Symmetries as Ground States of Local Superoperators: Hydrodynamic Implications

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Symmetry algebras of quantum many-body systems with locality can be understood using commutant algebras, which are defined as algebras of operators that commute with a given set of local operators. In this work, we show that these symmetry algebras can be expressed as frustration-free ground states of a local superoperator, which we refer to as a "super-Hamiltonian." We demonstrate this for conventional symmetries such as $\mathbb{Z}_2$, $U(1)$, and $SU(2)$, where the symmetry algebras map to various kinds of ferromagnetic ground states, as well as for unconventional ones that lead to weak ergodicity-breaking phenomena of Hilbert-space fragmentation (HSF) and quantum many-body scars. In addition, we show that the low-energy excitations of this super-Hamiltonian can be understood as approximate symmetries, which in turn are related to slowly relaxing hydrodynamic modes in symmetric systems. This connection is made precise by relating the super-Hamiltonian to the superoperator that governs the operator relaxation in noisy symmetric Brownian circuits and this physical interpretation also provides a novel interpretation for Mazur bounds for autocorrelation functions. We find examples of gapped (gapless) super-Hamiltonians indicating the absence (presence) of slow modes, which happens in the presence of discrete (continuous) symmetries. In the gapless cases, we recover hydrodynamic modes such as diffusion, tracer diffusion, and asymptotic scars in the presence of $U(1)$ symmetry, HSF, and a tower of quantum scars, respectively. In all, this demonstrates the power of the commutant-algebra framework in obtaining a comprehensive understanding of exact symmetries and associated approximate symmetries and hydrodynamic modes, and their dynamical consequences in systems with locality.

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I. INTRODUCTION

Developing an understanding of symmetries in their most general form has been a recent quest in many different parts of physics. The definition of symmetries in most of the quantum many-body physics literature implicitly assumes some kinds of restriction imposed on the symmetry operators, e.g., they are usually on-site unitary symmetries with nice group structures or lattice symmetries such as translation, rotation, reflection, etc. However, several recent works have demonstrated that generalized symmetries beyond what is usually studied in much of the literature can naturally arise in several physical settings. In the context of equilibrium physics, several new types of symmetries have recently been studied in the context of various physical lattice models or quantum field theories [1,2]. Examples include "higher-form symmetries," where the symmetry operators live on manifolds of some nonzero codimension [3–6]; "noninvertible" or "categorical" symmetries, where the symmetries are not representations of groups but of categories [7–9]; "MPO symmetries," where the symmetry operators are matrix product operators (MPOs) [10–12]; or even more exotic symmetries that appear in the study of fractons [13–16], where the symmetries depend on the system size. In the context of nonequilibrium physics, a general framework for symmetries based on "commutant algebras" has naturally appeared in the study of dynamical phenomena known as weak ergodicity breaking [17–20]. For example, systems exhibiting Hilbert-space fragmentation

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(HSF) [21–24] have symmetry algebras that grow exponentially with system size [25] and systems exhibiting quantum many-body scars (QMBSs) [17,19,20,26–30] have nonlocal symmetries such as projectors onto certain pure states [31]. The discovery of such wide varieties of symmetries motivates the search for certain characterizing properties of symmetry operators that are allowed in physical quantum many-body systems.

A hint comes from the framework where symmetry algebras can be understood as *commutant algebras*, i.e., as the associative algebra of operators that commute with a given set of *local* operators. In our previous works, we have explored this framework in detail and demonstrated that it can be used to understand regular symmetries and symmetry sectors in a wide variety of standard Hamiltonians [32], as well as to discover novel symmetries that explain the phenomena of QMBSs [31] and HSF [25]. Further, in Ref. [33], we have introduced numerical methods to calculate commutant algebras. One such method has been based on the property of these operator algebras that when operators are viewed as states in a doubled Hilbert space, these algebras are the ground-state spaces of certain frustration-free local superoperators. Mapping the determination of symmetry algebras to a ground-state problem has led to efficient algorithms to determine symmetries, which have used ideas from tensor-network methods for determining ground states in general [34] as well as specialized methods for determining frustration-free ground states [35–37].

In this work, we explore the analytical aspects and consequences of the idea that symmetries are ground states of local superoperators, which we refer to as "super-Hamiltonians." This allows us to analytically understand several properties of symmetric systems with locality. We work out the explicit super-Hamiltonians, which have interpretations in terms of simple ladder or bilayer Hamiltonians, and we solve for their ground states, which map precisely onto commutant algebras. This allows us to obtain *a priori* bizarre connections between $\mathbb{Z}_2$, $U(1)$, and $SU(2)$ symmetry algebras and ferromagnetic states of various kinds, which can all be expressed as ground states of frustration-free Hamiltonians. In addition, we illustrate the commutant algebras in certain unconventional symmetries, including some of the examples of HSFs and QMBSs discussed in Refs. [25,31].

In addition to a clear understanding of *exact* symmetries, which are understood as "white" or "black" properties of the system—i.e., a given symmetry either exists or it does not—this mapping to ground states also introduces a grayscale and provides a precise language for discussing approximate symmetries. These approximate symmetries are naturally defined as operators that are in the low-energy spectrum of the super-Hamiltonians, of which *exact* symmetries are the ground states. Since the exact symmetries in many of the examples map onto ferromagnetic states, the low-energy excitations are given by spin waves, which map back onto approximate symmetries. These are approximately conserved quantities, which can be loosely viewed as long-wavelength textures in the densities of the exactly conserved quantities and hence are conserved up to times that diverge with the system size (e.g., taking the longest wavelengths fitting into the system). This feature of approximate symmetries also illustrates their connection to hydrodynamic modes, as we discuss below.

This connection to approximate symmetries is made precise by a remarkable physical relation between the super-Hamiltonian and noisy Brownian-circuit models similar to those studied in the context of noisy spin chains [38–42], or as toy models for quantum chaos [43–52]. In particular, the low-energy spectrum of the superoperator is related to the relaxation rates of noise-averaged autocorrelation functions toward their Mazur-bound values dictated by symmetry [53,54], which leads to two main insights. First, it provides an alternative physical meaning to the Mazur-bound value, usually interpreted as a lower bound on the autocorrelation function of an operator evolving under a single physical Hamiltonian with a given set of symmetries. Second, it shows that the *approximate symmetries* that appear as low-energy excitations above the ground state of the super-Hamiltonian correspond to slowly relaxing hydrodynamic modes that govern late-time transport properties in symmetric systems with locality. For example, we are able to understand the approximately L^2 relaxation time in $U(1)$ symmetric systems, which occurs due to the presence of charge or spin diffusion, in terms of spin-wave excitations above ferromagnetic ground states of the superoperator Hamiltonian. In addition, we are also able to use this framework to understand hydrodynamic modes for unconventional symmetries such as QMBS, which coincide with slowly thermalizing initial states in such systems, recently referred to as *asymptotic* QMBSs [55]. With these insights, we connect exact symmetries understood in the commutant-algebra framework to approximate symmetries that are related to hydrodynamic relaxation modes and late-time transport, which have been of significant interest lately in systems with various kinds of symmetries [56–64].

This paper is organized as follows. In Sec. II, we review key concepts in the study of commutant algebras and their connection to ground states of local super-Hamiltonians. In Sec. III, we work out several examples in the context of conventional and unconventional symmetries. Then, in Sec. IV, we illustrate the connection between the low-energy spectrum of the super-Hamiltonians and operator relaxation to Mazur bounds, which can be made concrete in Brownian- or noisy-circuit models. We also exhibit the approximate conserved quantities in the context of various kinds of symmetries. Finally, we conclude with open questions in Sec. VI.

II. COMMUTANT ALGEBRAS AND GROUND STATES

We first review the connection between commutant algebras and frustration-free ground states of local superoperator Hamiltonians, which was first introduced in the context of numerical methods to detect symmetries in [33]. Here, we only review aspects necessary for this work and more comprehensive discussions can be found in our previous papers [25,31–33].

A. Definitions

The essential idea of commutant algebras is to think of symmetries in terms of a pair of operator algebras $(\mathcal{A}, \mathcal{C})$, referred to as the local algebra and the commutant algebra, respectively, which are centralizers of each other in the algebra of all operators on the full (finite-dimensional) Hilbert space. As the name suggests, the *local algebra* $\mathcal{A}$ is generated by a set of Hermitian local operators $\{\widehat{H}_\alpha\}$, which can either be strictly local or extensive local, and we denote it as $\mathcal{A} = \langle\langle\{\widehat{H}_\alpha\}\rangle\rangle$. In the case in which all the operators $\widehat{H}_\alpha$ are strictly local, the algebra $\mathcal{A}$ is also commonly referred to as a *bond algebra* [65–67]. The *commutant algebra* $\mathcal{C}$, by definition, is the centralizer of $\mathcal{A}$, i.e., the set of all operators that commute with the $\{\widehat{H}_\alpha\}$, which is the symmetry algebra for *all* Hamiltonians in $\mathcal{A}$, i.e., those that can be expressed in terms of linear combinations of products of $\{\widehat{H}_\alpha\}$.

For the Hamiltonians constructed out of the generators of $\mathcal{A}$, symmetry sectors and dynamically disconnected Krylov subspaces due to the symmetry algebra $\mathcal{C}$ can be understood in terms of their representation theory of von Neumann algebras. Thinking of symmetries in this commutant-algebra framework leads to a comprehensive understanding of symmetry algebras, symmetric operators, and associated symmetry quantum number sectors, we refer to Refs. [25,31,32] for concrete examples in a variety of systems.

B. Liouvillians and super-Hamiltonians

Given the local algebra $\mathcal{A}$, determining the commutant $\mathcal{C}$ is not always straightforward in practice. Hence in Ref. [33], we have introduced two numerical methods to numerically construct the full commutant algebra $\mathcal{C}$ given a set of local terms $\{\widehat{H}_\alpha\}$ that generate the local algebra $\mathcal{A}$. The method relevant for this work is the "Liouvillian method," which starts by interpreting operators $\widehat{O}$ on the Hilbert space $\mathcal{H}$ as vectors $|\widehat{O})$. In particular, operators on the Hilbert space $\mathcal{H}$, which themselves form a Hilbert space denoted as $\mathcal{L}(\mathcal{H})$, can be mapped onto states on the doubled Hilbert space $\mathcal{H}\otimes\mathcal{H}$ via the mapping

$$\widehat{O} = \sum_{\mu,\nu} o_{\mu\nu}|v_\mu\rangle\langle v_\nu| \iff |\widehat{O}) = \sum_{\mu,\nu} o_{\mu\nu}\,|v_\mu\rangle\otimes|v_\nu\rangle, \tag{1}$$

where $\{|v_\mu\rangle\}$ is an orthonormal basis for $\mathcal{H}$, which we take to be the computational basis of product states. For example, for a spin-1/2 system, we have

$$|\mathbb{1})_j = |\uparrow\rangle_j\otimes|\uparrow\rangle_j + |\downarrow\rangle_j\otimes|\downarrow\rangle_j\,, \tag{2}$$

where j labels a site [the identity operator in a many-body system is $\hat{\mathbb{1}} \iff \otimes_j|\mathbb{1})_j$]. In this work, we will sometimes interchangeably use $|\bullet\rangle$ and $|\bullet)$ when referring to operators as states on a doubled Hilbert space and the meaning should be obvious from the context. Adapting the definition of Eq. (1) together with the conventional inner product in the doubled space implies that the inner product in the operator Hilbert space is defined as [68]

$$(\widehat{A}|\widehat{B}) := \mathrm{Tr}(\widehat{A}^\dagger\widehat{B}). \tag{3}$$

The action of the commutator of an operator $\widehat{O}$ with an operator $\widehat{H}_\alpha$ can be represented as

$$[\widehat{H}_\alpha, \widehat{O}] \iff \overbrace{\left(\widehat{H}_\alpha\otimes\mathbb{1} - \mathbb{1}\otimes\widehat{H}_\alpha^T\right)}^{\widehat{\mathcal{L}}_\alpha :=}|\widehat{O}), \tag{4}$$

where the transpose is taken in the computational basis. In Eq. (4), $\widehat{\mathcal{L}}_\alpha$ is referred to as the *Liouvillian* corresponding to the term $\widehat{H}_\alpha$, which is the superoperator that represents the adjoint action of that term, i.e., $\widehat{\mathcal{L}}_\alpha\,|\bullet) := [\widehat{H}_\alpha, \bullet]$.

Given an algebra $\mathcal{A} = \langle\langle\{\widehat{H}_\alpha\}\rangle\rangle$, the operators in the commutant $\mathcal{C}$ by definition commute with each of the $\widehat{H}_\alpha$. Hence, as vectors in the doubled Hilbert space, they span the common kernel of the Liouvillian superoperators $\{\widehat{\mathcal{L}}_\alpha\}$, which is also the null space of the positive semidefinite superoperator defined as

$$\widehat{\mathcal{P}} := \sum_\alpha \overbrace{\widehat{\mathcal{L}}_\alpha^\dagger\widehat{\mathcal{L}}_\alpha}^{\widehat{\mathcal{P}}_\alpha :=}, \quad \widehat{\mathcal{P}}|\widehat{O}) = 0 \iff \widehat{\mathcal{L}}_\alpha|\widehat{O}) = 0\ \forall\alpha, \tag{5}$$

where the second condition follows since all the $\widehat{\mathcal{P}}_\alpha$ are positive semidefinite. As discussed in Ref. [33], this provides an efficient numerical method to compute the full commutant $\mathcal{C}$, given the generators $\{\widehat{H}_\alpha\}$. In addition, in the absence of exact symmetries, the low-energy spectrum of $\widehat{\mathcal{P}}$ can be treated as approximate symmetries, as we discuss later in this work. In Appendix D, we comment on the dependence of the super-Hamiltonians on the choice of the generators of the bond algebra $\mathcal{A}$, which does not affect the exact ground states and is not of any concern when using this method to find the commutant $\mathcal{C}$.

C. Ladder-bilayer interpretation

In order to study the $\widehat{\mathcal{P}}$ in typical cases, where $\mathcal{H}$ is a tensor-product Hilbert space with qudits arranged on some lattice, it is convenient to view the Hilbert space $\mathcal{H}\otimes\mathcal{H}$

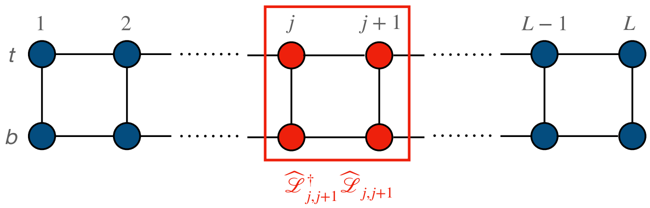

FIG. 1. The doubled Hilbert space $\mathcal{H}\otimes\mathcal{H}$ for one-dimensional (1D) systems depicted as a ladder with the two legs labeled by $\{t, b\}$. The super-Hamiltonian term $\widehat{\mathcal{L}}^{\dagger}_{j,j+1}\widehat{\mathcal{L}}_{j,j+1}$ arising from a nearest-neighbor bond-algebra term $\widehat{h}_{j,j+1}$ is a nearest-neighbor term along the ladder, i.e., it acts nontrivially only on rungs j and $j+1$.

as two copies of the original lattice with the sites aligned, i.e., a site j on the first copy is taken to be neighbor of the site j on the second copy. This geometry corresponds to a ladder [in one dimension (1D); see Fig. 1] or a bilayer [in two dimensions (2D)]; hence we will refer to the first copy of the Hilbert space as the "top" leg or layer, the second as the "bottom" leg or layer, and the link between the two legs or layers as "rungs." We denote operators on the doubled Hilbert space that act only on one leg or layer using the shorthand notations

$$\widehat{O}_{\alpha;t} := \widehat{O}_{\alpha}\otimes\mathbb{1}, \quad \widehat{O}_{\alpha;b} := \mathbb{1}\otimes\widehat{O}_{\alpha}, \tag{6}$$

where using leg or layer labels $\{t, b\}$ as "local indices," as often done when writing local Hamiltonians in compact form (we label them using subscripts $\{t, b\}$).

In most examples that we study, $\mathcal{A}$ is a bond algebra, i.e., the operators $\{\widehat{H}_{\alpha}\}$ are strictly local operators, e.g., nearest-neighbor terms of the form $\{\widehat{h}_{j,j+1}\}$. In such cases, the superoperators $\widehat{\mathcal{P}}_{\alpha} := \widehat{\mathcal{L}}^{\dagger}_{\alpha}\widehat{\mathcal{L}}_{\alpha}$ are also strictly local operators on the ladder or bilayer, with the same range along the ladder or bilayer as the $\{\widehat{H}_{\alpha}\}$, e.g., nearest-neighbor terms $\{\widehat{h}_{j,j+1}\}$ give rise to superoperator terms $\{\widehat{\mathcal{L}}^{\dagger}_{j,j+1}\widehat{\mathcal{L}}_{j,j+1}\}$, as shown in Fig. 1. As a consequence, $\widehat{\mathcal{P}}$ of Eq. (5) is an extensive local operator on the ladder or bilayer, and has a natural interpretation as a superoperator Hamiltonian, which we refer to as a "super-Hamiltonian." The definition of the $\widehat{\mathcal{L}}_{\alpha}$ in Eq. (4) can be used to directly obtain an expression for $\widehat{\mathcal{P}}$ of Eq. (5) in terms of the $\{\widehat{H}_{\alpha}\}$, which reads

$$\widehat{\mathcal{P}} = \sum_{\alpha}\widehat{\mathcal{P}}_{\alpha} = \sum_{\alpha}\left(\widehat{H}^{2}_{\alpha;t} + (\widehat{H}^{*}_{\alpha;b})^{2} - 2\widehat{H}_{\alpha;t}\widehat{H}^{*}_{\alpha;b}\right), \tag{7}$$

where we have used the fact that the $\hat{H}_{\alpha}$ are Hermitian. This super-Hamiltonian object is the key focus of this work and in the subsequent sections we will study several examples of this superoperator in various bond and commutant algebras.

Symmetries, which are the operators in the commutant $\mathcal{C}$ and satisfy $\widehat{\mathcal{L}}_{\alpha}|\cdot) = 0$, hence are the *frustration-free ground states* of the local superoperator $\widehat{\mathcal{P}}$, since they are ground states of each term $\widehat{\mathcal{P}}_{\alpha}$ of $\widehat{\mathcal{P}}$. The dimension of the commutant, $\dim(\mathcal{C})$, is given by the number of independent ground states of $\widehat{\mathcal{P}}$.

Finally, we note that this super-Hamiltonian $\widehat{\mathcal{P}}$ also has an interpretation as the dissipator of a Lindbladian if we treat the $\widehat{H}_{\alpha}$ as jump operators of a Lindblad master equation, since the action of $\widehat{\mathcal{P}}$ on an operator $|\widehat{O})$ reads

$$\widehat{\mathcal{P}}|\widehat{O}) \iff -\sum_{\alpha}\left(2\widehat{H}_{\alpha}\widehat{O}\widehat{H}_{\alpha} - \{\widehat{H}^{2}_{\alpha}, \widehat{O}\}\right), \tag{8}$$

which corresponds to the dissipative part of the Lindbladian. Indeed, similar mappings are also commonly used in the literature on Lindblad systems [69,70]. In Appendix C, we further discuss formal symmetry properties of the super-Hamiltonians viewed as ladder-bilayer systems; in particular, we encounter ones known as strong symmetries in the Lindblad context and consider how the ground states of the super-Hamiltonians relate to these.

III. EXAMPLES OF SUPER-HAMILTONIANS

We now study super-Hamiltonians $\widehat{\mathcal{P}}$ in the context of several conventional and unconventional symmetries studied in earlier literature [25,31,32] and show that the respective commutant algebras can be understood as its ground states. Note that the super-Hamiltonian corresponding to a given symmetry is not unique and it depends on the choice of generators of the corresponding bond algebra. The ground states of all such super-Hamiltonians are the same by definition but the excited states can differ. We will restrict to a simple choice of local generators of the bond algebras, which lead to simple local super-Hamiltonians, and we expect the qualitative features of low-energy excited states to be the same for any other choice of local generators of the bond algebras. We also restrict examples to 1D systems and higher-dimensional examples proceed in similar ways.

A. Global symmetry

We start with the case of global symmetries, studied in Ref. [32], and we separately show examples of discrete and continuous symmetries. Note that we only illustrate the Hamiltonians for 1D systems but that the results generalize to higher dimensions as well.

1. $\mathbb{Z}_2$ symmetry

As an example of a discrete symmetry, we focus on $\mathbb{Z}_2$ symmetry in spin-1/2 systems, where we know that the pair of bond and commutant algebras is given by [32]

$$\mathcal{A}_{\mathbb{Z}_2} = \langle\langle \{X_j X_{j+1}\}, \{Z_j\} \rangle\rangle,$$
$$\mathcal{C}_{\mathbb{Z}_2} = \left\langle\!\!\left\langle \prod_j Z_j \right\rangle\!\!\right\rangle = \text{span}\left\{ \mathbb{1}, \prod_{j=1}^{L} Z_j \right\}, \tag{9}$$

where the X_j and Z_j are Pauli matrices on site j. We can use the generators of $\mathcal{A}_{\mathbb{Z}_2}$ to construct the corresponding superoperator $\widehat{\mathcal{P}}_{\mathbb{Z}_2}$, which when interpreted as a Hamiltonian on a ladder reads, following Eq. (7),

$$\widehat{\mathcal{P}}_{\mathbb{Z}_2} = 2\sum_j [1 - Z_{j;t}Z_{j;b}] + 2\sum_j [1 - X_{j,t}X_{j;b}X_{j+1;t}X_{j+1;b}], \tag{10}$$

where $X_{j;\ell}$ and $Z_{j,\ell}$ are now the Pauli matrices on the ℓth leg of the jth site of the ladder system representing the doubled Hilbert space. Note that all the terms of $\widehat{\mathcal{P}}_{\mathbb{Z}_2}$ commute among themselves; hence its spectrum is completely solvable.

We can directly solve for the ground states of $\widehat{\mathcal{P}}_{\mathbb{Z}_2}$ by starting with configurations that "satisfy," i.e., minimize, the energy under each of the terms individually. First, we note that "rung term" $\{1 - Z_{j;t}Z_{j;b}\}$ in $\widehat{\mathcal{P}}_{\mathbb{Z}_2}$ is satisfied when both the spins on the rung at site j are aligned; hence we can work in the space of composite spins on the rungs, defined as

$$|\tilde{\uparrow}\rangle := \left| \begin{matrix}\uparrow \\ \uparrow\end{matrix} \right\rangle, \quad |\tilde{\downarrow}\rangle := \left| \begin{matrix}\downarrow \\ \downarrow\end{matrix} \right\rangle, \tag{11}$$

where the top and bottom spins in the ket are states of the top and bottom legs, respectively. Within the 2^L-dimensional space spanned by all product configurations of the "composite spins" $|\tilde{\uparrow}\rangle$ and $|\tilde{\downarrow}\rangle$, the action of $\widehat{\mathcal{P}}_{\mathbb{Z}_2}$ reads

$$\widehat{\mathcal{P}}_{\mathbb{Z}_2|\text{comp}} = 2\sum_j [1 - \widetilde{X}_j \widetilde{X}_{j+1}], \tag{12}$$

where $\widetilde{X}_j$ is the composite-spin Pauli matrix on site j; this is because the action of $X_{j;t}X_{j;b}$ flips the composite spins and hence can be mapped to $\tilde{X}_j$. Equation (12) is simply the classical Ising ferromagnet and its two ground states, $|G_{\rightarrow}\rangle$ and $|G_{\leftarrow}\rangle$, are given by

$$|G_{\rightarrow}\rangle = |\tilde{\rightrightarrows}\,\tilde{\rightrightarrows}\cdots\tilde{\rightrightarrows}\,\tilde{\rightrightarrows}\rangle, \quad |G_{\leftarrow}\rangle = |\tilde{\leftleftarrows}\,\tilde{\leftleftarrows}\cdots\tilde{\leftleftarrows}\,\tilde{\leftleftarrows}\rangle,$$
$$|\tilde{\rightrightarrows}\rangle := \frac{|\tilde{\uparrow}\rangle + |\tilde{\downarrow}\rangle}{\sqrt{2}}, \quad |\tilde{\leftleftarrows}\rangle := \frac{|\tilde{\uparrow}\rangle - |\tilde{\downarrow}\rangle}{\sqrt{2}}. \tag{13}$$

In the operator language, the composite spins on rung j map to physical spin projectors on site j as $|\tilde{\uparrow}\rangle_j = ||\uparrow\rangle\langle\uparrow|)_j$ and $|\tilde{\downarrow}\rangle_j = ||\downarrow\rangle\langle\downarrow|)_j$. Hence the composite spins of Eq. (13) map as

$$|\tilde{\rightrightarrows}\rangle = \frac{1}{\sqrt{2}}|\mathbb{1})_j, \quad |\tilde{\leftleftarrows}\rangle = \frac{1}{\sqrt{2}}|Z)_j \tag{14}$$

and the normalized ground states are

$$|G_{\rightarrow}\rangle = \frac{1}{2^{\frac{L}{2}}}|\mathbb{1}), \quad |G_{\leftarrow}\rangle = \frac{1}{2^{\frac{L}{2}}}|\prod_{j=1}^{L} Z_j), \tag{15}$$

which are precisely the two linearly independent operators that span the commutant algebra for the $\mathbb{Z}_2$ symmetry, shown in Eq. (9). Hence the ground-state space of the superoperator $\widehat{\mathcal{P}}_{\mathbb{Z}_2}$ precisely maps to the commutant algebra $\mathcal{C}_{\mathbb{Z}_2}$. In Appendix C 1, we further discuss the fate of the formal inherited symmetries of the super-Hamiltonian—here, independent $\mathbb{Z}_2$ symmetries associated with each leg—in the corresponding quantum "phase" that contains these ground states and show that they can be understood in terms of particular spontaneous symmetry breaking.

As an extension, it is easy to see that all Pauli-string bond algebras, i.e., those generated by Pauli strings, have super-Hamiltonians that are composed of commuting projectors. This is because the Pauli strings themselves square to 1, making the first two terms in Eq. (7) constants, while the Pauli strings appear in pairs in the last term, which commute with any other pair of Pauli strings. Hence the ground states of these super-Hamiltonians can be solved to reproduce the respective commutants (which in such cases are also generated by physical Pauli strings [32]). Since the commutants in such cases are also generated by Pauli strings [32], we expect them to correspond to discrete symmetries.

2. $U(1)$ symmetry

We next illustrate a continuous symmetry, turning to the commutant of the spin-1/2 $U(1)$ bond algebra, given in 1D by [25,32]

$$\mathcal{A}_{U(1)} = \langle\langle \{X_j X_{j+1} + Y_j Y_{j+1}\}, \{Z_j\}\rangle\rangle,$$
$$\mathcal{C}_{U(1)} = \left\langle\!\!\left\langle \sum_{j=1}^{L} Z_j \right\rangle\!\!\right\rangle = \text{span}\left\{\sum_{j_1<\cdots<j_m} Z_{j_1}\cdots Z_{j_m}\right\}. \quad (16)$$

We can then use the generators of $\mathcal{A}_{U(1)}$ to construct the superoperator $\widehat{\mathcal{P}}_{U(1)}$ using Eq. (7). When expressed on the two-leg ladder, after simplification it reads

$$\widehat{\mathcal{P}}_{U(1)} = 2\sum_j [1 - Z_{j;t}Z_{j;b}] + 2\sum_j \left[2 - \sum_{\ell\in\{t,b\}} Z_{j,\ell}Z_{j+1,\ell} - (X_{j;t}X_{j+1;t} + Y_{j;t}Y_{j+1;t})(X_{j;b}X_{j+1;b} + Y_{j;b}Y_{j+1;b})\right]. \quad (17)$$

Although $\widehat{\mathcal{P}}_{U(1)}$ is not composed of commuting terms, we can solve for their exact ground states. Note that similar to the $\mathbb{Z}_2$ case, each of the first-rung terms proportional to $1 - Z_{j;t}Z_{j;b}$ commutes with all other terms in $\widehat{\mathcal{P}}_{U(1)}$ and is satisfied when spins on both legs at site j are aligned, which then justifies working in terms of the composite spins of Eq. (11). It is also easy to see that $\widehat{\mathcal{P}}_{U(1)}$ leaves the subspace spanned by the 2^L product configurations of the composite spins invariant and, within that subspace, $\widehat{\mathcal{P}}_{U(1)}$ acts as

$$\widehat{\mathcal{P}}_{U(1)|\text{comp}} = 8\sum_j (|\tilde{\uparrow}\tilde{\downarrow}\rangle - |\tilde{\downarrow}\tilde{\uparrow}\rangle)(\langle\tilde{\uparrow}\tilde{\downarrow}| - \langle\tilde{\downarrow}\tilde{\uparrow}|)_{[j,j+1]}$$
$$= 4\sum_j [1 - (\widetilde{X}_j\widetilde{X}_{j+1} + \widetilde{Y}_j\widetilde{Y}_{j+1} + \widetilde{Z}_j\widetilde{Z}_{j+1})], \quad (18)$$

where $\widetilde{X}_j$, $\widetilde{Y}_j$, and $\widetilde{Z}_j$ are the composite-spin Pauli operators on site j, defined in the obvious way. Up to an overall factor, this is simply the ferromagnetic Heisenberg model reviewed in Appendix A, here in terms of the composite spins. Its ground-state space is hence the $(L+1)$-dimensional ferromagnetic multiplet of the composite spins; these are obtained by replacing the regular spins of the usual ferromagnet shown in Eq. (A5) by the composite spins. Using the correspondence between states on the ladder and operators of Eq. (14), the states of the composite-spin ferromagnetic multiplet translate into operators of the form

$$|Q_m^z) \sim \sum_{j_1<\cdots<j_m} |Z_{j_1}Z_{j_2}\cdots Z_{j_m}), \quad (19)$$

where we have ignored an overall constant. These are precisely the $L+1$ linearly independent operators forming a basis in the commutant algebra $\mathcal{C}_{U(1)}$ corresponding to the $U(1)$ symmetry [25,32], shown in Eq. (16).

While the above operator mapping is more evident in the $\hat{x}$ basis of the composite spins, the same multiplet can be described in terms of the $\hat{z}$ basis of composite spins, analogous to Eq. (A5). Since the composite-spin states $|\tilde{\uparrow}\rangle_j$ and $|\tilde{\downarrow}\rangle_j$ map onto physical spin projectors $|\uparrow\rangle\langle\uparrow|_j$ and $|\downarrow\rangle\langle\downarrow|_j$ in the operator language, this $\hat{z}$ basis for the ground-state space of $\widehat{\mathcal{P}}_{U(1)}$ corresponds to projectors onto the $L+1$ spin sectors labeled by different values of the physical spin S^z_{tot}. Indeed, for Abelian symmetries, the projectors onto symmetry sectors form an orthogonal basis for the commutant algebra [25].

Finally, in Appendix C 2, we consider the formal inherited symmetries of the super-Hamiltonian—here, independent $U(1)$ symmetries associated with each leg—and show that the ground-state manifold can be understood using particular spontaneous symmetry breaking.

3. SU(2) symmetry

As an example of a non-Abelian symmetry, we now illustrate the commutant of the spin-1/2 $SU(2)$ bond algebra, given by

$$\mathcal{A}_{SU(2)} = \langle\langle\{\vec{S}_j\cdot\vec{S}_{j+1}\}\rangle\rangle = \langle\langle\{P^{(2)}_{[j,j+1]}\}\rangle\rangle,$$
$$P^{(2)}_{[j,j+1]} := 2\vec{S}_j\cdot\vec{S}_{j+1} + \frac{1}{2},$$
$$\mathcal{C}_{SU(2)} = \langle\langle S^x_{\text{tot}}, S^y_{\text{tot}}, S^z_{\text{tot}}\rangle\rangle, \quad (20)$$

where $P^{(2)}_{[j,j+1]}$ here is the spin-1/2 permutation operator between sites j and $j+1$, i.e.,

$$P^{(2)}_{[j,j+1]}|\sigma\sigma'\rangle_{[j,j+1]} = |\sigma'\sigma\rangle_{[j,j+1]}, \quad (21)$$

and the $\{S^\alpha_{\text{tot}}\}$ are the total spin operators. (This bond algebra contains the Heisenberg Hamiltonian reviewed in Appendix A.) As we will see, the expression in terms of the permutation operators is more convenient for solving the corresponding super-Hamiltonian and in this form the treatment immediately generalizes to bond algebras for $SU(q)$ symmetry for any $q \geq 2$, which are generated by permutation operators for q-level systems, which we denote by $P^{(q)}_{[j,j+1]}$.

In the following, we denote the permutation operators without superscripts to mean that they hold for any q and that $q=2$ corresponds to the $SU(2)$ case. Using $(P_{[j,j+1]})^2 = 1$, the super-Hamiltonian associated with $\{P_{[j,j+1]}\}$ in the ladder representation has the form

$$\widehat{\mathcal{P}}_{SU(q)} = 2\sum_j (1 - P_{[j,j+1];t}P_{[j,j+1];b}) = 2\sum_j (1 - P^{\text{rung}}_{[j,j+1]}), \tag{22}$$

where $P^{\text{rung}}_{[j,j+1]}$ is the permutation operator for the rungs j and $j+1$, i.e., $P^{\text{rung}}_{[j,j+1]} \left| \begin{matrix} \sigma\ \sigma' \\ \tau\ \tau' \end{matrix} \right\rangle_{[j,j+1]} = \left| \begin{matrix} \sigma'\ \sigma \\ \tau'\ \tau \end{matrix} \right\rangle_{[j,j+1]}$. Note that this can also be viewed as the permutation operator $P^{(q^2)}_{[j,j+1]}$ acting on the q^2-level systems associated with each of the rungs j and $j+1$. The permutation operator $P^{(q^2)}_{[j,j+1]}$ possesses an $SU(q^2)$ symmetry and the super-Hamiltonian $\widehat{\mathcal{P}}_{SU(q)}$ is then equivalent to a chain of q^2-level systems with nearest-neighbor ferromagnetic $SU(q^2)$-invariant interactions.

The ground states of $\widehat{\mathcal{P}}_{SU(q)}$ are the ferromagnetic states of this chain of q^2-level systems, which can be tabulated as follows. Given a fixed number of on-site states of each type $N_1, N_2, \ldots, N_{q^2}$ with constraint $N_1 + N_2 + \cdots + N_{q^2} = L$, we define $|\Psi_{N_1,N_2,\ldots,N_{q^2}}\rangle$ to be an equal-weight superposition of all possible configurations with precisely the given number of on-site states of each type. This is analogous to the spin-1/2 ferromagnetic multiplet that appears as the ground state of the super-Hamiltonian in the $U(1)$ case. These states are invariant under permutations $P^{(q^2)}_{[j,j+1]}$ and it is easy to prove that they completely span the ground-state manifold of $\widehat{\mathcal{P}}_{SU(q)}$. Their count is

$$\dim(\mathcal{C}_{SU(q)}) = \sum_{N_1,N_2,\ldots,N_{q^2}=0}^{L} \delta_{N_1+N_2+\ldots N_{q^2}=L} = \binom{L+q^2-1}{q^2-1}, \tag{23}$$

which, for the $q=2$ case, precisely matches the known description of the commutant for the $SU(2)$ symmetry [25,32].

B. Hilbert-space fragmentation

We now turn to examples of HSF, where the dimension of the commutant scales exponentially with the system size [25,71,72], which leads to exponentially many disconnected Krylov subspaces [18,19,21–23], which are analogues of quantum number sectors for conventional symmetries.

We start with an example of classical fragmentation, the $t-J_z$ model [73–75], which is a model of two species of spins, ↑ and ↓, which are allowed to hop but are not allowed to cross. Schematically, there are three possible states at any given site—spin ↑, spin ↓, and the vacant site 0—and the allowed "moves" can be denoted as $\uparrow 0 \leftrightarrow 0 \uparrow$ and $\downarrow 0 \leftrightarrow 0 \downarrow$. These moves satisfy a conservation of the full pattern of spins (i.e., ↑ and ↓, ignoring the 0) in one dimension, which results in a fragmented Hilbert space with exponentially many Krylov subspaces corresponding to exponentially many allowed patterns [24,25]. In Ref. [25], we have shown that these exponentially many subspaces are attributed to exponentially many conserved quantities in the commutant algebra.

Specifically, the bond algebra corresponding to the $t-J_z$ model, when viewed as a hard-core bosonic model for simplicity, is given by [25]

$$\begin{aligned} \mathcal{A}_{t-J_z} &= \langle\langle \{\widehat{T}^{\uparrow}_{[j,j+1]}\}, \{\widehat{T}^{\downarrow}_{[j,j+1]}\}, \{S^z_j\}\rangle\rangle, \\ \widehat{T}^{\sigma}_{[j,j+1]} &:= (|\sigma\ 0\rangle\langle 0\ \sigma| + \text{h.c.})_{[j,j+1]}, \\ S^z_j &:= (|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|)_j, \end{aligned} \tag{24}$$

where $\sigma \in \{\uparrow,\downarrow\}$, and the $\{T^{\sigma}_{[j,j+1]}\}$ are the nearest-neighbor hopping terms for the spin of type σ. The corresponding commutant, derived in Ref. [25], reads

$$\mathcal{C}_{t-J_z} = \text{span}\{N^{\sigma_1\sigma_2\cdots\sigma_k}, k = 0,1,\ldots,L\},$$
$$N^{\sigma_1\sigma_2\cdots\sigma_k} = \sum_{j_1<j_2<\cdots<j_k} N^{\sigma_1}_{j_1} N^{\sigma_2}_{j_2} \cdots N^{\sigma_k}_{j_k}, \quad \sigma_j \in \{\uparrow,\downarrow\}. \tag{25}$$

Note that most of the conserved quantities in the commutant $\mathcal{C}_{t-J_z}$ are functionally independent from the two obvious $U(1)$ symmetries, $N^\sigma = \sum_j N^\sigma_j$, $\sigma \in \{\uparrow,\downarrow\}$, which are the separate particle-number conservations of ↑ spins and ↓ spins. The commutant $\mathcal{C}_{t-J_z}$ can be generated by a distinct set of nonlocal operators, referred to as statistically localized integrals of motion (SLIOMs) [24,25]; however, for our purposes, working with the linearly independent basis for $\mathcal{C}_{t-J_z}$ is more convenient.

We can construct the super-Hamiltonian using the generators of $\mathcal{A}_{t-J_z}$ to understand the operators in $\mathcal{C}_{t-J_z}$ as its ground states. We first note that the super-Hamiltonian is of the form

$$\widehat{\mathcal{P}}_{t-J_z} = \sum_j (S^z_{j;t} - S^z_{j;b})^2 + \sum_j F(\{\widehat{T}^{\sigma}_{[j,j+1];\ell}\}), \tag{26}$$

where $F(\{\widehat{T}^{\sigma}_{[j,j+1];\ell}\})$ is the positive-semidefinite part of the super-Hamiltonian that comes from the generators $\{\widehat{T}^{\sigma}_{[j,j+1]}\}$ of $\mathcal{A}_{t-J_z}$. We now observe that the first sum in

Eq. (26) enforces that the ground states $|\Psi\rangle$ of $\widehat{\mathcal{P}}_{t-J_z}$ satisfy $S^z_{j;t}|\Psi\rangle = S^z_{j;b}|\Psi\rangle$; hence the ground states lie in the subspace spanned by product states of composite spins, defined here as

$$\left|\tilde{\uparrow}\right\rangle := \left|\begin{array}{c}\uparrow\\ \uparrow\end{array}\right\rangle,\quad \left|\tilde{0}\right\rangle := \left|\begin{array}{c}0\\ 0\end{array}\right\rangle,\quad \left|\tilde{\downarrow}\right\rangle := \left|\begin{array}{c}\downarrow\\ \downarrow\end{array}\right\rangle. \tag{27}$$

It is easy to check that these composite spins are left invariant under the action of $\widehat{\mathcal{P}}_{t-J_z}$ and that the effective Hamiltonian in this subspace reads

$$\widehat{\mathcal{P}}_{t-J_z|\mathrm{comp}} = 2\sum_{j,\sigma\in\{\uparrow,\downarrow\}}\left(|\tilde{\sigma}\ \tilde{0}\rangle - |\tilde{0}\ \tilde{\sigma}\rangle\right) \times \left(\langle\tilde{\sigma}\ \tilde{0}| - \langle\tilde{0}\ \tilde{\sigma}|\right)_{[j,j+1]}, \tag{28}$$

where we have used the expression for $\widehat{T}^{\sigma}_{[j,j+1]}$ in Eq. (24). This resembles the form of the ferromagnetic Heisenberg Hamiltonian in Eq. (18) and, in fact, as we discuss in Sec. V B 2, there is a mapping between eigenstates of $\widehat{\mathcal{P}}_{t-J_z|\mathrm{comp}}$ and those of the Heisenberg model discussed in Appendix A. For the purposes of the ground states, it is easy to show that they must have equal amplitudes on *all* product states that are "connected" by the nearest-neighbor "hops" of the composite spin $\tilde{\sigma}$. An orthogonal basis for the ground states is formed by states that are equal-weight superpositions of all spin configurations with a fixed pattern of composite spins $\tilde{\uparrow}$ and $\tilde{\downarrow}$. For example, the ground states of $\widehat{\mathcal{P}}_{t-J_z}$ with ordinary boundary conditions (OBCs) read

$$|G^{\sigma_1\cdots\sigma_k}\rangle = \sum_{j_1<j_2<\cdots<j_k} |\tilde{\sigma}_1(j_1)\ \tilde{\sigma}_2(j_2)\ \cdots\ \tilde{\sigma}_k(j_k)\rangle, \tag{29}$$

where the notation $\tilde{\sigma}_l(j_l)$ indicates that the composite spin $\tilde{\sigma}_l$ is at site j_l and the remaining sites are implicitly assumed to be $\tilde{0}$. We note that this form of the ground state is a consequence of the fact that $\widehat{\mathcal{P}}_{t-J_z}$ of Eq. (28) is of the Rokhsar-Kivelson (RK) form [also referred to as stoquastic or stochastic-matrix-form (SMF) decomposable] [76,77]. Such superoperators often appear in systems with "classical" symmetries, i.e., where all the symmetry operators are diagonal in the product-state basis, and we discuss these connections in Appendix B. Note that the symmetry of the action of the super-Hamiltonian $\widehat{\mathcal{P}}_{t-J_z|\mathrm{comp}}$ on $\tilde{\uparrow}$ and $\tilde{\downarrow}$ in Eq. (28) is in fact a composite-spin $SU(2)$ symmetry. As a consequence, $\widehat{\mathcal{P}}_{t-J_z|\mathrm{comp}}$ has the same form when written in terms of composite spins $\tilde{\rightrightarrows}$ and $\tilde{\leftleftarrows}$ in the $\hat{x}$ basis defined in Eq. (13), i.e., when $\tilde{\sigma}\in\{\tilde{\rightrightarrows},\tilde{\leftleftarrows}\}$ in Eq. (28).

We can map the ground states of $\widehat{\mathcal{P}}_{t-J_z}$ to operators by noting that a composite-spin configuration $|\tilde{s}\rangle_j$, $s\in\{\uparrow,0,\downarrow\}$, maps to on-site projectors $||s\rangle\langle s|)_j$. The ground states of $\widehat{\mathcal{P}}_{t-J_z}$ then correspond to the projectors onto the Krylov subspaces of the $t-J_z$ model, which are spanned by all configurations with a fixed pattern of spins. Thus, the ground-state space of $\widehat{\mathcal{P}}_{t-J_z}$ is equivalent to the subspace spanned by these projectors, which, one can argue, is the same as the subspace spanned by the (nonorthogonal) operators $N^{\sigma_1\sigma_2\cdots\sigma_k}$; hence it precisely reproduces the commutant algebra $\mathcal{C}_{t-J_z}$ of Eq. (25).

C. Quantum many-body scars

We finally analyze the commutants of QMBSs [17–20], using the super-Hamiltonian picture. In Ref. [31], we have built on several earlier works [26,78,79] to show that QMBSs can be defined as singlets of locally generated algebras. By this we mean that there are local operators such that their common eigenstates are the QMBSs. A simple example is when the QMBS eigenstates are simultaneous eigenstates of a set of strictly local operators; without loss of generality, projectors $\{R_{[j]}\}$ acting of a few sites neighboring j (with the range bounded by some fixed number) that annihilate the QMBSs, as proposed by Shiraishi and Mori in Ref. [26]. We denote the common kernel of $\{R_{[j]}\}$ as

$$\mathrm{span}\{|\Phi_n\rangle\} := \{|\psi\rangle : R_{[j]}|\psi\rangle = 0 \quad \forall\, j\}, \tag{30}$$

where $\{|\Phi_n\rangle\}$ is an orthonormal basis for the kernel, and these are the QMBS eigenstates. Several known examples of QMBSs, including embedded matrix product states [26], towers of states in the spin-1 XY [80], and the Hubbard model [78,79,81,82], as well as those in the Affleck-Kennedy-Lieb-Tasaki (AKLT) models [27,29,83,84], can be understood to be of this form.

For simplicity, we restrict ourselves to spin-1/2 systems. As discussed in Ref. [31], we expect the bond and commutant algebra

$$\begin{aligned}\mathcal{A}_{\mathrm{scar}} &= \langle\langle\{R_{[j]}\sigma^{\alpha}_k,\ k\in\Lambda_j,\ \alpha=0,x,y,z\}\rangle\rangle,\\ \mathcal{C}_{\mathrm{scar}} &= \langle\langle\{|\Phi_n\rangle\langle\Phi_m|\}\rangle\rangle,\end{aligned} \tag{31}$$

where the $\{\sigma^{\alpha}_k\}$ are the on-site Pauli matrices with $\sigma^0_k = \mathbb{1}_k$, $R_{[j]}$ is a projector acting on few sites near j, and the index k runs over a set of sites Λ_j that does not have overlap with the support of $R_{[j]}$ but is in the vicinity [85]. Note that there are many different choices of generators for $\mathcal{A}_{\mathrm{scar}}$, which then determine the super-Hamiltonian, and we have chosen one set that is convenient for calculations. The algebra $\mathcal{A}_{\mathrm{scar}}$ is claimed to be the *exhaustive* algebra of operators with $\{|\Phi_n\rangle\}$ as degenerate eigenstates with eigenvalue 0; i.e., any such operator can be expressed as sums of products of generators of $\mathcal{A}_{\mathrm{scar}}$. We will verify this claim using the super-Hamiltonian formalism for specific examples below. In examples of towers of QMBSs, a "lifting" term [81,83] can be added to the generators of these algebras to obtain the exhaustive algebra of operators $\{|\Phi_n\rangle\}$ as nondegenerate eigenstates (for more details, see Ref. [31]).

Following Eq. (7), the expression for the super-Hamiltonian $\widehat{\mathcal{P}}_{\text{scar}}$ reads

$$\widehat{\mathcal{P}}_{\text{scar}} = \sum_{j,k,\alpha}[R_{[j];t} + R_{[j];b} - 2R_{[j];t}R_{[j];b}\,\eta^{\alpha}\sigma^{\alpha}_{k;t}\sigma^{\alpha}_{k;b}],$$
$$= 4\sum_{j,k}(R_{[j];t} - R_{[j];b})^2 + 8\sum_{j,k}R_{[j];t}R_{[j];b}[1 - |\iota\rangle\langle\iota|]_k, \tag{32}$$

where for simplicity we have assumed that $R^T_{[j]} = R_{[j]}$ (equivalently, $R^*_{[j]} = R_{[j]}$) in the computational basis, and we have defined $\eta^0 = \eta^x = \eta^z = 1$ and $\eta^y = -1$, and the state

$$|\iota\rangle_k := \frac{1}{\sqrt{2}}\left(\left|\begin{array}{c}\uparrow\\ \uparrow\end{array}\right\rangle_k + \left|\begin{array}{c}\downarrow\\ \downarrow\end{array}\right\rangle_k\right). \tag{33}$$

Note that $|\iota\rangle_k$ is identical to the composite spin $|\rightleftarrows\rangle_k$ of Eq. (13) and hence in the operator language it maps onto $\frac{1}{\sqrt{2}}|\mathbb{1})_k$.

Since all the individual terms in Eq. (32) are positive semidefinite, any ground state $|\Psi\rangle$ of $\widehat{\mathcal{P}}_{\text{scar}}$ should satisfy

$$(R_{[j];t} - R_{[j];b})^2\,|\Psi\rangle = 0 \implies R_{[j];t}\,|\Psi\rangle = R_{[j];b}|\Psi\rangle, \tag{34}$$

$$R_{[j];t}R_{[j];b}[1 - |\iota\rangle\langle\iota|]_k\,|\Psi\rangle = 0 \implies R_{[j],\ell}[1 - |\iota\rangle\langle\iota|]_k\,|\Psi\rangle = 0, \tag{35}$$

where $\ell \in \{t, b\}$, $k \in \Lambda_j$, and we have used Eq. (34) in the second step of Eq. (35). Using these equations, it is easy to see the presence of the following two types of ground states:

$$|G_{m,n}\rangle := |\Phi_m\rangle_t \otimes |\Phi_n\rangle_b\,, \quad |G_{\mathbb{1}}\rangle = \bigotimes_k |\iota\rangle_k. \tag{36}$$

In the operator language, $|G_{m,n}\rangle$ maps onto $||\Phi_m\rangle\langle\Phi_n|)$ (using $|\Phi_n\rangle$ with real-valued amplitudes in the computational basis corresponding to the earlier assumption of real-valuedness of $R_{[j]}$ in this basis) and $|G_{\mathbb{1}}\rangle$ is proportional to the global identity operator $|\mathbb{1})$. These operators are all in the commutant $\mathcal{C}_{\text{scar}}$ shown in Eq. (31).

1. Isolated QMBSs

Proving that these are the *only* ground states of $\widehat{\mathcal{P}}_{\text{scar}}$ is more challenging and we do that in specific cases in Appendix E. There, we consider an example of an *isolated* QMBS [19], where $R_{[j]} := R_j = (1 - \sigma^z_j)/2 = |\downarrow\rangle\langle\downarrow|_j$; hence, following Eq. (30), the only scar state is

$$|\Phi\rangle := |\uparrow\uparrow\,\ldots\,\uparrow\rangle\,. \tag{37}$$

We start with a set of bond generators of the form of Eq. (31) with $\Lambda_j = \{j-1, j+1\}$ and follow Eq. (32) to construct the super-Hamiltonian $\widehat{\mathcal{P}}_{\text{iso}}$ corresponding to this case [see the explicit expression in Eq. (E2)]. As we show in Appendix E 1, in the same composite-spin subsector of interest, we obtain the spin model

$$\widehat{\mathcal{P}}_{\text{iso}|\text{comp}} = 2\sum_j[(1 - \widetilde{Z}_j)(1 - \widetilde{X}_{j+1}) + (1 - \widetilde{X}_j)(1 - \widetilde{Z}_{j+1})], \tag{38}$$

where the $\{\widetilde{X}_j, \widetilde{Z}_j\}$ are the composite-spin Pauli operators also used in Eq. (18). As we show in Appendix E 1, this Hamiltonian can be mapped to a frustration-free model that lies on the so-called Peschel-Emery line in the vicinity of a transverse-field Ising model [86–88]. From these studies, this is known to possess only two ground states both in periodic boundary conditions (PBCs) and OBCs, which in the composite-spin language read

$$|G_{\mathbb{1}}\rangle = |\rightleftarrows\,\rightleftarrows\,\cdots\,\rightleftarrows\,\rightleftarrows\rangle = \frac{1}{2^{\frac{L}{2}}}|\mathbb{1}),$$
$$|G_{\text{QMBS}}\rangle = |\tilde{\uparrow}\,\tilde{\uparrow}\,\cdots\,\tilde{\uparrow}\,\tilde{\uparrow}\rangle = ||\Phi\rangle\langle\Phi|). \tag{39}$$

This proves the above claim and hence shows the existence of the following pairs of algebras of the form of Eq. (31) that are centralizers of each other:

$$\mathcal{A}_{\text{iso}} := \langle\langle\{R_j\sigma^{\alpha}_{j+1}, \sigma^{\alpha}_j R_{j+1}\}\rangle\rangle, \quad \mathcal{C}_{\text{iso}} := \langle\langle|\Phi\rangle\langle\Phi|\rangle\rangle. \tag{40}$$

Any operator constructed out of the generators of $\mathcal{A}_{\text{iso}}$ contains the state $|\Phi\rangle$ as an eigenstate, which can be a QMBS if it is in the bulk of the spectrum [31].

Finally, in Appendix C 3, we consider the relation between the ground-state manifold and the inherited formal symmetries of the super-Hamiltonian in this case.

2. Tower of QMBSs

Similarly, as an example of a tower of QMBSs, we consider $R_{[j]} := R_{j,j+1} = \frac{1}{4} - \vec{S}_j \cdot \vec{S}_{j+1} = \frac{1}{2}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)(\langle\uparrow\downarrow| - \langle\downarrow\uparrow|)_{j,j+1}$, where the common kernel contains the entire ferromagnetic tower of states as scars [89]:

$$|\Phi_{n,0}\rangle := \frac{1}{n!}\binom{L}{n}^{-\frac{1}{2}}(S^{z-}_{\text{tot}})^n\,|\uparrow\uparrow\ \ldots\ \uparrow\rangle,\quad 0 \leq n \leq L, \tag{41}$$

where S^{z-}_{tot} is the total spin lowering operator in the $\hat{z}$ direction. We again start with bond generators of the form of Eq. (31) with $\Lambda_j \in \{j-1, j+2\}$ and follow Eq. (32) to construct the super-Hamiltonian $\widehat{\mathcal{P}}_{\text{tower}}$ corresponding to this case [see the explicit expression in Eq. (E13)] and then solve for all of its ground states to show the existence of the following pairs of algebras that are centralizers of each other:

$$\begin{aligned}\mathcal{A}_{\text{tower}} &:= \langle\langle\{R_{j,j+1}\sigma^\alpha_{j+2}, \sigma^\alpha_{j-1}R_{j,j+1}\}\rangle\rangle,\\ \mathcal{C}_{\text{tower}} &= \langle\langle\{|\Phi_{n,0}\rangle\langle\Phi_{m,0}|\}\rangle\rangle.\end{aligned} \tag{42}$$

Any operator constructed out of the generators of $\mathcal{A}_{\text{tower}}$ contains the ferromagnetic multiplet as degenerate QMBSs [31]. These results also generalize to algebras corresponding to the case in which the QMBSs are nondegenerate towers. However, we discuss this case in Appendix H, due to more care being required in the Brownian-circuit setup and analysis.

IV. SUPER-HAMILTONIAN SPECTRUM: APPROXIMATE SYMMETRIES AND SLOW MODES

While the ground states of the superoperator $\widehat{\mathcal{P}}$ correspond to the symmetry operators in the commutant algebra, we now show that the low-energy excitations correspond to operators that are "approximate" symmetries until long times under local dynamics. To make this idea precise, we show that the superoperator $\widehat{\mathcal{P}}$ acts as a dissipator for operators in ensemble-averaged noisy Brownian circuits. Several types of Brownian circuits have been studied in the literature, e.g., in the context of information scrambling [43–45], the SYK model [47–50,52], in quantum generalizations of certain classical processes such as the symmetric simple exclusion process (SSEP) [38,39,41,42,90], and in the context of transport with symmetries [51,62]; and similar connections between superoperators and ensemble-averaged dissipative dynamics of operators have been noted in some of them.

A. Algebra-based Brownian circuits

Given a bond algebra $\mathcal{A} = \langle\langle\{\widehat{H}_\alpha\}\rangle\rangle$, we consider an associated *Brownian circuit* that consists of time evolution with the Hamiltonian $H = \sum_\alpha J^{(t)}_\alpha \widehat{H}_\alpha$ for a short time of Δt. At each time step, the $\{J^{(t)}_\alpha\}$ are chosen to be uncorrelated random variables from a fixed distribution and we are eventually interested in ensemble-averaged quantities such as correlation functions. Operators under this circuit evolve in the adjoint language as

$$\begin{aligned}|\widehat{O}(t+\Delta t)) &= e^{i\sum_\alpha J^{(t)}_\alpha \widehat{\mathcal{L}}_\alpha \Delta t}|\widehat{O}(t))\\ &= |\widehat{O}(t)) + i\Delta t\sum_\alpha J^{(t)}_\alpha\widehat{\mathcal{L}}_\alpha|\widehat{O}(t)) - \frac{(\Delta t)^2}{2}\sum_{\alpha,\beta}J^{(t)}_\alpha J^{(t)}_\beta\widehat{\mathcal{L}}_\alpha\widehat{\mathcal{L}}_\beta|\widehat{O}(t)) + \mathcal{O}((\Delta t)^3),\end{aligned} \tag{43}$$

where $\widehat{\mathcal{L}}_\alpha$ is the Liouvillian corresponding to $\widehat{H}_\alpha$, defined in Eq. (4). If the $\{J^{(t)}_\alpha\}$ at different times t are chosen to be uncorrelated random variables, and we are only interested in ensemble-averaged quantities that are linear in the operator $\widehat{O}(t)$, e.g., correlation functions such as $\text{Tr}(\widehat{A}^\dagger\widehat{O}(t)\rho_0)$ for some fixed operator $\widehat{A}$ and density matrix ρ_0, we can average the operator over the probability distribution of the random variables directly. Using Gaussian distributions for $J^{(t)}_\alpha$,

$$P(\{J^{(t)}_\alpha\}) \sim \exp\left(-\sum_{t,\alpha}\frac{(J^{(t)}_\alpha)^2}{2\sigma^2_\alpha}\right),\quad \sigma^2_\alpha = \frac{2\kappa_\alpha}{\Delta t}, \tag{44}$$

where σ^2_α is the variance of $J^{(t)}_\alpha$, we obtain

$$\overline{|\widehat{O}(t+\Delta t))} = \overline{|\widehat{O}(t))} - \Delta t\sum_\alpha\kappa_\alpha\widehat{\mathcal{L}}^\dagger_\alpha\widehat{\mathcal{L}}_\alpha\overline{|\widehat{O}(t))} + \mathcal{O}((\Delta t)^2), \tag{45}$$

where $\overline{\cdots}$ denotes the average over all the $\{J^{(t)}_\alpha\}$ performed independently at all times t and we have used the properties that $\overline{J^{(t)}_\alpha} = 0$, $\overline{J^{(t)}_\alpha J^{(t')}_\beta} = \sigma^2_\alpha\delta_{\alpha,\beta}\delta_{t,t'}$ and also the Hermiticity of the super-Hamiltonian with respect to the Frobenius scalar product of Eq. (3). Note that in the continuous time limit when $\Delta t \to 0$, the distribution of Eq. (44) is referred to as shot noise [91] and the Brownian circuits can be understood using the language of stochastic processes and Itô calculus [92,93], which we will not discuss here. For our purposes, it is sufficient to note that in the continuous time limit, we obtain

$$\frac{d}{dt}|\overline{\widehat{O}(t)}) = -\sum_{\alpha}\kappa_\alpha\widehat{\mathcal{L}}_\alpha^\dagger\widehat{\mathcal{L}}_\alpha|\overline{\widehat{O}(t)})$$
$$\implies \quad |\overline{\widehat{O}(t)}) = e^{-\kappa\widehat{\mathcal{P}}t}|\widehat{O}(0)), \tag{46}$$

where for simplicity we assume $\kappa_\alpha = \kappa$, we use that $\overline{\widehat{O}(0)} = \widehat{O}(0)$, since it is independent of all the $J_\alpha^{(t)}$, and $\widehat{\mathcal{P}}$ is defined in Eqs. (5) and (7). Ensemble-averaged quantities linear in the time-evolved operator $\widehat{O}(t)$, e.g., two-point correlation functions with a fixed second operator, can then be expressed in terms of the ensemble-averaged operator $|\overline{\widehat{O}(t)})$. We note in passing that ensemble averages of higher-order functionals of the operator $\widehat{O}(t)$, e.g., higher-point correlation functions or Rényi entropies, can also be studied using similar methods; these usually involve studying effective Hamiltonians on more copies or replicas of the original Hilbert space (for discussions of such techniques, see, e.g., Refs. [47,50]).

B. Correlation functions

With this understanding, ensemble-averaged correlation functions can also be studied using the eigenstates and spectrum of $\widehat{\mathcal{P}}$. The two-point dynamical correlation functions of operators $\widehat{A}$ and $\widehat{B}$ at infinite temperature are defined as [94]

$$C_{\widehat{B},\widehat{A}}(t) := \frac{1}{D}\mathrm{Tr}(\widehat{B}(0)^\dagger\widehat{A}(t)) = \frac{1}{D}(\widehat{B}(0)|\widehat{A}(t)), \tag{47}$$

where $D := \mathrm{Tr}(\mathbb{1}) = \dim(\mathcal{H})$. After ensemble averaging, this can be written in terms of eigenstates of $\widehat{\mathcal{P}}$ as

$$\begin{aligned}\overline{C_{\widehat{B},\widehat{A}}(t)} &= \frac{1}{D}(\widehat{B}(0)|\overline{\widehat{A}(t)}) = \frac{1}{D}(\widehat{B}(0)|e^{-\kappa\widehat{\mathcal{P}}t}|\widehat{A}(0))\\ &= \frac{1}{D}\sum_{\mu}(\widehat{B}|\lambda_\mu)(\lambda_\mu|\widehat{A})e^{-\kappa p_\mu t}\\ &= \frac{1}{D}\sum_{E}e^{-\kappa E t}\sum_{\nu_E=1}^{N_E}(\widehat{B}|\lambda_{\nu_E}(E))(\lambda_{\nu_E}(E)|\widehat{A}),\end{aligned} \tag{48}$$

where the $\{|\lambda_\mu)\}$ are the orthonormal eigenstates of $\widehat{\mathcal{P}}$ with eigenvalues $\{p_\mu\}$, with real $p_\mu \geq 0$, since $\widehat{\mathcal{P}}$ is positive semidefinite. In the last step, we have reorganized the sum in terms of energy eigenvalues of $\widehat{\mathcal{P}}$ and their degeneracies, where the $\{|\lambda_{\nu_E}(E))\}$ are eigenstates with eigenvalue E and N_E is the degeneracy at that energy; this form is convenient to work with in examples that we study in Sec. V.

As $t \to \infty$ for a finite system size L, we obtain the equilibrium value of the ensemble-averaged two-point correlation function,

$$\overline{C_{\widehat{B},\widehat{A}}(\infty)} := \lim_{t\to\infty}\overline{C_{\widehat{B},\widehat{A}}(t)} = \frac{1}{D}\sum_{\mu}\delta_{p_\mu,0}(\widehat{B}|\lambda_\mu)(\lambda_\mu|\widehat{A}). \tag{49}$$

The information of the late-time transport associated with the symmetry is stored in the nature of the approach to the infinite-time quantity, which for a finite system is given by $\overline{C_{\widehat{B},\widehat{A}}(t)} - \overline{C_{\widehat{B},\widehat{A}}(\infty)}$. However, for such purposes, we are usually interested in the $L \to \infty$, in which case we usually have $\overline{C_{\widehat{B},\widehat{A}}(\infty)} \to 0$ and it is sufficient to focus on $\overline{C_{\widehat{B},\widehat{A}}(t)}$.

C. Autocorrelation functions and Mazur bounds

With the understanding of correlation functions, we illustrate a novel interpretation for the Mazur bounds of autocorrelation functions, studied extensively in the literature [24,25,53,54,95–98]. The autocorrelation function of an operator $\widehat{A}$ is defined as

$$C_{\widehat{A}}(t) := \frac{1}{D}(\widehat{A}(0)|\widehat{A}(t)), \tag{50}$$

and using Eq. (48) its ensemble-averaged value can be written as

$$\begin{aligned}\overline{C_{\widehat{A}}(t)} &= \frac{1}{D}(\widehat{A}(0)|e^{-\kappa\widehat{\mathcal{P}}t}|\widehat{A}(0)) = \frac{1}{D}\sum_{\mu}|(\lambda_\mu|\widehat{A})|^2e^{-\kappa p_\mu t}\\ &= \sum_{E}e^{-\kappa E t}\underbrace{\frac{1}{D}\sum_{\nu_E=1}^{N_E}|(\lambda_{\nu_E}(E)|\widehat{A})|^2}_{W_{\widehat{A}}(E):=},\end{aligned} \tag{51}$$

where in the second line we have expressed the sum in terms of energies (possibly degenerate) and corresponding eigenstates of $\widehat{\mathcal{P}}$, similar to Eq. (48). Note that $W_{\widehat{A}}(E)$ can also be viewed as the weight of the state $|\widehat{A})$ in the subspace spanned by energy eigenstates $\{|\lambda_{\nu_E}(E))\}$, i.e., the degenerate subspace of eigenstates with eigenvalue E. This shows that only eigenstates $|\lambda_\mu)$ that have nonzero overlap with the operator $|\widehat{A})$ contribute to $\overline{C_{\widehat{A}}(t)}$. At $t = 0$, this is simply the total weight of the initial operator $\widehat{A}(0)$, given by

$$\overline{C_{\widehat{A}}(0)} = \frac{1}{D}(\widehat{A}|\widehat{A}) = \sum_{E}W_{\widehat{A}}(E). \tag{52}$$

For operators of interest, such as a local operator, this starts at a value of $O(1)$; e.g., for a Pauli matrix in a spin-$\frac{1}{2}$ system, we have $(Z_j|Z_j)/D = 1$ in our definition of the operator scalar product. As $t \to \infty$ for a finite system size L, its equilibrium value is

$$\overline{C_{\widehat{A}}(\infty)} := \lim_{t\to\infty} \overline{C_{\widehat{A}}(t)} = \frac{1}{D}\sum_{\mu} \delta_{p_\mu,0} |(\lambda_\mu|\widehat{A})|^2. \quad (53)$$

Noting that $\{|\lambda_\mu)\}_{p_\mu=0}$, i.e., the ground states of $\widehat{\mathcal{P}}$ among the above eigenstates, is an orthonormal basis for the commutant algebra $\mathcal{C}$, the right-hand side of Eq. (53) is precisely the Mazur bound [25,53,54]. Hence the Mazur bound can also be interpreted as the saturation value of the ensemble-averaged autocorrelation function in Brownian circuits, in addition to the conventional interpretation as a lower bound for the time-averaged autocorrelation function for a static Hamiltonian. Note that there is an extra factor of $(1/D)$ in Eq. (53) due to the different definition of operator overlap in Eq. (3) from that commonly used in the related literature.

The nature of decay of the autocorrelation to the Mazur bound of Eq. (53) also reveals information about the slowest operators or hydrodynamic modes in the system. The deviation of the autocorrelation function from the Mazur bound for a finite system is $\overline{C_{\widehat{A}}(t)} - \overline{C_{\widehat{A}}(\infty)}$. Unlike the Mazur bound, which is usually computed for finite L, in the decay of autocorrelations we are usually interested in the limit $L \to \infty$ and finite but long times t. Since the Mazur bound $\overline{C_{\widehat{A}}(\infty)}$ vanishes in the $L \to \infty$ limit for all the examples that we study, we can still restrict our study to $\overline{C_{\widehat{A}}(t)}$. In this limit, it is clear that the behavior is dominated by the nature of the low-energy excitations of $\widehat{\mathcal{P}}$. However, the precise behavior also depends on their degeneracies as well as the weights of the operator of interest on these eigenstates.

If $\widehat{\mathcal{P}}$ is gapped in the thermodynamic limit, with gap $E_{\min} := \min_\mu \; E_\mu > 0$, using Eqs. (52) and (51), and assuming $\overline{C_{\widehat{A}}(\infty)} = 0$ in the thermodynamic limit, we obtain that $\overline{C_{\widehat{A}}(t)} \leq \overline{C_{\widehat{A}}(0)} \exp(-\kappa E_{\min} t)$; hence it decays exponentially fast, with a rate proportional to the inverse gap. When $\widehat{\mathcal{P}}$ is gapless, i.e., when $E_{\min} \to 0$ as $L \to \infty$, we need to be more careful in deriving the form of the decay. Since $\widehat{\mathcal{P}}$ is a local superoperator, the low-energy excited states are usually quasiparticles such as spin waves with dispersion relations of the form $E(k) \sim k^\gamma$. If the full weight of the operator $\widehat{A}$, or at least a majority of it lies within this quasiparticle band of states, we can heuristically write Eq. (51) as $\overline{C_{\widehat{A}}(t)} \sim \int \mathrm{d}k \, e^{-\kappa E(k) t} \sim t^{-(1/\gamma)}$. As we will see with concrete examples in Sec. V, for many conventional symmetries such as $U(1)$ or $SU(2)$, we obtain $E(k) \sim k^2$ and hence we obtain a power-law decay of the autocorrelation function, i.e., $\overline{C_{\widehat{A}}(t)} \sim (1/\sqrt{t})$. However, note that this argument is not rigorous and if the weight of the operator does not lie fully in the lowest quasiparticle band, this argument may not lead to the correct form of the late-time $\overline{C_{\widehat{A}}(t)}$; we demonstrate this with an example of Hilbert-space fragmentation in Sec. V. Hence, when $\widehat{\mathcal{P}}$ is gapless, it is important to carefully study the nature of the operator weight distribution $W_{\widehat{A}}(E)$ in Eq. (51) across the spectrum.

D. Approximate block-diagonalization

While exact symmetries lead to exact block diagonalizations of operators with those symmetries [25,32], it is natural to ask if approximate symmetries lead to approximate block diagonalizations of operators. While we have not been able to establish this in complete generality, here we nevertheless make some simple observations in this direction.

Given a super-Hamiltonian $\widehat{\mathcal{P}}$ of the form of Eq. (5) and a subspace $\mathcal{K}$ of the Hilbert space with dimension $D_{\mathcal{K}}$, the "energy" of its projector $\Pi_{\mathcal{K}}$ under $\widehat{\mathcal{P}}$ is a measure of how connected this subspace is to the rest of the Hilbert space under the action of the terms $\{\widehat{H}_\alpha\}$. To see this, note that the energy of $\Pi_{\mathcal{K}}$ is given by

$$\begin{aligned}\varepsilon_{\mathcal{K}} &:= \frac{(\Pi_{\mathcal{K}}|\widehat{\mathcal{P}}|\Pi_{\mathcal{K}})}{(\Pi_{\mathcal{K}}|\Pi_{\mathcal{K}})} \\ &= \frac{2}{D_{\mathcal{K}}}\sum_{\alpha}(\mathrm{Tr}[\widehat{H}_\alpha^2\Pi_{\mathcal{K}}] - \mathrm{Tr}[\Pi_{\mathcal{K}}\widehat{H}_\alpha\Pi_{\mathcal{K}}\widehat{H}_\alpha]) \\ &= \frac{2}{D_{\mathcal{K}}}\sum_{\alpha}\mathrm{Tr}[\Pi_{\mathcal{K}}\widehat{H}_\alpha\Pi_{\mathcal{K}^\perp}\widehat{H}_\alpha], \end{aligned} \quad (54)$$

where $\Pi_{\mathcal{K}^\perp}$ is the projector onto the subspace orthogonal to $\mathcal{K}$. The last line in Eq. (54) is precisely the sum of the norms of the "block off-diagonal" parts of the $\widehat{H}_\alpha$, i.e., the sums of squares of matrix elements between states in $\mathcal{K}$ and $\mathcal{K}^\perp$. This is consistent with the fact that if $\mathcal{K}$ is a symmetry sector or a closed Krylov subspace, $\Pi_{\mathcal{K}}$ has zero energy under $\widehat{\mathcal{P}}$, which implies that $\mathcal{K}$ is completely disconnected from the rest of the Hilbert space.

Hence the existence of a basis in which all the $\widehat{H}_\alpha$ have an "approximate block diagonal structure" [in the sense that $\varepsilon_{\mathcal{K}}$ of Eq. (54) is small] implies the existence of low-energy excitations in the corresponding super-Hamiltonian. Likewise, the existence of a *projector* that has a small energy under $\widehat{\mathcal{P}}$ implies the existence of a basis in which $\widehat{H}_\alpha$ (and hence the Hamiltonians formed by their linear combinations) have approximate block-diagonal forms. However, the existence of general low-energy excitations of the super-Hamiltonian, which is what we show for a number of cases in Sec. V B, does not itself guarantee the existence of low-energy projectors in general. We defer a careful exploration of this issue for future works.

V. EXAMPLES OF LOW-ENERGY EXCITATIONS

In this section, we construct the low-energy excited states of the super-Hamiltonian $\widehat{\mathcal{P}}$ for several of the examples discussed in Sec. III and discuss corollaries for dynamical properties. We illustrate examples of gapped and gapless super-Hamiltonians separately, since they lead to qualitatively different physics. Again, we restrict explicit illustrations to 1D systems, but the results carry over to higher-dimensional systems *mutatis mutandis*. Note that while the low-energy spectra depend on the precise choice of generators of the bond algebras $\mathcal{A}$ that leads to the super-Hamiltonians $\widehat{\mathcal{P}}$, the qualitative aspects should not depend on these details despite some choices leading to extraneous features that allow more tractability, as we argue in Appendix D.

A. Gapped super-Hamiltonians

1. $\mathbb{Z}_2$ symmetry

We begin by considering an example from Sec. III A 1. Since $\widehat{\mathcal{P}}_{\mathbb{Z}_2}$ of Eq. (10) is a commuting projector Hamiltonian, it is easy to see that it is gapped, since the lowest-energy excitations can be constructed only by "unsatisfying" one of the terms $Z_{j;t}Z_{j;b}$ or $X_{j;t}X_{j;b}X_{j+1;t}X_{j+1;b}$. For example, one of the lowest excited states can be constructed by "destroying" a single composite spin—say, at the rung j_0—by acting the operator $X_{j_0;b}$ on either of the ferromagnetic ground states $|G_{\rightarrow}\rangle$ or $|G_{\leftarrow}\rangle$ of Eq. (13). The resulting excitation "violates" the $Z_{j_0;t}Z_{j_0;b}$ term and hence the energy of the excitation from Eq. (10) is 4. In the original operator language, these excitations can be written as $|X_{j_0})$ and $|iY_{j_0}\prod_{j\neq j_0}Z_j)$, respectively (omitting numerical factors), which commute with all $\{X_jX_{j+1}\}$ terms and all $\{Z_j\}$ in the generators of the bond algebra $\mathcal{A}_{\mathbb{Z}_2}$, except with Z_{j_0}. Hence they can be also viewed as "local-charge-insertion" operators for the original $\mathbb{Z}_2$ symmetry generated by $\prod_j Z_j$.

Another type of excitation can be constructed by violating one of the $X_{j;t}X_{j;b}X_{j+1;t}X_{j+1;b}$ terms, which corresponds to a domain wall between the two ferromagnetic configurations $|G_{\leftarrow}\rangle$ and $|G_{\rightarrow}\rangle$. For a system with OBCs, such excitations have the same energy as the ones in the previous paragraph, whereas for PBCs, they have twice the energy, since domain walls necessarily appear in pairs in this case. In the operator language, considering OBCs for simplicity, we obtain operators of the form $|\prod_{k\leq j_0}Z_k)$ and $|\prod_{k>j_0}Z_k)$, which can be viewed as operators that create a charge of the "dual $\mathbb{Z}_2$ symmetry" obtained by applying a Kramers-Wannier duality transformation on the original system [99]. These are operators that commute with all the generators of $\mathcal{A}_{\mathbb{Z}_2}$ except $X_{j_0}X_{j_0+1}$. Both types of the low-energy excitations can be viewed as operators that create either a $\mathbb{Z}_2$ charge or a dual $\mathbb{Z}_2$ charge as discussed in Ref. [99] (note that the authors' convention has Z and X interchanged compared to our Sec. III A 1). In the "holographic" view of symmetry in 1D [8,99], these charge creation operators in turn correspond to e or m particles of the 2D toric code and it would be interesting to make further connections between their view of symmetries and our super-Hamiltonian perspective.

Finally, we note that the gapped nature following from the commuting-projector property of $\widehat{\mathcal{P}}_{\mathbb{Z}_2}$ extends to super-Hamiltonians for all Pauli-string algebras, which usually correspond to discrete symmetries. Hence this feature also carries over to higher-dimensional super-Hamiltonians corresponding to such symmetries, which can be interpreted as commuting projector Hamiltonians on a bilayer geometry.

2. Isolated QMBS

The case of an isolated QMBS, discussed in Sec. III C, also gives rise to a gapped super-Hamiltonian $\widehat{\mathcal{P}}_{\text{iso}}$ corresponding to the algebra $\mathcal{A}_{\text{iso}}$ of Eq. (40). This super-Hamiltonian, restricted to the composite-spin sector, is shown in Eq. (38). An intuition for the gap is that $\widehat{\mathcal{P}}_{\text{iso}}$ has exactly two linearly independent ground states shown in Eq. (39) and that there is no natural "smooth" low-energy excitation on top of the two ground states. While this model is not solvable, it is frustration free and has been studied in the earlier literature, and it has been proven to be gapped with OBCs [87]. We also numerically find evidence that it is gapped with PBCs (for more details, see Appendix E). We expect that similar phenomenology holds for other examples of isolated QMBSs and it would be interesting to prove this in general, perhaps using some of the mathematical physics methods developed for such purposes [100–103].

B. Gapless super-Hamiltonians

We now move on to demonstrate interesting examples of gapless super-Hamiltonians, which lead to slowly relaxing hydrodynamic modes associated with the symmetry.

1. $U(1)$ symmetry

We start with the case of $U(1)$ symmetry. As discussed in Sec. III A, the ground states of the super-Hamiltonian $\widehat{\mathcal{P}}_{U(1)}$ of Eq. (17) are in the composite-spin sector, obtained by minimizing the energy under the rung terms $\{1 - Z_{j;t}Z_{j;b}\}$ defined on rungs of the ladder in Eq. (11). Since each of the rung terms $\{1 - Z_{j;t}Z_{j;b}\}$ commutes with all the terms in Eq. (17), any state not in the composite-spin sector has an energy of at least 4; hence we expect the lowest excited states of $\widehat{\mathcal{P}}_{U(1)}$ to be within the composite-spin sector. Within this sector, the effective Hamiltonian maps onto the ferromagnetic Heisenberg model of Eq. (18) and, as discussed in Appendix A, the lowest-energy eigenstates are spin waves on top of the ferromagnetic multiplet, shown in Eq. (A6). The energies of these states are given by $32\sin^2(k/2)$, where k is quantized as shown in Eq. (A7) or Eq. (A8), and hence the gap of $\widehat{\mathcal{P}}_{U(1)}$ scales as approximately $1/L^2$, showing that it is gapless. When mapped to the operator language, using Eq. (14), the spin-wave states of Eq. (A6) translate to

$$|\lambda_{m,k}) = \frac{1}{\sqrt{2^L \mathcal{M}_{m,k}}} \sum_{j_1<\cdots<j_m} \left(\sum_{\ell=1}^{m} c_{j_\ell,k}\right) |Z_{j_1}\cdots Z_{j_m}), \quad (55)$$

where $c_{j,k}$ is the form of the orbitals given in Eqs. (A7) and (A8) for PBCs and OBCs and $\mathcal{M}_{m,k}$ is a normalization factor shown in Eq. (A9).

With the exact form of the excited states, we can compute the ensemble-averaged correlation functions of local operators $\overline{C_{\widehat{B},\widehat{A}}(t)}$ and $\overline{C_{\widehat{A}}(t)}$, shown in Eqs. (48) and (53). We study correlators of the on-site operators $\{Z_j\}$, which in the composite-spin language read

$$|Z_j) = 2^{\frac{L}{2}} |\rightleftarrows \cdots \rightleftarrows \leftrightarrows_j \rightleftarrows \cdots \rightleftarrows\rangle. \quad (56)$$

Using this expression, it is clear that all the weight of this operator belongs to the Hilbert space spanned by the one-spin-flip spin waves $\{|\lambda_{1,k})\}$ of Eq. (55), including the $k=0$ case, which belongs to the ground-state manifold (for more details, see Appendix A), and $|Z_j)$ is orthogonal to every other eigenstate of $\widehat{\mathcal{P}}_{U(1)}$, both in and outside the composite-spin sector. Hence the full time evolution of ensemble-averaged correlation functions $\overline{C_{Z_j,Z_{j'}}(t)}$ and $\overline{C_{Z_j}(t)}$ can be computed just using these one-spin-flip spin waves.

The overlap of these operators on the single spin-wave eigenstates reads

$$(\lambda_{1,k}|Z_j) = \frac{c^*_{j,k} 2^{L/2}}{\sqrt{\mathcal{M}_{1,k}}}, \quad \frac{1}{D}|(\lambda_{1,k}|Z_j)|^2 = \frac{|c_{j,k}|^2}{\mathcal{M}_{1,k}} = |c_{j,k}|^2. \quad (57)$$

When $k=0$, the last expression is precisely the Mazur bound, $\overline{C_{Z_j}(\infty)} = 1/L$, which has also been computed in Ref. [25]. From now on, we specialize to PBCs for simplicity [104]. Using Eqs. (51) and (57), and the PBC parameters discussed in Appendix A, the time dependence of the autocorrelation reads

$$\overline{C_{Z_j}(t)} = \frac{1}{L}\sum_k e^{-32\kappa \sin^2\left(\frac{k}{2}\right)t} = \int_0^{2\pi} \frac{dk}{2\pi}\, e^{-16\kappa[1-\cos(k)]t}$$
$$= e^{-16\kappa t}\mathcal{I}_0(16\kappa t) \approx \frac{1}{\sqrt{2\pi \times 16\kappa t}} \quad \text{at large } t, \quad (58)$$

where k in the first sum is quantized as $2\pi n/L$ for $0 \leq n \leq L-1$ and we have taken the $L\to\infty$ limit to go from the sum to the integral, and $\mathcal{I}_0$ is the modified Bessel function of the first kind. Note that the late-time dependence can also be easily recovered by simply substituting the "slow-mode" dispersion to be approximately k^2, which leads directly to $\overline{C_{Z_j}(t)} \sim (1/\sqrt{t})$, consistent with diffusive systems.

It is also possible to recover the Gaussian spatial spreading nature of the two-point correlation of Eq. (48), which in the $L\to\infty$ limit leads to the integral

$$\overline{C_{Z_j,Z_{j'}}(t)} = \int_0^{2\pi} \frac{dk}{2\pi}\, e^{-16\kappa[1-\cos(k)]t} e^{ik(j-j')}$$
$$= e^{16\kappa t}\mathcal{I}_{j-j'}(16\kappa t) \quad (59)$$

$$\overset{\kappa t\gg 1}{\approx} \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{-8\kappa k^2 t} e^{ik(j-j')} = \frac{e^{-\frac{(j-j')^2}{32\kappa t}}}{\sqrt{32\pi\kappa t}}, \quad (60)$$

where in the second line $\mathcal{I}_\nu$ is the modified Bessel function of the first kind of index $\nu = j - j'$, while the last line shows the behavior for $\kappa t \gg 1$. The Gaussian nature of the correlation function in Eq. (60), with a variance growing linearly in t, is consistent with the prediction of diffusion.

It is easy to see that similar results hold in higher dimensions and the complete weight of the local spin operator $|Z_j)$ is within the one-spin-flip spin-wave band. This allows us to compute the ensemble-averaged correlation functions, recovering the standard results expected from diffusion, e.g., the decay of autocorrelations as approximately $t^{-(d/2)}$ in d dimensions.

Moreover, as discussed in Appendix D, the super-Hamiltonian can be different if one starts with a different set of generators of the bond algebra $\mathcal{A}_{U(1)}$ and it need not be solvable or $SU(2)$ symmetric. However, the ground states are always the same by construction and on physical grounds we expect the approximately k^2 dispersion of the low-energy excitations to be the same as long as the generator set is chosen to be local, since it still represents a $U(1)$ symmetric Brownian circuit. Indeed, the gap in longer-range Hamiltonians with this set of ground states has been studied numerically in Ref. [59] and has been shown to be consistent with approximately $1/L^2$ scaling for a system of size L, and the approximately k^2 form of low-energy excitations has been argued based on mappings to a field theory [105].

2. Hilbert-space fragmentation

We now discuss the low-energy excitations of the super-Hamiltonian $\widehat{\mathcal{P}}_{t-J_z}$ of Eq. (26) in the case with $t-J_z$ fragmentation and we show that its low-energy excitations can be used to understand slow modes and late-time behavior of the $t-J_z$ model. Due to the conservation of the pattern of spins, the $t-J_z$ model at late times is expected to exhibit *tracer diffusion* for typical initial states [106,107], which is the phenomenology exhibited in one dimension by a single "tracer" particle that is not allowed to cross its neighbors [108–110]. This leads to an approximately $t^{-\frac{1}{4}}$ prediction for the nature of decay of spin autocorrelation functions at late times [111].

To determine the low-energy excitations, it is sufficient to work in the composite-spin sector defined by spins of the form of Eq. (27), since there is necessarily a gap to other sectors due to the first term in Eq. (26). The super-Hamiltonian restricted to composite spins, $\mathcal{P}_{t-J_z|\mathrm{comp}}$ of Eq. (28), in the ket-bra notation then closely resembles the Heisenberg Hamiltonian $\mathcal{P}_{U(1)|\mathrm{comp}}$ of Eq. (18). In fact, apart from the degeneracies, the spectra of these Hamiltonians are identical. To see that, we first note that $\widehat{\mathcal{P}}_{t-J_z|\mathrm{comp}}$ acting on states with n "spinful particles" in OBCs preserves the pattern of spins $\tilde{\sigma}_1, \tilde{\sigma}_2, \ldots, \tilde{\sigma}_n$ on these particles, ordered from left to right. Working in a sector with such a fixed pattern, the action of $\widehat{\mathcal{P}}_{t-J_z|\mathrm{comp}}$ does not differentiate in any way between the spins $\tilde{\uparrow}$ and $\tilde{\downarrow}$ on these particles and we can simply label the states in this sector by marking each particle location as "state" $\tilde{1}$, obtaining a Hilbert space of L qubits $\tilde{1}/\tilde{0}$ with precisely n qubits in state $\tilde{1}$. The $\tilde{0}$ and $\tilde{1}$ can further be mapped onto the Hilbert space of spins $\uparrow$ and $\downarrow$ and the Hamiltonian $\widehat{\mathcal{P}}_{t-J_z|\mathrm{comp}}$ of Eq. (28) after these identifications precisely maps to the ferromagnetic Heisenberg model H_{Heis} of Eq. (A2), up to an overall factor. Once the eigenstates of the Heisenberg model are written in terms of the spins $\uparrow$ and $\downarrow$, they can first be mapped to $\tilde{0}$ and $\tilde{1}$, respectively, and then the $\tilde{1}$ can be replaced by the specific pattern of spins in the given sector in the $t-J_z$ model, i.e., $\tilde{\uparrow}$ and $\tilde{\downarrow}$, to obtain an eigenstate of $\widehat{\mathcal{P}}_{t-J_z|\mathrm{comp}}$, and hence that of $\widehat{\mathcal{P}}_{t-J_z}$. Hence any eigenstate of the Heisenberg model with n $\uparrow$ and $(L-n)$ $\downarrow$ corresponds to 2^n degenerate eigenstates of $\widehat{\mathcal{P}}_{t-J_z}$. This also maps the $L+1$ ground states of the ferromagnetic Heisenberg model in Eq. (A5) to the total of $\sum_{n=0}^{L} 2^n = 2^{L+1}-1$ ground states of $\widehat{\mathcal{P}}_{t-J_z|\mathrm{comp}}$ in Eq. (29). Moreover, due to the composite-spin $SU(2)$ symmetry of $\widehat{\mathcal{P}}_{t-J_z|\mathrm{comp}}$ discussed in Sec. III B, this mapping also holds in terms of composite-spin states $\{\widetilde{\rightrightarrows}, \widetilde{\leftleftarrows}\}$ defined in Eq. (13) instead of $\{\tilde{\uparrow}, \tilde{\downarrow}\}$; this is useful in the discussion in Appendix F. The entire list of mappings can be summarized as follows:

$$(\tilde{\uparrow}/\tilde{\downarrow}, \tilde{0})/(\widetilde{\rightrightarrows}/\widetilde{\leftleftarrows}, \tilde{0}) \longleftrightarrow (\tilde{1}, \tilde{0}) \longleftrightarrow (\downarrow, \uparrow)/(\leftarrow, \rightarrow), \tag{61}$$

where the leftmost states are in the $t-J_z$ composite-spin Hilbert space and the rightmost ones are in the spin-1/2 Hilbert space.

To understand the behavior of autocorrelation functions, we restrict to a local operator $\widehat{A} = S_j^z$, defined in Eq. (24). We then need to compute the behavior of $\overline{C_{S_j^z}(t)}$, which according to Eq. (51) requires the computation of the overlaps between S_j^z and the eigenstates of $\widehat{\mathcal{P}}_{t-J_z}$. First, since the ground states of $\widehat{\mathcal{P}}_{t-J_z}$ are precisely the operators of the commutant $\mathcal{C}_{t-J_z}$, the total weight of $|S_j^z)$ on all these ground states is the Mazur bound. This bound has been computed exactly for the spin operator S_j^z in Ref. [25] and it has been shown to decay with the system size as approximately $(1/\sqrt{L})$ for OBCs in the bulk of the chain and remain $O(1)$ at the boundaries, even as $L \to \infty$. Since we are interested in the bulk transport properties, we focus on the behavior of $\overline{C_{S_j^z}(t)}$ at large t for j in the middle of the chain as $L \to \infty$.

We discuss the computation of the overlap with other low-energy eigenstates of $\widehat{\mathcal{P}}_{t-J_z}$ in Appendix F. Unlike the case for $U(1)$ symmetry discussed in Sec. V B 1, where the weight of the local spin operator was completely within the spin-wave band of excitations of $\widehat{\mathcal{P}}_{U(1)}$, the weight distribution of the S_j^z operator seems to be significantly more complicated; in particular, a significant portion of the weight appears to lie in states of higher energy. Since the operator $|S_j^z)$ corresponds to the composite spin $|\widetilde{\leftleftarrows}\rangle_j$ on the ladder, it is easy to see that it has nonzero overlap only on the eigenstates of $\widehat{\mathcal{P}}_{t-J_z|\mathrm{comp}}$. Although the Heisenberg model, and hence $\widehat{\mathcal{P}}_{t-J_z|\mathrm{comp}}$, is completely integrable, its eigenstates do not have a simple form, which hinders a fully analytical computation of these overlaps. Nevertheless, with a combination of analytical and numerical results, we are able to deduce the existence of tracer diffusion from the spectrum of $\widehat{\mathcal{P}}_{t-J_z|\mathrm{comp}}$.

We first express $\overline{C_{\widehat{A}}(t)}$ from Eq. (51) as

$$\begin{aligned}
\overline{C_{\widehat{A}}(t)} &= \int \mathrm{d}E\, w_{\widehat{A}}(E)\, e^{-\kappa E t} = \int \mathrm{d}E\, \frac{\mathrm{d}\Omega_{\widehat{A}}}{\mathrm{d}E}\, e^{-\kappa E t},\\
w_{\widehat{A}}(E) &:= \lim_{\Delta E \to 0} \frac{1}{\Delta E} \sum_{E' \in (E, E+\Delta E)} W_{\widehat{A}}(E'),\\
\Omega_{\widehat{A}}(E) &:= \int_0^E \mathrm{d}E'\, w_{\widehat{A}}(E') = \sum_{E'<E} W_{\widehat{A}}(E'),
\end{aligned} \tag{62}$$

where $w_{\widehat{A}}(E)$ can be interpreted as the "density" of the weight at a given energy E and $\Omega_{\widehat{A}}(E)$ is the cumulative weight on eigenstates at energies below E. This expression is valid for finite sizes with discrete levels (possibly degenerate but all are included in the formal sum) but it is written in anticipation of the thermodynamic limit $L \to \infty$ where, for a local observable $\widehat{A}$, we expect $\Omega_{\widehat{A}}(E)$ to converge to an L-independent function that, at $E \to \infty$, gives the total weight $(1/D)(\widehat{A}|\widehat{A})$, which is a fixed $O(1)$ number. Note that, in fact, we expect most of the weight to be spread over a finite range of E, since $|\widehat{A})$ for any strictly local operator $\widehat{A}$ can be viewed as a result of an action of a superoperator $|\widehat{A})(\mathbb{1}|$ that is strictly local in the ladder formulation on one of the ground states $|\mathbb{1})$ of the super-Hamiltonian, which can deposit only finite energy of the latter. As an example of immediate interest to us, a simple calculation gives $(S_j^z|\widehat{\mathcal{P}}_{t-J_z}|S_j^z)/(S_j^z|S_j^z) = 4/3$.

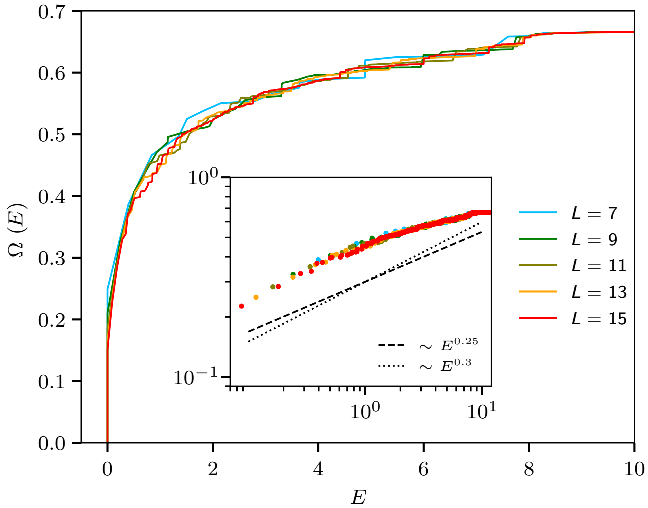

FIG. 2. The cumulative weight $\Omega(E)$ of the operator $|S_j^z)$, with $j = ((L+1)/2)$ in the middle of an OBC chain, on the eigenstates of the super-Hamiltonian $\widehat{\mathcal{P}}_{t-J_z|\text{comp}}$ of Eq. (28) as a function of the energy for various system sizes L. Note that $\Omega(E)$ appears to be converging to an L-independent function. At large E, it approaches the total weight of the operator, which is $\frac{2}{3}$. With our choices of overall factors in $\widehat{\mathcal{P}}_{t-J_z|\text{comp}}$, the Heisenberg model to which it maps has a one-magnon bandwidth of 8 and a total bandwidth of $\mathcal{O}(2.6L)$, which are the natural energy scales to judge the horizontal axis. The inset shows the same plot on a log-log scale to extract its scaling as $E \to 0$. The form appears to be $\Omega(E) \sim E^\gamma$, with $\gamma \in [0.25, 0.3]$.

We can then use the behavior of $\Omega_{\widehat{A}}(E)$ at small E to deduce the behavior of $\overline{C_{\widehat{A}}(t)}$ at large t. For example, if we have $\Omega_{\widehat{A}}(E) \sim E^\gamma$ as $E \to 0$, according to Eq. (62) we have

$$\overline{C_{\widehat{A}}(t)} \sim \gamma \int dE\, E^{\gamma-1} e^{-\kappa E t} \sim t^{-\gamma} \quad \text{for large } t. \tag{63}$$

For the operator $|S_j^z)$, the weights on the eigenstates of $\widehat{\mathcal{P}}_{t-J_z|\text{comp}}$ can formally be written down in terms of eigenstates of the Heisenberg model; we present the details in Appendix F. The cumulative weight distribution $\Omega_{S_j^z}(E)$ can then be computed numerically and its form for $j = ((L+1)/2)$ and odd system sizes with OBCs is shown in Fig. 2. The nature of this distribution as $E \to 0$ is consistent with $\gamma \in [0.25, 0.3]$ in Eq. (63), which is consistent with the scaling expected from the tracer diffusion [107].

3. Asymptotic QMBS

We now demonstrate that asymptotic QMBS, introduced in Ref. [55], can be understood in terms of low-energy excitations of the super-Hamiltonians corresponding to towers of QMBSs, e.g., $\widehat{\mathcal{P}}_{\text{scar}}$ of Eq. (32) for the bond algebra of Eq. (42). In models with a tower of exact QMBSs, asymptotic QMBSs are low-entanglement states orthogonal to exact QMBSs that have a vanishing energy variance in the $L \to \infty$. As a consequence of their low variance, their relaxation time diverges with the system size, a property that does not happen for generic low-entanglement states under local Hamiltonian dynamics [112].

Simple examples of asymptotic QMBSs [55] in the context of models with the ferromagnetic tower of QMBSs, which correspond to the algebras of Eq. (42), are

$$|\Phi_{n,k}\rangle := \frac{1}{\sqrt{\mathcal{N}_{n,k}}} S_k^- |\Phi_{n+1,0}\rangle, \quad 1 \le n \le L-1, \tag{64}$$

where $S_k^- := \sum_j c_{j,k} S_j^-$, with $c_{j,k}$ chosen such that $\langle \Phi_{n,k} | \Phi_{n',0}\rangle = \delta_{n,n'}\delta_{k,0}$ and the variance of $|\Phi_{n,k}\rangle$ decreases with increasing system size L, and $\mathcal{N}_{n,k}$ is a normalization constant that can be tedious to compute. A simple way to satisfy these conditions is to choose $|\Phi_{n,k}\rangle$ to be a spin wave on top of the ferromagnet, with $k \ll 2\pi$ such that $k \to 0$ as $L \to \infty$; hence the similarity between $|\Phi_{m,k}\rangle$ and $|\lambda_{m,k}^z\rangle$ of Eq. (A6). For example, the use of $(c_{j,k}, k)$ of the form of Eq. (A7) with $k = (2\pi/L)$ leads to an energy variance of approximately $1/L^2$ and a fidelity-relaxation time scale of approximately L, similar to the example discussed in Ref. [55].

We now show that the behavior of asymptotic QMBSs of the form of Eq. (64) can be understood from the low-energy excitations of the super-Hamiltonian $\widehat{\mathcal{P}}_{\text{tower}}$ for the tower of QMBSs. We start with a subspace of states of the form $|\psi\rangle_t \otimes |\Phi_{m,0}\rangle_b$, where the state on the bottom leg of the ladder is $|\Phi_{m,0}\rangle$, an exact QMBS shown in Eq. (41). Using the fact that $R_{[j,j+1];b}|\Phi_{m,0}\rangle_b = 0$ and $R_{[j,j+1];\ell}^2 = R_{[j,j+1];\ell}$, $\widehat{\mathcal{P}}_{\text{tower}}$ of the form of Eq. (32) keeps this subspace invariant and acts within it as

$$\widehat{\mathcal{P}}_{\text{tower}} = 8\sum_j R_{[j,j+1];t} = 2L - 8\sum_j (\vec{S}_{j;t} \cdot \vec{S}_{j+1;t}), \tag{65}$$

which, up to an overall factor, is simply the ferromagnetic Heisenberg Hamiltonian of Appendix A, acting on the top leg. Hence, excitations within this subspace are spin waves on the top leg of the ladder of the form

$$|\Phi_{n,k}\rangle_t \otimes |\Phi_{m,0}\rangle_b = ||\Phi_{n,k}\rangle\langle\Phi_{m,0}|), \tag{66}$$

where $|\Phi_{n,k}\rangle$ is a spin wave on top of the Heisenberg ferromagnet, e.g., as defined in Eq. (64). These states on the ladder have energy as shown in Eq. (A10); in particular, the dispersion scales as approximately k^2. Hence the relaxation of the autocorrelation function of any operator to its Mazur bound is expected to occur on time scales of approximately L^2, provided that the operator has a nonzero overlap with the slowly relaxing mode.

To apply the general theory of autocorrelation functions to study the asymptotic QMBSs, we first note some general properties that hold for an initial state $|\psi\rangle$ evolving under a Brownian circuit corresponding to an algebra $\mathcal{A} = \langle\langle\{\widehat{H}_\alpha\}\rangle\rangle$. The autocorrelation function of an operator $\widehat{A} = |\psi\rangle\langle\Upsilon|$, for any normalized state $|\Upsilon\rangle$, is given by

$$\mathrm{Tr}[\widehat{A}(0)^\dagger \widehat{A}(-t)] = \langle\psi(0)|\psi(t)\rangle\langle\Upsilon(t)|\Upsilon(0)\rangle, \tag{67}$$

where $\widehat{A}(-t)$ can be viewed as the time-evolved operator under the Brownian circuit with bond generators of opposite sign; in this context, $\widehat{A}(-t) = |\psi(t)\rangle\langle\Upsilon(t)|$, with $|\psi(t)\rangle$ and $|\Upsilon(t)\rangle$ being the time-evolved states under a single realization of the Brownian-circuit couplings $\{J_\alpha^{(t')}\}$ (for precise details, see Appendix G). Given that the algebra $\mathcal{A}$ admits a *singlet* $|\Upsilon\rangle$, i.e., that is an eigenstate of each of the $\{\widehat{H}_\alpha\}$ (so that $|\Upsilon(t)\rangle = e^{-i\varphi t}|\Upsilon(0)\rangle$ for some φ that, in general, depends on the random couplings), we can use Eq. (67) to lower bound the ensemble-averaged fidelity as follows:

$$\overline{\mathcal{F}(t)} := \overline{|\langle\psi(0)|\psi(t)\rangle|^2} \geq \left|\overline{\langle\psi(0)|\psi(t)\rangle}\right|^2$$
$$\geq \left|\overline{\langle\psi(0)|\psi(t)\rangle e^{i\varphi t}}\right|^2 = \left|\overline{\mathrm{Tr}[\widehat{A}(0)^\dagger \widehat{A}(-t)]}\right|^2. \tag{68}$$

In the case of asymptotic QMBSs, we can choose $\widehat{A} = |\Phi_{n,k}\rangle\langle\Phi_{m,0}|$ with normalized $|\Phi_{m,0}\rangle$ and $|\Phi_{n,k}\rangle$, which satisfy the required conditions. This operator $|\widehat{A})$ is precisely that in Eq. (66) and is hence an eigenstate of the super-Hamiltonian $\widehat{\mathcal{P}}_{\rm tower}$ with eigenvalue $p_k = 8[1 - \cos(k)]$. Using Eq. (68), we have

$$\overline{|\langle\Phi_{n,k}(0)|\Phi_{n,k}(t)\rangle|^2} \geq \left|\overline{(\widehat{A}(0)|\widehat{A}(-t))}\right|^2 = e^{-2\kappa p_k t}. \tag{69}$$

Thus, the average fidelity decays on a time scale of approximately L^2. This qualitatively recovers that the fidelity of asymptotic QMBSs decays on time scales that grow with the system size. However, note that this scaling differs quantitatively from the approximately L scaling of the fidelity-decay time scale seen in Hamiltonian systems with asymptotic QMBSs [55]. We hypothesize on reasons for this difference between the Brownian-circuit and Hamiltonian systems in Appendix G and it appears to be related to the quantum Zeno effect due to stochasticity in the Brownian circuit.

The behavior of the overlap in Eq. (69) can also be understood from a direct analysis of the evolution of states under Brownian-circuit dynamics, which we discuss in Appendix G. Considerations discussed there also lead us to the following conjecture on the existence of asymptotic QMBS in Hamiltonians with exact QMBS.

Conjecture 1. Consider a space $\mathcal{S} = \mathrm{span}\{|\Phi_n\rangle\}$ that can be expressed as the exhaustive common kernel of a set of strictly local projectors. Any local Hamiltonian that realizes this subspace as the exact QMBS subspace also has asymptotic QMBSs if $\mathcal{S}$ cannot be expressed as the ground-state space of a gapped frustration-free Hamiltonian. Furthermore, the gapless excitations of any such Hamiltonian are the asymptotic QMBSs.

The same phenomenology generalizes to cases in which the QMBSs are nondegenerate and we discuss this in more detail in Appendix H.

Finally, we remark that even though one can construct low-energy excitations of $\widehat{\mathcal{P}}_{\rm tower}$ with dispersion approximately k^2 that is similar to the dispersion of the low-energy excitations of $\widehat{\mathcal{P}}_{U(1)}$ in the $U(1)$-symmetry case, there is generally no diffusion of local operators in systems with only QMBSs. This is due to the exponentially small overlap of local operators on these low-energy modes, similar to the result that QMBSs have an exponentially small contribution to the Mazur bound of general local operators, as demonstrated in Ref. [31].

4. Other continuous symmetries

The strategy of studying the low-energy excitations of the super-Hamiltonians can be applied to more general symmetries and we briefly discuss two cases here.

First, this can be applied to non-Abelian symmetries such as $SU(q)$ for $q \geq 2$. As discussed in Sec. III A 3, the simplest super-Hamiltonians in such cases are Heisenberg-like models with $SU(q^2)$ symmetry [see Eq. (22)] and the ground states are $SU(q^2)$ ferromagnets. We can then straightforwardly also obtain exact low-energy excitations of such Hamiltonians by creating spin waves on top of these generalized ferromagnetic states, e.g.,

$$\sum_{j=1}^{L} e^{ikj} S_j^{\tilde{m},\tilde{m}'} |\tilde{m},\ldots,\tilde{m},\tilde{m},\tilde{m},\ldots,\tilde{m}\rangle, \tag{70}$$

where $S_j^{\tilde{m},\tilde{m}'}$ is the operator that changes the state on the (rung) site j from $\tilde{m}$ to $\tilde{m}'$. This state can be shown to have energy $4(1-\cos(k)) \sim k^2$ at small k, similar to the spin waves of the spin-1/2 Heisenberg model. This enables computations of various correlation functions, including autocorrelation functions of local operators similar to the $U(1)$ case discussed in Sec. V B 1, and we get similar answers, e.g., diffusion due to the similar nature of the super-Hamiltonians in both cases.

Second, the analysis simplifies in the case of "classical" symmetries, where the super-Hamiltonians map onto RK-type Hamiltonians, as discussed in Appendix B. Similar RK-type Hamiltonians appear in the study of spectral form factors in Floquet random circuits with symmetries or constraints [59,113,114], and the *Thouless time* is determined by the scaling of the inverse of the gap of the corresponding Hamiltonian with the system size. For example, RK-type Hamiltonians that appear in the context of dipole and multipole symmetries have been studied in Ref. [59]. Their low-energy physics can be understood using Lifshitz-like field theories, which leads to an approximately k^4 dispersion of the low-energy mode for dipole-moment-conserving systems, and approximately $k^{2(m+1)}$ for systems conserving the mth moment. With the appropriate choice of bond generators, the same set of RK-type Hamiltonians would appear as super-Hamiltonians in our analysis [115] and using heuristic arguments based on the dispersion relation of the low-energy modes, we obtain that autocorrelations should decay as approximately $1/t^{(d/(2(m+1)))}$ in d-dimensional systems with mth-multipole-moment conservation, indicating subdiffusion. This Brownian-circuit approach to determine transport phenomena has also recently been applied to short-range and long-range dipole-conserving Hamiltonians [62], where the low-energy excitations of the effective super-Hamiltonians (referred to there as Lindbladians) have yielded results consistent with those obtained from other methods [58,59,63,64].

We close this discussion with a general remark on the low-energy excited states of general super-Hamiltonians. Note that the identity operator $|\mathbb{1})$ is always a ground state of any super-Hamiltonian, since it always belongs to the commutant algebra. In the ladder language, this corresponds to a "homogeneous" product state, e.g., $|\widetilde{\rightarrow}\widetilde{\rightarrow}\cdots\widetilde{\rightarrow}\rangle$ for spin-1/2 systems. Given that the super-Hamiltonian is a local superoperator, it is natural to expect that its low-energy spectrum should be well approximated by a "quasiparticle" trial state that aids in determining the late-time transport. This happens in all of the cases that we have studied; however, exploring the validity of this statement or coming up with counterexamples would be an interesting avenue for future work.

VI. CONCLUSIONS AND OUTLOOK

In this work, we have shown that many examples of both conventional and unconventional symmetries can be understood as ground states of local superoperators interpreted as Hamiltonians acting on a doubled ladder Hilbert space; hence we have referred to them as "super-Hamiltonians." This originates from the understanding of symmetries as commutants of bond algebras generated by local operators, as illustrated in Refs. [25,31–33]. For conventional symmetries such as $\mathbb{Z}_2$, $U(1)$, $SU(2)$, the symmetry algebras can be interpreted as various kinds of ferromagnetic states of appropriate super-Hamiltonians. Unconventional symmetries such as fragmentation and QMBSs have also led to frustration-free Hamiltonians with solvable ground states.

We have then shown that the low-energy spectra of the super-Hamiltonians can be interpreted as approximate symmetries associated with the exact symmetries. We have done this by showing that super-Hamiltonians obtained in this way are effective Hamiltonians that describe noise-averaged dynamics in noisy symmetric Brownian circuits constructed using the bond-algebra generators. This gives a physical interpretation for the super-Hamiltonians and connects their low-energy excited states to slowly relaxing hydrodynamic modes of the symmetric Brownian circuits. This also gives a novel interpretation for the Mazur bound [53,54], usually interpreted as a lower bound for the time-averaged autocorrelation function, as the saturation value of the ensemble-averaged autocorrelation function of Brownian circuits. The approach to this saturation value is governed by the low-energy spectra of the super-Hamiltonians; hence their low-energy eigenstates beyond the ground states have interpretations as approximate symmetries.

We have then explicitly solved for the low-energy spectra of the super-Hamiltonians and discussed the dynamical consequences of the associated slowly relaxing modes. Using this framework, we have first recovered well-known facts that while conventional discrete symmetries such as $\mathbb{Z}_2$ have gapped super-Hamiltonians, and hence no associated slow modes, conventional continuous symmetries such as $U(1)$ and $SU(2)$ have gapless super-Hamiltonians and the corresponding slow modes lead to diffusion. However, we have shown that this framework works much more generally, for understanding slow modes associated with unconventional symmetries such as fragmentation and QMBSs as well. While isolated QMBSs have gapped super-Hamiltonians and hence no associated slow modes, towers of QMBSs have *asymptotic scars* of the type discussed in Ref. [55] as slow modes. Hilbert-space fragmentation in the $t - J_z$ model has slow modes, which can be used to understand tracer diffusion in such systems, as pointed out in earlier works [106,107]. On a technical note, the quantitative understanding of the slow relaxation of certain observables in some cases, such as the $t - J_z$ fragmentation, has required a careful analysis of the full low-energy spectrum (including appropriate weights for observables), rather than the simple scaling of the gap that has been sufficient for such purposes in earlier works [59,62]. In all, our work connects studies of the commutant algebra focusing on exact conserved quantities (ground states of the corresponding super-Hamiltonians) to studies of hydrodynamic and transport properties controlled by approximately conserved quantities (low-lying excitations of the super-Hamiltonians) in symmetric systems. While we have restricted illustrations to 1D systems, the results and phenomenology generalize straightforwardly to higher-dimensional systems.

It would be interesting to explore the applicability of this method to other generalized symmetries being studied in the literature, e.g., subsystem symmetries [60,116–119], spatially modulated symmetries [61], and categorical or MPO symmetries [11,12,120], and understand if they can be viewed as ground states of local super-Hamiltonians. The low-energy spectrum of the corresponding super-Hamiltonians should reveal the late-time dynamical properties of such systems and of the associated hydrodynamic modes, which would also be interesting to explore in other models of Hilbert-space fragmentation [21,25,72,121–124], and lattice gauge theories with strictly local symmetries [125,126]. It would also be interesting to try to reproduce in this language the sector-dependent hydrodynamic behavior observed in many models of HSF based on pattern conservation or "irreducible strings" [106,127]. Of course, strictly speaking, some symmetries, such as dynamical symmetries [32,128], are not ground states of local Hermitian superoperators, since they correspond to algebras generated by also including extensive local terms (for more details, see Ref. [32]) but we hope that a generalization of this story might capture many more examples. In addition, lattice symmetries or those that appear in the context of integrability [129] have so far not been explored in the commutant framework, which would be an interesting direction to pursue.

The fact that the super-Hamiltonians can be understood as frustration-free Hamiltonians—and, moreover, of the RK form in many cases—also opens up many directions of exploration. First, such Hamiltonians are easy to analyze and this method might provide a better systematic approach to prove the exhaustion of commutant algebras, which has turned out to be tedious using brute-force methods [31,32]. Second, they are also amenable to standard techniques for proving gaps or their absence [100–103, 130–132] and the understanding of which symmetries have a gap is important for understanding the nature of late-time transport in symmetric systems. Third, RK Hamiltonians have connections to several standard concepts in classical master equations and also to spectral graph theory [59,76,77,133], and it would be interesting to exploit this property to study the low-energy excited states using existing methods such as classical stochastic circuits, similar to those used in the literature [58,72,134,135], and potentially also quantum Monte Carlo techniques [136]. Finally, many of these super-Hamiltonians also have interesting continuum limits and their low-energy physics can be understood in terms of field theories. For example, several types of RK Hamiltonians map onto Lifshitz field theories that are easy to analyze [59,137–139]. Given that many generalized symmetries are studied in the context of quantum field theories in the continuum [1,2], it is natural to wonder if the novel symmetries there, e.g., noninvertible symmetries understood via category theory, can be understood as "ground states" in any sense. Some aspects of the super-Hamiltonian constructions, e.g., working in a doubled Hilbert space and studying the low-energy physics, resemble the Schwinger-Keldysh formalism [140–143] and it would be useful to elucidate these connections further.

More speculatively, connecting symmetry algebras to ground states should also help impose some general constraints on symmetry operators, e.g., perhaps they necessarily have MPO forms or some restrictions on their operator entanglement. Moreover, the fact that symmetry, which is a property of the Hilbert space, is connected to ground-state properties of a local operator, is consistent with the conjecture that symmetries are related to topological orders—a ground-state property—in one higher dimension [8,99,144]. The commutant framework along with this ground-state mapping might be a framework in which to make such a correspondence more precise in lattice systems.

The language of super-Hamiltonians also introduces a precise language with which to discuss approximate symmetries. While we have illustrated this only for approximate symmetries that accompany exact symmetries, it would be very interesting to identify bond algebras without exact symmetries, but with approximate symmetries that appear as low-energy excited states of the super-Hamiltonians, which could lead to slow dynamics and phenomena such as prethermalization. Furthermore, as we have shown in Sec. IV D, approximate symmetries are also potentially related to approximate block-diagonal structures and hence the language of super-Hamiltonians might help explain the origin of approximate symmetries in certain systems in the literature; e.g., the so-called PXP model [17] is known to exhibit approximate QMBS and approximate block-diagonal structures [145].

On a different note, since algebra-based Brownian circuits have played a crucial role in understanding and/or interpreting the super-Hamiltonian spectrum, this seems like a useful setting to explore more. For example, it is likely that several results on Haar-random circuits can be reproduced using the seemingly more tractable Brownian circuits and, indeed, there have been many interesting works studying the properties of "generic" Brownian circuits using "effective Hamiltonians," which are super-Hamiltonians of the kind we study in this work [43,45, 47]. On the other hand, Brownian circuits with symmetries have been much less studied and the large class of "algebra-based" circuits that we have introduced in this work, which are defined using the bond algebra corresponding to the symmetry, might prove to be useful toy models that are easier to study than symmetric Haar-random circuits for a number of reasons. First, defining the latter requires knowledge of the irreducible representations [113,146–148], i.e., the block-diagonal structure of each gate, whereas Brownian circuits only require the generators of the corresponding bond algebra. Second, in contrast to rigid Haar-random circuits, the class of Brownian circuits that we study possesses a lot of tunable parameters in the choice of their generators, which might lead to more analytically tractable super-Hamiltonians that provide better physical insights. Finally, while computations in Haar-random circuits map onto questions in classical statistical mechanics, computations in Brownian circuits map onto the low-energy physics of effective super-Hamiltonians, which, although they are equivalent to questions in classical statistical mechanics, are nevertheless more directly accessible using analytical and numerical treatments developed in the context of quantum many-body systems. For example, the hydrodynamic modes associated with the symmetries arise more “naturally” as “low-energy excitations” on top of simple ground states, which can be studied using a variety of variational methods. Hence this should be a nice analytical tool with which to explore the physics of symmetric systems, including those with unconventional symmetries, and this can be contrasted from the physics of systems without any symmetry, by studying bond-algebra generators that have a trivial commutant of only the identity operator.

Finally, it is important to better understand the precise connections between the dynamics of Brownian circuits and more general Hamiltonian or Floquet systems. While the microscopic physics is expected to be different, *universal* properties such as hydrodynamic modes, that arise solely due to the symmetry and locality of the systems, should appear in both kinds of systems, even though they are analytically tractable only in Brownian circuits. It would be interesting to check if these modes survive under “relaxation” of the structure of Brownian circuits and making this closer to non-Markovian Hamiltonian systems in various ways, e.g., by incorporating temporally correlated noise.

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APPENDIX A: THE FERROMAGNETIC HEISENBERG MODEL

In this appendix, we define a canonical form for the ferromagnetic Heisenberg model and set the conventions we use to describe it and its eigenstates. This appears repeatedly in the analysis of various super-Hamiltonians that we have studied in the main text. It is a spin-$\frac{1}{2}$ Hamiltonian acting on a system of size L, with the local degrees of freedom $|\uparrow\rangle_j$ and $|\downarrow\rangle_j$ in the $\hat{z}$ basis or $|\rightarrow\rangle_j$ and $|\leftarrow\rangle_j$ in the $\hat{x}$ basis. We use the convention that these are related as

$$|\rightarrow\rangle_j := \frac{|\uparrow\rangle_j + |\downarrow\rangle_j}{\sqrt{2}}, \quad |\leftarrow\rangle_j := \frac{|\uparrow\rangle_j - |\downarrow\rangle_j}{\sqrt{2}}. \tag{A1}$$

The standard forms of the Heisenberg Hamiltonian that we use in this work are given by

$$\begin{aligned} H_{\text{Heis}} &= \sum_{j=1}^{L_{\max}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)(\langle\uparrow\downarrow| - \langle\downarrow\uparrow|)_{[j,j+1]} \\ &= \sum_{j=1}^{L_{\max}} (|\rightarrow\leftarrow\rangle - |\leftarrow\rightarrow\rangle)(\langle\rightarrow\leftarrow| - \langle\leftarrow\rightarrow|)_{[j,j+1]} \\ &= \sum_{j=1}^{L_{\max}} (1 - P^{(2)}_{j,j+1}) \\ &= \frac{1}{2}\sum_{j=1}^{L_{\max}} [1 - (X_j X_{j+1} + Y_j Y_{j+1} + Z_j Z_{j+1})] \\ &= 2\sum_{j=1}^{L_{\max}} \left[\frac{1}{4} - \vec{S}_j \cdot \vec{S}_{j+1}\right], \end{aligned} \tag{A2}$$

where the $\{X_j, Y_j, Z_j\}$ are the Pauli operators on site j, $\{S^x_j, S^y_j, S^z_j\}$ are the spin operators on site j, which are the Pauli matrices multiplied by a factor of $\frac{1}{2}$, and $P^{(2)}_{j,j+1}$ is the operator that permutes the states on sites j and $j+1$ defined in Eq. (21). Moreover, $L_{\max} = L - 1$ with OBCs and $L_{\max} = L$ with PBCs, with the subscripts taken to be modulo L. We also define raising and lowering operators,

$$\begin{aligned} S^{z-}_{\text{tot}} &:= \sum_{j=1}^{L} S^{z-}_j = \sum_{j=1}^{L} |\downarrow\rangle\langle\uparrow|_j, \\ S^{x-}_{\text{tot}} &:= \sum_{j=1}^{L} S^{x-}_j = \sum_{j=1}^{L} |\leftarrow\rangle\langle\rightarrow|_j, \\ S^{z+}_{\text{tot}} &:= (S^{z-}_{\text{tot}})^\dagger, \quad S^{x+}_{\text{tot}} := (S^{x-}_{\text{tot}})^\dagger, \end{aligned} \tag{A3}$$

all of which commute with H_{Heis}.

This Hamiltonian has $(L+1)$-fold degenerate frustration-free ground states, which we often refer to as the “ferromagnetic multiplet.” To obtain an orthonormal basis for this multiplet, we can start with the fully polarized state with all spins in either the $+\hat{z}$ or $+\hat{x}$ direction, which read

$$|F^z_0\rangle := |\uparrow\uparrow\cdots\uparrow\rangle, \quad |F^x_0\rangle := |\rightarrow\rightarrow\cdots\rightarrow\rangle, \tag{A4}$$

and repeatedly act with the corresponding lowering operators, S^{z-}_{tot} or S^{x-}_{tot} of Eq. (A3), respectively, to obtain $L+1$ linearly independent states of the form

$$|F_m^\alpha\rangle = \frac{1}{\sqrt{\binom{L}{m}}} \sum_{j_1<\cdots<j_m} S_{j_1}^{\alpha-} S_{j_2}^{\alpha-} \cdots S_{j_m}^{\alpha-} |F_0^\alpha\rangle, \quad 0 \le m \le L, \quad \alpha \in \{z, x\}. \tag{A5}$$

Beyond the ground-state space, the low-energy excitations of H_{Heis} are well known to be spin waves. These spin waves are $(L-1)$-fold degenerate (corresponding to degeneracy of a multiplet with total spin of $L/2-1$) and a complete orthonormal basis for these degenerate eigenstates can be chosen as

$$|\lambda_{m,k}^\alpha\rangle = \frac{1}{\sqrt{\mathcal{M}_{m,k}}} \sum_{j_1<\cdots<j_m} \left[\left(\sum_{\ell=1}^{m} c_{j_\ell,k}\right) S_{j_1}^{\alpha-} \cdots S_{j_m}^{\alpha-} |F_0^\alpha\rangle\right], \quad 1 \le m \le L-1, \quad \alpha \in \{z, x\}, \tag{A6}$$

where k labels orthonormal "orbitals" $c_{j,k}$ in the single-magnon problem, e.g., k is the plane-wave momentum in the PBC case,

$$c_{j,k}^{\mathrm{PBC}} := \frac{1}{\sqrt{L}} e^{ikj}, \quad k = \frac{2\pi n}{L}, \quad 1 \le n \le L-1; \tag{A7}$$

or k is the appropriate standing-wave "momentum" in the OBC case,

$$c_{j,k}^{\mathrm{OBC}} := \sqrt{\frac{2}{L}} \cos[k(j-1/2)], \quad k = \frac{\pi n}{L}, \quad 1 \le n \le L-1. \tag{A8}$$

Further unpacking Eq. (A6), integer $1 \le m \le L-1$ labels states in the given $SU(2)$ multiplet with fixed k and $\mathcal{M}_{m,k}$ is a normalization factor chosen so that $|\lambda_{m,k}\rangle$ is normalized, with the precise form shown in Eq. (A9) below. However, for much of the description, we can keep the spin-wave orbitals general by only requiring orthonormality among themselves as well as orthogonality to a completely uniform "$k=0$ orbital," obtained for convenience by setting $k=0$ in Eq. (A7) or Eq. (A8) for PBCs or OBCs, respectively. The states in Eq. (A6) formally corresponding to the latter in fact belong to the ferromagnetic multiplet, i.e., the ground states of the Heisenberg model, and we have $|\lambda_{m,0}^\alpha\rangle = |F_m^\alpha\rangle$ of Eq. (A5). In all, the normalization factor reads

$$\mathcal{M}_{m,k} = \begin{cases} \binom{L-2}{m-1}, & \text{if } k \neq 0, \\ \frac{2m^2}{L}\binom{L}{m}, & \text{if } k = 0. \end{cases} \tag{A9}$$

The above spin-wave excitation solutions are directly obtained by solving the Heisenberg Hamiltonian in the one-spin-flip Hilbert space spanned by the states of the form $|\uparrow\ \cdots\ \uparrow\downarrow\uparrow\ \cdots\ \uparrow\rangle$ or $|\rightarrow\ \cdots\ \rightarrow\leftarrow\rightarrow\ \cdots\ \rightarrow\rangle$, which gives $|\lambda_{1,k}^z\rangle$ or $|\lambda_{1,k}^x\rangle$, respectively, with total spin $L/2-1$ for $k \neq 0$, and then by repeatedly acting with the lowering operator $S_{\mathrm{tot}}^{\alpha-}$ on the state $|\lambda_{1,k}^\alpha\rangle$ for $\alpha \in \{z, x\}$. The energies of these states are given by

$$H_{\mathrm{Heis}}|\lambda_{m,k}^\alpha\rangle = 4\sin^2\left(\frac{k}{2}\right)|\lambda_{m,k}^\alpha\rangle, \quad 1 \le m \le L-1, \quad \alpha \in \{z, x\}; \tag{A10}$$

hence the gap of H_{Heis} is given by $4\sin^2(\pi/2L)$ or $4\sin^2(\pi/L)$ for OBCs or PBCs. This gap scales as approximately $1/L^2$, showing that H_{Heis} is gapless in the thermodynamic limit.

Moreover, the H_{Heis} is known to be completely integrable and the expressions for the eigenstates can in principle be derived using the Bethe ansatz. However, they are in general not simple to use for our purposes and we refer interested readers to one of the numerous review articles on the subject for more details [149–151].

APPENDIX B: ROKHSAR-KIVELSON-TYPE SUPER-HAMILTONIANS FROM CLASSICAL SYMMETRIES

In this appendix, we discuss cases in which the superoperator $\widehat{\mathcal{P}}$ is an RK-type Hamiltonian, also known as a stochastic-matrix-form (SMF) decomposible or stoquastic Hamiltonian. Abstractly, these are Hamiltonians that are defined over a Hilbert space spanned by a set of classical configurations, which could be a set of product states or something more complex, such as nonintersecting dimer coverings of a lattice. Given this configuration space, one can define a set of local transitions that relate two different configurations; this defines a local Hamiltonian. With these sets of local configurations and transitions defined, a simple RK-type Hamiltonian is defined as

$$H_{\mathrm{RK}} = \sum_{\langle C,C'\rangle} \widehat{Q}_{C,C'}, \quad \widehat{Q}_{C,C'} := \left(|C\rangle - |C'\rangle\right)\left(\langle C| - \langle C'|\right), \tag{B1}$$

where "$\langle C, C'\rangle$" in the sum indicates that the configurations C and C' are connected by some local moves. Noting that each term $\widehat{Q}_{C,C'}$ in H_{RK} is positive semidefinite, it is easy to solve for its ground states $\{|G^{(\mathcal{K})}\rangle\}$, which are given by

$$\widehat{Q}_{C,C'}(|C\rangle + |C'\rangle) = 0,\ \forall C, C' \quad \Longrightarrow \quad |G^{(\mathcal{K})}\rangle = \frac{1}{\sqrt{N_{\mathcal{K}}}} \sum_{C\in\mathcal{K}} |C\rangle, \tag{B2}$$

where $\mathcal{K}$ defines a Krylov subspace of $N_{\mathcal{K}}$ configurations connected by the local moves and there is one ground state corresponding to each Krylov subspace. As a simple example, the ferromagnetic Heisenberg Hamiltonian of Eq. (A2) is a Hamiltonian of the RK form of Eq. (B1), where the configuration space is the space of all product states, and the local moves that connect different configurations are given by nearest-neighbor swap $\uparrow\downarrow \leftrightarrow \downarrow\uparrow$ in the $\hat{z}$ basis or $\rightarrow\leftarrow \leftrightarrow \leftarrow\rightarrow$ in the $\hat{x}$ basis. The $(L+1)$ ferromagnetic ground states of the Heisenberg model are also ground states of the form of Eq. (B2), where each Krylov subspace $\mathcal{K}$ consists of product states with the same total spin (since they can all be connected via the aforementioned local moves, assuming a connected lattice of sites) and there are $(L+1)$ such Krylov subspaces. Such RK-type Hamiltonians have been extensively studied in the literature and they naturally appear in several different physically relevant contexts. Examples include dimer models [138,152,153], Markov processes satisfying detailed balance [76,137], and in the study of various kinds of random circuits [59,113,114,154]. There are also generalizations of Hamiltonians of this type and we refer readers to Ref. [76] for further discussions.

Turning to the problem of finding commutant algebras, here we consider families of Hamiltonians defined on a q-level Hilbert space, which are comprised of terms that relate some set of classical product-state configurations $\{|C\rangle\}$ that form a basis of the Hilbert space. In particular, we work with strictly local terms defined as

$$T^{(\vec{\tau},\vec{\tau}')}_{[j,j+r]} := (|\vec{\tau}\rangle\langle\vec{\tau}'| + |\vec{\tau}'\rangle\langle\vec{\tau}|)_{[j,j+r]}, \quad S^z_j := \sum_{\sigma} s_\sigma |\sigma\rangle\langle\sigma|_j, \quad 1 \le \sigma \le q, \quad N^{\vec{\tau}}_{[j,j+r]} := |\vec{\tau}\rangle\langle_{[j,j+r]}|, \tag{B3}$$

where S^z_j is spin operator and s_σ is the spin of level σ and $(\vec{\tau}, \vec{\tau}')$ denotes a pair of strictly local $(r+1)$-site orthogonal product configurations that are "connected" according to some local rules that we leave general in this appendix. The generators of several standard examples of bond algebras can be cast in this form, e.g., $X_jX_{j+1} + Y_jY_{j+1} = 2\, T^{(\uparrow\downarrow,\downarrow\uparrow)}_{[j,j+1]}$. With these definitions, we can write down *classical* bond and commutant algebras as

$$\mathcal{A}_{\mathrm{cl}} = \langle\langle \{S^z_j\}, \{T^{(\vec{\tau},\vec{\tau}')}_{[j,j+r]}\}\rangle\rangle, \quad \mathcal{C}_{\mathrm{cl}} = \langle\langle \{F_\alpha(\{S^z_j\})\}\rangle\rangle, \tag{B4}$$

where $\{F_\alpha(\cdots)\}$ denotes some set of polynomials [that depends on the specific set of connections $(\vec{\tau}, \vec{\tau}')$ in $\mathcal{A}_{\mathrm{cl}}$], essentially stating that the operators in the commutant $\mathcal{C}_{\mathrm{cl}}$ are diagonal in the computational basis. Note that the diagonal form of operators in $\mathcal{C}_{\mathrm{cl}}$ directly follows from the inclusion of $\{S^z_j\}$ in the generators of the bond algebra, as we have shown in Appendix A of Ref. [25] (here, implicitly assuming that powers of S^z on a qubit generate the space of all $q \times q$ diagonal matrices). We refer to commutants of the form of $\mathcal{C}_{\mathrm{cl}}$ as classical symmetries, since they lead to block decompositions of Hamiltonians in $\mathcal{A}_{\mathrm{cl}}$ that are completely understood in the product-state basis. The super-Hamiltonian of Eq. (7) corresponding to $\mathcal{A}_{\mathrm{cl}}$ is then of the form

$$\begin{aligned}\widehat{\mathcal{P}}_{\mathrm{cl}} &= \sum_j (S^z_{j;t} - S^z_{j;b})^2 + \sum_{j,(\vec{\tau},\vec{\tau}')} \left[(T^{(\vec{\tau},\vec{\tau}')}_{[j,j+r];t})^2 + (T^{(\vec{\tau},\vec{\tau}')}_{[j,j+r];b})^2 - 2T^{(\vec{\tau},\vec{\tau}')}_{[j,j+r];t} T^{(\vec{\tau},\vec{\tau}')}_{[j,j+r];b} \right] \\ &= \sum_j (S^z_{j;t} - S^z_{j;b})^2 + \sum_{j,(\vec{\tau},\vec{\tau}')} \left[N^{\vec{\tau}}_{[j,j+r];t} + N^{\vec{\tau}'}_{[j,j+r];t} + N^{\vec{\tau}}_{[j,j+r];b} + N^{\vec{\tau}'}_{[j,j+r];b} - 2T^{(\vec{\tau},\vec{\tau}')}_{[j,j+r];t} T^{(\vec{\tau},\vec{\tau}')}_{[j,j+r];b} \right].\end{aligned} \tag{B5}$$

The minimization of energy under the first term in Eq. (B5) ensures that the ground state is in the subspace spanned by composite spins defined on the rungs of the ladder (or bilayer) as $|\tilde{\sigma}\rangle := \left| \begin{matrix}\sigma \\ \sigma\end{matrix} \right\rangle$. $\mathcal{P}_{\mathrm{cl}}$ restricted to this composite-spin subspace then reads

$$\widehat{\mathcal{P}}_{\text{cl}|\text{comp}} = 2\sum_{j,(\vec{\tau},\vec{\tau}')}\Big[|\widetilde{\vec{\tau}}\rangle\langle\widetilde{\vec{\tau}}|_{[j,j+r]} + |\widetilde{\vec{\tau}}'\rangle\langle\widetilde{\vec{\tau}}'|_{[j,j+r]} - |\widetilde{\vec{\tau}}\rangle\langle\widetilde{\vec{\tau}}'|_{[j,j+r]} - |\widetilde{\vec{\tau}}'\rangle\langle\widetilde{\vec{\tau}}|_{[j,j+r]}\Big]$$
$$= 2\sum_{j,(\vec{\tau},\vec{\tau}')}(|\widetilde{\vec{\tau}}\rangle - |\widetilde{\vec{\tau}}'\rangle)(\langle\widetilde{\vec{\tau}}| - \langle\widetilde{\vec{\tau}}'|)_{[j,j+r]}, \quad \text{(B6)}$$

where $|\widetilde{\vec{\tau}}\rangle$ is the $(r+1)$-site composite-spin configuration that contains identical $\vec{\tau}$ placed on both the top and bottom legs of the ladder (similar to $|\tilde{\sigma}\rangle$). Note that several superoperators studied in the main text can be brought to this form [see, e.g., Eqs. (18) and (28)]. Equation (B6) can alternatively be expressed in terms of overall classical product configurations $\{|C\rangle\}$ as

$$\widehat{\mathcal{P}}_{\text{cl}|\text{comp}} = \sum_{\langle C,C'\rangle}(|\widetilde{C}\rangle - |\widetilde{C}'\rangle)(\langle\widetilde{C}| - \langle\widetilde{C}'|), \quad |\widetilde{C}\rangle := \left|\begin{array}{c} C \\ C \end{array}\right\rangle, \quad \text{(B7)}$$

which is precisely a Hamiltonian of the RK form of Eq. (B1), and the corresponding analysis of the ground states can be immediately reused with tildes playing a dummy role (since the $\widetilde{C}$ are in one-to-one correspondence with the C). Hence the ground states of $\widehat{\mathcal{P}}_{\text{cl}}$ are of the form of Eq. (B2) and the number of ground states is the number of Krylov subspaces of classical configurations connected by the moves $\vec{\tau} \leftrightarrow \vec{\tau}'$. In the operator language, noting that the composite spins $|\tilde{\sigma}\rangle$ map onto projectors $||\sigma\rangle\langle\sigma|)$, the ground state $|G^{(\mathcal{K})}\rangle \sim \sum_{C\in\mathcal{K}}|\widetilde{C}\rangle$ of $\widehat{\mathcal{P}}_{\text{cl}|\text{comp}}$ maps onto $|P_{\mathcal{K}})$, where $P_{\mathcal{K}}$ is the projector onto the Krylov subspace $\mathcal{K}$. In the case of classical conventional symmetries such as $U(1)$, these Krylov subspaces are equivalent to conventional symmetry quantum number sectors.

APPENDIX C: FORMAL SYMMETRIES OF SUPER-HAMILTONIANS

In this appendix, we discuss some additional formal properties of the constructed super-Hamiltonians viewed as ladder bilayer systems, as described in Sec. II C and Eq. (7). By construction, such super-Hamiltonians have symmetries that descend directly from the symmetries of the bond algebra terms $\{\widehat{H}_\alpha\}$: Using Eq. (7), it is easy to see that for any $\widehat{C} \in \mathcal{C}$ (i.e., the commutant of the algebra $\mathcal{A} = \langle\langle\{\widehat{H}_\alpha\}\rangle\rangle$), we have that $\widehat{C}_t \otimes \mathbb{1}_b$ and $\mathbb{1}_t \otimes \widehat{C}_b^T$ commute with the super-Hamiltonian. In studies of Lindbladians [155,156], these symmetries associated independently with each leg or layer are often referred to as *strong symmetries*, which commute with all the jump operators.

There are additional symmetries of the super-Hamiltonian that do not play significant roles in our analysis, e.g., the super-Hamiltonians also commute with an *antiunitary* operator composed of the exchange operation between the two legs and complex conjugation in the computational basis. While there seem to be no other obvious symmetries, for each $\widehat{C} \in \mathcal{C}$, the corresponding $|\widehat{C})$ is an exact zero-eigenvalue eigenstate of each $\widehat{\mathcal{L}}_\alpha$; hence the $|\widehat{C})(\widehat{C}'|$ with $\widehat{C}, \widehat{C}' \in \mathcal{C}$ are additional conserved quantities of the super-Hamiltonians, which can be understood by thinking about symmetries in terms of $\{\widehat{\mathcal{L}}_\alpha\}$. This character of the physical symmetries $|\mathcal{C})$ with respect to families of super-Hamiltonians resembles exact scar states [31] but here occurring in the superoperator space.

Furthermore, since the super-Hamiltonians contain only positive-semidefinite terms $\{\widehat{\mathcal{L}}_\alpha^\dagger\widehat{\mathcal{L}}_\alpha\}$, they have an additional "quantitative feature" that all $|\widehat{C})$ are exact ground states by the very construction. As we discuss below, in examples with conventional Abelian symmetries, this quantitative feature leads to the symmetries being broken in the ground states in a particular way that preserves some combinations of the symmetries; i.e., the super-Hamiltonians can be loosely viewed as being in a particular partial symmetry-breaking phase. Below, we will illustrate the formal symmetries and their fate in the ground states of the super-Hamiltonians for several conventional symmetries from Sec. III and we also discuss the extension of the concepts to the unconventional case of isolated QMBS.

1. Global $\mathbb{Z}_2$ symmetry

We start with the case of the global $\mathbb{Z}_2$ symmetry considered in Sec. III A 1. Here, the inherited "strong" symmetries can be expressed in terms of unitaries $U_t^{\mathbb{Z}_2} := \prod_j Z_{j;t}$ and $U_b^{\mathbb{Z}_2} := \prod_j Z_{j;b}$, which are $\mathbb{Z}_2$ symmetries associated with each individual leg. The two exact ground states $|G_\rightarrow\rangle$ and $|G_\leftarrow\rangle$ of Eq. (15) break the individual $\mathbb{Z}_2$ symmetries while preserving the combined symmetry $U_t^{\mathbb{Z}_2} U_b^{\mathbb{Z}_2}$:

$$U_\ell^{\mathbb{Z}_2} |G_\rightarrow\rangle = |G_\leftarrow\rangle , \quad U_\ell^{\mathbb{Z}_2} |G_\leftarrow\rangle = |G_\rightarrow\rangle , \quad \ell \in \{t, b\}; \qquad U_t^{\mathbb{Z}_2} U_b^{\mathbb{Z}_2} |G_a\rangle = |G_a\rangle , \quad a \in \{\rightarrow, \leftarrow\}. \tag{C1}$$

Alternatively, this $\mathbb{Z}_2$ symmetry breaking can be detected by checking if the ground-state space contains states from both $+1$ and -1 quantum numbers of the symmetry, which is satisfied by $U_t^{\mathbb{Z}_2}$ and $U_b^{\mathbb{Z}_2}$. This symmetry breaking is also detected by a local order parameter $X_{j;t}X_{j;b}$ (charged under both $U_t^{\mathbb{Z}_2}$ and $U_b^{\mathbb{Z}_2}$), which clearly has perfect long-range order in $|G_\rightarrow\rangle$ and $|G_\leftarrow\rangle$. On the other hand, the combined symmetry $U_t^{\mathbb{Z}_2} U_b^{\mathbb{Z}_2}$ in fact acts as an identity in the entire composite-spin sector, so it is not broken in this sector and hence in the ground states. While we can loosely say that this pattern of the $(\mathbb{Z}_2)_t \times (\mathbb{Z}_2)_b$ symmetry breaking down to a single remaining $\mathbb{Z}_2$ is responsible for the appearance of the 2D ground-state manifold, further structures or energetics in the super-Hamiltonian by construction are responsible for the specific ground-state wave functions and their exact degeneracy at finite system sizes, landing the system at a "fine-tuned point" inside of this particular symmetry-breaking phase.

2. Global $U(1)$ symmetry

We next consider the case of the global $U(1)$ symmetry from Sec. III A 2. Here, the inherited strong symmetries of the super-Hamiltonians are $U(1)$ symmetries associated with each individual leg, which can be implemented with unitaries $U_t^{U(1)}(\theta_t) := \exp(i\theta_t \sum_j Z_{j;t})$ and $U_b^{U(1)}(\theta_b) := \exp(i\theta_b \sum_j Z_{j;b})$ with $\theta_t, \theta_b \in [0, 2\pi)$. The ground states of the super-Hamiltonian form the ferromagnetic manifold of the composite spins and these $U(1)$ symmetries act nontrivially in this manifold, e.g., in the case of the basis $|Q_m^z\rangle \sim |\widetilde{F}_m^x\rangle$ of Eq. (19), and are hence broken. Alternatively, consider a different ground-state basis $\{|\widetilde{F}_m^z\rangle\}$, i.e., a composite-spin version of the ferromagnetic states in Eq. (A5) with polarization axis $\alpha = z$. The inherited $U(1)$ symmetries act on these as

$$U_\ell^{U(1)}(\theta_\ell)|\widetilde{F}_m^z\rangle = e^{i\theta_\ell(L-2m)}|\widetilde{F}_m^z\rangle, \quad \ell \in \{t, b\} \tag{C2}$$

and the presence of such nontrivial eigenvalues in the ground-state manifold signifies $U(1)$ symmetry breaking. On the other hand, the combined symmetry $U_t^{U(1)}(\theta)U_b^{U(1)}(-\theta)$ acts trivially in the composite-spin subspace and hence in the ground-state manifold; hence it is not broken, i.e., we have only partial breaking of the formal $U(1)_t \times U(1)_b$ symmetry. In physical terms, this symmetry breaking represents a quantum phase in which the charges from the top and bottom legs are bound and the resulting composite particle is condensed, while they are individually gapped.

To give a more precise description of the character of the condensate, we note that $|\widetilde{F}_m^z\rangle$ is an equal-weight superposition of all configurations with m bosonic composite particles represented by the $\tilde{\downarrow}$ composite spins in the "vacuum" of $\tilde{\uparrow}$ composite spins, and that such a wave function represents a "perfect superfluid"— i.e., a Bose-Einstein condensate (BEC)—of such bosons. Indeed, correlations of a local order parameter $\widetilde{S}_j^+ := S_{j;t}^+ S_{j;b}^+ = |\tilde{\uparrow}\rangle\langle\tilde{\downarrow}|$ [charged with respect to both $U(1)$] in terms of the boson density ρ read

$$\langle\widetilde{F}_m^z|\widetilde{S}_j^+\widetilde{S}_{j'}^-|\widetilde{F}_m^z\rangle = \frac{\binom{L-2}{m-1}}{\binom{L}{m}} = \frac{m(L-m)}{L(L-1)} \xrightarrow{L\to\infty} \rho(1-\rho), \quad \rho := \frac{m}{L}. \tag{C3}$$

The key observation is that this is nonzero for any $0 < \rho < 1$ and is independent of the separation between the points j and j', and this is true in any dimension. This is unlike generic superfluid wave functions, where the correlations would approach the nonzero limit in a power-law fashion in dimension $d > 1$. A related BEC versus generic superfluid difference also shows up in the excitation spectrum: quadratically dispersing excitations for our super-Hamiltonians as discussed in Sec. V B 1 versus linearly dispersing Goldstone modes in generic superfluids.

The perfect superfluid order revealed by all these perspectives, as well as the exact degeneracy among the ground states, are due to the further structures or energetics present in the super-Hamiltonian by construction, as discussed in the introductory part of this appendix.

3. Isolated QMBS

It is also curious to examine the fate of the inherited symmetries in the case of the isolated scar of Sec. III C 1. Here, the inherited symmetries can be viewed as highly nonlocal $\mathbb{Z}_2$ symmetries associated with each leg and specified by unitaries $U_t^{\text{iso}} := (\mathbb{1} - 2|\Phi\rangle\langle\Phi|)_t \otimes \mathbb{1}_b$ and $U_b^{\text{iso}} := \mathbb{1}_t \otimes (\mathbb{1} - 2|\Phi\rangle\langle\Phi|)_b$. Simple analysis shows that in the ground-state manifold spanned by the (nonorthonormal) basis of Eq. (39), both $U_t^{\rm iso}$ and $U_b^{\rm iso}$ take eigenvalues ± 1, while $U_t^{\rm iso} U_b^{\rm iso}$ acts trivially (true in the full composite-spin sector). This structure of the eigenvalues of the symmetries is similar to the super-Hamiltonians constructed in the case of the global $\mathbb{Z}_2$ symmetry considered in Sec. C 1 and one may loosely say that $U_t^{\rm iso}$ and $U_b^{\rm iso}$ are broken while $U_t^{\rm iso} U_b^{\rm iso}$ is preserved. However, in this case there is no local order parameter that would have nontrivial charge under these symmetries and that could detect this "symmetry breaking," which is then not a very useful concept here.

APPENDIX D: EXTRANEOUS FEATURES OF SPECIFIC SUPER-HAMILTONIANS

Many of the super-Hamiltonians in the main text have some additional extraneous features, sometimes allowing full or partial solvability beyond the expected exact ground states. These features in fact depend on the specific choice of the bond-algebra generators $\{\widehat{H}_\alpha\}$ used to define the super-Hamiltonian $\widehat{\mathcal{P}}$ in Eq. (5) and in this appendix we comment on this dependence.

Most importantly, for a fixed bond algebra $\mathcal{A}$, the formal superoperator-space symmetries of the set of the Liouvillian superoperators $\{\widehat{\mathcal{L}}_\alpha\}$ can depend on the choice of the generators $\{\widehat{H}_\alpha\}$. That is, for different sets of generators of $\mathcal{A} = \langle\langle\{\widehat{H}_\alpha\}\rangle\rangle = \langle\langle\{\widehat{H}'_\beta\}\rangle\rangle$, the corresponding sets of the Liouvillian superoperators $\{\widehat{\mathcal{L}}_\alpha\}$ and $\{\widehat{\mathcal{L}}'_\beta\}$ can generate different superoperator algebras and their commutants can be different and larger than the set of formal symmetries discussed in Appendix C, which are always present. Examples of this include instances in which the composite-spin subspace is invariant under the action of the super-Hamiltonians (many cases in the main text), the appearance of the $SU(2)$ symmetry in the case of $\widehat{\mathcal{P}}_{U(1)|\text{comp}}$ in Eq. (18), or the appearance of the $SU(q^2)$ symmetry in the case of $\widehat{\mathcal{P}}_{SU(q)}$ in Eq. (22). Furthermore, the couplings with which $\{\widehat{\mathcal{L}}_\alpha^2\}$ enter in $\widehat{\mathcal{P}}$ also matter for some extraneous features as well as for lattice symmetries of the super-Hamiltonians. Nevertheless, for the problem of finding the commutant of $\mathcal{A}$, we are guaranteed that the commutant is the exact ground-state manifold of any super-Hamiltonian constructed from any set of generators of $\mathcal{A}$, so there is no issue here.

On the other hand, one may worry whether the low-energy spectra of such specific "more symmetric" super-Hamiltonians correspond to slow dynamical modes in more generic systems. We expect that this is true, namely, that possible additional features in the super-Hamiltonians do not change the qualitative character of the low-energy excitations, which we think is tied to the structure of the exact ground-state space, and we demonstrate this in some cases in the main text, while here we provide more general comments.

First, in all cases, we can progressively suppress the additional features by adding more terms from the bond algebra (e.g., combining different sets of generators) and this would add more positive-semidefinite terms to the super-Hamiltonian. By construction, the exact ground-state manifold would remain unchanged, while all excitation energies would only increase. In particular, the gapped cases would remain gapped (which we can then consider as a proof of a generic gap), while in the gapless cases the presented low excitation energies of specific super-Hamiltonians would at least provide exact lower bounds on the excitation energies of the modified super-Hamiltonians. We further expect that the corresponding specific eigenstates could be used as trial states and would also provide variational upper bounds on the excitation energies of the modified super-Hamiltonians that would retain the same qualitative character as before, e.g., would give similar dispersion laws. In the main text, we have shown evidence for this in the case of $\widehat{\mathcal{P}}_{U(1)}$. In some cases, the additional features allowing solvability beyond the exact ground states are like the integrability of the Heisenberg chain. It is well established that low-energy excitations of such integrable models capture qualitatively the physics of more generic models in the same phase.

Finally, our main confidence that the additional features are not qualitatively important comes from the fact that the specific super-Hamiltonians arise naturally as descriptions of properties of concrete Brownian circuits that by themselves do not look fine tuned, e.g., each random instance is not solvable or special in any way. The additional features in the super-Hamiltonians can be loosely thought of as coming from some choices of taking the simplest generators as well as convenient distributions of the random couplings, and such choices should not affect qualitative long-time hydrodynamic properties in the Brownian circuits.

APPENDIX E: DETAILS ON QUANTUM MANY-BODY SCAR SUPER-HAMILTONIANS

In this appendix, we provide some details on the ground states of the super-Hamiltonians $\widehat{\mathcal{P}}_{\text{scar}}$ that appear in the study of QMBS.

1. Isolated QMBS

In the case of a single isolated QMBS given by $|\Phi\rangle = |\uparrow\uparrow\ \ldots\ \uparrow\rangle$, we can choose $R_{[j]} := R_j = (1-\sigma_j^z)/2 = |\downarrow\rangle\langle\downarrow|_j$. It is easy to see that the common kernel of the R_j contains a single state $|\Phi\rangle$. We start with the bond algebra

$$\mathcal{A}_{\text{iso}} := \langle\langle\{R_j\sigma_{j+1}^\alpha, \sigma_j^\alpha R_{j+1}\}\rangle\rangle, \tag{E1}$$

where for simplicity we choose PBCs for the bond-algebra generators, although similar results carry forward for the OBC case. The full super-Hamiltonian of Eq. (32) reads

$$\begin{aligned}\widehat{\mathcal{P}}_{\text{iso}} &= 8\sum_j (R_{j;t} - R_{j;b})^2 + 8\sum_j \big(R_{j;t}R_{j;b}[1-|\iota\rangle\langle\iota|]_{j+1} + [1-|\iota\rangle\langle\iota|]_j R_{j+1;t}R_{j+1;b}\big)\\ &= 2\sum_j(\sigma_{j;t}^z - \sigma_{j;b}^z)^2 + 8\sum_j\big(|\downarrow\rangle\langle\downarrow|_{j;t}|\downarrow\rangle\langle\downarrow|_{j;b}[1-|\iota\rangle\langle\iota|]_{j+1} + [1-|\iota\rangle\langle\iota|]_j|\downarrow\rangle\langle\downarrow|_{j+1;t}|\downarrow\rangle\langle\downarrow|_{j+1;b}\big).\end{aligned} \tag{E2}$$

Note that the $(\sigma_{j;t}^z - \sigma_{j;b}^z)^2$ terms enforce that the ground state is in the sector of composite spins defined in Eq. (11), similar to the $\mathbb{Z}_2$ and $U(1)$ cases.

a. Ground states

We do not need to use the full structure of the super-Hamiltonian to obtain the ground states. According to Eq. (35), the ground-state space satisfies

$$\begin{aligned}&|\downarrow\rangle\langle\downarrow|_{j;\ell}[1-|\iota\rangle\langle\iota|]_{j+1}|\Psi\rangle = 0, \quad [1-|\iota\rangle\langle\iota|]_j|\downarrow\rangle\langle\downarrow|_{j+1;\ell}|\Psi\rangle = 0, \quad \ell\in\{t,b\}\\ &\implies |\tilde{\downarrow},\widetilde{\leftleftarrows}\rangle\langle\tilde{\downarrow},\widetilde{\leftleftarrows}|_{j,j+1}|\Psi\rangle = 0, \quad |\widetilde{\leftleftarrows},\tilde{\downarrow}\rangle\langle\widetilde{\leftleftarrows},\tilde{\downarrow}|_{j,j+1}|\Psi\rangle = 0,\end{aligned} \tag{E3}$$

where in the second line we have expressed the conditions in terms of composite spins defined in Eqs. (11) and (14), and in replacing $[1-|\iota\rangle\langle\iota|]$ by $|\widetilde{\leftleftarrows}\rangle\langle\widetilde{\leftleftarrows}|$ we have used that the ground states $|\Psi\rangle$ are in the composite-spin sector. These conditions highly constrain the structure of $|\Psi\rangle$. In particular, suppose that $|\Psi\rangle$ is decomposed as

$$|\Psi\rangle = \sum_\alpha |u^\alpha\rangle_{[j,j+1]}\otimes|v^\alpha\rangle_{\text{rest}}, \tag{E4}$$

where the supports of each part of the wave function along the ladder are indicated in the subscript, with "rest" denoting the complement to $[j,j+1]$, and $\{|v^\alpha\rangle_{\text{rest}}\}$ form a linearly independent set. Such a decomposition always exists, Schmidt decomposition being one example, but we will only require the linear independence of $\{|v^\alpha\rangle_{\text{rest}}\}$ and not orthonormality. The conditions of Eq. (E3) and the linear independence of $\{|v_{\text{rest}}^\alpha\rangle\}$ imply that both of the following should hold:

$$\begin{aligned}&|\tilde{\downarrow},\widetilde{\leftleftarrows}\rangle\langle\tilde{\downarrow},\widetilde{\leftleftarrows}|_{j,j+1}|u^\alpha\rangle_{j,j+1} = 0, \quad |\widetilde{\leftleftarrows},\tilde{\downarrow}\rangle\langle\widetilde{\leftleftarrows},\tilde{\downarrow}|_{j,j+1}|u^\alpha\rangle_{j,j+1} = 0\\ &\implies |u^\alpha\rangle_{j,j+1}\in\text{span}\{|\tilde{\uparrow},\tilde{\uparrow}\rangle_{j,j+1}, |\widetilde{\rightrightarrows},\widetilde{\rightrightarrows}\rangle_{j,j+1}\}.\end{aligned} \tag{E5}$$

Hence, $|\Psi\rangle$ can be written as

$$|\Psi\rangle = |\tilde{\uparrow},\tilde{\uparrow}\rangle_{j,j+1}\otimes|\Upsilon\rangle_{\text{rest}} + |\widetilde{\rightrightarrows},\widetilde{\rightrightarrows}\rangle_{j,j+1}\otimes|\Theta\rangle_{\text{rest}}, \tag{E6}$$

with some states, $|\Upsilon\rangle_{\text{rest}}$ and $|\Theta\rangle_{\text{rest}}$, on the complement to $[j,j+1]$. Moving on to requiring Eq. (E5) on the next pair of sites $[j+1,j+2]$, since $|\tilde{\uparrow}\rangle_j$ and $|\widetilde{\rightrightarrows}\rangle_j$ are linearly independent, we can apply a similar argument independently to $|\tilde{\uparrow}\rangle_{j+1}\otimes|\Upsilon\rangle_{\text{rest}}$ and $|\widetilde{\rightrightarrows}\rangle_{j+1}\otimes|\Theta\rangle_{\text{rest}}$. For example, we obtain that

$$\begin{aligned}&|\tilde{\uparrow}\rangle_{j+1}\otimes|\Upsilon\rangle_{\text{rest}} = |\tilde{\uparrow},\tilde{\uparrow}\rangle_{j+1,j+2}\otimes|\Upsilon'\rangle_{\text{rest}'} + |\widetilde{\rightrightarrows},\widetilde{\rightrightarrows}\rangle_{j+1,j+2}\otimes|\Theta'\rangle_{\text{rest}'}\\ &\implies |\tilde{\uparrow}\rangle_{j+1}\otimes|\Upsilon\rangle_{\text{rest}} = |\tilde{\uparrow},\tilde{\uparrow}\rangle_{j+1,j+2}\otimes|\Upsilon'\rangle_{\text{rest}'},\end{aligned} \tag{E7}$$

where we have used linear independence of $|\tilde{\uparrow}\rangle_{j+1}$ and $|\widetilde{\rightrightarrows}\rangle_{j+1}$ and "rest′" denotes the complement to $[j,j+1,j+2]$. In all, requiring Eq. (E5) on all pairs of neighboring sites, we can conclude that $|\Psi\rangle$ is spanned by $|\tilde{\uparrow},\tilde{\uparrow},\ldots,\tilde{\uparrow}\rangle$ and

$|\tilde{\rightrightarrows}, \tilde{\rightrightarrows}, \ldots, \tilde{\rightrightarrows}\rangle$, which correspond to operators $|\mathbb{1})$ and $||\Phi\rangle\langle\Phi|)$. In the original language, this means that the commutant of Eq. (E1) is given by

$$\mathcal{C}_{\text{iso}} = \langle\langle|\Phi\rangle\langle\Phi|\rangle\rangle, \tag{E8}$$

where the $\mathbb{1}$ is implicit in the notation $\langle\langle\cdots\rangle\rangle$. Note that while we have included two types of bond-algebra generators in Eq. (E1), it is usually possible to choose a subset of them and still recover the same commutant of Eq. (E8), although the analytical analysis might not be so straightforward.

b. Gap and low-energy excitations

We then study the low-energy spectrum of $\widehat{\mathcal{P}}_{\text{iso}}$ of Eq. (E2). Since $(\sigma^z_{j;t} - \sigma^z_{j;b})^2$ commutes with all other terms in the Hamiltonian, configurations of composite spins form a closed subspace for $\widehat{\mathcal{P}}_{\text{iso}}$. The violation of such a composite spin costs a constant amount of energy; hence we can work in the space of composite spins to determine if $\widehat{\mathcal{P}}_{\text{iso}}$ may have a smaller gap. Restricted to the space of composite spins, the action of $\widehat{\mathcal{P}}_{\text{iso}}$ reads

$$\begin{aligned}&\widehat{\mathcal{P}}_{\text{iso|comp}} = 2\sum_j[(1-\widetilde{Z}_j)(1-\widetilde{X}_{j+1}) + (1-\widetilde{X}_j)(1-\widetilde{Z}_{j+1})],\\ &\widetilde{Z} := |\tilde{\uparrow}\rangle\langle\tilde{\uparrow}| - |\tilde{\downarrow}\rangle\langle\tilde{\downarrow}|, \quad \widetilde{X} := |\tilde{\uparrow}\rangle\langle\tilde{\downarrow}| + |\tilde{\downarrow}\rangle\langle\tilde{\uparrow}|.\end{aligned} \tag{E9}$$

This can be simplified to

$$\begin{aligned}\widehat{\mathcal{P}}_{\text{iso|comp}} &= 2\sum_j[2 - (\widetilde{X}_j + \widetilde{Z}_j + \widetilde{X}_{j+1} + \widetilde{Z}_{j+1}) + \widetilde{X}_j\widetilde{Z}_{j+1} + \widetilde{Z}_j\widetilde{X}_{j+1}]\\ &= 2\sum_j[2 - \sqrt{2}(\widetilde{X}'_j + \widetilde{X}'_{j+1}) + \widetilde{X}'_j\widetilde{X}'_{j+1} - \widetilde{Y}'_j\widetilde{Y}'_{j+1}], \quad \widetilde{X}'_j := \frac{\widetilde{X}_j + \widetilde{Z}_j}{\sqrt{2}}, \quad \widetilde{Y}'_j := \frac{\widetilde{X}_j - \widetilde{Z}_j}{\sqrt{2}},\end{aligned} \tag{E10}$$

where in the second step we have performed a basis transformation for the composite spins. For even system size and PBCs, this further maps to

$$\widehat{\mathcal{P}}_{\text{iso|comp}} = 4\left[L - \sqrt{2}\sum_j\widetilde{X}'_j + \frac{1}{2}\sum_j(\widetilde{X}'_j\widetilde{X}'_{j+1} + \widetilde{Y}'_j\widetilde{Y}'_{j+1})\right], \tag{E11}$$

where we have used the bipartiteness of the lattice to transform spins on even sites such that $\widetilde{Y}'_j \to (-1)^j\widetilde{Y}'_j$ by rotating around the $\widetilde{X}'$ axis. Note that this is simply the antiferromagnetic XX model with a longitudinal field with the specific value or, equivalently, the transverse-field Ising model with nearest-neighbor interactions with the specific field and interaction values, such that the product states $|\tilde{\rightrightarrows}\ \tilde{\rightrightarrows}\ \cdots\ \tilde{\rightrightarrows}\rangle$ and $|\tilde{\uparrow}\ \tilde{\uparrow}\ \cdots\ \tilde{\uparrow}\rangle$ (in the original $\tilde{X}, \tilde{Z}$ axes) are exact ground states.

This model has in fact been studied in the earlier literature. For example, Eq. (E11) is known to be dual to the well-studied axial next-nearest-neighbor Ising (ANNNI) models and in the phase diagram obtained in Ref. [157], this appears to be in the gapped ferromagnetic phase. Our own exact diagonalization study of the specific model in Eq. (E11) in PBCs indeed finds that there are two exactly degenerate ground states that spontaneously break the $\mathbb{Z}_2$ symmetry generated by $\prod_j \widetilde{X}'_j$, separated by a gap between 0.5 and 1. from the rest of the spectrum (thus, the lowest-energy excitation of $\widehat{\mathcal{P}}_{\text{iso}}$ indeed lies in the composite-spin sector). Moreover, the Hamiltonian of Eq. (E10) exactly maps onto a frustration-free model that lies on the so-called Peschel-Emery line [86,88]. A relatively recent work [87] proves that the OBC version of the Hamiltonian in Eq. (E10) is gapped, by performing a Jordan-Wigner mapping to an interacting Majorana chain in OBCs and exhibiting a deformation path to a gapped free-fermion Hamiltonian without closing a gap. The same argument can also be carried out directly in the spin model in the PBC system as well.

2. Tower of QMBSs

We now illustrate the example of the ferromagnetic tower of QMBSs. Consider $R_{[j]} := R_{j,j+1} = \frac{1}{4} - \vec{S}_j \cdot \vec{S}_{j+1} = \frac{1}{2}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)(\langle\uparrow\downarrow| - \langle\downarrow\uparrow|)_{j,j+1} = \frac{1}{2}(1 - P^{(2)}_{j,j+1})$, where the target scar manifold contains the entire ferromagnetic tower of QMBSs $|\Phi_n\rangle \sim (S^-_{\text{tot}})^n |\uparrow\uparrow \ \ldots \ \uparrow\rangle$, $0 \le n \le L$. We start with the bond algebra similar to Eq. (E1),

$$\mathcal{A}_{\text{tower}} := \langle\langle\{R_{j,j+1}\sigma^\alpha_{j+2}, \sigma^\alpha_{j-1}R_{j,j+1}\}\rangle\rangle, \tag{E12}$$

and for simplicity choose PBCs for the bond-algebra generators. Any operator constructed out of the generators of $\mathcal{A}_{\text{tower}}$ contains the ferromagnetic multiplet as degenerate QMBS [31].

The full super-Hamiltonian in this case reads

$$\widehat{\mathcal{P}}_{\text{tower}} = 8\sum_j (R_{[j,j+1];t} - R_{[j,j+1];b})^2 + 8\sum_j \big(R_{[j,j+1];t}R_{[j,j+1];b}[1 - |\iota\rangle\langle\iota|]_{j+1} + [1 - |\iota\rangle\langle\iota|]_j R_{[j+1,j+2];t}R_{[j+1,j+2];b}\big). \tag{E13}$$

Since $(R_{[j,j+1];t} - R_{[j,j+1];b})^2 = \frac{1}{2}(1 - P^{(2)}_{[j,j+1],t}P^{(2)}_{[j,j+1],b})$, the ground states must be symmetric under exchange of the states on the nearby rungs:

$$P^{\text{rung}}_{[j,j+1]}|\Psi\rangle = |\Psi\rangle \iff [1 - P^{\text{rung}}_{[j,j+1]}]|\Psi\rangle = 0, \quad P^{\text{rung}}_{[j,j+1]} := P^{(2)}_{[j,j+1];t}P^{(2)}_{[j,j+1];b}. \tag{E14}$$

Furthermore, according to Eq. (35), the ground-state space satisfies

$$R_{[j,j+1];\ell}[1 - |\iota\rangle\langle\iota|]_{j+2}|\Psi\rangle = 0, \quad [1 - |\iota\rangle\langle\iota|]_j R_{[j+1,j+2];\ell}|\Psi\rangle = 0, \quad \ell \in \{t, b\}. \tag{E15}$$

As in the isolated QMBSs, these conditions highly constrain the structure of the wave function $|\Psi\rangle$. Generalizing Eq. (E4), given a region A and its complement A^c, suppose that we decompose $|\Psi\rangle$ as

$$|\Psi\rangle = \sum_\alpha |u^\alpha\rangle_A \otimes |v^\alpha\rangle_{A^c}, \tag{E16}$$

where the supports of each part of the wave function are indicated in the subscript and the $\{|v^\alpha\rangle_{A^c}\}$ are linearly independent. Then, if $|\Psi\rangle$ is annihilated by some operators acting entirely within the region A, it follows that each $|u^\alpha\rangle_A$ is annihilated by the same operators and we can write

$$|\Psi\rangle = \sum_\gamma |e^\gamma\rangle_A \otimes |w^\gamma\rangle_{A^c}, \tag{E17}$$

where $\{|e^\gamma\rangle_A\}$ is a complete basis in the common kernel of the annihilators acting within A and the $\{|w^\gamma\rangle_{A^c}\}$ are some new states on A^c that are not required to be linearly independent. The above are precise statements about the constraints on local "parts" of such a $|\Psi\rangle$ and are used repeatedly (often implicitly) below.

Before proceeding with proofs, we introduce some shorthand notation. On a segment $[j, j+r]$, we denote the common kernel of all $\{R_{[k,k+1];\ell}\}$ with support completely within the segment as $K_{[j,j+r]}$. It is easy to see that it is spanned by $\{|\Phi_m\rangle_{[j,j+r];t} \otimes |\Phi_n\rangle_{[j,j+r];b}\}$, where $\{|\Phi_m\rangle_{[j,j+r]}\}$ is the full QMBS set in the original problem on the segment $[j, j+r]$. We further denote a complete basis of $K_{[j,j+r]}$ as $\{|\Gamma^a\rangle_{[j,j+r]}\}$. On the same segment, the common kernel of all $\{[1 - |\iota\rangle\langle\iota|]_k\}$ acting within the segment is spanned by a single state, for which we introduce a shorthand $|\iota\rangle_{[j,j+r]} := \otimes_{k=j}^{j+r}|\iota\rangle_k$.

Lemma 1. The common kernel of $R_{[j,j+1];\ell}$ and $[1 - |\iota\rangle\langle\iota|]_k$ is trivial for any $\ell \in \{t, b\}$ if $k \in \{j, j+1\}$. As corollaries, $R_{[j,j+1]}|\iota\rangle_{[j,j+1]} \neq 0$ and $|\iota\rangle_{[j,j+r]} \notin K_{[j,j+r]}$ for any r.

Proof. We illustrate the proof for $k = j$; the proof for $k = j + 1$ follows in the same way. Any state $|v\rangle_{[j,j+1]}$ on rungs j and $j + 1$ in the kernel of $[1 - |\iota\rangle\langle\iota|]_j$ can, without loss of generality, be expressed as

$$|v\rangle_{[j,j+1]} = |\iota\rangle_j \otimes |\uparrow\rangle_{j+1;\ell} \otimes |w^{\uparrow}\rangle_{j+1;\bar{\ell}} + |\iota\rangle_j \otimes |\downarrow\rangle_{j+1;\ell} \otimes |w^{\downarrow}\rangle_{j+1;\bar{\ell}}, \tag{E18}$$

where $|w^{\uparrow}\rangle_{j+1;\bar{\ell}}$ and the $|w^{\downarrow}\rangle_{j+1;\bar{\ell}}$ are some states on the site $(j + 1; \bar{\ell})$, where $\bar{\ell}$ is the complement of ℓ in $\{t, b\}$. The action of $R_{[j,j+1];\ell}$ on $|v\rangle_{[j,j+1]}$ of this form reads

$$R_{[j,j+1];\ell}\, |v\rangle_{[j,j+1]} = \frac{1}{2\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)_{[j,j+1];\ell} \otimes (-|\downarrow\rangle_{j;\bar{\ell}} \otimes |w^{\uparrow}\rangle_{j+1,\bar{\ell}} + |\uparrow\rangle_{j;\bar{\ell}} \otimes |w^{\downarrow}\rangle_{j+1,\bar{\ell}}). \tag{E19}$$

It is then easy to see that $R_{[j,j+1];\ell}$ vanishes on $|v\rangle_{[j,j+1]}$ only if $|w^{\uparrow}\rangle_{j+1;\bar{\ell}} = |w^{\downarrow}\rangle_{j+1;\bar{\ell}} = 0$, which in turn means that $|v\rangle_{[j,j+1]} = 0$.

It then directly follows that $R_{[j,j+1]}\, |\iota\rangle_{[j,j+1]} \neq 0$, since $|\iota\rangle_{[j,j+1]}$ is clearly in the kernel of $[1 - |\iota\rangle\langle\iota|_j]$. This can also be directly verified; a simple calculation gives $R_{[j,j+1];t}\, |\iota\rangle_{[j,j+1]} = \frac{1}{4}\, |S\rangle_{[j,j+1];t} \otimes |S\rangle_{[j,j+1];b} \neq 0$, where $|S\rangle$ is a singlet state for two spins involved. This also means that $|\iota\rangle_{[j,j+r]} \notin K_{[j,j+r]}$, and $|\iota\rangle_{[j,j+r]}$ is linearly independent from $\{|\Gamma^a\rangle_{[j,j+r]}\}$. ∎

Lemma 2. All the states satisfying Eqs. (E14) and (E15) in region $[j, j + r]$ are in span$\{|\Gamma^a\rangle_{[j,j+r]}, |\iota\rangle_{[j,j+r]}\}$.

Proof. Clearly, $\{|\Gamma^a_{[j,j+r]}\rangle\}$ and $|\iota\rangle_{[j,j+r]}$ are annihilated by all $\{1 - P^{\text{rung}}_{k,k+1}\}$ and $\{R_{[k,k+1];\ell}(1 - |\iota\rangle\langle\iota|)_{k+2}\}$ acting within the segment; hence they satisfy Eqs. (E14) and (E15). We now show that any other state $|\Psi\rangle$ satisfying these conditions is spanned by the above states.

Starting with $r = 1$, i.e., two rungs of the ladder at $j, j + 1$, we show that just requiring annihilation by $[1 - P^{\text{rung}}_{[j,j+1]}]$ enforces $|\Psi\rangle$ to lie in the span of $K_{[j,j+1]}$ and $|\iota\rangle_{[j,j+1]}$. The kernel of $[1 - P^{\text{rung}}_{[j,j+1]}]$ is ten-dimensional, consisting of all states that are symmetric under the rung exchanges, and its complete basis is $\{|T_m\rangle_{[j,j+1];t} \otimes |T_n\rangle_{[j,j+1];b}\,, m, n \in \{0, \pm 1\}\} \cup |S\rangle_{[j,j+1];t} \otimes |S\rangle_{[j,j+1];b}$, where $|T_m\rangle$ and $|S\rangle$ are, respectively, triplet and singlet states for two spins involved. For two spins, the triplet space $\{|T_m\rangle_{[j,j+1]}\}$ is the same as $\{|\Phi_m\rangle_{[j,j+1]}\}$, so the nine states $\{|T_m\rangle_{[j,j+1];t} \otimes |T_n\rangle_{[j,j+1];b}\}$ can be replaced by $\{|\Gamma^a\rangle_{[j,j+1]}\}$. Since $|\iota\rangle_{[j,j+1]}$ is also annihilated by $[1 - P^{\text{rung}}_{[j,j+1]}]$ and is linearly independent of $\{|\Gamma_a\rangle_{[j,j+1]}\}$, it can replace $|S\rangle_{j,j+1;t} \otimes |S\rangle_{j,j+1;b}$ in the constructed complete basis of the kernel of $1 - P^{\text{rung}}_{j,j+1}$.

Moving to $r = 2$, for the appropriate parts of $|\Psi\rangle$, we can write

$$|\Psi\rangle_{[j,j+2]} = \sum_a |\Gamma^a\rangle_{[j,j+1]} \otimes |v^a\rangle_{j+2} + |\iota\rangle_{[j,j+1]} \otimes |w\rangle_{j+2}\,, \tag{E20}$$

where $\{|v^a\rangle_{j+2}\}$ and $\{|w\rangle_{j+2}\}$ are some states on the rung at $j + 2$. Now consider annihilation by $R_{[j,j+1];\ell}(1 - |\iota\rangle\langle\iota|)_{j+2}$. Since $R_{[j,j+1];\ell}$ annihilates all $\{|\Gamma^a\rangle_{[j,j+1]}\}$, we deduce that

$$R_{[j,j+1];\ell}(1 - |\iota\rangle\langle\iota|)_{j+2}\ \ |\iota\rangle_{[j,j+1]} \otimes |w\rangle_{j+2} = 0. \tag{E21}$$

However, since $R_{[j,j+1];\ell}\, |\iota\rangle_{[j,j+1]} \neq 0$ following Lemma 1, we have $|w\rangle_{j+2} = c\, |\iota\rangle_{j+2}$; hence

$$\sum_a |\Gamma^a\rangle_{[j,j+1]} \otimes |v^a\rangle_{j+2} = |\psi\rangle_{[j,j+2]} - c\, |\iota\rangle_{[j,j+2]}\,. \tag{E22}$$

Since $|\psi\rangle_{[j,j+2]}$ must be symmetric under the exchange of rungs $j + 1$ and $j + 2$ according to Eq. (E14), and $|\iota\rangle_{[j,j+2]}$ clearly is, the left-hand side should be symmetric under this and all other rung exchanges in $[j, j + 2]$. Since the left-hand side is annihilated by $R_{[j,j+1];\ell}$, it must then also be annihilated by $R_{j+1,j+2;\ell}$, i.e., it belongs to the span of $\{|\Gamma_{a'}\rangle_{[j,j+2]}\}$. This completes the proof for the segment $[j, j + 2]$.

Essentially the same steps can then be used for an inductive proof going from $[j, j + r]$ to $[j, j + r + 1]$, considering annihilation by $R_{[j+r-1,j+r];\ell}(1 - |\iota\rangle\langle\iota|)_{j+r+1}$ to peel off the $|\iota\rangle_{[j,j+r+1]}$ contribution, and then for the remainder deducing the symmetry under the rung exchange at $j + r$ and $j + r + 1$ and proving that it belongs to $K_{[j,j+r+1]}$. ∎

Hence the only states on the full chain satisfying Eqs. (E14) and (E15) are

$$|\Psi\rangle = |\Phi_m\rangle_t \otimes |\Phi_n\rangle_b = ||\Phi_m\rangle\langle\Phi_n|) \quad \text{or} \quad |\Psi\rangle = \otimes_{j=1}^{L} |\iota\rangle_j = \frac{1}{2^{\frac{L}{2}}}|\mathbb{1}). \tag{E23}$$

In the original language, this means that the commutant of Eq. (E1) is given by

$$\mathcal{C}_{\text{tower}} = \langle\langle\{|\Phi_m\rangle\langle\Phi_n|\}\rangle\rangle, \tag{E24}$$

where the $\mathbb{1}$ is implicit in the notation $\langle\langle\cdots\rangle\rangle$.

APPENDIX F: EIGENSTATES OF THE $t-J_z$ SUPER-HAMILTONIAN

In this appendix, we provide details on the eigenstates of the super-Hamiltonian $\widehat{\mathcal{P}}_{t-J_z}$ and on the computation of the weights of the operator $|S_j^z)$ on these eigenstates. As discussed in Sec. V B 2, understanding the cumulative weight function $\Omega_{S_j^z}(E)$ is key to understanding the late-time behavior of the ensemble-averaged autocorrelation function $\overline{C_{S_j^z}(t)}$. However, we are only able to analytically determine the weights corresponding to the ground states and the "spin-wave" excited states, both of which ultimately vanish in the thermodynamic limit. It is then crucial to include contributions from the higher excited states to analytically reproduce the results of Fig. 2; hence we also discuss the general setup for this weight calculation, although we are only able to implement this numerically in general.

1. General setup

To determine the low-energy excitations of $\widehat{\mathcal{P}}_{t-J_z}$ that have a nonzero overlap on $|S_j^z)$, it is sufficient to work in the composite-spin sector on the ladder (for discussions on this, see Sec. V B 2). A convenient computational basis in terms of the composite spins $\{\tilde{\uparrow}, \tilde{\downarrow}, \tilde{0}\}$ defined in Eq. (27) is of the form

$$\{|\tilde{\tau}_{j_1}^{(1)}\tilde{\tau}_{j_2}^{(2)}\cdots\tilde{\tau}_{j_m}^{(m)}\rangle\}, \quad \tilde{\tau}^{(q)} \in \{\tilde{\rightleftarrows}, \tilde{\leftrightarrows}\}, \quad |\tilde{\rightleftarrows}\rangle := \frac{|\tilde{\uparrow}\rangle + |\tilde{\downarrow}\rangle}{\sqrt{2}}, \quad |\tilde{\leftrightarrows}\rangle := \frac{|\tilde{\uparrow}\rangle - |\tilde{\downarrow}\rangle}{\sqrt{2}}, \tag{F1}$$

where the subscripts indicate the positions of the $\tilde{\tau}^{(q)}$ and the rest of the sites are assumed to be occupied by the $\tilde{0}$. Note that we are working in the basis with the spins $\tilde{\rightleftarrows}$ and $\tilde{\leftrightarrows}$ instead of $\tilde{\uparrow}$ and $\tilde{\downarrow}$, since we are ultimately interested in the overlap with the operator $|S_j^z)$, which maps onto the composite spin $|\tilde{\leftrightarrows}\rangle_j$ on the ladder. The ground-state space of $\widehat{\mathcal{P}}_{t-J_z|\text{comp}}$, shown in Eq. (29), in this basis is spanned by the equal-weight superpositions of the states with fixed pattern $\tau^{(1)}, \tau^{(2)}, \ldots, \tau^{(m)}$, i.e.,

$$|G^{\tau^{(1)}\cdots\tau^{(m)}}\rangle = \sum_{j_1<\cdots<j_m} |\tilde{\tau}_{j_1}^{(1)}\cdots\tilde{\tau}_{j_m}^{(m)}\rangle. \tag{F2}$$

In the operator language, these ground states are the "words" defined in Appendix B in Ref. [25], which have been shown to form an orthogonal basis for the commutant algebra $\mathcal{C}_{t-J_z}$ for OBCs and have been used to compute exact Mazur bounds.

We then study the excited states of $\widehat{\mathcal{P}}_{t-J_z|\text{comp}}$ in the computational basis of Eq. (F1). We start with an eigenstate of the Heisenberg model H_{Heis} of Appendix A with m $\downarrow$, that has the form

$$|\lambda; m\rangle = \sum_{j_1<j_2<\cdots<j_m} C^{\lambda}_{j_1,j_2,\ldots,j_m} |\downarrow_{j_1}\downarrow_{j_2}\cdots\downarrow_{j_m}\rangle, \tag{F3}$$

where λ denotes the energy and m denotes the number of $\downarrow$ in the eigenstate, the subscripts of the $\downarrow$ denote their positions, and the rest of the sites are assumed to be $\uparrow$. Utilizing the mapping between H_{Heis} and $\widehat{\mathcal{P}}_{t-J_z|\text{comp}}$ and summarized in Eq. (61), we can write down 2^m degenerate eigenstates of the latter corresponding to each eigenstate $|\lambda; m\rangle$ of the former;

these are of the form

$$|\lambda;\tau^{(1)}\cdots\tau^{(m)}\rangle=\sum_{j_1<\cdots<j_m}C^{\lambda}_{j_1,\ldots,j_m}|\tilde{\tau}^{(1)}_{j_1}\cdots\tilde{\tau}^{(m)}_{j_m}\rangle,\quad \tau^{(\ell)}\in\{\rightarrow,\leftarrow\},\tag{F4}$$

labeled by the fixed pattern $\tau^{(1)},\tau^{(2)},\ldots,\tau^{(m)}$. In all, given that there are $\binom{L}{m}$ Heisenberg eigenstates with $m\downarrow$, we obtain a total of $\sum_{m=0}^{L}2^m\binom{L}{m}=3^L$ eigenstates of $\widehat{\mathcal{P}}_{t-J_z|\text{comp}}$, which, as expected, covers the entire Hilbert space of the composite spins.

To compute the overlap between these eigenstates and the operator $|S^z_j)$, we note that this operator maps onto the following state on the ladder:

$$|S^z_j):=|\zeta_j\rangle=\prod_{k=1}^{j-1}(\sqrt{2}\,|\widetilde{\rightrightarrows}\rangle_k+|\tilde{0}\rangle_k)\otimes\sqrt{2}\,|\widetilde{\leftleftarrows}\rangle_j\otimes\prod_{k=j+1}^{L}(\sqrt{2}\,|\widetilde{\rightrightarrows}\rangle_k+|\tilde{0}\rangle_k).\tag{F5}$$

The only configurations $|\tilde{\tau}^{(1)}_{j_1}\cdots\tilde{\tau}^{(m)}_{j_m}\rangle$ with which $|\zeta_j\rangle$ has nonzero overlap are the ones where $j_\ell=j$ for some $1\le\ell\le m$, with $\tau^{(\ell)}=\leftarrow$ and $\tau^{(k)}=\rightarrow$ for $k\neq\ell$. Hence the only eigenstates of the form of Eq. (F4) that have a nonzero overlap with $|\zeta_j\rangle$ are those of the form

$$|\lambda;\rightarrow^{\alpha}\leftarrow\rightarrow^{\beta}\rangle=\sum_{j_1<\cdots<j_{\alpha+\beta+1}}C^{\lambda}_{j_1,\ldots,j_{\alpha+\beta+1}}|\widetilde{\rightrightarrows}_{j_1}\cdots\widetilde{\rightrightarrows}_{j_\alpha}\widetilde{\leftleftarrows}_{j_{\alpha+1}}\widetilde{\rightrightarrows}_{j_{\alpha+2}}\cdots\widetilde{\rightrightarrows}_{j_{\alpha+\beta+1}}\rangle,\tag{F6}$$

where the sum is over $\{j_\ell\}$ for $1\le\ell\le\alpha+\beta+1$ and "τ^k" in the pattern label for $\tau\in\{\rightarrow,\leftarrow\}$ denotes that τ is repeated k times. That is, in an m-spin pattern, there is precisely one $\leftarrow$ the position of which is parametrized by α,β with $\alpha+1+\beta=m$. This is a total of $\sum_{m=1}^{L}m\binom{L}{m}=L\times2^{L-1}$ eigenstates that can have a nonzero overlap with the $|\zeta_j\rangle$. We find that the overlap is generically nonzero for all such states and that such eigenstates are not necessarily only in the very low-energy part of the spectrum, e.g., in the one-magnon band. Using Eqs. (F5) and (F6), the weight of $|\zeta_j\rangle$ on the eigenstate $|\lambda;\rightarrow^{\alpha}\leftarrow\rightarrow^{\beta}\rangle$ is given by [for the definition, see Eq. (51)]

$$\frac{1}{3^L}|\langle\zeta_j|\lambda;\rightarrow^{\alpha}\leftarrow\rightarrow^{\beta}\rangle|^2=\frac{2^{\alpha+\beta+1}}{3^L}\left|\sum_{j_1<\cdots<j_{\alpha+\beta+1}}C^{\lambda}_{j_1,\ldots,j_{\alpha+\beta+1}}\delta_{j_{\alpha+1},j}\right|^2.\tag{F7}$$

We are unable to use Eq. (F7) to proceed analytically without any approximations. However, given the eigenstates of the Heisenberg model numerically, it is easy to use this expression to numerically compute the weights. We have employed this method to compute the cumulative weight $\Omega_{S^z_j}(E)$ shown in Fig. 2.

2. Spin-wave contribution

We apply the results of Sec. F 1 to compute the overlap of $|S^z_j)$ on the spin-wave excited states of $\widehat{\mathcal{P}}_{t-J_z|\text{comp}}$. In the notation of Eq. (F3), the spin-wave excited states for the OBC Heisenberg model, shown in Eq. (A6), read

$$|\lambda_k;m\rangle:=\frac{1}{\sqrt{\mathcal{M}_{m,k}}}\sum_{j_1<\cdots<j_m}\sum_{\ell=1}^{m}c_{j_\ell,k}|\downarrow_{j_1}\downarrow_{j_2}\cdots\downarrow_{j_m}\rangle\quad\Longrightarrow\quad C^{\lambda_k}_{j_1,\ldots,j_m}=\frac{1}{\sqrt{\mathcal{M}_{m,k}}}\times\sum_{\ell=1}^{m}c_{j_\ell,k},\tag{F8}$$

where $k\in(\pi n/L)$ for $1\le n\le L-1$, and $c_{j_\ell,k}$ and $\mathcal{M}_{m,k}$ are shown in Eqs. (A8) and (A9), respectively. The corresponding eigenstates of $\widehat{\mathcal{P}}_{t-J_z|\text{comp}}$ are of the form $|\lambda_k;\tau^{(1)}\cdots\tau^{(m)}\rangle$, which can be explicitly written down using Eq. (F4).

To compute the weight in Eq. (F7), we first compute

$$
\begin{aligned}
G^{\alpha,\beta}_{j,k} &:= \sum_{j_1<\cdots<j_{\alpha+\beta+1}} c^{\lambda_k}_{j_1,\ldots,j_{\alpha+\beta+1}}\delta_{j_{\alpha+1},j} \\
&= \frac{1}{\sqrt{\mathcal{M}_{\alpha+\beta+1,k}}}\sum_{1\le j_1<\cdots<j_\alpha\le j-1}\left[\sum_{j+1\le j_{\alpha+2}<\cdots<j_{\alpha+\beta+1}\le L}\left(\sum_{\ell=1}^{\alpha}c_{j_\ell,k}+c_{j,k}+\sum_{\ell=\alpha+2}^{\alpha+\beta+1}c_{j_\ell,k}\right)\right] \\
&= \frac{1}{\sqrt{\mathcal{M}_{\alpha+\beta+1,k}}}\left[F^{1,j-1}_{\alpha,k}\binom{L-j}{\beta}+c_{j,k}\binom{j-1}{\alpha}\binom{L-j}{\beta}+\binom{j-1}{\alpha}F^{j+1,L}_{\beta,k}\right],
\end{aligned}
\tag{F9}
$$

where we have defined

$$
F^{l,r}_{m,k} := \sum_{l\le j_1<\cdots<j_m\le r}\left(\sum_{\ell=1}^{m}c_{j_\ell,k}\right) = \binom{r-l}{m-1}\sum_{j_\ell=l}^{r}c_{j_\ell,k}. \tag{F10}
$$

Using Eqs. (F9) and (F10) and the OBC expression for $c_{j,k}$ in Eq. (A8), we obtain, for $k \neq 0$,

$$
\begin{aligned}
F^{l,r}_{m,k} &= \sqrt{\frac{2}{L}}\times\frac{\sin(kr)-\sin(k(l-1))}{2\sin(\frac{k}{2})}\times\binom{r-l}{m-1}, \\
G^{\alpha,\beta}_{j,k} &= \frac{\sqrt{2}\binom{j-1}{\alpha}\binom{L-j}{\beta}}{\sqrt{L\times\mathcal{M}_{\alpha+\beta+1,k}}}\times\left[\frac{\alpha}{2(j-1)}\frac{\sin[k(j-1)]}{\sin(\frac{k}{2})}+\cos\left[k(j-\frac{1}{2})\right]+\frac{\beta}{2(L-j)}\frac{\sin(kL)-\sin(kj)}{\sin(\frac{k}{2})}\right],
\end{aligned}
\tag{F11}
$$

while for $k = 0$ we have

$$
F^{l,r}_{m,k=0} = \frac{\sqrt{2}}{\sqrt{L}}(r-l+1)\times\binom{r-l}{m-1}, \quad G^{\alpha,\beta}_{j,k=0} = \frac{\sqrt{2}\binom{j-1}{\alpha}\binom{L-j}{\beta}}{\sqrt{L\times\mathcal{M}_{\alpha+\beta+1,k=0}}}\times[\alpha+1+\beta]. \tag{F12}
$$

For simplicity, we henceforth assume that L is odd and that $j = ((L+1)/2)$. We then set $k = (n\pi/L)$ and compute the total weight of the state $|\zeta_{((L+1)/2)}\rangle$ on all eigenstates of the form $|\lambda_k; \rightarrow^\alpha \leftarrow\!\!\rightarrow^\beta\rangle$ for all values of α and β. Using Eq. (F7), this weight is

$$
\begin{aligned}
W_{S^z_{\frac{L+1}{2}}}\left(k=\frac{n\pi}{L}\right) &:= \frac{1}{3^L}\sum_{\alpha=0}^{\frac{L-1}{2}}\sum_{\beta=0}^{\frac{L-1}{2}}2^{\alpha+\beta+1}|G^{\alpha,\beta}_{\frac{L+1}{2},k}|^2 \\
&= \sum_{\alpha=0}^{\frac{L-1}{2}}\sum_{\beta=0}^{\frac{L-1}{2}}\frac{2^{\alpha+\beta+1}\binom{\frac{L-1}{2}}{\alpha}^2\binom{\frac{L-1}{2}}{\beta}^2}{3^L\times\binom{L}{\alpha+\beta+1}}\times
\begin{cases}
1, & \text{if } n=0,\\
\frac{2(L-1-\alpha-\beta)}{(L-1)(\alpha+\beta+1)}, & \text{if } n\neq 0 \text{ even},\\
\frac{2(\alpha-\beta)^2}{(L-1)(L-\alpha-\beta-1)(\alpha+\beta+1)}\cot^2(\frac{k}{2}), & \text{if } n \text{ odd},
\end{cases}
\end{aligned}
\tag{F13}
$$

where we have used Eq. (F11) and the normalization factors of Eq. (A9). Note that $W_{S^z_{((L+1)/2)}}(k=0)$ is the same as the Mazur bound computed in Ref. [25], done here in the composite-spin language.

The expression in Eq. (F13) for general k can be analyzed in detail using a saddle-point analysis for large L, similar to the calculation for the Mazur bound demonstrated in Ref. [25], but for our purposes it is sufficient to schematically extract the L dependence. To obtain this, we substitute $\alpha = Lp$ and $\beta = Lq$ to convert the sums into integrals over p and q. In Appendix G of Ref. [25], the expression for the Mazur bound $W_{S^z_{((L+1)/2)}}(k=0)$ has been shown to be of the form (the computation has been done there for a general $x = j/L$, whereas here we will set $x = 1/2$ and remove it from the arguments of the functions involved)

$$W_{S^z_{\frac{L+1}{2}}}(k=0)=\sqrt{L}\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}dp\;dq\;C(p,q)\exp\left(LF(p,q)\right)\approx\frac{2\pi\;C(p_0,q_0)}{\sqrt{L\;\det H(p_0,q_0)}}\exp\left(LF(p_0,q_0)\right), \tag{F14}$$

where $C(p,q)$ and $F(p,q)$ are some L-independent functions, and a saddle-point approximation has been performed in the second step, which we unpack below. $H(p,q)$ is the Hessian of $F(p,q)$ and the "saddle" is given by the point at which $((\partial F/\partial p),(\partial F/\partial q))|_{(p,q)=(p_0,q_0)}=(0,0)$, which turns out to be at $(p_0,q_0)=(\frac{1}{3},\frac{1}{3})$. It also turns out that $F(p_0,q_0)=0$; hence we obtain the Mazur-bound scaling of approximately $L^{-\frac{1}{2}}$ [25]. Note that both $H(p,q)$ and (p_0,q_0) are completely determined by $F(p,q)$.

Since the $n\neq 0$ expressions in Eq. (F13) differ from the $n=0$ case only by factors that are polynomial in L, we can express them in terms of p and q and, analogously, write down the form of the leading-order terms in the saddle-point approximation:

$$\begin{aligned}W_{S^z_{\frac{L+1}{2}}}\left(k=\frac{n\pi}{L}\right)&\approx\frac{2\pi\;C(p_0,q_0)}{\sqrt{L\;\det H(p_0,q_0)}}\exp\left(LF(p_0,q_0)\right)\\&\times\begin{cases}\frac{2(1-p_0-q_0)}{L(p_0+q_0)}, & \text{if } n\neq 0 \text{ even},\\[2ex] \frac{2(p_0-q_0)^2}{L(1-p_0-q_0)(p_0+q_0)}\cot^2(\frac{k}{2}), & \text{if } n \text{ odd},\end{cases}\end{aligned} \tag{F15}$$

where the saddle (p_0,q_0) is unchanged, since the function $F(p,q)$ in the exponent is unchanged from the $n=0$ case.

We can then use Eq. (F15) with the fact that $F(p_0,q_0)=0$ to determine the scaling of $W_{S^z_{((L+1)/2)}}(k)$. For $n\neq 0$ even, we obtain a scaling of approximately $L^{-\frac{3}{2}}$ and adding the contributions over all even n, we obtain a total scaling of approximately $L^{-\frac{1}{2}}$. For n odd, since $p_0=q_0$, the leading-order term shown in Eq. (F15) vanishes. Since the subleading terms in the saddle-point approximation are suppressed by a factor of L, we obtain a scaling of approximately $L^{-\frac{5}{2}}\cot^2(k/2)$. Adding these contributions over all odd $n=2l+1$, we obtain approximately $L^{-\frac{5}{2}}\sum_{l=0}^{((L-1)/2)}\cot^2[(l+\frac{1}{2})(\pi/L)]\sim L^{-\frac{1}{2}}$, since it is dominated by a few values of k close to 0, where $\cot^2(k/2)\sim k^{-2}$. In all, the total weight of the operator $S^z_{((L+1)/2)}$ on the ground states and the "single-magnon" spin-wave states scales as approximately $L^{-\frac{1}{2}}$, which is vanishing in the thermodynamic limit. This completes the demonstration that the contribution from these excitations to $\Omega_{S^z_j}(E)$ of Eq. (62) vanishes when $L\to\infty$ and more complicated excitations need to be considered to understand its form, which at present we have only done numerically.

APPENDIX G: ASYMPTOTIC QMBSs IN BROWNIAN CIRCUITS

In this appendix, we discuss asymptotic QMBSs in Brownian circuits with exact QMBSs. We start with the assumption that there is a set of local projectors $\{R_{[j]}\}$ such that the common kernel of these projectors is spanned by the exact QMBS $\{|\Phi_n\rangle\}$; this is sometimes referred to as the Shiraishi-Mori condition [26,31]. In other words, the subspace spanned by these QMBS states can be expressed as the exhaustive ground-state space of a frustration-free Hamiltonian, i.e.,

$$\sum_j R_{[j]}|\psi\rangle=0\quad\Longleftrightarrow\quad|\psi\rangle\in\mathcal{S}=\mathrm{span}\{|\Phi_n\rangle\}. \tag{G1}$$

Several examples of QMBSs, including those in the spin-1 XY model [80], the Hubbard model and its deformations [81,82], and also those in the spin-1 AKLT model [27] can be understood in this way [19,20,31]. With this, we can generically write down a bond algebra $\mathcal{A}_{\mathrm{scar}}$ of the form of Eq. (42), which has generators of the form $\{R_{[j]}\sigma^\alpha_k\}$, such that its centralizer is $\mathcal{C}_{\mathrm{scar}}$ spanned by $\{|\Phi_m\rangle\langle\Phi_n|\}$ (for a precise statement, see Ref. [85]). This structure guarantees that the QMBSs are degenerate eigenstates of all operators constructed out of the generators of $\mathcal{A}_{\mathrm{scar}}$.

Here, we consider Brownian circuits built out of the generators of $\mathcal{A}_{\mathrm{scar}}$, and directly work with the evolution of states under this circuit, as opposed to operators discussed in Sec. IV A. Exact QMBSs are stationary states under such circuits, since they are by definition eigenstates of each "gate" of the circuit. Here, we show that working with states directly also shows the existence of asymptotic QMBSs that relax slowly in such circuits. Denoting the generators of the bond algebra to be $\{\widehat{H}_\alpha\}$, the time evolution of a state $|\psi(t)\rangle$ by a time step Δt can be written, in direct analogy to Eq. (43), as

$$|\psi(t+\Delta t)\rangle = e^{-i\sum_\alpha J_\alpha^{(t)}\widehat{H}_\alpha \Delta t}|\psi(t)\rangle = |\psi(t)\rangle - i\Delta t\sum_\alpha J_\alpha^{(t)}\widehat{H}_\alpha|\psi(t)\rangle - \frac{(\Delta t)^2}{2}\sum_{\alpha,\beta}J_\alpha^{(t)}J_\beta^{(t)}\widehat{H}_\alpha\widehat{H}_\beta|\psi(t)\rangle + \mathcal{O}((\Delta t)^3). \tag{G2}$$

After ensemble averaging with the distributions of Eq. (44), the expression for the state in the continuum time limit reads, in analogy to Eq. (46),

$$\overline{|\psi(t)\rangle} = e^{-\kappa\widehat{P}t}|\psi(0)\rangle, \quad \Longrightarrow \quad \overline{\langle\psi(0)|\psi(t)\rangle} = \langle\psi(0)|e^{-\kappa\widehat{P}t}|\psi(0)\rangle, \quad \widehat{P} := \sum_\alpha \widehat{H}_\alpha^2. \tag{G3}$$

As a consequence, the decay of the ensemble-averaged overlap is governed by the spectrum of $\widehat{P}$. While we are unable to directly compute the ensemble-averaged fidelity using this approach, it can be lower bounded in terms of the overlap as

$$\overline{\mathcal{F}(t)} = \overline{|\langle\psi(0)|\psi(t)\rangle|^2} \geq \left|\overline{\langle\psi(0)|\psi(t)\rangle}\right|^2. \tag{G4}$$

We now restrict to specific examples of QMBSs, where bond generators are of the form of Eq. (31). Then, we have

$$\widehat{P} = \sum_{j,k,\alpha}(R_{[j]}\sigma_k^\alpha)^2 = C\sum_j R_{[j]}, \tag{G5}$$

where C is an overall constant that depends on the number of α and k in the generators (assumed to be the same for each j for simplicity) and we have used the fact that $R_{[j]}$ is a projector. Strikingly, one can see that this is precisely the frustration-free Hamiltonian that has appeared in Eq. (G1). This already shows that exact QMBSs never decay, since they are ground states of $\widehat{P}$. The slowly relaxing states, or the asymptotic QMBSs, are then the low-energy excitations of $\widehat{P}$, provided that it is gapless. In the case of the ferromagnetic tower of QMBSs discussed in Sec. III C 2, we have $R_{[j]} = \frac{1}{4} - \vec{S}_j\cdot\vec{S}_{j+1}$; hence $\widehat{P}$ is just the ferromagnetic Heisenberg model of Eq. (A2) up to an overall factor. The asymptotic QMBSs $\{|\Phi_{n,k}\rangle\}$ are then simply spin waves on top of the ferromagnet shown in Eq. (A6); this explains their form in Eq. (64). Since these states with small k have energy approximately $p_k \sim k^2$ under $\widehat{P}$, their ensemble-averaged overlap $\overline{\langle\Phi_{n,k}(0)|\Phi_{n,k}(t)\rangle}$ decays on time scales of approximately L^2, which due to Eq. (G4) is also a lower bound for the time scale for the fidelity, consistent with Eq. (69). This method hence more directly reproduces the asymptotic QMBSs found from the super-Hamiltonian perspective in Sec. V B 3 and explains the significance of the corresponding super-Hamiltonian eigenstates, i.e., those associated with spin waves on only one leg of the ladder that appeared there.

The appearance of the frustration-free Hamiltonian of Eq. (G1) in Eq. (G5), and the physical interpretation of its eigenstates as the decay modes of the overlap in Eq. (G3), leads us to the Conjecture 1 on the conditions for the existence of asymptotic QMBS, restated here for clarity.

Conjecture 2. Consider a space $\mathcal{S} = \text{span}\{|\Phi_n\rangle\}$ that can be expressed as the exhaustive common kernel of a set of strictly local projectors. Any local Hamiltonian that realizes this subspace as the exact QMBS subspace also has asymptotic QMBSs if $\mathcal{S}$ cannot be expressed as the ground-state space of a gapped frustration-free Hamiltonian. Furthermore, the gapless excitations of any such Hamiltonian are the asymptotic QMBSs.

One might also note that the form of the decay of the overlap of the asymptotic QMBSs obtained using Eq. (G3) is necessarily a simple exponential of the form $\exp(-ct/L^2)$, where the asymptotic QMBS is an eigenstate of $\widehat{P}$ with eigenvalue approximately $k^2 \sim c/L^2$. Since the ensemble-averaged fidelity is lower bounded by this [see Eq. (68)], this predicts a fidelity-decay time scale that scales as approximately L^2. This is different from the fidelity-decay time scale of asymptotic QMBSs predicted and observed in Hamiltonian systems in Ref. [55]. The fidelity of an initial state under Hamiltonian evolution at short times is of the form $\exp(-\Delta H^2 t^2)$ [158], where ΔH^2 is the variance of the energy in the initial state, $\Delta H^2 \equiv |\psi_0\rangle H^2\langle\psi_0| - (|\psi_0\rangle H\langle\psi_0|)^2$. Given that the variance of asymptotic QMBS scales as $\Delta H^2 \sim 1/L^2$ [55], the fidelity decay is of the form $\exp(-c' t^2/L^2)$, which predicts a decay time scale approximately L. The fidelity in the Brownian circuits hence decays parametrically slower than in the Hamiltonian evolution. This is reminiscent of the quantum Zeno effect, where unitary evolution is suppressed by external factors such as repeated measurements or fast-fluctuating stochasticity. It would be interesting to make this connection precise in future work, while here we can give a rough argument showing the reconcilability of these results, which also sheds some light on the quantitative relations between the Brownian-circuit and Hamiltonian dynamics.

Note that the derivation of the ensemble-averaged state dynamics from Eqs. (G2)–(G3) has formally required taking $\Delta t \to 0$ limit while taking the variance of the couplings $J_\alpha^{(t)}$ to diverge as $\sigma_J^2 = 2\kappa/\Delta t$ at fixed κ. We can, in fact, also use the obtained results in the circuit setups, where σ_J^2 is kept fixed while we take Δt sufficiently small—which, however, then enters the characteristic rate κ in all results: $\kappa = \sigma_J^2 \Delta t/2$. This already shows that if the applied Hamiltonians have typical couplings of a given strength of approximately σ_J and hence typical dynamic rates of approximately σ_J, changing the Hamiltonian randomly after every small time interval Δt suppresses the dynamic rates to approximately $\kappa \sim \sigma_J \cdot \sigma_J \Delta t$, assuming that $\sigma_J \Delta t \ll 1$. For an initial state $|\psi(0)\rangle$ that is an eigenstate of $\widehat{P}$ of Eq. (G3) with a bounded eigenvalue p: $\widehat{P}\,|\psi(0)\rangle = p\,|\psi(0)\rangle$, the ensemble-averaged Eq. (G2) gives

$$\overline{|\psi(\Delta t)\rangle} \approx \left(1 - \frac{(\Delta t)^2}{2}\sigma_J^2 p\right)|\psi(0)\rangle \quad\Longrightarrow\quad \overline{|\psi(t)\rangle} \approx \left(1 - \frac{(\Delta t)^2}{2}\sigma_J^2 p\right)^{\frac{t}{\Delta t}}|\psi(0)\rangle \approx e^{-\frac{1}{2}\sigma_J^2(\Delta t)pt}\,|\psi(0)\rangle\,, \tag{G6}$$

which matches the result in Eq. (G3) with $\kappa = \frac{1}{2}\sigma_J^2 \Delta t$ as claimed, and the approximations used are controlled as long as $(((\Delta t)^2)/2)\sigma_J^2 p \ll 1$.

This is also roughly consistent with the expected fidelity decay under a fixed Hamiltonian, controlled by its variance ΔH^2 in the state [158], which we expect to apply for individual evolution steps over time Δt:

$$|\langle\psi(0)|\psi(\Delta t)\rangle|^2 = e^{-\Delta H^2(\Delta t)^2}, \quad \Delta H^2 := \langle\psi(0)|\,H^2\,|\psi(0)\rangle - \langle\psi(0)|\,H\,|\psi(0)\rangle^2\,. \tag{G7}$$

With $H = \sum_\alpha J_\alpha^{(t)} H_\alpha$, an elementary calculation averaging over the independent Gaussian $J_\alpha^{(t)}$ with variance σ_J^2 gives $\overline{|\psi_0\rangle\,H^2\,\langle\psi_0|} = \sigma_J^2\,|\psi_0\rangle\,\widehat{P}\,\langle\psi_0| = \sigma_J^2 p$ if $\widehat{P}\,|\psi_0\rangle = p\,|\psi_0\rangle$. This upper bounds the ensemble-averaged variance of the energy, $\overline{\Delta H^2}$, and can be viewed as a reasonable estimate of a typical value of the variance of the energy in $|\psi_0\rangle$ for Hamiltonians drawn from this distribution. We can then recover the qualitative form of Eq. (G6) if we conjecture that the result of the multiple time steps of the Brownian circuit is to have roughly the same fidelity-suppression factor for each step Δt: in this case, the total suppression factor is $e^{-\Delta H_{\mathrm{typ}}^2(\Delta t)^2\cdot(t/\Delta t)}$ qualitatively matching Eq. (G6).

While this is not a precise quantitative argument, we can justify using the same suppression factor after each time step Δt as follows. At any given instance of the Brownian circuit, we have

$$|\psi(\Delta t)\rangle = |\psi(0)\rangle\,\langle\psi(0)|\psi(\Delta t)\rangle + |\delta\psi^\perp\rangle, \tag{G8}$$

where the first term is the projection onto $|\psi(0)\rangle$, while $|\delta\psi^\perp\rangle$ is the deviation. While the asymptotic QMBS property of $|\psi(0)\rangle$ is common for all $H^{(t)}$ (i.e., is essentially nonrandom across them), the deviation $|\delta\psi^\perp\rangle$ is particular to the Hamiltonian applied at that step, which is hence random across different steps, and it is plausible that the chances of the $|\delta\psi^\perp\rangle$ part returning close to $|\psi(0)\rangle$ under the subsequent steps are small. For the purposes of calculating the fidelity, we are only interested in keeping track of the $|\psi(0)\rangle$ projection and we obtain a similar suppression factor at each step.

Finally, note that the presentation here has focused on the asymptotic QMBSs appearing due to exact towers of QMBSs, where the $p \sim 1/L^2$ scaling of the super-Hamiltonian energy of an initial state implies the divergence of its fidelity decay time. Mathematically, the same arguments also go through in the case of exact isolated QMBSs such as those in Sec. V A 2, where the super-Hamiltonian energy of any initial state $|\psi(0)\rangle$ orthogonal to the exact QMBS is at least a constant, implying a constant fidelity decay time. Nevertheless, this still signifies some hidden "nonthermalness" in the "thermal" sector that occurs even due to a single exact scar state [159,160], although it is less dramatic than in the asymptotic QMBSs. It would be interesting to understand whether this framework can be used to quantify such nonthermalness in more detail.

APPENDIX H: SUPER-HAMILTONIANS FOR EQUALLY SPACED TOWER OF QMBSs

We now discuss different choices of super-Hamiltonians suitable for analyzing cases with a tower of QMBSs and we show that the physics of asymptotic QMBSs is captured in this setup. As discussed in Ref. [31], we can describe such a tower of exact ferromagnetic scar states split in energy by the Zeeman-field term using the algebras

$$\mathcal{A}_{\text{tower-lift}} := \langle\langle\{R_{j,j+1}\sigma^{\alpha}_{j+2}, \sigma^{\alpha}_{j-1}R_{j,j+1}\}, \sum_j \sigma^z_j\rangle\rangle, \quad \mathcal{C}_{\text{tower-lift}} = \langle\langle\{|\Phi_{n,0}\rangle\langle\Phi_{n,0}|\}\rangle\rangle. \tag{H1}$$

This seems to necessarily require the addition of an extensive-local term $Z_{\text{tot}} := \sum_j \sigma^z_j$ to the generators of $\mathcal{A}_{\text{tower}}$. In order to capture the commutant $\mathcal{C}_{\text{tower-lift}}$ as the ground states of a super-Hamiltonian, we need to sacrifice either locality or Hermiticity of the super-Hamiltonian and we discuss both options below.

1. Nonlocal super-Hamiltonian

We can naively follow the discussion in Sec. II B and construct a super-Hamiltonian using Eq. (5). While the resulting super-Hamiltonian $\widehat{\mathcal{P}}_{\text{tower-lift}}$ is Hermitian, it becomes nonlocal, with the addition of the term $[\sum_j (\sigma^z_{j;t} - \sigma^z_{j;b})]^2$. This term changes the energy of the ground states $|\Phi_{m,0}\rangle \otimes |\Phi_{n,0}\rangle$ of $\widehat{\mathcal{P}}_{\text{tower}}$ from 0 to $(m-n)^2$. Hence the ground states of the new super-Hamiltonian $\widehat{\mathcal{P}}_{\text{tower-lift}}$ with this nonlocal term now require $m = n$ and are hence $\{|\Phi_{n,0}\rangle_t \otimes |\Phi_{n,0}\rangle_b\}$, which shows that the above algebras are indeed centralizers of each other. The operators $|\Phi_{n,k}\rangle\langle\Phi_{n,0}|$ are still exact low-energy eigenstates of $\widehat{\mathcal{P}}_{\text{tower-lift}}$ with eigenvalue $p_k = 8[1-\cos(k)]$, which allows us to understand the asymptotic QMBSs. Equation (69) holds for $\widehat{A} = |\Phi_{n,k}\rangle\langle\Phi_{n,0}|$, since $|\Phi_{n,0}\rangle$ is still a singlet of $\mathcal{A}_{\text{tower-lift}}$, which rigorously lower bounds the fidelity decay time in the case of nondegenerate towers as well.

Note that this quantity is not straightforward to bound from the direct consideration of the dynamics of states discussed in Appendix G. The effective Hamiltonian $\widehat{P}$ shown in Eq. (G3) acquires an additional term $(\sum_j \sigma^z_j)^2$, which then shows that

$$\overline{\langle\Phi_{n,k}(0)|\Phi_{n,k}(t)\rangle} = e^{-\kappa[p_k+(L-2n)^2]t}, \tag{H2}$$

which decays rapidly when $n \neq (L/2)$, i.e., when the eigenvalue $Z_{\text{tot}} = L - 2n \neq 0$. This is also the case for the overlap $\overline{\langle\Phi_{n,0}(0)|\Phi_{n,0}(t)\rangle}$, even though $|\Phi_{n,0}\rangle$ is an exact QMBS, and it is an effect of averaging over the random phases acquired by the action of Z_{tot}. While these are mathematically correct properties of the Brownian circuit with a random fluctuating Zeeman field, they are not useful for understanding the fidelity properties of the asymptotic QMBS.

2. Non-Hermitian super-Hamiltonian

We now discuss an alternative super-Hamiltonian for the algebra of Eq. (H1) that preserves locality but sacrifices Hermiticity. This naturally appears in a Brownian circuit where the coefficient of the magnetic field is constant and *not* random. We can then redo the analysis of Eqs. (43)–(46) to derive an effective Hamiltonian that describes the ensemble-averaged operator dynamics. Given a Brownian circuit evolving under a set of operators $\{\widehat{H}_\alpha\}$ with random coefficients $\{J^{(t)}_\alpha\}$ chosen from the distribution of Eq. (44), and an operator $\widehat{G}$ with a constant $\mathcal{O}(1)$ coefficient K, analogous to Eq. (45), we obtain

$$\overline{|\widehat{O}(t+\Delta t))} = \overline{|\widehat{O}(t))} + i\Delta t K\widehat{\mathcal{L}}_{\widehat{G}}\overline{|\widehat{O}(t))} - \Delta t\sum_\alpha \kappa_\alpha \widehat{\mathcal{L}}^\dagger_\alpha\widehat{\mathcal{L}}_\alpha\overline{|\widehat{O}(t))} + \mathcal{O}((\Delta t)^2), \tag{H3}$$

where $\widehat{\mathcal{L}}_{\widehat{G}} := \widehat{G}_t \otimes \mathbb{1}_b - \mathbb{1}_t \otimes \widehat{G}_b$ is the Liouvillian corresponding to $\widehat{G}$ (assumed to be Hermitian and real valued in the working basis for simplicity). In the continuous time limit, we then obtain

$$\frac{d}{dt}\overline{|\widehat{O}(t))} = -\left[\sum_\alpha \kappa_\alpha\widehat{\mathcal{L}}^\dagger_\alpha\widehat{\mathcal{L}}_\alpha - iK\widehat{\mathcal{L}}_{\widehat{G}}\right]\overline{|\widehat{O}(t))}, \implies \overline{|\widehat{O}(t))} = e^{-(\kappa\widehat{\mathcal{P}} - iK\widehat{\mathcal{L}}_{\widehat{G}})t}|\widehat{O}(0)).$$

Hence the physics of this system can be understood using the non-Hermitian super-Hamiltonian $\widehat{\mathcal{P}}_{\text{n-h}} = \kappa\widehat{\mathcal{P}} - iK\widehat{\mathcal{L}}_{\widehat{G}}$. Operators in the commutant $\mathcal{C}_{\text{ext-loc}}$ of the algebra $\mathcal{A}_{\text{ext-loc}} = \langle\langle\{\widehat{H}_\alpha\}, \widehat{G}\rangle\rangle$ are guaranteed to have zero eigenvalue under $\widehat{\mathcal{P}}_{\text{n-h}}$. Moreover, any zero-eigenvalued operator of $\widehat{\mathcal{P}}_{\text{n-h}}$ is in the commutant $\mathcal{C}_{\text{ext-loc}}$. A simple proof is as follows. Starting from $\widehat{\mathcal{P}}_{\text{n-h}}|\Psi) = 0$, we have $(\Psi|\widehat{\mathcal{P}}_{\text{n-h}}|\Psi) = \kappa(\Psi|\widehat{\mathcal{P}}|\Psi) - iK(\Psi|\widehat{\mathcal{L}}_{\widehat{G}}|\Psi) = 0$, which, since $\widehat{\mathcal{P}}$ and $\widehat{\mathcal{L}}_G$ are Hermitian, means that $(\Psi|\widehat{\mathcal{P}}|\Psi) = 0$ and $(\Psi|\widehat{\mathcal{L}}_G|\Psi) = 0$. Since $\widehat{\mathcal{P}}$ is positive semidefinite, we can conclude that $\widehat{\mathcal{P}}|\Psi) = 0$, which also means that $\widehat{\mathcal{L}}_G|\Psi) = 0$, showing that $|\Psi)$ is in the commutant $\mathcal{C}_{\text{ext-loc}}$.

Applying this to asymptotic QMBSs, where the $\{\widehat{H}_\alpha\}$ are the generators of $\mathcal{A}_{\text{tower}}$ of Eq. (42) and $\widehat{G} = \sum_j \sigma_j^z$, we have $\widehat{\mathcal{L}}_{\widehat{G}} = \sum_j (\sigma_{j;t}^z - \sigma_{j;b}^z)$. Since $\widehat{\mathcal{P}}$ has the form of the dissipator of a Lindblad master equation [see Eq. (8)] with jump operators $\{\widehat{H}_\alpha\}$, the full $\widehat{\mathcal{P}}_{\text{n-h}}$ has the form of a full Lindbladian with the Hamiltonian $\widehat{G}$ and the jump operators $\{\widehat{H}_\alpha\}$. Hence the eigenvalues of the $\widehat{\mathcal{P}}_{\text{n-h}}$ are guaranteed to have non-negative real parts. It is easy to verify that the eigenstates $|\Phi_{n,k}\rangle_t \otimes |\Phi_{m,0}\rangle_b$ of $\widehat{\mathcal{P}}$, discussed in Eq. (66), continue to be eigenstates of $\widehat{\mathcal{P}}_{\text{n-h}}$, and we have

$$\widehat{\mathcal{P}}_{\text{n-h}}||\Phi_{n,k}\rangle\langle\Phi_{m,0}|) = [\kappa p_k + 2iK(n-m)]||\Phi_{n,k}\rangle\langle\Phi_{m,0}|), \tag{H4}$$

where $p_k = 8[1-\cos(k)]$ and we have used that $|\Phi_{n,0}\rangle$ and $|\Phi_{n,k}\rangle$ are eigenstates of Z_{tot} with eigenvalue $L - 2n$. We can then follow the same arguments as in Sec. V B 3 to obtain exact results—lower bounds—for the ensemble-averaged fidelity, by relating it to the autocorrelation function of an operator $\widehat{A} = |\Phi_{n,k}\rangle\langle\Phi_{m,0}|$, which leads to Eq. (69), consistent with the expected slow decay of asymptotic QMBSs.

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variance approximately $1/L^2$. Asymptotic QMBSs have been explicitly constructed in Ref. [55] in a particular class of spin-1 *XY*-like models hosting an exact QMBS tower qualitatively similar to the ferromagnetic multiplet, and the same arguments and trial states hold when the QMBS states are actually the frustration-free ground states rather than in the middle of the spectrum. These trial states then provide upper bounds for the nearby excitation energies dispersing as approximately k^2. Alternatively, the trial states can also be constructed using standard tools such as the Feynman-Bijl ansatz on top of the ferromagnetic multiplet [62].

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