Vector space Ramsey numbers and weakly Sidorenko affine configurations

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Abstract

For $B \subseteq \mathbb{F}_q^m$, the n-th affine extremal number of B is the maximum cardinality of a set $A \subseteq \mathbb{F}_q^n$ with no subset which is affinely isomorphic to B. Furstenberg and Katznelson proved that for any $B \subseteq \mathbb{F}_q^m$, the n-th affine extremal number of B is $o(q^n)$ as $n \to \infty$. By counting affine homomorphisms between subsets of $\mathbb{F}_q^n$, we derive new bounds and give new proofs of some previously known bounds for certain affine extremal numbers. At the same time, we establish corresponding supersaturation results. We connect these bounds to certain Ramsey-type numbers in vector spaces over finite fields. For $s, t \geq 1$, let $R_q(s,t)$ denote the minimum n such that in every red-blue coloring of the one-dimensional subspaces of $\mathbb{F}_q^n$, there is either a red s-dimensional subspace or a blue t-dimensional subspace of $\mathbb{F}_q^n$. The existence of these numbers is a special case of a well-known theorem of Graham, Leeb, Rothschild. We improve the best known upper bounds on $R_2(2,t)$, $R_3(2,t)$, $R_2(t,t)$, and $R_3(t,t)$.

1 Introduction

We consider bounds for Ramsey-type and Turán-type problems in the setting of vector spaces over finite fields. In this paper, we use $\left[{V \atop t}\right]$ to denote the collection of all t-dimensional linear subspaces of a vector space V. The following theorem is a special case of a classical theorem of Graham, Leeb, and Rothschild [17], which establishes the existence of the Ramsey numbers we consider.

Theorem 1.1 (Graham, Leeb, Rothschild)**.** *Let $\mathbb{F}_q$ be any finite field. For any positive integers $t_1, \dots, t_k$, there exists a minimum $n =: R_q(t_1, \dots, t_k)$ such that for every k-coloring $f : \left[{\mathbb{F}_q^n \atop 1}\right] \to [k]$ of the 1-dimensional linear subspaces of $\mathbb{F}_q^n$, there exist $i \in [k]$ and a linear subspace $U \subseteq \mathbb{F}_q^n$ of dimension t_i, such that $\left[{U \atop 1}\right]$ is monochromatic in color i.*

In the case $t_1 = \cdots = t_k = t$, we write $R_q(t_1, \dots, t_k) = R_q(t; k)$. The bounds for $R_q(t_1, \dots, t_k)$ implied by early proofs of Theorem 1.1 (see [17], [30]) are quite large due to repeated use of the Hales-Jewett Theorem [19]. In the case $q = 2$, the problem can be reduced to the disjoint unions problem for finite sets, considered by Taylor [31], which gives the following bound.

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Theorem 1.2 (Taylor)**.** *The number $R_2(t;k)$ is at most a tower of height $2k(t-1)$ of the form*

$$R_2(t;k) \leq k^{3^{k^{\cdot^{\cdot^{\cdot^{3}}}}}}.$$

For comparison, lower bounds for $R_2(t;k)$ attained from applying the techniques from [2] such as the Lovász Local Lemma to a uniform random coloring are only on the order of $\Omega\left(\frac{2^t}{t}\log_2 k\right)$.

We improve the bound of Theorem 1.2 by bringing the height of the tower down to $(k-1)(t-1)+o(t)$, and we prove a corresponding bound over $\mathbb{F}_3$.

Theorem 1.3. *There exists a constant $C_0 \approx 13.901$ such that for $\sigma_2 = 2$ and $\sigma_3 = C_0$, the following holds for $q \in \{2,3\}$. For any $k \geq 2$ and any $t_k \geq \cdots \geq t_1 \geq 2$, $R_q(t_1,\ldots,t_k)$ is at most a tower of height $\sum_{i=1}^{k-1}(t_i-1)+1$ of the form*

$$R_q(t_1,\ldots,t_k) \leq \sigma_q^{\sigma_q^{\cdot^{\cdot^{\cdot^{\sigma_q^{3t_k}}}}}}.$$

More recently, Nelson and Nomoto [23] considered the off-diagonal version of this problem over $\mathbb{F}_2$ with two colors while investigating χ-boundedness of certain classes of binary matroids, and they proved the following bound.

Theorem 1.4 (Nelson, Nomoto)**.** *For every $t \geq 2$,*

$$R_2(2,t) \leq (t+1)2^t.$$

In this case, standard probabilistic arguments give a lower bound for $R_2(2,t)$ which is only linear in t. Specifially, the Lovász Local Lemma gives $R_2(2,t) \geq (2-o(1))t$ as $t \to \infty$, and more generally,

$$R_q(s,t) \geq \left(\frac{q^s - q}{(q-1)(s-1)} - o(1)\right)t \qquad \text{as } t \to \infty$$

for fixed q and s. Nelson and Nomoto asked if a subexponential upper bound is possible. While the answer to that question remains to be seen, we provide the following exponential improvement for $R_2(2,t)$ and similarly give the first exponential upper bound for $R_3(2,t)$.

Theorem 1.5. *There exists a constant $C_0 \approx 13.901$ such that as $t \to \infty$,*

$$R_2(2,t) = O\big(t6^{t/4}\big);$$
$$R_3(2,t) = O\big(tC_0^t\big).$$

These improved bounds come from some simple observations about affine extremal numbers and their supersaturation properties, which are analogous to bipartite Turán numbers in graph theory. The n-th *affine extremal number* of a family $\mathcal{B}$ of affine configurations $\{B_i \subseteq \mathbb{F}_q^{m_i}\}_{i\in I}$, denoted $\mathrm{ex}_{\mathrm{aff}}(n,\mathcal{B})$, is the maximum size of a subset $A \subseteq \mathbb{F}_q^n$ with no affine copy of any B_i (see Section 2 for a more complete definition). The asymptotic study of affine extremal numbers dates back at least to the following theorem of Furstenberg and Katznelson [13].

Theorem 1.6 (Furstenberg, Katznelson)**.** *Let $\mathbb{F}_q$ be any finite field. For any $t \geq 0$,*

$$\mathrm{ex}_{\mathrm{aff}}(n, \mathbb{F}_q^t) = o(q^n) \qquad as \qquad n \to \infty.$$

Since any $\mathcal{B}$-free set is $\mathbb{F}_q^t$-free for some t, Theorem 1.6 says that affine extremal numbers are always $o(q^n)$. Furstenberg and Katznelson went on to prove a density version of the Hales-Jewett Theorem [14], from which Theorem 1.6 is immediate. Alternative proofs of these results can be found in [25] and [24], respectively.

The projective version of this problem is even older, beginning with the following result of Bose and Burton [6].

Theorem 1.7 (Bose, Burton)**.** *Let $\mathbb{F}_q$ be a finite field, and let $t \geq 1$. Let $\mathcal{A}$ be a subset of $\begin{bmatrix} \mathbb{F}_q^n \\ 1 \end{bmatrix}$ for which there is no linear t-dimensional subspace $U \subseteq \mathbb{F}_q^n$ with $\begin{bmatrix} U \\ 1 \end{bmatrix} \subseteq \mathcal{A}$. Then*

$$|\mathcal{A}| \leq \frac{q^n - q^{n-t+1}}{q-1},$$

with equality if and only if $\mathcal{A} = \begin{bmatrix} \mathbb{F}_q^n \\ 1 \end{bmatrix} \setminus \begin{bmatrix} W \\ 1 \end{bmatrix}$ for some linear $(n-t+1)$-dimensional linear subspace $W \subseteq \mathbb{F}_q^n$.

Remark. It is sometimes convenient to identify a set $\mathcal{A} \subseteq \begin{bmatrix} \mathbb{F}_q^n \\ 1 \end{bmatrix}$ of projective points with a set $A \subseteq \mathbb{F}_q^n \setminus \{0\}$ of vectors, given by

$$A = \bigcup_{\ell \in \mathcal{A}} \ell \setminus \{0\}.$$

We call a set A of this form *projectively determined*. Similarly, we can identify any k-coloring of $\begin{bmatrix} \mathbb{F}_q^n \\ 1 \end{bmatrix}$ with a *projectively determined* k-coloring of $\mathbb{F}_q^n \setminus \{0\}$, meaning that each color class is a projectively determined set of vectors. Moving forward, we will work from the perspective of projectively determined sets and colorings whenever we discuss results of a projective nature, such as Theorems 1.3, 1.5, and 1.7.

The problem of determining projective extremal numbers asymptotically for general projective configurations over $\mathbb{F}_q$ was almost entirely solved by Geelen and Nelson [15], who proved a theorem analogous to the Erdős-Stone-Simonivits Theorem for graphs. Their theorem gives precise asymptotics for the extremal number of any projective configuration, except for those which exclude a linear hyperplane, which are usually called "affine". Up to a constant factor, the projective extremal numbers of these "affine" projective configurations reduce to affine extremal numbers of the type discussed in this paper. Therefore, what can be said of their projective extremal numbers is that they are degenerate by Theorem 1.6, which is far from an asymptotic determination, similar to the case of extremal numbers of bipartite graphs.

It is unknown in general (see [16], Open Problem 32) whether the $o(q^n)$ bound in Theorem 1.6 can be taken to be of the form $O\left((q^{1-\delta})^n\right)$ for some $\delta = \delta(q,t) > 0$. However, for $q = 2$ and $q = 3$, we have the following respective results of Bonin and Qin [5], and of Fox and Pham [11].

Theorem 1.8 (Bonin, Qin)**.** *There exists an absolute constant c such that for every $t \geq 1$, every subset of $\mathbb{F}_2^n$ of size at least $(2^{1-c2^{-t}})^n$ contains an affine t-space.*

Theorem 1.9 (Fox, Pham)**.** *There exist absolute constants c and C_0, with $C_0 \approx 13.901$ such that for every $t \geq 1$, every subset of $\mathbb{F}_3^n$ of size at least $\left(3^{1-cC_0^{-t}}\right)^n$ contains an affine t-space.*

The proof of Theorem 1.8 is entirely self-contained and is no more than a page. Theorem 1.9, on the other hand, is the culmination of several breakthroughs related to the Cap Set Problem, starting with the advances in polynomial methods from Croot, Lev, and Pach [8] and the subsequent proof of the Cap Set Theorem by Ellenberg and Gijswijt [9], which says that $\mathrm{ex}_{\mathrm{aff}}(n, \mathbb{F}_3^1) \leq (3^{1-\delta})^n$ for some $\delta > 0$. Fox and Lovász [10] then used this result to give improved bounds on Green's Arithmetic Triangle Removal Lemma [18]. Fox and Pham observed that this improvement implies a supersaturation version of the Cap Set Theorem, from which they derived Theorem 1.9, which is a multidimensional extension of the Cap Set Theorem. It is unknown whether the constant C_0 given in the theorem is tight, as probabilistic lower bounds for $\mathrm{ex}_{\mathrm{aff}}(n, \mathbb{F}_3^t)$ are on the order of $\left(3^{1-3^{-(1+o(1))t}}\right)^n$ [11].

The argument of Fox and Pham over $\mathbb{F}_3$ is essentially the same as Bonin and Qin's proof over $\mathbb{F}_2$. The key ingredient to both is supersaturation of affine lines, which is trivial over $\mathbb{F}_2$ and highly non-trivial over $\mathbb{F}_3$. We include this argument here in a more general form in Section 4. We also give a new proof of these results which additionally asserts a strong form of supersaturation, giving a quantitative improvement to a supersaturation result of Gijswijt [16] for certain affine configurations.

Our supersaturation results arise naturally from counting affine homomorphisms which are maps preserving affine configurations. (See Section 2 for details of the notation and terminology used here). We use $\hom_{\mathrm{aff}}(B, A)$ to denote the number of affine homomorphisms $B \to A$. We say that an affine configuration $B \subseteq \mathbb{F}_q^m$ is *C-weakly Sidorenko* if, for any $A \subseteq \mathbb{F}_q^n$ of density α,

$$\hom_{\mathrm{aff}}(B, A) \geq \alpha^C N^{\mathrm{rank}_{\mathrm{aff}}(B)},$$

where $N := q^n$. By taking A to be a p-random subset of $\mathbb{F}_q^n$ for some fixed $p \in (0, 1)$, we see that B cannot be C-weakly Sidorenko for $C < |B|$. In the case that B is C-weakly Sidorenko with $C = |B|$, we simply say that B is *Sidorenko*; that is, B is Sidorenko if

$$\hom_{\mathrm{aff}}(B, A) \geq \alpha^{|B|} N^{\mathrm{rank}_{\mathrm{aff}}(B)}$$

for any $A \subseteq \mathbb{F}_q^n$ of density α, with $N := q^n$.

The notion of Sidorenko affine configurations originates from Saad and Wolf [26], who gave an equivalent definition in the language of linear forms. They proved that an affine configuration $\{x_1, \ldots, x_k\}$ with a single relation $\sum_{i=1}^k \lambda_i x_i = 0$ is Sidorenko whenever the coefficients λ_i can be partitioned into zero-sum pairs. They conjectured that these are the only affine configurations with a single relation which are Sidorenko. Fox, Pham, and Zhao showed that the conjecture is true in spirit, but in reality the correct statement is that $\{x_1, \ldots, x_k\}$ with a single relation $\sum_{i=1}^k \lambda_i x_i = 0$ is Sidorenko if and only if the *nonzero* coefficients λ_i can be partitioned into zero-sum pairs [12]. In particular, these results tell us that for each $k \geq 2$, the *circuit* of length $2k$ over $\mathbb{F}_2$, which we define to be the affine configuration $C_{2k} := \{0, e_1, \ldots, e_{2k-2}, \sum_{i=1}^{2k-2} e_i\} \subseteq \mathbb{F}_2^{2k-2}$, where $e_1, \ldots, e_{2k-2}$ are the standard basis vectors in $\mathbb{F}_2^{2k-2}$, is Sidorenko. We also have trivially that $\mathbb{F}_2^1$ is Sidorenko, as is any affinely independent affine configuration over any finite field. It is unknown whether there exist affine configurations over $\mathbb{F}_2$ which are not Sidorenko. Over $\mathbb{F}_3$, Fox and Pham [11] pointed out that a result of Fox and Lovász [10] implies that for $C_0 \approx 13.901$, for any affine configuration $A \subseteq \mathbb{F}_3^n$ of size αN, where $N := 3^n$, there are at least $\alpha^{C_0} N^2$ triples in A^3 of the form $(x, x+d, x+2d)$. In other words, $\mathbb{F}_3^1$

is C_0-weakly Sidorenko. It is unclear whether or not $\mathbb{F}_3^1$ is C-weakly Sidorenko for some $C < C_0$. What we can say is that known lower bounds on $\mathrm{ex}_{\mathrm{aff}}(n, \mathbb{F}_3^1)$ by Tyrell [32] imply that $\mathbb{F}_3^1$ is not C-weakly Sidorenko for any $C < 4.63$. We summarize these for future reference in the following lemma.

Lemma 1.10. *Let $\sigma_2 := 2$ and $\sigma_3 := C_0 \approx 13.901$ as in Theorem 1.9.*

(a) $\mathbb{F}_2^1$ is σ_2-weakly Sidorenko and hence Sidorenko.

(b) For each $k \geq 2$, $C_{2k} \subseteq \mathbb{F}_2^{2k-2}$ is Sidorenko.

(c) $\mathbb{F}_3^1$ is σ_3-weakly Sidorenko.

We make the following simple observation which allows us to construct new weakly Sidorenko affine configurations from old ones, which we prove in Section 3.

Theorem 1.11. *Suppose that $B_1 \subseteq \mathbb{F}_q^{m_1}$ is C_1-weakly Sidorenko and $B_2 \subseteq \mathbb{F}_q^{m_2}$ is C_2-weakly Sidorenko. Then $B_1 \times B_2$ is C_1C_2-weakly Sidorenko. In particular, if B_1 and B_2 are both Sidorenko, then so is $B_1 \times B_2$.*

Recently, some attention has been given to classifying Sidorenko affine configurations, usually motivated by Ramsey multiplicity problems in additive settings. In addition to the work of [26] and [12] already mentioned, Kamčev, Liebenau, and Morrison proved that affine configurations admitting a certain type of tree-like structure are Sidorenko [21]. They additionally gave a necessary condition for an affine configuration to be Sidorenko, which always holds trivially over $\mathbb{F}_2$. Altman defined a local weakening of the Sidorenko property and described a particular family of affine configurations (none over $\mathbb{F}_2$) which are not Sidorenko [3]. See [4], [7], [20], [22], [33], and [34] for further work in the area.

We now give a vector space analogue of Sidorenko's Conjecture [29] on graph homomorphisms, which says that for any bipartite graph H on v vertices with e edges, and any graph G with N vertices and $\alpha N^2/2$ edges, the number of homomorphisms from H to G is at least $\alpha^e N^v$.

Conjecture 1.12. *Every affine configuration over $\mathbb{F}_2$ is Sidorenko.*

It is not hard to check that $\mathbb{F}_q^1$ is not Sidorenko for any $q > 2$, so the conjecture only makes sense over $\mathbb{F}_2$. We also note that for $q \in \{2, 3\}$, every affine configuration is C-weakly Sidorenko for some C. Indeed, if $B \subseteq \mathbb{F}_q^n$ has affine rank r, then we can embed B into $\mathbb{F}_q^{r-1}$, which is σ_q^{r-1}-weakly Sidorenko by Lemma 1.10 and Theorem 1.11, with σ_q as in Lemma 1.10. It follows that B is σ_q^{r-1}-weakly Sidorenko as well. We ask if the same holds for general q.

Question 1.13. *Is every affine configuration over any finite field C-weakly Sidorenko for some C?*

By Theorem 1.11, this is equivalent to asking if for every prime power q, there exists σ_q such that $\mathbb{F}_q^1$ is σ_q-weakly Sidorenko.

2 Preliminaries

The objects we consider are subsets of finite-dimensional vector spaces over a fixed finite field $\mathbb{F}_q$. Such a subset $A \subseteq \mathbb{F}_q^n$ has linear structure as well as affine structure, which we define precisely below. Depending on which type of structure we are considering, we call A a *linear configuration* or an *affine configuration*.

A *linear relation* on $A = \{x_1, \ldots, x_k\} \subseteq \mathbb{F}_q^n$ is an equation of the form $\sum_{i=1}^k \lambda_i x_i = 0$, where $\lambda_1, \ldots, \lambda_k \in \mathbb{F}_q$. If, in addition, we have $\sum_{i=1}^k \lambda_i = 0$, then the relation is called *affine*. The relation is *trivial* if each $\lambda_i = 0$. If A has no nontrivial affine relations, then A is called *affinely independent*. A maximal affinely independent subset of A is called an *affine basis* for A. The size of any affine basis for A is an invariant of A, called its *affine rank*, which we denote by $\mathrm{rank}_{\mathrm{aff}}(A)$.

Given two configurations $A \subseteq \mathbb{F}_q^n$ and $B \subseteq \mathbb{F}_q^m$, a function $\varphi : B \to A$ is an *affine homomorphism* if φ preserves affine relations; that is, for any $\lambda_1, \ldots, \lambda_k \in \mathbb{F}_q$ and $x_1, \ldots, x_k \in B$ with $\sum_{i=1}^k \lambda_i = 0$ and $\sum_{i=1}^k \lambda_i x_i = 0$, we have $\sum_{i=1}^k \lambda_i \varphi(x_i) = 0$. Equivalently, φ is an affine homomorphism if φ extends to an affine map $\mathbb{F}_q^m \to \mathbb{F}_q^n$. We say that a homomorphism $\varphi : B \to A$ is an *isomorphism* if φ is bijective and φ^{-1} is a homomorphism. If $B = A$, we call φ an *automorphism* of B, the set of which we denote by $\mathrm{Aut}_{\mathrm{aff}}(B)$. The affine isomorphism class of $A \subseteq \mathbb{F}_q^n$ is called its *affine structure*. A homomorphism φ which is an isomorphism onto its image is called *non-degenerate*, which means that φ is injective and preserves relations as well as non-relations.

Each of the affine notions above has a naturally-defined linear counterpart by considering linear relations instead of affine relations. In particular, the *linear structure* of $A \subseteq \mathbb{F}_q^n$ is its linear isomorphism class, which characterizes the linear relations and non-relations among elements of A. We denote the linear rank of A by $\mathrm{rank}(A)$.

We say that $A \subseteq \mathbb{F}_q^n$ contains an *affine copy* of $B \subseteq \mathbb{F}_q^m$ if there is a non-degenerate affine homomorphism $B \to A$. If $\mathcal{B} = \{B_i\}_{i \in I}$ is a family of affine configurations $B_i \subseteq \mathbb{F}_q^{m_i}$, we say that A is $\mathcal{B}$-free if A contains no affine copy of any B_i. The largest size $\mathrm{ex}_{\mathrm{aff}}(n, \mathcal{B})$ of an affine $\mathcal{B}$-free subset of $\mathbb{F}_q^n$ is called the n-th *affine extremal number* of $\mathcal{B}$. If $\mathcal{B} = \{B\}$, we write $\mathrm{ex}_{\mathrm{aff}}(n, \{B\}) = \mathrm{ex}_{\mathrm{aff}}(n, B)$.

For $A \subseteq \mathbb{F}_q^n$ and $B \subseteq \mathbb{F}_q^m$, define the *product* of A and B to be the set

$$A \times B := \{(x, y) \in \mathbb{F}_q^{n+m} : x \in A, y \in B\}.$$

The affine and linear structures of $A \times B$ are determined by the respective affine and linear structures of A and B.

For $A \subseteq \mathbb{F}_q^n$, define the *direction set* of A to be the set

$$A^{\rightarrow} := \{d \in \mathbb{F}_q^n : \text{there exists } x \in \mathbb{F}_q^n \text{ such that } x + \lambda d \in A \text{ for all } \lambda \in \mathbb{F}_q\}.$$

Note that for $q = 2$, $A^{\rightarrow}$ is just the sumset $A + A = \{x + y : x, y \in A\}$. Also note that the linear structure of $A^{\rightarrow}$ is entirely determined by the affine structure of A since, for any $x \in \mathbb{F}_q^n$, the translate $A + x$ has the same direction set as A does, and any linear isomorphism applied to A will also preserve the linear structure of $A^{\rightarrow}$. Additionally, it should be clear from the definitions that for any $A \subseteq \mathbb{F}_q^n, B \subseteq \mathbb{F}_q^m$, we have

$$(A \times B)^{\rightarrow} = A^{\rightarrow} \times B^{\rightarrow}.$$

We define a few more linear and affine invariants. For nonempty $A \subseteq \mathbb{F}_q^n$, let $\omega(A)$ denote the dimension of the largest linear subspace of $\mathbb{F}_q^n$ contained in $A \cup \{0\}$. Define $\omega^{\rightarrow}(A) := \omega(A^{\rightarrow})$, which is determined by the affine structure of A. Let $\omega_{\mathrm{aff}}(A)$ denote the dimension of the largest affine subspace of $\mathbb{F}_q^n$ contained in

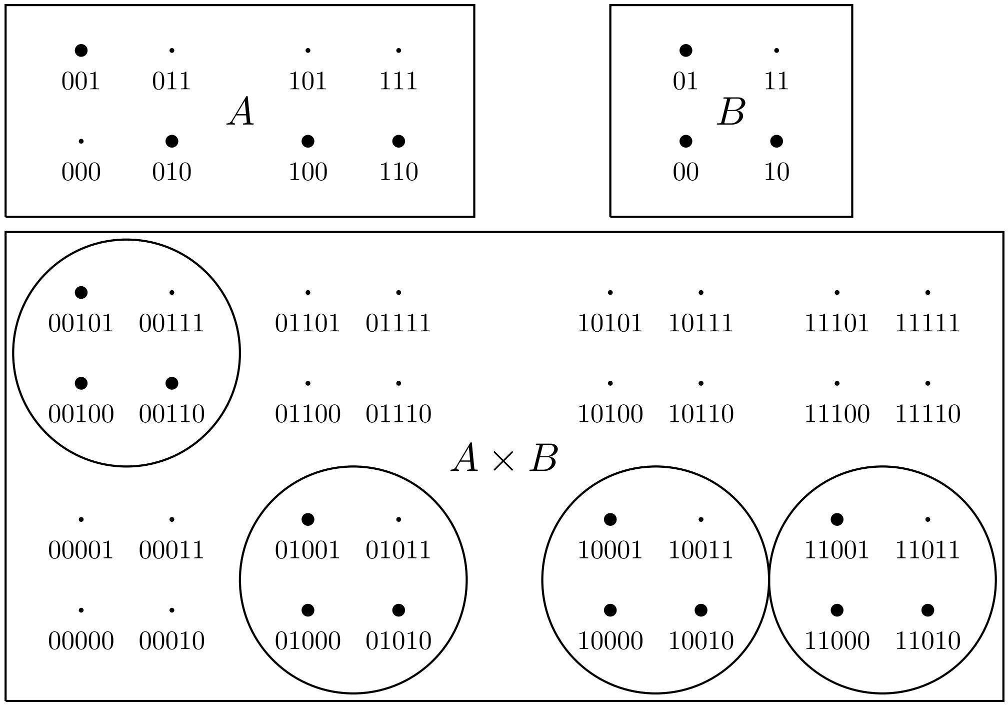

Figure 1: Example showing the product of configurations $A \subseteq \mathbb{F}_2^3$ and $B \subseteq \mathbb{F}_2^2$.

A. We also have

$$\omega(A) \le \omega_{\text{aff}}(A) \le \omega^{\to}(A)$$

since any linear subspace is an affine subspace, and the direction set of any affine subspace is a linear subspace of the same dimension. The following proposition shows that these invariants interact well with the product operation. Part (b) will be especially useful for our purposes.

Proposition 2.1. *Let $A \subseteq \mathbb{F}_q^n$ and $B \subseteq \mathbb{F}_q^m$ be nonempty. Then we have the following.*

(a) If $0 \in A$ and $0 \in B$, then $\omega(A \times B) = \omega(A) + \omega(B)$.

(b) $\omega^{\to}(A \times B) = \omega^{\to}(A) + \omega^{\to}(B)$.

(c) $\omega_{\text{aff}}(A \times B) = \omega_{\text{aff}}(A) + \omega_{\text{aff}}(B)$.

(d) If $0 \in A$ and $0 \in B$, then $\operatorname{rank}(A \times B) = \operatorname{rank}(A) + \operatorname{rank}(B)$.

(e) $\operatorname{rank}_{\text{aff}}(A \times B) = \operatorname{rank}_{\text{aff}}(A) + \operatorname{rank}_{\text{aff}}(B) - 1$.

Proof. We first prove (a). Let $k = \omega(A)$ and $\ell = \omega(B)$. Since A contains a linear copy of $\mathbb{F}_q^k$ and B contains a linear copy of $\mathbb{F}_q^\ell$, $A \times B$ contains a linear copy of $\mathbb{F}_q^k \times \mathbb{F}_q^\ell = \mathbb{F}_q^{k+\ell}$. Thus $\omega(A \times B) \ge k + \ell$.

Conversely, suppose that W is a linear subspace in $A \times B$ with basis $\{(x_1, y_1), \dots, (x_t, y_t)\}$, with $t = \omega(A \times B)$. Then $\operatorname{span}\{x_1, \dots, x_t\} \subseteq A$ and $\operatorname{span}\{y_1, \dots, y_t\} \subseteq B$, which means that $\operatorname{rank}\{x_1, \dots, x_t\} \le k$ and $\operatorname{rank}\{y_1, \dots, y_t\} \le \ell$. Now W is contained in the $\le k + \ell$ dimensional subspace of $\mathbb{F}_q^{n+m}$ spanned by

$(x_1, 0), \ldots, (x_t, 0), (0, y_1), \ldots, (0, y_t)$, so

$$\omega(A \times B) = \dim W \leq k + \ell.$$

Part (b) now follows. Since $0 \in A^{\rightarrow}$ and $0 \in B^{\rightarrow}$, we have by part (a) that

$$\omega((A \times B)^{\rightarrow}) = \omega(A^{\rightarrow} \times B^{\rightarrow}) = \omega(A^{\rightarrow}) + \omega(B^{\rightarrow}).$$

Now we prove (c). First, let U be an affine space in A of dimension $\omega_{\mathrm{aff}}(A)$, and let W be an affine space in B of dimension $\omega_{\mathrm{aff}}(B)$. Let $x \in U$ and $y \in W$. By translating A by $-x$ and B by $-y$, we may assume that $0 \in A$ and that $\omega_{\mathrm{aff}}(A) = \omega(A)$, and also that $0 \in B$ and that $\omega_{\mathrm{aff}}(B) = \omega(B)$. Now by part (a),

$$\omega_{\mathrm{aff}}(A \times B) \geq \omega(A \times B) = \omega(A) + \omega(B) = \omega_{\mathrm{aff}}(A) + \omega_{\mathrm{aff}}(B).$$

Conversely, let V be an affine space in $A \times B$ of dimension $\omega_{\mathrm{aff}}(A \times B)$, and let $z' = (x', y') \in V$. Then $x' \in A$ and $y' \in B$, so by translating A by $-x'$ and B by $-y'$, we can assume that $0 \in A$, that $0 \in B$, and that V is a linear space in $A \times B$. Thus by part (a),

$$\omega_{\mathrm{aff}}(A \times B) = \omega(A \times B) = \omega(A) + \omega(B) \leq \omega_{\mathrm{aff}}(A) + \omega_{\mathrm{aff}}(B).$$

For part (d), note that, given a linear basis $\{x_1, \ldots, x_k\}$ for A and a linear basis $\{y_1, \ldots, y_\ell\}$ for B, the set $\{(x_1, 0), \ldots, (x_k, 0), (0, y_1), \ldots, (0, y_\ell)\}$ is a linear basis for $A \times B$.

Finally, we consider part (e). By translating, we can assume $0 \in A$ and $0 \in B$, which implies $0 \in A \times B$ as well. Then $\mathrm{rank}(A) = \mathrm{rank}_{\mathrm{aff}}(A) - 1$, since for any linear basis $\{x_1, \ldots, x_k\}$ for A, it is easily seen that $\{x_1, \ldots, x_k, 0\}$ is an affine basis for A. Similarly, we have $\mathrm{rank}(B) = \mathrm{rank}_{\mathrm{aff}}(B) - 1$, and $\mathrm{rank}(A \times B) = \mathrm{rank}_{\mathrm{aff}}(A \times B) - 1$. Now by part (d), we have

$$\begin{aligned} \mathrm{rank}_{\mathrm{aff}}(A \times B) &= \mathrm{rank}(A \times B) + 1 \\ &= \mathrm{rank}(A) + \mathrm{rank}(B) + 1 \\ &= \mathrm{rank}_{\mathrm{aff}}(A) - 1 + \mathrm{rank}_{\mathrm{aff}}(B) - 1 + 1 \\ &= \mathrm{rank}_{\mathrm{aff}}(A) + \mathrm{rank}_{\mathrm{aff}}(B) - 1. \end{aligned}$$

□

3 Homomorphic Supersaturation

We first prove a simple lemma that shows that the number of degenerate affine homomorphisms $B \to A$ is small compared to the total number of affine homomorphisms $B \to A$.

Lemma 3.1. *Let $B \subseteq \mathbb{F}_q^m$ and $A \subseteq \mathbb{F}_q^n$ be affine configurations, with B nonempty. Write $r = \mathrm{rank}_{\mathrm{aff}}(B) \geq 1$, $N = q^n$, and $|A| = \alpha N$. Then the number of degenerate affine homomorphisms $B \to A$ is less than $(q\alpha N)^{r-1}$.*

Proof. If $\{x_0, \ldots, x_{r-1}\}$ is an affine basis for B, then an affine homomorphism $f : B \to A$ is degenerate iff $\{f(x_0), \ldots, f(x_{r-1})\} \subseteq A$ is affinely dependent. There are $(q^{r-1} - 1)/(q - 1) < q^{r-1}$ possible nontrivial affine relations among the r elements $f(x_0), \ldots, f(x_{r-1})$, up to scaling, each of the form $\sum_{i=0}^{r-1} \lambda_i f(x_i) = 0$, with

$\sum_{i=0}^{r-1} \lambda_i = 0$ and some $\lambda_i \neq 0$. Once such a relation is established, then the entire function f is determined by the values it takes on $\{x_0, \ldots, x_{r-1}\} \setminus \{x_i\}$, so there are at most $(\alpha N)^{r-1}$ such f's with the given relation. Altogether, this gives the desired count. □

We now show that the property of B being C-weakly Sidorenko immediately gives an upper bound on the extremal number of B, and that affine configurations larger than the given bound have supersaturation of affine copies of B. In particular, when B is Sidorenko, we have the strongest possible form of supersaturation of copies of B for affine configurations $A \subseteq \mathbb{F}_q^n$ above a certain threshold, namely the same number asymptotically as a p-random subset of $\mathbb{F}_q^n$ with $p = |A|/q^n$.

Lemma 3.2. *Let $B \subseteq \mathbb{F}_q^m$ be C-weakly Sidorenko, with $\mathrm{rank}_{\mathrm{aff}}(B) =: r \geq 1$. Then for every n,*

$$\mathrm{ex}_{\mathrm{aff}}(n, B) < q^{n-(n-r+1)/(C-r+1)}.$$

Moreover, if $A \subseteq \mathbb{F}_q^n$ with $|A| = Dq^{(1-1/(C-r+1))n}$ for some $D > 0$, then A contains more than

$$\left(1 - \frac{q^{r-1}}{D^{C-r+1}}\right) \frac{\alpha^C N^r}{|\mathrm{Aut}_{\mathrm{aff}}(B)|}$$

subsets affinely isomorphic to B, where $N = q^n$ and $|A| = \alpha N$.

Proof. We prove the supersaturation result first. If $A \subseteq \mathbb{F}_q^n$ has $|A| = Dq^{(1-1/(C-r+1))n}$, then by Lemma 3.1, the number of degenerate affine homomorphisms $B \to A$ is less than

$$(q\alpha N)^{r-1} = \frac{q^{r-1}}{D^{C-r+1}} \alpha^C N^r.$$

Since B is C-weakly Sidorenko, we have more than

$$\left(1 - \frac{q^{r-1}}{D^{C-r+1}}\right) \alpha^C N^r$$

non-degenerate affine homomorphisms $B \to A$. For each subset $B' \subseteq A$ which is affinely isomorphic to B, there are exactly $|\mathrm{Aut}_{\mathrm{aff}}(B)|$ non-degenerate affine homomorphisms mapping B onto B', so we must have more than

$$\left(1 - \frac{q^{r-1}}{D^{C-r+1}}\right) \frac{\alpha^C N^r}{|\mathrm{Aut}_{\mathrm{aff}}(B)|}$$

such subsets.

In particular, if

$$|A| \geq q^{n-(n-r+1)/(C-r+1)},$$

then

$$\frac{q^{r-1}}{D^{C-r+1}} \leq 1,$$

so A must contain an affine copy of B, giving our desired bound on $\mathrm{ex}_{\mathrm{aff}}(n, B)$. □

We now show that the property of being weakly Sidorenko is preserved under taking products.

Proof of Theorem 1.11. Let $A \subseteq \mathbb{F}_q^n$ be an affine configuration with density α, and let $N = q^n$. Fix respective affine bases $\{x_0, \dots, x_{r_1-1}\}$ and $\{y_0, \dots, y_{r_2-1}\}$ for B_1 and B_2. For $\mathbf{u} = (u_1, \dots, u_{r_1-1}) \in (\mathbb{F}_q^n)^{r_1-1}$, we use $\mathrm{span}_{B_1}(\mathbf{u})$ to denote the set

$$\left\{ \sum_{i=1}^{r_1-1} \lambda_i u_i \in \mathbb{F}_q^n : x_0 + \sum_{i=1}^{r_1-1} \lambda_i (x_i - x_0) \in B_1 \right\}.$$

We similarly define

$$\mathrm{span}_{B_2}(\mathbf{v}) := \left\{ \sum_{i=1}^{r_2-1} \lambda_i v_i \in \mathbb{F}_q^n : y_0 + \sum_{i=1}^{r_2-1} \lambda_i (y_i - y_0) \in B_2 \right\}$$

for $\mathbf{v} = (v_1, \dots, v_{r_2-1}) \in (\mathbb{F}_q^n)^{r_2-1}$, and we further define

$$A_{\mathbf{v}} := \{ z \in \mathbb{F}_q^n : z + \mathrm{span}_{B_2}(\mathbf{v}) \subseteq A \}.$$

Note that

$$\begin{aligned} \hom_{\mathrm{aff}}(B_1, A) &= \# \left\{ (z, \mathbf{u}) \in \mathbb{F}_q^n \times (\mathbb{F}_q^n)^{r_1-1} : z + \mathrm{span}_{B_1}(\mathbf{u}) \subseteq A \right\}; \\ \hom_{\mathrm{aff}}(B_2, A) &= \# \left\{ (z, \mathbf{v}) \in \mathbb{F}_q^n \times (\mathbb{F}_q^n)^{r_2-1} : z + \mathrm{span}_{B_2}(\mathbf{v}) \subseteq A \right\}. \end{aligned}$$

We can thus express $\hom_{\mathrm{aff}}(B_1 \times B_2, A)$ as

$$\begin{aligned} \hom_{\mathrm{aff}}(B_1 \times B_2, A) &= \# \left\{ (z, \mathbf{u}, \mathbf{v}) \in \mathbb{F}_q^n \times (\mathbb{F}_q^n)^{r_1-1} \times (\mathbb{F}_q^n)^{r_2-1} : z + \mathrm{span}_{B_1}(\mathbf{u}) + \mathrm{span}_{B_2}(\mathbf{v}) \subseteq A \right\} \\ &= \# \left\{ (z, \mathbf{u}, \mathbf{v}) \in \mathbb{F}_q^n \times (\mathbb{F}_q^n)^{r_1-1} \times (\mathbb{F}_q^n)^{r_2-1} : z + \mathrm{span}_{B_1}(\mathbf{u}) \subseteq A_{\mathbf{v}} \right\} \\ &= \sum_{\mathbf{v} \in (\mathbb{F}_q^n)^{r_2-1}} \# \left\{ (z, \mathbf{u}) \in \mathbb{F}_q^n \times (\mathbb{F}_q^n)^{r_1-1} : z + \mathrm{span}_{B_1}(\mathbf{u}) \subseteq A_{\mathbf{v}} \right\} \\ &= \sum_{\mathbf{v} \in (\mathbb{F}_q^n)^{r_2-1}} \hom_{\mathrm{aff}}(B_1, A_{\mathbf{v}}). \end{aligned}$$

Since B_1 is C_1-weakly Sidorenko, this is at least

$$\begin{aligned} \sum_{\mathbf{v} \in (\mathbb{F}_q^n)^{r_2-1}} \left(\frac{|A_{\mathbf{v}}|}{N} \right)^{C_1} N^{r_1} &= N^{r_1 - C_1} \sum_{\mathbf{v} \in (\mathbb{F}_q^n)^{r_2-1}} |A_{\mathbf{v}}|^{C_1} \\ &\geq N^{r_1 - C_1} N^{r_2 - 1} \left(\frac{1}{N^{r_2-1}} \sum_{\mathbf{v} \in (\mathbb{F}_q^n)^{r_2-1}} |A_{\mathbf{v}}| \right)^{C_1} \end{aligned}$$

by Jensen's inequality. Since B_2 is C_2-weakly Sidorenko, and $\sum_{\mathbf{v}\in(\mathbb{F}_q^n)^{r_2-1}} |A_{\mathbf{v}}| = \hom_{\text{aff}}(B_2, A)$, we have

$$\begin{aligned}\hom_{\text{aff}}(B_1 \times B_2, A) &\geq N^{r_1-C_1} N^{r_2-1} \left(\frac{1}{N^{r_2-1}} \hom_{\text{aff}}(B_2, A)\right)^{C_1} \\ &\geq N^{r_1-C_1} N^{r_2-1} \left(\alpha^{C_2} N\right)^{C_1} \\ &= \alpha^{C_1 C_2} N^{r_1+r_2-1}.\end{aligned}$$

By Proposition 2.1, $\text{rank}_{\text{aff}}(B_1 \times B_2) = r_1 + r_2 - 1$, so this is our desired bound, and the proof is complete. $\square$

4 Unified Proofs of Theorem 1.8 and Theorem 1.9

The following is the same argument used in [5] and [11], but stated in our language in a unified and generalized way. For an affine configuration B and a family $\mathcal{F}$ of affine configurations, we use $B \times \mathcal{F}$ to denote the family $\{B \times F : F \in \mathcal{F}\}$.

Lemma 4.1. *Let $B \subseteq \mathbb{F}_q^m$ with $r := \text{rank}_{\text{aff}}(B) \geq 1$, and let $\mathcal{F}$ be any family of affine configurations. Let $n \geq r \geq 1$, $N = q^n$, and $\text{ex}_{\text{aff}}(n, B \times \mathcal{F}) = \alpha N$, and let $c(B, n, \alpha)$ denote the minimum number of non-degenerate affine homomorphisms $B \to A$ for an affine configuration $A \subseteq \mathbb{F}_q^n$ of density α. Then*

$$c(B, n, \alpha) \leq q^{r-1} N^{r-1} \text{ex}_{\text{aff}}(n - r + 1, \mathcal{F}).$$

Proof. Let $A \subseteq \mathbb{F}_q^n$ be affine $(B \times \mathcal{F})$-free with density α. Let S be the set of non-degenerate affine homomorphisms $B \to A$, which has size at least $c(B, n, \alpha)$ by assumption. Fix an affine basis $\{x_0, \ldots, x_{r-1}\}$ for B. For each $f \in S$ and for each $1 \leq i \leq r - 1$, define $u_i(f) := f(x_i) - f(x_0)$, and define

$$\mathbf{u}(f) := (u_1(f), \ldots, u_{r-1}(f)).$$

Note that the components of $\mathbf{u}(f)$ are linearly independent elements of $\mathbb{F}_q^n$ since f is non-degenerate. For each ordered $(r-1)$-tuple $\mathbf{u} = (u_1, \ldots, u_{r-1}) \in (\mathbb{F}_q^n)^{r-1}$ with linearly independent components, let $S_{\mathbf{u}} = \{f \in S : \mathbf{u}(f) = \mathbf{u}\}$. By the Pigeonhole Principle, there exists some $\mathbf{u} = (u_1, \ldots, u_{r-1})$ with

$$|S_{\mathbf{u}}| \geq \frac{c(B, n, \alpha)}{N^{r-1}}.$$

Now we choose a linear subspace $W_{\mathbf{u}} \subseteq \mathbb{F}_q^n$ of codimension $r - 1$ with $\mathbb{F}_q^n = W_{\mathbf{u}} \oplus \text{span}\{u_1, \ldots, u_{r-1}\}$. We take $W_{\mathbf{u}}^{(1)}, \ldots, W_{\mathbf{u}}^{(q^{r-1})}$ to be the distinct translates of $W_{\mathbf{u}}$, and for each $1 \leq j \leq q^{r-1}$, we define

$$S_{\mathbf{u}}^{(j)} = \{f \in S_{\mathbf{u}} : f(x_0) \in W_{\mathbf{u}}^{(j)}\}.$$

Again, by the Pigeonhole Principle, there exists some j with

$$|S_{\mathbf{u}}^{(j)}| \geq \frac{c(B, n, \alpha)}{q^{r-1} N^{r-1}}.$$

We now define

$$A_{\mathbf{u}}^{(j)} = \left\{ f(x_0) : f \in S_{\mathbf{u}}^{(j)} \right\} \subseteq W_{\mathbf{u}}^{(j)}.$$

Note that the map $S_{\mathbf{u}}^{(j)} \to A_{\mathbf{u}}^{(j)}$ given by $f \mapsto f(x_0)$ is a bijection with inverse

$$y \mapsto \left[f(x_i) = \begin{cases} y & \text{if } i = 0 \\ y + u_i & \text{otherwise} \end{cases} \right].$$

In particular,

$$|A_{\mathbf{u}}^{(j)}| \geq \frac{c(B, n, \alpha)}{q^{r-1} N^{r-1}}.$$

On the other hand, if we have an affine copy F' of some member $F \in \mathcal{F}$ in $A_{\mathbf{u}}^{(j)}$, then

$$G := \left\{ f(x) : x \in B, f \in S_{\mathbf{u}}^{(j)}, f(x_0) \in F' \right\}$$

is an affine copy of $B \times F \in B \times \mathcal{F}$ in A, contrary to assumption. Thus $A_{\mathbf{u}}^{(j)} \subseteq W_{\mathbf{u}}^{(j)}$ is $\mathcal{F}$-free, so we have

$$\frac{c(B, n, \alpha)}{q^{r-1} N^{r-1}} \leq |A_{\mathbf{u}}^{(j)}| \leq \mathrm{ex}_{\mathrm{aff}}(n - r + 1, \mathcal{F}). \qquad \square$$

We now apply Lemma 4.1 iteratively to recover the results of Bonin-Qin and Fox-Pham.

Proof of Theorem 1.8 and Theorem 1.9. For $q \in \{2, 3\}$, let σ_q be as in Lemma 1.10. We will prove by induction on t that for all $n \geq t$,

$$\mathrm{ex}_{\mathrm{aff}}(n, \mathbb{F}_q^t) < q^{n - n/((\sigma_q - 1)\sigma_q^{t-1}) + 2}. \tag{1}$$

For $t = 1$, let $N = q^n$, and let $A \subseteq \mathbb{F}_q^n$ be $\mathbb{F}_q^1$-free of size $\alpha N = \mathrm{ex}_{\mathrm{aff}}(n, \mathbb{F}_q^1)$. By Lemma 1.10, there are at least $\alpha^{\sigma_q} N^2$ affine homomorphisms $\mathbb{F}_q^1 \to A$, all of which are degenerate since A is $\mathbb{F}_q^1$-free. But the degenerate affine homomorphisms $\mathbb{F}_q^1 \to A$ are precisely the constant maps, so we have $\alpha^{\sigma_q} N^2 \leq \alpha N$, and hence

$$\alpha N \leq N^{1 - 1/(\sigma_q - 1)}.$$

Now assume $t \geq 2$. Let $N = q^n$, let $\mathrm{ex}_{\mathrm{aff}}(n, \mathbb{F}_q^t) = \alpha N$, and let $A \subseteq \mathbb{F}_q^t$ be an affine configuration of density α. Again, by Lemma 1.10, there are at least $\alpha^{\sigma_q} N^2 - \alpha N$ non-degenerate affine homomorphisms $\mathbb{F}_q^1 \to A$. Therefore, by Lemma 4.1, with $B = \mathbb{F}_q^1$, $\mathcal{F} = \{\mathbb{F}_q^{t-1}\}$, and $c(B, n, \alpha) \geq \alpha^{\sigma_q} N^2 - \alpha N$, we have

$$\alpha^{\sigma_q} N^2 - \alpha N \leq qN \mathrm{ex}_{\mathrm{aff}}(n - 1, \mathbb{F}_q^{t-1}).$$

By the inductive hypothesis, this gives

$$\alpha^{\sigma_q} N^2 - \alpha N < qN q^{n - 1 - (n-1)/((\sigma_q - 1)\sigma_q^{t-2}) + 2} = q^{2 + 1/((\sigma_q - 1)\sigma_q^{t-2})} N^{2 - 1/((\sigma_q - 1)\sigma_q^{t-2})}.$$

We can assume that $\alpha^{\sigma_q} N^2 \geq 2\alpha N$, as otherwise the claim holds already. Now we have

$$\frac{1}{2}\alpha^{\sigma_q} N^2 < q^{2+1/((\sigma_q-1)\sigma_q^{t-2})} N^{2-1/((\sigma_q-1)\sigma_q^{t-2})},$$

and hence

$$\alpha N < \left(2q^{2+1/((\sigma_q-1)\sigma_q^{t-2})}\right)^{1/\sigma_q} N^{1-1/((\sigma_q-1)\sigma_q^{t-1})} \leq q^2 N^{1-1/((\sigma_q-1)\sigma_q^{t-1})}. \qquad \square$$

We observe that our supersaturation results from Section 3 give an alternative proof of Theorems 1.8 and 1.9. This method additionally establishes a strong supersaturation result for affine subspaces of $\mathbb{F}_q$ for $q \in \{2, 3\}$.

Theorem 4.2. *For $q \in \{2,3\}$, let σ_q be as in the statement of Lemma 1.10, and let $t \geq 0$. Then for any n,*

$$\mathrm{ex}_{\mathrm{aff}}(n, \mathbb{F}_q^t) < q^{n-(n-t)/(\sigma_q^t - t)}.$$

Moreover, if $A \subseteq \mathbb{F}_q^n$ with $|A| = Dq^{(1-1/(\sigma_q^t - t))n}$ for some $D > 0$, then A contains more than

$$\left(1 - \frac{q^t}{D^{\sigma_q^t - t}}\right) \frac{\alpha^{\sigma_q^t} N^{t+1}}{|\mathrm{Aut}_{\mathrm{aff}}(\mathbb{F}_q^t)|}$$

affine t-spaces, where $N = q^n$ and $|A| = \alpha N$.

Proof. The claim is simply a special case of Lemma 3.2. By Lemma 1.10, $\mathbb{F}_q^1$ is σ_q-weakly Sidorenko, so by Theorem 1.11, $\mathbb{F}_q^t$ is σ_q^t-weakly Sidorenko, with $\mathrm{rank}_{\mathrm{aff}}(\mathbb{F}_q^t) = t + 1$. $\square$

We take a moment to compare our supersaturation results to a supersaturation result of Gijswijt ([16] Proposition 22). His result applies to any affine configuration $B \subseteq \mathbb{F}_q^m$ whose n-th affine extremal number is bounded above by $(q^{1-\delta})^n$ for some constant $\delta > 0$. He proves that affine configurations in $\mathbb{F}_q^n$ with density $\alpha \gg q^{-\delta n}$ have $\Omega(\alpha^{(r-1+2\delta)/\delta} N^r)$ affine copies of B, where $r = \mathrm{rank}_{\mathrm{aff}}(B)$ and $N = q^n$. Our result Lemma 3.2 improves this count to $\Omega(\alpha^{(\delta(r-1)+1)/\delta} N^r)$ affine copies of B when B is C-weakly Sidorenko and we take $\delta = 1/(C - r + 1)$. In particular, for $q \in \{2,3\}$, Gijswijt's result guarantees only $\alpha^{(1+o(1))t\sigma_q^t} N^{t+1}$ affine t-spaces when $|A|$ is above the Bonin-Qin threshold (for $q = 2$) or Fox-Pham threshold (for $q = 3$). Now Theorem 4.2 improves this to $\Omega(\alpha^{\sigma_q^t} N^{t+1})$ affine t-spaces, which is tight for $q = 2$ by considering a random affine configuration of density α.

5 Proof of Theorem 1.3

We first prove a general upper bound for the two-color Ramsey number $R_q(s,t)$ for $q \in \{2,3\}$, from which Theorem 1.3 is easily derived. The proof uses nothing more than Theorem 1.7 and our explicit forms of Theorem 1.8 and Theorem 1.9 for bounds on $\mathrm{ex}_{\mathrm{aff}}(n, \mathbb{F}_q^t)$.

Theorem 5.1. *For $q \in \{2,3\}$, let σ_q be as in Lemma 1.10. For any $t \geq s \geq 2$, $R_q(s,t)$ is at most a tower of height s of the form*

$$R_q(s,t) \leq \sigma_q^{\sigma_q^{\cdot^{\cdot^{\cdot^{\sigma_q^{2t}}}}}}.$$

Proof. We induct on s for fixed t. Clearly, we have $R_q(1,t) = t$. Now for $s \geq 2$, let $r = R_q(s-1,t)$, and let $n = t\sigma_q^r$. Suppose we have a projectively determined red-blue coloring of $\mathbb{F}_q^n \setminus \{0\}$ with red set R and blue set B satisfying $\omega(R) < s$ and $\omega(B) < t$. By Theorem 1.7, the fact that $\omega(B) < t$ implies that $|R| \geq q^{n-t+1} - 1$, with equality if and only if $R \cup \{0\}$ is a linear $(n-t+1)$-space. But $\omega(R) < s \leq n-t+1$, so we must have $|R| \geq q^{n-t+1}$. Also, by Theorem 4.2,

$$\mathrm{ex}_{\mathrm{aff}}(n, \mathbb{F}_q^r) \leq q^{n-(n-r)/(\sigma_q^r - r)} \leq q^{n-n/\sigma_q^r} = q^{n-t} < |R|,$$

so R contains an affine r-space A. Note that $0 \notin A$ since $0 \notin R$. Let W be the translate of A containing 0, which is a linear r-space. Then by our choice of r and because we've assumed $\omega(B) < t$, there exists a linear $(s-1)$-space $U' \subseteq W$ with $U' \setminus \{0\}$ entirely red. Now because the coloring is projectively determined, for any $u \in A$ and $\lambda \in \mathbb{F}_q \setminus \{0\}$, the set $W + \lambda u = \lambda A$ is entirely red. But then $U := \mathrm{span}\{U', u\}$ is a linear s-space contained in $U' \cup \bigcup_{\lambda \in \mathbb{F}_q \setminus \{0\}} \lambda A$, so $U \setminus \{0\}$ is entirely red, a contradiction. Thus

$$R_q(s,t) \leq n = t\sigma_q^r.$$

By induction, $R_q(s,t)$ is at most a tower of height s of the form

$$R_q(s,t) \leq t\sigma_q^{t\sigma_q^{\cdot^{\cdot^{\cdot^{t\sigma_q^t}}}}}.$$

To obtain the friendlier-looking bound stated in the theorem, it suffices to show that $\log_{\sigma_q}^{(s-1)}(R_q(s,t)) \leq 2t$, where $\log_b^{(k)}(x)$ denotes the k-th iterated logarithm (to base b) of x, defined by

$$\log_b^{(k)}(x) := \begin{cases} x & \text{if } k = 0, \\ \log_b\left(\log_b^{(k-1)}(x)\right) & \text{if } k \geq 1 \text{ and } \log_b^{(k-1)}(x) > 0, \\ -\infty & \text{if } k \geq 1 \text{ and } \log_b^{(k-1)}(x) \leq 0. \end{cases}$$

First note that

$$\log_{\sigma_q}(R_q(s,t)) \leq \log_{\sigma_q}\left(t\sigma_q^{t\sigma_q^{\cdot^{\cdot^{\cdot^{t\sigma_q^t}}}}}\right) = \log_{\sigma_q} t + t\sigma_q^{t\sigma_q^{\cdot^{\cdot^{\cdot^{t\sigma_q^t}}}}} \leq 2t\sigma_q^{t\sigma_q^{\cdot^{\cdot^{\cdot^{t\sigma_q^t}}}}},$$

where the height of the tower on the right is $s-1$. Now applying the logarithm again gives

$$\log_{\sigma_q}^{(2)}(R_q(s,t)) \leq \log_{\sigma_q}(2t) + t\sigma_q^{t\sigma_q^{\cdot^{\cdot^{\cdot^{t\sigma_q^t}}}}} \leq 2t\sigma_q^{t\sigma_q^{\cdot^{\cdot^{\cdot^{t\sigma_q^t}}}}},$$

where the height is now $s-2$. Continuing in this fashion, we obtain $\log_{\sigma_q}^{(s-2)}(R_q(s,t)) \leq 2t\sigma_q^t$, and thus

$$\log_{\sigma_q}^{(s-1)}(R_q(s,t)) \leq \log_{\sigma_q}(2t) + t \leq 2t$$

since $t \geq 2$. $\square$

Proof of Theorem 1.3. For $q \in \{2,3\}$, let σ_q be as in Lemma 1.10. We induct on k. The base case $k=2$ is given by Theorem 5.1.

For $k \geq 3$, we use the simple recurrence

$$R_q(t_1,\ldots,t_k) \leq R_q(t_1,\ldots,t_{k-2},R_q(t_{k-1},t_k)).$$

Indeed, consider a partition $\mathbb{F}_q^n \setminus \{0\} = B_1 \cup \cdots \cup B_k$ with $n = R_q(t_1,\ldots,t_{k-2},R_q(t_{k-1},t_k))$, where each set B_i is projectively determined. If $\omega(B_i) < t_i$ for all $i \leq k-2$, then we must have $\omega(B_{k-1}\cup B_k) \geq R_q(t_{k-1},t_k)$, and so $\omega(B_i) \geq t_i$ for some $i \geq k-1$.

With this observation, we obtain by the inductive hypothesis that

$$R_q(t_1,\ldots,t_k) \leq R_q(t_1,\ldots,t_{k-2},R_q(t_{k-1},t_k)) \leq \sigma_q^{\sigma_q^{\cdot^{\cdot^{\cdot^{\sigma_q^{3R_q(t_{k-1},t_k)}}}}}},$$

where the height of the tower is $\sum_{i=1}^{k-2}(t_i-1)+1$. Now by Theorem 5.1,

$$\log_{\sigma_q}^{(t_{k-1}-1)}(R_q(t_{k-1},t_k)) \leq 2t_k,$$

which implies

$$\log_{\sigma_q}^{(t_{k-1}-1)}(3R_q(t_{k-1},t_k)) \leq 2t_k + \log_{\sigma_q} 3 \leq 3t_k$$

since $t_k \geq 2$. This completes the inductive step. $\square$

6 A Reformulation of $R_q(2,t)$

We now reformulate the off-diagonal Ramsey problem as an affine extremal problem. We look at the $\mathbb{F}_2$ case first for the sake of exposition. Consider the *sumset* of $A \subseteq \mathbb{F}_2^n$, defined as

$$A + A := \{x+y : x,y \in A\},$$

and let $m_2(t)$ be the minimum n such that every set $A \subseteq \mathbb{F}_2^n$ of size at least 2^{n-t+1} satisfies $\omega(A+A) \geq t$; that is, $A+A$ contains a linear t-space. Nelson and Nomoto [23] observed that $m_2(t)$ is an upper bound for $R_2(2,t)$ for all $t \geq 2$ (see Lemma 6.2 for the argument). One way to bound $m_2(t)$ from above is via the following theorem of Sanders [27].

Theorem 6.1 (Sanders)**.** *Let A be a subset of $\mathbb{F}_2^n$ of density $\alpha < 1/2$. Then*

$$\omega(A+A) \geq n - \left\lceil n/\log_2 \frac{2-2\alpha}{1-2\alpha} \right\rceil.$$

Taking $\alpha = 2^{1-t}$ and $n = (t+1)2^t$, and noting that $n - \left\lceil n/\log_2 \frac{2-2\alpha}{1-2\alpha}\right\rceil \geq \alpha n/2 - 1 = t$ for this choice of parameters, Theorem 6.1 gives $m_2(t) \leq n$. This is how Theorem 1.4 is proved in [23].

Alternatively, we can take an affine extremal approach to bound $m_2(t)$, based on the simple observation that $\omega(A+A) \geq t$ if and only if A contains an affine copy B' of some affine configuration B with $\omega(B+B) \geq t$.

Indeed, as noted in Section 2, $\omega(B+B) = \omega^{\to}(B)$ is entirely determined by the affine structure of B, so $\omega(B'+B') = \omega(B+B)$. Therefore, if we define

$$\mathcal{B}_2^t := \{B \subseteq \mathbb{F}_2^m : m \geq 1,\, \omega(B+B) \geq t\},$$

then we have the alternative description of $m_2(t)$ as the minimum n such that $\mathrm{ex}_{\mathrm{aff}}(n, \mathcal{B}_2^t) < 2^{n-t+1}$. We see that this is finite by Theorem 1.6, and in fact, Theorem 1.8 immediately implies an improvement of Theorem 1.4 by a constant factor. Note that any set A which *properly* contains an affine $(t-1)$-space has $\omega(A+A) \geq t$, so using the explicit bound in (1), we have

$$\mathrm{ex}_{\mathrm{aff}}(n, \mathcal{B}_2^t) \leq \mathrm{ex}_{\mathrm{aff}}(n, \mathbb{F}_2^{t-1}) < 2^{(1-2^{2-t})n+2}$$

for $n \geq t$. In particular, if $n = (t+1)2^{t-2}$, then $\mathrm{ex}_{\mathrm{aff}}(n, \mathcal{B}_2^t) < 2^{n-t+1}$, so

$$R_2(2,t) \leq m_2(t) \leq (t+1)2^{t-2}.$$

We obtain further improvements on $R_2(2,t)$ by finding better upper bounds for $\mathrm{ex}_{\mathrm{aff}}(n, \mathcal{B}_2^t)$.

More generally, we define for an arbitrary finite field $\mathbb{F}_q$

$$\mathcal{B}_q^t := \{B \subseteq \mathbb{F}_q^m : m \geq 1, \omega^{\to}(B) \geq t\},$$

and we define $m_q(t)$ to be the minimum n such that $\mathrm{ex}_{\mathrm{aff}}(n, \mathcal{B}_q^t) < q^{n-t+1}$. Equivalently, $m_q(t)$ is the minimum n such that $\omega^{\to}(A) \geq t$ for every $A \subseteq \mathbb{F}_q^n$ of size at least q^{n-t+1}. We have the following.

Lemma 6.2. *Let $\mathbb{F}_q$ be any finite field. Then $R_q(2,t) \leq m_q(t)$ for all $t \geq 2$.*

Proof. Let $n = m_q(t)$. First, we show that $n \geq t+1$. Let H be a linear hyperplane in $\mathbb{F}_q^n$, which satisfies $\omega^{\to}(H) = n-1$. Since $|H| = q^{n-1} \geq q^{n-t+1}$, we have $\omega^{\to}(H) \geq t$ by our choice of n.

Now suppose we have a projectively determined red-blue coloring of $\mathbb{F}_q^n \setminus \{0\}$ with red set R and blue set B satisfying $\omega(R) < 2$ and $\omega(B) < t$. Since $\omega(B) < t$, we have by Theorem 1.7 that $|R| \geq q^{n-t+1} - 1$, with equality iff $R \cup \{0\}$ is a linear $(n-t+1)$-space. But $n-t+1 \geq 2 > \omega(R)$, so we can't have equality, and hence $|R| \geq q^{n-t+1}$. By our choice of n, $\omega^{\to}(R) \geq t > \omega(B)$, so there exists some nonzero $d \in R^{\to} \setminus B$. That is, $d \in R^{\to} \cap R$. Let $a \in R$ be such that $a + \lambda d \in R$ for every $\lambda \in \mathbb{F}_q$. Note that a and d are linearly independent since $0 \notin R$. Therefore, since R is projectively determined, $\mathrm{span}\{a, d\}$ is a linear 2-space contained in $R \cup \{0\}$, contradicting that $\omega(R) < 2$. □

We can now use the machinery from Section 3 to prove Theorem 1.5.

Proof of Theorem 1.5. Let $t \geq 1$, and let $k = \lceil t/4 \rceil$. Consider the affine configuration $C_6 \subseteq \mathbb{F}_2^4$, defined prior to Lemma 1.10. Suppose we have two pairs $\{x,y\}, \{x',y'\} \in \binom{C_6}{2}$ with $x+y = x'+y'$. Since every 4 distinct elements of C_6 are affinely independent, we must have that x, y, x', y' are not distinct. Then $x = x'$ without loss of generality, which implies $y = y'$ as well. Thus we have $\binom{6}{2} = 15$ distinct nonzero sums $x+y \in \mathbb{F}_2^4$ for $x, y \in C_6$ with $x \neq y$, which means that $C_6 + C_6 = \mathbb{F}_2^4$, and we have $\omega^{\to}(C_6) = 4$. By Proposition 2.1,

$\omega^{\to}(C_6^k) = 4k \geq t$ and $\mathrm{rank}_{\mathrm{aff}}(C_6^k) = 4k+1$. Recall that $\mathcal{B}_2^t = \{B : \omega^{\to}(B) \geq t\}$, so $C_6^k \in \mathcal{B}_2^t$. Furthermore, by Lemma 1.10 and Theorem 1.11, C_6^k is Sidorenko, so by Lemma 3.2,

$$\mathrm{ex}_{\mathrm{aff}}(n, \mathcal{B}_2^t) \leq \mathrm{ex}_{\mathrm{aff}}(n, C_6^k) < 2^{n-(n-4k)/(6^k-4k)} = 2^{n-t+1}$$

for $n = (t-1)(6^k - 4k) + 4k$. Thus

$$R_2(2,t) \leq m_2(t) \leq (t-1)(6^k - 4k) + 4k = O\big(t6^{t/4}\big)$$

by Lemma 6.2.

Similarly, since $\omega^{\to}(\mathbb{F}_3^t) = t$, we have by Theorem 4.2 that

$$\mathrm{ex}_{\mathrm{aff}}(n, \mathcal{B}_3^t) \leq \mathrm{ex}_{\mathrm{aff}}(n, \mathbb{F}_3^t) < 2^{n-(n-t)/(C_0^t-t)} = 2^{n-t+1}$$

for $n = (t-1)(C_0^t - t) + t$, with $C_0 \approx 13.901$ as in Theorem 1.9. Thus

$$R_3(2,t) \leq m_3(t) \leq (t-1)(C_0^t - t) + t = O\big(tC_0^t\big)$$

by Lemma 6.2. □

We remark that the constants implicit in the $O(\cdot)$ in our bounds for Theorem 1.5 are not optimized. Bounding the extremal numbers of $\mathcal{B}_2^t$ and $\mathcal{B}_3^t$ via iterative application of Lemma 4.1 gives the best results, but the computations are slightly more cumbersome.

We state a generalization of this argument, which can be used to further improve our off-diagonal Ramsey bounds by establishing homomorphic supersaturation of affine configurations. Unfortunately, this technique by itself can never give a subexponential bound for $R_q(2,t)$. Indeed, for any affine configuration $B \subseteq \mathbb{F}_q^m$ with $\omega^{\to}(B) \geq 1$, the map $f : B^2 \to \mathbb{F}_q^m$ given by $f(x,y) = y - x$ has $B^{\to}$ in its image so $|B^{\to}| \leq |B|^2$. Also, for any linear configuration A, we have $|A| \geq q^{\omega(A)}$, and hence

$$|B|^{1/\omega^{\to}(B)} \geq |B^{\to}|^{1/(2\omega^{\to}(B))} \geq q^{1/2}.$$

Thus the best possible upper bound that can come directly from Theorem 6.3 is $\Omega\big(tq^{t/2}\big)$.

Theorem 6.3. *Suppose that $B \subseteq \mathbb{F}_q^m$ is C-weakly Sidorenko, and let $p = \omega^{\to}(B) \geq 1$. Then as $t \to \infty$,*

$$R_q(2,t) = O\left(tC^{t/p}\right).$$

Proof. Let $n \geq t \geq 2$, and let $k = \lceil t/p \rceil$ and $r = \mathrm{rank}_{\mathrm{aff}}(B)$. By Theorem 1.11 and Proposition 2.1, B^k is C^k-weakly Sidorenko with $\omega^{\to}(B^k) = kp \geq t$ and $\mathrm{rank}_{\mathrm{aff}}(B^k) = (r-1)k+1$. Therefore, by Lemma 3.2,

$$\mathrm{ex}_{\mathrm{aff}}(n, \mathcal{B}_q^t) \leq \mathrm{ex}_{\mathrm{aff}}(n, B^k) < q^{n-(n-(r-1)k)/(C^k-(r-1)k)}$$

If we take

$$n = (t-1)(C^k - (r-1)k) + (r-1)k = O(tC^{t/p}),$$

then $\mathrm{ex}_{\mathrm{aff}}(n, \mathcal{B}_q^t) < q^{n-t+1}$. By Lemma 6.2, $R_q(2,t) \leq m_q(t) \leq n$. □

7 Concluding Remarks

We believe $m_q(t) = \min\{n : \mathrm{ex}_{\mathrm{aff}}(n, \mathcal{B}_q^t) < q^{n-t+1}\}$ to be polynomial in t for all q, which would imply that $R_q(2,t)$ is also polynomial by Lemma 6.2. For $q = 2$, this was asked by Peter Nelson [1] in the second Barbados graph theory workshop 2022 (Problem 17), and this remains open. For $q \neq 2, 3$, it is unknown whether $m_q(t)$ is even bounded by an exponential function. Such a bound would follow immediately from exponential improvements on the affine extremal number of $\mathbb{F}_q^1$ by Lemma 4.1, combined with the aforementioned supersaturation result of Gijswijt ([16], Proposition 22). In particular, if it is true that $\mathrm{ex}_{\mathrm{aff}}(n, \mathbb{F}_q^1) \leq (q^{1-\delta})^n$ for some $\delta > 0$, then we immediately obtain

$$R_q(2,t) \leq m_q(t) = O\left(t(2 + 1/\delta)^t\right).$$

It is also worth mentioning the natural relationship of affine extremal numbers to *affine Ramsey numbers*. We use $R_{\mathrm{aff}q}(t_1, \ldots, t_k)$ to denote the minimum n such that for every k-coloring $f : \mathbb{F}_q^n \to [k]$ of the points of $\mathbb{F}_q^n$, there exist $i \in [k]$ and an affine subspace $U \subseteq \mathbb{F}_q^n$ of dimension t_i, such that U is monochromatic in color i. If $t_1 = \cdots = t_k = t$, we write $R_{\mathrm{aff}q}(t_1, \ldots, t_k) = R_{\mathrm{aff}q}(t; k)$. Such Ramsey numbers clearly exist by Theorem 1.6 since the majority color class, say color i, has size at least q^n/k, which is greater than $\mathrm{ex}_{\mathrm{aff}}(n, \mathbb{F}_q^{t_i})$ for large n. In fact, any general upper bound for $\mathrm{ex}_{\mathrm{aff}}(n, \mathbb{F}_q^t)$ immediately implies upper bounds for affine Ramsey numbers. For $q \in \{2, 3\}$, Theorems 1.8 and 1.9 give

$$R_{\mathrm{aff}q}(t; k) \leq (\log_2 k)\sigma_q^t \qquad \text{for all } k \geq 2,\ t \geq 1;$$

$$R_{\mathrm{aff}q}(s, t) \leq (\log_q \sigma_q)(\sigma_q - 1)\sigma_q^{s-1} t \qquad \text{for all } s \text{ fixed, } t \text{ large,}$$

where σ_q is as in Lemma 1.10. Upper bounds on Hales-Jewett numbers (see [28], for example) also imply upper bounds on affine Ramsey numbers for general q, though these are of a much larger order of growth. For lower bounds, straightforward applications of the Lovász Local Lemma give the following:

$$R_{\mathrm{aff}q}(t; k) \geq (\log_q k)\frac{q^t}{t} \qquad \text{for all } k \text{ fixed, } t \text{ large;}$$

$$R_{\mathrm{aff}q}(s, t) \geq \left(\frac{q^s - 1}{s} - o(1)\right) t \qquad \text{for all } s \text{ fixed, as } t \to \infty.$$

It would be interesting to see new methods develop for obtaining upper bounds on affine Ramsey numbers.

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