebra (see Definition 3.6).

Proposition 3.5. *Let $\Bbbk$ be an infinite field, R a finitely generated $\Bbbk$-algebra of dimension d, and I an ideal of positive height. If R is positively graded and I is generated by forms of the same degree, set $f := 0 \in R$. Otherwise, let $f \in I$ be an element not contained in any minimal prime ideal of R of dimension d. If J is an ideal generated by d general elements of I, then $J + (f)$ is a reduction of I.*

Proof. Set $H = J + (f)$ and let $\bar{\ }$ denote the images in $\bar{R} = R/(f)$. Applying Lemma 3.4 to the images of H and I in $\bar{R}$ we observe that $\bar{H} :_{\bar{R}} \bar{I}^\infty = \bar{R}$, and hence H and I have the same radical. Since $HI^{n-1} : I^n \subseteq HI^n : I^{n+1}$ for every $n \in \mathbb{N}$, this sequence of ideals stabilizes for $n \gg 0$. Therefore, it suffices to prove that $H_\mathfrak{p}$ is a reduction of $I_\mathfrak{p}$ for every $\mathfrak{p} \in V(I)$. By Lemma 3.4 we have $(J : I^\infty) + I = R$ and therefore $I_\mathfrak{p}^n \subseteq J_\mathfrak{p}$ for $n \gg 0$ and every $\mathfrak{p} \in V(I)$. Let $\mathrm{gr}_I(R) = \bigoplus_{n \geqslant 0} I^n/I^{n+1}$. We can proceed as in [Xie 2012, Proposition 2.3] to show that the images in $I/I^2 = [\mathrm{gr}_I(R)]_1$ of the d generators of J form a filter regular sequence with respect to $\mathrm{gr}_I(R)_+ = \bigoplus_{n>0} I^n/I^{n+1}$. Therefore, J is generated by a superficial sequence of I. Then, $J \cap I^{n+1} = JI^n$ for $n \gg 0$ by [Huneke and Swanson 2006, Lemma 8.5.11]. Thus, $J_\mathfrak{p}$ is a reduction of $I_\mathfrak{p}$ for every $\mathfrak{p} \in V(I)$. Thus, $H_\mathfrak{p}$ is a reduction of $I_\mathfrak{p}$, which finishes the proof. □

Definition 3.6. With assumptions and notations as in Proposition 3.5, we say that $K_1, \dots, K_n$ are *general reductions* of I, if $K_i = J_i + (f)$, where $J_1, \dots, J_n$ are n ideals generated by d general elements of I as in Notation 3.1.

Remark 3.7. Notice that for each $r > 0$ we have $K_i' := J_i + (f^r)$ is a reduction of I by Proposition 3.5. Furthermore, for $\mathfrak{m} \in V(I)$ a fixed maximal ideal we have $(J_i)_\mathfrak{m}$ is a reduction of $I_\mathfrak{m}$ and $(K_i')_\mathfrak{m} = (J_i)_\mathfrak{m}$ if $r > r(I_\mathfrak{m})$ [Huneke and Swanson 2006, Theorem 8.6.6]. If in addition $\ell(I_\mathfrak{m}) = d$, then $(K_i')_\mathfrak{m}$ is a minimal reduction of $I_\mathfrak{m}$. In case $R_\mathfrak{m}$ is regular and $\dim(R_\mathfrak{m}) = d$, we also have $(K_i')_\mathfrak{m} = (J_i)_\mathfrak{m}$ for $r \geqslant d$ by [loc. cit., Corollary 13.3.4]. We also call the ideals K_i' *general reductions* of I for any choice of r.

Lemma 3.8. *Let R be a Cohen–Macaulay ring, s a nonnegative integer, and I an ideal satisfying G_{s+1}. Then I is weakly s-residually S_2 if and only if $I_\mathfrak{p}$ is weakly s-residually S_2 for every $\mathfrak{p} \in V(I)$.*

Proof. The property of being weakly s-residually S_2 localizes; see [Corso et al. 2001, Lemma 2.1(a)]. For the converse, let $i \leqslant s$, let $K = J : I$ be a geometric i-residual intersection of I, and let $\mathfrak{p} \in V(K)$. If $\mathfrak{p} \in V(I)$, then $R_\mathfrak{p}/K_\mathfrak{p}$ is S_2 by our assumption. If $\mathfrak{p} \notin V(I)$, then $K_\mathfrak{p} = J_\mathfrak{p}$ is a complete intersection, and therefore $R_\mathfrak{p}/K_\mathfrak{p}$ is Cohen–Macaulay. □

Theorem 3.9 extends [Corso et al. 2001, Theorem 4.5] from local rings to finitely generated algebras over a field. We recall that an ideal I is of *linear type* if the natural map between its symmetric algebra and Rees algebra is an isomorphism. If I is of linear type it has no proper reductions and hence $\operatorname{core}(I) = I$.

Theorem 3.9. *Let $\Bbbk$ be an infinite field, R a Cohen–Macaulay finitely generated $\Bbbk$-algebra of dimension d, and I an ideal of positive height. Assume that I satisfies G_d and is weakly $(d-2)$-residually S_2. Let $f \in I$ be as in Definition 3.6. Then there exist positive integers n and r such that*

$$\operatorname{core}(I) = K_1 \cap \cdots \cap K_n,$$

where $K_i = J_i + (f^r)$ for $1 \leqslant i \leqslant n$ are general reductions of I as in Remark 3.7. If in addition R is regular, then r can be chosen to be d.

Proof. For every $\mathfrak{p} \in V(I)$ the ideal $I_\mathfrak{p}$ is G_d and from Lemma 3.8 it follows that $I_\mathfrak{p}$ is weakly $(d-2)$-residually S_2. Hence [Chardin et al. 2001, Corollary 3.6(b)] and [Vasconcelos 1994, Theorem 2.3.2] show that $I_\mathfrak{p}$ is of linear type for every $\mathfrak{p} \in V(I)$ with $\operatorname{ht}(\mathfrak{p}) < d$. Clearly, $I_\mathfrak{p}$ is of linear type for every prime $\mathfrak{p} \notin V(I)$.

Let $\operatorname{Sym}(I)$ and $\mathscr{R}(I)$ be the symmetric algebra and the Rees algebra of I, respectively. Consider the following exact sequence

$$0 \to \mathscr{A} \to \operatorname{Sym}(I) \to \mathscr{R}(I) \to 0.$$

The ideal $\mathscr{A}$ is generated by homogeneous elements of degree at most e, for some nonnegative integer e. Therefore, $\operatorname{Supp}_R(\mathscr{A}) = \bigcup_{i=0}^{e} \operatorname{Supp}_R(\mathscr{A}_i)$ is a closed subset of $\operatorname{Spec}(R)$. It follows that the set of prime ideals $\mathfrak{p}$ such that $I_\mathfrak{p}$ is not of linear type consists only of finitely many maximal ideals, say $\mathfrak{m}_1, \ldots, \mathfrak{m}_t$. Notice that $\ell(I_{\mathfrak{m}_i}) = d$ for each $1 \leqslant i \leqslant t$; indeed, if $\ell(I_{\mathfrak{m}_i}) < d$, then $I_{\mathfrak{m}_i}$ is generated by $d-1$ elements according to [Corso et al. 2001, Lemma 2.1(g)] and hence $I_{\mathfrak{m}_i}$ would be of linear type by [Chardin et al. 2001, Corollary 3.6(b)] and [Vasconcelos 1994, Theorem 2.3.2].

The ideals $I_{\mathfrak{m}_i}$ have analytic spread d, satisfy G_d, and are weakly $(d-2)$-residually S_2, and hence are weakly $(d-1)$-residually S_2; see [Chardin et al. 2001, Proposition 3.4(a)]. Applying [Corso et al. 2001, Theorem 4.5] and Remark 3.7 to the finitely many ideals $I_{\mathfrak{m}_i}$, we obtain that $\operatorname{core}(I_{\mathfrak{m}_i}) = (K_1)_{\mathfrak{m}_i} \cap \cdots \cap (K_n)_{\mathfrak{m}_i}$ for some integer n, where $K_1, \ldots, K_n$ are general reductions of I with $r = 1 + \max\{r(I_{\mathfrak{m}_i}) \mid 1 \leqslant i \leqslant t\}$ or with $r = d$ in case R is regular.

We claim that $\operatorname{core}(I) = K_1 \cap \cdots \cap K_n$. Clearly, $\operatorname{core}(I) \subseteq K_1 \cap \cdots \cap K_n$, because $K_1, \ldots, K_n$ are reductions of I by Proposition 3.5. To show the reverse inclusion, let K be any reduction of I. We need to show that $K_1 \cap \cdots \cap K_n \subseteq K$, or equivalently $(K_1)_\mathfrak{p} \cap \cdots \cap (K_n)_\mathfrak{p} \subseteq K_\mathfrak{p}$ for every $\mathfrak{p} \in \operatorname{Spec}(R)$. If $\mathfrak{p} \notin \{\mathfrak{m}_1, \ldots, \mathfrak{m}_t\}$, then $I_\mathfrak{p}$ is of linear type. Hence $K_\mathfrak{p} = I_\mathfrak{p}$ and the assertion holds trivially. Otherwise, $(K_1)_\mathfrak{p} \cap \cdots \cap (K_n)_\mathfrak{p} = \operatorname{core}(I_\mathfrak{p}) \subset K_\mathfrak{p}$. □

Theorem 3.9 and its proof allows us to show that under the assumptions therein the core of I localizes; compare to [Corso et al. 2001, Theorem 4.8].

Corollary 3.10. *If the hypotheses of Theorem 3.9 hold, then* $\operatorname{core}(I_\mathfrak{p}) = (\operatorname{core}(I))_\mathfrak{p}$ *for every* $\mathfrak{p} \in \operatorname{Spec}(R)$.

Proof. In the proof of Theorem 3.9 we showed that $\operatorname{core}(I_{\mathfrak{m}_i}) = (K_1)_{\mathfrak{m}_i} \cap \cdots \cap (K_n)_{\mathfrak{m}_i}$, which is $(\operatorname{core}(I))_{\mathfrak{m}_i}$ by Theorem 3.9. If $\mathfrak{p} \notin \{\mathfrak{m}_1, \ldots, \mathfrak{m}_t\}$, then $I_\mathfrak{p}$ is of linear type, therefore $\operatorname{core}(I_\mathfrak{p}) = I_\mathfrak{p}$ and $(\operatorname{core}(I))_\mathfrak{p} = (K_1)_\mathfrak{p} \cap \cdots \cap (K_n)_\mathfrak{p} = I_\mathfrak{p}$. □

Remark 3.11. If in addition to the assumptions of Theorem 3.9 the ring R is a positively graded $\Bbbk$-algebra with maximal homogeneous ideal $\mathfrak{m}$ and the ideal I is homogeneous, then we can replace the assumption that I is weakly $(d-2)$-residually S_2 by the hypothesis that $I_\mathfrak{m}$ is weakly $(d-2)$-residually S_2. In addition, r can be chosen to be $1 + r(I_\mathfrak{m})$.

Proof. Following the proof of Theorem 3.9, it suffices to show that $I_\mathfrak{p}$ is of linear type whenever $\mathfrak{p} \in V(I)$ and $\mathfrak{p} \neq \mathfrak{m}$. Since I is homogeneous, the minimal prime ideals of $\operatorname{Supp}_R(\mathscr{A})$ are homogeneous and hence are all contained in $\mathfrak{m}$. On the other hand, if $\mathfrak{p} \subsetneq \mathfrak{m}$, then $I_\mathfrak{p}$ is of linear type by our assumption on $I_\mathfrak{m}$; see [Chardin et al. 2001, Corollary 3.6(b)] and [Vasconcelos 1994, Theorem 2.3.2]. □

In the case of ideals generated by forms of the same degree, we have the following simpler version of Theorem 3.9.

Corollary 3.12. *Let $\Bbbk$ be an infinite field, R a Cohen–Macaulay positively graded $\Bbbk$-algebra of dimension d, and $\mathfrak{m}$ the homogeneous maximal ideal of R. Let I be an ideal of positive height generated by homogeneous elements of the same degree δ. If I satisfies G_d and $I_\mathfrak{m}$ is weakly $(d-2)$-residually S_2, then there exists a positive integer n such that*

$$\operatorname{core}(I) = J_1 \cap \cdots \cap J_n,$$

where $J_1, \ldots, J_n$ are generated by d general elements of I with respect to a generating set of I contained in I_δ (see Notation 3.1). In particular,

$$\operatorname{core}(I) = \operatorname{gradedcore}(I).$$

Proof. We choose $f = 0$ as in Proposition 3.5, so that $J_i = K_i$. Now the assertion follows by Theorem 3.9 and Remark 3.11. □

Remark 3.13. The proof of Theorem 3.9 shows that under the assumptions of Corollary 3.12 either $\ell(I) = d$ or $\operatorname{core}(I) = I$.

Question 3.14. Is it possible to replace the assumptions in Corollary 3.12 that I satisfies G_d and $I_\mathfrak{m}$ is weakly $(d-2)$-residually S_2 by the hypotheses that I satisfies G_ℓ and $I_\mathfrak{m}$ is weakly $(\ell-1)$-residually S_2

for $\ell = \ell(I)$, to say that

$$\operatorname{core}(I) = J_1 \cap \cdots \cap J_n,$$

where $J_1, \ldots, J_n$ are generated by ℓ general elements of I with respect to a generating set of I contained in I_δ?

4. The core and the mono

In this section we generalize the main result of [Polini et al. 2007] to monomial ideals of higher dimension. We show that under suitable residual conditions, the core of a monomial ideal I coincides with the largest monomial ideal in a general reduction of I, provided I is of maximal analytic spread.

Proposition 4.1. *Let $\Bbbk$ be an infinite field, $\underline{x} = x_1, \ldots, x_r$, $\underline{y} = y_1, \ldots, y_s$, and $\underline{z} = z_1, \ldots, z_t$ be three sets of variables. Let H be an ideal of $\Bbbk[\underline{x}, \underline{y}, \underline{z}]$. For every $\underline{\lambda} = (\lambda_i) \in \mathbb{A}^t_{\Bbbk}$ let $\pi_{\underline{\lambda}} : \Bbbk[\underline{x}, \underline{y}, \underline{z}] \to \Bbbk[\underline{x}, \underline{y}]$ denote the evaluation map given by $z_i \mapsto \lambda_i$. Then for general $\underline{\lambda}$ we have $\pi_{\underline{\lambda}}(H \cap \Bbbk[\underline{x}, \underline{z}]) = \pi_{\underline{\lambda}}(H) \cap \Bbbk[\underline{x}]$.*

Proof. It is straightforward to see that $\pi_{\underline{\lambda}}(H \cap \Bbbk[\underline{x}, \underline{z}]) \subseteq \pi_{\underline{\lambda}}(H) \cap \Bbbk[\underline{x}]$.

To prove the reverse inclusion, we consider the lexicographic monomial order $<$ on the two polynomial rings $\Bbbk[\underline{z}][\underline{x}, \underline{y}]$ and $A := \Bbbk(\underline{z})[\underline{x}, \underline{y}]$ in the variables $\underline{x}, \underline{y}$ with $x_i < y_j$. Let $G = \{g_1, \ldots, g_m\} \subset H$ be a Gröbner basis of HA with respect to $<$.

Clearly $\pi_{\underline{\lambda}}(G)$ is a generating set of $\pi_{\underline{\lambda}}(H)$ for general $\underline{\lambda}$. We claim that for general $\underline{\lambda}$ the set $\pi_{\underline{\lambda}}(G)$ is also a Gröbner basis. By Buchberger's criterion it suffices to show that for every $i \neq j$ the S-pair $S_{ij} := S(\pi_{\underline{\lambda}}(g_i), \pi_{\underline{\lambda}}(g_j))$ is equal to an expression

$$h_1 \pi_{\underline{\lambda}}(g_1) + \cdots + h_m \pi_{\underline{\lambda}}(g_m), \tag{2}$$

where $h_k \in \Bbbk[\underline{x}, \underline{y}]$ and the initial monomials satisfy $\operatorname{in}_<(h_k \pi_{\underline{\lambda}}(g_k)) \leqslant \operatorname{in}_<(S_{ij})$ for every $1 \leqslant k \leqslant m$. Since G is a Gröbner basis of HA, there is an expression

$$S(g_i, g_j) = \tilde{h}_1 g_1 + \cdots + \tilde{h}_m g_m, \tag{3}$$

where $\tilde{h}_k \in A$ and $\operatorname{in}_<(\tilde{h}_k g_k) \leqslant \operatorname{in}_<(S(g_i, g_j))$ for every $1 \leqslant k \leqslant m$.

If $c_{g_k} \in \Bbbk[\underline{z}]$ is the coefficient of $\operatorname{in}_<(g_k)$, then $\pi_{\underline{\lambda}}(c_{g_k})$ is the coefficient of $\operatorname{in}_<(\pi_{\underline{\lambda}}(g_k))$ for general $\underline{\lambda}$. Hence $\operatorname{in}_<(\pi_{\underline{\lambda}}(g_k)) = \operatorname{in}_<(g_k)$ and $S_{ij} = \pi_{\underline{\lambda}}(S(g_i, g_j))$ since $S(g_i, g_j) \in \Bbbk[\underline{x}, \underline{y}, \underline{z}]$. Therefore, after clearing denominators in (3) and applying $\pi_{\underline{\lambda}}$ for general $\underline{\lambda}$, the desired expression for S_{ij} as in (2) follows.

Since $\pi_{\underline{\lambda}}(G)$ is a Gröbner basis of $\pi_{\underline{\lambda}}(H)$ for general $\underline{\lambda}$ and $<$ is an elimination order, it follows that $\pi_{\underline{\lambda}}(H) \cap \Bbbk[\underline{x}]$ is generated by $\pi_{\underline{\lambda}}(G) \cap \Bbbk[\underline{x}]$. Finally, for general $\underline{\lambda}$ we have

$$\pi_{\underline{\lambda}}(G) \cap \Bbbk[\underline{x}] = \pi_{\underline{\lambda}}(G \cap \Bbbk[\underline{x}, \underline{z}]) \subseteq \pi_{\underline{\lambda}}(H \cap \Bbbk[\underline{x}, \underline{z}]),$$

and the conclusion follows. □

The goal in this section is to show that under suitable assumptions on a monomial ideal I, the core of I can be obtained as the mono of a general reduction of I, namely $\operatorname{core}(I) = \operatorname{mono}(K)$, where K is a general reduction of I as in Definition 3.6 and $\operatorname{mono}(K)$ denotes the largest monomial ideal contained in K. In order to compute $\operatorname{mono}(K)$ we follow an algorithm due to Saito, Sturmfels, and Takayama [Saito et al. 2000, Algorithm 4.4.2].

For the proof of our main result we need a notion of mono of an ideal in a polynomial ring over an arbitrary Noetherian ring. Let A be a Noetherian ring. For an ideal L in the polynomial ring $A[x_1, \dots, x_d]$, the *multihomogenization* of L, denoted by $\widetilde{L}$, is the ideal of the polynomial ring $A[x_1, \dots, x_d, y_1, \dots, y_d]$ generated by

$$\left\{ \tilde{g} = g\left(\frac{x_1}{y_1}, \dots, \frac{x_d}{y_d}\right) y_1^{\deg_{x_1}(g)} \cdots y_d^{\deg_{x_d}(g)} \;\middle|\; g \in L \right\}.$$

We consider $A[x_1, \dots, x_d, y_1, \dots, y_d]$ with the $\mathbb{N}^d$-grading induced by $\deg(x_i) = \deg(y_i) = \boldsymbol{e}_i$. We note that $\tilde{g}$ is indeed multihomogeneous with $\deg(\tilde{g}) = (\deg_{x_1}(g), \dots, \deg_{x_d}(g)) \in \mathbb{N}^d$.

The next example illustrates the process of multihomogenization of an element.

Example 4.2. Let $g = c_1 x_1^2 x_2 + c_2 x_1 x_3^2 + c_3 x_2^3 x_3 \in A[x_1, x_2, x_3]$ for some $c_1, c_2, c_3 \in A$. Then $\tilde{g} = c_1 x_1^2 x_2 y_2^2 y_3^2 + c_2 x_1 x_3^2 y_1 y_2^3 + c_3 x_2^3 x_3 y_1^2 y_3$.

To obtain the multihomogenization of an ideal $L \subset A[x_1, \dots, x_d]$ it is enough to multihomogenize a given generating set $g_1, \dots, g_u$ of L and to saturate with respect to $Y = \prod_{j=1}^d y_j$, that is,

$$\widetilde{L} = (\tilde{g}_1, \dots, \tilde{g}_u) : Y^{\infty}. \tag{4}$$

Definition 4.3. Let A be a Noetherian ring and let L be an ideal in the polynomial ring $A[x_1, \dots, x_d]$. We define $\operatorname{mono}(L)$ to be the ideal generated by the elements in L of the form am, where $a \in A$ and m is a monomial.

Following [Saito et al. 2000, Algorithm 4.4.2], we obtain

$$\operatorname{mono}(L) = \widetilde{L} \cap A[x_1, \dots, x_d]. \tag{5}$$

Proposition 4.4. *Let $A = \Bbbk[z_1, \dots, z_t]$ be a polynomial ring over an infinite field $\Bbbk$ and L a proper ideal in the polynomial ring $A[x_1, \dots, x_d]$. For $\underline{\lambda} \in \mathbb{A}_{\Bbbk}^t$ let $\pi_{\underline{\lambda}} : A[x_1, \dots, x_d] \to \Bbbk[x_1, \dots, x_d]$ be the evaluation map given by $z_i \mapsto \lambda_i$. For general $\underline{\lambda} \in \mathbb{A}_{\Bbbk}^t$ we have the following*:

(a) $\widetilde{\pi_{\underline{\lambda}}(L)} = \pi_{\underline{\lambda}}(\widetilde{L})$.

(b) $\operatorname{mono}(\pi_{\underline{\lambda}}(L)) = \pi_{\underline{\lambda}}(\operatorname{mono}(L))$.

(c) $\operatorname{mono}(\pi_{\underline{\lambda}}(L))$ *does not depend on* $\underline{\lambda}$.

Proof. We begin with the proof of (a). We first notice that for any $g \in A[x_1, \dots, x_d]$ and general $\underline{\lambda}$ we have $\widetilde{\pi_{\underline{\lambda}}(g)} = \pi_{\underline{\lambda}}(\tilde{g})$. Write $L = (g_1, \dots, g_u)$, then

$$(\widetilde{\pi_{\underline{\lambda}}(g_1)}, \dots, \widetilde{\pi_{\underline{\lambda}}(g_u)}) = (\pi_{\underline{\lambda}}(\tilde{g}_1), \dots, \pi_{\underline{\lambda}}(\tilde{g}_u)) = \pi_{\underline{\lambda}}(\tilde{g}_1, \dots, \tilde{g}_u).$$

Therefore,

$$\widetilde{\pi_{\underline{\lambda}}(L)} = (\widetilde{\pi_{\underline{\lambda}}(g_1)}, \dots, \widetilde{\pi_{\underline{\lambda}}(g_u)}) : Y^{\infty} = \pi_{\underline{\lambda}}(\tilde{g}_1, \dots, \tilde{g}_u) : Y^{\infty}.$$

On the other hand,

$$\pi_{\underline{\lambda}}(\widetilde{L}) = \pi_{\underline{\lambda}}((\tilde{g}_1, \ldots, \tilde{g}_u) : Y^\infty).$$

Notice that

$$\pi_{\underline{\lambda}}(\tilde{g}_1, \ldots, \tilde{g}_u) \subseteq \pi_{\underline{\lambda}}((\tilde{g}_1, \ldots, \tilde{g}_u) : Y^\infty) \subseteq \pi_{\underline{\lambda}}(\tilde{g}_1, \ldots, \tilde{g}_u) : Y^\infty.$$

Thus to prove that $\widetilde{\pi_{\underline{\lambda}}(L)} = \pi_{\underline{\lambda}}(\widetilde{L})$ it suffices to show that $\pi_{\underline{\lambda}}((\tilde{g}_1, \ldots, \tilde{g}_u) : Y^\infty)$ is saturated with respect to Y, equivalently it suffices to show that Y is a nonzerodivisor on $\Bbbk[\underline{x}, \underline{y}]/\pi_{\underline{\lambda}}((\tilde{g}_1, \ldots, \tilde{g}_u) : Y^\infty) = \Bbbk[\underline{x}, \underline{y}]/\pi_{\underline{\lambda}}(\widetilde{L})$. The image of Y is not a unit in $\Bbbk[\underline{x}, \underline{y}]/\pi_{\underline{\lambda}}(\widetilde{L})$, hence $\Bbbk[\underline{x}, \underline{y}]/(\pi_{\underline{\lambda}}(\widetilde{L}), Y) \neq 0$.

Set

$$T = \frac{\Bbbk[\underline{z}]_{(\underline{z}-\underline{\lambda})}[\underline{x}, \underline{y}]}{\widetilde{L}}$$

and notice that $T/(\underline{z} - \underline{\lambda}) = \Bbbk[\underline{x}, \underline{y}]/\pi_{\underline{\lambda}}(\widetilde{L})$. By generic freeness [Eisenbud 1995, Theorem 14.4], for general $\underline{\lambda}$ the map $\Bbbk[\underline{z}]_{(\underline{z}-\underline{\lambda})} \to T/(Y)$ is flat and hence the elements $\underline{z} - \underline{\lambda}$ form a regular sequence on $T/(Y)$. For this also recall that

$$T/(Y, \underline{z} - \underline{\lambda}) = \Bbbk[\underline{x}, \underline{y}]/(Y, \pi_{\underline{\lambda}}(\widetilde{L})) \neq 0.$$

Since Y is a nonzerodivisor on T it follows that $Y, \underline{z} - \underline{\lambda}$ is a T-regular sequence. As this sequence consists of homogeneous elements in T, and T is a positively graded ring over a local ring, we obtain that $\underline{z} - \underline{\lambda}, Y$ is also a regular sequence [Matsumura 1986, Theorems 16.2 and 16.3]. We conclude that Y is a nonzerodivisor on $T/(\underline{z} - \underline{\lambda}) = \Bbbk[\underline{x}, \underline{y}]/\pi_{\underline{\lambda}}(\widetilde{L})$.

Part (b) is a direct consequence of (a) and Proposition 4.1.

Finally, part (c) follows from (b) because $\pi_{\underline{\lambda}}(\mathrm{mono}(L))$ does not depend on $\underline{\lambda}$ for general $\underline{\lambda}$. Indeed, if $\{a_i m_i\}$ is a finite generating set of $\mathrm{mono}(L)$, where $a_i \in \Bbbk[\underline{z}]$ and m_i are monomials in $x_1, \ldots, x_d$, then for any $\underline{\lambda} \in D\big(\prod_i a_i\big)$ the ideal $\pi_{\underline{\lambda}}(\mathrm{mono}(L))$ is independent of $\underline{\lambda}$. □

Corollary 4.5. *Let $\Bbbk$ be an infinite field, $R = \Bbbk[x_1, \ldots, x_d]$ a polynomial ring, and I a monomial ideal. For any $n \in \mathbb{N}$ let K and $K_1, \ldots, K_n$ be general reductions of I as in Remark 3.7. We have*

$$\mathrm{core}(I) \subseteq \mathrm{mono}(K) \subseteq K_1 \cap \cdots \cap K_n.$$

Proof. Clearly, $\mathrm{core}(I) \subseteq \mathrm{mono}(K)$, since K is a reduction of I by Proposition 3.5 and $\mathrm{core}(I)$ is a monomial ideal by [Corso et al. 2001, proof of Remark 5.1]. For the second inclusion, notice that $\mathrm{mono}(K) = \mathrm{mono}(K_i)$ for all $1 \leqslant i \leqslant n$ according to Proposition 4.4(c). □

Remark 4.6. If in Corollary 4.5, the ideal I is generated by monomials of degree δ and the elements $f_1, \ldots, f_u$ of Notation 3.1 are chosen to be homogeneous polynomials of degree δ and $f = 0$, then K is a homogeneous reduction of I. Therefore

$$\mathrm{gradedcore}(I) \subseteq \mathrm{mono}(K).$$

We now prove the main theorem of this section.

Theorem 4.7. *Let $\Bbbk$ be an infinite field, $R = \Bbbk[x_1, \dots, x_d]$ a polynomial ring, $\mathfrak{m} := (x_1, \dots, x_d)$, and I a monomial ideal. If I satisfies G_d and $I_\mathfrak{m}$ is weakly $(d-2)$-residually S_2, then*

$$\operatorname{core}(I) = \operatorname{mono}(K)$$

for K a general reduction of I with $r = d$ as in Remark 3.7.

Proof. The proof follows by Corollary 4.5, Theorem 3.9, and Remark 3.11. □

The following example shows that Theorems 3.9 and 4.7 do not hold without the assumption that I is G_d.

Example 4.8. Let $R = \mathbb{Q}[x_1, x_2, x_3]$ and $I = (x_1^3, x_1^2x_2, x_1x_3^2, x_3^3)$. The ideal I has height 2 and analytic spread 3. It is weakly 2-residually S_2 because every link of I is unmixed and hence Cohen–Macaulay. However, I does not satisfy G_3. Computation with Macaulay2 shows that there exist nonzero polynomials h and g in $\mathbb{Q}[\underline{z}]$ such that

$$\operatorname{mono}(J_{3,\underline{z}}) = (h)(x_1^2, x_1x_2, x_1x_3, x_2x_3, x_3^2)I + (hg)(x_1^2x_2^3, x_1x_2^2x_3^2, x_2^2x_3^3) =: (h)\mathfrak{A} + (hg)\mathfrak{B}.$$

For general $\underline{\lambda}$ we have $\mathfrak{A} + \mathfrak{B} = \pi_{\underline{\lambda}}(\operatorname{mono}(J_{3,\underline{z}})) = \operatorname{mono}(K)$, where $K := \pi_{\underline{\lambda}}(J_{3,\underline{z}})$ (see Proposition 4.4). Therefore $\operatorname{core}(I) \subseteq \operatorname{gradedcore}(I) \subseteq \mathfrak{A}+\mathfrak{B} = I\mathfrak{m}^2$ (see Remark 4.6). The ideal $H = (x_1^3, x_1^2x_2, x_1x_3^2+x_3^3)$ is a minimal reduction of I, since $I^3 = HI^2$. On the other hand, $\operatorname{mono}(K) = I\mathfrak{m}^2 \not\subseteq H$. Hence, $\operatorname{gradedcore}(I)$ is not equal to $\operatorname{mono}(K)$ for a general reduction K and thus $\operatorname{core}(I)$ is not a finite intersection of general reductions of I (see Corollary 4.5). In particular, neither Theorem 3.9 nor Theorem 4.7 hold.

The ideal in the next example is G_d (in fact G_∞), but $\ell(I) < d$ and $I_\mathfrak{m}$ is not weakly $(d-2)$-residually S_2 (see Remark 3.13 and Corollary 3.12). Again, $\operatorname{core}(I)$ is not a finite intersection of general minimal reductions of I and it is not the mono of a general minimal reduction of I.

Example 4.9. Let

$$R = \mathbb{Q}[x_1, x_2, x_3, x_4, x_5, x_6] \quad \text{and} \quad I = (x_1x_2, x_2x_3, x_3x_4, x_4x_1, x_4x_5, x_5x_6).$$

One easily verifies that the height of I is 3 and that it satisfies G_∞. However, $\ell(I) = 5$ and $I_\mathfrak{m}$ is not weakly 3-residually S_2. In [Fouli and Morey 2012, Example 4.8] it is shown that $\operatorname{core}(I) \neq \mathfrak{m}I$. Using Macaulay2 one verifies that $\operatorname{mono}(J) = \mathfrak{m}I$, for J a general minimal reduction of I, i.e., an ideal generated by five general elements of I with respect to the six monomial generators of I. Therefore $\operatorname{core}(I)$ is not equal to $\operatorname{mono}(J)$.

5. The core of monomial ideals generated in one degree

There is no known method to compute the core of a given ideal if the residual conditions required in the previous sections do not hold. Our goal in this section is to propose an approach to compute the core of monomial ideals generated in a single degree without any further assumptions.

In the previous section we established that for a monomial ideal I, core(I) and gradedcore(I) are contained in mono(K) for a general reduction K of I. This containment holds in general and it is an equality under appropriate residual conditions (see Corollary 4.5, Remark 4.6, Theorem 4.7). For a monomial ideal I generated in a single degree we construct an ideal that is contained in gradedcore(I) and we conjecture that equality holds in general (see Theorem 5.4 and Conjecture 5.5). We verify the conjecture for a specific ideal in Example 5.9. Furthermore, under the same residual conditions core(I) can be obtained as the intersection of finitely many general reductions; see Corollary 3.12. However, as seen in Example 4.8, *special* reductions are needed in the absence of the residual conditions. In this section we provide a method to find these special reductions.

Notation 5.1. Let $\Bbbk$ be an infinite field, $R = \Bbbk[x_1, \dots, x_d]$ a polynomial ring, and $\mathfrak{m} = (x_1, \dots, x_d)$ its homogeneous maximal ideal. Let I be a nonzero ideal generated by homogeneous elements of the same degree δ. Let $\mathscr{F} := \mathscr{F}(I)$ be the special fiber ring of I and $\mathscr{F}_+$ the ideal generated by the elements of $\mathscr{F}$ of positive degree. Notice that $\mathscr{F} \cong \Bbbk[I_\delta] \subseteq R$ because I is generated in a single degree. Let $\ell := \ell(I) = \dim(\mathscr{F})$ be the analytic spread of I.

Fix a generating sequence $\underline{f} = f_1, \dots, f_u$ of I contained in I_δ. Consider ℓu variables $\underline{z} = \{z_{ij} \mid 1 \leqslant i \leqslant \ell \text{ and } 1 \leqslant j \leqslant u\}$. Write $b_i = \sum_{j=1}^{u} z_{i,j} f_j$. Let $\mathscr{H} \subseteq J := J_{\ell,\underline{z}}(\underline{f})$ be the ideals generated by $b_1, \dots, b_\ell$ in the rings $\mathscr{F}[\underline{z}] \subseteq R[\underline{z}]$, respectively. For $\underline{\lambda} \in \mathbb{A}^{\ell u}_{\Bbbk}$, we write $\mathscr{H}_{\underline{\lambda}} = \pi_{\underline{\lambda}}(\mathscr{H}) \subseteq \mathscr{F}$ and $J_{\underline{\lambda}} = \pi_{\underline{\lambda}}(J) \subseteq R$, where $\pi_{\underline{\lambda}}$ denotes the evaluation map.

Notice that $\mathscr{F}_+ R = I$, $\mathscr{H} R[\underline{z}] = J$, and $\mathscr{H}_{\underline{\lambda}} R = J_{\underline{\lambda}}$. Moreover, $J_{\underline{\lambda}}$ is a reduction of I if and only if $I^{r+1} = J_{\underline{\lambda}} I^r$ for some $r \geqslant 0$ if and only if $\mathscr{F}_+^{r+1} = \mathscr{H}_{\underline{\lambda}} \mathscr{F}_+^r$ for some $r \geqslant 0$ if and only if $\mathscr{F}_+ \subseteq \sqrt{\mathscr{H}_{\underline{\lambda}}}$.

The following result describes the locus of the points $\underline{\lambda}$ for which $J_{\underline{\lambda}}$ is not a reduction of I. We also show that this locus is determined by a single irreducible polynomial of $\Bbbk[\underline{z}]$. We note that here we only assume I is homogeneous and not necessarily generated by monomials.

Proposition 5.2. *With assumptions as in Notation 5.1 let* $\mathscr{A} = (\mathscr{H} :_{\mathscr{F}[\underline{z}]} \mathscr{F}_+^\infty) \cap \Bbbk[\underline{z}]$:

(a) *The* $\Bbbk[\underline{z}]$*-ideal* $\mathscr{A}$ *defines the locus where* $J_{\underline{\lambda}}$ *is not a reduction of* I.

(b) *The ideal* $\mathscr{A}$ *is a prime ideal of height* 1. *Thus* $\mathscr{A} = (h)$, *where* h *an irreducible polynomial in* $\Bbbk[\underline{z}]$.

Proof. To prove (a) we write $T = \mathscr{F}[\underline{z}]/\mathscr{H}$ and consider the natural map $\phi : \operatorname{Proj}(T) \to \operatorname{Spec}(\Bbbk[\underline{z}])$. Clearly $\operatorname{Im}(\phi) \subseteq V(\mathscr{A})$; we claim that $V(\mathscr{A}) = \operatorname{Im}(\phi)$. For this, we first note that $\mathscr{H} :_{\mathscr{F}[\underline{z}]} \mathscr{F}_+^\infty$ is a prime ideal of height ℓ by Lemma 3.3. Therefore, we have an inclusion of domains

$$U := \Bbbk[\underline{z}]/\mathscr{A} \hookrightarrow V := \mathscr{F}[\underline{z}]/(\mathscr{H} :_{\mathscr{F}[\underline{z}]} \mathscr{F}_+^\infty).$$

Since

$$\dim(V \otimes_U \operatorname{Quot}(U)) = \operatorname{ht}(\mathscr{F}_+ V) \geqslant 1, \tag{6}$$

by semicontinuity of fiber dimension [Eisenbud 1995, Theorem 14.8(b)] we have that $\dim(V \otimes_U \kappa(P)) \geqslant 1$ for every $P \in \operatorname{Spec}(U)$. Therefore $P \in \operatorname{Im}(\phi)$ for every $P \in \operatorname{Spec}(U)$, whence the claim follows.

A point $\underline{\lambda} \in \mathbb{A}_{\Bbbk}^{\ell u}$ belongs to $\operatorname{Im}(\phi)$ if and only if $\dim(T \otimes_{\Bbbk[\underline{z}]} (\Bbbk[\underline{z}]/(\underline{z}-\underline{\lambda}))) > 0$. Since

$$T \otimes_{\Bbbk[\underline{z}]} (\Bbbk[\underline{z}]/(\underline{z}-\underline{\lambda})) \cong \mathscr{F}/\mathscr{H}_{\underline{\lambda}},$$

the last condition is equivalent to $\mathscr{F}_+ \nsubseteq \sqrt{\mathscr{H}_{\underline{\lambda}}}$, which means that $J_{\underline{\lambda}}$ is not a reduction of I.

For part (b), it remains to show that $\operatorname{ht}(\mathscr{A}) = 1$. We first observe that

$$\dim(V) = \dim(\mathscr{F}[\underline{z}]) - \operatorname{ht}(\mathscr{H} :_{\mathscr{F}[\underline{z}]} \mathscr{F}_+^{\infty}) = (\ell + \dim(\Bbbk[\underline{z}])) - \ell = \dim(\Bbbk[\underline{z}]).$$

We think of points in $\mathbb{A}_{\Bbbk}^{\ell u}$ as $\ell \times u$ matrices. If $\underline{\lambda}_0 \in \mathbb{A}_{\Bbbk}^{\ell u}$ is a matrix whose first $\ell - 1$ rows are general and whose last row consists of zeros, then $\operatorname{ht}(\mathscr{H}_{\underline{\lambda}_0}) = \ell - 1$. Therefore, $\mathscr{F}/\mathscr{H}_{\underline{\lambda}_0} = V \otimes_U (U/(\underline{z} - \underline{\lambda}_0))$ has dimension 1, which shows that $\dim(V \otimes_U \operatorname{Quot}(U)) = 1$ by [Eisenbud 1995, Theorem 14.8(b)] and (6). Thus,

$$1 = \operatorname{trdeg}_U(V) = \operatorname{trdeg}_{\Bbbk}(V) - \operatorname{trdeg}_{\Bbbk}(U) = \dim(V) - \dim(U) = \dim(\Bbbk[\underline{z}]) - \dim(U) = \operatorname{ht}(\mathscr{A}),$$

completing the proof. □

Following Notation 5.1, one can see that every homogeneous reduction of I contains a reduction generated by ℓ homogeneous elements of degree δ, which is necessarily of the form $J_{\underline{\lambda}}$ for some $\underline{\lambda} \in \mathbb{A}_{\Bbbk}^{\ell u}$.

Corollary 5.3. *With assumptions as in Notation 5.1 and Proposition 5.2, we have*

$$\operatorname{gradedcore}(I) = \bigcap_{\underline{\lambda} \notin V(\mathscr{A})} J_{\underline{\lambda}}.$$

In the following result we show that for a monomial ideal I, $(\operatorname{mono}(J) :_{R[\underline{z}]} (h)^{\infty}) \cap R$ is contained in every homogeneous reduction of I. In fact, we conjecture that this ideal is equal to $\operatorname{gradedcore}(I)$ (see Conjecture 5.5). Here we think of J as an ideal in the polynomial ring $A[x_1, \dots, x_d]$ with $A = \Bbbk[\underline{z}]$ (see Definition 4.3). For a vector $\boldsymbol{w} = (w_1, \dots, w_d) \in \mathbb{N}^d$ we denote by $\boldsymbol{x}^{\boldsymbol{w}}$ the monomial $x_1^{w_1} \cdots x_d^{w_d}$.

Theorem 5.4. *In addition to the assumptions of Notation 5.1 we suppose that $f_1, \dots, f_u$ are monomials. If $h \in \Bbbk[\underline{z}]$ is as in Proposition 5.2(b), then*

$$(\operatorname{mono}(J) :_{R[\underline{z}]} (h)^{\infty}) \cap R \subseteq \operatorname{gradedcore}(I).$$

Proof. Let $\boldsymbol{x}^{\boldsymbol{v}} \in (\operatorname{mono}(J) :_{R[\underline{z}]} (h)^{\infty}) \cap R$. Then $\boldsymbol{x}^{\boldsymbol{v}} h^N \in \operatorname{mono}(J)$ for $N \gg 0$. By Proposition 5.2 for each $\underline{\lambda}$ such that $J_{\underline{\lambda}}$ is a reduction of I we have $\pi_{\underline{\lambda}}(h) \neq 0$. Hence, setting $Y = \prod y_i$ as in (4) and using (5), we obtain

$$\begin{aligned}
\boldsymbol{x}^{\boldsymbol{v}} \in \pi_{\underline{\lambda}}(\operatorname{mono}(J)) &= \pi_{\underline{\lambda}}(((\tilde{b}_1, \dots, \tilde{b}_\ell) :_{R[\underline{z},\underline{y}]} Y^{\infty}) \cap R[\underline{z}]) \\
&\subseteq \pi_{\underline{\lambda}}((\tilde{b}_1, \dots, \tilde{b}_\ell) :_{R[\underline{z},\underline{y}]} Y^{\infty}) \cap R \\
&\subseteq (\pi_{\underline{\lambda}}(\tilde{b}_1, \dots, \tilde{b}_\ell) :_{R[\underline{y}]} Y^{\infty}) \cap R \\
&= \operatorname{mono}(J_{\underline{\lambda}}) \subseteq J_{\underline{\lambda}}.
\end{aligned}$$

Taking the intersection over all such $\underline{\lambda}$ we obtain $\boldsymbol{x}^{\boldsymbol{v}} \in \operatorname{gradedcore}(I)$, as desired. □

We propose the following conjecture based on the previous result and computational evidence.

Conjecture 5.5. *Let I and h be as in Theorem 5.4. Then*

$$\text{gradedcore}(I) = (\text{mono}(J) :_{R[\underline{z}]} (h)^\infty) \cap R.$$

In our next result, we show that the content ideal of $\text{mono}(J)$ is principal and that it is generated by the irreducible polynomial h from Proposition 5.2(b). We use this result to verify Conjecture 5.5 for specific examples at the end of the section. Before we proceed we need to fix more notation.

Notation 5.6. In addition to the assumptions of Notation 5.1 we suppose that $f_1, \dots, f_u$ are monomials. Let $\boldsymbol{v}_1, \dots, \boldsymbol{v}_r \in \mathbb{N}^d$ be distinct vectors such that

$$\text{mono}(J) = C_1(\boldsymbol{x}^{\boldsymbol{v}_1}) + \cdots + C_r(\boldsymbol{x}^{\boldsymbol{v}_r}),$$

where $C_1, \dots, C_r$ are ideals of $\Bbbk[\underline{z}]$ (see Definition 4.3). The ideal $\mathscr{C} = C_1 + \cdots + C_r$ is called the *content ideal* of $\text{mono}(J)$. We note that the set of monomials $\mathscr{M} := \{\boldsymbol{x}^{\boldsymbol{v}_1}, \dots, \boldsymbol{x}^{\boldsymbol{v}_r}\}$ generates $\text{mono}(J_{\underline{\lambda}})$ for general $\underline{\lambda}$ by Proposition 4.4(b).

Let $\boldsymbol{x}^{\boldsymbol{v}} = \text{lcm}(f_1, \dots, f_u)$ and for each f_i let g_i be the monomial in $R[\underline{y}] = R[y_1, \dots, y_d]$ such that $\deg(g_i) = \boldsymbol{v}$ and $\deg_{x_j}(g_i) = \deg_{x_j}(f_i)$ for every i and j. Notice that $\sum_{j=1}^{u} z_{i,j} g_j$ is $\tilde{b}_i$, the multihomogenization of $b_i \in \Bbbk[\underline{z}][x_1, \dots, x_d]$. Let $\widehat{f_i}$ be the element of $\Bbbk[\underline{y}]$ such that $\deg_{y_j}(\widehat{f_i}) = \deg_{y_j}(g_i)$, that is $g_i = f_i \widehat{f_i}$, and set $\widehat{I} = (\widehat{f_1}, \dots, \widehat{f_u}) \subseteq \Bbbk[\underline{y}]$. The ideal $\widehat{I}$ is the *Newton complementary dual* of I defined in [Costa and Simis 2013]; see also [Ansaldi et al. 2021].

Theorem 5.7. *Let $\mathscr{C}$ be as in Notation 5.6 and let $h \in \Bbbk[\underline{z}]$ be as in Proposition 5.2*(b). *Then $\mathscr{C} = (h)$.*

Proof. We prove the result by constructing a $\Bbbk[\underline{z}]$-isomorphism

$$\eta : \frac{\Bbbk[\underline{z}]}{(h)} \xrightarrow{\sim} \frac{\Bbbk[\underline{z}]}{\mathscr{C}} :$$

For this we consider the following diagram, which we explain in the rest of the proof:

$$\begin{array}{ccccc}
\dfrac{T}{(\tilde b_1,\dots,\tilde b_\ell):Y^\infty} & \overset{(1)}{=} & \dfrac{T}{(\tilde b_1,\dots,\tilde b_\ell):G^\infty} & & \\
 & & \uparrow \psi & & \\
\dfrac{\mathscr{F}[\underline{z}]}{\mathscr{H}:\mathscr{F}_+^\infty} & \xrightarrow{\bar\varphi} & \dfrac{S}{(\tilde b_1,\dots,\tilde b_\ell):G^\infty} & \xrightarrow[\sim]{\bar\phi} & \dfrac{\mathscr{F}(\widehat{I})[\underline{z}]}{(\widehat{b_1},\dots,\widehat{b_\ell}):\widehat{F}^\infty}[\underline{x},\underline{x}^{-1}] \\
\uparrow & & \uparrow\chi & & \uparrow \\
\dfrac{A}{(h)} & & \dfrac{A[\underline{x},\underline{x}^{-1}]}{\text{mono}(J)A[\underline{x},\underline{x}^{-1}]} = \dfrac{A}{\mathscr{C}}[\underline{x},\underline{x}^{-1}] & \xrightarrow[\sim]{\Phi} & \dfrac{A}{(q)}[\underline{x},\underline{x}^{-1}] \\
 & \searrow^{\eta} & \uparrow & & \uparrow \\
 & & \dfrac{A}{\mathscr{C}} & & \dfrac{A}{(q)}
\end{array}$$

Write

$$A := \Bbbk[\underline{z}] \subseteq S := A[\underline{x}, \underline{x}^{-1}][g_1, \dots, g_u] \subseteq T := A[\underline{x}, \underline{x}^{-1}][y_1, \dots, y_d],$$

and set $Y = \prod_{j=1}^d y_j$, $F = \prod_{j=1}^u f_j$, $G = \prod_{j=1}^u g_j$, and $\widehat{F} = \prod_{j=1}^u \widehat{f}_j$.

The equality (1) at the top of the diagram follows from Lemma 3.3 since

$$Y \in \sqrt{(g_1, \dots, g_u)\Bbbk[\underline{x}, \underline{x}^{-1}][y_1, \dots, y_d]}$$

as x_i are units.

We continue by constructing the map ψ. The inclusion $S \subset T$ induces an A-algebra homomorphism

$$\psi : \frac{S}{(\tilde{b}_1, \dots, \tilde{b}_\ell) :_S G^\infty} \to \frac{T}{(\tilde{b}_1, \dots, \tilde{b}_\ell) :_T G^\infty}.$$

We claim ψ is injective. Since G is a nonzerodivisor modulo $(\tilde{b}_1, \dots, \tilde{b}_\ell) :_S G^\infty$, it suffices to show that $\psi \otimes_S S_G$ is injective. Write $z'_i = z_{i,1} + g_1^{-1} \sum_{j=2}^u z_{i,j} g_j \in S_G$ for $1 \leqslant i \leqslant \ell$. Notice that

$$((\tilde{b}_1, \dots, \tilde{b}_\ell) :_S G^\infty)_G = (\tilde{b}_1, \dots, \tilde{b}_\ell) S_G = (z'_1, \dots, z'_\ell) S_G$$

and similarly

$$((\tilde{b}_1, \dots, \tilde{b}_\ell) :_T G^\infty)_G = (z'_1, \dots, z'_\ell) T_G.$$

Consider the two rings

$$B := \Bbbk[\underline{x}, \underline{x}^{-1}][\underline{g}, G^{-1}][\{z_{i,j} \mid j \geqslant 2\}] \subseteq C := \Bbbk[\underline{x}, \underline{x}^{-1}][\underline{y}, G^{-1}][\{z_{i,j} \mid j \geqslant 2\}].$$

One has $S_G = B[z'_1, \dots, z'_\ell]$ and $T_G = C[z'_1, \dots, z'_\ell]$, and $z'_1, \dots, z'_\ell$ are variables over B and C. Clearly,

$$\frac{B[z'_1, \dots, z'_\ell]}{(z'_1, \dots, z'_\ell)} \hookrightarrow \frac{C[z'_1, \dots, z'_\ell]}{(z'_1, \dots, z'_\ell)},$$

which proves the claim.

Next we deal with the map $\bar{\varphi}$. Define a map of A-algebras

$$\varphi : \mathscr{F}[\underline{z}] \to S$$

given by $\varphi(f_i) = g_i$. To prove that φ is well-defined, let p be a polynomial with coefficients in A such that $p(f_1, \dots, f_u) = 0$ and fix $\boldsymbol{w} = (w_1, \dots, w_d) \in \mathbb{N}^d$. Let $p_{\boldsymbol{w}}$ be the sum of the terms p' of p such that $\deg(p'(f_1, \dots, f_u)) = \boldsymbol{w}$. Therefore

$$p_{\boldsymbol{w}}(g_1, \dots, g_u) = \boldsymbol{y}^{(\sum_i w_i/\delta)\boldsymbol{v} - \boldsymbol{w}} p_{\boldsymbol{w}}(f_1, \dots, f_u) = 0.$$

We conclude that $p(g_1, \dots, g_u) = 0$ showing that φ is well-defined. Notice that $\varphi(b_i) = \sum_{j=1}^u z_{i,j} g_j = \tilde{b}_i$; hence $\varphi(\mathscr{H}) \subseteq (\tilde{b}_1, \dots, \tilde{b}_\ell)$. Therefore, we have $\varphi(\mathscr{H} :_{\mathscr{F}[\underline{z}]} F^\infty) \subseteq (\tilde{b}_1, \dots, \tilde{b}_\ell) :_S G^\infty$. Now Lemma 3.3 shows that $\mathscr{H} :_{\mathscr{F}[\underline{z}]} \mathscr{F}_+^\infty = \mathscr{H} :_{\mathscr{F}[\underline{z}]} F^\infty$. It follows that φ induces a homomorphism of A-algebras

$$\bar{\varphi} : \frac{\mathscr{F}[\underline{z}]}{\mathscr{H} : \mathscr{F}_+^\infty} \to \frac{S}{(\tilde{b}_1, \dots, \tilde{b}_\ell) : G^\infty}.$$

Now we construct the isomorphism $\bar{\phi}$. Notice that $\widehat{f_j} = a_j g_j$, where $a_j = f_j^{-1} \in \Bbbk[\underline{x}, \underline{x}^{-1}]$ is a unit; in particular $\widehat{F}$ is equal to G times a unit in $\Bbbk[\underline{x}, \underline{x}^{-1}]$. Now

$$S = \Bbbk[\widehat{f_1}, \dots, \widehat{f_u}][\underline{z}][\underline{x}, \underline{x}^{-1}] = \mathscr{F}(\widehat{I})[\underline{z}][\underline{x}, \underline{x}^{-1}].$$

Consider the automorphism ϕ of S as an algebra over $\Bbbk[g_1, \dots, g_u][\underline{x}, \underline{x}^{-1}]$ that sends $z_{i,j}$ to $a_j z_{i,j}$. Notice that ϕ maps $A[\underline{x}, \underline{x}^{-1}]$ onto itself and sends $\tilde{b}_i$ to $\widehat{b_i} := \sum_{j=1}^{u} z_{i,j} \widehat{f_j}$. Hence ϕ induces an isomorphism

$$\bar{\phi} : \frac{S}{(\tilde{b}_1, \dots, \tilde{b}_\ell) : G^\infty} \longrightarrow \frac{\mathscr{F}(\widehat{I})[\underline{z}]}{(\widehat{b_1}, \dots, \widehat{b_\ell}) : \widehat{F}^\infty}[\underline{x}, \underline{x}^{-1}]$$

that maps the image of $A[\underline{x}, \underline{x}^{-1}]$ onto itself.

We now deal with the map χ. Recall that $(\tilde{b}_1, \dots, \tilde{b}_\ell) :_T Y^\infty = (\tilde{b}_1, \dots, \tilde{b}_\ell) :_T G^\infty$ by the equality (1) at the top of the diagram. Hence the inclusion $A[\underline{x}, \underline{x}^{-1}] \subseteq S \subseteq T$ induces the natural embedding

$$\chi : \frac{A[\underline{x}, \underline{x}^{-1}]}{((\tilde{b}_1, \dots, \tilde{b}_\ell) :_T Y^\infty) \cap A[\underline{x}, \underline{x}^{-1}]} \hookrightarrow \frac{S}{(\tilde{b}_1, \dots, \tilde{b}_\ell) : G^\infty}.$$

On the other hand,

$$\begin{aligned}((\tilde{b}_1, \dots, \tilde{b}_\ell) :_T Y^\infty) \cap A[\underline{x}, \underline{x}^{-1}] &= \big(((\tilde{b}_1, \dots, \tilde{b}_\ell) :_{A[\underline{x},\underline{y}]} Y^\infty) \cap A[\underline{x}]\big) A[\underline{x}, \underline{x}^{-1}] \\ &= \operatorname{mono}(J) A[\underline{x}, \underline{x}^{-1}] = \mathscr{C} A[\underline{x}, \underline{x}^{-1}],\end{aligned}$$

where the penultimate equality holds by (5).

We continue by establishing the isomorphism Φ. Since the isomorphism $\bar{\phi}$ maps the image $A[\underline{x}, \underline{x}^{-1}]$ onto itself, it follows that this map restricts to an isomorphism

$$\Phi : \frac{A}{\mathscr{C}}[\underline{x}, \underline{x}^{-1}] \to \frac{A}{((\widehat{b_1}, \dots, \widehat{b_\ell}) : \widehat{F}^\infty) \cap A}[\underline{x}, \underline{x}^{-1}].$$

By Proposition 5.2(b) and Lemma 3.3, the ideal $((\widehat{b_1}, \dots, \widehat{b_\ell}) :_{\mathscr{F}(\widehat{I})[\underline{z}]} \widehat{F}^\infty) \cap A$ is generated by an irreducible polynomial q.

Finally we construct the desired map η. Recall that $(\mathscr{H} : \mathscr{F}_+^\infty) \cap A = (h)$ by Proposition 5.2(b). Since $\bar{\varphi}$ is a homomorphism of A-algebras, it induces an epimorphism of A-algebras

$$\eta : \frac{A}{(h)} \twoheadrightarrow \frac{A}{\mathscr{C}}.$$

It follows that $(h) \subseteq \mathscr{C}$. On the other hand,

$$\operatorname{ht}(\mathscr{C}) = \operatorname{ht}(\mathscr{C} A[\underline{x}, \underline{x}^{-1}]) = \operatorname{ht}(q A[\underline{x}, \underline{x}^{-1}]) = \operatorname{ht}(q) = 1,$$

where the second equality holds because of the isomorphism Φ. Since (h) is a prime ideal of height 1, we conclude that $\mathscr{C} = (h)$, finishing the proof. □

The variation of the coefficient ideals occurring in $\operatorname{mono}(J)$ provides a tool to distinguish between the monomials of $\mathscr{M}$ and possibly single out the relevant ones:

Discussion 5.8. Adopt the assumptions of Theorem 5.7. We are now in a position to single out the set $\mathscr{N} = \{\boldsymbol{x}^{\boldsymbol{v}_i} \in \mathscr{M} \mid \sqrt{C_i} = (h)\}$ of monomials with "maximal" coefficient ideals and consider the sum of "nonmaximal" coefficient ideals $\mathscr{D} = \sum_i C_i$, where $\boldsymbol{x}^{\boldsymbol{v}_i}$ ranges over set $\mathscr{M} \setminus \mathscr{N}$. Notice that the monomials in $\mathscr{N}$ generate the ideal $(\mathrm{mono}(J) :_{R[\underline{z}]} (h)^\infty) \cap R$ in Conjecture 5.5.

We believe that the ideal $\mathscr{D}$ defines the closed subset of $\mathbb{A}^{\ell u}_{\Bbbk}$ that identifies the "general special" reductions $J_{\underline{\lambda}}$ needed to describe the graded core. Namely, we conjecture that:

(a) $\mathrm{gradedcore}(I) = J_{\underline{\lambda}}^{n+1} : I^n$ for $n \gg 0$ if λ is general in $V(\mathscr{D})$.

(b) $\mathrm{gradedcore}(I)$ can be obtained by intersecting the mono of a general minimal reduction with finitely many $J_{\underline{\lambda}}$ with λ general in $V(\mathscr{D})$.

We now verify Conjecture 5.5 and Conjecture (b) in Discussion 5.8 for a specific example.

Example 5.9. Let $I = (x_1^3, x_1^2x_2, x_1x_3^2, x_3^3) \subseteq R = \mathbb{Q}[x_1, x_2, x_3]$ be as in Example 4.8 and $\mathscr{M}, \mathscr{N}, \mathscr{D}$ as in Discussion 5.8. Recall that $\mathrm{gradedcore}(I)$ is not a finite intersection of general reductions of I and is not $\mathrm{mono}(K)$, for a general reduction K. However, as it turns out, $\mathrm{gradedcore}(I)$ is a finite intersection of special reductions of I.

There exist relatively prime nonconstant polynomials h and g in $\mathbb{Q}[\underline{z}]$ such that

$$\mathrm{mono}(J) = (h)(x_1^2, x_1x_2, x_1x_3, x_2x_3, x_3^2)I + (hg)(x_1^2x_2^3, x_1x_2^2x_3^2, x_2^2x_3^3) =: (h)\mathfrak{A} + (hg)\mathfrak{B}.$$

We note that

$$\mathscr{M} = \{x_1^2, x_1x_2, x_1x_3, x_2x_3, x_3^2, x_1^2x_2^3, x_1x_2^2x_3^2, x_2^2x_3^3\}, \quad \mathscr{N} = \{x_1^2, x_1x_2, x_1x_3, x_2x_3, x_3^2\}, \quad \mathscr{D} = (hg).$$

By Proposition 5.2 and Theorem 5.7 the polynomial h is irreducible and defines the locus where $J_{\underline{\lambda}}$ is not a reduction of I. As we have seen in Example 4.8 $\mathrm{core}(I) \subseteq \mathrm{gradedcore}(I) \subseteq \mathfrak{A} + \mathfrak{B} = I\mathfrak{m}^2$. Since h and g are relative prime, we obtain $(\mathrm{mono}(J) : (h)^\infty) \cap R = \mathfrak{A}$, with J is as in Notation 5.1. Hence $\mathfrak{A} \subseteq \mathrm{gradedcore}(I)$ according to Theorem 5.4.

Next we search for special reductions that are needed to compute the graded core.

Computation with Macaulay2 shows that

$$g = z_{1,4}z_{2,3}z_{3,2} - z_{1,3}z_{2,4}z_{3,2} - z_{1,4}z_{2,2}z_{3,3} + z_{1,2}z_{2,4}z_{3,3} + z_{1,3}z_{2,2}z_{3,4} - z_{1,2}z_{2,3}z_{3,4}.$$

Consider

$$\underline{\lambda}_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \underline{\lambda}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then $\pi_{\underline{\lambda}_0}(g) = \pi_{\underline{\lambda}_1}(g) = 0$, $\pi_{\underline{\lambda}_0}(h) \neq 0$, and $\pi_{\underline{\lambda}_1}(h) \neq 0$. Therefore $\underline{\lambda}_0$ and $\underline{\lambda}_1$ belong to $V(\mathscr{D})$, and the ideals

$$J_{\underline{\lambda}_0} = (x_1^3, x_1^2x_2, x_1x_3^2 + x_3^3) \quad \text{and} \quad J_{\underline{\lambda}_1} = (x_1^3, x_1^2x_2 + x_1x_3^2, x_3^3)$$

are special reductions of I. Thus

$$\mathfrak{A} \subseteq \operatorname{gradedcore}(I) \subseteq I\mathfrak{m}^2 \cap \operatorname{mono}(J_{\underline{\lambda}_0} \cap J_{\underline{\lambda}_1}) = \mathfrak{A},$$

where mono$(-)$ is computed using the command `monomialSubideal` in Macaulay2. We conclude that gradedcore$(I) = \mathfrak{A}$, which verifies Conjecture 5.5 and Conjecture (b) in Discussion 5.8.

The following theorem gives an instance where the graded core equals the mono of a general minimal reduction without any residual conditions.

Theorem 5.10. *Using Notation 5.6, if* $\sqrt{C_i} = \mathscr{C}$ *for every* i, *then* gradedcore$(I) = \operatorname{mono}(J_{\underline{\lambda}})$ *for general* $\underline{\lambda}$.

Proof. The assumption implies that $\mathscr{M} = \mathscr{N}$ for $\mathscr{M}$ and $\mathscr{N}$ as in Discussion 5.8. Now we use the inclusions

$$(\mathscr{N}) = (\operatorname{mono}(J) :_{R[\underline{z}]} (h)^\infty) \cap R \subseteq \operatorname{gradedcore}(I) \subseteq \operatorname{mono}(J_{\underline{\lambda}}) = (\mathscr{M})$$

that follow from Discussion 5.8, Theorem 5.4, and Notation 5.6. □

6. The core of lex-segment ideals

In this section we investigate the core of a special class of monomial ideals, lex-segment ideals. Throughout R denotes a polynomial ring $\Bbbk[x_1, \dots, x_d]$ over a field $\Bbbk$ and I denotes a homogeneous ideal.

We begin by recalling some basic facts about *lex-segment* ideals; for a thorough treatment see [Miller and Sturmfels 2005] or [Herzog and Hibi 2011]. Let H_M denote the Hilbert function of a finitely generated graded R-module M. Write $R = \bigoplus_{i \geqslant 0} R_i$ and consider the lexicographic monomial order with $x_1 > x_2 > \cdots > x_d$. Let L_i be the subspace of R_i generated by the largest $H_I(i)$ monomials and set $L = \bigoplus_{i \geqslant 0} L_i$. The vector space L is an ideal, and any ideal constructed this way is called a *lex-segment ideal*. Lex-segment ideals are *strongly stable*, i.e., if $u \in L$ is a monomial and $x_j \mid u$ for some j, then $x_i \frac{u}{xj} \in L$ for every $i < j$. However, there are strongly stable ideals that are not lex-segment.

The purpose of this section is to tackle the following conjecture.

Conjecture 6.1. *Let* $R = \Bbbk[x_1, \dots, x_d]$ *be a polynomial ring over a field* $\Bbbk$ *of characteristic zero and* $\mathfrak{m} = (x_1, \dots, x_d)$ *the homogeneous maximal ideal of* R. *If* L *is a lex-segment ideal of height* $g \geqslant 2$ *generated in degree* $\delta \geqslant 2$, *then*

$$\operatorname{core}(L) = L\mathfrak{m}^{d(\delta-2)+g-\delta+1}.$$

Remark 6.2. We have strong evidence supporting this conjecture. The case $\delta = 2$ was shown in [Smith 2011, Theorem 5.1], and the case $g = d$, i.e., I is a power of $\mathfrak{m}$, was shown in [Corso et al. 2002, Proposition 4.2]. The case $d \leqslant 3$ is Corollary 6.14. Moreover, a large number of cases were verified with Macaulay2. In fact, we developed an algorithm based on Theorem 4.7, Proposition 6.5, and Remark 6.10 that tested the conjecture for every lex-segment ideal in the following cases: $d = 4$ and $\delta \leqslant 12$; $d = 5$ and $\delta \leqslant 5$; $d = 6$ and $\delta \leqslant 3$. Furthermore, in Theorems 6.9, 6.11 and 6.13 we obtain other partial results towards the conjecture.

For a monomial ideal $I \subset R = \Bbbk[x_1, \dots, x_d]$, we denote by $\Gamma(I)$ the set of monomials in I and by $G(I)$ the minimal set of monomial generators $\{\boldsymbol{x}^{\boldsymbol{v}_1}, \dots, \boldsymbol{x}^{\boldsymbol{v}_u}\}$ of I. For a set of monomials W in R, we denote by $\log(W) \subseteq \mathbb{N}^d$ the set of exponents of the monomials in W. For $\boldsymbol{w} = (w_1, \dots, w_d) \in \mathbb{N}^d$, we define $\min(\boldsymbol{w})$ and $\max(\boldsymbol{w})$ to be the smallest and largest i such that $w_i \neq 0$, respectively; we also set $|\boldsymbol{w}| = \sum_i w_i$.

The following technical results are needed in the proofs of the main results of this section. The first one gives a characterization of the analytic spread and height of strongly stable ideals.

Proposition 6.3. *Let $R = \Bbbk[x_1, \dots, x_d]$ be a polynomial ring over a field $\Bbbk$ and I a strongly stable ideal*:

(a) $\operatorname{ht}(I) = \max\{\min(\boldsymbol{v}) \mid \boldsymbol{v} \in \log(G(I))\}$.

(b) *If in addition I is generated in a single degree then*

$$\ell(I) = \max\{\max(\boldsymbol{v}) \mid \boldsymbol{v} \in \log(G(I))\}.$$

Proof. To prove part (a) let $r = \max\{\min(\boldsymbol{v}) \mid \boldsymbol{v} \in \log(G(I))\}$. It is clear that $I \subseteq (x_1, \dots, x_r)$ and so $\operatorname{ht}(I) \leqslant r$. On the other hand, let $\mathfrak{p} \in V(I)$ and let $\boldsymbol{v} \in \log(G(I))$. If $i \leqslant \min(\boldsymbol{v})$, then $x_i^{|\boldsymbol{v}|} \in I$ since I is strongly stable. Therefore $x_i \in \mathfrak{p}$ for every $1 \leqslant i \leqslant r$, and the conclusion follows.

We now prove part (b). Let $s = \max\{\max(\boldsymbol{v}) \mid \boldsymbol{v} \in \log(G(I))\}$. Notice that $G(I)$ consists of monomials in the variables $x_1, \dots, x_s$. Hence $\ell(I) \leqslant s$. On the other hand, since I is strongly stable and generated in one degree, say δ, it follows that $x_1^{\delta-1}(x_1, \dots, x_s) \subseteq I$. Therefore

$$\ell(I) = \operatorname{trdeg}_{\Bbbk}(\Bbbk[I_\delta]) \geqslant \operatorname{trdeg}_{\Bbbk}(\Bbbk[x_1^{\delta-1}x_1, \dots, x_1^{\delta-1}x_s]) = s. \qquad \square$$

Remark 6.4. If L is a lex-segment ideal of height $g \geqslant 2$ generated in degree $\delta \geqslant 2$, then $\ell(L) = d$ and the minimal number of generators of L is at least $d + 1$. Indeed, in this case $x_2^\delta \in L$ and then $x_1^{\delta-1}(x_1, \dots, x_d) \subset L$. The conclusion about $\ell(L)$ now follows from Proposition 6.3(b).

The following proposition allows us to use the results of [Corso et al. 2001] and [Polini and Ulrich 2005] for the computation of cores of lex-segment ideals. Some of the techniques in the proof originate from [Smith 2011, Theorem 3.3]. Recall that an ideal I of height g is said to satisfy AN_s^-, where s is an integer, if for every $g \leqslant i \leqslant s$ and every geometric i-residual intersection K of I the ring R/K is Cohen–Macaulay. Notice that if $I_\mathfrak{p}$ satisfies AN_s^- for every $\mathfrak{p} \in V(I)$, then I satisfies AN_s^-.

Let $a_1, \dots, a_n$ be homogeneous elements of R and I the ideal they generate. Write H_i for the i-th Koszul homology of $a_1, \dots, a_n$. The ideal I satisfies *sliding depth* if $\operatorname{depth}(H_i) \geqslant d - n + i$ for every i, where we use the convention $\operatorname{depth}(0) = \infty$; see [Herzog et al. 1985].

Proposition 6.5. *Let $R = \Bbbk[x_1, \dots, x_d]$ be a polynomial ring over a field $\Bbbk$, $\mathfrak{m} = (x_1, \dots, x_d)$ the maximal homogeneous ideal of R, and L a lex-segment ideal. Then $L^{\text{sat}} = L : \mathfrak{m}^\infty$ satisfies G_∞, sliding depth, and AN_{d-1}^-. Moreover, L satisfies G_d and AN_{d-1}^-.*

Proof. We may assume that $L \neq 0$ and $L \neq R$. Write $g = \operatorname{ht}(L)$. We claim that L^{sat} satisfies G_∞ and sliding depth. Let δ be the largest degree of a monomial generator of L. We use induction on δ. If $\delta = 1$, then $L =$

$(x_1, \dots, x_g)$, and the claim holds trivially. Assume $\delta \geqslant 2$ and the claim holds for every lex-segment ideal generated in degrees smaller than δ. We may assume $g < d$, as otherwise $L^{\text{sat}} = R$. Set $S := \Bbbk[x_g, \dots, x_d]$. By Proposition 6.3(a) we can write $L = I + x_g L'$ for some ideals I and L' such that $I \subseteq (x_1, \dots, x_{g-1})$, the generators of L' involve only the variables $x_g, \dots, x_d$, and $L' \cap S$ is a lex-segment ideal in S generated in degrees smaller than δ. Clearly, $x_g^\delta \in L$ and then $(x_1, \dots, x_{g-1})\mathfrak{m}^{\delta-1} \subseteq I$. We conclude that

$$(x_1, \dots, x_{g-1})\mathfrak{m}^{\delta-1} + x_g L' \subseteq L \subseteq (x_1, \dots, x_{g-1}) + x_g L'.$$

Therefore

$$\begin{aligned} L^{\text{sat}} &= (x_1, \dots, x_{g-1}) + ((x_g L' \cap S) :_S (x_g, \dots, x_d)^\infty) R \\ &= (x_1, \dots, x_{g-1}) + x_g((L' \cap S) :_S (x_g, \dots, x_d)^\infty) R \quad \text{since } g < d. \end{aligned}$$

It follows from the induction hypothesis that L^{sat} satisfies G_∞. Now, since $x_1, \dots, x_{g-1}$ is a regular sequence and the image of L^{sat} in $S \cong R/(x_1, \dots, x_{g-1})$ is $x_g((L' \cap S) :_S (x_g, \dots, x_d)^\infty)$, by [Herzog et al. 1985, Lemma 3.5] the ideal L^{sat} satisfies sliding depth if and only if $x_g((L' \cap S) :_S (x_g, \dots, x_d)^\infty)$ satisfies sliding depth. Since x_g is a regular element, the latter is equivalent to $((L' \cap S) :_S (x_g, \dots, x_d)^\infty)$ satisfying sliding depth. The conclusion now follows from the induction hypothesis.

Now for every $\mathfrak{p} \in V(L^{\text{sat}})$, the ideal $L^{\text{sat}}_{\mathfrak{p}}$ satisfies G_∞ and sliding depth. It follows from [Herzog et al. 1985, Theorem 3.3] that this ideal is AN^-_{d-1}. Hence L^{sat} satisfies AN^-_{d-1}.

Notice that the ideals L and L^{sat} are equal locally at every prime ideal $\mathfrak{p} \neq \mathfrak{m}$. Hence the property G_d passes from L^{sat} to L. According to [Ulrich 1994, Remark 1.12] the property AN^-_{d-1} passes from $L^{\text{sat}}_{\mathfrak{p}}$ to $L_{\mathfrak{p}}$ because the two ideals coincide locally in codimension $d-1$. Hence L satisfies AN^-_{d-1}. □

Let R be a Cohen–Macaulay *local ring with a graded canonical module ω_R; see [Bruns and Herzog 1993, Section 3.6]. For a graded R-module M, we denote by $M^\vee = \operatorname{Hom}_R(M, \omega_R)$ the *ω-dual* of M. The following proposition and its proof are essentially contained in [Ulrich 1994, Lemma 2.1] (see also [Chardin et al. 2001, Lemma 4.9]), we include it here in its graded version.

Proposition 6.6. *Let R be a Cohen–Macaulay *local ring with a graded canonical module ω_R. Let I be a homogeneous ideal. Let $x \in I$ be a homogeneous regular element and $J = (x) : I$. Then*

$$\omega_{R/J} \cong ((I\omega_R)^{\vee\vee}/x\omega_R)(\deg(x)).$$

Proof. We may assume that $J \neq R$. There are homogeneous isomorphisms

$$\begin{aligned} J &= x(R :_{\operatorname{Quot}(R)} I) \\ &\cong x \operatorname{Hom}_R(I, R) \\ &\cong x \operatorname{Hom}_R(I, \operatorname{Hom}_R(\omega_R, \omega_R)) \\ &\cong x \operatorname{Hom}_R(I \otimes_R \omega_R, \omega_R) \\ &\cong x \operatorname{Hom}_R(I\omega_R, \omega_R), \qquad \text{as } \operatorname{Ker}(I \otimes_R \omega_R \twoheadrightarrow I\omega_R) \text{ is torsion.} \end{aligned}$$

We conclude that $J \cong x(I\omega_R)^\vee$, and therefore

$$J^\vee \cong (x(I\omega_R)^\vee)^\vee \cong x^{-1}(I\omega_R)^{\vee\vee}. \tag{7}$$

Dualizing the exact sequence

$$0 \to J \to R \to R/J \to 0$$

into ω_R, one obtains an exact sequence

$$\mathrm{Hom}_R(R/J, \omega_R) \to R^\vee \to J^\vee \to \mathrm{Ext}^1_R(R/J, \omega_R) \to 0. \tag{8}$$

Since $\mathrm{ht}(J) = 1$, we have $\mathrm{Hom}_R(R/J, \omega_R) = 0$ and $\mathrm{Ext}^1_R(R/J, \omega_R) \cong \omega_{R/J}$. Thus (7) and (8) yield

$$0 \to \omega_R \to x^{-1}(I\omega_R)^{\vee\vee} \to \omega_{R/J} \to 0.$$

Hence

$$\omega_{R/J} \cong (x^{-1}(I\omega_R)^{\vee\vee})/\omega_R \cong ((I\omega_R)^{\vee\vee}/x\omega_R)(\deg(x)),$$

as desired. □

For a graded module $M = \bigoplus_{i\in\mathbb{Z}} M_i$ we denote by $\mathrm{indeg}(M)$ the *initial degree* of M, i.e., $\mathrm{indeg}(M) = \inf\{i \mid M_i \neq 0\}$.

Lemma 6.7. *Let R be a standard graded Cohen–Macaulay ring over a field $\Bbbk$ with* $\dim(R) = d$, ω_R *the graded canonical module of R, and I a homogeneous ideal. Assume that I satisfies G_{d-1} and is weakly $(d-2)$-residually S_2. Let n and $\delta \geqslant 0$ be integers, and consider the following statements*:

(i) $\mathrm{indeg}(\omega_{R/((a_1,\dots,a_{d-1}):I)}) \geqslant -n$ *for some $(d-1)$-residual intersection*

$$(a_1, \dots, a_{d-1}) : I$$

of I such that each a_i is homogeneous of degree δ.

(ii) $\mathrm{indeg}(\omega_{R/((a_1,\dots,a_{d-1}):I)}) \geqslant -n$ *for every $(d-1)$-residual intersection*

$$(a_1, \dots, a_{d-1}) : I$$

of I such that each a_i is homogeneous of degree δ.

(iii) $\mathrm{indeg}(\omega_{R/((a_1,\dots,a_d):I)}) \geqslant -n$ *for every d-residual intersection*

$$(a_1, \dots, a_d) : I$$

of I such that each a_i is homogeneous of degree δ.

Then (i) *is equivalent to* (ii). *Moreover*, *if* $\mathrm{indeg}(I) \geqslant \delta$, *then* (ii) *implies* (iii).

Proof. For a Noetherian graded $\Bbbk$-algebra T, we denote by $\mathrm{HS}_T(t)$ the Hilbert series of T. We may assume that the field $\Bbbk$ is infinite:

(i) ⇒ (ii) Set $\underline{a} = \{a_1, \dots, a_{d-1}\}$ and let $\bar{R} = R/((\underline{a}) : I)$. Since $\bar{R}$ is Cohen–Macaulay of dimension 1 (see [Chardin et al. 2001, Proposition 3.4(a)]), we may write $\mathrm{HS}_{\bar{R}}(t) = Q_{\bar{R}}(t)/(1-t)$ for some $Q_{\bar{R}}(t) \in \mathbb{Z}[t]$. By [Bruns and Herzog 1993, Corollary 4.4.6(a)] and the assumption in (i), we have

$$\deg(Q_{\bar{R}}(t)) = 1 - \mathrm{indeg}(\omega_{\bar{R}}) \leqslant 1 + n. \tag{9}$$

By [Chardin et al. 2001, Proposition 3.1 and Theorem 2.1(b)], $\mathrm{HS}_{R/((\underline{a}):I)}(t)$ is the same for every $(d-1)$-residual intersection $((\underline{a}):I)$ of I such that each a_i is homogeneous of degree δ. The conclusion now follows by applying (9) again.

(ii) $\Rightarrow$ (iii) We assume that $\operatorname{indeg}(I) \geqslant \delta$. Let $(\underline{a}, a_d) : I = (a_1, \dots, a_{d-1}, a_d) : I$ be a d-residual intersection of I. By [Ulrich 1994, Corollary 1.6(a)] we may assume that $(\underline{a}) : I$ is a geometric $(d-1)$-residual intersection of I. Write $\overline{}$ for images in $\bar{R} = R/((\underline{a}) : I)$. From [Chardin et al. 2001, Propositions 3.1 and 3.3, and Lemma 2.4(b)] it follows that $\bar{R}$ is Cohen–Macaulay of dimension 1, $\overline{a_d} \in \bar{I}$ is a homogeneous $\bar{R}$-regular element of degree δ, and $\overline{(\underline{a}, a_d) :_R I} = \overline{(a_d)} :_{\bar{R}} \bar{I}$. Hence by Proposition 6.6 and the fact that $I\omega_{\bar{R}}$ is a maximal Cohen–Macaulay $\bar{R}$-module, we have

$$\omega_{R/((\underline{a},a_d):I)} = \omega_{\bar{R}/((\overline{a_d}):\bar{I})} \cong ((I\omega_{\bar{R}})^{\vee\vee}/a_d\omega_{\bar{R}})(\delta) \cong (I\omega_{\bar{R}}/a_d\omega_{\bar{R}})(\delta),$$

where $(-)^{\vee} = \operatorname{Hom}_{\bar{R}}(-, \omega_{\bar{R}})$. Therefore, $\operatorname{indeg}(\omega_{R/((\underline{a},a_d):I)}) \geqslant -n$ as desired. □

Remark 6.8. Let $R = \Bbbk[x_1, \dots, x_d]$ be a polynomial ring over a field $\Bbbk$ and L a lex-segment ideal of height g generated in degree δ. Let $R' = \Bbbk[x_1, \dots, x_{d-1}]$. Then $L \cap R'$ is a lex-segment ideal of R' generated in degree δ and $\operatorname{ht}(L') = \min\{g, d-1\}$.

Proof. If $g = d$, then $L = (x_1, \dots, x_d)^\delta$ and the result is clear. Hence we may assume $g < d$. Let $\{\boldsymbol{x}^{\boldsymbol{v}_1}, \dots, \boldsymbol{x}^{\boldsymbol{v}_u}\}$ be the minimal monomial generating set of L. Thus $L \cap R'$ is generated by the monomials $\boldsymbol{x}^{\boldsymbol{v}_i}$ such that $x_d \nmid \boldsymbol{x}^{\boldsymbol{v}_i}$, and then it is a lex-segment ideal of R'. Finally, by Proposition 6.3(a) we have $\operatorname{ht}(L') = g$. □

In the following we prove one inclusion of Conjecture 6.1 in full generality.

Theorem 6.9. *Let $R = \Bbbk[x_1, \dots, x_d]$ be a polynomial ring over an infinite field $\Bbbk$ and $\mathfrak{m} = (x_1, \dots, x_d)$ the maximal homogeneous ideal of R. If L is a lex-segment ideal of height $g \geqslant 2$ generated in degree $\delta \geqslant 2$, then*

$$L\mathfrak{m}^{d(\delta-2)+g-\delta+1} \subseteq \operatorname{core}(L).$$

Proof. Recall that by Proposition 6.5 the ideal L satisfies G_d and AN^-_{d-1}. According to Corollary 3.12, we have that $\operatorname{core}(L)$ is the intersection of finitely many reductions generated by d general elements of L with respect to a generating set of L contained in L_δ. Let $\underline{a} = a_1, \dots, a_d$ be such general elements. To prove the statement of the theorem, it suffices to show that $\mathfrak{m}^{d(\delta-2)+g-\delta+1} \subseteq (\underline{a}) : L$. The latter is equivalent to

$$H_{R/((\underline{a}):L)}(n) = 0 \quad \text{for every } n \geqslant d(\delta-2)+g-\delta+1. \tag{10}$$

Since L is G_d, by [Polini and Xie 2013, Lemma 3.1(a)] and Remark 6.4, $(\underline{a}) : L$ is a d-residual intersection of L. Therefore the ring $R/((\underline{a}) : L)$ is Artinian. Hence (10) is equivalent to

$$\operatorname{indeg}(\omega_{R/((\underline{a}):L)}) \geqslant -(d(\delta-2)+g-\delta).$$

We claim that there exists a $(d-1)$-residual intersection of L, $(b_1, \dots, b_{d-1}) : L$, such that each b_i is homogeneous of degree δ and $\operatorname{indeg}(\omega_{R/((b_1,\dots,b_{d-1}):L)}) \geqslant -(d(\delta-2)+g-\delta)$. The result will follow from this claim and the implication (i) $\Rightarrow$ (iii) in Lemma 6.7.

We now prove the claim by induction on $\sigma(L) = d - g \geqslant 0$. If $\sigma(L) = 0$, then $L = \mathfrak{m}^\delta$ and the claim is satisfied by taking $b_i = x_i^\delta$ for every i; see [Bruns and Herzog 1993, Corollary 3.6.14].

For the induction step assume $\sigma(L) > 0$ and set $R' = \Bbbk[x_1, \dots, x_{d-1}]$ and $L' = L \cap R'$. By Remark 6.8 the ideal L' is a lex-segment ideal of height g generated in degree δ. In particular, $\sigma(L') = d-1-g < \sigma(L)$. By induction hypothesis there exists a $(d-2)$-residual intersection $(\underline{b}) :_{R'} L' = (b_1, \dots, b_{d-2}) :_{R'} L'$ such that each b_i is homogeneous of degree δ and

$$\operatorname{indeg}(\omega_{R'/((\underline{b}):_{R'}L')}) \geqslant -((d-1)(\delta-2)+g-\delta). \tag{11}$$

Therefore the implication (i) $\Rightarrow$ (ii) in Lemma 6.7 shows that every $(d-2)$-residual intersection $(\underline{b}) :_{R'} L'$ such that each b_i is homogeneous of degree δ has this property. Hence we may choose this residual intersection to be a geometric residual intersection, which exists by Proposition 6.5 and [Ulrich 1994, proof of Lemma 1.4].

To pass back to L we consider the saturation L^{sat} and write $(L^{\text{sat}})' = L^{\text{sat}} \cap R'$. From [Eisenbud 1995, Proposition 15.24] we have $L^{\text{sat}} = L : (x_d)^\infty$, hence the minimal monomial generators of L^{sat} are not divisible by x_d, i.e.,

$$L^{\text{sat}} = (L^{\text{sat}})'R. \tag{12}$$

By Proposition 6.5, L^{sat} satisfies G_∞ and AN_{d-1}^-, hence so does $(L^{\text{sat}})'$. The homogeneous inclusion $(L^{\text{sat}})'/L' \hookrightarrow L^{\text{sat}}/L$ implies that the Hilbert function of $(L^{\text{sat}})'/L'$ is eventually zero, hence

$$\operatorname{ht}(L' :_{R'} (L^{\text{sat}})') \geqslant d-1. \tag{13}$$

Notice that $(\underline{b}) :_{R'} (L^{\text{sat}})' \subset (\underline{b}) :_{R'} L'$. Since $(\underline{b}) :_{R'} L'$ is a geometric $(d-2)$-residual intersection, (13) implies that $(\underline{b}) :_{R'} (L^{\text{sat}})'$ is a geometric $(d-2)$-residual intersection. The ideal $(\underline{b}) :_{R'} (L^{\text{sat}})'$ is unmixed of height $d-2$ by [Ulrich 1994, Proposition 1.7(a)]. Therefore, by (13),

$$(\underline{b}) :_{R'} (L^{\text{sat}})' = (\underline{b}) :_{R'} L'. \tag{14}$$

Let $\overline{}$ denote images in the ring $\overline{R'} = R'/((\underline{b}) :_{R'} (L^{\text{sat}})') = R'/((\underline{b}) :_{R'} L')$. Notice that $\overline{R'}$ is Cohen–Macaulay of dimension 1. Since $(\underline{b}) :_{R'} L'$ is a geometric $(d-2)$-residual intersection of L' we have that $\operatorname{ht}(\overline{L'}) = 1$. Hence there exists an $\overline{R'}$-regular element $b_{d-1} \in L'_\delta$. Thus by (14) the ideal $(\underline{b}, b_{d-1}) :_{R'} (L^{\text{sat}})'$ is a $(d-1)$-residual intersection. From [Ulrich 1994, Proposition 1.7(f)] it follows that $\overline{(\underline{b}, b_{d-1}) :_{R'} (L^{\text{sat}})'} = \overline{(b_{d-1})} :_{\overline{R'}} \overline{(L^{\text{sat}})'}$. Moreover, $(L^{\text{sat}})'\omega_{\overline{R'}}$ is a maximal Cohen–Macaulay $\overline{R'}$-module. Hence Proposition 6.6 implies

$$\omega_{R'/((\underline{b},b_{d-1}):_{R'}(L^{\text{sat}})')} \cong ((L^{\text{sat}})'\omega_{\overline{R'}}/b_{d-1}\omega_{\overline{R'}})(\delta).$$

Using (12) we see that

$$\omega_{R/((\underline{b},b_{d-1}):_R L^{\mathrm{sat}})} \cong \omega_{R'/((\underline{b},b_{d-1}):_{R'}(L^{\mathrm{sat}})')} \otimes_{R'} R(-1).$$

Hence

$$\begin{aligned}\operatorname{indeg}(\omega_{R/((\underline{b},b_{d-1}):_R L^{\mathrm{sat}})}) &= \operatorname{indeg}(\omega_{R'/((\underline{b},b_{d-1}):_{R'}(L^{\mathrm{sat}})')}) + 1\\ &= \operatorname{indeg}(((L^{\mathrm{sat}})'\omega_{\overline{R'}}/b_{d-1}\omega_{\overline{R'}})(\delta)) + 1\\ &\geqslant \operatorname{indeg}(\omega_{\overline{R'}}) - \delta + 2\\ &\geqslant -((d-1)(\delta-2)+g-\delta) - \delta + 2 \qquad \text{by (11)}\\ &= -(d(\delta-2)+g-\delta).\end{aligned}$$

Finally, since $(\underline{b}, b_{d-1}) :_{R'} (L^{\mathrm{sat}})'$ is a $(d-1)$-residual intersection, (12) shows that $(\underline{b}, b_{d-1}) :_R L^{\mathrm{sat}}$ is a $(d-1)$-residual intersection. This ideal is unmixed of height $d-1$ by [Ulrich 1994, Proposition 1.7(a)], and moreover $\operatorname{ht}(L :_R L^{\mathrm{sat}}) \geqslant d$. Therefore $(\underline{b}, b_{d-1}) :_R L^{\mathrm{sat}} = (\underline{b}, b_{d-1}) :_R L$ is a $(d-1)$-residual intersection of L, completing the proof. □

Remark 6.10. We remark that in order to show the reverse containment in Theorem 6.9, if $\Bbbk$ is a field of characteristic zero, it is enough to show that $x_1^{d(\delta-2)+g} \notin J$ for some reduction J of L. To see this, notice that

$$L \cap \mathfrak{m}^{d(\delta-2)+g+1} = L\mathfrak{m}^{d(\delta-2)+g-\delta+1}$$

because L is generated in degree δ. Therefore it suffices to show that $\operatorname{core}(L) \subseteq \mathfrak{m}^{d(\delta-2)+g+1}$. Since $\operatorname{core}(L)$ is a strongly stable monomial ideal [Smith 2011, Proposition 2.3], this containment is equivalent to $x_1^{d(\delta-2)+g} \notin \operatorname{core}(L)$.

The next two theorems settle Conjecture 6.1 in some particular cases. In the first theorem, we show that Conjecture 6.1 holds for the smallest and largest lex-segment ideals for fixed d, g, and δ.

Theorem 6.11. *Let $R = \Bbbk[x_1, \dots, x_d]$ be a polynomial ring over a field $\Bbbk$ of characteristic zero and $\mathfrak{m} = (x_1, \dots, x_d)$ the maximal homogeneous ideal of R. Let g and δ be integers such that $2 \leqslant g \leqslant d$ and $\delta \geqslant 2$. Let L be one of the following lex-segment ideals*

(1) $(x_1, \dots, x_{g-1})\mathfrak{m}^{\delta-1} + (x_g)^{\delta}$, *or*

(2) $(x_1, \dots, x_g)\mathfrak{m}^{\delta-1}$.

Then $\operatorname{core}(L) = L\mathfrak{m}^{d(\delta-2)+g-\delta+1}$.

We need the following lemma for the proof of Theorem 6.11.

Lemma 6.12. *Let $R = \Bbbk[x_1, \dots, x_d]$ be a polynomial ring over a field $\Bbbk$ of characteristic zero and L a lex-segment ideal generated in degree $\delta \geqslant 2$. If J is any reduction of L, then for every $n \geqslant 0$ we have*

$$\operatorname{core}(L) \subseteq J^{n+1} : L^n.$$

Proof. Since L satisfies G_d and AN^-_{d-1} according to Proposition 6.5, by Corollary 3.10 it suffices to show that $\operatorname{core}(L_\mathfrak{p}) \subseteq K^{n+1} :_{R_\mathfrak{p}} L^n_\mathfrak{p}$ for every $\mathfrak{p} \in \operatorname{Spec}(R)$ and every reduction K of $L_\mathfrak{p}$. We may further assume that K is a minimal reduction of $L_\mathfrak{p}$, and in particular $\mu(K) \leqslant \dim(R_\mathfrak{p})$. By [Ulrich 1994, Lemma 1.10(b)] the ideal $L_\mathfrak{p}$ satisfies G_d and AN^-_{d-1}, and by [Johnson and Ulrich 1996, Remark 2.7] we have $\operatorname{ht}(K :_{R_\mathfrak{p}} L_\mathfrak{p}) \geqslant \dim(R_\mathfrak{p})$. Therefore K satisfies G_∞. Hence by [Ulrich 1994, Remark 1.12 and Corollary 1.8(c)], K satisfies sliding depth. Now the proof of [Polini and Ulrich 2005, Theorem 4.4] shows that for all $n \geqslant 0$

$$\operatorname{core}(L_\mathfrak{p}) \subseteq K^{n+1} :_{R_\mathfrak{p}} \sum_{y \in L_\mathfrak{p}} (K, y)^n = K^{n+1} :_{R_\mathfrak{p}} L^n_\mathfrak{p},$$

where the last equality holds since $\Bbbk$ has characteristic zero and then $L^n_\mathfrak{p} = \sum_{y \in L_\mathfrak{p}} (y^n)$. □

Proof of Theorem 6.11. We write $L_1 = (x_1, \dots, x_{g-1})\mathfrak{m}^{\delta-1} + (x_g^\delta)$ and $L_2 = (x_1, \dots, x_g)\mathfrak{m}^{\delta-1}$. Let

$$J_1 = (x_1^\delta, \dots, x_g^\delta) + (x_1, \dots, x_{g-1})(x_{g+1}^{\delta-1}, \dots, x_d^{\delta-1})$$

and

$$J_2 = (x_1^\delta, \dots, x_g^\delta) + (x_1, \dots, x_g)(x_{g+1}^{\delta-1}, \dots, x_d^{\delta-1}).$$

We claim that J_1 is a reduction of L_1 and J_2 is a reduction of L_2. To see this, notice that by [Huneke and Swanson 2006, Proposition 8.1.7] the ideal $(x_1, \dots, x_{g-1})(x_1^{\delta-1}, \dots, x_d^{\delta-1}) + (x_g^\delta)$ is a reduction of L_1, and this ideal is equal to

$$(x_1, \dots, x_{g-1})(x_1^{\delta-1}, \dots, x_g^{\delta-1}) + (x_g^\delta) + (x_1, \dots, x_{g-1})(x_{g+1}^{\delta-1}, \dots, x_d^{\delta-1}).$$

Clearly the ideal $(x_1^\delta, \dots, x_g^\delta)$ is a reduction of $(x_1, \dots, x_{g-1})(x_1^{\delta-1}, \dots, x_g^{\delta-1}) + (x_g^\delta) \subset (x_1, \dots, x_g)^\delta$. Therefore again by [Huneke and Swanson 2006, Proposition 8.1.7] and transitivity of reductions we conclude J_1 is a reduction of L_1. Likewise, J_2 is a reduction of L_2.

Now by Remark 6.10 and Lemma 6.12, it suffices to show that

$$x_1^{d(\delta-2)+g} \notin J_1^d : L_1^{d-1} \quad \text{and} \quad x_1^{d(\delta-2)+g} \notin J_2^d : L_2^{d-1}.$$

Let $\alpha = x_1^{2d-g-1} x_2^{\delta-1} \cdots x_g^{\delta-1} x_{g+1}^{\delta-2} \cdots x_d^{\delta-2}$ and notice that $\alpha \in L_1^{d-1} \subseteq L_2^{d-1}$. We now show that $x_1^{d(\delta-2)+g}\alpha \notin J_2^d$, and hence $x_1^{d(\delta-2)+g}\alpha \notin J_1^d$, which finishes the proof. Indeed, suppose by contradiction that

$$\beta := x_1^{d(\delta-2)+g}\alpha = x_1^{d\delta-1} x_2^{\delta-1} \cdots x_g^{\delta-1} x_{g+1}^{\delta-2} \cdots x_d^{\delta-2} \in J_2^d.$$

Since none of the minimal monomial generators of J_2, other than x_1^δ, divides β, we must have $x_1^{d\delta}$ divides β, a contradiction. □

The next theorem shows that Conjecture 6.1 holds for any δ if $g = d - 1 \geqslant 2$.

Theorem 6.13. *Let $R = \Bbbk[x_1, \dots, x_d]$ be a polynomial ring over a field $\Bbbk$ of characteristic zero, $\mathfrak{m} = (x_1, \dots, x_d)$ the maximal homogeneous ideal of R, and L a lex-segment ideal generated in degree $\delta \geqslant 2$.*

Assume $d \geqslant 3$ and that L has height $g = d-1$. Then

$$\operatorname{core}(L) = L\mathfrak{m}^{d(\delta-2)+g-\delta+1}.$$

Proof. By Remark 6.10 it suffices to show that $x_1^{d(\delta-1)-1} \notin \operatorname{core}(L)$. Thus by Lemma 6.12, it is enough to prove that $x_1^{d(\delta-1)-1} \notin J^d : L^{d-1}$ for some reduction J of L. Since $g = d-1$, it follows that $L = (x_1, \dots, x_{d-2})\mathfrak{m}^{\delta-1} + x_{d-1}L'$, where L' is a lex-segment ideal in the variables x_{d-1} and x_d generated in degree $\delta - 1$. By Theorem 6.11 we may assume that

$$L = (x_1, \dots, x_{d-2})\mathfrak{m}^{\delta-1} + (x_{d-1}^{\delta}, x_{d-1}^{\delta-1}x_d, \dots, x_{d-1}^{\delta-i}x_d^i),$$

with $1 \leqslant i \leqslant \delta - 2$. Therefore $K = (x_1^\delta, \dots, x_{d-1}^\delta, x_{d-1}^{\delta-i}x_d^i) + (x_1, \dots, x_{d-2})x_d^{\delta-1}$ is a reduction of L according to [Singla 2007, Proposition 2.1].

We claim that

$$J = (x_1^\delta - x_{d-1}^{\delta-i}x_d^i, x_2^\delta, \dots, x_{d-1}^\delta) + (x_1, \dots, x_{d-2})x_d^{\delta-1}$$

is a reduction of L and that $x_1^{d(\delta-1)-1} \notin J^d : L^{d-1}$.

First we show that J is a reduction of L. Let

$$K' = (x_1^\delta, x_1x_d^{\delta-1}, x_{d-1}^\delta, x_{d-1}^{\delta-i}x_d^i) \quad \text{and} \quad J' = (x_1^\delta - x_{d-1}^{\delta-i}x_d^i, x_1x_d^{\delta-1}, x_{d-1}^\delta).$$

Let $\mathscr{F}(K')$ denote the special fiber ring of K'. Then $\mathscr{F}(K') \cong \Bbbk[T_1, T_2, T_3, T_4]/\mathscr{D}$, for some homogeneous ideal $\mathscr{D}$, where the isomorphism is induced by the map sending T_1 to x_1^δ, T_2 to $x_1x_d^{\delta-1}$, T_3 to x_{d-1}^δ, and T_4 to $x_{d-1}^{\delta-i}x_d^i$. Notice that $T_1^iT_4^{\delta(\delta-1)} - T_2^{\delta i}T_3^{(\delta-1)(\delta-i)} \in \mathscr{D}$. Therefore the ring

$$\mathscr{F}(K')/J'\mathscr{F}(K') \cong \mathscr{F}(K')/(T_1 - T_4, T_2, T_3)\mathscr{F}(K')$$

is Artinian; here we denote by $J'\mathscr{F}(K')$ the $\mathscr{F}(K')$-ideal generated by the image of J' in $[\mathscr{F}(K')]_1$. Thus J' is a reduction of K', and hence J is a reduction of K. We conclude that J is a reduction of L, proving the claim.

Next we show $x_1^{d(\delta-1)-1} \notin J^d : L^{d-1}$. Let $\alpha = x_1^d x_2^{\delta-1} \cdots x_{d-1}^{\delta-1}x_d^{\delta-2}$ and notice that $\alpha \in L^{d-1}$. We claim that $\alpha x_1^{d(\delta-1)-1} \notin J^d$, which will complete the proof.

Let

$$H = (x_1^\delta - x_{d-1}^{\delta-i}x_d^i, x_{d-1}^\delta) + (x_1, \dots, x_{d-2})x_d^{\delta-1} \subset J.$$

Clearly $\alpha x_1^{d(\delta-1)-1} \in J^d$ if and only if $\alpha x_1^{d(\delta-1)-1} \in H^d$. Thus we focus the rest of the proof on showing that $\alpha x_1^{d(\delta-1)-1} \notin H^d$. For this we consider the sequence $\underline{f} = f_1, f_2, f_3$, where

$$f_1 = x_1^{2d\delta-\delta}, \quad f_2 = x_{d-1}^{d\delta}, \quad f_3 = x_d^{d(\delta-1)},$$

and note that it suffices to show

$$C := (\underline{f}) : \alpha x_1^{d(\delta-1)-1} \not\supseteq (\underline{f}) : H^d.$$

It is easy to see that

$$C = (x_1^{(d-1)\delta+1}, x_{d-1}^{(d-1)\delta+1}, x_d^{(d-1)(\delta-1)+1}).$$

Consider the element

$$\beta := (x_1^{\delta} + d x_{d-1}^{\delta-i} x_d^{i})M, \quad \text{where } M = x_1^{(d-2)\delta} x_{d-1}^{(d-1)\delta} x_d^{(d-1)(\delta-1)}.$$

Since

$$x_1^{\delta} M = x_1^{(d-1)\delta} x_{d-1}^{(d-1)\delta} x_d^{(d-1)(\delta-1)} \notin C,$$

we have that $\beta \notin C$. Thus it is enough to show $\beta \in (\underline{f}) : H^d$. Since $x_{d-1}^{\delta} M \in (\underline{f})$ and $x_d^{\delta-1} M \in (\underline{f})$, it remains to see that $\beta (x_1^{\delta} - x_{d-1}^{\delta-i} x_d^{i})^d \in (\underline{f})$.

Notice that

$$\begin{aligned}
\beta (x_1^{\delta} - x_{d-1}^{\delta-i} x_d^{i})^d &= \beta \sum_{j=0}^{d} (-1)^j \binom{d}{j} x_1^{\delta(d-j)} x_{d-1}^{(\delta-i)j} x_d^{ij} \\
&= x_1^{(d-1)\delta} x_{d-1}^{(d-1)\delta} x_d^{(d-1)(\delta-1)} \sum_{j=0}^{d} (-1)^j \binom{d}{j} x_1^{\delta(d-j)} x_{d-1}^{(\delta-i)j} x_d^{ij} \\
&\qquad + d x_1^{d\delta-2\delta} x_{d-1}^{d\delta-i} x_d^{(d-1)(\delta-1)+i} \sum_{j=0}^{d} (-1)^j \binom{d}{j} x_1^{\delta(d-j)} x_{d-1}^{(\delta-i)j} x_d^{ij}.
\end{aligned}$$

To simplify the notation, set

$$\beta_1 := x_1^{(d-1)\delta} x_{d-1}^{(d-1)\delta} x_d^{(d-1)(\delta-1)} \quad \text{and} \quad \beta_2 := d x_1^{d\delta-2\delta} x_{d-1}^{d\delta-i} x_d^{(d-1)(\delta-1)+i},$$

and for $0 \leqslant j \leqslant d$ set

$$h_j := (-1)^j \binom{d}{j} x_1^{\delta(d-j)} x_{d-1}^{(\delta-i)j} x_d^{ij}.$$

It is easy to see that $\beta_1 h_0 \in (\underline{f})$ and $\beta_1 h_1 + \beta_2 h_0 = 0$. We prove below that $\beta_1 h_j \in (\underline{f})$ for every $j \geqslant 2$ and that $\beta_2 h_j \in (\underline{f})$ for every $j \geqslant 1$.

Consider the terms $\beta_1 h_j$ with $j \geqslant 2$. In $\beta_1 h_j$ the degree of x_{d-1} is $(d-1)\delta + (\delta - i)j$ and the degree of x_d is $(d-1)(\delta-1) + ij$. Hence to show that $\beta_1 h_j \in (\underline{f})$ we need to prove that for every $2 \leqslant j \leqslant d$ one of the following inequalities holds:

$$(d-1)\delta + (\delta - i)j \geqslant d\delta \quad \text{or} \quad (d-1)(\delta-1) + ij \geqslant d(\delta-1). \tag{15}$$

The first inequality is equivalent to $i \leqslant \delta(j-1)/j$ and the second one is equivalent to $i \geqslant (\delta-1)/j$. Since $\delta(j-1)/j \geqslant (\delta-1)/j$ for $j \geqslant 2$, we have that one of the inequalities in (15) must hold for any such j.

Finally, consider the terms $\beta_2 h_j$, with $j \geqslant 1$. In $\beta_2 h_j$ the degree of x_{d-1} is $d\delta - i + (\delta - i)j$ and the degree of x_d is $(d-1)(\delta-1) + i + ij$. Hence, to show that $\beta_2 h_j \in (\underline{f})$ we need to prove that for every $1 \leqslant j \leqslant d$ one of the following inequalities holds:

$$d\delta - i + (\delta - i)j \geqslant d\delta \quad \text{or} \quad (d-1)(\delta-1) + i + ij \geqslant d(\delta-1). \tag{16}$$

The first inequality is equivalent to $i \leqslant \delta j/(j+1)$ and the second one is equivalent to $i \geqslant (\delta-1)/(j+1)$. Since $\delta j/(j+1) \geqslant (\delta-1)/(j+1)$ for $j \geqslant 1$, we have that one of the inequalities in (16) must hold for any such j. $\square$

Corollary 6.14. *Conjecture 6.1 holds if $d \leqslant 3$.*

Proof. The result follows from Theorem 6.13 and [Corso et al. 2002, Proposition 4.2]. □

The next fact follows directly by combining several results in the literature. We state it here for completeness and to provide a reference. For more information about integral closures and reductions see [Huneke and Swanson 2006] .

Proposition 6.15. *Let $R = \Bbbk[x_1, \dots, x_d]$ be a polynomial ring over a field $\Bbbk$ and L a lex-segment ideal generated in degree $\delta \geqslant 1$. The ideal L is normal, i.e., L^n is integrally closed for every $n \in \mathbb{N}$.*

Proof. The proof follows from [Herzog et al. 2005, Theorem 5.1], [De Negri 1999, Proposition 2.14 and its proof], and [Sturmfels 1996, Proposition 13.15]. □

The following two results provide upper bounds for the core of lex-segment ideals generated in a single degree.

Theorem 6.16. *Let $R = \Bbbk[x_1, \dots, x_d]$ be a polynomial ring over a field $\Bbbk$ of characteristic zero. If L is a lex-segment ideal of height $g \geqslant 2$ generated in degree $\delta \geqslant 2$, then*

$$\operatorname{core}(L) \subseteq \operatorname{adj}(L^g),$$

where $\operatorname{adj}(L^g)$ *denotes the adjoint of L^g as in* [Lipman 1994, Definition 1.1].

Proof. Let J be any minimal reduction of L. Let $A = R[Jt, t^{-1}]$ and $B = R[Lt, t^{-1}]$ be the extended Rees algebras of J and L, and ω_A and ω_B be their graded canonical modules. Notice that

$$\omega_A \subset (\omega_A)_{t^{-1}} = \omega_{A_{t^{-1}}} = \omega_{R[t,t^{-1}]}.$$

Thus making the identification $\omega_{R[t,t^{-1}]} = R[t, t^{-1}]$ we obtain an embedding $\omega_A \subset R[t, t^{-1}]$ so that $(\omega_A)_{t^{-1}} = R[t, t^{-1}]$. Therefore $[\omega_A]_n t^{-n} = R$ for n sufficiently small.

By Proposition 6.15 L is a normal monomial ideal, thus B is a normal Cohen–Macaulay algebra and a direct summand of a polynomial ring according to [Bruns and Gubeladze 2009, Theorems 6.10 and 4.43]. Therefore, B has rational singularities [Boutot 1987, Théorème]. We conclude that

$$[\omega_B]_i t^{-i} = \operatorname{adj}(L^i) \tag{17}$$

for every $i \geqslant 0$ by [Hyry 2001, proof of Corollary 3.5]. Let $K := \operatorname{Quot}(R)$ be the field of fractions of R and let $r := r_J(L)$ be the reduction number of L with respect to J. We have the following isomorphisms of graded A-modules

$$\begin{aligned}\omega_B \cong \operatorname{Hom}_A(B, \omega_A) \cong \omega_A :_{K(t)} B = \omega_A :_{R[t,t^{-1}]} B = \omega_A :_{R[t,t^{-1}]} (R \oplus Lt \oplus \cdots \oplus L^r t^r)\\ = \bigcap_{i=0}^{r} (\omega_A :_{R[t,t^{-1}]} L^i) t^{-i} = \bigcap_{i=0}^{r} \left(\left(\bigoplus_{j \in \mathbb{Z}} [\omega_A]_j \right) :_{R[t,t^{-1}]} L^i \right) t^{-i}\\ = \bigcap_{i=0}^{r} \bigoplus_{j \in \mathbb{Z}} ([\omega_A]_j t^{-j} :_R L^i) t^{j-i} = \bigoplus_{s \in \mathbb{Z}} \left(\bigcap_{i=0}^{r} ([\omega_A]_{s+i} t^{-s-i} :_R L^i) \right) t^s.\end{aligned}$$

In particular,

$$[\omega_B]_g t^{-g} = \bigcap_{i=0}^{r} ([\omega_A]_{g+i} t^{-g-i} :_R L^i). \tag{18}$$

By Proposition 6.5 and [Ulrich 1994, Proposition 1.11, Remark 1.12, and Corollary 1.8(c)], J satisfies G_∞ and sliding depth. Thus by [Herzog et al. 1983, Theorem 6.1] we have that $\operatorname{gr}_J(R) = \bigoplus_{n\geqslant 0} J^n/J^{n+1}$ is Cohen–Macaulay. Moreover $\{J^{n+1} : L^n\}_{n\in\mathbb{N}}$ forms a decreasing sequence of ideals. Indeed, since $\operatorname{gr}_J(R)$ is Cohen–Macaulay, we have $J^{n+i} : J^n = J^i$ for every nonnegative integers i and n. Therefore

$$J^{n+1} : L^n = (J^{n+2} : J) : L^n = J^{n+2} : JL^n \supseteq J^{n+2} : L^{n+1}.$$

Computing a-invariants we have $a(A) = a(\operatorname{gr}_J(R)) + 1$, as $\operatorname{gr}_J(R) \cong A/(t^{-1})$. Furthermore, $a(\operatorname{gr}_J(R)) = -g$ by [Simis et al. 1995, Theorem 3.5]. Therefore $[\omega_A]_{g-1} t^{1-g} = R$, which implies $At^{g-1} \subseteq \omega_A$. Hence by Lemma 6.12 for all $0 \leqslant i \leqslant r$ we have

$$\operatorname{core}(L) \subseteq J^{r+1} :_R L^r \subseteq J^{i+1} :_R L^i = [At^{g-1}]_{g+i} t^{-g-i} :_R L^i \subseteq [\omega_A]_{g+i} t^{-g-i} :_R L^i.$$

Thus $\operatorname{core}(L) \subseteq \bigcap_{i=0}^{r} ([\omega_A]_{g+i} t^{-g-i} :_R L^i) = [\omega_B]_g t^{-g} = \operatorname{adj}(L^g)$ by (18) and (17), as desired. □

Corollary 6.17. *Let $R = \Bbbk[x_1, \dots, x_d]$ be a polynomial ring over a field $\Bbbk$ of characteristic zero and $\mathfrak{m} = (x_1, \dots, x_d)$ the maximal homogeneous ideal of R. If L is a lex-segment ideal of height $g \geqslant 2$ generated in degree $\delta \geqslant 2$, then*

$$\operatorname{core}(L) \subseteq L\mathfrak{m}^{(g-1)\delta-d+1}.$$

Proof. The ideal L^g is integrally closed by Proposition 6.15 and it is generated in a single degree. Hence [Howald 2001, Main Theorem] implies that if $\boldsymbol{x}^{\boldsymbol{v}} \in \operatorname{adj}(L^g)$ then $\boldsymbol{x}^{\boldsymbol{v}} x_1 x_2 \cdots x_d \in L^g\mathfrak{m}$. Therefore $\operatorname{adj}(L^g) \subseteq \mathfrak{m}^{g\delta-d+1}$. Finally, by Theorem 6.16 we conclude that $\operatorname{core}(L) \subseteq L \cap \operatorname{adj}(L^g) \subseteq L \cap \mathfrak{m}^{g\delta-d+1} = L\mathfrak{m}^{(g-1)\delta-d+1}$. □

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